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ON THE COMPUTATION OF THE SCREW PARAMETERS OF A RIGID-BODY MOTION. PART II: INFINITESIMALLY-SEPARATED POSITIONS

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ON THE COMPUTATION OF THE SCREW PARAMETERS OF A RIGID-BODY MOTION. PART II: INFINITESIMALLY-SEPARATED POSITIONS.

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ABSTRACT

This paper is a continuation of a previous one concerned with the determination of the screw parameters of a rigid-body motion for finitely-separated positions. A series of results paralleling those presented in the aforementioned paper are derived, which are then applied to devise algorithms for the computation of the said parameters for infinitesimally-separated positions. These algorithms are then realised within a subprogram which is outlined. A set of numerical examples is included.

ÜBER DIE BERECHNUNG DER SCHRAUBPARAMETER DER BEWEGUNG EINES STARKÖRPERS. TEIL II: UNENDLICH BENACHBARTEN LAGEN

ZUSAMMENFASSUNG

Dieser Bericht ist eine Erweiterung eines Vorhergehenden, der sich um die Bestimmung der Schraubparameter bei endlich benachbarten Lagen handelt. Eine Reihe Ergebnisse, die ähnlich diese im früheren Bericht hinstellt, wird abgeleitet und weiter auf den Aufbau von Algorithmen zur Berechnung der obengenannten Parameter bei unendlichen benachbarten Lagen angewandt. Diese Algorithmen werden innerhalb eines Unterprogrammes verwirklicht, das kurz beschrieben wird. Zuletzt wird eine Reihe numerischen Beispiele eingeschlossen.

NOMENCLATURE

$\underset{\sim}{r}$: lower-case underlined character, a n-dimensional vector.

$\underset{\sim}{A}$: upper-case underlined character, a mxn matrix

$\underset{\sim}{r}^T, \underset{\sim}{A}^T$: the transpose of a vector or, correspondingly, of a matrix.

$\phi'(\underset{\sim}{r})$: the gradient of ϕ with respect to $\underset{\sim}{r}$, a n-dimensional vector

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$\phi''(\tilde{r})$: the Hessian of ϕ with respect to $\tilde{r}$, a nxn matrix

$\tilde{r}^T\tilde{u}, \tilde{r}.\tilde{u}$: the inner product of $\tilde{r}$ and $\tilde{u}$

$||\tilde{r}||$: the Euclidean norm of $\tilde{r}$, i.e. $\sqrt{\tilde{r}^T\tilde{r}}$

INTRODUCTION

The computation of the screw parameters of a rigid-body motion for finitely-separated positions was dealt with in a previous paper [1][2]. The present paper is concerned with the computation of the said parameters for infinitesimally-separated positions. It is shown that the well-known general formulae for the computation of those parameters fail under certain cases which, very special as they may be, can arise rather frequently, however.

DETERMINATION OF THE PARAMETERS OF THE INSTANTANEOUS SCREW. GENERAL CASE.

The interest of counting with accurate means of computing the screw parameters of rigid-body motions within the realm of linkage synthesis becomes apparent from the work reported by some researchers [2], [3], [4]. In [2, pp.56-59], formulae are derived to compute the parameters of the instantaneous screw. These fail, however, in some particular cases, as is shown within the paper. Some results are derived first in the following.

<u>*Theorem 1*</u>. *Any rigid-body motion is equivalent to a screw motion, composed of a velocity $\tilde{v}_0$ and a spin $\tilde{\omega}$ parallel to $\tilde{v}_0$. The locus of those points of the body having a uniform velocity $\tilde{v}_0$ are located on a line parallel to $\tilde{\omega}$, called the instantantaneous screw axis, and $\tilde{v}_0$ is of minimum magnitude. Letting $\tilde{a}$ be the position vector of an arbitrary point of the body, whose velocity is denoted by $\tilde{v}_A$, the point of the screw axis lying closest to the origin is given by the position vector $\tilde{r}_0$, computed as*

$$\tilde{r}_0 = \tilde{a} + \frac{1}{||\tilde{\omega}||}\left[\tilde{\omega} \times \tilde{v}_A - (\tilde{\omega}.\tilde{a})\tilde{\omega}\right] \tag{1}$$

[2] Numbers in brackets designate references at the end of the paper.

Proof

In [5] it is shown that the rotation of a rigid body is given by the angular velocity matrix $\underset{\sim}{\Omega}$ obtained as

$$\underset{\sim}{\Omega}=\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^{T} \tag{2}$$

$\underset{\sim}{Q}$ being the orthogonal matrix of the motion associated with finite rotations. Using standard index notation [6], the components of the angular velocity vector $\underset{\sim}{\omega}$ are next defined in terms of the components of $\underset{\sim}{\Omega}$ as

$$\omega_{i}=-\varepsilon_{ijk}\Omega_{jk} \tag{3}$$

The velocity of a point P of a rigid body rotating about the origin with an angular velocity $\underset{\sim}{\omega}$ can then be expressed as

$$\underset{\sim}{v}_{P}=\underset{\sim}{\omega}\times\underset{\sim}{r}_{P} \tag{4}$$

whereas the velocity of that point, if the body undergoes a <u>general motion</u>, i.e. one under which the velocity of no point of the body vanishes instantaneously, is given by

$$\underset{\sim}{v}_{P}=\underset{\sim}{v}_{A}+\underset{\sim}{\Omega}\left[\underset{\sim}{r}_{P}-\underset{\sim}{a}\right] \tag{5}$$

