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FACULTAD DE INGENIERÍA

**Dynamical analysis, computation of
Invariant Manifolds, and low-energy
transfer design in the Sun-Earth system
using solar sail propulsion**

TESIS

Que para obtener el título de
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P R E S E N T A

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Abstract

A complete numerical routine is presented for the computation of heteroclinic connections in the planar circular restricted three-body problem (CR3BP) with a solar sail parameterized by the sail lightness number β , based on [1] and extending the methods of [2]. The critical Jacobi constant $C_J^*(\beta)$, as a function of β , is identified, below which transfer maneuvers between a Lyapunov orbit around SL_2 and a Lyapunov orbit around SL_1 can be realized with theoretical zero- Δv using heteroclinic connections. Results are obtained for several values of β in the Sun-Earth system, with the orbit amplitude defined by reference to the orbit of the James Webb Space Telescope. The results provide numerical evidence of the existence of these heteroclinic connections and document an increasing x -amplitude disparity between the two connected orbits as β grows, reaching a ratio $\Delta x_{\gamma_2}^*/\Delta x_{\gamma_1}^* = 11.78$ at $\beta = 0.01$, suggesting that the set of Lyapunov orbits admissible for heteroclinic connection appears to shrink toward an empty set at some specific value of β . The flight times of these theoretical zero- Δv maneuvers are of the order of 470–505 days.

Se presenta una rutina numérica completa para el cálculo de conexiones heteroclínicas en el caso plano del problema restringido circular de tres cuerpos (CR3BP) con una vela solar parametrizada por el número de ligereza β , basada en [1] y extendiendo la metodología de [2]. Se identifica el valor crítico de la constante de Jacobi $C_J^*(\beta)$, en función de β , por debajo del cual las maniobras de transferencia entre una órbita de Lyapunov alrededor de SL_2 y otra alrededor de SL_1 pueden realizarse con costo teóricamente nulo ($\Delta v = 0$) a través de las conexiones heteroclínicas. Se obtienen resultados para varios valores de β en el sistema Sol-Tierra, con la amplitud de órbita definida tomando como referencia la órbita del telescopio espacial James Webb. Los resultados muestran numéricamente la existencia de estas

conexiones heteroclínicas y muestran una creciente diferencia de la amplitud en x entre las dos órbitas conectadas al aumentar β , alcanzando una relación $\Delta x_{\gamma_2}^*/\Delta x_{\gamma_1}^* = 11.78$ en $\beta = 0.01$, lo que sugiere que el conjunto de órbitas de Lyapunov admisibles para la conexión heteroclínica se reduce hasta volverse vacío en cierto valor de β . Los tiempos de vuelo de estas maniobras teóricas de $\Delta v = 0$ son del orden de 470–505 días.

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Introduction

Modern space mission design is increasingly based on exploiting the natural dynamics of multi-body gravitational environments in order to reduce propellant consumption (Δv) and extend mission lifetimes. While the well-known two-body (Keplerian) model, together with perturbative methods, is well suited for missions in low, medium, or geostationary Earth orbit, it becomes inadequate for missions operating in dynamically richer environments such as the cislunar region or the vicinity of the libration points of the Sun–Earth and Earth–Moon systems. In these regimes, the circular restricted three-body problem (CR3BP) provides a more accurate dynamical framework that captures a better approximation of the motion and its invariant structures, including the existence of equilibrium points, families of periodic and quasi-periodic orbits, and the invariant manifold structures that organize transport in phase space.

The CR3BP has been studied intensively since the foundational works of Euler, Lagrange, and Poincaré. In particular, the collinear libration points L_1 and L_2 have been of great interest for space missions over the last few years. Although unstable in the Lyapunov sense, they give rise to families of periodic orbits (Lyapunov, Halo, and vertical orbits) and quasi-periodic orbits (Lissajous and quasi-Halo orbits) whose stable and unstable invariant manifolds form tubular structures that act as separatrices of the flow. The geometric foundations of this framework were laid by Conley [3] and McGehee [4], who characterized the hyperbolic structure near the collinear equilibria and identified the role of invariant manifold tubes as conduits for transit orbits between the regions of the primary bodies. These ideas were further developed into a systematic and highly precise computational framework by the Barcelona school — most notably by Simó [5, 6] and by Gómez, Jorba, Masdemont, and Simó [7, 8] — who computed families of halo and Lyapunov orbits together with their associated stable and unstable manifolds, estab-

lishing the geometric foundation that supports modern low-energy trajectory design. Based on these works, Koon, Lo, Marsden, and Ross [9] demonstrated numerically the existence of heteroclinic connections between Lyapunov orbits around L_1 and L_2 : trajectories that are asymptotic to one periodic orbit in backward time and to another in forward time. Linking these connection orbits leads to dynamical chains that build the path for low-energy transfers between the regions of the primary bodies. Canalias and Masdemont [10] worked on classifying these connections, and their geometric properties were studied in detail by Gidea and Masdemont [11] using symbolic dynamics.

More recently, Calleja et al. [12] developed boundary value problem formulations for the systematic computation of invariant manifolds and connecting orbits in the CR3BP, providing a robust numerical framework applicable to a wide class of Hamiltonian systems. The natural extension to the spatial CR3BP reveals a substantially richer invariant structure; the work of Delshams, de la Llave, and Seara [13] provides the theoretical framework to analyze more general heteroclinic phenomena that emerge in the three-dimensional setting, and how chaotic transport theory can be used to build theoretical zero- Δv trajectories.

Solar radiation pressure is a significant force acting on small bodies in the solar system, as well as gravity. Solar sails exploit this force by using a large, highly reflective membrane to transform photon momentum into thrust without propellant. This opens the possibility of missions that are unfeasible with conventional propulsion systems. When the effect of the solar sail is incorporated into the CR3BP, the equilibrium points are displaced, and the dynamics of the problem is modified. For the particular flat case, perfectly reflecting sail oriented perpendicular to the Sun–sail line, the system remains Hamiltonian, with a reduction of the Sun’s gravitational parameter by a factor $(1 - \beta)$, where β is the sail lightness number. Solar sails dynamics, in the restricted three-body problem, have been studied by McInnes [14] and, from a dynamical systems perspective, by Farrés [1], who computed families of artificial equilibria, analyzed their stability, and studied the dynamics on the center manifold. Mission concepts and station keeping strategies around L_1 , L_2 and even the triangular equilibrium points have been studied by Farrés [1, 15]. More recently, Heiligers [16] explored homoclinic and heteroclinic connections for solar sail periodic orbits in the Earth–Moon system, though in a non-autonomous setting where classical tools such as the Jacobi constant do not directly apply. The objective of this thesis is to study the existence and detection of heteroclinic connections between Lyapunov orbits around

the displaced equilibrium points SL_1 and SL_2 of the Sun–Earth CR3BP with a solar sail, in the planar Hamiltonian case. The main contributions are:

- The computation of families of Lyapunov periodic orbits around SL_1 and SL_2 for different values of the sail lightness number β , using a shooting method and numerical continuation.
- The numerical detection of heteroclinic connections between these orbits via Poincaré sections above the Earth position, extending the methodology of Koon et al. (2000) to the solar sail case.
- The identification of a critical value $C_J^*(\beta)$ of the Jacobi constant at which the stable and unstable manifolds first become tangent, denoting the energy threshold from which heteroclinic transfers begin at the first return map.

The work is organized as follows. Chapter 1 presents the theoretical framework: Hamiltonian mechanics, the CR3BP and its invariant structure, and the development of the solar sail model. Chapter 2 develops the Hamiltonian case of the solar sail, deriving the modified equations of motion, the displaced equilibrium points SL_1 , SL_2 , and the associated saddle $\times$ center linearization. Chapter 3 describes the numerical methods for computing and continuing Lyapunov periodic orbits and their invariant manifolds using Floquet theory. Chapter 4 addresses the detection of heteroclinic connections through Poincaré sections and identifies the critical value $C_J^*(\beta)$. Chapter 5 presents the results and discusses their implications for low-energy trajectory design.

Chapter 1

Theoretical Framework

This chapter presents the theoretical framework to analyze solar sail trajectories in the CR3BP. The starting point is classical mechanics; in particular, Lagrangian and Hamiltonian formulations will be used, which allow us to appreciate the dynamical structures that arise naturally in the system. Classical mechanics provides tools to model the behavior of mechanical systems. For historical reasons, the motion of celestial bodies has been a basis of its development, giving rise to celestial mechanics. The N-body problem has been studied intensively by renowned mathematicians, who have provided us with powerful analytical tools. After Kepler’s solution of the two-body problem, the next model of interest was the three-body problem, for which no general analytical solution is known to date. In particular, Euler’s attempt to describe lunar dynamics led him to propose a simplification known as the restricted three-body problem. Although this model did not fully resolve the lunar problem, it is today a fundamental tool in the design of modern space missions. A notable example is the James Webb Space Telescope [17], which orbits the L_2 equilibrium point of the Sun–Earth system — a solution that emerges naturally from the restricted three-body problem [18]. Finally, the solar sail is introduced as a propulsion system within the CR3BP. Unlike conventional propulsion, solar radiation pressure continuously modifies the dynamical structure of the problem, displacing the equilibrium points and modifying the associated invariant manifolds [1, 14]. This opens the possibility of designing low-energy trajectories around displaced orbits that do not exist in the classical problem.

1.1 Classical mechanics

Although classical mechanics can be formulated from Newton's laws, for the purposes of this work, it is convenient to adopt a more general formulation. Classical mechanics can be described from a variational point of view by introducing a set of generalized coordinates $q = (q_1, q_2, \dots, q_n)$ that describe the configuration of a system with n degrees of freedom.

The above formulation is the so-called Lagrangian formulation, that can be derived from the principle of least action, or Hamilton's principle. Hamilton's principle characterizes the dynamics of a system by a differentiable manifold (configuration space) and a function on its tangent bundle called the Lagrangian [19, 20] function defined as:

$$L = L(q, \dot{q}, t) = K(q, \dot{q}) - U(q, t), \quad (1.1)$$

where K and U represent the generalized kinetic and potential energy of the system, respectively. Hamilton's principle states that the evolution of the system is such that the action $S = \int_{t_0}^{t_1} L(q, \dot{q}) dt$ is stationary, i.e., $\delta S = 0$. This leads to the Euler–Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, n \quad (1.2)$$

a system of n second-order differential equations from which the equations of motion of any mechanical system can be derived given its kinetic and potential energy.

An alternative, and particularly useful, way to describe a mechanical system is in terms of the generalized coordinates and conjugate momenta, defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$. This description is called the Hamiltonian formulation and turns out to be very useful in celestial mechanics.

The Hamiltonian of the system is obtained through the Legendre transformation:

$$H = \left(\sum_{i=1}^n p_i \dot{q}_i \right) - L \quad (1.3)$$

Under this formalism, the equations of motion take the form of a system of $2n$ first-order differential equations:

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{cases} \quad i = 1, 2, \dots, n, \quad (1.4)$$

known as Hamilton's equations. When the Hamiltonian does not depend explicitly on time, it becomes a conserved quantity of the system.

If we denote $z = (q, p)$ as the generalized coordinates and conjugate momenta of the system, the space of all states z is called the phase space of the system [19]. Hamilton's equations can then be written as:

$$\dot{z} = J\nabla H(z) \quad (1.5)$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the canonical symplectic matrix. The advantage of studying a dynamical system under this formalism lies not only in the reduction of the order of the equations, but also in the possibility of treating the problem from a geometric point of view [19]. Hamilton's equations allow to describe the dynamics as a flow on a symplectic manifold $(\mathcal{M}, \Omega)$, where $\mathcal{M}$ is a differentiable manifold and Ω is a non-degenerate 2-form that is preserved under the flow, as established by Liouville's theorem. This geometric structure leads to the identification and classification of invariant objects such as equilibrium points, periodic orbits, quasi-periodic tori, and stable and unstable manifolds, which qualitatively describe the evolution of the system and, in a certain sense, organize its dynamics. This theoretical framework is especially well-suited for celestial mechanics, where gravitational models naturally lead to Hamiltonian systems with integrals of motion, symmetries, and invariant structures. In particular, the restricted three-body problem can be formulated as a Hamiltonian system in a synodic or rotating frame, which will allow us to analyze its structure of equilibrium points, periodic orbits, and invariant manifolds — all fundamental for the design of space trajectories.

1.2 Celestial mechanics and the N-body problem

Celestial mechanics studies the motion and behavior of bodies under gravitational interaction. Historically, this discipline has been developed by math-

ematicians and physicists such as Kepler, Newton, Euler, Lagrange, and Poincaré, and its fundamental problems have motivated the development of a large part of modern mathematics — from the calculus of variations to the theory of dynamical systems, passing through the works of Kolmogorov, Arnold, and Moser on the stability of Hamiltonian systems. At the same time, the tools developed in this context are now fundamental for the design of space missions.

In particular, the N-body problem is concerned with the motion of n mass particles attracting each other according to Newton's law of gravitation [21]. By Newton's second law, the motion of the i -th body in a system of N bodies is described by the following equation:

$$\vec{F}_i = \sum_{\substack{j=1 \\ j \neq i}}^N G \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij} \quad (1.6)$$

where G is the universal gravitational constant and $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$ is the relative position vector between bodies i and j .

Equivalently, the equations of motion can be written in terms of the gradient of the potential energy of the system [21]:

$$m_i \ddot{\vec{r}}_i = \nabla_i U = \frac{\partial U}{\partial \vec{r}_i} \quad (1.7)$$

The potential energy of the N-body system is:

$$U = \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{G m_i m_j}{\|\vec{r}_{ij}\|} \quad (1.8)$$

A fundamental case is the two-body problem, which when expressed in relative coordinates [22, 23] reduces to the motion of a particle under a central potential:

$$F = -\frac{\mu}{\|\vec{r}\|^3} \vec{r} \quad (1.9)$$

where $\mu = G(m_1 + m_2)$.

Equation (1.9) is completely integrable; Kepler also prove that the integration of equation (1.9) falls to the conic section. Elliptic and circular orbits

are bounded trajectories, while parabolic and hyperbolic are not. The integrability of the two-body problem is due to the existence of sufficiently many integrals of motion that reduce the dynamics to an effective one-dimensional equation. However, when considering systems with $N > 2$ bodies, the trajectories no longer correspond in general to simple analytical solutions. Although global quantities such as the center of mass, total linear momentum, total angular momentum, and total energy are conserved, these integrals are not sufficient to guarantee the integrability of the system.

The natural case study in this non-integrable regime is the three-body problem, which describes the motion of three point masses m_1, m_2, m_3 interacting under the influence of a gravitational field. Equations (1.7) for $N = 3$ give:

$$m_i \ddot{\vec{r}}_i = \sum_{j \neq i} G \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij}, \quad i = 1, 2, 3. \quad (1.10)$$

Equation (1.10) constitutes a system of 18 nonlinear first-order ordinary differential equations. Poincaré proved that the dynamics in the three-body problem becomes extremely complicated compared with the two-body problem.

Poincaré’s work marked a turning point in classical mechanics. L. Euler introduced the restricted three-body problem, a model of two primary massive bodies that orbit around their common center of mass in a Keplerian way, and a third body with negligible mass that is influenced by the two primaries but does not affect those bodies. In studying the restricted three-body problem, Poincaré introduced geometric and topological tools — such as the sections that today bear his name and the concept of homoclinic connection — which laid the foundations of the qualitative theory of differential equations and the modern theory of dynamical systems. It is precisely this qualitative approach, based on the identification of invariant structures in phase space, that will be adopted in this work for trajectory design.

Nevertheless, the three-body problem does admit particular exact solutions. Euler found solutions in which the three bodies remain collinear throughout their motion, while Lagrange proved the existence of equilateral triangle configurations that are also preserved in time [22, 23]. These two types of solutions give rise to the five equilibrium points of the restricted problem [2, 24], which will be central to the development of the following chapters.

1.3 Circular restricted three-body problem

In the circular restricted three-body problem (CR3BP) [2, 24], two primary bodies of masses m_1 and m_2 are considered to move in circular orbits around their center of mass under mutual gravitational interaction, while a third body of negligible mass moves under the gravitational influence of the primaries without affecting their motion.

The dimensionless mass parameter is defined as:

$$\mu = \frac{m_2}{m_1 + m_2} \quad (1.11)$$

where by convention $m_2 < m_1$, so that $\mu \in [0, \frac{1}{2})$. A normalized unit system is chosen such that the distance between the primaries equals one, the orbital period of the primaries is 2π , and the gravitational constant is normalized ($G = 1$). In this system, the masses of the primaries are rewritten as:

$$\mu_1 = Gm_1 = m_1 = 1 - \mu, \quad \mu_2 = Gm_2 = m_2 = \mu.$$

Following [24], let (X, Y, Z) be the coordinates of an inertial sidereal reference frame with the center of mass at the origin, with X along the line joining the primaries m_1, m_2 at $t = 0$. The coordinates (X, Y, Z) denote the position of the third body in this reference frame, and (X_1, Y_1, Z_1) , (X_2, Y_2, Z_2) denote the coordinates of the primaries respectively.

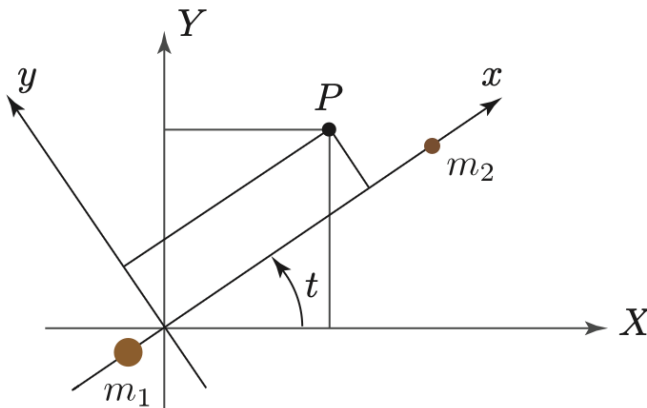


Figure 1.1: CR3BP in sidereal frame of reference. Reproduced from [2], Fig. 2.3.1.

The dynamics of the primaries is known since they are part of the circular Kepler problem, while the equations of motion of the third body are:

$$\begin{aligned}\ddot{X} &= \mu_1 \frac{(X_1 - X)}{r_1^3} + \mu_2 \frac{(X_2 - X)}{r_2^3} \\ \ddot{Y} &= \mu_1 \frac{(Y_1 - Y)}{r_1^3} + \mu_2 \frac{(Y_2 - Y)}{r_2^3} \\ \ddot{Z} &= \mu_1 \frac{(Z_1 - Z)}{r_1^3} + \mu_2 \frac{(Z_2 - Z)}{r_2^3}\end{aligned}\tag{1.12}$$

with

$$r_1^2 = (X_1 - X)^2 + (Y_1 - Y)^2 + (Z_1 - Z)^2, \quad r_2^2 = (X_2 - X)^2 + (Y_2 - Y)^2 + (Z_2 - Z)^2.\tag{1.13}$$

The natural choice of reference frame for the CR3BP is the synodic (or rotating) frame, which rotates with the same angular velocity ω as the primaries. In this frame, the primaries remain fixed on the x -axis at positions:

$$\vec{r}_1 = (-\mu, 0, 0), \quad \vec{r}_2 = (1 - \mu, 0, 0).\tag{1.14}$$

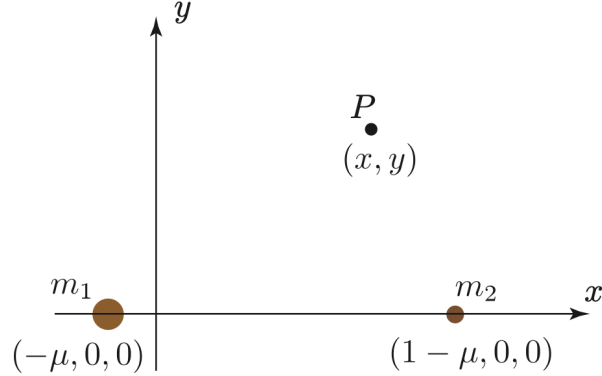


Figure 1.2: CR3BP in synodic frame of reference. Reproduced from [2], Fig. 2.3.2.

If (x, y, z) denotes the coordinates of the third body in the synodic frame, the distances to each primary are:

$$r_1^2 = (x + \mu)^2 + y^2 + z^2, \quad r_2^2 = (x - (1 - \mu))^2 + y^2 + z^2. \quad (1.15)$$

The relationship between the sidereal coordinates (X, Y, Z) and those of the synodic frame (x, y, z) is given by a rotation at angular velocity ω :

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (1.16)$$

Differentiating (1.16) twice with respect to time, we obtain:

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2\omega\dot{y} - \omega^2 x \\ \ddot{y} + 2\omega\dot{x} - \omega^2 y \\ \ddot{z} \end{pmatrix} \quad (1.17)$$

The terms $\omega\dot{x}, \omega\dot{y}$ are known as Coriolis accelerations, and $\omega^2 x, \omega^2 y$ as centrifugal accelerations.

Substituting (1.16) and (1.17) into (1.12) with $\omega = 1$, the equations of motion of the third body in the synodic reference frame take the form:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= x + \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} &= y + \frac{\partial U}{\partial y} \\ \ddot{z} &= \frac{\partial U}{\partial z}, \end{aligned} \quad (1.18)$$

where $U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$ is the potential energy of the system. These equations can be rewritten in compact form by introducing the effective potential:

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + U, \quad (1.19)$$

equations of motion are expressed as:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} \end{aligned} \quad (1.20)$$

As discussed in Section 1.1, Hamiltonian formalism allows us to study the dynamics from a geometric point of view, identifying invariant structures in phase space, in particular periodic orbits and their associated stable and unstable manifolds, which will be computed and analyzed in Chapters 3 and 4. It is therefore natural to reformulate the CR3BP in this language. Starting from coordinates in the rotating frame, the kinetic energy of the system is:

$$T = \frac{1}{2}((\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2).$$

In this way, we can write the Lagrangian of the system with no time dependence:

$$L = \frac{1}{2}((\dot{x} - y)^2 + (\dot{y} + x)^2 + \dot{z}^2) - U(x, y, z). \quad (1.21)$$

Applying the Euler–Lagrange equations one recovers system (1.17). The conjugate momenta are:

$$p_x = \frac{\partial L}{\partial \dot{q}_1} = \dot{x} - y, \quad p_y = \frac{\partial L}{\partial \dot{q}_2} = \dot{y} + x, \quad p_z = \dot{z},$$

and the Hamiltonian of the system is:

$$H = \left(\sum_{i=1}^n p_i \dot{q}_i \right) - L = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + (p_x y - p_y x) - U, \quad (1.22)$$

which is a conserved quantity of the system. From (1.22), Hamilton's equations can be derived:

$$\begin{aligned} \dot{x} &= p_x + y \\ \dot{y} &= p_y - x \\ \dot{z} &= p_z \\ \dot{p}_x &= p_y + \frac{\partial H}{\partial x} \\ \dot{p}_y &= -p_x + \frac{\partial H}{\partial y} \\ \dot{p}_z &= \frac{\partial H}{\partial z} \end{aligned} \quad (1.23)$$

It is important to note that the conjugate momenta in the rotating frame do not coincide with the velocities. This is a relevant distinction when interpreting the phase space of the problem. In the context of astrodynamics, the conserved quantity used is:

$$C_J = -2H = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad (1.24)$$

called the Jacobi Constant. The value C_J remains constant along any trajectory of the system and constitutes the only integral of motion of the CR3BP. The Jacobi constant imposes geometric constraints on the motion of the third body. If we fix a value of C_J and consider all points (x, y, z) at which the velocity magnitude of the third body vanishes ($\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 0$), we obtain the so-called zero-velocity curves, defined by:

$$C_J = 2\Omega(x, y, z).$$

These curves bound the regions of configuration space accessible to the third body at a given energy level, meaning that motion is confined to the regions where $C_J \leq 2\Omega$. The forbidden regions are known as Hill regions. As the energy of the third body increases (i.e., C_J decreases), the forbidden regions shrink, and eventually the regions around the collinear equilibrium points open up, allowing transit between the vicinity of one primary and the other. Understanding the existence and geometry of these regions is fundamental for the design of low-energy transfer trajectories.

