



UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
PROGRAMA DE MAESTRÍA Y DOCTORADO EN INGENIERÍA
MECÁNICA – MECATRÓNICA

CINEMÁTICA Y DINÁMICA DE SISTEMA MECÁNICO APLICADOR DE PINTURA

TESIS
QUE PARA OPTAR POR EL GRADO DE:
MAESTRO EN INGENIERÍA

PRESENTA:
LUIS ÁNGEL LÓPEZ CEDILLO

TUTOR PRINCIPAL
DR. FRANCISCO CUENCA JIMÉNEZ

Ciudad Universitaria, CD. MX. FEBRERO 2017

Jurado Asignado:

Presidente: Dr. González Villela Víctor Javier

Secretario: Dr. Ramírez Reivich Alejandro C.

Vocal: Dr. Cuenca Jiménez Francisco

1er. Suplente: Dr. Velázquez Villegas Fernando

2do. Suplente: Dr. Díaz Hernández Octavio

Lugar donde se realizó la tesis: Facultad de Ingeniería

Tutor de Tesis:

Dr. Francisco Cuenca Jiménez

Firma

Capítulo 1 Generalidades.....	6
1.1 Introducción.	6
1.2 Objetivo.....	6
1.3 Fundamentos de Robots.	6
1.3.1 Definición de robot industrial.	6
1.3.2 Estructura mecánica del robot.	7
1.4 Estado del arte.	8
1.5 Marco teórico.	10
1.6 Justificación.....	11
1.7 Metodología.	12
1.8 Arquitectura del robot.	12
Capítulo 2 Análisis cinemático.....	14
2.1 Introducción	14
2.2 Grados de libertad.	15
2.3 Solución algebraica.	16
2.4 Cálculo de posición para la cadena 0.....	16
2.4.1 Solución del ángulo θ_{21}	16
2.4.2 Solución del ángulo θ_{43}	22
2.4.3 Solución del ángulo θ_{54}	23
2.4.4 Solución del ángulo θ_{1817}	23
2.4.5 Solución del ángulo θ_{1716}	25
2.4.6 Solución del ángulo θ_{1615}	26
2.5 Cálculo de velocidad para la cadena 0.	27
2.5.1 Solución de la velocidad para ángulo θ_{21}	27
2.5.2 Solución de la velocidad para ángulo θ_{43}	28
2.5.3 Solución de la velocidad, ángulo θ_{54}	29
2.5.4 Solución de la velocidad, ángulo θ_{1817}	29
2.5.5 Solución de la velocidad, ángulo θ_{1716}	30
2.5.6 Solución de la velocidad, ángulo θ_{1615}	31
2.6 Calculo de la aceleración para la cadena 0.....	32
2.6.1 Solución de la aceleración, ángulo θ_{21}	32
2.6.2 Solución de la aceleración, ángulo θ_{43}	33
2.6.3 Solución de la aceleración, ángulo θ_{54}	34
2.6.4 Solución de la aceleración, ángulo θ_{1817}	34

2.6.5 Solución de la aceleración, ángulo θ_{1716}	35
2.6.6 Solución de la aceleración, ángulo θ_{1615}	35
2.7 Cálculo de posición para las cadenas i.....	36
2.7.1 Solución del ángulo θ_{21i} de las cadenas i.....	36
2.7.2 Solución del ángulo θ_{65i} de las cadenas i.....	41
2.7.3 Solución del ángulo θ_{76i} de las cadenas i.....	42
2.7.4 Solución del ángulo θ_{1110i} de las cadenas i.....	43
2.7.5 Solución del ángulo θ_{109i} de las cadenas i.....	44
2.7.6 Solución del ángulo θ_{98i} de las cadenas i.....	46
2.8 Cálculo de velocidad de las cadenas i.....	47
2.8.1 Solución de la velocidad ángulo θ_{21i} de las cadenas i.....	47
2.8.2 Solución de la velocidad ángulo θ_{65i} de las cadenas i.....	48
2.8.3 Solución de la velocidad ángulo θ_{76i} de las cadenas i.....	49
2.8.4 Solución de la velocidad ángulo θ_{1110i} de las cadenas i.....	50
2.8.5 Solución de la velocidad ángulo θ_{109i} de las cadenas i.....	52
2.8.6 Solución de la velocidad ángulo θ_{98i} de las cadenas i.....	53
2.9 Cálculo de la aceleración para las cadenas i.....	54
2.9.1 Solución de la aceleración ángulo θ_{21i} de las cadenas i.....	54
2.9.2 Solución de la aceleración ángulo θ_{65i} de las cadenas i.....	55
2.9.3 Solución de la aceleración ángulo θ_{76i} de las cadenas i.....	57
2.9.4 Solución de la aceleración ángulo θ_{1110i} de las cadenas i.....	57
2.9.5 Solución de la aceleración ángulo θ_{109i} de las cadenas i.....	57
2.9.6 Solución de la aceleración ángulo θ_{109i} de las cadenas i.....	58
Capítulo 3 Análisis Dinámico Formulación Euler-Lagrange	59
3.1 Introducción	59
3.2 Velocidad de Centros de Gravedad y Velocidad Angular para cadena 0.....	60
3.2.1 Velocidad de Centro de Gravedad del Cuerpo 1	63
3.2.2 Velocidad Angular del Cuerpo 1	64
3.2.3 Velocidad de Centro de Gravedad del Cuerpo 2	64
3.2.4 Velocidad Angular del Cuerpo 2	65
3.2.5 Velocidad de Centro de Gravedad del Cuerpo 3	66
3.2.6 Velocidad Angular del Cuerpo 3	67
3.2.7 Velocidad de Centro de Gravedad del Cuerpo 4	67
3.2.8 Velocidad Angular del Cuerpo 4	68

3.2.9 Velocidad de Centro de Gravedad del Cuerpo 5	69
3.3 Función Lagrangiana	70
3.3.1 Desarrollo del Primer Término de la Ecuación de Lagrange para la cadena 0.	71
3.3.2. Desarrollo del Segundo Término de la Ecuación de Lagrange para la cadena 0.....	85
3.4 Velocidad de Centros de Gravedad y Velocidad Angular para las cadenas i.	115
3.4.1 Velocidad de Centro de Gravedad del Cuerpo 1i.....	116
3.4.2 Velocidad Angular del Cuerpo 1i.....	117
3.4.3 Velocidad de Centro de Gravedad del Cuerpo 2i.....	117
3.4.4 Velocidad Angular del Cuerpo 2i.....	118
3.5 Función Lagrangiana para cadenas i.....	119
3.5.1 Desarrollo del Primer Término de la Ecuación de Lagrange para las cadenas i.	119
3.5.2. Desarrollo del Segundo Término de la Ecuación de Lagrange para las cadenas i.	124
3.6 Fuerzas Generalizadas.....	131
3.7 Sustitución de los términos de la ecuación de Lagrange.....	133
Capítulo 4 Resultados.....	136
4.1 Gráficas de Posición	136
4.2 Gráficas de Velocidad	140
4.3 Gráficas de Aceleración	144
4.4 Solución del método Euler-Lagrange	148
Capítulo 5 Conclusiones	150
Bibliografía	151
Apéndice A.....	153
Apéndice B.....	162
Apéndice C	163
Apéndice D.....	192
Apéndice E	222

Capítulo 1

Generalidades

1.1 Introducción.

En este trabajo se presenta primero un marco teórico acerca de los robots paralelos industriales. Posteriormente se da una descripción del proyecto y se comienza con el análisis cinemático del mismo.

1.2 Objetivo.

Analizar la cinemática y dinámica de un robot paralelo espacial de 6 grados de libertad para la obtención de un modelo.

