ON THE COMPUTATION OF THE SCREW PARAMETERS OF A RIGID-BODY MOTION . PART I: FINITELY-SEPARATED POSITIONS

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ON THE COMPUTATION OF THE SCREW PARAMETERS OF A RIGID-BODY MOTION. PART I: FINITELY-SEPARATED POSITIONS.

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ABSTRACT

A set of results concerning the determination of the screw parameters of a rigid-body motion are derived. These results are then applied to the computation of the screw parameters with the aid of an algorithm whose description is included. The algorithm considers all possible particular cases that could arise in this context.

ÜBER DIE BERECHNUNG DER SCHRAUBPARAMETER DER BEWEGUNG EINES STARKÖRPERS.
TEIL I: ENDLICH BENACHBARTE LAGEN.

J. Angeles

Zusammenfassung. Eine Menge Ergebnisse bezüglich auf die Bestimmung der Schraubparameter der Bewegung eines Starkörpers wird abgeleitet. Diese Ergebnisse dienen zur Berechnung der Schraubparameter der Bewegung mit Hilfe eines Algorithmus, dessen Beschreibung eingeschlossen wird. Der Algorithmus berücksichtigt alle mögliche Sonderfälle, die in diesem Zusammenhang entstehen könnten.

NOMENCLATURE

r: lower-case underlined character, a n-dimensional vector

A: upper-case underlined character, a mxn matrix

r^T, A^T: the transpose of a vector or, correspondingly, of a matrix

$\phi'(r)$: the gradient of ϕ with respect to r, a n vector

$\phi''(r)$: the Hessian of ϕ with respect to r, a nxn matrix

$r^T r, r.u$: the inner product of r and u

$||r||$: the Euclidean norm of vector r, i.e. $\sqrt{r^T r}$

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INTRODUCTION

According to Chasles' theorem [1][2], given a rigid body R undergoing
a motion which carries it from configuration R_1 to configuration R_2,
there exists a set of points of R_1 lying on a line L whose displacement
vectors have a minimum magnitude. Alternatively, this theorem states
that the displacement vectors of the points of line L are all identical
to each other and parallel to the axis of the rotation component of
the motion. Moreover, line L and the axis of rotation are parallel.

The screw parameters defining the motion are then: i) line L, ii) the
magnitude of the displacement and iii) the angle of rotation. Besides
the purely scientific interest of computing the screw parameters, this
computation is also required in some practical applications such as
linkage synthesis [2] and robotics. The computation of the screw
parameters for the general case is well known [3,4]. However, two items
have not yet been taken into account, namely: i) the case when the
displacements of three noncollinear points of the body are coplanar,
and the displacement differences are parallel, and ii) the determination
of the sign of the angle of rotation directly from the rotation matrix.
These items are discussed in what follows.

DEFINITION OF THE SCREW PARAMETERS.

It is widely known [5] that the screw parameters of a rigid-body motion
taking the body from R_1 to R_2 are defined if and only if the positions
of three noncollinear points of R are known in R_1 and R_2. In general,
the displacements of these points are noncoplanar, and the general
algorithm to compute the parameters makes use of this fact. Now, the
involved displacements are necessarily coplanar if the motion is a pure
rotation; furthermore, in this case, these displacements are normal to
the axis of rotation. In the latter case, the determination of the screw
parameters is even easier to perform. There are cases, however, for
which the three displacements are coplanar and yet the motion is general
i.e. contains a combination of translation and rotation. In this case

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[2] Numbers in brackets designate references at the end of the paper.

and other particular ones, which are discussed within the paper, the general algorithm cannot be applied. The paper indicates how to handle these particular cases.

The screw parameters of the motion are first defined precisely. Let A,B and C be three noncollinear points of the body whose position vectors in R_1 are denoted by a_1, b_1 and c_1, respectively, and those in R_2 by a_2, b_2 and c_2.

These two triads of vectors are supposed henceforth to be known. The motion from R_1 to R_2 can be thought of as occurring in two stages: in the first one the body is translated without rotation in such a way that C_1 is carried into C_2. Letting R_i be this intermediate configuration, the position vectors of the three involved points in R_i are then a_i, b_i, c_i, and $c_i = c_2$. The matrix Q will be dealt with later on. For the moment it is assumed that this matrix is known. Since Q is known, the position vector of any point P in R_2, labelled p_2, can be written in terms of p_1 – its position vector in R_1 –, a_1, a_2 and Q, as follows:

$$p_2 = a_2 + Q(p_1 - a_1) \tag{2}$$

If P is a point on L, its position vector in R_1 can be found via a minimisation procedure, i.e. minimising the square of the magnitude of u, the displacement of P, where

$$u = p_2 - p_1 = a_2 + Q(p_1 - a_1) - p_1 \tag{3}$$

The function to minimise is then

$$\phi(u) = u^T u \tag{4}$$

which attains an extremum at a point where its gradient with respect to p_1 vanishes. This gradient is, by application of the "chain rule",

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$$\phi'(p_1) = (\frac{\partial u}{\partial p_1})^T \frac{\partial \phi}{\partial u}$$

where, from eq. (3),

$$\frac{\partial u}{\partial p_1} = Q - I$$

and from eq. (4)

$$\frac{\partial \phi}{\partial u} = 2u$$

Hence, a necessary condition for ϕ to have an extremum is

$$\phi'(p_1) = 2(Q-I)^T u = 0 \tag{5}$$

which states that at those points whose displacements are parallel to the axis of rotation, the magnitude of the displacement vector attains an extremum. That this extremum is a minimum can be proved by computing the Hessian matrix $\phi''(p_1)$. In fact, from eq. (5),

$$\phi''(p_1) = 2(Q-I)^T(Q-I) \tag{6}$$

which is a positive semidefinite matrix, i.e., the extremum satisfies the sufficient condition to be a minimum. In the latter expression notice that the quadratic form associated to $\phi''(p_1)$ vanishes in the direction of the axis of rotation, i.e. along the direction of the real eigenvector of Q, which indicates that along this direction, ϕ remains a minimum.