Now, assume the velocity of a point P of a rigid body, $\underset{\sim}{v}_{P}$, as well as the involved angular velocity matrix are known. The location of those points of the body having a minimum-magnitude velocity is now determined. Denoting by $\underset{\sim}{r}_{0}$ the position vector of one of these points, its velocity being $\underset{\sim}{v}_{0}$, one can substitute in eq. (5) $\underset{\sim}{a}$ and $\underset{\sim}{v}_{A}$ for $\underset{\sim}{r}_{0}$ and $\underset{\sim}{v}_{0}$ and solve for $\underset{\sim}{v}_{0}$, thus obtaining

$$\underset{\sim}{v}_{0}=\underset{\sim}{v}_{P}+\underset{\sim}{\Omega}\left[\underset{\sim}{r}_{0}-\underset{\sim}{r}_{P}\right] \tag{6}$$

The location of the points of minimum-magnitude velocity is now accomplished via a minimisation procedure. In fact, define a quadratic objective function ϕ, whose minimisation leads to the solution of the problem at hand,

$$\phi=\left|\left|\underset{\sim}{v}_{0}\right|\right|^{2}=\underset{\sim}{v}_{0}^{T}\underset{\sim}{v}_{0} \tag{7}$$

which is, in fact, quadratic in $\underset{\sim}{r}_0$, for $\underset{\sim}{v}_0$ is linear in it, as can be seen from eq. (6). Then, an extremum of ϕ is found by setting the gradient $\phi'(\underset{\sim}{r}_0)$ equal to zero. Thus, applying the "chain rule",

$$\phi'(\underset{\sim}{r}_0)=\left[\frac{\partial \underset{\sim}{v}_0}{\partial \underset{\sim}{r}_0}\right]^T \frac{\partial \phi}{\partial \underset{\sim}{v}_0} \tag{8}$$

which readily leads to

$$\underset{\sim}{\Omega}^T \underset{\sim}{v}_0 = \underset{\sim}{0} \tag{9}$$

which shows that the minimum-magnitude velocity lies in the null space of $\underset{\sim}{\Omega}^T$ and, since $\underset{\sim}{\Omega}$ is skew symmetric [5,p.21], it also lies in that of $\underset{\sim}{\Omega}$, which is a line in space parallel to vector $\underset{\sim}{\omega}$, for eq. (9) can alternatively be written as

$$\underset{\sim}{\omega} \times \underset{\sim}{v}_0 = \underset{\sim}{0} \tag{9 a}$$

That the foregoing extremum is in fact a minimum, and not a maximum or a saddle point, becomes apparent from the positive semidefiniteness of the Hessian matrix ϕ'', which results to be

$$\phi''=-2\left[\underset{\sim}{\Omega}^2\right]^T$$

Eq. (6) can be in turn rewritten in terms of $\underset{\sim}{\omega}$ as

$$\underset{\sim}{v}_0=\underset{\sim}{v}_P+\underset{\sim}{\omega}\times\left[\underset{\sim}{r}_0-\underset{\sim}{r}_P\right] \tag{6 a}$$

Substituting expression (6 a) for $\underset{\sim}{v}_0$ in eq. (9 a) and imposing the condition on $\underset{\sim}{r}_0$ that it be the position vector of a point of the screw axis lying closest to the origin, i.e. that $\underset{\sim}{r}_0$ be perpendicular to $\underset{\sim}{\omega}$, one obtains expresssion (1), q.e.d.

From the foregoing result, one obtains readily the following.

<u>Corollary 1</u>. *The velocities of all points of a rigid body undergoing an arbitrary motion have identical projections along the instantaneous screw axis.*

Corollary 2 If at least one point of a rigid body has a velocity whose orthogonal projection on its spin vanishes, the body undergoes a pure rotation.

Corollary 3 The difference vector of the velocities of any two points of a rigid body undergoing an arbitrary motion is perpendicular to the instantaneous screw axis.

One more useful result is next proved.

Theorem 2 If the velocities of three noncollinear points of a rigid body are identical, the body undergoes a pure translation.