1.3 Fundamentos de Robots.

1.3.1 Definición de robot industrial.

Existen diversas definiciones acerca de lo que es un robot industrial según la institución de la cual provenga dicha definición. Algunas de estas son:

Según la **Asociación de Industrias Robóticas (RIA)**:

- Un robot industrial es un manipulador multifuncional reprogramable, capaz de mover materias, piezas, herramientas o dispositivos especiales, según trayectorias variables, programadas para realizar tareas diversas (Barrientos et al., 2007).

Según la **Organización Internacional de Estándares (ISO)**:

- Manipulador funcional reprogramable con varios grados de libertad, capaz de manipular materias, piezas, herramientas o dispositivos especiales según trayectorias variables programadas para realizar tareas diversas (Barrientos et al., 2007).

Al leer las dos definiciones anteriores se observa que son muy parecidas aunque en la de la ISO se especifica que debe tener varios grados de libertad. Las dos hacen referencia a que es un manipulador aunque no se da la definición de éste. Esto no ocurre en la definición de la **Asociación Francesa de Normalización (AFNOR)** que primero define lo que es un manipulador:

- Manipulador: mecanismo formado generalmente por elementos en serie, articulados entre sí, destinado al agarre y desplazamiento de objetos. Es multifuncional y puede ser gobernado directamente por un operador humano o mediante un dispositivo lógico (Barrientos et al., 2007).

- **Robot:** manipulador automático servo-controlado, reprogramable, polivalente, capaz de orientar y posicionar piezas útiles o dispositivos especiales, siguiendo trayectorias variables reprogramables, para la ejecución de tareas variadas. Normalmente tiene la forma de uno o varios brazos terminados en una muñeca. Su unidad de control incluye un dispositivo de memoria y ocasionalmente de percepción del entorno. Normalmente su uso es el de realizar una tarea de manera cíclica, pudiéndose adaptar a otra sin cambios permanentes en su material (Barrientos et al., 2007).

1.3.2 Estructura mecánica del robot.

Mecánicamente un robot está formado por una serie de elementos o eslabones interconectados entre sí por medio de articulaciones o juntas, que permiten el movimiento y establecen el número de grados de libertad entre dos elementos consecutivos. Los grados de libertad se pueden definir como el número de movimientos independientes que tiene un eslabón con respecto a otro.

El movimiento de las articulaciones puede ser de rotación o de traslación o una combinación de ambas, por lo que existen seis diferentes tipos de articulación (Barrientos et al., 2007):

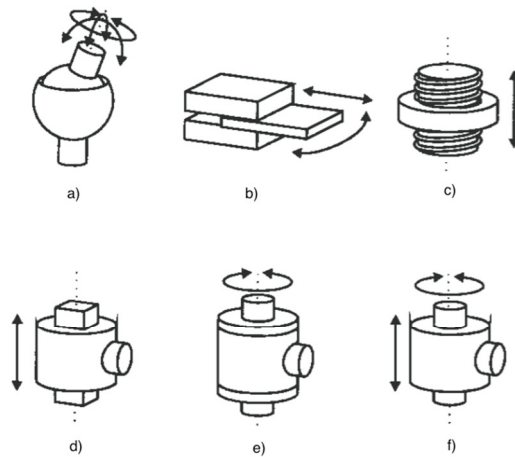


Figura 1-1 Tipos de articulación.

- a) **Esférica (3 GDL):** Permite tres giros (θ_x θ_y θ_z) entre eslabones.
- b) **Plana (3 GDL):** Permite dos traslaciones (x , y) y un giro (θ) entre eslabones.
- c) **Tornillo (1 GDL):** Permite una traslación (x) y un giro (θ) pero están relacionados.
- d) **Prismática (1 GDL):** Permite una traslación (x) entre eslabones.
- e) **Rotacional (1 GDL):** Permite un giro (θ) entre eslabones.
- f) **Cilíndrica (2 GDL):** Permite un giro (θ) y una traslación (x) entre sus eslabones.

El empleo de diferentes combinaciones de articulaciones, da lugar a diferentes configuraciones, con características a tener en cuenta tanto en el diseño y construcción del robot como en su aplicación. A partir de estas combinaciones se puede establecer una clasificación que se muestra en la imagen siguiente (Barrientos et al., 2007):

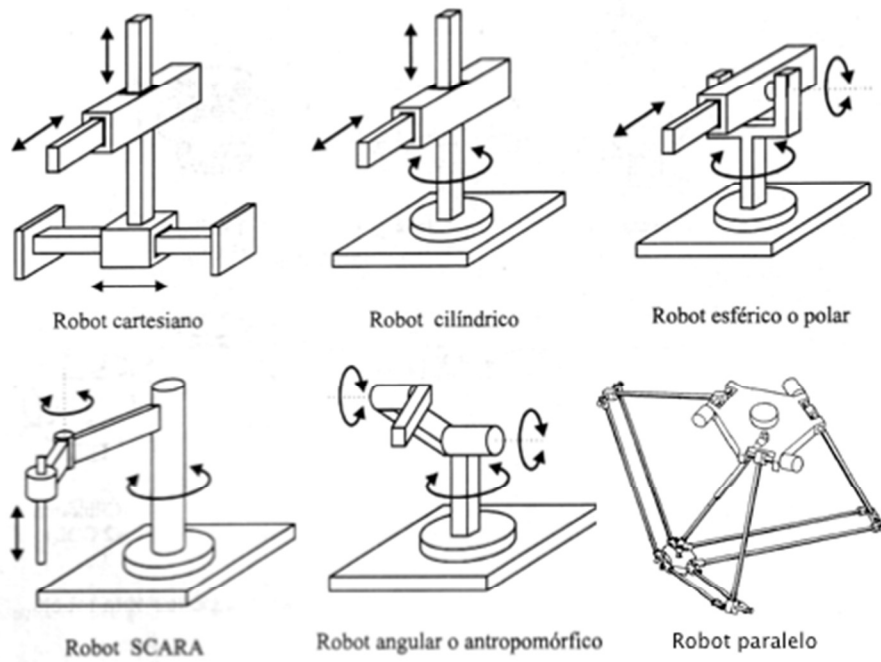


Figura 1-2 Clasificación de robots industriales.

De esta manera un robot paralelo está formado por un efector y una base fija unidos por al menos dos cadenas cinemáticas independientes en donde el número de actuadores es igual número de grados de libertad (Merlet, 2000).

1.4 Estado del arte.

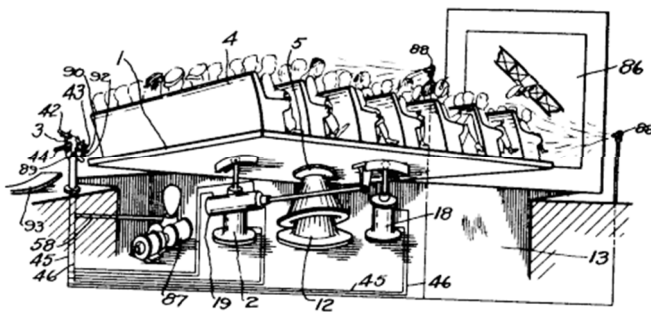


Figura 1-3 Plataforma diseñada por James Gwinnett.

Los robots paralelos fueron originalmente propuestos para lidiar con las desventajas de los robots seriales siendo estas su limitada capacidad de carga, poca precisión y baja rigidez (Angeles, 1997).

El primer robot espacial paralelo fue una plataforma de movimiento dedicada a la industria del entretenimiento y diseñada por James E. Gwinnett (Gwinnett, 1931). Su dispositivo estaba basado en un robot esférico paralelo y fue patentado en 1931.