In order to compute the vector p_1 whose displacement satisfies eq. (5), expression (3) is substituted into this equation, thus obtaining

$$(Q-I)^T \left[(Q-I)p_1 + a_2 - Qa_1\right] = 0$$

or

$$(Q-I)^T(Q-I)p_1 = (Q-I)^T(Qa_1 - a_2) \tag{7}$$

which is a linear system of equations. Vector p_1, however, cannot be solved for from eq. (7) because the system is singular. Indeed, from the fact that 1 is an eigenvalue of any proper orthogonal matrix Q [6], it follows that $Q - I$ is of rank 2. Hence, eq. (7) contains exactly two linearly independent equations, thus making it impossible to solve for p_1. This means that eq. (7) holds not for one vector p_1, but for an infinity of such vectors. These vectors represent the position vectors of all points lying on a line, the screw axis L. In order to find a particular point p_1 through which line L passes, an additional condition should be imposed on this point, for instance that it be the point of L closest to the origin. This point has a position vector whose magnitude is a minimum, which can be computed by writing the two linearly independent equations of (7) in the form

$$A p_1 = b \tag{8}$$

where A is a 2 x 3 matrix. The minimum-magnitude solution to (8) can be found making use of the pseudo-inverse [7] of A in the form

$$p_1 = A^T (A A^T)^{-1} b \tag{9}$$

i.e. eq. (9) defines the postion vector of that point of L, p_1, closest to the origin. Hence, the screw axis L passes through point P of R, whose position vector in R_1 is p_1, as given by eq. (9), and has the direction of the axis of rotation, i.e. is parallel to vector e, the eigenvector of Q associated with the eigenvalue 1 of Q. In what follows it will be shown how to compute vector e without having to compute Q explicitly. For this, the following result will be resorted to.

Theorem 1. The displacement vectors of all points of a rigid body moving from configuration R_1 to configuration R_2 have identical projections along the screw axis.

Proof

Let u_A be the displacement of A, and p_1 and p_2 the position vectors of point P on the screw axis in R_1 and R_2, respectively. Then

$$u_A = a_2 - a_1 = (Q-I)a_1 + p_2 - Qp_1$$

The projection of u_A along L is given by

$$e^T u_A = e^T(Q-I)a_1 + e^T p_2 - e^T Q p_1 \tag{10}$$

But e is the eigenvector of Q associated with the eigenvalue 1. Hence

$$Qe = Q^T e = e$$

from which, eq. (10) becomes

$$e^T u_A = e^T(p_2 - p_1)$$

which is independent of a_1, thereby proving the constancy of this projection, q. e. d.

A direct consequence of Theorem 1 is the following:

Corollary 1. If the displacement of at least one point of a rigid body in motion has a zero projection along the screw axis, the motion is a pure rotation.

Before discussing the computation of e, that of θ is presented. there are two possibilities, either vector e is known, in which case use is made of Rodrigues' formula [8]:

$$a_2 - a_1 = \tan\frac{\theta}{2} e \times (a_1 + a_2) \tag{11a}$$

or else matrix Q is known, in which case θ is computed as [9]:

$$\theta = \cos^{-1}\left[\frac{1}{2}(Tr\underset{\sim}{Q}-1)\right] \tag{11b}$$

Due to the fact that the cos function is an even function, i.e. $\cos(\theta)=\cos(-\theta)$, eq.(11b) provides only the absolute value of the angle, not its sign. The sign is found invoking the following:

Theorem 2. Let a rigid body undergo a pure rotation about a fixed point 0 and let $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ be the initial and the final position vectors of a point of the body (measured from 0) not lying on the axis of rotation. Furthermore, let θ and $\underset{\sim}{e}$ be the angle of rotation and a unit vector parallel to the axis of rotation. Then

$$sgn(\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 \cdot \underset{\sim}{e}) = sgn(\theta) \tag{12}$$

Proof

Vector $\underset{\sim}{r}_2$ can be written as [10]:

$$\underset{\sim}{r}_2 = \left[(1-\cos\theta)\underset{\sim\sim}{ee}+\cos\theta\underset{\sim}{1}+\sin\theta\underset{\sim}{1}\times\underset{\sim}{e}\right]\cdot\underset{\sim}{r}_1$$

where dyadic notation [11] has been introduced. Then

$$\underset{\sim}{r}_1\times\underset{\sim}{r}_2 = (1-\cos\theta)(\underset{\sim}{e}\cdot\underset{\sim}{r}_1)\underset{\sim}{r}_1\times\underset{\sim}{e}+\sin\theta\left[r_1^2\underset{\sim}{e}-(\underset{\sim}{r}_1\cdot\underset{\sim}{e})\underset{\sim}{r}_1\right]$$

where

$$r_1^2 \equiv ||\underset{\sim}{r}_1||^2$$

Thus,

$$\underset{\sim}{r}_1\times\underset{\sim}{r}_2\cdot\underset{\sim}{e} = \sin\theta\left[r_1^2-(\underset{\sim}{r}_1\cdot\underset{\sim}{e})^2\right] =$$

$$= \sin\theta\left[r_1^2-r_1^2\cos^2(\underset{\sim}{r}_1,\underset{\sim}{e})\right]$$

$$= r_1^2\sin\theta\sin^2(\underset{\sim}{r}_1,\underset{\sim}{e})$$

Hence,

$$\text{sgn}(\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 \cdot \underset{\sim}{e}) = \text{sgn}(\sin\theta)$$