Proof

Let v_A, v_B and v_C be the respective velocities of points A, B and C. Referring these to the velocity of an arbitrary point P, one obtains

$$v_A = v_P + \Omega[a-p]$$

$$v_B = v_P + \Omega[b-p]$$

$$v_C = v_P + \Omega[c-p]$$

Subtracting the third equation from the other ones yields

$$v_A - v_C = \Omega[a-c] = 0$$

$$v_B - v_C = \Omega[b-c] = 0$$

Which implies that both $a-c$ and $b-c$ lie the null space of Ω. This space, however, is of dimension 1, as is clear from eqs.(9) and

(9 a). Since points A,B and C are noncollinear, vectors $\underset{\sim}{a} - \underset{\sim}{c}$ and $\underset{\sim}{b} - \underset{\sim}{c}$ are linearly independent and hence cannot be simultaneously in the null space of $\underset{\sim}{\Omega}$, unless $\underset{\sim}{\Omega} = \underset{\sim}{0}$, the motion thus reducing to a pure translation, q.e.d.

The problem that still remains, however, is that of computing $\underset{\sim}{\omega}$. This is done readily, provided the velocity of three noncollinear points of the body are known, and these velocities happen to be such that their two arising independent differences are nonparallel . In fact, let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be the position vectors of three noncollinear points of the body, their corresponding velocities being $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$.

From the fact that

$$\underset{\sim}{v}_{C/A} = \underset{\sim}{\omega} \times (\underset{\sim}{c} - \underset{\sim}{a}) \tag{10}$$

then, cross multiplying the foregoing relative velocity times $\underset{\sim}{v}_{B/A} \times$, one obtains

$$\underset{\sim}{v}_{B/A} \times v_{C/A} = \underset{\sim}{v}_{B/A} \times \left[\underset{\sim}{\omega} \times (\underset{\sim}{c} - \underset{\sim}{a})\right] =$$

$$= \left[\underset{\sim}{v}_{B/A} \cdot (\underset{\sim}{c} - \underset{\sim}{a})\right] \underset{\sim}{\omega} - (\underset{\sim}{v}_{B/A} \cdot \underset{\sim}{\omega}) (\underset{\sim}{c} - \underset{\sim}{a}) \tag{11}$$

But, as can be readily shown,

$$\underset{\sim}{v}_{B/A} \cdot \underset{\sim}{\omega} = 0$$

Hence, one can solve for $\underset{\sim}{\omega}$ from eq. (11) as

$$\underset{\sim}{\omega} = \frac{\underset{\sim}{v}_{B/A} \times \underset{\sim}{v}_{C/A}}{\underset{\sim}{v}_{B/A} \cdot (\underset{\sim}{c} - \underset{\sim}{a})} \tag{12}$$

provided the relative velocities $\underset{\sim}{v}_{B/A}$ and $\underset{\sim}{v}_{C/A}$ (and hence, vectors $\underset{\sim}{v}_{B/A}$ and $\underset{\sim}{c} - \underset{\sim}{a}$) do not happen to be parallel. As will be shown next, the said relative velocities are parallel if and only if, for noncollinear points A, B and C, these happen to lie in a plane parallel to the axis of the instantaneous screw or of pure rotation, if the motion has a zero sliding velocity . Henceforth the term "instantaneous screw axis" will be employed regardless of whether the motion is general or a pure rotation.

DISCUSSION OF PARTICULAR CASES.

In order to proceed further, the following result is next proved.

Theorem 3 *The nonidentical velocities of three points of a rigid body are coplanar if and only if one of the three following conditions is met:*

i) the motion is a pure rotation

ii) the motion is general, but the points are collinear

iii) the motion is general and the points are noncollinear, but lie in a plane parallel to the axis of the instantaneous screw.

Proof. In what follows the three points are referred to as A, B and C, their position vectors being labelled correspondingly $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, whereas their velocities, $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$. The angular velocity involved is either referred to as matrix $\underset{\sim}{\Omega}$ or as vector $\underset{\sim}{\omega}$.

Sufficiency is first proved.

i) If the motion is a pure rotation, the velocity of any point with position vector $\underset{\sim}{r}$ is given by

$$\underset{\sim}{v} = \underset{\sim}{\Omega}\underset{\sim}{r}$$

wich states that $\underset{\sim}{v}$ lies in the range of $\underset{\sim}{\Omega}$, which is of dimension 2, given that its null space is of dimension 1. This means that all velocity vectors lie in a plane perpendicular to the null space of $\underset{\sim}{\Omega}$, i.e. perpendicular to the axis of rotation, thereby showing that these velocities are coplanar.

ii) Let A, B and C be three collinear points of the rigid body and $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be their respective position vectors. The velocities of these points, referred to an arbitrary point with position vector $\underset{\sim}{p}$ are

$$\underset{\sim}{v}_A = \underset{\sim}{v}_P + \underset{\sim}{\Omega}\,(\underset{\sim}{a}-\underset{\sim}{p})$$
$$\underset{\sim}{v}_B = \underset{\sim}{v}_P + \underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{p})$$
$$\underset{\sim}{v}_C = \underset{\sim}{v}_P + \underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{p})$$

Since the points are collinear, their position vectors are related by

$$\underset{\sim}{c} - \underset{\sim}{a} = \alpha\,(\underset{\sim}{b}-\underset{\sim}{a})$$