The critical points of the effective potential, defined by $\nabla\Omega = 0$, correspond to the equilibrium points of the system, denoted $L_1, L_2, \dots, L_5$ [2, 24]. Three of them, L_1 , L_2 and L_3 , lie on the line joining the primaries (collinear points) and were discovered by Euler. The other two, L_4 and L_5 , form equilateral triangles with the primaries and correspond to the solutions found by Lagrange.

The dynamics in the vicinity of the collinear points L_1 and L_2 is particularly relevant to this work. These points are unstable in the Lyapunov sense, meaning that an arbitrarily small perturbation can drive the third body away from the equilibrium. However, this instability — far from being an obstacle — can be exploited by means of the stable and unstable manifolds associated with the periodic orbits that exist around these points. It is by identifying these manifolds that low-energy transfer trajectories can be designed between the different regions on the phase space, as will be developed in the following chapters.

1.4 Solar sail model

A solar sail is a space propulsion system consisting of a very thin membrane with a large surface area and high reflectivity, which uses the momentum transferred by photons from solar radiation to generate thrust [1, 14]. Unlike conventional propulsion systems that depend on a limited supply of fuel, a solar sail produces continuous acceleration as long as it is exposed to solar radiation. Although the magnitude of this acceleration is small compared to that of conventional thrusters, it allows the design of missions that would not be feasible with these traditional propulsion systems. A case of particular interest is the possibility of displacing the equilibrium points of the CR3BP by incorporating the solar sail force into the model. The Geostorm Warning Mission [1, 14] is an example of an application of this concept, where the proposal is to place a solar sail at an artificial equilibrium point displaced toward the Sun relative to the classical L_1 point of the Sun–Earth system, which would allow to reduce the warning time for solar storms compared to a conventional observatory at L_1 , such as SOHO. Other mission proposals includes the displacement of the equilibrium points above the ecliptic plane and counteracting the Earth’s gravity in order to place a satellite over one of the Earth’s pole [25].

A solar sail generates thrust from the momentum transferred by solar radiation photons. Following the work of McInnes and Farrés [1, 14], the pressure P exerted on the sail surface is expressed as the energy flux divided by the speed of light:

$$P = \frac{W}{c}, \quad (1.25)$$

where W is the energy flux (energy per unit area per unit time) at a distance r from the Sun. This flux decreases with the square of the distance:

$$W = W_E \left(\frac{R_E}{r} \right)^2, \quad (1.26)$$

where R_E is the Sun–Earth distance (1 AU), and $W_E = \frac{L_s}{(4\pi R_E^2)}$ is the energy flux measured at one astronomical unit, with L_s the luminosity of the Sun. The accepted mean value is $W_E = 1368 \frac{J}{m^2 s}$, giving a solar radiation pressure at 1 AU of $P \approx 4.56 \times 10^{-6} \frac{N}{m^2}$. Considering a perfectly reflecting sail, the observed pressure can double this value, as the photons are totally

reflected. When the sail is not perfectly reflecting, a reflectivity coefficient $\eta < 1$ is introduced. Moreover, the angle of incident photons will be the same as the angle of reflected photons.

Since a solar sail is an orientable surface, the acceleration it experiences depends on its area and orientation. The latter is defined by a normal vector to the sail surface ($\vec{n}$).

For a flat, perfectly reflecting sail, the total force from incident and reflected photons is:

$$\vec{F}_{sail} = 2PA\langle\vec{u}_i, \vec{n}\rangle^2\vec{n}, \quad (1.27)$$

where A is the sail area and $\langle\vec{u}_i, \vec{n}\rangle$ is the dot product between the direction of incident photons and the sail normal.

Combining with the energy flux expression (1.23), the total sail force can be written as:

$$\vec{F}_{sail} = \frac{2AW_ER_E^2}{cr^2}\langle\vec{u}_i, \vec{n}\rangle^2\vec{n}. \quad (1.28)$$

For heliocentric orbits, the direction of the incident photons coincides with the radial vector from the Sun to the sail ($\vec{r}$), and the sail acceleration can be expressed in terms of the Sun's gravitational acceleration:

$$\vec{a}_{sail} = \beta \frac{Gm_s}{r^2}\langle\hat{r}, \vec{n}\rangle^2\vec{n}, \quad (1.29)$$

where m_s is the Sun's mass, $\hat{r}$ is the unit vector from the Sun to the sail, and β is a dimensionless parameter known as the sail lightness number.

The parameter β is defined as the ratio of the acceleration due to solar radiation pressure to the gravitational acceleration of the Sun. Both quantities decrease with the inverse square of the distance, this fact makes the β parameter dependent only on the physical properties of the sail. We can relate β with W_E in the following way:

$$\beta = \frac{\sigma^*}{\sigma}, \quad \text{where } \sigma^* = \frac{L_s}{2\pi GM_s c} \approx 1.53 \frac{g}{m^2}. \quad (1.30)$$

Here, $\sigma = \frac{m}{A}$ denotes the sail loading parameter, defined as the total spacecraft mass per unit sail area. The performance of the solar sail is thus completely parameterized by β (or equivalently σ).

When $\beta = 1$, the sail force exactly compensates the Sun's gravitational attraction. For $\beta > 1$, radiation pressure would dominate over solar gravity. With current technology, achievable values of $\beta \approx 0.05$, are considered, corresponding to a sail loading of $\sigma \approx 30 \frac{g}{m^2}$. As an example, this would require a square sail of approximately $75m \times 75m$ for a spacecraft of $170kg$ total mass.

1.5 Solar sail in the CR3BP equations of motion

To describe the motion of a solar sail in the Sun–Earth system, the sail acceleration is incorporated into the CR3BP. Using the normalized synodic coordinate system from Section 1.3, and considering a flat, perfectly reflecting sail, the acceleration due to the sail is:

$$\vec{a}_{sail} = \beta \frac{1 - \mu}{r_{PS}^2} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}, \quad (1.31)$$

where r_{PS} is the distance from the Sun to the sail, $\vec{r}_s$ is the Sun–sail direction. The equations of motion of the CR3BP with solar sail are:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Omega}{\partial x} - \beta \frac{1 - \mu}{r_{PS}^2} \langle \vec{r}_s, \vec{n} \rangle^2 n_x \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Omega}{\partial y} - \beta \frac{1 - \mu}{r_{PS}^2} \langle \vec{r}_s, \vec{n} \rangle^2 n_y \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} - \beta \frac{1 - \mu}{r_{PS}^2} \langle \vec{r}_s, \vec{n} \rangle^2 n_z, \end{aligned} \quad (1.32)$$

where Ω is the effective potential of the classical CR3BP defined in (1.19).

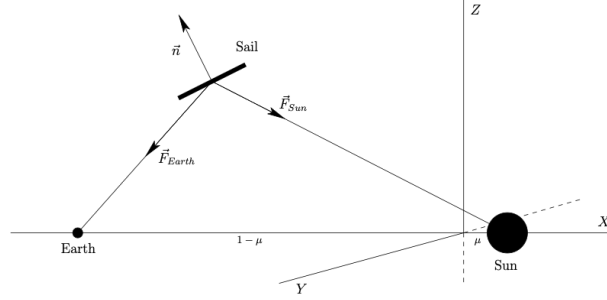


Figure 1.3: Solar sail in synodic frame of reference. Reproduced from [1], Fig. 1.3.

Note that if $\beta = 0$, or if the sail orientation is perpendicular to the Sun direction ($\vec{r}_s \perp \vec{n}$), the sail effect vanishes and the classical CR3BP equations (1.18) are recovered.

The orientation of the solar sail determines the magnitude and direction of the thrust. This orientation is defined by the normal vector to the sail surface $\vec{n}$, and it can be parameterized by two angles α and δ :

- α is the in-plane angle between the Sun–sail line and the normal vector $\vec{n}$, both projected onto the orbital (ecliptic) plane.

- δ is the difference between the angle between the Sun–sail line and the ecliptic plane and the angle between the normal vector $\vec{n}$ and the ecliptic plane.

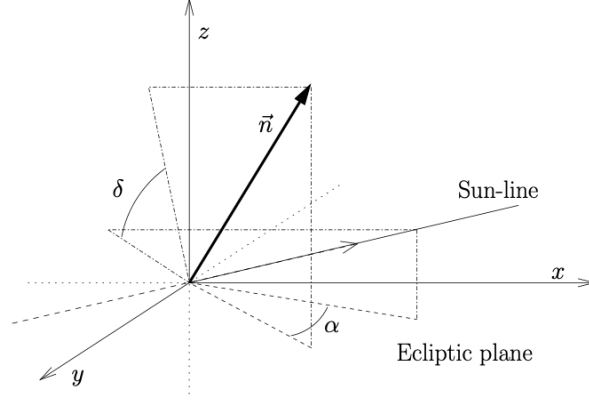


Figure 1.4: Angle representation of the solar sail orientation in the CR3BP. Reproduced from [1], Fig. 1.2.

In spherical coordinates relative to the Sun–sail direction r_s , the components of the normal vector are:

$$n_x = \cos(\phi + \alpha) \cos(\psi + \delta), \quad n_y = \sin(\phi + \alpha) \cos(\psi + \delta), \quad n_z = \sin(\psi + \delta), \quad (1.33)$$

where

$$\phi(x, y) = \arctan\left(\frac{y}{x - \mu}\right) \quad \text{and} \quad \psi(x, y, z) = \arctan\left(\frac{z}{\sqrt{(x - \mu)^2 + y^2}}\right),$$

are the angles that define the Sun–sail direction $r_s = [x - \mu, y, z]/r_{PS}$ in spherical coordinates. It can be shown that the squared dot product between the direction of incident photons and the normal vector:

$$\langle \vec{r}_s, \vec{n} \rangle^2 = \left[\frac{(r_2^2 \cos \alpha + z^2) \cos \delta - z r_2 (1 - \cos \alpha) \sin \delta}{r_{ps}} \right]^2,$$

with $r_2 = \sqrt{(x - \mu)^2 + y^2}$.

Note that even with a fixed orientation, the sail force term varies with the position of the sail.

The condition $\langle \vec{r}_s, \vec{n} \rangle \geq 0$ (the normal vector cannot point toward the Sun) imposes constraints on the admissible values of α and δ . If $\delta = \pm \frac{\pi}{2}$,

then $\vec{r}_s \perp \vec{n}$ and we recover the classical CR3BP. If $\alpha = \delta = 0$, the sail is perpendicular to the Sun–sail line and the force reaches its maximum in the radial direction.

In general, the solar sail force cannot be expressed as the gradient of a potential function, so system (1.32) is non-Hamiltonian. However, the two particular cases mentioned above preserve the Hamiltonian structure: when the sail is perpendicular to the Sun–sail line ($\alpha = \delta = 0$) and when it is parallel to it ($\delta = \pm \frac{\pi}{2}$). The first case is of greatest practical interest, since the sail force acts in the radial direction opposite to the Sun’s gravity. The detailed analysis of this Hamiltonian case and its dynamical consequences — the displacement of the equilibrium points, the modification of the Jacobi constant, and the zero-velocity curves — is the subject of the following chapter.

Chapter 2

Development and Analysis of the Hamiltonian Case of the Solar Sail

This chapter focuses on the case in which the solar sail maintains an orientation perpendicular to the Sun–sail line at all times, i.e., $\alpha = \delta = 0$. Recall that the sail force also depends on the position of the sail. Considering that we are going to work with periodic orbits, the Sun-sail direction changes as the spacecraft moves along the orbit. This means that the solar sail still requires active attitude control in order to keep the sail normally aligned with the Sun-sail line as the spacecraft moves. As shown at the end of the previous chapter, this is one of the particular cases in which the system preserves the Hamiltonian structure, allowing the dynamical-systems tools developed in Section 1.1 to be applied directly.

Although a sail permanently oriented toward the Sun is not the most general case from a practical standpoint, it is useful as a first approach for the dynamical analysis. Being Hamiltonian, this model allows the invariant structures of the system to be identified rigorously, a prerequisite for the study of more general cases — which can be treated as perturbations of the Hamiltonian case.

The scope of this work is limited to this fully Hamiltonian case. In what follows, the equations of motion for the planar case are derived, the displaced equilibrium points and their dependence on β are computed, the Jacobi constant and the zero-velocity curves of the modified system are analyzed, and the system is linearized around the collinear equilibrium points to character-

ize their stability.

The case $\alpha = 0$ allows us to rewrite the equations of motion as

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial \tilde{\Omega}}{\partial x} - \beta \frac{(1-\mu)}{r_1^3} \frac{(x-\mu)z}{r_2} \cos^2 \delta \sin \delta \\ \ddot{y} + 2\dot{x} &= \frac{\partial \tilde{\Omega}}{\partial y} - \beta \frac{(1-\mu)}{r_1^3} \frac{(y)z}{r_2} \cos^2 \delta \sin \delta \\ \ddot{z} &= \frac{\partial \tilde{\Omega}}{\partial z} - \beta \frac{(1-\mu)}{r_1^3} r_2 \cos^2 \delta \sin \delta,\end{aligned}\tag{2.1}$$

where $\tilde{\Omega}(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)(1-\beta \cos^3 \delta)}{r_1} + \frac{\mu}{r_2}$.

Once we have (2.1), it becomes clear that the two particular cases in which the dynamical system becomes Hamiltonian are when $\delta = 0$ or $\delta = \pm \frac{\pi}{2}$. Physically, these correspond to the solar sail oriented perpendicular and parallel, respectively, to the Sun–Earth line of sight.

2.1 Equations of motion of the model with the sail perpendicular to the Sun-sail line

The case of interest is $\alpha = 0$, $\delta = 0$, this is when the sail is perpendicular to the Sun–sail line and $\langle \hat{r}_s, \hat{n} \rangle = 1$ for all time. The sail force acts in the radial direction opposite to solar gravity, which is equivalent to reducing the solar gravitational attraction by a factor of $(1 - \beta)$. The equations of motion for this case are:

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial \tilde{\Omega}}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial \tilde{\Omega}}{\partial y} \\ \ddot{z} &= \frac{\partial \tilde{\Omega}}{\partial z},\end{aligned}\tag{2.2}$$

with the effective potential expressed as:

$$\tilde{\Omega}(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)(1-\beta)}{r_1} + \frac{\mu}{r_2}.\tag{2.3}$$

Note the similarity with the classical CR3BP equations (1.20), the only difference is the factor $(1 - \beta)$ in the solar gravitational term. When $\beta = 0$, the classical CR3BP is recovered.

For the purposes of this work, the model is restricted to the planar case, i.e., $z = 0$. In this way:

$$\begin{aligned}\ddot{x} - 2\dot{y} &= \frac{\partial \tilde{\Omega}}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial \tilde{\Omega}}{\partial y},\end{aligned}\tag{2.4}$$

where $\tilde{\Omega}(x, y)$ has the form (2.3) and:

$$r_1^2 = (x + \mu)^2 + y^2, \quad r_2^2 = (x - 1 + \mu)^2 + y^2.\tag{2.5}$$

Introducing the change of variable $\dot{x} = v_x$ and $\dot{y} = v_y$, (2.4) can be rewritten as a first-order system with state vector $X = (x, y, v_x, v_y)$:

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{X} &= f(X), \quad \dot{v}_x = 2v_y + \frac{\partial \tilde{\Omega}}{\partial x} \\ \dot{v}_y &= -2v_x + \frac{\partial \tilde{\Omega}}{\partial y},\end{aligned}\tag{2.6}$$

2.2 Equilibrium points

The system of differential equations described in (2.6) represents a non-linear dynamical system, from which, as we already mentioned, non-general solutions are known. A fundamental step towards understanding the dynamics is finding the equilibrium points of the system. To find the equilibrium points of a mechanical system, all velocities and accelerations are required to vanish, i.e. $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$. If we denote the system of equations describing the model as:

$$\dot{X} = f(X)\tag{2.7}$$

an equilibrium point X^* satisfies

$$f(X^*) = 0, \quad (2.8)$$

The physical meaning of the equilibrium points in this context implies that a solar sail placed at this position with $|v| = 0$, will remain in that position in the absence of external perturbations. The system under analysis has 5 equilibrium points that lie very close to the Lagrange points of the CR3BP, with the exception that they are displaced by the effect of the solar sail. In this work, the displaced equilibrium points are called SL_i for $i \in \{1, 2, 3, 4, 5\}$. In particular, the points known as SL_1 and SL_2 are of great interest due to their applicability in space missions.

These equilibrium points belong to the group of collinear equilibrium points, which satisfy the condition $y = 0$ in (2.8), leading to:

$$\left. \frac{\partial \tilde{\Omega}}{\partial x} \right|_{y=0} = x + \frac{(1-\mu)(1-\beta)(x+\mu)}{r_1^3} + \frac{\mu(x-1+\mu)}{r_2^3} = 0. \quad (2.9)$$

One can note that this is a case analogous to the classical CR3BP; with the difference being the presence of the factor $(1-\beta)$ that displaces the position of these points. We denote by SL_1 and SL_2 the displaced equilibrium points corresponding to L_1 and L_2 of the classical CR3BP.

To solve (2.9), we rewrite the equation following the method described by Szebehely [26]. For the case of SL_1 , it is known in advance that the point lies between the Sun at $x_{\text{Sun}} = -\mu$ and the Earth at $x_{\text{Earth}} = 1 - \mu$, defining the following change of variables:

$$r'_1 = x_1 + \mu, \quad r'_2 = (1 - \mu) - x_1. \quad (2.10)$$

$$\gamma_1 = 1 - \mu - x_1 = r'_2, \quad r'_1 = 1 - \gamma_1. \quad (2.11)$$

Here, γ_1 represents the distance between SL_1 and the secondary body (the Earth). Equation (2.9) can be rewritten as:

$$1 - \mu - \gamma_1 + \frac{(1-\mu)(1-\beta)(x_1 + \mu)}{r_1'^3} + \frac{\mu(x_1 - 1 + \mu)}{r_2'^3} = 0. \quad (2.12)$$

Substituting the change of variables into (2.12), a fifth-degree polynomial in γ_1 is obtained:

$$\gamma_1^5 + \gamma_1^4(\mu - 3) + \gamma_1^3(3 - 2\mu) + \gamma_1^2(\beta\mu - \beta - \mu) + 2\mu\gamma_1 - \mu = 0 \quad (2.13)$$

This polynomial can be solved numerically via an iterative scheme:

$$\gamma_{1[n+1]} = \left[\frac{\mu(\gamma_{1[n]} - 1)^2 + \beta\gamma_{1[n]}^2(1 - \mu)}{3 - 2\mu + \gamma_{1[n]}(\gamma_{1[n]} + \mu - 3)} \right]^{1/3}, \quad (2.14)$$

or directly using Newton's method. An analogous procedure, with the substitution $\gamma_2 = x_2 - 1 + \mu$, allows the position of SL_2 to be computed:

$$\gamma_{2[n+1]} = \left[\frac{\mu(\gamma_{2[n]} + 1)^2 + \beta\gamma_{2[n]}^2(\mu - 1)}{3 - 2\mu + \gamma_{2[n]}(\gamma_{2[n]} - \mu + 3)} \right]^{1/3}, \quad (2.15)$$

Table 2.1 shows the position of SL_1 and SL_2 for different values of β and a fixed $\mu = 3.0034806 \times 10^{-6}$ that correspond to the mass parameter of the Sun-Earth system. It is observed that as β increases, both points shift toward the Sun, as expected since the effective solar attraction is reduced.

β	$x\text{-}SL_1$	$x\text{-}SL_2$
0.0	0.9900265938647256	1.0100341164283044
1.0×10^{-4}	0.9900153963152573	1.0100230917600819
5.0×10^{-4}	0.9899703506303805	1.0099792315608427
1.0×10^{-3}	0.9899134642089551	1.009924939094473
5.0×10^{-3}	0.9894343238056129	1.0095111688268406
0.01	0.9887710828741909	1.0090418188063162
0.05	0.9804352316019486	1.0065942063417972
0.1	0.9646842211706248	1.0051234952570731

Table 2.1: The x component of SL_1 and SL_2 w.r.t β

Figures 2.1 and 2.2 illustrate the data from Table 2.1, showing how both SL_1 and SL_2 shift toward the Sun as β increases. The displacement is monotone and becomes more pronounced for larger values of β , reflecting the progressive reduction of the effective solar gravitational attraction.

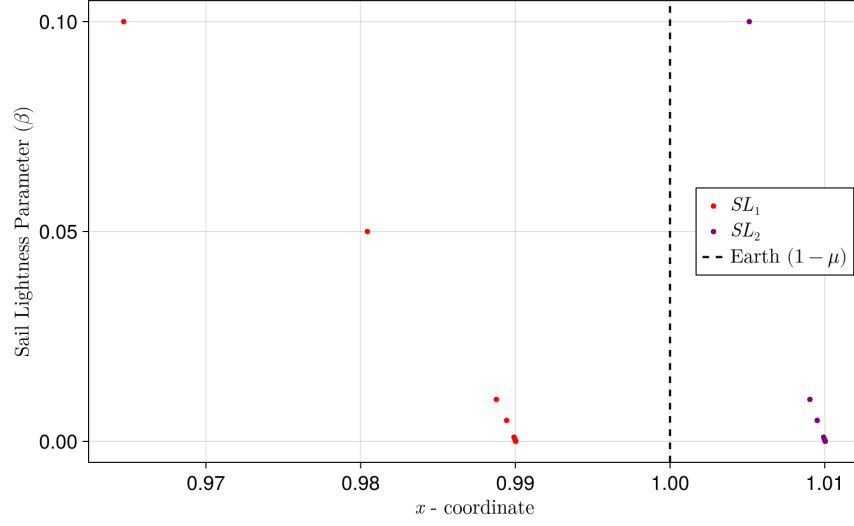


Figure 2.1: SL_1 and SL_2 displacement w.r.t. β .

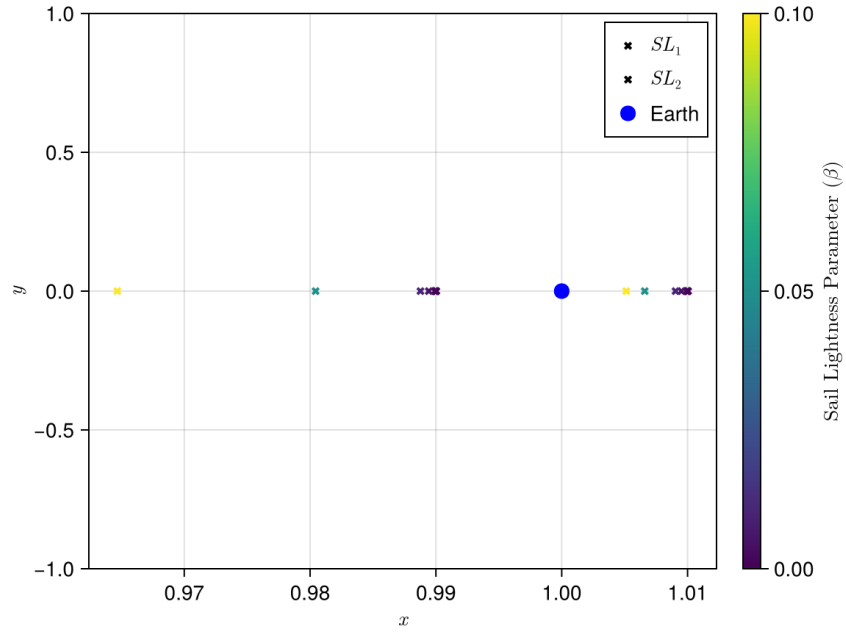


Figure 2.2: SL_1 and SL_2 displacement w.r.t. β in the configuration space.