But since sin is an odd function, i.e. $\sin(\theta) = -\sin(-\theta)$, then

$$\text{sgn}(\sin\theta) = \text{sgn}(\theta)$$

Thus, one obtains finally

$$\text{sgn}(\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 \cdot \underset{\sim}{e}) = \text{sgn}(\theta), \text{q.e.d.}$$

Given $\underset{\sim}{e}$, θ can be computed even if $\underset{\sim}{Q}$ is not known. In fact, since θ is involved only in the rotation part of the motion, one can assume with no loss of generality that the motion is a pure rotation about a point which, with no loss of generality, will be assumed to be the origin. Moreover it will be assumed that the initial and the final position vectors of a point A, $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$, respectively, of the rigid body, are known. Let A' be the orthogonal projection of A_1 (point A in its reference configuration) on the axis of rotation. Clearly, A' is also the orthogonal projection of A_2 (point A in its final configuration). Then,

$$\theta = \angle A_1 A' A_2$$

Letting $\underset{\sim}{a}'$ be the position vector of A', this angle can be computed as

$$\theta = \cos^{-1} \frac{(\underset{\sim}{a}_1 - \underset{\sim}{a}')^T (\underset{\sim}{a}_2 - \underset{\sim}{a}')}{||\underset{\sim}{a}_1 - \underset{\sim}{a}'||^2} \tag{13}$$

which, nevertheless, does not provide the sign of θ; this, however, can be obtained from eq. (12). Vector $\underset{\sim}{a}'$ can be easily computed as

$$\underset{\sim}{a}' = \underset{\sim}{a}_1 - \underset{\sim}{a}_1 \cdot \underset{\sim}{e}\,\underset{\sim}{e} \tag{14}$$

Returning to the computation of $\underset{\sim}{e}$ if only the position vectors of three noncollinear points in both their initial and their final configurations are known, this can be performed by application of Theorem 1. According to this theorem, all that is needed is to find that direction along which the displacement vectors $\underset{\sim}{u}_A$, $\underset{\sim}{u}_B$ and $\underset{\sim}{u}_C$ of the three given noncollinear points have identical projections. If the displacement vectors are noncoplanar, this can be achieved as follows: Attach all three vectors to each other at their tails, so that their tips determine a plane. Vector $\underset{\sim}{e}$ is then found as the unit normal to the resulting plane, i.e. as

$$\underset{\sim}{v} = (\underset{\sim}{u}_A - \underset{\sim}{u}_C) \times (\underset{\sim}{u}_B - \underset{\sim}{u}_C) \tag{15a}$$

$$\underset{\sim}{e} = \underset{\sim}{v} / ||\underset{\sim}{v}|| \tag{15b}$$

If, on the other hand, the given displacement vectors are coplanar, then either the differences $\underset{\sim}{u}_A - \underset{\sim}{u}_C$ and $\underset{\sim}{u}_B - \underset{\sim}{u}_C$ are parallel or they are not. If they are not, the motion is necessarily a pure rotation and vector $\underset{\sim}{e}$ is perpendicular to the plane defined by the displacements. This vector, thus, can be computed as

$$\underset{\sim}{v} = \underset{\sim}{u}_A \times \underset{\sim}{u}_B \tag{15c}$$

$$\underset{\sim}{e} = \underset{\sim}{v} / ||\underset{\sim}{v}|| \tag{15d}$$

If, however, the aforementioned differences are parallel, then the motion is not necessarily a pure rotation, but, in this instance, $\underset{\sim}{e}$ lies in the plane of the three given points, as will be shown in Theorem 3. In this case the computation of $\underset{\sim}{e}$ proceeds by application of Corollary 5, which is stated after Theorem 3 has been proved.

If θ and $\underset{\sim}{e}$ are known, on the other hand, the rotation matrix $\underset{\sim}{Q}$ can be expressed in dyadic form as [9]:

$$\underset{\sim}{Q} + (1 - \cos\theta)\underset{\sim\sim}{ee} + \cos\theta\underset{\sim}{1} + \sin\theta\underset{\sim}{1} \times \underset{\sim}{e} \tag{16}$$

Finally the computation of the position vector of that
point of L lying closest to the origin can be performed either
by eq.(9) or by application of the formula

$$\underset{\sim}{\rho}_0 = \frac{1}{2}\cot\frac{\theta}{2}\underset{\sim}{e}x\underset{\sim}{u}_A - \frac{1}{2}\underset{\sim}{e}x\left[\underset{\sim}{e}x(\underset{\sim}{a}_1 + 2\underset{\sim}{u}_A)\right] \tag{17}$$

as derived in $\left[5, \text{pp.}11-12\right]$. Eqs. (11), (12),(15) and (17) define
the screw parameters and provide, in general, formulae for their
computation

DISCUSSION OF PARTICULAR CASES

*Theorem 3. The nonidentical displacements of three points of a
rigid body are coplanar if and only if one of the following
conditions is met:*

i)   the motion is a pure rotation
ii)   the motion is general, but the points are collinear
*iii)   the motion is general and the points are not collinear,
        but lie in a plane parallel to the screw axis.*

Proof.

(Sufficiency)

i)   If the motion is a pure rotation and the origin of
        coordinates is located along the axis of rotation, the
        displacement $\underset{\sim}{u}$ of any point with position vector $\underset{\sim}{r}$ is then

$$\underset{\sim}{u} = \underset{\sim}{Q}\underset{\sim}{r} - \underset{\sim}{r} = (\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{r}$$

Now let $\underset{\sim}{e}$ be a unit vector parallel to the axis of rotation.
Then

$$\underset{\sim}{e}^T\underset{\sim}{u} = \underset{\sim}{e}^T(\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{r} = \left[(\underset{\sim}{Q}^T - \underset{\sim}{I})\underset{\sim}{e}\right]^T\underset{\sim}{r}$$

But

$$(\underset{\sim}{Q}^T - \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{Q}^T\underset{\sim}{e} - \underset{\sim}{e} = \underset{\sim}{e} - \underset{\sim}{e} = 0$$

Now let e be a unit vector parallel to the axis of rotation.