Now, adding $\underset{\sim}{\Omega}\underset{\sim}{a}$ to $\underset{\sim}{v}_C$ and subtracting it simultaneously from the same expression, one obtains

$$\underset{\sim}{v}_C = \underset{\sim}{v}_P + \underset{\sim}{\Omega}\,(\underset{\sim}{a}-\underset{\sim}{p}) + \underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a})$$

whose first two terms can be readily identified with $\mathbf{v}_A$. Moreover, substituting $\mathbf{c}-\mathbf{a}$ in the third term of the latter equation by $\alpha(\mathbf{b}-\mathbf{a})$, as given above, leads to

$$\mathbf{v}_C = \mathbf{v}_A + \alpha\boldsymbol{\Omega}\,(\mathbf{b}-\mathbf{a})$$

But

$$\boldsymbol{\Omega}\,(\mathbf{b}-\mathbf{a}) = \mathbf{v}_B - \mathbf{v}_A$$

Hence, the expression for $\mathbf{v}_C$ is transformed into

$$\mathbf{v}_C = \mathbf{v}_A + \alpha\,(\mathbf{v}_B - \mathbf{v}_A)$$

or, equivalently,

$$\mathbf{v}_C = (1-\alpha)\,\mathbf{v}_A + \alpha\,\mathbf{v}_B$$

thereby proving the linear dependence of $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_C$, i.e. that vectors $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_C$ are coplanar.

iii) The velocities of the three given points, A, B and C, are referred to that of a point P on the screw axis. These velocities take on the form appearing in ii. Thus, the velocity of P, $\mathbf{v}_P$, is parallel to the screw axis. On the other hand, the fact that A, B and C lie in a plane parallel to the screw axis allows one to establish the following relationship

$$\mathbf{c}-\mathbf{a} = \alpha\,(\mathbf{b}-\mathbf{a}) + \beta\,\mathbf{v}_P$$

or, equivalently,

$$\underset{\sim}{c} = (1-\alpha)\, \underset{\sim}{a} + \alpha \underset{\sim}{b} + \beta\, \underset{\sim}{v}_P$$

Substituting the latter expression in $\underset{\sim}{v}_C$ as given in ii leads to

$$\underset{\sim}{v}_C = \underset{\sim}{v}_P + \underset{\sim}{\Omega}\, (\underset{\sim}{c}-\underset{\sim}{p}) =$$

$$= \underset{\sim}{v}_P + \underset{\sim}{\Omega}\, (\underset{\sim}{a}-\underset{\sim}{p}) - \alpha\underset{\sim}{\Omega}\, (\underset{\sim}{b}-\underset{\sim}{a}) + \beta\Omega \underset{\sim}{v}_P$$

whose two first terms can be readily identified as $\underset{\sim}{v}_A$, its fourth term vanishing because it lies in the null space of $\underset{\sim}{\Omega}$. Hence

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A - \alpha\underset{\sim}{\Omega}\, (\underset{\sim}{b}-\underset{\sim}{a})$$

But

$$\underset{\sim}{\Omega}\, (\underset{\sim}{b}-\underset{\sim}{a}) = \underset{\sim}{v}_B - \underset{\sim}{v}_A$$

Thus, the latter expression for $\underset{\sim}{v}_C$ is transformed into

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A - \alpha\, (\underset{\sim}{v}_B - \underset{\sim}{v}_A)$$

which shows the linear dependence of the three given velocity vectors, i.e. that they are coplanar.

Necessity is now proved.

Assuming that the velocities $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ of three given points A, B and C are coplanar, the following relationship holds

$$\det\, (\underset{\sim}{v}_A, \underset{\sim}{v}_B, \underset{\sim}{v}_C) = 0$$

Referring $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ to $\underset{\sim}{v}_A$ one has

$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{a})$$

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A + \underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a})$$

Thus, the above expression for the determinant becomes

$$\det\left[\underset{\sim}{v}_A,\ \underset{\sim}{v}_A + \underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{a}),\ \underset{\sim}{v}_A + \underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a})\right] = 0$$

Subtracting the first column of this determinant from the remaining ones does not change the value of the determinant. Hence

$$\det\left[\underset{\sim}{v}_A,\ \underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{a}),\ \underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a})\right] = 0$$

which is equivalent to

$$\underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{a})\ \mathrm{x}\underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a})\cdot \underset{\sim}{v}_A = 0$$

Introducing Gibbs' notation, and expanding the resulting expression,

$$\underset{\sim}{\Omega}\,(\underset{\sim}{b}-\underset{\sim}{a})\ \mathrm{x}\underset{\sim}{\Omega}\,(\underset{\sim}{c}-\underset{\sim}{a}) = \left[\underset{\sim}{\omega}\mathrm{x}(\underset{\sim}{b}-\underset{\sim}{a})\right]\ \mathrm{x}\ \left[\underset{\sim}{\omega}\mathrm{x}(\underset{\sim}{c}-\underset{\sim}{a})\right] =$$