2.3 Jacobi constant

As discussed in Section 1.3, the Jacobi constant is the only integral of motion of the CR3BP and bounds the accessible regions of configuration space. For the Hamiltonian case of the solar sail, the Jacobi constant is modified by replacing the classical effective potential with the modified potential (2.3):

$$C_J = 2\tilde{\Omega}(x, y) - (\dot{x}^2 + \dot{y}^2). \quad (2.16)$$

The value of C_J evaluated at the equilibrium points SL_1 and SL_2 determines the energy levels at which the passages allowing transit between the primary regions open. Table 2.2 presents these values for different β .

β	$C_J(SL_1)$	$C_J(SL_2)$
0.0	3.00089069382695	3.000886689145631
1.0×10^{-4}	3.000688679129159	3.00068867613475
5.0×10^{-4}	2.999880597410451	2.99989660254725
1.0×10^{-3}	2.9988704432783098	2.9989064624346238
5.0×10^{-3}	2.9907870507951784	2.9909834823945043
0.01	2.9806770014572277	2.981075412454562
0.05	2.899463418241557	2.901684210995871
0.1	2.796669895126338	2.802258976131804

Table 2.2: C_J values (w.r.t β) evaluated at SL_1 and SL_2

A notable feature is that the relationship between $C_J(SL_1)$ and $C_J(SL_2)$ depends on the value of β . For small values of β , the order in which the regions open is the same as in the classical CR3BP, first the region around SL_1 opens (between the Sun and the Earth), and then that of SL_2 (exterior to the Earth). However, beyond a certain value of β , this order reverses: the region around SL_2 opens first. This inversion has direct consequences for trajectory design, as it determines which transfers are accessible at a given energy level. For a fixed value of C_J , equation (2.16) defines a 4D energy surface $M_J \subset \mathbb{R}^4$ in the planar phase space $(x, y, \dot{x}, \dot{y})$. Different values of C_J correspond to different energy surfaces, each of these defining different accessible regions of configuration space, including the neighborhoods of the equilibrium points.

2.4 Zero-velocity curves

Finding the zero-velocity curves for different values of C_J close to the case where SL_1 and SL_2 are open gives insight into the possible motions that can be reached by the solar sail.

The zero-velocity curves are obtained by imposing $\dot{x}^2 + \dot{y}^2 = 0$ in equation (2.16):

$$C_J - 2\tilde{\Omega}(x, y) = 0. \quad (2.17)$$

Given a value of C_J , one can find curves that separate the regions of possible motion (Hill's regions, where $2\tilde{\Omega} \geq C_J$) from the forbidden regions (forbidden realm). Figure 2.3 shows schematically the five possible cases according to the energy level, and Figures 2.4 and 2.5 show the zero-velocity curves for the Sun–Earth system near the regions around SL_1 and SL_2 with $\beta = 0$ and for different values of β , respectively.

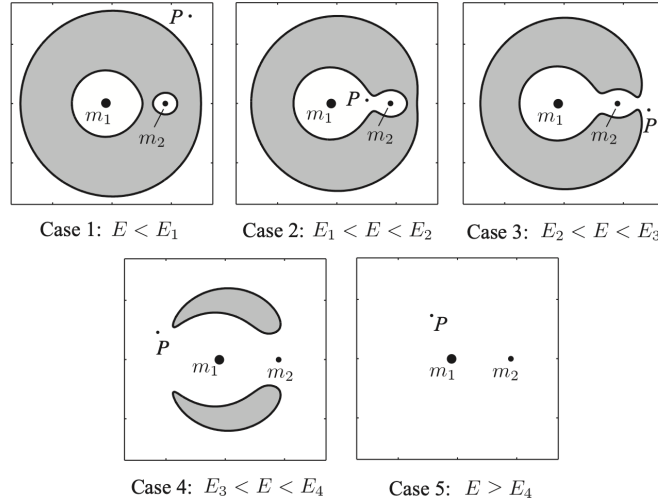
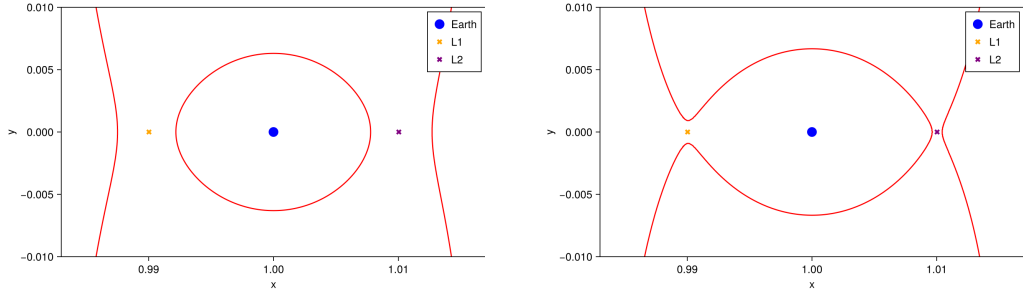
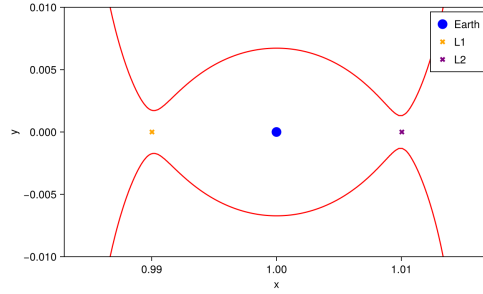


Figure 2.3: Hill's regions in the CR3BP. Reproduced from Koon et al. [2], Fig. 2.4.2.

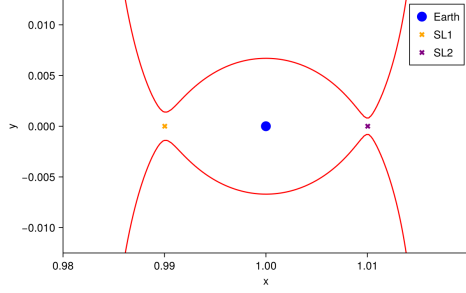


(a) Zero-velocity curves in the case $C_J > C_J(L_1)$ (Earth's region is bounded) (b) Zero-velocity curves in the case $C_J(L_1) > C_J > C_J(L_2)$ (L_1 region is open)

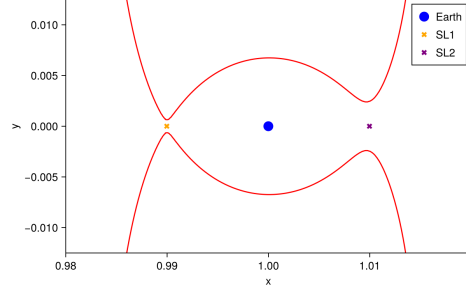


(c) Zero-velocity curves in the case $C_J(L_2) > C_J$ (L_1 and L_2 regions are open)

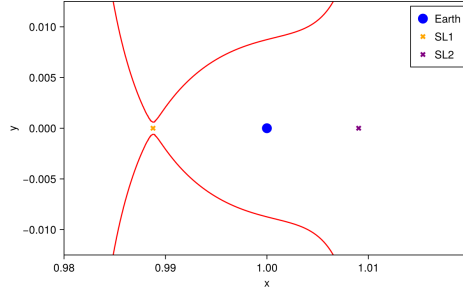
Figure 2.4: Zero-velocity curves for Sun-Earth system before and after L_1 and L_2 are open



(a) Zero-velocity curves in the case $C_J < C_J(SL_2), C_J(SL_1)$ for $\beta = 0.0$ (SL_1 and SL_2 are open)



(b) Zero-velocity curves in the case $C_J < C_J(SL_2), C_J(SL_1)$ for $\beta = 0.0005$ (SL_1 and SL_2 are open)



(c) Zero-velocity curves in the case $C_J < C_J(SL_2), C_J(SL_1)$ for $\beta = 0.01$ (SL_1 and SL_2 are open)

Figure 2.5: Zero-velocity curves with different values of β after SL_1 and SL_2 are open.

The case of interest for this work is the one in which both regions around SL_1 and SL_2 are open, since from this configuration onward, the periodic orbits around both points begin to coexist, and the associated invariant manifolds can generate heteroclinic connections between them, leading to low-energy transfers between two relevant regions for space missions. The dynamical consequence is that both neck regions are simultaneously open, and the Hill's region, which in Case 1 consists of three disconnected pieces, becomes a single connected domain that allows the transition between five sub-regions: the Sun realm, the SL_1 neighborhood, the Earth realm, the SL_2 neighborhood, and the exterior. A trajectory can travel between these sub-regions only by passing through one of the two necks. The Lyapunov periodic orbits around SL_1 and SL_2 , with their stable and unstable invariant man-

ifolds, are the dynamical structures that organize this transport, and their heteroclinic intersections are the zero- Δv transfers between both families of orbits.

2.5 Linearization of the equations of motion around the SL_1 and SL_2 equilibrium points

After locating the equilibrium points of interest, a natural way to continue understanding the dynamics of the model is to study the local dynamics in the neighborhood of these equilibrium points, SL_1 and SL_2 in this case. The local dynamics give insight into the behavior of the third body near this region. To analyze the local dynamics, system (2.6) is linearized around these points.

Let X^* be an equilibrium point of system (2.6), i.e., $f(X^*) = 0$, and let $\xi = X - X^*$ be a small perturbation. Since X^* is a fixed point, $\dot{\xi} = \dot{X} = f(X)$. Expanding f in Taylor series around X^* :

$$\dot{\xi} = f(X^* + \xi) = f(X^*) + D_X f(X^*) \xi + \mathcal{O}(\|\xi\|^2). \quad (2.18)$$

Since X^* is an equilibrium point, $f(X^*) = 0$. Neglecting higher-order terms, the linearized system is:

$$\dot{\xi} = D_X f(X^*) \xi, \quad (2.19)$$

where $D_X f(X^*)$ is the Jacobian matrix of the vector field evaluated at the equilibrium point. For the planar case (2.6), we obtain the following 4×4 Jacobian matrix:

$$D_X f(X^*) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \tilde{\Omega}_{xx} & \tilde{\Omega}_{xy} & 0 & 2 \\ \tilde{\Omega}_{xy} & \tilde{\Omega}_{yy} & -2 & 0 \end{pmatrix}. \quad (2.20)$$

Where $\tilde{\Omega}_{xx}$, $\tilde{\Omega}_{xy}$ and $\tilde{\Omega}_{yy}$ denote the second partial derivatives of the effective potential evaluated at the equilibrium point. Considering collinear points ($y = 0$), one has $\tilde{\Omega}_{xy} = 0$, which simplifies the analysis. The next step is to compute the eigenvalues of the linearized system, for this, setting:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ \tilde{\Omega}_{xx} & 0 & -\lambda & 2 \\ 0 & \tilde{\Omega}_{yy} & -2 & -\lambda \end{vmatrix} = 0, \quad (2.21)$$

it is possible to derive the characteristic polynomial by expanding it using cofactors in the first row, obtaining a biquadratic equation:

$$p(\lambda) = \lambda^4 + (4 - \tilde{\Omega}_{xx} - \tilde{\Omega}_{yy})\lambda^2 + \tilde{\Omega}_{xx}\tilde{\Omega}_{yy} = 0. \quad (2.22)$$

Note that substituting $\alpha = \lambda^2$ into the equation reduces it to a quadratic equation:

$$p(\alpha) = \alpha^2 + (4 - \tilde{\Omega}_{xx} - \tilde{\Omega}_{yy})\alpha + \tilde{\Omega}_{xx}\tilde{\Omega}_{yy} = 0. \quad (2.23)$$

At the collinear points, $\tilde{\Omega}_{xx} > 0$ and $\tilde{\Omega}_{yy} < 0$, so $\tilde{\Omega}_{xx}\tilde{\Omega}_{yy} < 0$ and the two roots are real with opposite signs, $\alpha_1 > 0$ and $\alpha_2 < 0$ [2].

The resulting eigenvalues are of the form:

$$\lambda_{1,2} = \pm\sqrt{\alpha_1} = \pm\lambda \quad (\text{real}), \quad \lambda_{3,4} = \pm\sqrt{\alpha_2} = \pm i\omega \quad (\text{purely imaginary}). \quad (2.24)$$

This structure corresponds to Saddle $\times$ Center dynamics [2], as in the classical CR3BP. The saddle component is associated with the stable and unstable directions of the equilibrium point, while the center component gives rise to the families of Lyapunov periodic orbits that will be studied in Chapter 3.

Tables 2.3 and 2.4 show the eigenvalues of $D_x f$ evaluated at SL_1 and SL_2 , respectively, for the values of β listed in Table 2.1.

β	λ_1	λ_2	λ_3	λ_4
0.0	-2.532354	$-2.086267i$	$2.086267i$	2.532354
1.0×10^{-4}	-2.528319	$-2.083805i$	$2.083805i$	2.528319
5.0×10^{-4}	-2.512203	$-2.073979i$	$2.073979i$	2.512203
1.0×10^{-3}	-2.492114	$-2.061753i$	$2.061753i$	2.492114
5.0×10^{-3}	-2.333832	$-1.966294i$	$1.966294i$	2.333832
0.01	-2.142964	$-1.853461i$	$1.853461i$	2.142964
0.05	-1.050488	$-1.281313i$	$1.281313i$	1.050488
0.1	-0.687601	$-1.459267i$	$1.459267i$	0.687601

Table 2.3: SL_1 eigenvalues in the linearized system

β	λ_1	λ_2	λ_3	λ_4
0.0	-2.484216	$-2.056953i$	$2.056953i$	2.484216
1.0×10^{-4}	-2.488199	$-2.059373i$	$2.059373i$	2.488199
5.0×10^{-4}	-2.504151	$-2.069076i$	$2.069076i$	2.504151
1.0×10^{-3}	-2.524142	$-2.081257i$	$2.081257i$	2.524142
5.0×10^{-3}	-2.685941	$-2.180665i$	$2.180665i$	2.685941
0.01	-2.892194	$-2.309301i$	$2.309301i$	2.892194
0.05	-4.604388	$-3.429303i$	$3.429303i$	4.604388
0.1	-6.699310	$-4.858374i$	$4.858374i$	6.699310

Table 2.4: SL_2 eigenvalues in the linearized system

Each eigenvalue has an associated eigenvector. We denote them as follows:

$$\vec{u}_{+\lambda}, \quad \vec{u}_{-\lambda}, \quad \vec{u}_\omega + i\vec{v}_\omega, \quad \vec{u}_\omega - i\vec{v}_\omega. \quad (2.25)$$

The solution to the linearized system (2.19) has the form:

$$Y(t) = X^* + A_0 [\cos(\omega t + \varphi)\vec{u}_\omega + \sin(\omega t + \varphi)\vec{v}_\omega] + B_0 e^{\lambda t} \vec{u}_{+\lambda} + C_0 e^{-\lambda t} \vec{u}_{-\lambda} \quad (2.26)$$

These eigenvectors, together with the linear solution (2.26), are used to estimate the initial conditions for the Lyapunov orbits and the stable and unstable manifolds. It is necessary to consider that the linear solution works as a good estimation for non-linear solutions when the third body is close to the equilibrium point, since this is a local analysis.

Chapter 3

Computation and Stability Analysis of Periodic Orbits Around SL_1 and SL_2

The previous chapter shows that the equilibrium points SL_1 and SL_2 of the CR3BP with solar sail present local dynamics of the Saddle $\times$ Center type. By Lyapunov's Center Theorem, the Center component implies the existence of continuous families of periodic orbits emanating from these points. This theorem was generalized by Moser [27] in the sense that, under certain non-resonance conditions, the periodic solutions of the linearized system extend to the full nonlinear system and were then used by Conley [3] in the CR3BP. At the same time, the Saddle component generates stable and unstable directions that, in a certain sense, organize the global transport in phase space.

In this chapter, we develop the numerical tools required to compute these invariant structures. Lyapunov's Center Theorem is first introduced, and the family of planar periodic orbits emanating from SL_1 and SL_2 equilibrium points is defined. The shooting method for correcting the initial guess of the periodic orbits, called differential correction, is then presented, using the State Transition Matrix (STM) and Newton's method to obtain accurate initial conditions for periodic orbits up to a tolerance of 10^{-12} . This method is combined with a numerical continuation strategy to generate complete families of orbits of increasing amplitude. Finally, the stability of these orbits is analyzed via Floquet theory, with the resulting Floquet multipliers providing the information needed for the computation of the invariant manifolds that will be developed in the following chapter.

3.1 Lyapunov periodic orbits

The linearization around the fixed points SL_1 and SL_2 (Section 2.5) is characterized by a Saddle $\times$ Center dynamics. In particular, the center-type dynamics — corresponding to the purely imaginary eigenvalues — is related to an oscillatory motion that implies the existence of periodic solutions of the linearized system with frequency ω around the equilibrium point.

Moser’s generalization of Lyapunov’s Center Theorem [27] states that, under certain non-resonance conditions, these periodic solutions of the linearized system extend to the full nonlinear system. This means that there exists a family of periodic orbits of the nonlinear system emanating from the equilibrium point. The period of the orbits tends to $T_0 = 2\pi/\omega$ as the Jacobi constant approaches its value at the equilibrium points.

Although the spatial CR3BP with solar sail admits a rich variety of periodic and quasi-periodic orbits, in the planar case, the relevant family consists of periodic orbits known as Lyapunov orbits. The system possesses a symmetry with respect to the x -axis, which implies reversibility of the vector field $f(X)$ under the transformation $(t, x, y, v_x, v_y) \rightarrow (-t, x, -y, -v_x, v_y)$ [1]. As a consequence, Lyapunov orbits cross the x -axis perpendicularly.

This symmetry property is fundamental for the shooting method presented below, as it reduces the problem of finding a complete periodic orbit to the problem of finding a half-period solution that satisfies the perpendicular crossing condition.

3.2 Differential correction and numerical computation of periodic orbits

Following the methodology described in Koon et al. [2], the process of numerically generating Lyapunov orbits requires, as a first step, a good initial guess of the initial condition. Particularly in the planar case, the estimation obtained from the solution of the linearized system around the equilibrium point is enough for solutions close to this point.

From the linear solution (2.24), a linear periodic orbit can be found by setting $B_0 = C_0 = 0$, this means that we are setting up the linear solution for which the saddle component is eliminated. Note that in this way, an initial condition for a linear periodic orbit at $t_0 = 0$, assuming $\varphi = 0$, takes the form:

$$Y(t) \Big|_{t=t_0} = X^* + A_0 [\cos(\omega t_0) \vec{u}_\omega + \sin(\omega t_0) \vec{v}_\omega] = X^* + A_0 \vec{u}_\omega. \quad (3.1)$$

The set of eigenvectors is normalized at the first component in order to use the constant $A_0 = \epsilon$ as an amplitude parameter. $\epsilon \sim 10^{-6}$ is small enough to remain within the region of validity of the linearized system. The vector $\vec{u}_\omega$ is the direction of the Center-type eigenvector — more specifically, its real part. In order to compute the non-linear periodic orbit, we will use an initial condition as the initial guess for the differential correction method:

$$X_0^g = X^* + \epsilon \vec{u}_\omega. \quad (3.2)$$

As we already mentioned, to give this a physical meaning, the eigenvectors from Section 2.5 are normalized at their first entry; in this way, ϵ serves directly as an amplitude scaling parameter for the initial periodic orbit. The real part of the Center eigenvector $\vec{u}_\omega$ has the form $\vec{u}_\omega = (1, 0, 0, v_{y(\omega)})$, so the initial condition takes the form:

$$X_0^g = (x_{SLi} + \epsilon, 0, 0, \epsilon \cdot v_{y(\omega)}). \quad (3.3)$$

This initial estimate is valid only in a very small neighborhood of the equilibrium point. To obtain the exact periodic orbit of the nonlinear system, a differential correction method is applied [2].

The objective is to find an initial condition $(x_0, 0, 0, v_{y0})$ and a half-period $T/2$ such that the trajectory of the nonlinear system crosses the x -axis perpendicularly, i.e., an initial condition with $y(T/2) = 0$ and $v_x(T/2) = 0$ at the crossing. By the symmetry of the system, this guarantees that the complete orbit is periodic with period $T = 2(T/2)$.

To implement the correction, one needs to know how a perturbation in the initial condition propagates along the trajectory. This information is contained in the State Transition Matrix (STM) [2, 28].

Consider first the linearized system (2.19). A linear system of differential equations of the form $\dot{Y} = A(t)Y$ satisfies the superposition principle: any linear combination of solutions is also a solution. Taking n linearly independent solutions as columns of a matrix, one obtains the fundamental solution matrix $\Psi(t)$, which satisfies:

$$\dot{\Psi}(t) = A(t)\Psi(t). \quad (3.4)$$

From the fundamental matrix, the STM is defined as $\Phi(t) = \Psi(t)\Psi(t_0)^{-1}$, and it corresponds to the family of fundamental matrix solutions at each time $t \in J$ defined on an open interval. The STM maps a given state at t_0 to the final state t via the flow $\phi(t, x_0) : x(t_0) \mapsto x(t)$. If $X(t_0) = X_0$, then $X(t) = \Phi(t, t_0)X_0$. The STM satisfies the following properties:

$$\Phi(t_0, t_0) = I, \quad \Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0). \quad (3.5)$$

The first corresponds to the initial condition (the identity does not transform the state), and the second is a composition property, which states that propagating from t_0 to t_2 is equivalent to propagating from t_0 to t_1 and then from t_1 to t_2 ; for a detailed treatment, see [28].

Although system (2.4) is nonlinear, the propagation of small perturbations along a reference trajectory can be studied linearly. Defining the flow of the differential equation as $\phi(t, X_0)$ — the trajectory passing through X_0 at $t = 0$ — a perturbation $X_0 + \delta X_0$ in the initial condition propagates along the flow and produces a perturbation:

$$\delta X(t) = \phi(t, X_0 + \delta X_0) - \phi(t, X_0). \quad (3.6)$$

Expanding in a Taylor series and neglecting higher order terms leads to:

$$\delta X(t) = \frac{\partial \phi(t, X_0)}{\partial X_0} \delta X_0 = \Phi(t, t_0) \delta X_0. \quad (3.7)$$

The STM satisfies $\Phi(t, t_0) = \frac{\partial \phi(t, X_0)}{\partial X_0}$, and, in the nonlinear system satisfies the variational equation:

$$\dot{\Phi}(t, t_0) = D_x f(X(t))\Phi(t, t_0), \quad \Phi(t_0, t_0) = I, \quad (3.8)$$

where $D_x f$ is the Jacobian matrix of the vector field evaluated along the reference trajectory $X(t)$. This equation has the same structure as the linear case, with $A(t) = D_x f(X(t))$, the difference being that the coefficient matrix now depends on the particular trajectory being followed. The STM is integrated numerically together with the equations of motion (2.5), forming a coupled system of $4 + 16 = 20$ ordinary differential equations in the planar case: 4 for the system state and 16 for the entries of the STM.