Una década después un nuevo robot paralelo fue inventado para el pintado automatizado con spray por Willard L.V. Pollard (Pollard, 1940) y es conocido como el primer robot paralelo industrial. Su invento tiene 5 grados de libertad, consta de tres cadenas cinemáticas y fue patentado en 1942 (US Patent No. 2,286,571).

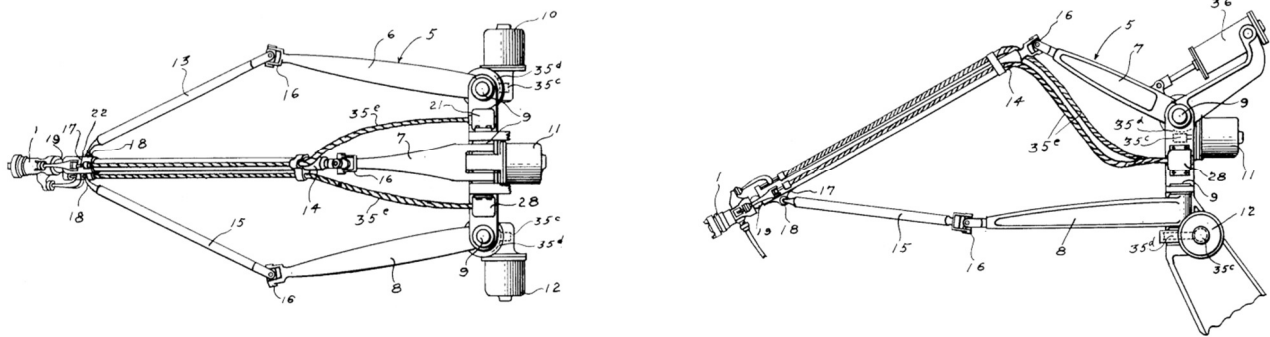


Figura 1-4 Robot patentado por Willard L.G. Pollard Jr.

En Octubre 29 de 1934 el hijo de Willard L.V. Pollard, Willard L.G. Pollard Jr. (Pollard, 1942), presentó una patente de una máquina para pintado con spray. La patente consistía en dos partes, una era el sistema de control el cual consistía en películas perforadas en las cuales la velocidad de rotación de los motores era proporcional a la densidad de los orificios. La otra parte era un manipulador mecánico el cual estaba basado en un pantógrafo con dos motores como actuadores en la base. La patente fue emitida el 27 de agosto de 1940.

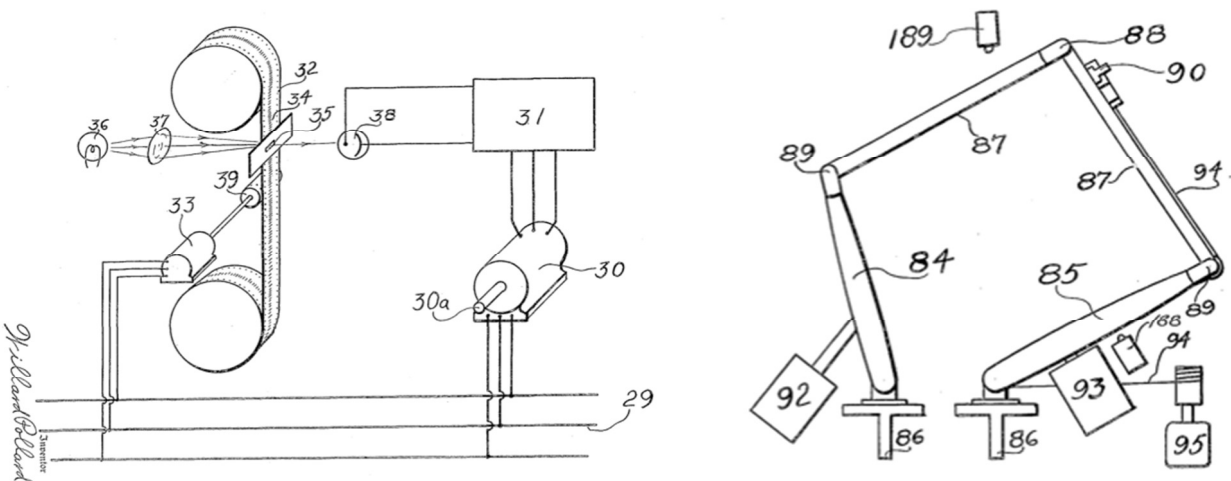


Figura 1-5 Ilustraciones presentes en la patente de DeVilbiss.

Mientras tanto en 1937 le fue concedida a una licencia a la compañía DeVilbiss, que más tarde en 1941 se convertiría en el primer proveedor de robots industriales, completando su primer prototipo bajo la dirección de Harold Roselund (Roselund,

1944). La patente de este robot fue emitida el 14 de Marzo de 1944 y utilizaba el sistema de control diseñado por Pollard Jr. Es por esto que el crédito del primer robot industrial construido se le atribuye tanto a Ruselund como a Pollard Jr.

Existen otros robots paralelos presentes en la historia como el hexápodo de Gough diseñado a partir de la necesidad de medir las deflexiones causadas por fuerzas aplicadas en cualquier dirección de un neumático (Gough y Whitehall, 1962). Una configuración similar fue un mecanismo de 8 barras con propósitos diversos desde espaciales hasta simuladores de vuelo (Stewart, 1965). Debido a la similitud entre ambos mecanismos las plataformas basadas en octaedros hexápodos son conocidas como plataformas Gough-Stewart (Gallardo, 2016). A pesar de los trabajos previos la primer patente para un mecanismo octaedro hexápodo fue concedida a Klaus Cappel en 1967 (Cappel, 1967).

En 1980 Raymond Clavel presentó un mecanismo original que hoy se conoce como "Robot Delta" el cual hace uso de paralelogramos para restringir las rotaciones entre la plataforma fija y la móvil (Clavel 1988). La patente para dicho robot la obtuvo en 1990 (Clavel, 1990). El robot delta y variantes del mismo son de los más usados para tareas de tomar y colocar en la actualidad. Una extensión a seis grados de libertad del robot delta fue desarrollada por Pierrot François y compañía, el robot fue conocido como "Hexa" (Pierrot et al., 1991). Entre 1993 y 1996 Gosselin y compañía desarrollaron un robot esférico para la rápida orientación de una cámara llamado "Agile Eye" (Gosselin et al., 1994). En 2003 Kim y Tsai desarrollaron un robot cartesiano donde el movimiento en cada eje es independiente. A este tipo de robot se le conoce como robot paralelo lineal (Kim y Tsai, 2003).

Muchos otros robots con aplicaciones específicas han sido diseñados con base en los antes mencionados, algunos de estos son; un robot para la emulación de operaciones espaciales basado en una plataforma de Stewart (Nguyen et al., 1993), mecanismo de 6 GDL, robot para lelo dedicado a operaciones de excavación subterránea (Arai et al., 1991), un delta modificado para resucitación cardiopulmonar (Yangmin y Qingsong, 2005) o un robot de 6 grados de libertad para el aislamiento de vibraciones (Geng y Haynes, 1993).

1.5 Marco teórico.

El desarrollo de la cinemática de robots se puede dividir en dos enfoques, la cinemática directa y la inversa. La cinemática directa consiste en describir la posición y orientación del efector final dados los valores de las variables de las juntas (Spong, 1989). El enfoque de cinemática inversa consiste en descubrir el comportamiento de las juntas dada una trayectoria específica (Spong, 1989).