Then

$$\underset{\sim}{e}^T \underset{\sim}{u} = \underset{\sim}{e}^T (\underset{\sim}{Q} - \underset{\sim}{I}) \underset{\sim}{r} = \left[(\underset{\sim}{Q}^T - \underset{\sim}{I}) \underset{\sim}{e}\right]^T \underset{\sim}{r}$$

But

$$(\underset{\sim}{Q}^T - \underset{\sim}{I}) \underset{\sim}{e} = \underset{\sim}{Q}^T \underset{\sim}{e} - \underset{\sim}{e} = \underset{\sim}{e} - \underset{\sim}{e} = \underset{\sim}{0}$$

Thus, all displacement vectors are perpendicular to the axis of rotation; hence, they lie in a plane perpendicular to this axis, thereby showing that they are coplanar.

ii) Let A, B and C be the given three collinear points of the rigid body undergoing a general motion. Let $\underset{\sim}{a}_1$, $\underset{\sim}{b}_1$ and $\underset{\sim}{c}_1$ be their respective position vectors in the reference configuration, and let $\underset{\sim}{a}_2$, $\underset{\sim}{b}_2$ and $\underset{\sim}{c}_2$ be those position vectors in the displaced configuration. Due to the collinearity of the three points, vectors $\underset{\sim}{b}_1 - \underset{\sim}{a}_1$ and $\underset{\sim}{c}_1 - \underset{\sim}{a}_1$ are parallel, i.e.

$$\underset{\sim}{c}_1 - \underset{\sim}{a}_1 = \alpha (\underset{\sim}{b}_1 - \underset{\sim}{a}_1) \tag{18}$$

where α is a scalar. Applying eq. (3) to describe the displacements of B and C, then

$$\underset{\sim}{u}_B = \underset{\sim}{a}_2 + \underset{\sim}{Q}(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) - \underset{\sim}{b}_1 =$$

$$= \underset{\sim}{a}_2 - \underset{\sim}{a}_1 + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) =$$

$$= \underset{\sim}{u}_A + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) \tag{19}$$

Similarly,

$$\underset{\sim}{u}_C = \underset{\sim}{u}_A + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{c}_1 - \underset{\sim}{a}_1) \tag{20}$$

Introducing eq. (18) into eq. (20),

$$\underset{\sim}{u}_C = \underset{\sim}{u}_A + \alpha(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{a}_1)$$

But, from eq. (19),

$$(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) = \underset{\sim}{u}_B - \underset{\sim}{u}_A$$

Hence, the latter expression for $\underset{\sim}{u}_C$ is transformed into

$$\underset{\sim}{u}_C = \underset{\sim}{u}_A + \alpha(\underset{\sim}{u}_B - \underset{\sim}{u}_A)$$

thereby proving that the displacement vectors are coplanar

iii) If point S lies on the screw axis then, from eq. (3), the displacements of points A, B and C can be written as

$$\underset{\sim}{u}_A = \underset{\sim}{u}_S + (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{a}_1 - \underset{\sim}{s}_1) \tag{21a}$$

$$\underset{\sim}{u}_B = \underset{\sim}{u}_S + (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{s}_1) \tag{21b}$$

$$\underset{\sim}{u}_C = \underset{\sim}{u}_S + (\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{c}_1 - \underset{\sim}{s}_1) \tag{21c}$$

where $\underset{\sim}{u}_S$ is the displacement of S and $\underset{\sim}{s}_1$ is its position vector in the reference configuration.

If points A, B and C lie in a plane parallel to the screw axis, then vectors $\underset{\sim}{b}_1 - \underset{\sim}{a}_1, \underset{\sim}{c}_1 - \underset{\sim}{a}_1$ and $\underset{\sim}{u}_S$ (parallel to the screw axis) are coplanar. Hence, they are related by

$$\underset{\sim}{c}_1 - \underset{\sim}{a}_1 = \alpha(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) + \beta \underset{\sim}{u}_S$$

or

$$\underset{\sim}{c}_1 = (1-\alpha)\underset{\sim}{a}_1 + \alpha \underset{\sim}{b}_1 + \beta \underset{\sim}{u}_S$$

Substituting this expression into eq. (21c), after cancellations and rearrangements, yields

$$\underset{\sim}{u}_C = \underset{\sim}{u}_A - \alpha(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{a}_1-\underset{\sim}{b}_1) \tag{22}$$

But, subtracting (21b) from (21a),

$$\underset{\sim}{u}_B - \underset{\sim}{u}_A = (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{a}_1-\underset{\sim}{b}_1)$$

Thus, eq. (22) can be rewritten as

$$\underset{\sim}{u}_C = (1-\alpha)\underset{\sim}{u}_A + \beta\underset{\sim}{u}_B$$

thereby showing that the displacement vectors are coplanar

(Necessity)

If u_A, u_B and u_C are coplanar, then

$$\det(\underset{\sim}{u}_A, \underset{\sim}{u}_B, \underset{\sim}{u}_C) = 0 \tag{23}$$

Introducing eqs. (19) and (20) and then subtracting the first column from the second and the third ones, eq. (23) becomes

$$\det(\underset{\sim}{u}_A, (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1-\underset{\sim}{a}_1), (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}_1-\underset{\sim}{a}_1)) = 0 \tag{24}$$

or, in Gibbs notation,

$$(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1-\underset{\sim}{a}_1) \times (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}_1-\underset{\sim}{a}_1) \cdot \underset{\sim}{u}_A = 0$$