The properties (3.5) are preserved in the nonlinear system and will be used implicitly in the differential correction that follows and explicitly in the computation of the monodromy matrix.

Let $X_0^g = (x_0, 0, 0, v_{y0})$ (representing equation (3.3)) be the first guess of the initial condition that corresponds to a Lyapunov orbit. The system is integrated together with the STM until the crossing with $y = 0$ at time t_1 , yielding the state $X(t_1) = (x_1, 0, v_{x1}, v_{y1})$ and the STM $\Phi(t_1, 0)$. In general, $v_{x1} \neq 0$. A correction δv_{y0} to the initial y -velocity is derived such that the new crossing with $y = 0$ satisfies $v_{x1} = 0$. The numerical implementation of this differential correction requires one technical remark. The initial guess $X_{0g} = (x_0, 0, 0, v_{y0})$ lies, by construction, on the symmetry axis $\{y = 0\}$ that defines the section used by the shooting condition. A direct application of an event-detection routine therefore registers a false detection at $t_1 \approx 0$, either because $y(0) = 0$ or because floating-point numbers produce a sign change between $t = 0$ and the first integration step. The intended half-period crossing is the one for which the trajectory returns to $\{y = 0\}$ from $y > 0$, that is, with $\dot{y}(t_1) = v_{y1} < 0$. Therefore, the crossing event is defined by the conditions $y(t) = 0$, $\dot{y}(t) < 0$, which automatically excludes the trivial crossing at the origin. An equivalent, integrator-dependent alternative is to restrict the event search to $t > t^*$, with t^* on the order of a few integration time-steps.

For a perturbation $\delta X_0 = (0, 0, 0, \delta v_{y0})$ in the initial condition, the first-order change in the final state is:

$$\delta X(t_1) = \Phi(t_1, 0) \delta X_0. \quad (3.9)$$

However, the crossing time also changes: the new trajectory crosses $y = 0$ at a slightly different time $t_1 + \delta t_1$. The first order total perturbation in the final state, including the variation of the crossing time, is:

$$\delta X(t_1 + \delta t_1) = \frac{\partial \phi(t_1, X_0)}{\partial X_0} \delta X_0 + \frac{\partial \phi(t_1, X_0)}{\partial t_1} \delta t_1, \quad (3.10)$$

where, $\frac{\partial \phi(t, X_0)}{\partial X_0} = \Phi(t, 0)$ (when $\delta t_1 = 0$) as mentioned already, and it can be verified that $\frac{\partial \phi(t_1, X_0)}{\partial t_1} = \frac{d\phi(t_1, X_0)}{dt} = f(X(t_1))$, obtaining the equation:

$$\delta X(t_1 + \delta t_1) = \Phi(t_1, 0) \delta X_0 + f(X(t_1)) \delta t_1, \quad (3.11)$$

where $f(X(t_1))$ is the vector field evaluated at the crossing state. The condition that the crossing still occurs at $y = 0$ imposes that the y -component of the total perturbation vanishes:

$$\delta y(t_1 + \delta t_1) = \Phi_{24} \delta v_{y0} + \dot{y}(t_1) \delta t_1 = 0, \quad (3.12)$$

where Φ_{24} is the $(2, 4)$ -element of the STM and $\dot{y}(t_1) = v_{y1}$ is the y -velocity at the crossing. Solving for the change in the crossing time:

$$\delta t_1 = -\frac{\Phi_{24}}{v_{y1}} \delta v_{y0}$$

The desired condition is that v_x vanishes at the new crossing. Expanding the perturbation in v_x :

$$\delta v_x(t_1 + \delta t_1) = \Phi_{34} \delta v_{y0} + \dot{v}_x(t_1) \delta t_1 = \Phi_{34} \delta v_{y0} - \frac{\dot{v}_x(t_1) \Phi_{24}}{v_{y1}} \delta v_{y0}.$$

Imposing that the total v_x vanishes at the new crossing, i.e., $v_{x1} + \delta v_x(t_1 + \delta t_1) = 0$:

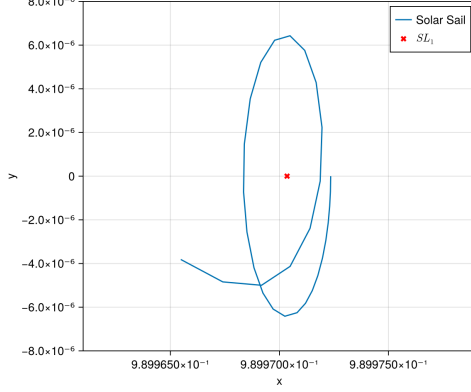
$$v_{x1} + \left(\Phi_{34} - \frac{\dot{v}_x(t_1) \Phi_{24}}{v_{y1}} \right) \delta v_{y0} = 0.$$

Solving for the correction:

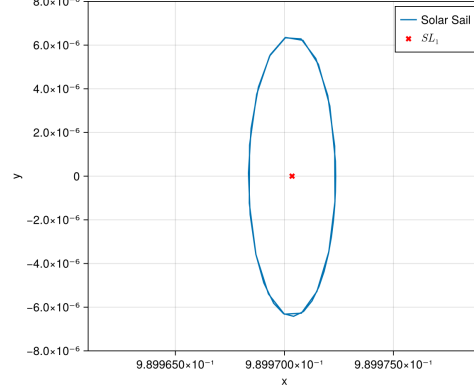
$$\delta v_{y0} = -\frac{v_{x1}}{\Phi_{34} - \frac{\dot{v}_x(t_1) \Phi_{24}}{v_{y1}}}. \quad (3.13)$$

where Φ_{34} and Φ_{24} are elements of the STM evaluated at t_1 , and $\dot{v}_x(t_1)$ is evaluated directly from the equations of motion (2.4). The initial y -velocity is updated as $v_{y0[n]} = v_{y0[n-1]} - \delta v_{y0}$, and the procedure is repeated until $|v_x(t_1)| < \varepsilon$, with ε of order 10^{-12} .

Figure 3.1 shows the result of applying this differential correction method numerically.



(a) Initial guess for orbit around SL_1



(b) Orbit around SL_1

Figure 3.1: Lyapunov orbit around SL_1 found with Differential Correction

3.3 Generation and numerical continuation of Lyapunov orbits

Differential correction yields a periodic orbit in the nonlinear system with an error of order $\varepsilon = 10^{-12}$ from a nearby initial guess. However, the linear estimation is valid only for very small amplitudes. To generate complete families of Lyapunov orbits with increasing amplitudes, a numerical continuation scheme is employed.

The idea is to use already computed orbits with small amplitudes to predict the initial condition of the next orbit in the family by extrapolation, and then apply differential correction to refine it.

For the first few members of the family, a linear extrapolation is used, this is, given two consecutive orbits with initial conditions $X_0^{(k)}$ and $X_0^{(k+1)}$, the next is predicted as:

$$X_0^{(k+2)} = X_0^{(k+1)} + \left(X_0^{(k+1)} - X_0^{(k)} \right)$$

Once sufficiently many orbits are available (four in this implementation), a cubic spline interpolation is used coordinate-by-coordinate to extrapolate

the initial condition of the next orbit. This higher-order predictor improves the convergence of the differential correction.

Each prediction is then refined with the differential correction algorithm described above. The process is repeated iteratively to generate a family of N orbits of increasing amplitude, as guaranteed by Lyapunov's Center Theorem.

Something that needs to be clear is that this continuation gives a discrete family of Lyapunov orbits. For the objectives of this work, this fact has consequences when searching for zero- Δv trajectories since one needs to match the energies of the connecting orbits. Figure (3.2) shows the families of Lyapunov orbits around SL_1 and SL_2 for the Sun–Earth system with $\beta = 0.0005$, generated by this procedure. The orbits grow in amplitude progressively from the neighborhood of each equilibrium point.

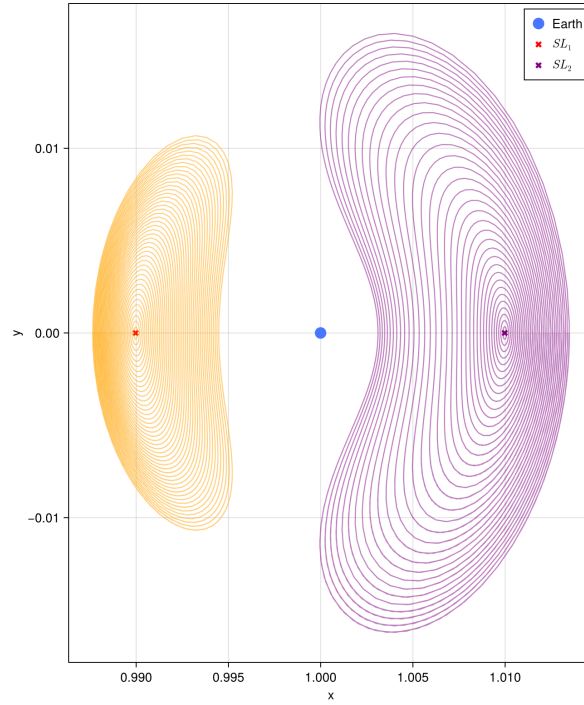


Figure 3.2: Families of Lyapunov orbits.

3.4 Stability analysis of Lyapunov orbits via Floquet Theory

Once the periodic orbits have been computed, their stability must be analyzed in order to characterize the dynamical structure in their neighborhood. For systems with periodic coefficients, this analysis is carried out via Floquet theory.

For this goal, we need to analyze the trajectories in a neighborhood of the Lyapunov orbit, this mean the trajectories that has initial conditions perturbed away from a solution corresponding to the periodic orbit. Given a initial condition X_0 of the real periodic orbit, the perturbed initial condition $X_0 + \delta X_0$ will be displaced after one period:

$$\delta X(T) = \phi(T, X_0 + \delta X_0) - \phi(T, X_0).$$

As we already saw in the previous section, the displacement is given, to first order, by:

$$\delta X(T) = \Phi(T) \delta X_0.$$

One can note that $\Phi(T) = \Phi(T, 0)$ is the STM integrated along a periodic orbit of period T . This matrix is called the monodromy matrix M , and is defined as:

$$M = \Phi(T, 0) = \frac{\partial \phi(T, X_0)}{\partial X_0} \quad (3.14)$$

The monodromy matrix can be obtained numerically integrating the system (2.4) together with the STM over one period. The relevance of the monodromy matrix is that give us information about how initial perturbations δX_0 decay or grow. Floquet theory states that, for a T-periodic system, as in the particular case of Lyapunov orbits in (2.4), if $\Phi(t)$ is a fundamental matrix solution, then:

$$\Phi(t) = P(t) e^{Rt}, \quad (3.15)$$

where $P(t)$ is T-periodic and R is a constant matrix. This representation is called a Floquet normal form, and one can note that if:

$$\Phi(0 + T) = \Phi(0) e^{RT} = e^{RT},$$

then, it satisfies the expression:

$$\Phi(kT) = e^{RkT} = M^k,$$

since $e^{RT} = M$.

The eigenvalues of the monodromy matrix, $\lambda_1, \dots, \lambda_4$, are called Floquet multipliers, and each k_i defined by $\lambda_i = e^{k_i T}$ is called the Floquet exponent. Both are used to determine the stability of the periodic orbit. For a planar Hamiltonian system with 2 degrees of freedom, the multipliers satisfy the symplectic constraint in the eigenvalues, if λ is a multiplier, so is $1/\lambda$. Furthermore, since the Jacobi constant is an integral of motion, there always exists a pair of trivial multipliers $\lambda_3 = \lambda_4 = 1$, associated respectively with the direction tangent to the orbit and the direction toward the next periodic orbit in the family, see [2]. The two remaining multipliers, λ_1 and $\lambda_2 = 1/\lambda_1$, determine the stability of the orbit.

For Lyapunov orbits around the collinear equilibria, the non-trivial multipliers are typically real with $|\lambda_1| > 1$ and $|\lambda_2| = 1/|\lambda_1| < 1$. This indicates the hyperbolic character of the orbit, that is, perturbations in the direction of the eigenvector associated with λ_1 grow exponentially, while those associated with λ_2 decay. The corresponding eigenvectors define the unstable and stable directions of the periodic orbit in a small neighborhood, respectively.

It is worth noting the relationship between the monodromy matrix and the Poincaré map. If a section transversal to the flow is defined (for example, the plane $y = 0$), the periodic orbit defines a fixed point of the associated Poincaré map. The nontrivial multipliers of the monodromy matrix M match the eigenvalues of the derivative of the reduced Poincaré map, and the stability analysis of the periodic orbit reduces to the stability analysis of a fixed point of a discrete map. This correspondence allows the stable and unstable manifold theorem for hyperbolic fixed points of diffeomorphisms to be applied directly, as discussed in the following section.

3.5 Stable and unstable manifolds of Lyapunov orbits

The theoretical framework of this section follows the classical treatment of invariant manifolds for hyperbolic fixed points of diffeomorphisms; see Wiggins [29] and Guckenheimer and Holmes [30] for the general theory, and Koon

et al. [2] and Chicone [28] for treatments oriented toward celestial mechanics and differential equations, respectively.

The Floquet multipliers provide the necessary elements to compute the invariant manifolds associated with a periodic orbit — the central objects for the trajectory design of Chapter 4.

The previous section showed that the non-trivial Floquet multipliers of Lyapunov orbits around the collinear equilibria are typically real, with $|\lambda_1| > 1$ and $|\lambda_2| = 1/|\lambda_1| < 1$. It was also established that the monodromy matrix can be interpreted as the linearization of the Poincaré map at the fixed point corresponding to the periodic orbit. This section uses that structure to define the invariant manifolds of periodic orbits and to present their computational method.

The eigenvectors of the monodromy matrix M associated with the non-trivial multipliers define two linear invariant subspaces. The eigenvector e_s associated with the multiplier $|\lambda_2| < 1$ generates the stable subspace E^s , where perturbations in this direction decay exponentially under iterations of the Poincaré map (i.e., after each period T of the orbit). The eigenvector e_u associated with the multiplier $|\lambda_1| > 1$ generates the unstable subspace E^u , where perturbations in this direction grow exponentially.

Although γ is a periodic orbit and not a fixed point of the flow, the identification of its invariant manifolds reduces to the classical stable and unstable manifold theorem for diffeomorphisms through a Poincaré section. Taking a section Σ transverse to γ , the periodic orbit intersects Σ at a single point $\bar{X} \in \gamma$, which is a fixed point of the first-return map $P : \Sigma \rightarrow \Sigma$. The linearized derivative $DP(\bar{X})$ can be identified with the restriction to Σ of the monodromy matrix M , so the non-trivial Floquet eigenvectors of M generate the linear subspaces E^s, E^u in $T_{\bar{X}}\Sigma$ (see [28]). The classical stable–unstable manifold theorem for diffeomorphisms applied to the fixed point $\bar{X}$ of P yields local invariant manifolds $W_{loc}^s(\bar{X}), W_{loc}^u(\bar{X}) \subset \Sigma$ tangent at $\bar{X}$ to E^s, E^u respectively. The stable and unstable manifolds of the periodic orbit γ are then recovered by transporting these manifolds along γ under the flow,

$$W^s(\gamma) = \bigcup_{t \in \mathbb{R}} \phi(t, W_{loc}^s(\bar{X})), \quad W^u(\gamma) = \bigcup_{t \in \mathbb{R}} \phi(t, W_{loc}^u(\bar{X})). \quad (3.16)$$

Extending the same concept, the manifolds of the orbit are obtained from the manifolds of a fixed point. The resulting invariant structures are two-dimensional cylinders emanating from γ , and the hyperbolic characteristic of

trajectories is now to be understood relative to the entire orbit rather than to a single equilibrium.

The stable and unstable manifold theorem for diffeomorphisms establishes that, given a hyperbolic fixed point of a diffeomorphism, there exist local invariant manifolds W_{loc}^s and W_{loc}^u tangent to E^s and E^u at the fixed point, defined as:

$$W_{\text{loc}}^s = \{X_s \in U \mid \phi(t, X_s) \rightarrow X^* \text{ when } t \rightarrow \infty, \text{ and } \phi(t, X_s) \in U \quad \forall t \geq 0\},$$

$$W_{\text{loc}}^u = \{X_u \in U \mid \phi(t, X_u) \rightarrow X^* \text{ when } t \rightarrow -\infty, \text{ and } \phi(t, X_u) \in U \quad \forall t \leq 0\},$$

where $U \subset \mathbb{R}^4$ is a neighborhood of a hyperbolic fixed point $X^* \in \gamma$, with γ as the periodic orbit under study. The tangency of the local manifolds to the linear subspaces means that, in a sufficiently small neighborhood of the periodic orbit, the invariant manifolds are well approximated by E^s and E^u .

Global manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are obtained by extending the local manifolds under the flow:

$$W^s(\gamma) = \bigcup_{t \leq 0} \phi(t, W_{\text{loc}}^s(\gamma)), \quad W^u(\gamma) = \bigcup_{t \geq 0} \phi(t, W_{\text{loc}}^u(\gamma)).$$

In the planar case, phase space is 4-dimensional and the energy surface fixed by C_J is 3-dimensional. Lyapunov orbits are closed curves (dimension 1) within this surface. Their stable and unstable manifolds are therefore 2-dimensional surfaces that, topologically, have the structure of a cylinder [2, 5]. These invariant tubes extend from the neighborhood of the periodic orbit into distant regions of phase space and act as separatrices of the flow in the sense that trajectories passing through the interior of a tube are transit orbits that cross from one region of configuration space to another, while trajectories exterior to the tube are non-transit orbits that return to their region of origin [3, 9].

The tangency between the local manifold and the linear subspace implies that a sufficiently small perturbation in the direction of the eigenvector places the initial condition on (or very close to) the local invariant manifold. Given a periodic orbit γ_i , the eigenvectors of the monodromy matrix computed over one period $M = \Phi(T, 0)$ represent the directions of the tangent spaces to the invariant manifolds at the starting point. The eigenvector associated

with the smallest absolute value of the Floquet multipliers computed over one period approximates the stable manifold in a small neighborhood of the initial point. The same holds for using the greatest absolute value of the Floquet multipliers to approximate the unstable manifold. In order to globalize the invariant manifolds, the eigenvectors corresponding to the stable and unstable directions need to be normalized in the following manner:

$$e_s(t_k) = \frac{\Phi(t_k, 0) \cdot e_s^0}{|\Phi(t_k, 0) \cdot e_s^0|}, \quad e_u(t_k) = \frac{\Phi(t_k, 0) \cdot e_u^0}{|\Phi(t_k, 0) \cdot e_u^0|}, \quad (3.17)$$

where e_s^0, e_u^0 are the stable and unstable directions corresponding to the monodromy matrix computed at the first point over one period in the periodic orbit, the point when $y = 0$ in this scheme. Once we have computed the normalized directions, the stable and unstable directions at any point within the periodic orbit can be obtained by transporting this normalized vector through the STM evaluated at each of these points parameterized by t_k . Initial conditions are then generated as:

$$X_0^{\pm s} = X(t_k) \pm \epsilon \cdot e_s, \quad X_0^{\pm u} = X(t_k) \pm \epsilon \cdot e_u, \quad (3.18)$$

where the displacement $\epsilon \sim 10^{-6}$ is small enough for the linear approximation to be valid and $t_k \in [0, T)$.

To obtain the global stable manifold, the initial conditions $X_0^{\pm s}$ are integrated backward in time. For the unstable manifold, the points $X_0^{\pm u}$ are integrated forward in time. The $\pm$ signs produce the two branches of each manifold, extending in opposite directions from the periodic orbit.

Applying this procedure for N points distributed along the periodic orbit yields a numerical approximation of each manifold as a collection of trajectories. Figure (3.3) shows the stable and unstable manifold tubes of a Lyapunov orbit around SL_1 for the Sun–Earth system with $\beta = 0.0005$.

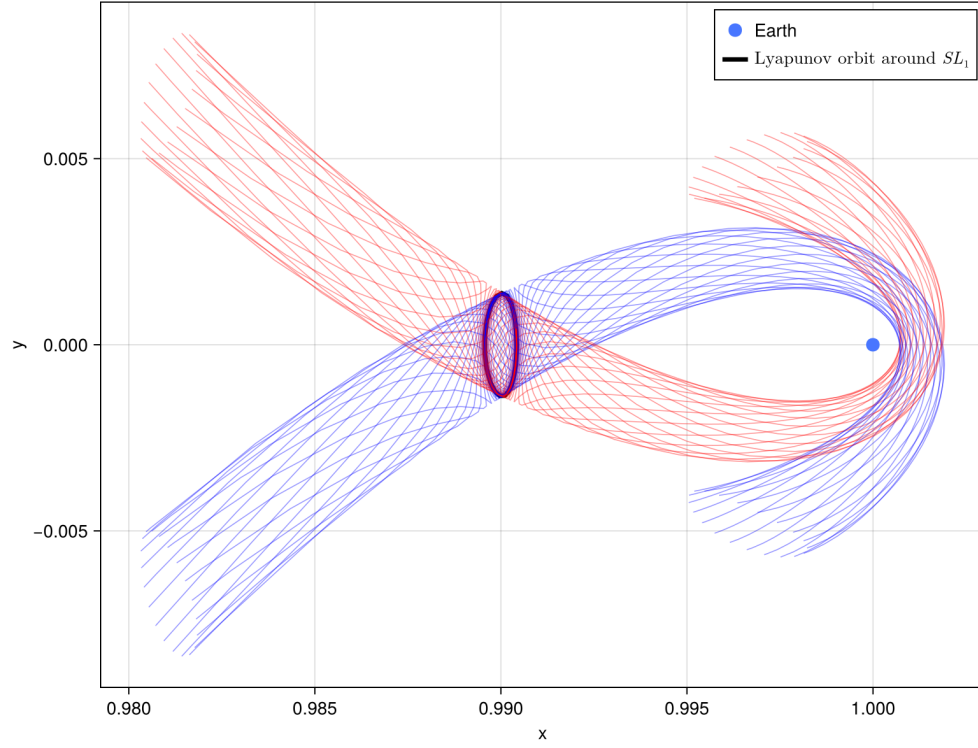


Figure 3.3: Projection of the tube manifolds for Lyap. Orbit at $C_J = 2.999874$ and $\beta = 0.0005$

Chapter 4

Trajectory Design Using Heteroclinic Connections

In the preceding chapters, the necessary tools were developed to compute Lyapunov periodic orbits and their invariant manifolds around the displaced equilibrium points SL_1 and SL_2 of the CR3BP with solar sail. In this chapter, these tools are used to address the central problem of this work: the design of low-energy transfer trajectories between the regions of SL_1 and SL_2 by means of heteroclinic connections. The idea is to take advantage of the hyperbolicity of the Lyapunov orbits, keeping in mind that the stable and unstable manifolds approach (and leave) the Lyapunov orbits in a very fast manner. This character allows to place the solar sail in the stable or unstable manifold in order to travel along the phase space.