$$= \left[\underset{\sim}{\omega}\mathrm{x}(\underset{\sim}{b}-\underset{\sim}{a})\,.\ (\underset{\sim}{c}-\underset{\sim}{a})\right]\underset{\sim}{\omega} - \left[\underset{\sim}{\omega}\mathrm{x}(\underset{\sim}{b}-\underset{\sim}{a})\,.\underset{\sim}{\omega}\right]\ (\underset{\sim}{c}-\underset{\sim}{a})$$

where the expression in brackets in the second term of the rightmost hand side clearly vanishes. Hence

$$\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a})\ \mathrm{x}\underset{\sim}{\Omega}\ (\underset{\sim}{c}-\underset{\sim}{a})\,.\underset{\sim}{v}_A = \left[\underset{\sim}{\omega}\mathrm{x}(\underset{\sim}{b}-\underset{\sim}{a})\,.(\underset{\sim}{c}-\underset{\sim}{a})\right]\underset{\sim}{\omega}\,.\underset{\sim}{v}_A$$

which vanishes under one of the following conditions:

i) $\underset{\sim}{\omega}\cdot\underset{\sim}{v}_A=0$

which implies, under Corollary 2.9.1, that the motion is a pure rotation.

ii) $(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{c}-\underset{\sim}{a})=0$

which means that points A, B and C are collinear

iii) $\underset{\sim}{\omega}\times(\underset{\sim}{b}-\underset{\sim}{a}),(\underset{\sim}{c}-\underset{\sim}{a})=0$

which indicates that vectors $\underset{\sim}{\omega}$, $\underset{\sim}{b}-\underset{\sim}{a}$ and $\underset{\sim}{c}-\underset{\sim}{a}$ are coplanar, q.e.d.

a direct consequence of the foregoing result is the following.

Corollary 4. *Assume a rigid body under motion and thoose any three noncollinear points A, B and C of the body. Letting $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_0$ be the three involved velocities, then the difference vectors $\underset{\sim}{v}_A-\underset{\sim}{v}_C$ and $\underset{\sim}{v}_B-\underset{\sim}{v}_C$ (and, consequently, $\underset{\sim}{v}_A-\underset{\sim}{v}_B$) are parallel if and only if the points lie in a plane parallel to the screw axis.*

More results related to the computation of the screw parameters are the following.

Corollary 5. *The velocities of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation.*

Corollary 6. *If two, and only two, velocities of three noncollinear points of a rigid body are parallel, then either i) the parallel velocities are identical and belong to points lying on a line parallel to the screw axis, or ii) the parallel velocities are different from each other, in which case the motion is a pure rotation whose axis is parallel to the line connecting the two points of parallel velocities.*

Corollary 7. *Given three noncollinear points, A, B and C, of a rigid body in motion, such that $\underset{\sim}{v}_C=\underset{\sim}{0}$ and $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ are parallel i.e. $\underset{\sim}{v}_B=\beta\underset{\sim}{v}_A$, then the body undergoes a pure rotation and its axis passes through C and is parallel to vector $\underset{\sim}{b}-\underset{\sim}{c}-\beta(\underset{\sim}{a}-\underset{\sim}{c},\underset{\sim}{a},\underset{\sim}{b}$*

and $\underset{\sim}{c}$ being the position vectors of A, B and C, respectively. The computation of the parameters of the instantaneous screw involves, then, the computation of the angular velocity vector $\underset{\sim}{\omega}$, the position vector of the point on the screw axis lying closest to the origin, $\underset{\sim}{r}_0$ and the sliding velocity, i.e. the orthogonal projection of the velocity field along the axis of the instantaneous screw. This computation is next described.

COMPUTATIONAL ALGORITHM

The computational algorithm is based upon the "tree diagram" shown in Fig 1. In that diagram, a particular case is referred to as a sequence of digits indicating which case one has at hand for successive levels. For instance, case 1.1.1.2 corresponds to case 1 of the first level, case 1 of the second level, case 1 of the third level and case 2 of the fourth level, i.e. the case at hand is the following: no vector vanishes, vectors are coplanar, differences are parallel, but at least one of these differences does not vanish. Furthermore, the diagram shows that, for this case, formula 1, which is next introduced, is to be applied.

The computation of vector $\underset{\sim}{\omega}$ is crucial in the present algorithm. Once this has been performed, the remaining parameters are computed by application of identical formulae. Vector $\underset{\sim}{\omega}$, however, is computed by application of one of two different formulae.

The computation of $\underset{\sim}{\omega}$ is based upon case 2. For cases 1 and 3, $\underset{\sim}{\omega}$ is computed after the motion is transformed into case 2, as described next.

Before proceeding further, two results need be proved, which is done next. All over, the notation introduced in Theorem 3 is resorted to.

Theorem 4. Given a rigid body motion defined by the velocities of three noncollinear points, such that no velocity vanishes, define *a new motion by assigning new velocity vectors $\underset{\sim}{v}'_A, \underset{\sim}{v}'_B$ and $\underset{\sim}{v}'_C$ to points A,B and C, respectively. If these vectors are defined as*

$$\underset{\sim}{v}'_A=\underset{\sim}{v}_A-\underset{\sim}{v}_C,\ \underset{\sim}{v}'_B=\underset{\sim}{v}_B-\underset{\sim}{v}_C,\ \underset{\sim}{v}'_C=\underset{\sim}{0}$$

then the angular velocity of the new motion is identical to the given one.