Existen algunos métodos para calcular el comportamiento cinemático del robot, como son; el método algebraico, el método iterativo, el método matricial que puede ser abordado por medio de transformaciones homogéneas o por la convención de Denavit-Hatenberg. El enfoque de transformaciones homogéneas consiste en expresar

los movimientos del robot en términos de matrices de rotación y traslación (Tsai, 1999). Para el caso de un robot espacial se definen 6 transformaciones, tres para las rotaciones y tres para las traslaciones. La convención de Denavit-Hatemberg consiste en usar una matriz de transformación compuesta por dos rotaciones y dos traslaciones (Spong, 1989). La teoría de tornillos también puede ser usada para el cálculo cinemático. Esta teoría permite expresar desplazamientos, velocidades, fuerzas y torques en un espacio tridimensional, usando la idea de que el movimiento de un cuerpo rígido puede ser representado por la unión de una rotación alrededor de un eje y una traslación sobre el mismo, donde el eje es coincidente con el objeto (Gallardo, 2016).

El cálculo dinámico consiste en calcular las fuerzas y torques necesarios para que el mecanismo alcance una posición deseada en un tiempo dado (Spong, 1989). El desarrollo de un modelo dinámico es importante para la simulación del robot ya que de esta manera es posible predecir el comportamiento del mismo antes de ser construido. Otro aspecto de importancia es que el modelo obtenido es usado para el diseño de una estrategia de control. Un controlador eficiente requiere de un modelo dinámico realista para tener un desempeño óptimo. Así mismo el análisis dinámico ayuda al diseño de los actuadores, eslabones y rodamientos, mediante la obtención de las fuerzas y momentos de reacción en las juntas (Tsai, 1999).

Al igual que con la cinemática, el problema de la dinámica puede ser abordado de dos maneras. Una es la dinámica directa, la cual consiste encontrar la respuesta del robot ante una serie de fuerzas y torques aplicados en función del tiempo. La otra es la dinámica inversa que se centra en calcular las fuerzas y torques necesarios para que el robot siga una trayectoria deseada (Tsai, 1999).

Las ecuaciones dinámicas de movimiento pueden ser formuladas de distintas maneras. Un enfoque es la aplicación del método de Newton-Euler, lo cual deriva en la obtención de ecuaciones que contemplan tanto las fuerzas aplicadas como las fuerzas de restricción. Otro método comúnmente es el de Euler-Lagrange el cual elimina las fuerzas de restricción y se enfoca únicamente en las fuerzas o torques encargados del movimiento (Tsai, 1999). Alternativamente un tercer método puede ser usado el cual se conoce como método de Kane, el cual al igual que el de Euler-Lagrange, prescinde de las fuerzas de restricción y se enfoca en el uso de fuerzas generalizadas para su formulación (Kane y Levinson, 1985).

1.6 Justificación.

El robot diseñado por Pollard fue el primer mecanismo paralelo en ser propuesto para propósitos industriales, desafortunadamente nunca llegó a ser construido. Por su configuración mecánica se puede considerar como el ancestro del robot delta sin embargo su análisis resulta ser más complicado debido a la presencia de juntas esféricas en el órgano terminal. Debido a esta mayor dificultad, la mayoría de los estudios se han centrado en analizar y mejorar configuraciones basadas en el delta. A

causa de esta falta de documentación referente a su análisis, resulta ser un buen caso de estudio para evaluar el comportamiento que tendría el robot en procesos industriales.

1.7 Metodología.

En el desarrollo de la tesis se seguirá la siguiente metodología:

Análisis Cinemático

1. Análisis de Posición
2. Análisis de Velocidad
3. Análisis de Aceleración

Análisis Dinámico

Formulación Euler-Lagrange

1. Energía Cinética
2. Energía Potencial
3. Función Lagrangiana
4. Fuerzas Generalizadas

1.8 Arquitectura del robot.

El modelo de robot a analizar está basado en el diseño de Willard L.G. Pollard.

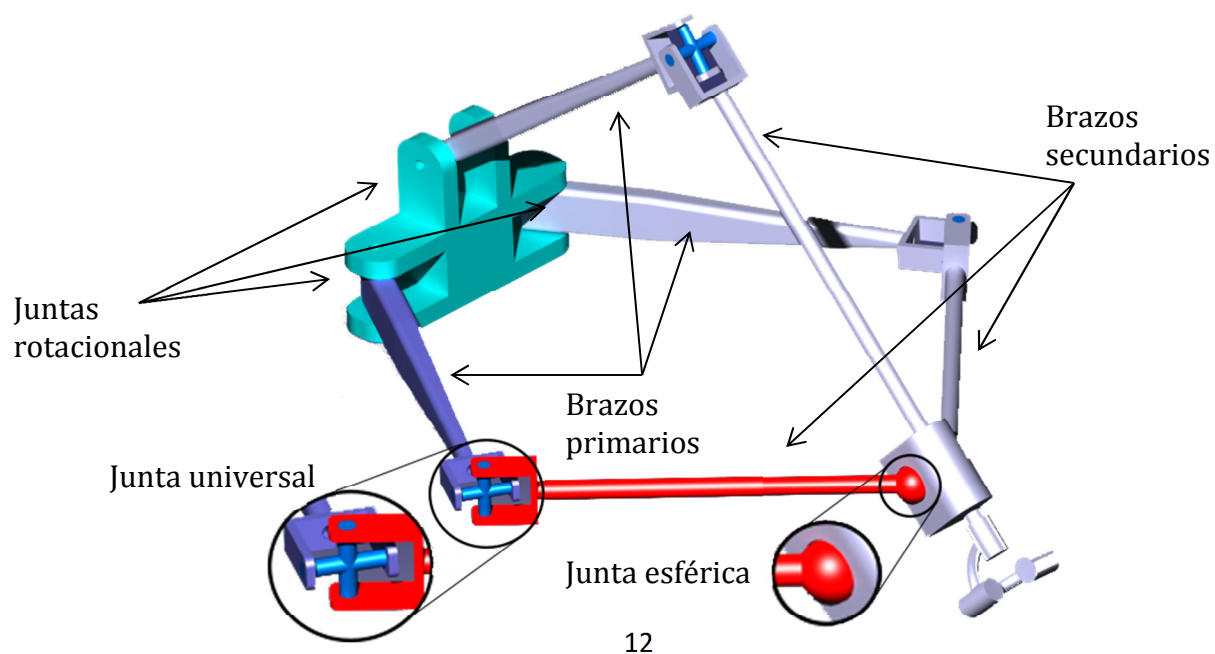


Figura 1-6 Modelo del robot basado en el diseño de Willard L.G. Pollard Jr.

El robot cuenta con tres brazos primarios unidos a una base mediante juntas rotacionales. Estos a su vez están unidos cada uno a tres brazos secundarios mediante juntas universales. Dos de los tres brazos convergen en un punto mediante juntas esféricas y el tercer brazo secundario está unido rígidamente a este mismo punto. En el punto donde convergen los tres brazos va montado el órgano terminal mediante una junta esférica. En la base, donde van montados los brazos primarios irán acoplados a estos los actuadores encargados de dar la posición al órgano terminal y en el lugar en el cual convergen los brazos secundarios irán los otros tres actuadores encargados de dar la orientación a la herramienta.

Capítulo 2

Análisis cinemático

2.1 Introducción

Se usaron Transformaciones Homogéneas para el análisis cinemático (Stejskal y Valáek, 1996). Este método consiste en crear sistemas de referencia locales a lo largo de cadenas cinemáticas con el fin de aplicar una serie de rotaciones y traslaciones por medio de matrices de transformación.