In the context of dynamical systems, specific trajectories have been studied intensively with the aim of understanding the geometric mechanisms that give rise to chaotic transport. These trajectories are the so-called homoclinic and heteroclinic orbits. A trajectory is said to be homoclinic if it is asymptotic to a periodic orbit as $t \rightarrow \pm\infty$; a heteroclinic connection is a trajectory that is asymptotic to a periodic orbit γ_1 as $t \rightarrow +\infty$ and to a different periodic orbit γ_2 as $t \rightarrow -\infty$. In the context of the CR3BP, homoclinic and heteroclinic connections between Lyapunov orbits around L_1 and L_2 were widely studied by Llibre et al. [5] who carried out an study on the transversality of the corresponding invariant manifolds and computational framework for studying the dynamics near the collinear points and their associated invariant manifolds was further developed by Jorba and Masdemont [31]. The

heteroclinic connections were demonstrated explicitly and numerically by Koon et al. [9], and then studied and classified by Canalias and Masdemont [10] and Gidea and Masdemont [11]. These works constitute the foundation of the invariant-manifold mission design paradigm. The existence of such connections implies the possibility of transferring a spacecraft between the vicinities of L_1 and L_2 without propulsion cost (theoretical $\Delta v = 0$), since the trajectory follows naturally the dynamical structure of the phase space. The transversality, in a certain sense, guarantees the possibility of having these connections and its structural stability. Following Farrés [1], we extended this to the case of the CR3BP with solar sail, where the equilibrium points are displaced by the effect of radiation pressure. It is shown how to detect numerically the heteroclinic connections between Lyapunov orbits around SL_1 and SL_2 via Poincaré sections, the critical value of the Jacobi constant below which these connections exist at the first return map is analyzed, and the robustness of transversal connections against perturbations is discussed.

4.1 Stable and unstable manifolds of the Lyapunov orbit around SL_1 and SL_2

In Section 3.5 the local invariant manifolds of the Lyapunov orbits were introduced and the numerical method for extending them beyond the neighborhood of the periodic orbit was presented. In this section the global structure of these manifolds is analyzed and the geometric setting for the search for heteroclinic connections is established.

The global invariant manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are obtained by extending the local manifolds under the flow. Formally:

$$W^s(\gamma) = \bigcup_{t \leq 0} \phi(t, W_{loc}^s(\gamma)), \quad W^u(\gamma) = \bigcup_{t \geq 0} \phi(t, W_{loc}^u(\gamma)),$$

where $\phi(t, \cdot)$ denotes the flow of the system. That is, the global stable manifold is the collection of all trajectories that eventually enter the local stable manifold (and therefore converge to γ), and the global unstable manifold is the collection of all trajectories that at some point departed from the local unstable manifold (and therefore originate from γ).

Following section 3.5 we have now a methodology to compute numerically the stable and unstable manifolds of Lyapunov orbits around SL_1 and

SL_2 using the Floquet multipliers and their associated eigenvectors. These manifolds represent the natural dynamical paths leading to and from the periodic orbits, where the trajectories on $W^s(\gamma)$ converge to γ exponentially, while those on $W^u(\gamma)$ depart from it at a rate determined by the non-trivial Floquet multiplier.

In the planar case, the phase space has dimension four, and the energy surface (fixed by the Jacobi constant C_J) has dimension three. The Lyapunov orbits are closed curves (dimension 1) within this surface. The stable and unstable manifolds are two-dimensional surfaces. Topologically, these manifolds have the structure of a cylinder and extend from the neighborhood of the periodic orbit toward distant regions of the phase space [2, 10], and they act as separatrices of the flow. The central interest of this work lies in the appearance of transversal intersections between the manifolds of the Lyapunov orbits around SL_1 and SL_2 that share the same value of the Jacobi constant C_J . This condition is necessary for the coexistence of the orbits and the appearance of heteroclinic connections since a trajectory of the Hamiltonian system preserves C_J throughout its entire evolution.

Figure (4.1) shows the coexistence of the invariant manifolds of two Lyapunov orbits around SL_1 and SL_2 for a given value of C_J . From now on, we will denote as γ_i , the Lyapunov orbit around SL_i .

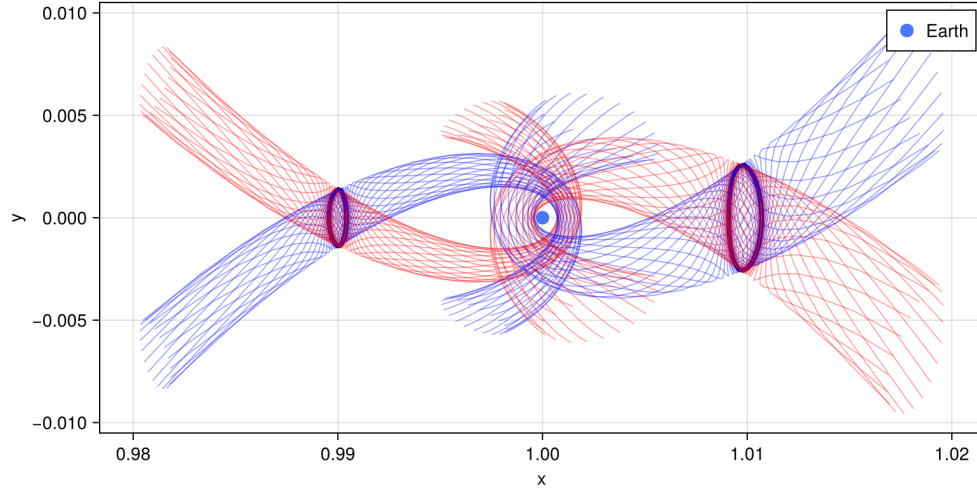


Figure 4.1: Projection of the tube manifolds for Lyapunov Orbit around SL_1 and SL_2 at $C_J = 2.999874$ and $\beta = 0.0005$

In the region between both equilibrium points, near the Earth, the stable and unstable tubes of γ_1 and γ_2 can extend far enough to meet each other. The proximity of the invariant manifolds to the Earth has been applied to low-cost maneuvers of missions around L_1 and L_2 . In the same context, a solar sail can be placed into a periodic orbit γ_1 and γ_2 with a low-cost maneuver using the proximity of these trajectories to the Earth. Once the Jacobi constant is fixed, both manifolds are two-dimensional surfaces within a three-dimensional energy surface, a generic intersection has codimension

1 and manifests as a curve in the phase space. To detect this intersection, an appropriate Poincaré section is employed, which reduces the problem to finding intersections of curves in a two-dimensional plane, as described in the following section. When the manifolds of orbits with different values of C_J are considered, a zero- Δv connection is no longer possible in general, and propulsive maneuvers need to be used to accomplish the connection. It is worth noting that the invariant tubes lead to a complex folding as they pass near the Earth, which can make the detection of intersections delicate. The Poincaré section must be chosen transversal to the flow so that every trajectory crosses it exactly once in a consistent direction [8, 9].

Each stable and unstable manifold can be classified into a positive and negative branch, where the positive one has an initial velocity $v_x > 0$ going to the right and the negative one has an initial velocity $v_x < 0$, going to the left. This can be identified and computed easily once we have $v_x > 0$ in the next way:

$$W^{s\pm} = \bigcup_{t \leq 0} \phi(t, X_0^{\pm s}), \quad W^{u\pm} = \bigcup_{t \geq 0} \phi(t, X_0^{\pm u})$$

With this in mind, for γ_1 we are looking for the positive branches W^{s+} , W^{u+} , and for γ_2 the negative ones W^{s-} , W^{u-} , since we are interested in working in the vicinity around the Earth.

4.2 Poincaré sections

To detect intersections between the invariant manifolds, the Poincaré section is implemented numerically, which reduces the dimension of the problem by projecting trajectories onto a surface transverse to the flow. In principle the intersections of invariant manifolds is a four-dimensional problem. Recall from Section 2.3 the Jacobi constant equation:

$$C_J(x, y, v_x, v_y) = -(v_x^2 + v_y^2) + (x^2 + y^2) + \frac{2(1-\mu)(1-\beta)}{r_1} + \frac{2\mu}{r_2}.$$

In order to reduce this, a C_J value is fixed and a Poincaré section is set at $x = 1 - \mu$, reducing the problem to an intersection of curves projected in a two-dimensional plane.

Considering the neighborhood containing the Earth and the SL_1 and SL_2 regions, two sections of interest can be defined as:

$$\Sigma_1 = \{(x, y, v_x, v_y) \mid x = 1 - \mu, \quad y > 0, \quad v_x < 0\},$$

$$\Sigma_2 = \{(x, y, v_x, v_y) \mid x = 1 - \mu, \quad y < 0, \quad v_x > 0\},$$

which correspond to the position of the secondary body (the Earth) on the x -axis and the upper and lower half-planes, respectively. This choice is natural for two reasons: first, the invariant manifolds of γ_1 and γ_2 extend toward the region of the Earth and cross this plane as can be seen in Figure 4.1; second, by fixing $x = 1 - \mu$ and the value of C_J , the state is determined by two free coordinates (y, v_y) (the component v_x is obtained from the Jacobi constant), which allows the manifolds to be visualized as curves in the (y, v_y) plane. It can be note in Figure 4.1 that for Σ_1 surface, W^{s+} and W^{u-} from γ_1 and γ_2 , respectively, need to be integrated, and analogously for Σ_2 , W^{s-} and W^{u+} from γ_1 and γ_2 , respectively. In the numerical computations used that were performed here we make use of Σ_1 even though computations using Σ_2 could have been used as well.

As mentioned in section 3.3, the computed families of Lyapunov orbits around SL_1 and SL_2 are discrete, this means that the C_J values of each orbit will not generally coincide. The latter is an important fact since the C_J needs to be the same in both orbits in order to have a heteroclinic connection. For this, a numerical interpolation for C_J is implemented to obtain a good initial guess for Lyapunov orbits from a given C_J value, this initial guess is then refined, yielding an error around 10^{-10} .

Once we have appropriate Lyapunov orbits fulfilling the Jacobi constant requirement, the numerical procedure continues interpolating a sufficient number of points on the Lyapunov orbits. For each orbit, N points are distributed along the orbit. The stable and unstable eigenvectors are obtained, and initial conditions on the corresponding manifold are generated via a displacement of magnitude $\epsilon = 10^{-6}$ (as described in Section 3.4). The trajectories are integrated until their first crossing with the section Σ_1 .

For section Σ_1 , the positive branch of the stable manifold of γ_1 is integrated backward in time, since trajectories on $W^{s+}(\gamma_1)$ originate from the region near the Earth and converge to the orbit when integrated forward.

Analogously, for the negative branch of the unstable manifold of γ_2 , its trajectories are integrated forward in time, since trajectories on $W^{u-}(\gamma_2)$ depart from the orbit and head toward the region of the Earth. In both cases the crossing with $x = 1 - \mu$ is recorded for $y > 0$ and $v_x < 0$, and the state of the trajectories at this section is recovered.

The numerical detection of the crossing with the section is carried out by means of an event-detection function during numerical integration, with a tolerance of order 10^{-12} . The result is a set of points (y, v_y) on the section Σ_1 for each manifold. If the stable manifold of γ_1 and the unstable manifold of γ_2 intersect on the Poincaré section, the intersection points correspond to initial conditions of heteroclinic trajectories.

Figure (4.2) shows the numerical computation of invariant manifolds on the Poincaré surface. Figure (4.3) provides a better view of the intersections between both unstable and stable invariant manifolds.

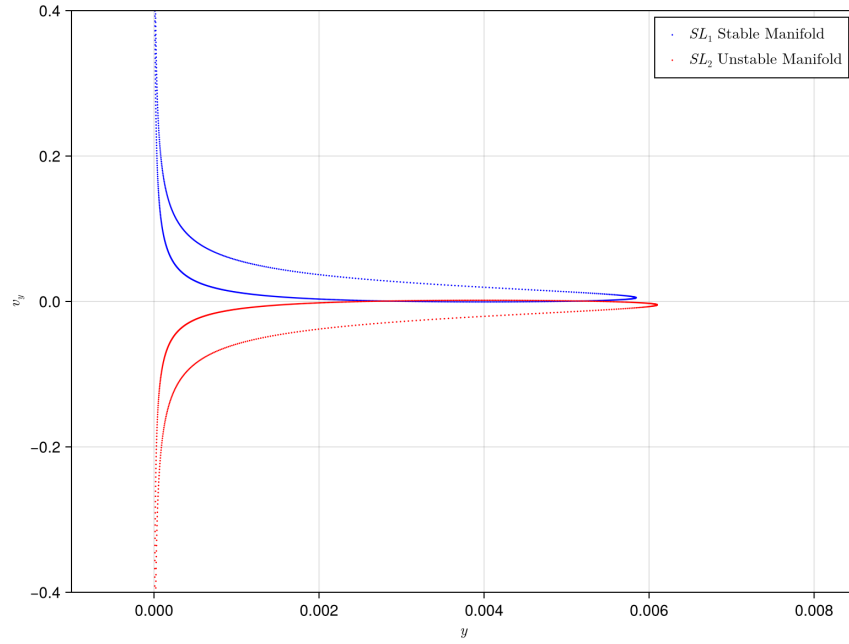


Figure 4.2: Poincaré Map

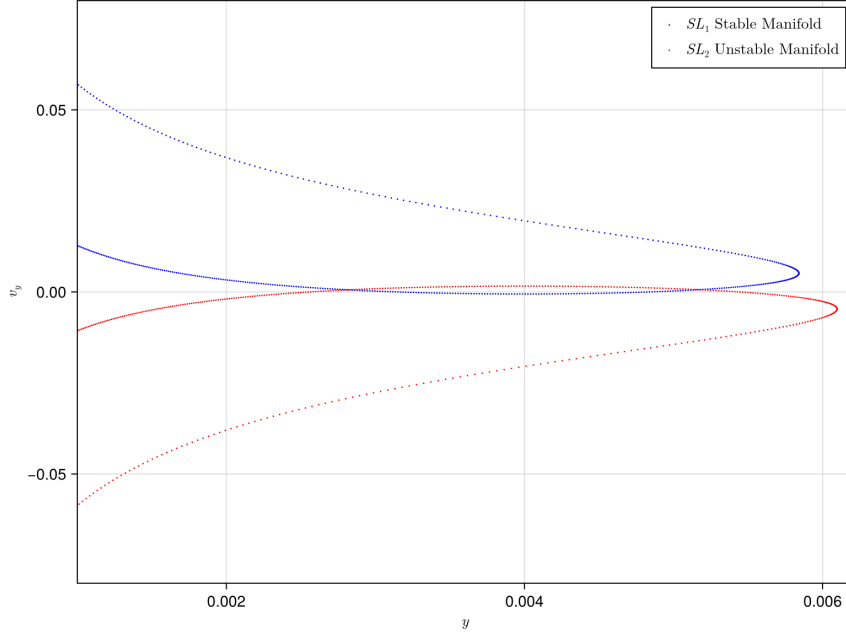


Figure 4.3: Poincaré Map (Zoom)

4.3 Heteroclinic connections between stable and unstable manifolds

In the context of dynamical systems, two types of orbits are of great interest: homoclinic and heteroclinic orbits. Homoclinic orbits are trajectories that converge asymptotically to a hyperbolic periodic orbit γ as $t \rightarrow \pm\infty$, while heteroclinic orbits are trajectories that converge asymptotically to a hyperbolic periodic orbit γ_1 as $t \rightarrow +\infty$ and converge asymptotically to another hyperbolic periodic orbit γ_2 as $t \rightarrow -\infty$. The implications in the CR3BP with solar sail are the existence of trajectories that can depart from a periodic orbit γ_1 and return after a certain time τ , or that can depart and be naturally inserted into a second periodic orbit γ_2 in a time τ ; that is, we can exploit the natural dynamics of the system and achieve a theoretical $\Delta v = 0$ to perform orbital maneuvers of interest in modern space missions in the vicinities of SL_1 and SL_2 .

In this work the orbits of interest are the heteroclinic orbits; more specif-

ically, what we seek are heteroclinic connections.

A heteroclinic connection between a Lyapunov orbit γ_1 and a Lyapunov orbit γ_2 is a trajectory $\phi(t, x_0)$ such that:

$$\lim_{t \rightarrow \pm\infty} \text{dist}(\phi(t, x_0), \gamma_1) = 0, \quad \lim_{t \rightarrow \mp\infty} \text{dist}(\phi(t, x_0), \gamma_2) = 0.$$

This trajectory belongs simultaneously to the stable manifold of γ_1 and to the unstable manifold of γ_2 . In the Poincaré sections defined in the previous section, detecting such a connection reduces to finding points at which the following pairs of curves have a non-empty intersection:

$$\begin{aligned} (W^{s+}(\gamma_1) \cap \Sigma_1) &\cap (W^{u-}(\gamma_2) \cap \Sigma_1), \\ (W^{u+}(\gamma_1) \cap \Sigma_2) &\cap (W^{s-}(\gamma_2) \cap \Sigma_2). \end{aligned}$$

The detection algorithm implemented consists of two stages: a coarse estimate and a refinement of the detection.

The procedure is described using Σ_1 . In the coarse estimate, the curves $(W^{s+}(\gamma_1) \cap \Sigma_1)$ and $(W^{u-}(\gamma_2) \cap \Sigma_1)$ are represented as discrete sets of points (y_k, v_{y_k}) on the Poincaré section. Intersections between consecutive segments parameterized by two consecutive (y_k, v_{y_k}) points of both curves are sought. For each pair of segments — one from the unstable manifold and one from the stable manifold — it is verified whether they cross by solving the corresponding 2×2 linear system for the intersection of two line segments. A bounding-box filter is applied beforehand to avoid unnecessary computations between distant segments. This can be obtained by defining boxes with the two consecutive points mentioned and verifying if there is a non-empty intersection. Once this condition is fulfilled, the approximate intersection is computed, providing an approximate intersection, therefore, an approximate heteroclinic connection. Figure (4.4) shows the results of the first detection algorithm. Figure (4.4) provides a better view into the first intersection from left to right, making evident that this first algorithm is an approximation.

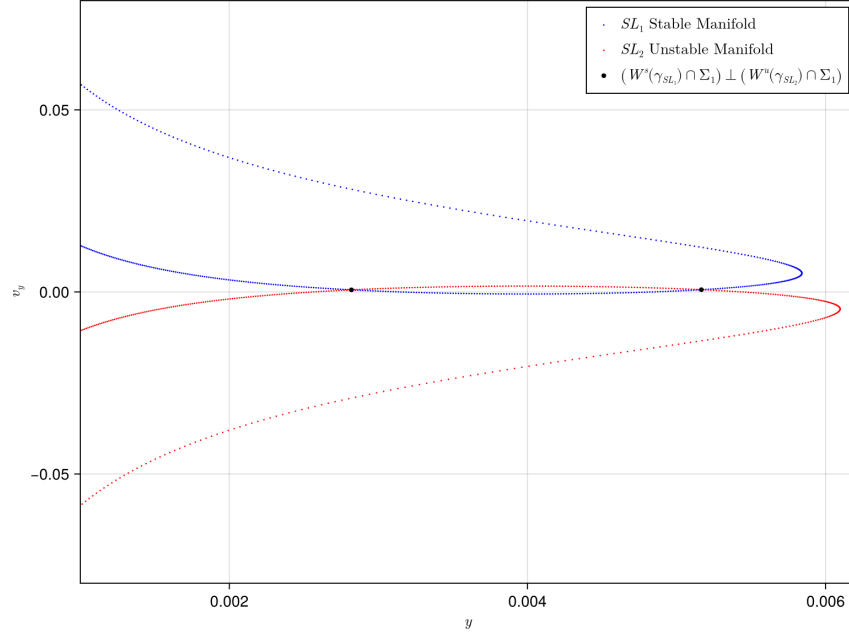


Figure 4.4: Detection of transversal intersections between invariant manifolds

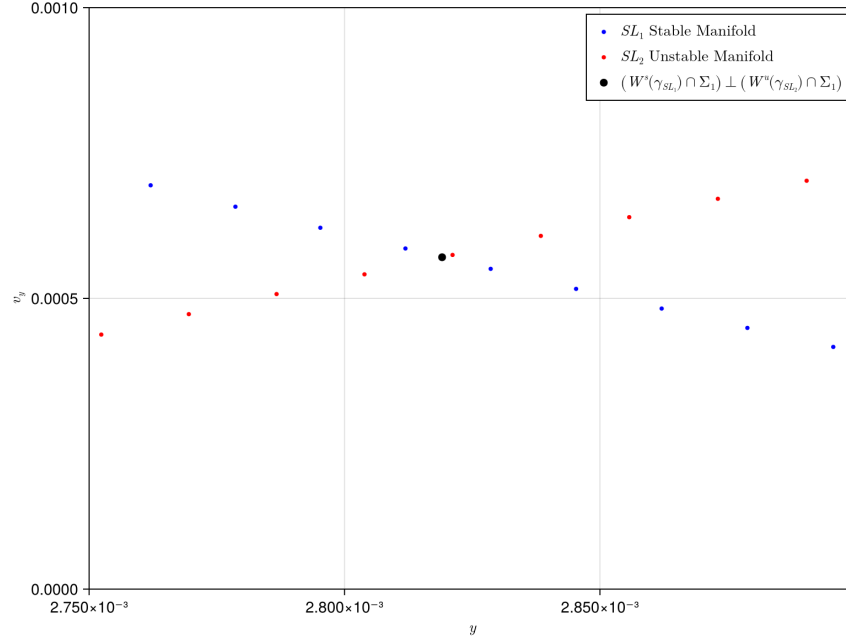


Figure 4.5: Zoom at the first detection of transversal intersection

Using the first approximation, (see Figure (4.4)), it is possible to compute an approximate heteroclinic trajectory. Since the intersection provides a $(y, \dot{y})$ coordinate while $x = 1 - \mu$ and C_J are fixed, the remaining velocity component $\dot{x}$ is recovered by solving (2.12), yielding a complete initial condition for the heteroclinic trajectory. The result is shown in Figure 4.6.

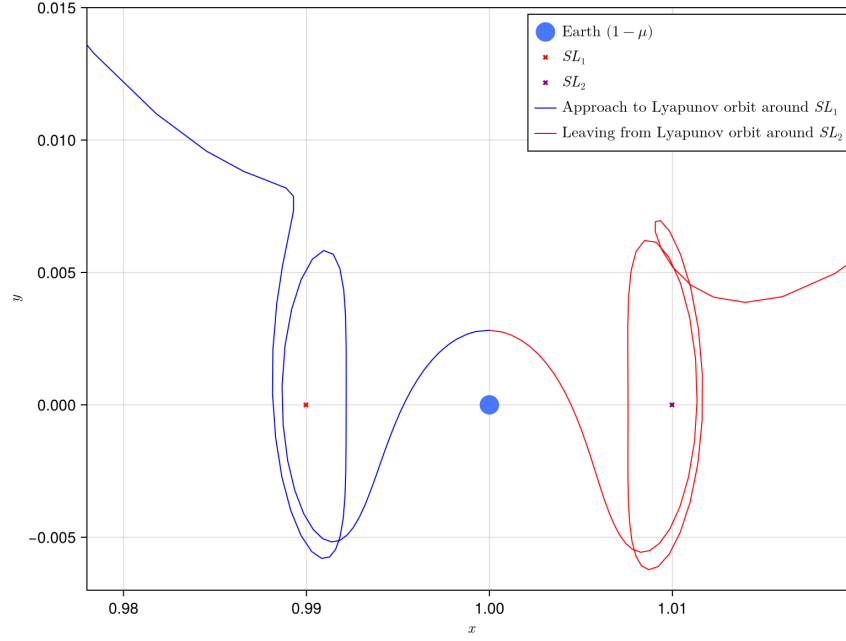


Figure 4.6: Approximated heteroclinic trajectory

What follows is to refine the neighborhood of the detection point. Once a coarse intersection is detected, it is refined by increasing the density of points on the manifold in the neighborhood of the detected intersection. To this end, the time interval of the periodic orbit that generates the trajectories near the intersection is selected and subdivided; the eigenvectors are recomputed, new initial conditions on the invariant manifold are generated, and the resulting trajectories are integrated to the Poincaré section. The intersection detection is then repeated on these refined curves, obtaining a more precise approximation of the heteroclinic connection.