Proof:

From eq. (5), $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ can be written as

$$\underset{\sim}{v}_A=\underset{\sim}{v}_C+\Omega(\underset{\sim}{a}-\underset{\sim}{c}) \tag{10a}$$

$$\underset{\sim}{v}_B=\underset{\sim}{v}_C+\Omega(\underset{\sim}{b}-\underset{\sim}{c}) \tag{10b}$$

Hence,

$$\underset{\sim}{v}_A-\underset{\sim}{v}_C=\underset{\sim}{0}+\Omega(\underset{\sim}{a}-\underset{\sim}{c})$$

and

$$\underset{\sim}{v}_B-\underset{\sim}{v}_C=\underset{\sim}{0}+\Omega(\underset{\sim}{b}-\underset{\sim}{c})$$

and recalling the definitions of $\underset{\sim}{v}'_A$, $\underset{\sim}{v}'_B$ and $\underset{\sim}{v}'_C$ the latter equations become, then,

$$\underset{\sim}{v}'_A=\underset{\sim}{v}'_C+\Omega(\underset{\sim}{a}-\underset{\sim}{c}) \tag{11a}$$

$$\underset{\sim}{v}'_B=\underset{\sim}{v}'_C+\Omega(\underset{\sim}{b}-\underset{\sim}{c}) \tag{11b}$$

which are expressions analogous to those of eqs. (10 a and b), except that $\underset{\sim}{v}'_A, \underset{\sim}{v}'_B$ and $\underset{\sim}{v}'_C$ have been placed instead of $\underset{\sim}{v}_A, \underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$, respectively. $\Omega, \underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ remain, however, thereby completing the proof.

Theorem 5. *Given a rigid body motion defined by the velocities of three noncollinear points, such that two of these, and only two, vanish, define a new motion by assigning new velocity vectors $\underset{\sim}{v}'_A, \underset{\sim}{v}'_B$ and $\underset{\sim}{v}'_C$ to points A,B and C, respectively. Letting*

$\underset{\sim}{v}_C$ be the unique nonvanishing velocity, define

$$\underset{\sim}{v}'_A=-\underset{\sim}{v}_C,\ \underset{\sim}{v}'_B=-\underset{\sim}{v}_C,\ \underset{\sim}{v}'_C=\underset{\sim}{0}$$

then the angular velocity of the new motion is identical to that of the given one.

Proof:

Since eqs. (10 a and b) are valid regardless of whether any involved velocity vanishes, one can apply them to the case at hand, i.e. for $\underset{\sim}{v}_A=\underset{\sim}{v}_B=\underset{\sim}{0}$ and $\underset{\sim}{v}_C\neq\underset{\sim}{0}$. Thus,

$$\underset{\sim}{0}=\underset{\sim}{v}_C+\underset{\sim}{\Omega}(\underset{\sim}{a}-\underset{\sim}{c}) \qquad (12a)$$

$$\underset{\sim}{0}=\underset{\sim}{v}_C+\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{c}) \qquad (12b)$$

Hence,

$$-\underset{\sim}{v}_C=\underset{\sim}{0}+\underset{\sim}{\Omega}(\underset{\sim}{a}-\underset{\sim}{c})$$

and

$$-\underset{\sim}{v}_C=\underset{\sim}{0}+\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{c})$$

Recalling the definitions of $\underset{\sim}{v}'_A$, $\underset{\sim}{v}'_B$ and $\underset{\sim}{v}'_C$, the latter equations become

$$\underset{\sim}{v}'_A=\underset{\sim}{v}'_C+\underset{\sim}{\Omega}(\underset{\sim}{a}-\underset{\sim}{c}) \qquad (13a)$$

$$\underset{\sim}{v}'_B=\underset{\sim}{v}'_C+\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{c}) \qquad (13b)$$

and the proof follows by introducing the argument resorted to for proving Theorem 4.

As an application of the foregoing results one can compute $\underset{\sim}{\omega}$ for the given motion by first defining motions leading to case 2 of Fig 1.

Next the computation of ω for that case is discussed. Since this contains two subcases, each is discussed separately.

Case 2.1 One vector, and only one, vanishes, the remaining ones being parallel. Let

$$v_B=\beta v_A \neq 0, v_C=0 \tag{14}$$

Then, according to Corollary 7, ω is given as

$$\omega=\alpha\left[(b-c)-\beta(a-c)\right] \tag{15}$$

Since the axis of rotation passes through C, v_A can be written as

$$v_A=\omega x(a-c) \tag{16}$$

Upon substitution of ω as given in (15), eq. (16) is transformed into

$$v_A=\alpha(b-c)x(a-c)$$

and hence

$$\alpha=\frac{||v_A||}{||(b-c)x(a-c)||}\ \mathrm{sgn}\left[(b-c)x(a-c).v_A\right] \tag{17}$$

Eqs. (15) and (17) constitute what is referred to as "formula 1" in Fig 1.