Una matriz homogénea de transformación es una matriz de 4x4 cuyo propósito es proyectar un vector de posición de un sistema de coordenadas a otro (Tsai, 1999). En general se puede dividir en 4 submatrices de manera siguiente:

$$\mathbf{T}_B^A = \begin{bmatrix} \mathbf{R}_B^A(3 \times 3) & \mathbf{q}(3 \times 1) \\ \boldsymbol{\gamma}(1 \times 3) & \rho(1 \times 1) \end{bmatrix}$$

Donde la submatriz superior izquierda de 3x3 denota la orientación de un marco móvil con respecto a un marco de referencia. La submatriz superior derecha de 3x1 denota la posición del origen del marco móvil con respecto al marco de referencia. La submatriz inferior izquierda denota una transformación de perspectiva y el elemento inferior derecho es un factor de escalamiento (Tsai, 1999), (Nevatia et al., 1941).

De esta manera se definen 6 transformaciones:

- Rotación alrededor en el eje X ($\mathbf{T}_{z1}$).
- Rotación alrededor en el eje Y ($\mathbf{T}_{z2}$).
- Rotación alrededor en el eje Z ($\mathbf{T}_{z3}$).
- Traslación sobre el eje X ($\mathbf{T}_{z4}$).
- Traslación sobre el eje Y ($\mathbf{T}_{z5}$).
- Traslación sobre el eje Z ($\mathbf{T}_{z6}$).

Por convención se nombra a cada una de ellas con $\mathbf{T}$ y se les da un subíndice para distinguirlas.

$$\mathbf{T}_{z1}(x) = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\mathbf{T}_{z2}(y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z3}(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z4}(\theta_x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z5}(\theta_y) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z6}(\theta_z) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Grados de libertad.

Los grados de libertad es el número de entradas independientes necesarias para definir la configuración de un mecanismo. Para un robot tridimensional el cálculo de los grados de libertad puede ser realizado por medio de la fórmula de Chebyshev-Grübler-Kutzbach de manera siguiente:

$$L = 6(b - g - 1) + \sum_k f_k$$

Donde b es el número de cuerpos, g es el número de juntas y f_k es el número de grados de libertad de la junta k . Para este caso particular los valores son los siguientes:

$$\begin{aligned} b &= 8 \\ g &= 9 \\ \sum_k f_k &= 18 \end{aligned}$$

Al sustituir los valores anteriores se tiene:

$$\begin{aligned} L &= 6(8 - 9 - 1) + 18 \\ L &= 6 \end{aligned}$$

Por lo tanto el robot tiene 6 grados de libertad

2.3 Solución algebraica.

Para la solución algebraica se usó el enfoque de cinemática inversa, es decir que se calcularon la posición, velocidad y aceleración de los ángulos de cada una de las juntas del robot en función de una trayectoria dada. De esta manera las variables de entrada son la posición y orientación del efector definidas con la notación $x, y, z, \psi, \theta, \phi$.

2.4 Cálculo de posición para la cadena 0.

Para el cálculo de la posición de la cadena 0 se tiene como variables a calcular los ángulos $\theta_{21}, \theta_{43}, \theta_{54}, \theta_{1817}, \theta_{1716}, \theta_{1615}$. Para comenzar con el análisis se estableció un punto de origen sobre a base del robot a partir del cual se comenzaron a escribir las matrices de transformación. Además se definieron tres cadenas cinemáticas que se muestran en la figura siguiente.

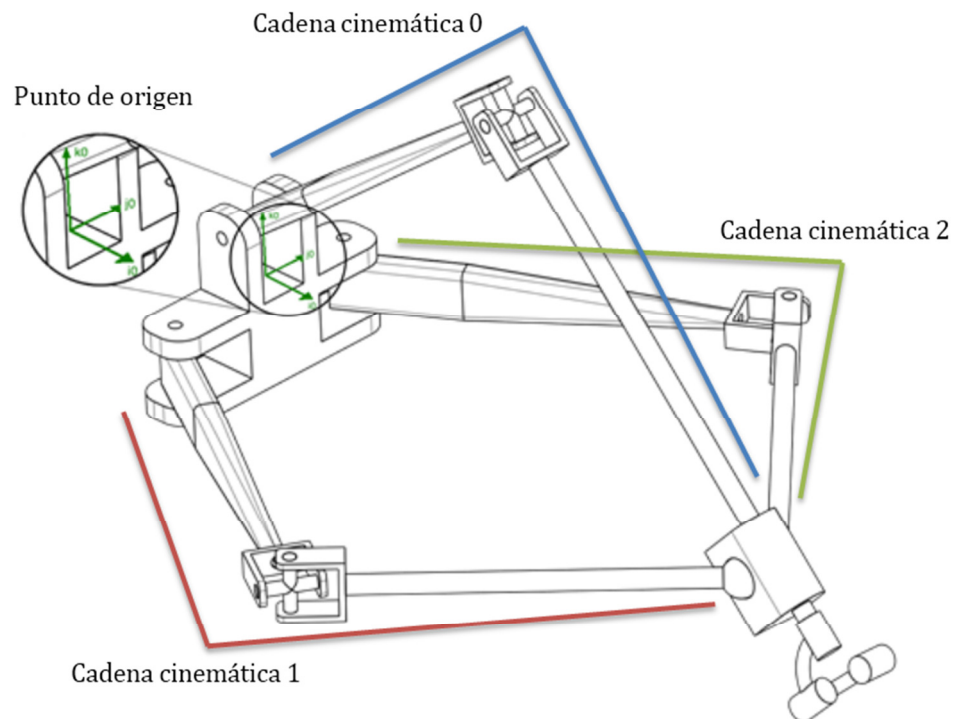


Figura 2-1 Cadenas cinemáticas.

2.4.1 Solución del ángulo θ_{21} .

Para la solución del ángulo se busca que éste quede expresado en términos de longitudes constantes y ángulos constantes o conocidos. De ésta manera se forman lazos vectoriales a lo largo de la cadena cinemática 0.

Se generan las bases locales a partir de la base inercial por medio de transformaciones homogéneas. En las figuras 2-2 a 2-5 se muestra la secuencia en la cual se fueron generando las bases locales.

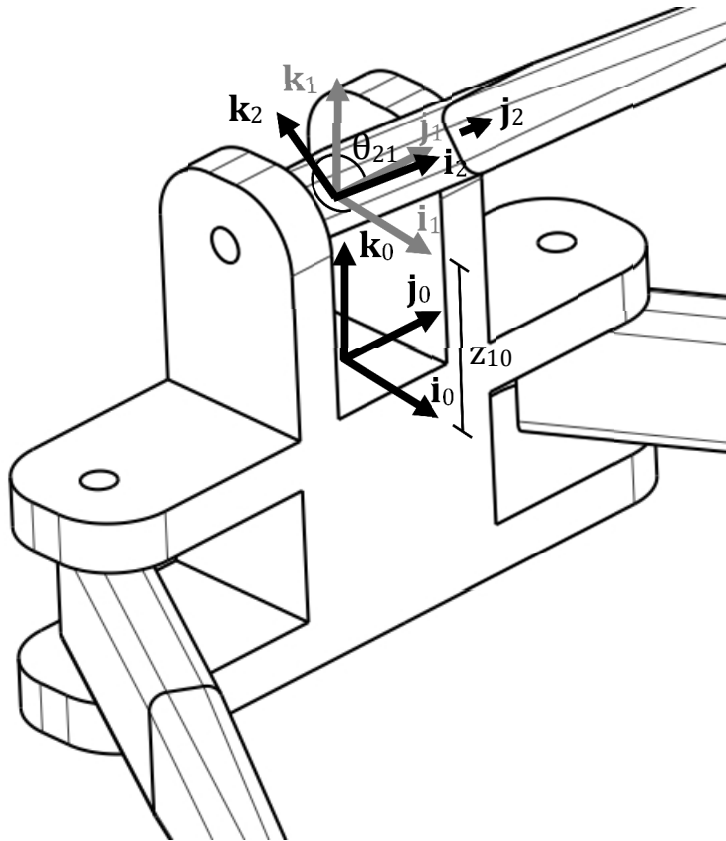


Figura 2-2 Bases de la 0 a la 2.