Figure 4.7 shows the refined manifold and the new intersection detected, which corresponds to the same connection identified in Figure 4.5. Following the same process described above to recover the complete initial condition yields a precise heteroclinic orbit, shown in Figure 4.8.

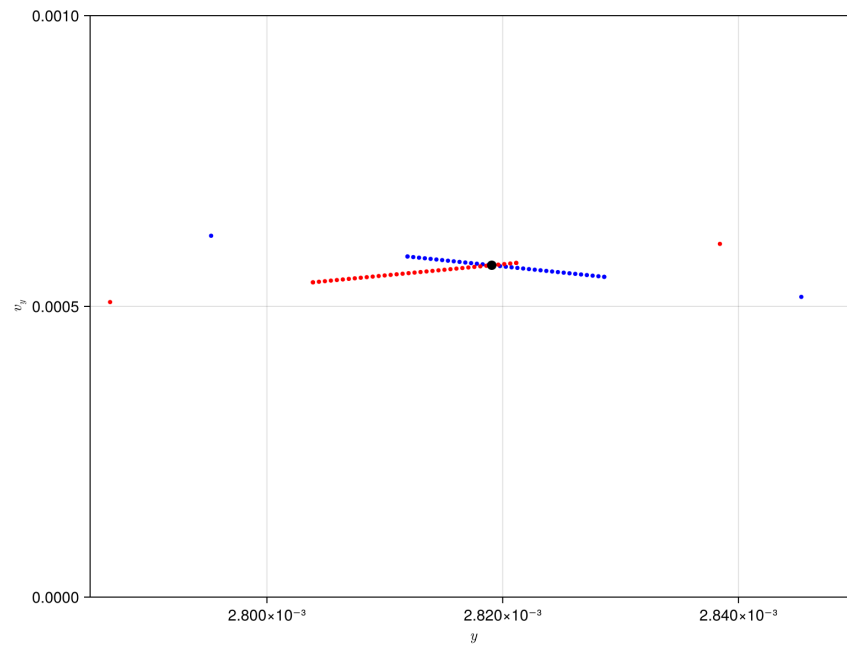


Figure 4.7: Zoom at the new detection of transversal intersection

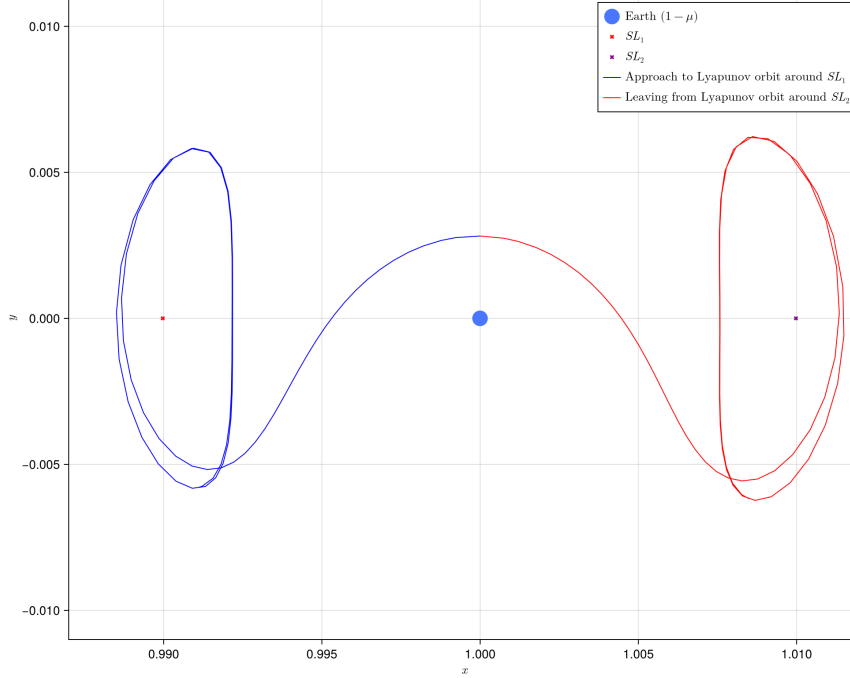


Figure 4.8: Heteroclinic trajectory

The heteroclinic trajectory just described corresponds to a zero- Δv transfer from SL_2 to SL_1 . A symmetric methodology applied to the other Poincaré section yields a zero- Δv transfer in the reverse direction, from SL_1 to SL_2 . More generally, additional intersections exist at the n -th return to the Poincaré section, producing trajectories that pass through the neighborhoods of the Sun and the Earth more than once. The family of connections at the first return is of particular practical interest, since it corresponds to the shortest transfer time between the two regions. With this in mind, the next section explores the critical value $C_j^*(\beta)$ at which heteroclinic connections first appear at the first return; this threshold also has implications for the structural stability of the connections, since a transversal intersection persists under small perturbations of the energy.

It is worth remarking that a true heteroclinic orbit has a theoretically infinite time of flight, as it asymptotically approaches a periodic orbit in both forward and backward time. In practice, the numerically computed trajectory is a pseudo-orbit; the shadowing lemma guarantees the existence of a true orbit of the system that remains arbitrarily close to it for all time [29].

4.4 Heteroclinic tangencies between stable and unstable manifolds

Not every pair of Lyapunov orbits around SL_1 and SL_2 admits a heteroclinic connection at the first return to the Poincaré section. The existence of an intersection between the invariant manifolds on Σ depends on the value of the Jacobi constant C_J , which defines both the amplitude of the periodic orbits and the extent of their associated invariant manifolds. This fact has physical implications in the time of flight associated with the transfer maneuver, making it crucial for applications in space missions.

For sufficiently large values of C_J (low energy, small-amplitude orbits), the associated stable and unstable manifolds associated to SL_1 and SL_2 Lyapunov orbits do not reach one another at the first crossing of Σ_i , this means that no first-return heteroclinic connection exists. As C_J decreases (increasing energy, larger-amplitude orbits), the manifold tubes expand and the corresponding curves on Σ_i approach each other. At a critical value $C_J^*(\beta)$, the two curves become tangent on Σ_i , denoting their first contact, with no transversal crossing. For values of C_J below this threshold, the curves intersect transversally, yielding structurally stable heteroclinic connections that persist under small perturbations.

It is well known that the classical CR3BP has a rich variety of periodic and quasi-periodic orbits. Although in this work, only planar Lyapunov orbits are considered, this is a first step towards the computation and exploration of the translation of the periodic and quasi-periodic in the CR3BP with solar sail, which results in inclined periodic and quasi-periodic orbits. Heteroclinic connections can be found between different types of orbits, as the maneuver used in Genesis mission. In this context, the identification of the heteroclinic tangencies, besides being of interest from the point of view of dynamical systems, is worthwhile since it allows us to use it as a design criterion in this kind of trajectory design for space mission maneuvers.

It is important to note that the critical value $C_J^*(\beta)$ determined here corresponds to the tangency at the *first* return to the Poincaré section. Heteroclinic connections involving multiple returns — in which a manifold tube winds around the primary body several times before intersecting the other manifold — can exist for larger values of C_J (lower energy). A systematic classification of such many-return connections, organized by the number of revolutions, was carried out by Canalias and Masdemont [10] and Gidea and

Masdemont [11] for the classical CR3BP; the same qualitative structure is expected to persist in the solar sail case.

To locate the critical value $C_J^*(\beta)$ at which the first-return tangency occurs, a bisection procedure is applied over C_J . Two values are identified in order to define an interval, $C_J^{(a)}$, for which no intersection is detected on Σ at the first return, and $C_J^{(b)} < C_J^{(a)}$, for which a transversal intersection exists. At each bisection step, the midpoint value

$$C_J^{\text{mid}} = \frac{C_J^{(a)} + C_J^{(b)}}{2}$$

is evaluated as follows:

1. The Lyapunov orbits around SL_1 and SL_2 corresponding to C_J^{mid} are obtained by interpolation over the previously computed families and refined by differential correction.
2. The stable and unstable manifolds are propagated to the Poincaré section Σ_i .
3. The coarse-detection algorithm described in Section 4.3 is applied to determine whether an intersection exists at the first return.
4. The bracketing interval is updated: if no intersection is found, $C_J^{(a)} \leftarrow C_J^{\text{mid}}$; otherwise, $C_J^{(b)} \leftarrow C_J^{\text{mid}}$.

The procedure is iterated until $|C_J^{(a)} - C_J^{(b)}| < \varepsilon$ for a prescribed tolerance ε , yielding the critical value $C_J^*(\beta)$.

Figures 4.9 and 4.10 show the Poincaré map for the selected Jacobi constants as limits.

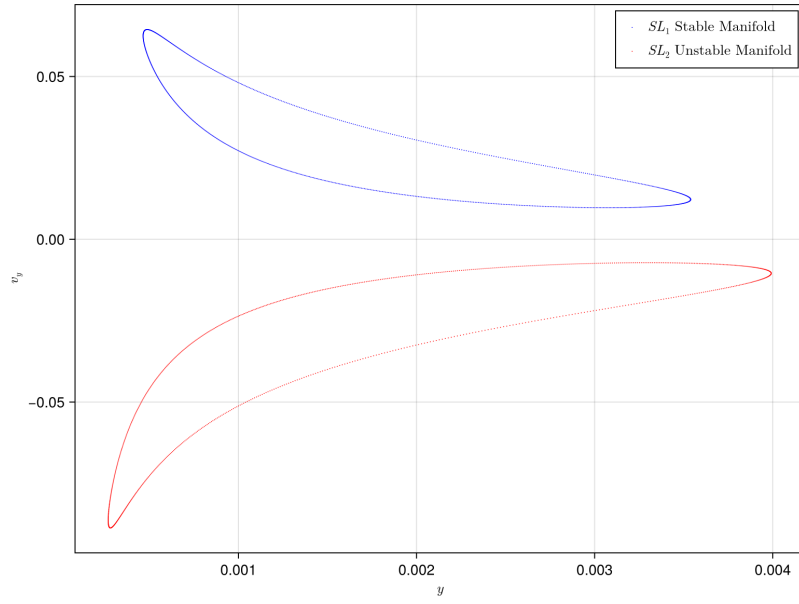


Figure 4.9: Poincaré map $x = 1 - \mu$ where no intersection is detected

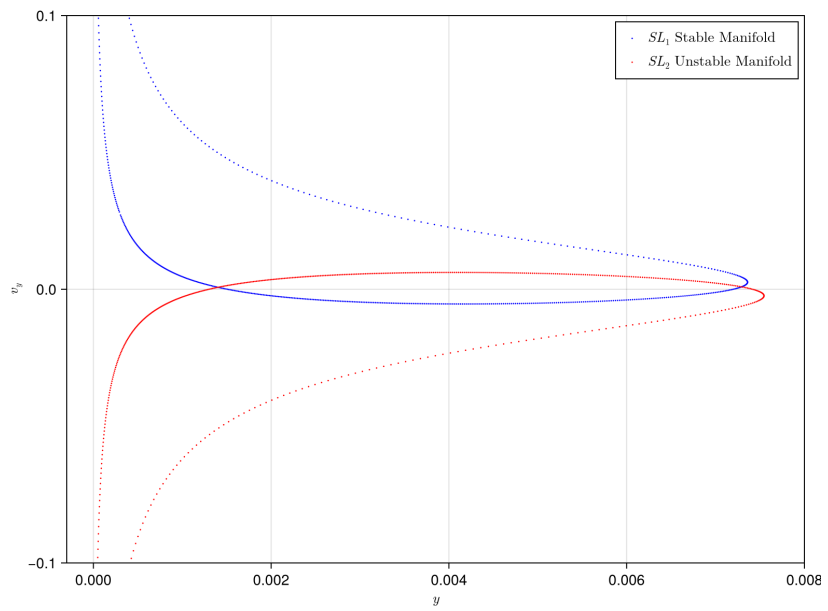


Figure 4.10: Poincaré map $x = 1 - \mu$ where transversal intersection exist

Once we have an interval $[C_J^{(a)}, C_J^{(b)}]$ defined, we proceed with the first guess for a tangency. Figure 4.11 shows the first guess of a tangent intersection detected with the coarse-detection algorithm.

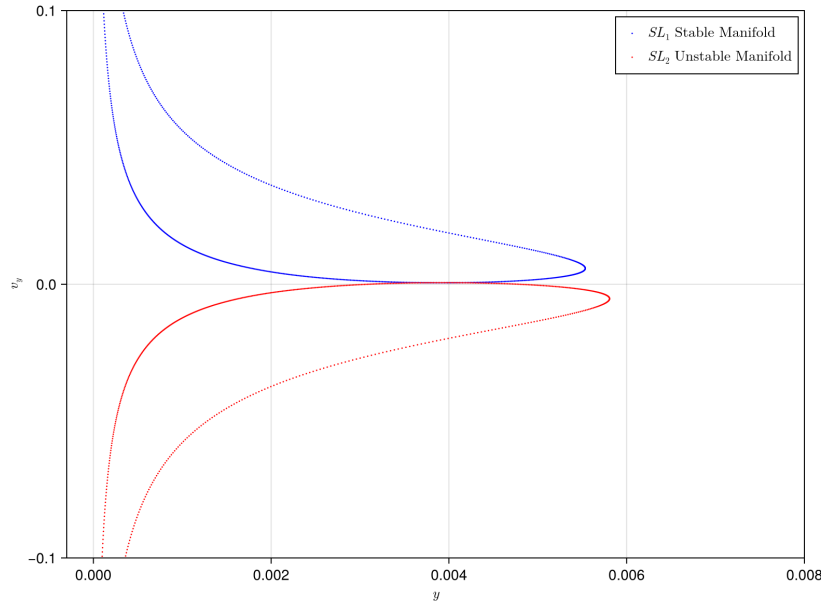


Figure 4.11: Poincaré map for the tangent intersection guess

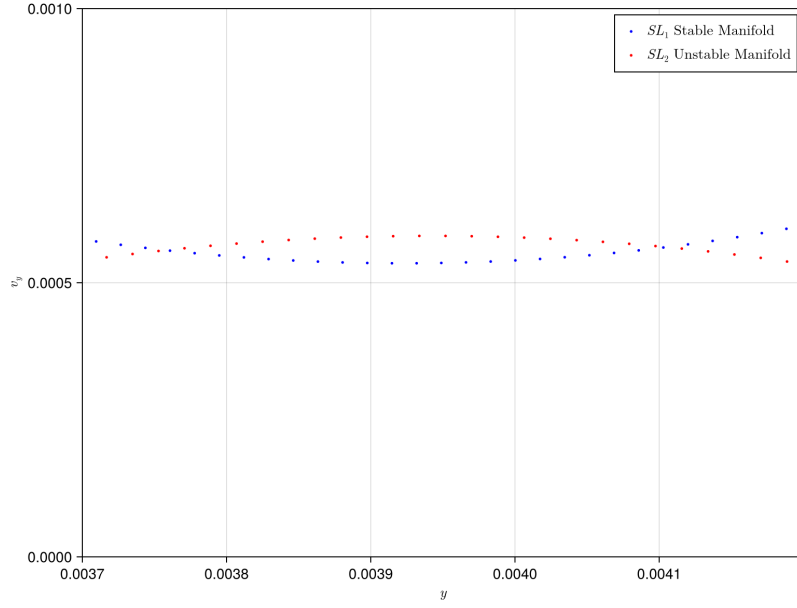


Figure 4.12: Zoom at tangent intersection neighborhood

The coarse algorithm provides a new interval $[C_J^{(a)}, C_J^{(b)}]$, that will be used in the bisection algorithm. Figure 4.13 shows the tangent intersection detected with a tolerance $\varepsilon = 1e^{-12}$.

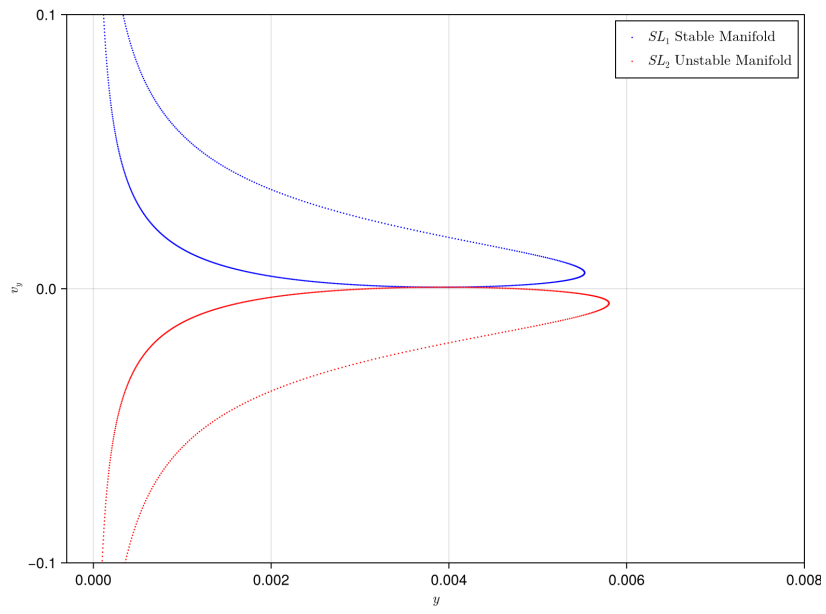


Figure 4.13: Poincaré map for the tangent intersection

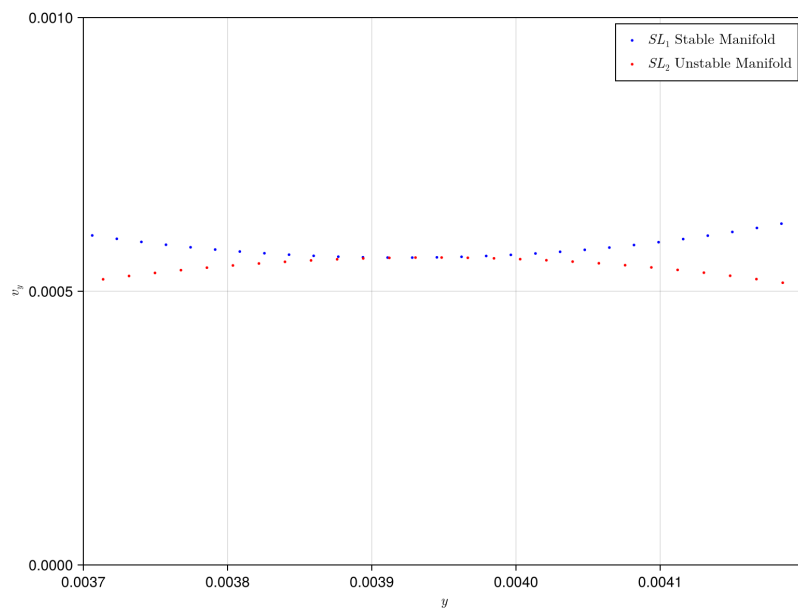


Figure 4.14: Zoom at tangent intersection neighborhood

Table 4.1 shows that $C_J^*(\beta)$ decreases as β increases. A stronger sail displaces SL_1 and SL_2 further from the classical libration points, shrinking the Hill regions and allowing the periodic orbits to exist at lower energy levels; the invariant manifold tubes, therefore, reach the Poincaré section Σ at lower values of C_J .

β	$C_J^*(\beta)$
0.0	3.000780076750054
0.0001	3.0005800771844626
0.0005	2.999779877045203
0.001	2.9987791729250732
0.005	2.9907551700262145
0.01	2.980676698714027

Table 4.1: Critical value of the Jacobi constant $C_J^*(\beta)$ at which tangent intersection between the stable and unstable invariant manifolds of the Lyapunov orbits around SL_1 and SL_2 is detected for each sail lightness number β .

Chapter 5

Results and Discussion

The preceding chapters established the analytical and numerical scheme required to study the Hamiltonian case of the planar CR3BP with solar sail, the modified equations of motion and displaced equilibria SL_1, SL_2 (Chapter 2), the differential corrector and continuation method for the Lyapunov families together with their Floquet analysis (Chapter 3), and the detection procedure for heteroclinic connections via Poincaré sections together with the bisection algorithm that locates the critical Jacobi constant $C_J^*(\beta)$ (Chapter 4). This chapter consolidates the numerical results produced, discusses their dynamical and mission-design implications, and translates the dimensionless output into physical units relevant to a Sun-Earth solar sail mission.

The presentation is organized as follows. Section 5.1 gives a unified description of the complete algorithm, so that the reader has the full sequence of numerical steps used to produce every figure in this work. Section 5.2 reports the families of Lyapunov periodic orbits around SL_1 and SL_2 for several values of the lightness number β , together with the corresponding zero-velocity curves and the resulting range of admissible Jacobi constants. Section 5.3 discusses the critical Jacobi constant $C_J^*(\beta)$ and the heteroclinic tangencies that bound the existence of connections. Section 5.4 catalogs the families of heteroclinic connections detected for different (β, C_J) regimes. Section 5.5 converts the results into dimensional Sun-Earth units and estimates the propulsive cost (Δv) of the resulting transfers. Section 5.6 discusses the limitations of the perpendicular-sail assumption and outlines the natural extensions of the present work. Section 5.7 summarizes the main findings.

5.1 Summary of the numerical implementation

The complete procedure used in this work to detect low-energy trajectories between the regions of SL_1 and SL_2 in the planar Hamiltonian solar sail CR3BP is summarized below. The individual steps were derived and justified in Chapters 2-4; here they are collected in a single routine so that the reader can follow how the numerical results of this chapter were produced.

1. **Selection of the sail parameter.** A value of the lightness number $\beta \in [0, \beta_{\max}]$ is fixed. For the Sun-Earth system the upper bound used in this work is $\beta_{\max} = 0.01$.
2. **Computation of the displaced equilibria SL_1, SL_2 .** The positions of the collinear equilibria of the modified potential (Section 2.2) are obtained by a one-dimensional Newton iteration on the x -axis. The spectrum of the linearized system (Section 2.5) is then evaluated at each equilibrium and used both to visualize the changes on the local dynamics (saddle $\times$ center type) and to seed the Lyapunov family.
3. **Generation of the Lyapunov families.** Two small-amplitude initial guesses are obtained from the center eigenvector at the equilibrium and refined with the differential corrector of Section 3.2. A spline extrapolation based continuation is then applied to span the family up to a sufficiently large amplitude at which the shooting method retains the prescribed tolerance and the orbits remain physically meaningful. The output is, for each equilibrium, a discrete family $\{\gamma_i^k\}$, $k \in \{1, \dots, N\}$ of periodic orbits indexed by the continuation parameter, together with their period and Jacobi constant $C_J(\gamma_i^k)$, $i \in \{1, 2\}$ represents the relative Lagrange point (SL_i).
4. **Energy matching by spline interpolation in C_J .** For a target Jacobi constant C_J^{tgt} , the two orbits $\gamma_i(C_J^{\text{tgt}}) \in \text{family}(SL_i)$ sharing the same energy are obtained by cubic spline interpolation in the initial conditions of the discrete families as a function of C_J . The resulting initial guess is then refined by the differential corrector, yielding the energy-matched orbit with an error of order $\sim 10^{-12}$ in C_J .