But the cross product can be expressed as [12]:

$$(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}_1-\underset{\sim}{a}_1) \times (\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}_1-\underset{\sim}{a}_1) = \alpha\underset{\sim}{e}$$

where

$$\alpha = 2(1-\cos\theta)\underset{\sim}{e} \times (\underset{\sim}{b}_1-\underset{\sim}{a}_1) \cdot (\underset{\sim}{c}_1-\underset{\sim}{a}_1)$$

θ and e being the angle of rotation and the unit vector parallel to the axis of rotation. The double product thus can vanish if any one of the following conditions is met:

a) $\underset{\sim}{e} \cdot \underset{\sim}{u}_A = 0$

This conditon implies, according to Corollary 1, that all points of the rigid body have a zero projection along the screw axis, i.e. the body undergoes a pure rotation, thereby proving i)

b) $\alpha = 0$

which is satisfied under one of the following conditions:

b.1) $1 - \cos\theta = 0$

which implies $\theta = 0°$, i.e. the motion reduces to a pure translation. This case, however, has been discarded in the present analysis, for the displacements have been assumed to be nonidentical.

b.2) $\underset{\sim}{e} x (\underset{\sim}{b}_1 - \underset{\sim}{a}_1) \cdot (\underset{\sim}{c}_1 - \underset{\sim}{a}_1) = 0$

which in turn holds under one of the following conditions:

b.2.1) $(\underset{\sim}{b}_1 - \underset{\sim}{a}_1) x (\underset{\sim}{c}_1 - \underset{\sim}{a}_1) = 0$ i.e. the points are collinear, thereby proving that ii) holds

b.2.2) $\underset{\sim}{e}, \underset{\sim}{b}_1 - \underset{\sim}{a}_1$ and $\underset{\sim}{c}_1 - \underset{\sim}{a}_1$ are coplanar, i.e. all three points lie in a plane parallel to the screw axis, q.e.d.

From Theorem 3 a series of results follows, which will be only stated. Within this context, if the given motion is one reducing to a pure rotation, the screw axis is usually referred to as the "axis of rotation". Henceforth, however, no distinction will be made and so the term "screw axis" will be employed, regardless of whether the motion is general or reduces to a pure rotation.

Corollary 2. Assume that a rigid body undergoes an arbitrary motion and choose any three noncollinear points of the body, A, B and C. The necessary and sufficient condition for the displacement differences $\underset{\sim}{u}_A - \underset{\sim}{u}_C$ and $\underset{\sim}{u}_B - \underset{\sim}{u}_C$ (and, consequently, $\underset{\sim}{u}_A - \underset{\sim}{u}_B$) to be

parallel is that the points lie in a plane parallel to the screw axis.

Corollary 3. The displacements of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation.

Corollary 4. If two, and only two, of the displacements of three noncollinear points of a rigid body are parallel, then either i) the parallel vectors are identical, in which case they correspond to points lying on a line parallel to the screw axis, the motion thus being either general or a pure rotation, or ii) the parallel vectors are different, in which case the motion is necessarily a pure rotation.

Keeping the notation introduced in Theorem 3, one has additionally the following:

Corollary 5. Let $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$ be parallel, i.e., let $\underset{\sim}{u}_B = \beta \underset{\sim}{u}_A$, and $\underset{\sim}{u}_C = \underset{\sim}{0}$. The motion is then a pure rotation about an axis passing through C and parallel to vector $\underset{\sim}{b}_1 - \underset{\sim}{c}_1 - \beta(\underset{\sim}{a}_1 - \underset{\sim}{c}_1)$.

Finally, if out of the three given displacements, $\underset{\sim}{u}_A, \underset{\sim}{u}_B$ and $\underset{\sim}{u}_C$, two of these vanish, then the motion is a pure rotation about an axis passing through those points whose displacements vanish. This, however, need not be stated as a Corollary, since it is obvious.

COMPUTATIONAL ALGORITHM

The computational algorithm is based upon the "tree diagram" shown in Fig 1. In that diagram, a particular case is referred to as a sequence of digits indicating which case one has at hand, for successive levels. For instance, case 1.1.1.2 corresponds to Case 1 of the first level, case 1 of the second level, case 1 of the third level and case 2 of the fourth level, i.e. the case at hand is the following: no vector vanishes, all vectors are coplanar, differences are parallel, but at least one difference does not vanish. Furthermore, the diagram shows that, for this case, formula 1, which is next introduced,

is to be applied.

The computation of vector e is crucial in the present algorithm.
Once this has been performed, the remaining parameters are
computaed by application of identical formulae. Vector e, however,
is computed by application of one of two different formulae.

The computation of e is based upon case 2. For cases 1 and 3, e
is computed after the motion is transformed into case 2, as
described next.

Before proceeding further, two results need be proved, which is
done next. All over, the notation of Theorem 3 is resorted to.