Para llegar al sistema de referencia 2 se emplea una traslación sobre el eje Z y una rotación sobre el eje Y, debido a que la primera junta es de tipo rotacional.

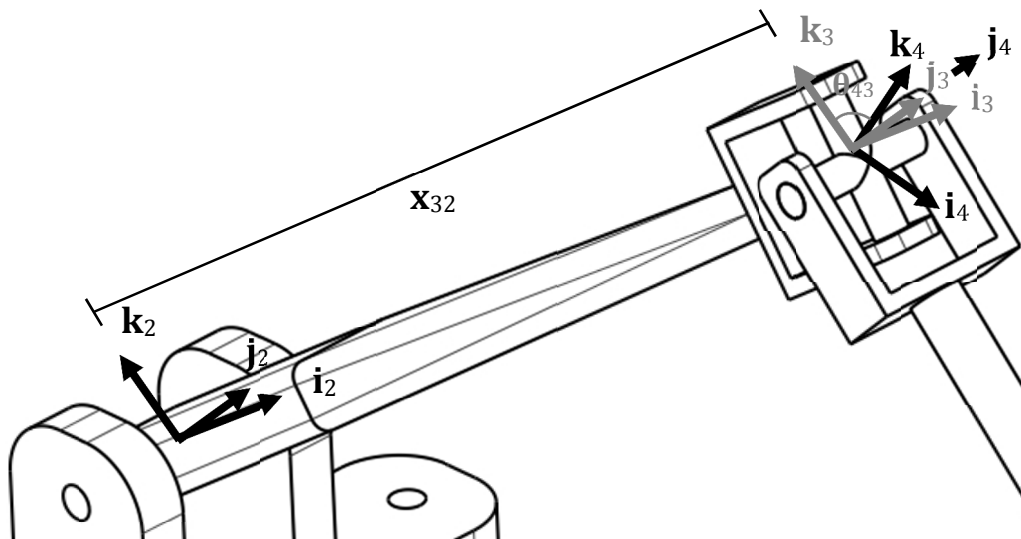


Figura 2-3 Bases de la 2 a la 4.

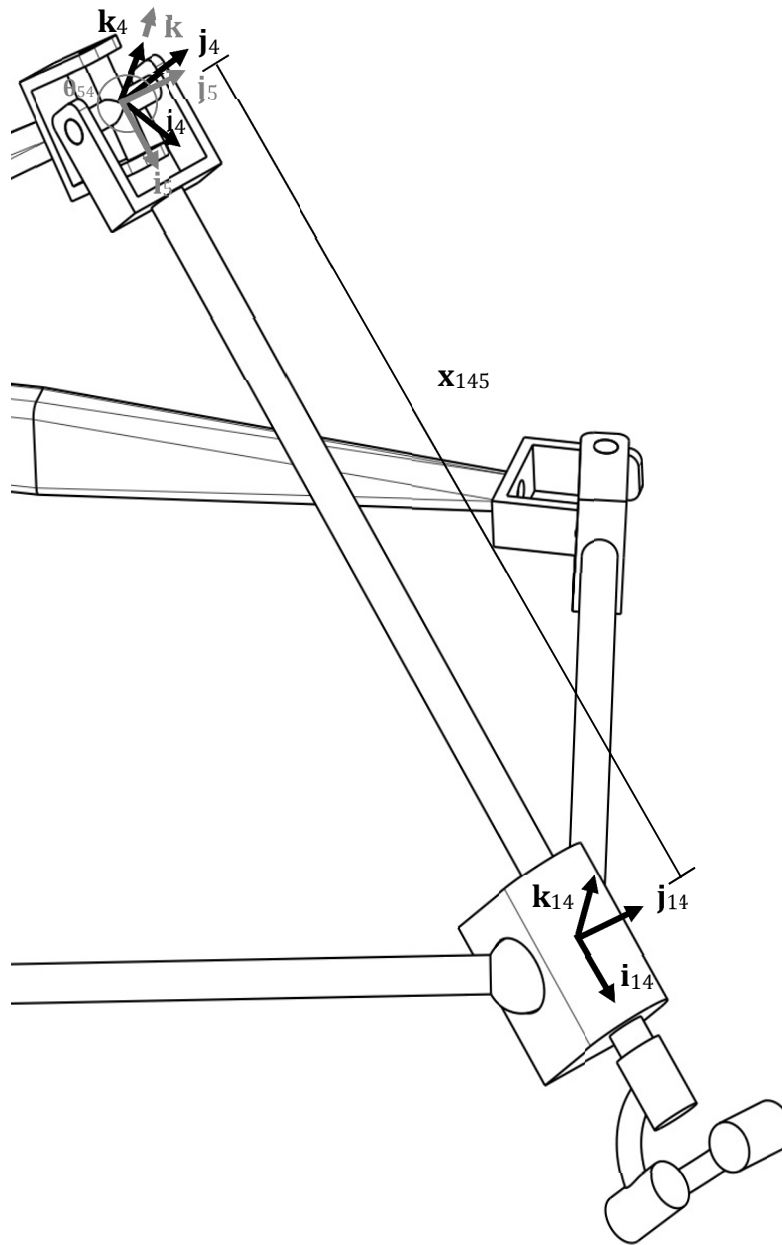


Figura 2-4 Bases de la 4 a la 14

Después se emplea una traslación sobre el eje X y dos rotaciones, por tratarse de una junta universal, sobre el eje Y y sobre el eje Z . De esta manera se llega a la base 5. Para llegar a la base 14 se emplea una traslación sobre el eje X .