Case 2.2 One vector, and only one, vanishes, the remaining ones being nonparallel. Let

$$v_C=0,\ v_A x v_B \neq 0 \tag{18}$$

Since the axis of rotation passes through C, v_A can be written as

$$\mathbf{v}_A = \omega \times (a - c)$$

and hence

$$\mathbf{v}_B \times \mathbf{v}_A = [\mathbf{v}_B \cdot (a-c)]\omega - (\mathbf{v}_B \cdot \omega)(a-c) \tag{19}$$

where the scalar product $\mathbf{v}_B \cdot \omega$ vanishes, due to the fact that the motion is a pure rotation. Thus, one can solve for ω from eq. (19) as

$$\omega = \frac{\mathbf{v}_B \times \mathbf{v}_A}{\mathbf{v}_B \cdot (a-c)} \tag{20}$$

where the denominator does not vanish by virtue of Corollary 4 and the nature of this case. Indeed, the denominator can be written as

$$\mathbf{v}_B \cdot (a-c) = \omega \times (b-c) \cdot (a-c)$$

which vanishes if the given points lie in a plane parallel to the instantaneous screw axis. Since the differences $\mathbf{v}_A - \mathbf{v}_C (= \mathbf{v}_A)$ and $\mathbf{v}_B - \mathbf{v}_C (= \mathbf{v}_B)$ are nonparallel, by virtue of Corollary 4, the points lie in a plane not parallel to the instantaneous screw axis, and the denominator does not vanish.

Eq. (20) constitutes what is referred to as "formula 2" in Fig 1.

Once ω is known, the position vector, r_0, of the point on the axis of the instantaneous screw lying closest to the origin, is computed by application of eq.(1). The sliding velocity s is computed simply as the orthogonal projection of the velocity field on the screw axis, i.e. as

$$s = \mathbf{v}_A \cdot \omega / ||\omega|| \tag{21}$$

thereby completing the computation of all the screw parameters defining a rigid-body motion for infinitesimally separated positions.

A computer subprogram, INSCRU [7], was written, that realises the foregoing algorithm. This subprogram was written in the Fortran IV dialect of the Burroughs 6700 computer of the U. of Mexico, UNAM, and is available upon request.

A series of examples is next presented.

EXAMPLE 1

POINT	COORDINATES X	Y	Z
A =	0.00000	1.00000	0.00000
B =	1.00000	3.00000	-0.50000
C =	0.00000	3.00000	1.00000
VA =	0.00000	3.00000	0.00000
VB =	0.00000	0.00000	-12.00000
VC =	3.42000	7.86000	-9.72000

NUMBER OF VANISHING VELOCITIES IS : 0

VELOCITIES ARE NONCOPLANAR

THE SPIN HAS THE FOLLOWING COMPONENTS
(VECTOR SPIN) : -4.86000 2.28000 -0.57000
THE POINT ON THE INSTANTANEOUS-SCREW AXIS CLOSEST TO
THE ORIGIN HAS THE FOLLOWING X-, Y- AND Z COORDINATES
(VECTOR RHO) : 0.43890 0.82162 -0.45570
THE VELOCITY COMPONENT ALONG THE INSTANTANEOUS-SCREW AXIS :
(SPEED) : 1.26704

EXAMPLE 2

POINT	COORDINATES X	Y	Z
A =	1.00000	0.00000	0.00000
B =	2.00000	-1.00000	1.00000
C =	0.00000	1.00000	1.00000
VA =	1.20000	1.20000	1.50000
VB =	3.20000	3.20000	1.50000
VC =	-0.80000	-0.80000	1.50000

NUMBER OF VANISHING VELOCITIES IS : 0

VELOCITIES ARE COPLANAR AND DIFERENCES ARE PARALLEL

THE SPIN HAS THE FOLLOWING COMPONENTS
(VECTOR SPIN) : 0.00000 0.00000 2.00000
THE POINT ON THE INSTANTANEOUS-SCREW AXIS CLOSEST TO
THE ORIGIN HAS THE FOLLOWING X-, Y- AND Z COORDINATES
(VECTOR RHO) : 0.40000 0.60000 0.00000
THE VELOCITY COMPONENT ALONG THE INSTANTANEOUS-SCREW AXIS :
(SPEED) : 1.50000

EXAMPLE 3

POINT	X	Y	Z
A =	0.40000	0.60000	0.00000
B =	0.00000	0.00000	0.00000
C =	2.00000	1.00000	0.00000
VA =	0.00000	0.00000	0.00000
VB =	1.20000	-0.80000	0.00000
VC =	-0.80000	3.20000	0.00000