*Theorem 4. Given a rigid-body motion defined by the displacements
of three noncollinear points, such that no displacement vanishes,
define a new motion by assigning new displacement vector: u'_A, u'_B
and u'_C to points A,B and C, respectively. If these vectors are
defined as*

$$u'_A = u_A - u_C, \quad u'_B = u_B - u_C, \quad u'_C = 0$$

*then the matrix describing the rotation involved in the new motion
is identical to that in the given one.*

Proof:

From eq. (3), u_A cna be written as

$$u_C = a_2 + Q(C_1 - a_1) - C_1 \tag{25}$$

where, clearly, instead of p_1, one has written c_1. Adding a_1
to the right-hand side of eq. (25) and subtacting it from the
same side does not alter that equation. Hence,

$$u_C = a_2 - a_1 + Q(c_1 - a_1) - (a_1 - a_1)$$

which can readily be rewritten as

$$\underset{\sim}{u}_C = \underset{\sim}{u}_A + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{c}_1 - \underset{\sim}{a}_1)$$

or, alternatively, as

$$\underset{\sim}{u}_A = \underset{\sim}{u}_C + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{a}_1 - \underset{\sim}{c}_1) \qquad (26a)$$

Similarly,

$$\underset{\sim}{u}_B = \underset{\sim}{u}_C + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{c}_1) \qquad (26b)$$

From the latter equations one has immediately

$$\underset{\sim}{u}'_A \equiv \underset{\sim}{u}_A - \underset{\sim}{u}_C = (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{a}_1 - \underset{\sim}{c}_1)$$

and

$$\underset{\sim}{u}'_B \equiv \underset{\sim}{u}_B - \underset{\sim}{u}_C = (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{c}_1)$$

or, since $\underset{\sim}{u}'_C$ has been defined as zero,

$$\underset{\sim}{u}'_A = \underset{\sim}{u}'_C + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{a}_1 - \underset{\sim}{c}_1) \qquad (27a)$$

$$\underset{\sim}{u}'_B = \underset{\sim}{u}'_C + (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{b}_1 - \underset{\sim}{c}_1) \qquad (27b)$$

which are expressions formally identical to eqs. (26a and b), with $\underset{\sim}{u}_A, \underset{\sim}{u}_B$ and $\underset{\sim}{u}_C$ exchanged for $\underset{\sim}{u}'_A, \underset{\sim}{u}'_B$ and $\underset{\sim}{u}'_C$. Vectors $\underset{\sim}{a}_1, \underset{\sim}{b}_1$ and $\underset{\sim}{c}_1$ and matrix $\underset{\sim}{Q}$ remain in the second pair of expressions, however, which completes the proof.

Theorem 5. Given a rigid-body motion defined by the displacements of three noncollinear points, such that two of these, and only two, vanish, define a new motion by assigning new displacement vectors $\underset{\sim}{u}'_A, \underset{\sim}{u}'_B, \underset{\sim}{u}'_C$ to points A, B and C, respectively. Letting $\underset{\sim}{u}_C$ be the unique nonvanishing vector, define

$$\underset{\sim}{u}'_A = -\underset{\sim}{u}_C, \quad \underset{\sim}{u}'_B = -\underset{\sim}{u}_C, \quad \underset{\sim}{u}'_C = \underset{\sim}{0}$$

Then, the matrix describing the rotation involved in the new motion is identical to that in the given one.

Proof:

Since eqs. (26 a and b) are valid regardless of whether any
involved displacement vanishes, one can apply them to the case
at hand, i.e. for $u_A = u_B = 0$, $u_C \neq 0$. Thus,

$$0 = u_C + (Q-I)(a_1 - c_1) \tag{28a}$$

$$0 = u_C + (Q-I)(b_1 - c_1) \tag{28b}$$

and hence,

$$-u_C = 0 + (Q-I)(a_1 - c_1)$$

$$-u_C = 0 + (Q-I)(b_1 - c_1)$$

Recalling the given definitions of u'_A, u'_B and u'_C one obtains
readily

$$u'_A = u'_C + (Q-I)(a_1 - c_1) \tag{29a}$$

$$u'_B = u'_C + (Q-I)(b_1 - c_1) \tag{29b}$$

and the proof follows by introducing the argument resorted to
for proving Theorem 4.

As an application of the foregoing results, one can compute e
for the motion at hand by first defining new motions leading to
case 2 of Fig 1.

Next, the computation of e for case 2 is discussed. Since this
contains two subcases, each is discussed separately.

Case 2.1: One vector, and only one, vanishes, the remaining two
being parallel.
Let $u_B = \beta u_A$, $u_C = 0$
Then, according to Corollary 5, the screw axis is
parallel to vector

$$\underset{\sim}{w} = \underset{\sim}{b}_1 - \underset{\sim}{c}_1 - \beta(\underset{\sim}{a}_1 - \underset{\sim}{c}_1) \tag{30}$$

Vector $\underset{\sim}{e}$ is thus computed as

$$\underset{\sim}{e} = \underset{\sim}{w}/||\underset{\sim}{w}|| \tag{31}$$

Eqs. (30) and (31) constitute what is referred to as "formula 1", in Fig 1

Case 2.2: One vector, and only one, vanishes, the remaining two being nonparallel.

Let $\underset{\sim}{u}_C = 0$ and $\underset{\sim}{u}_A \times \underset{\sim}{u}_B \neq 0$

Then, the screw axis is perpendicular to both $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$ and hence, is given as

$$\underset{\sim}{w} = \underset{\sim}{u}_A \times \underset{\sim}{u}_B \tag{32}$$
$$\underset{\sim}{e} = \underset{\sim}{w}/||\underset{\sim}{w}|| \tag{33}$$

Eqs. (32) and (33) constitute what is referred to as "formula 2" in Fig 1

Once $\underset{\sim}{e}$ is known, θ is computed by application of formulae (14), (13) and (12), in this order, regardless of the particular case at hand.

Vector $\underset{\sim}{\rho}_0$ is computed by application of eq. (17), <u>provided $\theta \neq 180°$</u>. For $\theta = 180°$, the cotangent function contained in that equation becomes infinity and hence, cannot be computed. The computation of $\underset{\sim}{\rho}_0$ in this case then proceeds differently, as is shown next. Before devising means to compute $\underset{\sim}{\rho}_0$ for $\theta = 180°$, the following result is proved.