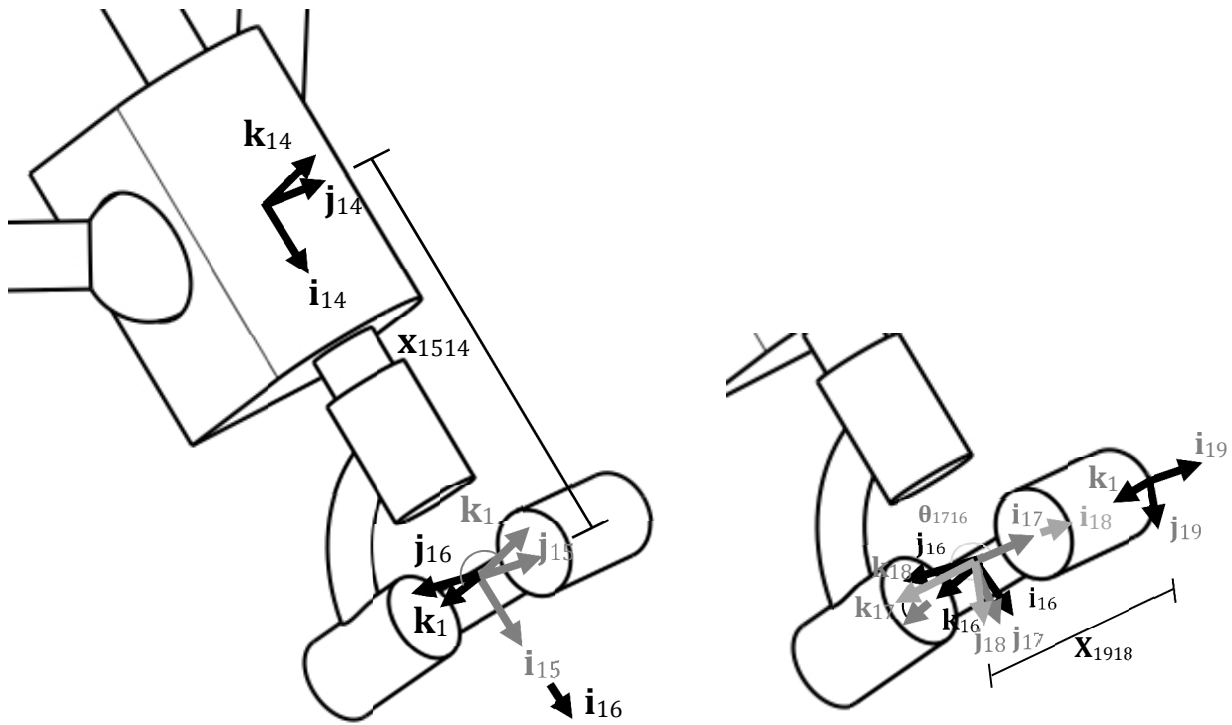


Figura 2-5 Bases de la14 a la 19.

Una traslación sobre el eje X se usa para llegar a la base 15. Una junta esférica es empleada para dar dirección al rociador por lo que es necesario definir tres traslaciones, que en este caso se eligió que fueran, sobre X , Z y X consecutivamente. Por último se usa una traslación sobre X para llegar a la base 19 donde se encuentra la punta del rociador.

De esta manera se forma la ecuación de lazo:

$$\mathbf{r}_{10} + \mathbf{r}_{32} + \mathbf{r}_{14,5} + \mathbf{r}_{15,14} = \mathbf{r}_{19,18} + \mathbf{r}_5 \quad (2.1)$$

Cuyos vectores se ilustran en la figura 2-6.

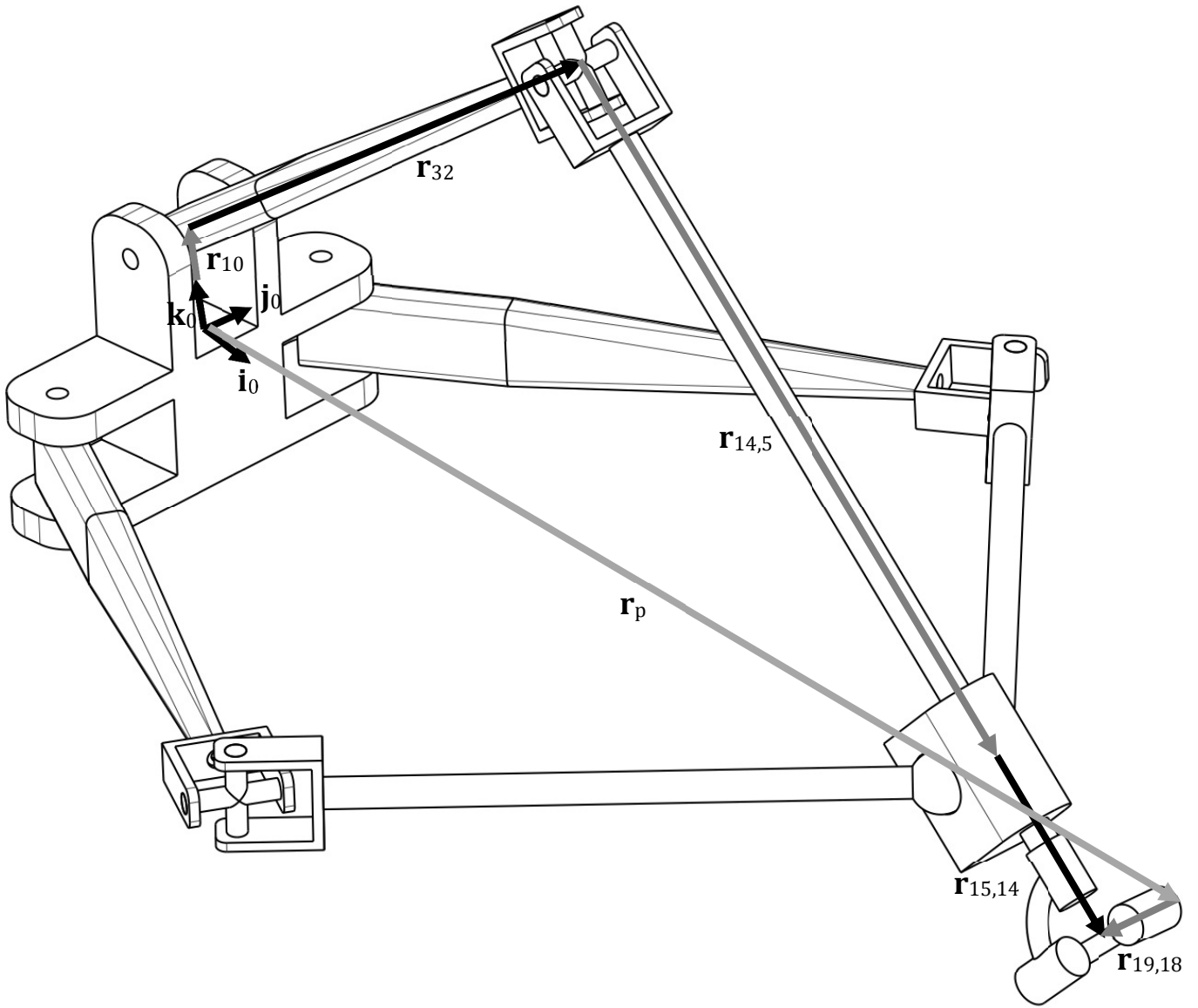


Figura 2-6 Lazo vectorial para la cadena 0.

Se renombran ambos lados de la ecuación (2.1) de la manera siguiente:

$$T_{03}T_{315} = T_{LD0} \quad (2.2)$$

Dónde:

$$\begin{aligned} T_{03} &= T_{z3}(z_{10})T_{z5}(\theta_{21})T_{z1}(x_{32}) \\ T_{315} &= T_{z5}(\theta_{43})T_{z6}(\theta_{54})T_{z1}(x_{65})T_{z1}(x_{1514}) \\ T_{LD0} &= T_{z1}(x)T_{z2}(y)T_{z3}(z)T_{z4}(\theta)T_{z5}(\psi)T_{z6}(\phi)T_{z1}(-x_{1918}) \end{aligned}$$

Ahora pasamos el término T_{03} , el cual contiene a θ_{21} , del lado derecho de la ecuación, además se posmultiplica ambos lados por un vector $\mathbf{n}$ con el fin de obtener la última columna de la matriz de transformación, con lo cual tenemos:

$$\mathbf{T}_{315}\mathbf{n} = \mathbf{T}_{03}^{-1}\mathbf{T}_{LD0}\mathbf{n} \quad (2.3)$$

Dónde:

$$\mathbf{n} = [0, 0, 0, 1]^T$$

Si renombramos el lado izquierdo de la ecuación (2.3) como $\mathbf{Ec}_1$ y el lado derecho como $\mathbf{Ec}_2$ se obtiene:

$$\mathbf{Ec}_1 = \mathbf{Ec}_2 \quad (2.4)$$

En esta ecuación se tienen como incógnitas a θ_{21} , θ_{43} y θ_{54} . Como nos interesa despejar a θ_{21} hay que eliminar las otras dos, lo cual se logra al obtener la magnitud de ambos lados de la ecuación:

$$\|\mathbf{Ec}_1\| = \|\mathbf{Ec}_2\| \quad (2.5)$$

$$\sqrt{\mathbf{Ec}_1^T \mathbf{Ec}_1} = \sqrt{\mathbf{Ec}_2^T \mathbf{Ec}_2} \quad (2.6)$$