5. **Monodromy and Floquet eigenvectors.** For each energy-matched orbit, the State Transition Matrix (STM) is integrated over one period in order to obtain the monodromy matrix $M = \Phi(T, 0)$. Its non-trivial Floquet multipliers and the associated eigenvectors are computed, and the stable and unstable directions at each of the points sampled along the orbit are then obtained by transporting these eigenvectors with the STM $\Phi(t, 0)$, as described in Section 3.5.
6. **Generation of the stable and unstable manifolds.** Initial conditions on the local stable and unstable manifolds are generated by displacing each periodic orbit point along the corresponding eigenvector by $\varepsilon = 10^{-6}$ in the non-dimensional units, both in the positive and negative branches.
7. **Poincaré section and event detection.** Trajectories are integrated until they reach the section $\Sigma_1 = \{x = 1 - \mu, y > 0, \dot{x} < 0\}$ (Section 4.2). The crossing is determined by an event-detection function with tolerance 10^{-12} , recovering the state $(y, \dot{y})$ at the section.
8. **Coarse intersection detection.** The discrete representations of the stable and unstable curves on Σ are scanned segment by segment with a bounding-box filter. Pairs of segments whose boxes overlap are tested for intersection by solving the corresponding 2×2 linear system. The output is the set of approximate intersection points, each one identifying an approximate heteroclinic trajectory.
9. **Local refinement.** Around each detected intersection, the time intervals of the Lyapunov orbits that produced the relevant manifold initial conditions are interpolated. New eigenvectors are computed by transporting the STM on the refined grid, the new trajectories are integrated to Σ_1 , and the coarse detection is used again on the densified manifold curves. The process is iterated until the residual in the four-dimensional ambient state of the two manifolds at the intersection point falls below 10^{-10} .
10. **Bisection over C_J for the tangency.** Once a regime with transversal intersections has been identified, two values $[C_J^{(a)}, C_J^{(b)}]$ are determined such that the coarse algorithm detects no intersection at the upper end of the bracket and at least one intersection at the lower end. Steps 4-8

are repeated at the midpoint, the bracket is updated, and the procedure terminates when $|C_J^{(a)} - C_J^{(b)}| < \varepsilon_C$ with $\varepsilon_C = 10^{-12}$, yielding the critical value $C_J^*(\beta)$.

The entire workflow was implemented in Julia using the `DifferentialEquations.jl` package with a Vern9 adaptive integrator and absolute/relative tolerances $(10^{-12}, 10^{-12})$. The algorithm is modular, and the only β -dependent quantities are the location of the equilibria, the family of Lyapunov orbits, and the Floquet eigenvectors. This workflow can be computationally optimized by implementing a parallelization of specific tasks related to the computation of the invariant manifolds; this is discussed in Section 5.6.

5.2 Families of Lyapunov orbits around SL_1 and SL_2

For each value of the sail lightness number $\beta \in \{0, 0.0001, 0.0005, 0.001, 0.005, 0.01\}$ the continuation procedure described in Section 3.3 was applied to construct the families of planar Lyapunov orbits around the displaced equilibria SL_1 and SL_2 . The coexisting families, restricted to their shared range of C_J , are shown in the synodic plane at the end of this section.

Admissible range of the Jacobi constant. One of the first steps towards finding heteroclinic connections is to delimit the range of the Jacobi in which Lyapunov orbits around both equilibria coexist. To this end, we compute the Jacobi constant values of the complete families around SL_1 and SL_2 and obtained the values shared by both families, that is, the maximum of the two minima and the minimum of the two maxima. For this work a routine was implemented that computes, by numerical continuation, 6000 orbits around SL_1 and 6000 orbits around SL_2 for each value of β . Since the numerical continuation is spline-based, the continuation step is not uniform and varies with β ; consequently the range of C_J covered by each family, and therefore the coexistence range, also varies with β and does not follow a concrete pattern, at least numerically. The values reported in Tables 5.1 and 5.2 are thus the ranges obtained computationally from this specific routine, and not a claim about the full theoretical extent of each family. Even so, this could suggest a complete behavior.

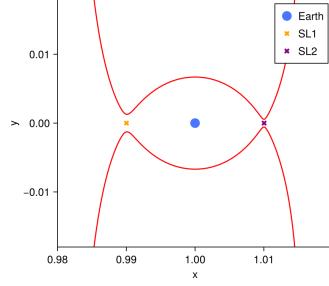
β	$C_J^{\min}(SL_1)-C_J^{\max}(SL_1)$	$C_J^{\min}(SL_2)-C_J^{\max}(SL_2)$
0.0	3.000355649837–3.000890693247	3.000056839349–3.000886688599
0.0001	2.999857221785–3.000688678551	2.99985583297–3.000688675585
0.0005	2.999353085017–2.999880596845	2.99905173682–2.999896601987
0.001	2.998350400601– 2.998870442727	2.998373620706– 2.998906461861
0.005	2.990324486041–2.990787050348	2.990389633693–2.990983481694
0.01	2.980280355476– 2.980677001114	2.980400133559–2.98107541156

Table 5.1: Minimum and maximum C_J of the Lyapunov families around SL_1 and SL_2 , obtained from the continuation routine described in the text.

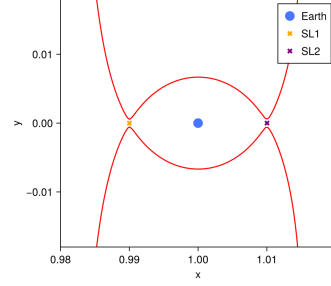
β	Coexistence interval of C_J
0.0	3.000355649837332–3.000886688599421
0.0001	2.99985722178565–3.0006886755857645
0.0005	2.999353085017204–2.999880596845497
0.001	2.9983736207064884–2.9988704427278248
0.005	2.9903896336935554–2.990787050348653
0.01	2.9804001335598596–2.9806770011144774

Table 5.2: Range of C_J in which the two Lyapunov families coexist, obtained as the intersection of the intervals in Table 5.1.

Zero-velocity curves. A geometrical intuition of the problem is obtained by computing and plotting the zero-velocity curves. As already discussed in Chapter 2, the shape of the zero-velocity curves and, in particular, the neck regions undergo significant changes that have important consequences for the dynamics of the solar sail and hence for the mission concept design. One of the main consequences, visible in Figure 5.3, is the large gap between the two neck regions at considerably large β . When one family has reached very large amplitudes, almost in collision with the Earth, the other family is only beginning to emerge from its equilibrium.

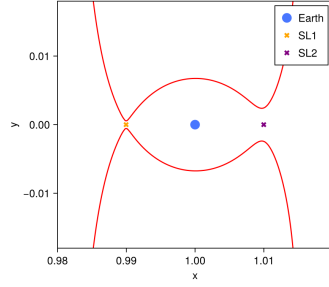


(a) Zero-velocity curves for $\beta = 0.0$

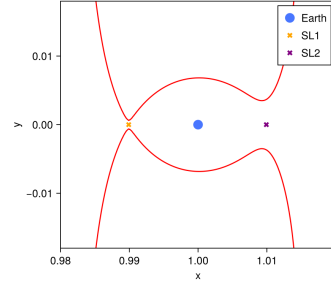


(b) Zero-velocity curves for $\beta = 0.0001$

Figure 5.1: Zero-velocity curves for $\beta = 0.0$ and $\beta = 0.0001$

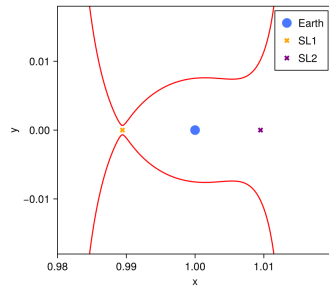


(a) Zero-velocity curves for $\beta = 0.0005$

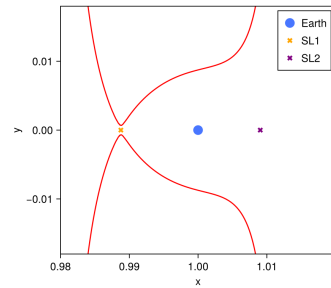


(b) Zero-velocity curves for $\beta = 0.001$

Figure 5.2: Zero-velocity curves $\beta = 0.0005$ and $\beta = 0.001$



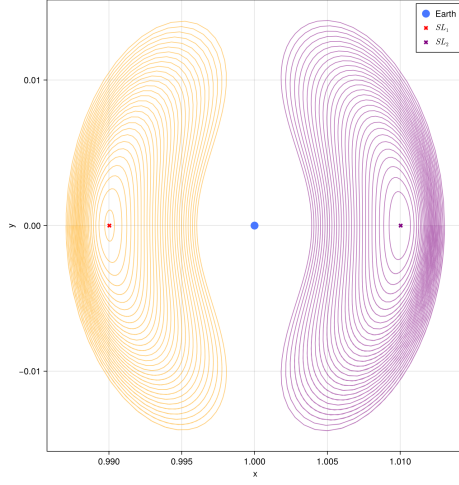
(a) Zero-velocity curves for $\beta = 0.005$



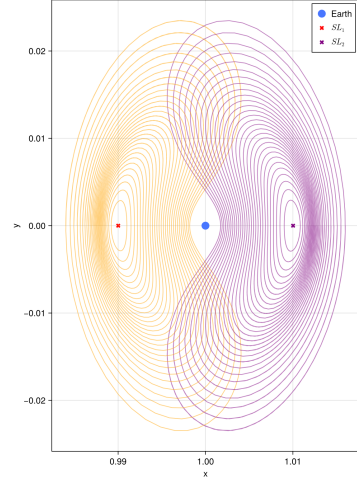
(b) Zero-velocity curves for $\beta = 0.01$

Figure 5.3: Zero-velocity curves $\beta = 0.005$ and $\beta = 0.01$

This becomes clearer in the following figures, where the difference in amplitude between energy-matched orbits grows as β increases.

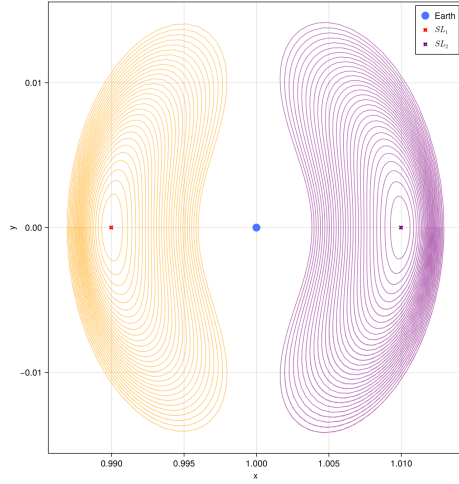


(a) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.0$

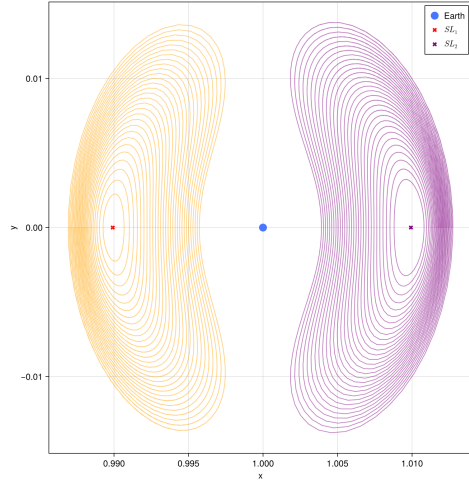


(b) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.0001$

Figure 5.4: Lyapunov orbit families

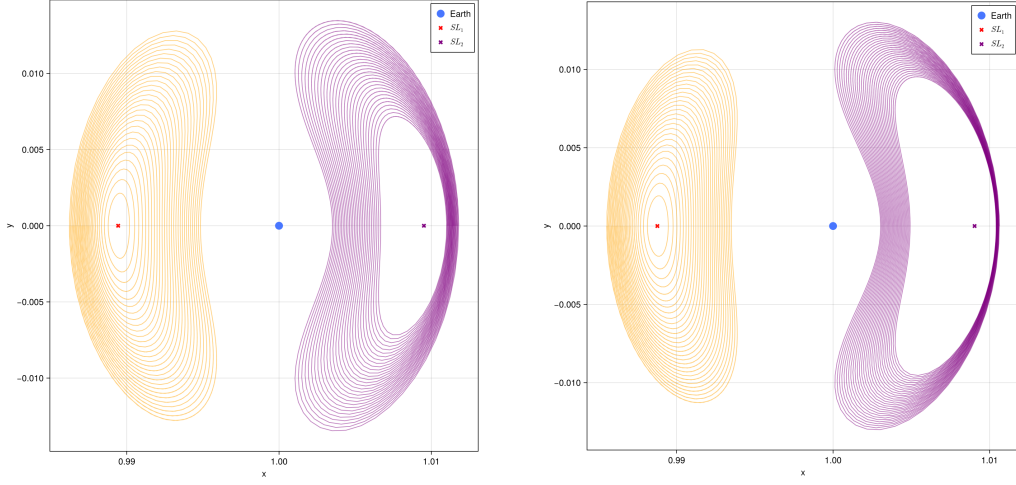


(a) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.0005$



(b) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.001$

Figure 5.5: Lyapunov orbit families



(a) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.005$ (b) Lyapunov orbit families around SL_1 , SL_2 with $\beta = 0.01$

Figure 5.6: Lyapunov orbit families

5.3 The critical Jacobi constant $C_J^*(\beta)$ and heteroclinic tangencies

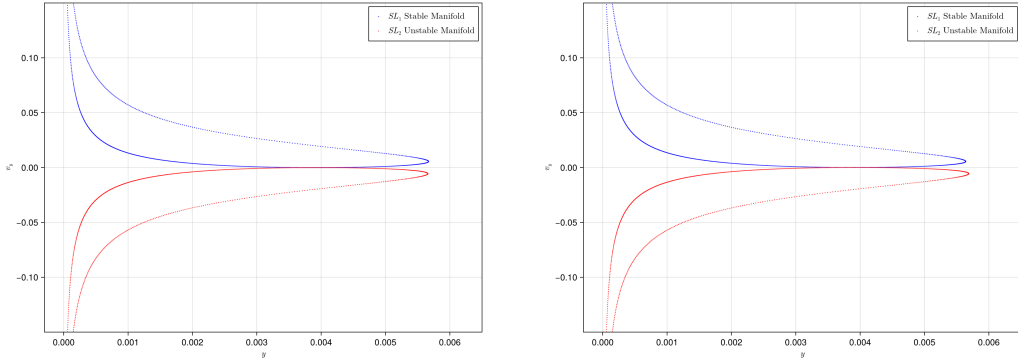
It is useful to determine the value of the Jacobi constant below which heteroclinic connections become available, and to express it in terms of a physical parameter, here, the x -amplitude of the Lyapunov orbit at which the heteroclinic tangency is detected and is used as the physical parameter. For C_J below this critical value the orbits amplitudes increase; the critical value therefore corresponds to a minimum x -amplitude at which a heteroclinic connection, and hence a zero- Δv transfer with a known x -amplitude is possible. This minimum amplitude depends on β , which ultimately translates into a minimum x -amplitude, corresponding to each orbit, for which zero- Δv transfer maneuvers are available as a function of β .

Table 5.3 reports the critical values $C_J^*(\beta)$ obtained with the bisection algorithm of Chapter 4.

β	$C_J^*(\beta)$
0.0	3.000780076750054
0.0001	3.0005800771844626
0.0005	2.999779877045203
0.001	2.9987791729250732
0.005	2.9907551700262145
0.01	2.980676698714027

Table 5.3: Critical Jacobi constant $C_J^*(\beta)$ below which heteroclinic connections between the Lyapunov families of SL_1 and SL_2 are detected, obtained with the bisection algorithm of Chapter 4.

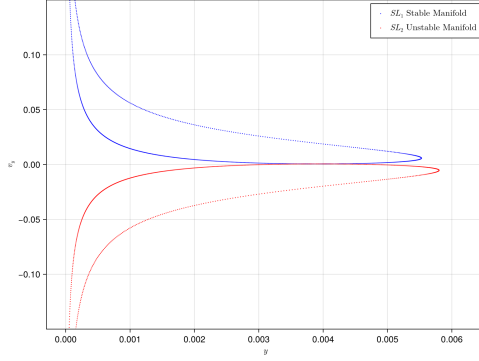
Further aspects can be observed by restricting the coexisting Lyapunov orbits in the sense of the Jacobi constant (Section 5.2) and by plotting the Poincaré sections at these tangencies. Figures 5.7, 5.8 and 5.9 show how the invariant manifolds intersect the Poincaré section Σ_1 as β varies.



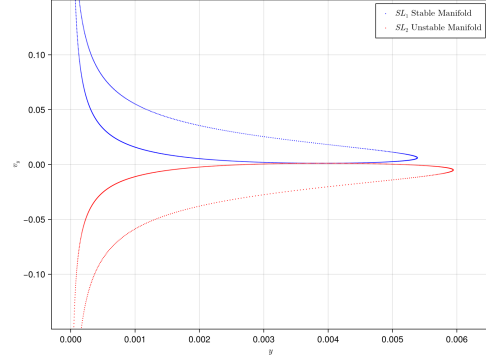
(a) Heteroclinic tangency with $\beta = 0.0$

(b) Heteroclinic tangency with $\beta = 0.0001$

Figure 5.7: Heteroclinic tangencies on the Poincaré section for $\beta = 0$ and $\beta = 0.0001$.

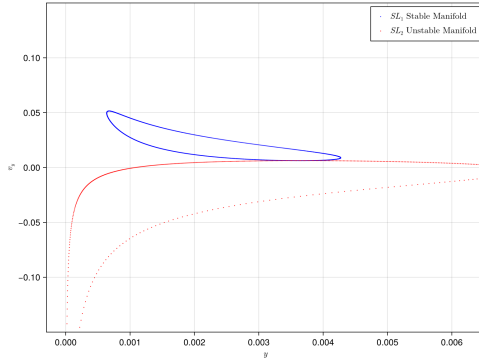


(a) Heteroclinic tangency with $\beta = 0.0005$

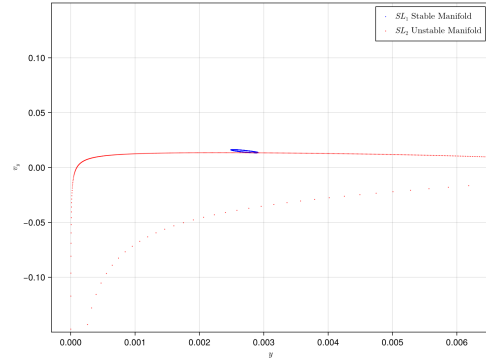


(b) Heteroclinic tangency with $\beta = 0.001$

Figure 5.8: Heteroclinic tangencies on the Poincaré section for $\beta = 0.0005$ and $\beta = 0.001$.



(a) Heteroclinic tangency with $\beta = 0.005$



(b) Heteroclinic tangency with $\beta = 0.01$

Figure 5.9: Heteroclinic tangencies on the Poincaré section for $\beta = 0.005$ and $\beta = 0.01$.

As Figure 5.9 shows, increasing β makes the tangency progressively harder to obtain. The reason is the growing size disparity between Lyapunov orbits that share the same Jacobi constant. There may be a value of β beyond which the heteroclinic tangency no longer exists and, consequently, heteroclinic trajectories are no longer available.

Minimum amplitudes at the tangency. The size disparity that limits the existence of the tangency can be made quantitative by reporting the x -

amplitudes of the two families exactly at $C_J = C_J^*(\beta)$, that is, the smallest amplitudes that the two families can adopt while still admitting a heteroclinic intersection. These minimum amplitudes are listed in Table 5.4.

β	$\Delta x_{\gamma_1}^*$ [DU]	$\Delta x_{\gamma_1}^*$ [km]	$\Delta x_{\gamma_2}^*$ [DU]	$\Delta x_{\gamma_2}^*$ [km]	$\Delta x_{\gamma_2}^*/\Delta x_{\gamma_1}^*$
0.0	0.001422	212 688	0.001442	215 646	1.01
0.0001	0.001416	211 863	0.001447	216 501	1.02
0.0005	0.001393	208 363	0.001469	219 721	1.05
0.001	0.001361	203 540	0.001493	223 303	1.09
0.005	0.000975	145 834	0.001569	234 822	1.61
0.01	0.000118	17 643	0.00139	207 942	11.78

Table 5.4: Minimum x -amplitudes of the Lyapunov families around SL_1 and SL_2 at the heteroclinic tangency $C_J = C_J^*(\beta)$ (cf. Table 5.3), in non-dimensional units (DU) and in kilometres (1 DU = $1.495\,978\,707 \times 10^8$ km). The last column gives the size ratio $\Delta x_{\gamma_2}^*/\Delta x_{\gamma_1}^*$ between the two energy-matched orbits at the tangency.

For $\beta \leq 0.0005$ the two orbits are essentially the same size (both around $\sim 210,000$ km, with a size ratio close to unity); from $\beta = 0.001$ onwards the symmetry breaks and the orbit around SL_2 grows substantially faster than the one around SL_1 . At $\beta = 0.01$ the trend reaches an extreme: the orbit around SL_1 has shrunk to $\Delta x_{\gamma_1}^* \approx 17,600$ km, while the orbit around SL_2 is around $\Delta x_{\gamma_2}^* \approx 207,942$ km, with a large size ratio between the two energy-matched orbits at the tangency.

5.4 Heteroclinic connections at different (β, C_J)

For each value of β considered in Section 5.2 the detection algorithm of Chapter 4 was applied at specific values of C_J within the coexistence interval reported in Table 5.2 and corresponding to x -amplitudes close to a reference x -amplitude. The resulting sets of heteroclinic connections at the first return to the Poincaré sections Σ_1 are shown in Figures 5.10-5.15.

The first-return family is of particular practical interest as it corresponds to the shortest transfer time and therefore to the mission-design scenario in which the spacecraft spends the least amount of time outside the controlled neighborhood of SL_1 and SL_2 . The dimensional values of the transfer times are reported in Section 5.5.