<u>*Theorem 6.*</u> *Given a rigid-body motion involving a $180°$-rotation, let S be a point of the screw axis and let A be an arbitrary point outside the screw axis. Moreover, let $\underset{\sim}{s}, \underset{\sim}{a}, \underset{\sim}{u}_S$ and $\underset{\sim}{u}_A$ be the respective position and displacement vectors of those points. Then the point located by the position vector $\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{u}_A$ lies on the screw axis.*

Proof:

If the point located by $a+\frac{1}{2}u_A$ is on the screw axis, then vector $a+\frac{1}{2}u_A-s$ is parallel to this axis. Thus, denoting by Q the involved rotation matrix, the following equality holds:

$$Q(a+\frac{1}{2}u_A-s)=a+\frac{1}{2}u_A-s \tag{34}$$

Thus, all that need be proved is eq. (34). This follows from the fact that, according to eq. (19), u_A can be written as

$$u_A=u_S+(Q-I)(a-s) \tag{35}$$

Now, $(Q-I)(a-s)$ lies naturally in the range of $(Q-I)$ (by definition) and hence is perpendicular to its null space, i.e., to u_S. Now, since $(Q-I)(a-s)$ is a vector perpendicular to u_S or equivalently, to the screw axis, its image under Q must be its negative, i.e.

$$Q(Q-I)(a-s)=-(Q-I)(a-s) \tag{36}$$

On the other hand,

$$Q(a+\frac{1}{2}u_A-s)=Q\left[a+\frac{1}{2}u_S+\frac{1}{2}(Q-I)(a-s)-s\right]=$$

$$=\frac{1}{2}u_S+Q\left[(a-s)+\frac{1}{2}(Q-I)(a-s)\right]=$$

$$=\frac{1}{2}u_S+Q(a-s)+\frac{1}{2}Q(Q-I)(a-s)$$

By virtue of eq. (36) the latter equation becomes

$$Q(a+\frac{1}{2}u_A-s)=\frac{1}{2}u_S+Q(a-s)-\frac{1}{2}(Q-I)(a-s)$$

Adding and subtracting $a-s$ to the left-hand side of the latter equation leads to

$$Q(a+\frac{1}{2}u_A-s)=\frac{1}{2}u_S+a-s+(Q-I)(a-s)-\frac{1}{2}(Q-I)(a-s)$$

$$\mathbf{a} + \frac{1}{2} \left[\mathbf{u_s} + (\mathbf{Q} - \mathbf{I})(\mathbf{a} - \mathbf{s}) \right] - \mathbf{s} =$$

$$= \mathbf{a} + \frac{1}{2} \mathbf{u_A} - \mathbf{s}, \text{q.e.d.}$$

Now, if $\mathbf{a} + \frac{1}{2} \mathbf{u_A}$ locates one point on the screw axis, the position vector $\boldsymbol{\rho}_0$, of the one point on the screw axis lying closest to the origin is the component of vector $\mathbf{a} + \frac{1}{2} \mathbf{u_A}$ perpendicular to the axis or, equivalently, to vector $\mathbf{e}$. Hence, $\boldsymbol{\rho}_0$ can be computed in this case as

$$\boldsymbol{\rho}_0 = \mathbf{a} + \frac{1}{2} \mathbf{u_A} - (\mathbf{a} + \frac{1}{2} \mathbf{u_A}) \cdot \mathbf{e}\mathbf{e} \tag{37}$$

The slide displacement s, being the orthogonal projection of the displacement field on the screw axis, is simply given as

$$s = \mathbf{u_A} \cdot \mathbf{e} \tag{38}$$

thereby completing the computation of the screw parameters defining the given rigid-body motion.

A computer subprogram, SCREW [13] was written, that realizes the foregoing algorithm. This subrprogram is written in the Fortran IV dialect of the Burroughs 6700 computer of the U. of Mexico and is available upon request.

Next, some examples are presented, which illustrate the program output.

EXAMPLE 1

| POINT | | COORDINATES | |
	X	Y	Z
A(1)=	1.00000	0.00000	0.00000
B(1)=	1.00000	1.00000	0.00000
C(1)=	2.00000	1.00000	-1.00000
A(2)=	2.00000	0.00000	-1.00000
B(2)=	2.00000	0.00000	0.00000
C(2)=	3.00000	-1.00000	0.00000

NUMBER OF VANISHING DISPLACEMENTS IS   0
VECTOR DIFFERENCES ARE PARALLEL

        THE SCREW AXIS HAS THE FOLLOWING DIRECTION COSINES
        (VECTOR E) :   -0.57735   0.57735   0.57735
        THE POINT ON THE SCREW AXIS CLOSEST TO THE ORIGIN HAS
        THE FOLLOWING X-,Y-AND Z COORDINATES
        (VECTOR RHO) :   0.50000   0.33333   0.16667
        THE ANGLE OF ROTATION (THETA) IS -336.57841 DEGREES
        THE DISPLACEMENT ALONG THE SCREW AXIS (DISPL) IS   -1.15470

EXAMPLE 2

| POINT | | COORDINATES | |
	X	Y	Z
A(1)=	0.00000	0.00000	1.00000
B(1)=	0.00000	1.00000	0.00000
C(1)=	1.00000	0.00000	0.00000
A(2)=	0.00000	1.00000	0.00000
B(2)=	1.00000	0.00000	0.00000
C(2)=	0.00000	0.00000	1.00000