NUMBER OF VANISHING VELOCITIES IS : 1

VELOCITIES ARE NONPARALLEL

THE SPIN HAS THE FOLLOWING COMPONENTS
(VECTOR SPIN) : 0.00000 0.00000 2.00000
THE POINT ON THE INSTANTANEOUS-SCREW AXIS CLOSEST TO
THE ORIGIN HAS THE FOLLOWING X-, Y- AND Z COORDINATES
(VECTOR RHO) : 0.40000 0.60000 0.00000
THE VELOCITY COMPONENT ALONG THE INSTANTANEOUS-SCREW AXIS :
(SPEED) : 0.00000

EXAMPLE 4

POINT	X	Y	Z
A =	0.00000	0.00000	0.00000
B =	1.00000	0.00000	0.00000
C =	2.00000	1.00000	0.00000
VA =	1.20000	1.20000	1.50000
VB =	1.20000	1.20000	1.50000
VC =	1.20000	1.20000	1.50000

NUMBER OF VANISHING VELOCITIES IS : 0

MOTION IS PURE TRANSLATION
THE VELOCITIES HAVE THE FOLLOWING X-,Y- AND Z- COORDINATES:
1.20000 1.20000 1.50000

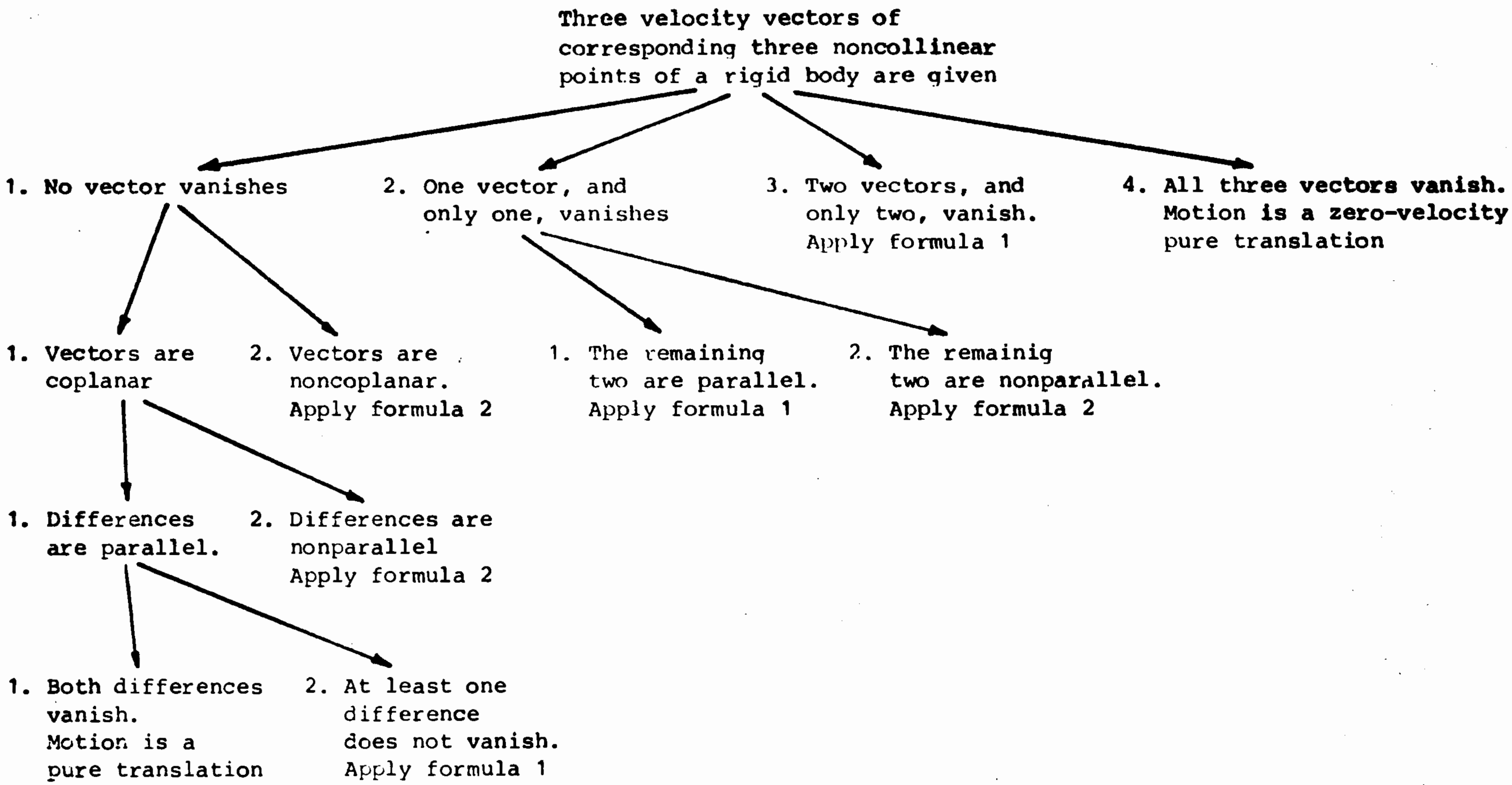

Fig 1. Tree diagram showing the different possible relationships amongst the velocities of three noncollinear points defining a rigid-body motion.

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