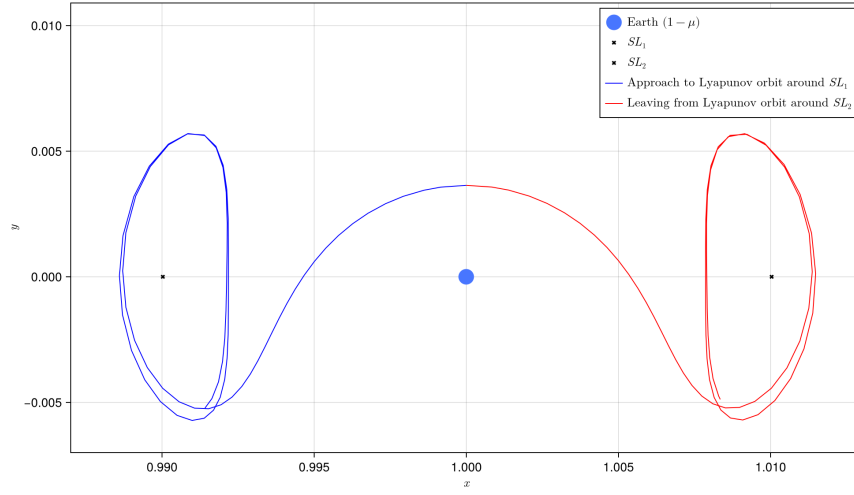


Figure 5.10: Heteroclinic orbit for $\beta = 0.0$ at $C_J \approx 3.00078$

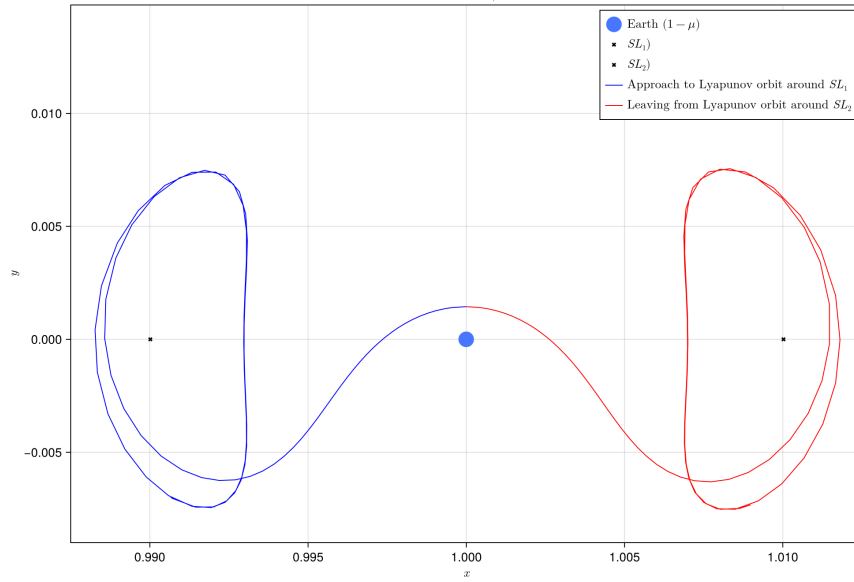


Figure 5.11: Heteroclinic orbit for $\beta = 0.0001$ at $C_J \approx 3.00057$

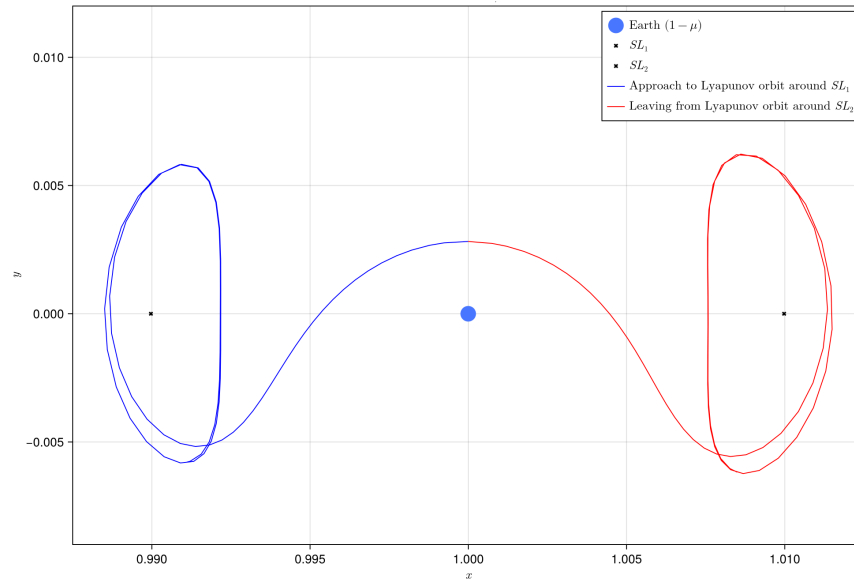


Figure 5.12: Heteroclinic orbit for $\beta = 0.0005$ at $C_J \approx 2.99977$

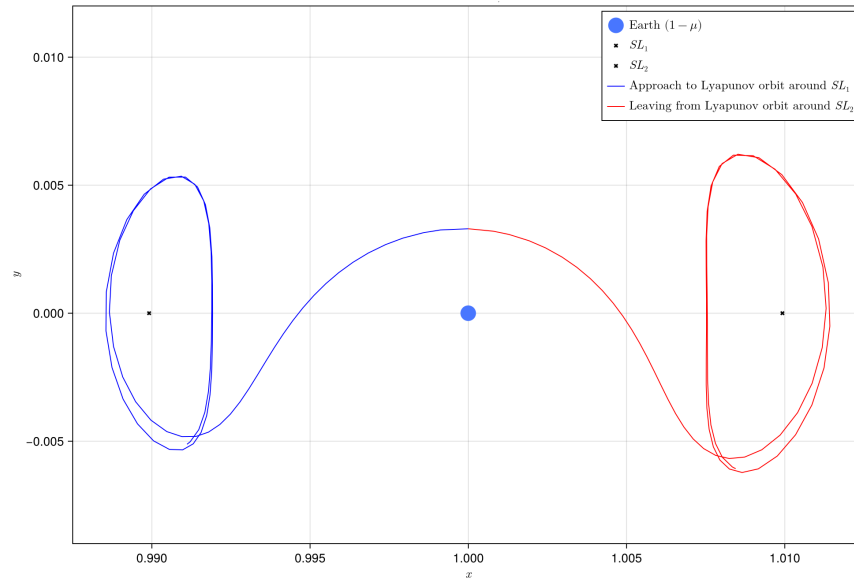


Figure 5.13: Heteroclinic orbit for $\beta = 0.001$ at $C_J \approx 2.99878$

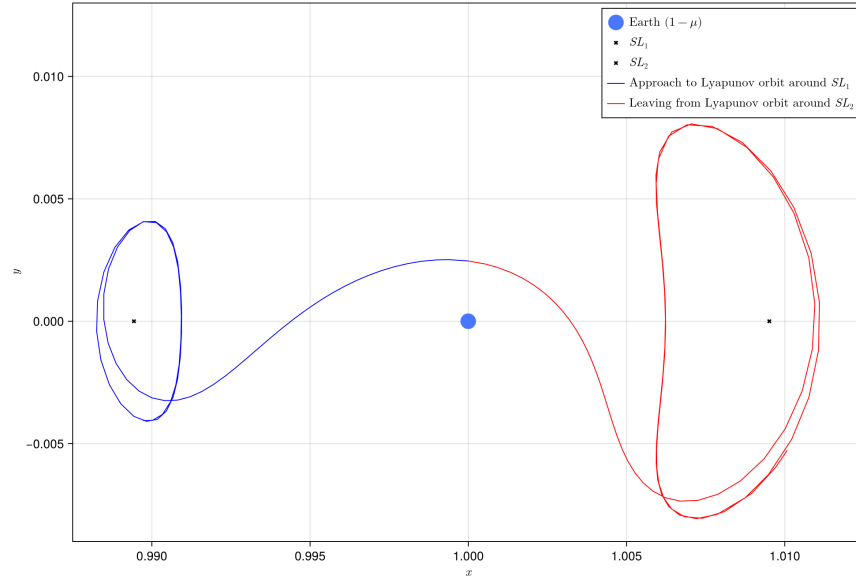


Figure 5.14: Heteroclinic orbit for $\beta = 0.005$ at $C_J \approx 2.99075$

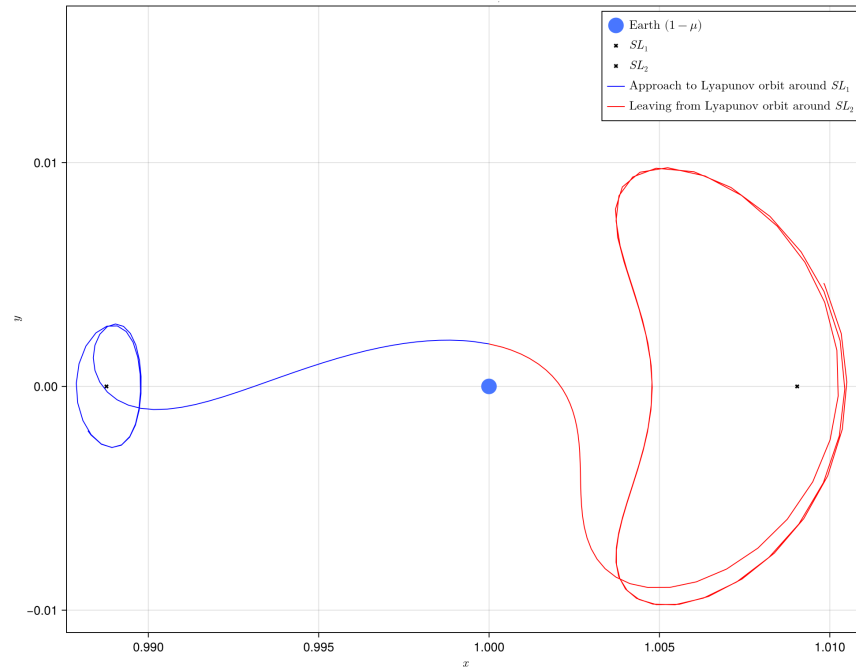


Figure 5.15: Heteroclinic orbit for $\beta = 0.01$ at $C_J \approx 2.98066$

The consequences discussed in Sections 5.2 and 5.3 can be observed in this section as well, and they impose relevant limits on the x -amplitudes of the Lyapunov orbits available for affordable heteroclinic connections. These figures make it clear that, beyond a certain value of β , heteroclinic connections may cease to be feasible, when the Lyapunov orbit around SL_1 is only beginning to emerge, the orbit around SL_2 already has a very large amplitude, possibly already reaching collision with the Earth, where the continuation of the Lyapunov family terminates.

5.5 Translation to dimensional units

The dynamical results of the previous sections are reported in the non-dimensional synodic units of the Sun-Earth CR3BP, in which the distance unit (d), the time unit (t), and the velocity unit (v) are

$$\text{DU} = a, \quad \text{TU} = \frac{1}{\omega}, \quad \text{VU} = \frac{d}{t} = \omega a, \quad (5.1)$$

with a the Sun-Earth semi-major axis and $\omega = \sqrt{G(m_{\text{Sun}} + m_{\text{Earth}})/a^3}$ the mean motion. Substituting the standard numerical values $a = 1.495\,978\,707 \times 10^{11}$ m and $G(m_{\text{Sun}} + m_{\text{Earth}}) = 1.327\,168\,5 \times 10^{20}$ m³ s⁻², considering $m_{\text{Sun}} = 1.98847 \times 10^{30}$ and $m_{\text{Earth}} = 5.9722 \times 10^{24}$ yields

$$\text{DU} \approx 1.496 \times 10^8 \text{ km}, \quad \text{TU} \approx 5.022 \times 10^6 \text{ s} \approx 58.13 \text{ days}, \quad \text{VU} \approx 29\,785.18 \text{ m s}^{-1}. \quad (5.2)$$

The Sun-Earth mass parameter is $\mu = m_{\text{Earth}}/(m_{\text{Sun}} + m_{\text{Earth}}) \approx 3.0034806 \times 10^{-6}$. These factors convert the non-dimensional outputs of the integrator into SI units through the relations $d' = \text{DU} \cdot d$, $t' = \text{TU} \cdot t$, and $v' = \text{VU} \cdot v$, in a form suitable for mission-design discussions.

Reference amplitude for the Lyapunov orbits. As a benchmark for Lyapunov orbits of practical interest in mission design, we adopt a reference x -amplitude representative of the orbit of the James Webb Space Telescope (JWST) about the Sun-Earth L_2 point. The JWST does not follow a planar Lyapunov orbit but a three-dimensional halo orbit. The operational trajectory documented by the Space Telescope Science Institute [32] shows that, in the rotating libration-point frame, the orbit's excursion along the Sun-Earth direction reaches values of the order of $1.4\text{--}2.0 \times 10^5$ km from L_2 ;

the trajectory diagram in that reference shows, for example, a representative operational position at $L + 200$ d with $x \approx 1.40 \times 10^5$ km.

Throughout this work, the x -amplitude of a planar Lyapunov orbit around SL_i is defined as the minimum distance from the orbit to the equilibrium point,

$$\Delta x_{\gamma_i} \equiv \min_{p \in \gamma_i} \|p|_{y=0} - SL_i\|. \quad (5.3)$$

We therefore take $\Delta x \approx 150,000$ km $\approx 1.003 \times 10^{-3}$ AU as the reference x -amplitude in the Sun-Earth Lyapunov families considered here.

Following Section 5.3, the critical Jacobi constant from which zero- Δv connections become available is known for each β . For each value of β , the Jacobi constant has been selected so that the resulting Lyapunov orbit around SL_1 attains an x -amplitude as close as possible to the JWST reference of $\Delta x \approx 150,000$ km, accepting whichever amplitude the energy-matched orbit around SL_2 happens to produce. Table 5.5 reports the results. For $\beta \leq 0.001$ the smallest amplitude reachable while preserving the heteroclinic connection is bounded below by the tangency, around 200,000–220,000 km, so the reference cannot be reached.

β	C_J (selected)	Δx_{γ_1} [DU]	Δx_{γ_1} [km]	Δx_{γ_2} [DU]	Δx_{γ_2} [km]
0.0	3.0007800171	0.001422	212 734	0.001442	215 695
0.0001	3.0005739439	0.001448	216 618	0.001479	221 310
0.0005	2.9997753649	0.001418	212 134	0.001491	223 033
0.001	2.9987784445	0.001365	204 214	0.001496	223 797
0.005	2.9907549269	0.000978	146 341	0.001570	234 900
0.01	2.9806580611	0.000888	132 791	0.001398	209 179

Table 5.5: Energy-matched Lyapunov orbits used for the heteroclinic connections of Section 5.4. For each β , the Jacobi constant has been selected to drive the orbit around SL_1 as close as possible to the reference x -amplitude of $\sim 150,000$ km defined in equation (5.3). The columns report the selected Jacobi constant (rounded to ten decimal places) and the resulting x -amplitudes of the orbits around SL_1 and SL_2 , in non-dimensional units (DU) and in kilometres (1 DU = $1.495\,978\,707 \times 10^8$ km).

As these orbits are not circumferences or symmetric with respect to the y -axis, the table does not show the size disparity between both orbits, this is more obvious in section 5.4.

Flight times of the heteroclinic transfers. The flight time of a heteroclinic trajectory connecting the Lyapunov families of SL_1 and SL_2 is the total integration time required for a trajectory on the unstable manifold of γ_2 to reach the Poincaré section Σ_1 and the subsequent integration time required to reach the target orbit γ_1 along the stable manifold of γ_1 . In non-dimensional units, the flight time decomposes as $T_{\text{flight}} = t_u + t_s$, with t_u the time along the unstable branch of γ_2 from the seed point on $W^u(\gamma_2)$ to the first intersection with Σ_1 , and t_s the time along the stable branch of γ_1 from Σ_1 backwards to the seed point on $W^s(\gamma_1)$. Converted to days through the time unit of equation (5.2), $T_{\text{flight}} [\text{days}] = T_{\text{flight}} [\text{TU}] \times 58.13$. Table 5.6 reports the flight times of the first-return heteroclinic connections of Figures 5.10-5.15.

β	t_s [TU]	t_u [TU]	T_{flight} [TU]	T_{flight} [days]
0.0	4.0065	4.0665	8.0730	469.3
0.0001	4.0283	4.0810	8.1093	471.4
0.0005	4.0384	4.0622	8.1007	470.9
0.001	4.0482	4.0325	8.0807	469.7
0.005	4.2518	3.9323	8.1841	475.7
0.01	4.6411	4.0457	8.6867	505.0

Table 5.6: Flight times of the first-return heteroclinic connections between the Lyapunov families of SL_2 and SL_1 , for the selected values of C_J reported in Table 5.5. t_s and t_u are the integration times along the stable branch of γ_1 and the unstable branch of γ_2 , respectively, from the manifold seed points to the Poincaré section Σ_1 .

The total flight time is remarkably stable across the range $\beta \in [0, 0.001]$, fluctuating around $T_{\text{flight}} \approx 8.08\text{--}8.11 \text{ TU} \approx 469\text{--}471$ days. For larger values of the sail lightness number, this time grows: at $\beta = 0.005$ the flight time reaches $T_{\text{flight}} \approx 8.18 \text{ TU} \approx 476$ days and at $\beta = 0.01$ it increases to $T_{\text{flight}} \approx 8.69 \text{ TU} \approx 505$ days. The increase is driven mostly by the stable branch t_s , which grows from $t_s \approx 4.01$ to $t_s \approx 4.64 \text{ TU}$ between $\beta = 0$ and $\beta = 0.01$, while the unstable branch t_u remains close to 4.0 TU for all the cases.

5.6 Limitations and future work

The simplifications adopted throughout this work, most notably the perpendicular sail (Hamiltonian) configuration, the restriction to the planar CR3BP, and the absence of additional gravitational perturbers, were chosen to isolate the role of the lightness number β in shaping the heteroclinic connection structure between SL_1 and SL_2 . Each of these simplifications, however, defines a natural direction in which the present analysis can be extended. The principal limitations are listed below, each paired with the future work it suggests.

Beyond the perpendicular-sail configuration. The Hamiltonian model used here assumes that the sail attitude is held exactly perpendicular to the Sun–spacecraft line at all times, a configuration that, in practice, requires active attitude control. A more realistic study should consider small deviations from perpendicularity, parameterized, for instance, by a sail tilt angle δ . Setting $\delta \neq 0$ breaks the Hamiltonian structure of the equations of motion: although the resulting Lyapunov-like orbits remain confined to a plane, that plane is inclined with respect to the ecliptic. The natural continuation of the present analysis is therefore to extend the differential corrector and the manifold computation to small but non-zero δ , treating (β, δ) as a two-parameter family and following how the critical tangency $C_J^*(\beta)$ changes. Connections between planar Lyapunov orbits of the Hamiltonian case and these tilted planar orbits of the non-Hamiltonian case, and, more generally, connections between planar Lyapunov orbits and three-dimensional quasi-periodic invariant tori, can be approached through boundary-value-problem formulations of the kind developed by Calleja et al. [12]. The companion work of Ariadna Farrés on solar-sail attitude perturbations [1] provides a natural starting point for the $\delta \neq 0$ continuation.

Perturbations and full-ephemeris dynamics. The transfers reported here are zero- Δv in the planar CR3BP with the Sun–Earth as the only celestial bodies. In an actual Sun–Earth mission the Moon, Jupiter, and other planets perturb the trajectory; in addition, station-keeping is required to maintain the spacecraft on either Lyapunov orbit. A natural sequence of extensions is to incorporate the Moon as a third perturber, then to move to a full-ephemeris model in which the nominal trajectories obtained here serve

as initial guesses for an optimal control problem that maintains the actual trajectory close to the nominal heteroclinic one.

Idealized solar radiation pressure. The acceleration of the sail in this work has been modeled as $(1 - \beta)$ times the solar gravitational acceleration, a model that implicitly assumes a perfectly flat, perfectly reflective sail. A more realistic model should consider non-perfect reflection due to the sail-surface imperfections.

Mission-design specifics: launch-window catalogues. In an operational setting, heteroclinic transfers are not computed once but many times in order to have a catalogue indexed by launch window. For a specific mission concept, a defined launch vehicle, a target science orbit, and an acceptable time of flight, the analysis of Section 5.5 can be extended into a systematic catalogue of transfer opportunities as a function of the launch date, using the planar Hamiltonian model as a first-cut filter before refining the candidates in full ephemeris. This is a direct mission-design application of the present work.

Computational improvements: parallelization and Jet Transport. The most time-consuming step of the algorithm in Section 5.1 is the integration of the $2N$ manifold trajectories required by the Poincaré detection at each energy level, where N is the number of orbit samples used to interpolate each Lyapunov orbit (in this work $N \approx 10^3$). Each of these trajectories is independent of the others until the intersection-detection phase, so the invariant manifold computation can be parallelized. Different initial conditions can be integrated on distinct processors, with each filament of the manifold assigned to different threads and then assembled by joining all the filaments produced in separate processors. This should lead to a significant computing time reduction. Beyond this, Jet Transport could be used to propagate not the individual seed trajectories but local Taylor expansions of the flow around each seed, allowing an analytic study of lobe dynamics and of the variational behavior of the heteroclinic connections under small displacements of the seed point.

Theoretical study of $C_J^*(\beta)$. The critical Jacobi constant $C_J^*(\beta)$ and the associated tangency amplitudes deserve a deeper theoretical treatment than

the purely numerical characterization of Sections 5.2 and 5.3. The data of Table 5.4 already suggest a collapse in the existence of heteroclinic connections: at $\beta = 0.01$ the orbit around SL_1 has shrunk to $\Delta x_{\gamma_1}^* \approx 17,600$ km, essentially the equilibrium point, and the size ratio between the two energy-matched orbits at the tangency reaches 11.78. Two specific questions emerge from this: (i) whether the rapid collapse of $\Delta x_{\gamma_1}^*$ near $\beta = 0.01$ signals the existence of a critical $\beta = \beta^*$ beyond which no heteroclinic intersection survives in the coexistence range; and (ii) whether this terminal β^* corresponds geometrically to the Earth-collision endpoint of the family around SL_2 , in the sense that the smallest energy-matched orbit around SL_1 that is still admissible requires a partner around SL_2 that has already reached the Earth's neighborhood. A refined grid of β , possibly combined with continuation in the second parameter δ introduced above, would clarify the structure of C_J^* as a one or two-parameter function and would establish if the collapse continues smoothly down to a definite threshold.

Other dynamical environments and propulsion models. The framework developed in this work is not specific to the Sun-Earth system. The same algorithm can be transported to the Earth-Moon CR3BP, where low-energy transfers are of growing interest for cislunar mission design and where the dependence on the sail parameter has been studied for related orbital regimes by Heiligers [16], or to more elaborate models such as the bicircular restricted four-body problem and the Hill restricted four-body problem. The lightness number β also admits a reinterpretation in terms of a continuous low-thrust acceleration of constant magnitude pointing along the Sun-spacecraft line, so the heteroclinic structure reported here may serve as a backbone for mission concepts based on low-thrust propulsion that mimic the sail-perpendicular acceleration profile.

5.7 Conclusions

A numerical routine was implemented, using theoretical tools from dynamical systems, to study the CR3BP with a solar sail. The resulting algorithm computes the displaced equilibrium points SL_1 and SL_2 , generates the families of planar Lyapunov orbits around them, determines their shared range of the Jacobi constant, propagates the associated stable and unstable invariant manifolds, and detects transversal intersections of these manifolds on a

Poincaré section, from which the corresponding heteroclinic orbit is recovered. Using this procedure, we provide numerical evidence of the existence of heteroclinic connections between planar Lyapunov orbits around SL_1 and SL_2 in the CR3BP with a solar sail, for several values of the sail lightness number β . Different zero- Δv transfer maneuvers were computed considering an itinerary from γ_2 to γ_1 and using a reference x -amplitude inspired by the orbit of the James Webb Space Telescope. The results show that as β increases, the x -amplitude disparity between the two connected orbits grows, reaching a ratio $\Delta x_{\gamma_2}^*/\Delta x_{\gamma_1}^* = 11.78$ at $\beta = 0.01$, a value at which the orbit around SL_1 is close to collapse to the equilibrium point, signaling the imminent end of the heteroclinic connection mechanism between SL_1 and SL_2 . The set of Lyapunov orbits available for heteroclinic connection consequently shrinks with β , suggesting that for some critical value of β this set becomes empty and no heteroclinic intersection persists. The dimensional flight times of these theoretical zero- Δv maneuvers fall in the range 470–505 days, illustrating the typical trade-off between energy cost and time of flight in low-energy mission design. More broadly, this work highlights the substantial computational effort that trajectory design demands in nontrivial dynamical models and shows how a deep understanding of the underlying dynamics can be turned into a constructive procedure for generating low-cost transfer maneuvers. The addition of solar-sail propulsion to the CR3BP enriches the range of possible space mission concepts by introducing β as a parameter, however, the present analysis indicates that this enrichment has a structural limit at sufficiently large β , beyond which the heteroclinic connection mechanism between SL_1 and SL_2 ceases to exist.

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