NUMBER OF VANISHING DISPLACEMENTS IS   0
VECTORS ARE COPLANAR AND DIFFERENCES ARE NONPARALLEL

        THE SCREW AXIS HAS THE FOLLOWING DIRECTION COSINES
        (VECTOR E) :   -0.57735   -0.57735   -0.57735
        THE POINT ON THE SCREW AXIS CLOSEST TO THE ORIGIN HAS
        THE FOLLOWING X-,Y-AND Z COORDINATES
        (VECTOR RHO) :   -0.00000   0.00000   0.0000(
        THE ANGLE OF ROTATION (THETA) IS   336.57841 DEGREES
        THE DISPLACEMENT ALONG THE SCREW AXIS (DISPL) IS   0.0000(

EXAMPLE 3

POINT	COORDINATES		
	X	Y	Z
A(1)=	2.00000	0.00000	0.00000
B(1)=	2.50000	0.50000	0.00000
C(1)=	3.00000	1.00000	0.00000
A(2)=	2.00000	2.00000	0.00000
B(2)=	2.50000	1.50000	0.00000
C(2)=	3.00000	1.00000	0.00000

POINTS ARE COLLINEAR.MOTION IS UNDEFINED

EXAMPLE 4

POINT	COORDINATES		
	X	Y	Z
A(1)=	2.00000	1.00000	0.00000
B(1)=	3.00000	1.00000	0.00000
C(1)=	2.00000	2.00000	0.00000
A(2)=	2.00000	1.00000	0.00000
B(2)=	3.00000	1.00000	0.00000
C(2)=	2.00000	1.00000	0.00000

MOTION IS NOT RIGID.LENGTH AC DOES NOT REMAIN CONSTANT.

Three displacement vectors of corresponding three noncollinear points of a rigid body are given

1. No vector vanishes

2. One vector, and only one, vanishes

3. Two vectors, and only two, vanish. Apply formula 1

4. All three vectors vanish. Motion **is a zero-displacement pure** translation.

1. Vectors are coplanar

2. Vectors are noncoplanar. **Apply formula 2**

1. the remaining two are parallel. Apply formula 1

2. The remaining two are nonparallel. Apply formula 2

1. Differences are parallel.

2. Differences are nonparallel Apply formula 2

1. Both differences vanish. Motion is a pure translation

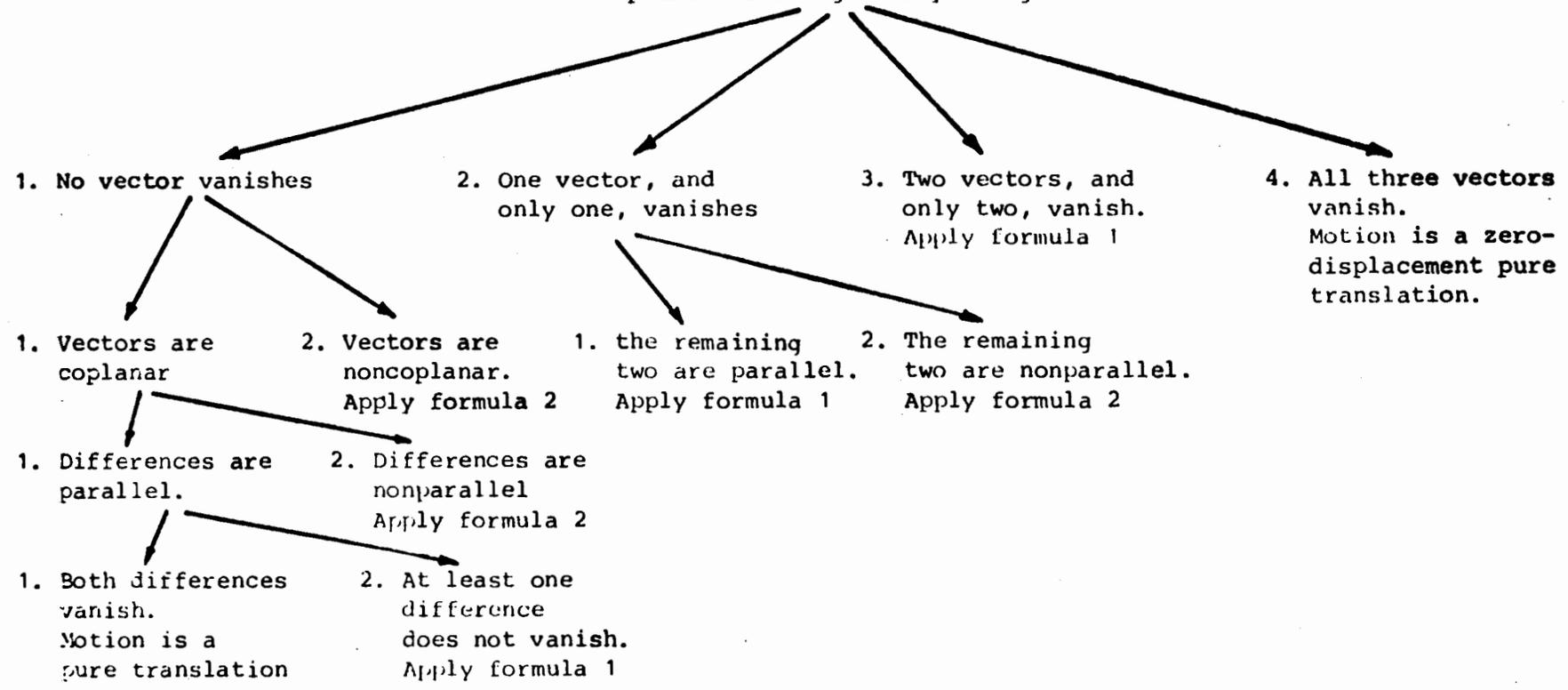**2. At least one** difference does not vanish. Apply formula 1

Fig 1. Tree diagram   showing the different possible relationships amongst the displacements of three noncollinear points defining a rigid-body motion.

ACKNOWLEDGEMENTS

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