Simplificado:

$$\mathbf{Ec}_1^T \mathbf{Ec}_1 - \mathbf{Ec}_2^T \mathbf{Ec}_2 = 0 \quad (2.7)$$

Al factorizar esta última ecuación en términos de $s\theta_{21}$ y $c\theta_{21}$ se obtiene una expresión de la forma:

$$C_{10} + A_{10}c\theta_{21} + B_{10}s\theta_{21} = 0 \quad (2.8)$$

Dónde:

$$\begin{aligned} A_{10} &= 2x_{32}(x - x_{19181}c\phi c\psi) \\ B_{10} &= -2x_{32}(z - z_{10} - x_{1918}s\theta s\phi + x_{1918}c\theta c\phi s\psi) \\ C_{10} &= -x_2 + x_{1514}^2 - x_{1918}^2 - x_{32}^2 + 2x_{1514}x_{65} + x_{65}^2 - y^2 - z^2 + 2zz_{10} - z_{10}^2 \\ &\quad + 2x_{1918}y c\theta s\phi + 2x_{1918}z s\theta s\phi - 2x_{1918}z_{10}s\theta s\phi \\ &\quad + 2x_{1918}c\phi(x c\psi + ((-z + z_{10}) * c\theta + y s\theta) * s\psi) \end{aligned} \quad (2.9)$$

De esta manera en la forma en la que se encuentra la ecuación (2.8) se puede despejar a θ_{21} como:

$$\theta_{21} = 2 \arctan \left(\frac{B_{10} \pm \sqrt{A_{10}^2 + B_{10}^2 - C_{10}^2}}{A_{10} - C_{10}} \right) \quad (2.10)$$

2.4.2 Solución del ángulo θ_{43} .

Del desarrollo anterior se puede obtener la forma matricial de ambos lados de la ecuación (2.4) como sigue:

$$\mathbf{Ec}_1 = \begin{bmatrix} (x_{1514} + x_{65}) c\theta_{43} c\theta_{54} \\ (x_{1514} + x_{65}) s\theta_{54} \\ -(x_{1514} + x_{65}) c\theta_{54} c\theta_{43} \\ 1 \end{bmatrix} \quad (2.11)$$

$$\mathbf{Ec}_2 = \begin{bmatrix} x - x_{32}c\theta_{21} - x_{1918}c\phi c\psi \\ y - x_{1918}(c\theta s\theta + c\phi s\phi s\psi) \\ z - z_{10} + x_{32}s\theta_{21} - x_{1918}s\theta s\phi + x_{1918}c\theta c\phi s\psi \\ 1 \end{bmatrix} \quad (2.12)$$

Se observa que $\mathbf{Ec}_2$ está formada por elementos constantes y por θ_{21} que es el ángulo que se ya se despejó, por lo que para un manejo más cómodo se renombraran sus elementos como X, Y, Z es decir:

$$\mathbf{Ec}_2 = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad (2.13)$$

De esta manera se manipulará más fácilmente el lado izquierdo de la ecuación, es decir $\mathbf{Ec}_1$, que contiene el ángulo que nos interesa despejar.

Entonces se tiene el sistema de ecuaciones siguiente:

$$(x_{1514} + x_{65}) c\theta_{43} c\theta_{54} = X \quad (2.14)$$

$$(x_{1514} + x_{65}) s\theta_{54} = Y \quad (2.15)$$

$$-(x_{1514} + x_{65}) c\theta_{54} s\theta_{43} = Z \quad (2.16)$$

Ahora de (2.14) se despeja $c\theta_{54}$ y se sustituye en (2.16) con lo que se obtiene:

$$-\frac{X x_{1514} \tan\theta_{43}}{x_{1514} + x_{65}} - \frac{X x_{65} \tan\theta_{43}}{x_{1514} + x_{65}} = Z \quad (2.17)$$

De (2.17) se despeja $\tan(\theta_{43})$:

$$\tan\theta_{43} = \frac{-Z}{X} \quad (2.18)$$

De esta manera:

$$\theta_{43} = \arctan\left(\frac{-Z}{X}\right) \quad (2.19)$$

Dónde:

$$\begin{aligned} X &= x - x_{32}c\theta_{21} - x_{1918}c\phi c\psi \\ Y &= y - x_{1918}(c\theta s\phi + c\phi s\theta s\psi) \\ Z &= z - z_{10} + x_{32}s\theta_{21} - x_{1918}s\theta s\phi + x_{1918}c\theta c\phi s\psi \end{aligned} \quad (2.20)$$

2.4.3 Solución del ángulo θ_{54} .

Para el despeje de este ángulo se utilizan las ecuaciones del desarrollo anterior, entonces de la ecuación (2.14) se despeja $c\theta_{54}$ y de la ecuación (2.15) se despeja $s\theta_{54}$ con lo cual se obtiene que:

$$\begin{aligned} c\theta_{54} &= \frac{X \sec\theta_{43}}{x_{1514} + x_{65}} \\ s\theta_{54} &= \frac{Y}{x_{1514} + x_{65}} \end{aligned} \quad (2.21)$$

De las ecuaciones (2.21) se llega a que:

$$\tan \theta_{54} = \frac{Yc\theta_{43}}{X} \quad (2.22)$$

Y por lo tanto:

$$\theta_{54} = \arctan\left(\frac{Yc\theta_{43}}{X}\right) \quad (2.23)$$

2.4.4 Solución del ángulo θ_{1817} .

Para la solución de los ángulos anteriores se formaron las ecuaciones de lazo sin considerar los giros de la junta esférica, lo cual se compensó al extraer la última columna de las transformaciones homogéneas. Es decir solo se expresó a los ángulos en términos de distancias.

Para despejar los ángulos de la junta esférica del órgano terminal se formarán las ecuaciones de lazo pero esta vez considerando tanto distancias como ángulos.

Al expresarse en transformaciones homogéneas:

$$\mathbf{T}_{03}\mathbf{T}_{315}\mathbf{T}_{1519} = \mathbf{T}_{z1}(x)\mathbf{T}_{z2}(y)\mathbf{T}_{z3}(z)\mathbf{T}_{z4}(\theta)\mathbf{T}_{z5}(\psi)\mathbf{T}_{z6}(\phi) \quad (2.24)$$

Dónde:

$$\begin{aligned} \mathbf{T}_{03} &= \mathbf{T}_{z3}(z_{10})\mathbf{T}_{z5}(\theta_{21})\mathbf{T}_{z1}(x_{32}) \\ \mathbf{T}_{315} &= \mathbf{T}_{z5}(\theta_{43})\mathbf{T}_{z6}(\theta_{54})\mathbf{T}_{z1}(x_{65})\mathbf{T}_{z1}(x_{1514}) \\ \mathbf{T}_{1519} &= \mathbf{T}_{z4}(\theta_{1615})\mathbf{T}_{z6}(\theta_{1716})\mathbf{T}_{z4}(\theta_{1817})\mathbf{T}_{z1}(x_{1918}) \end{aligned}$$

Renombrando el lado derecho de la ecuación (2.24):

$$\mathbf{T}_{03}\mathbf{T}_{315}\mathbf{T}_{1519} = \mathbf{T}_{LDn} \quad (2.25)$$

Dónde:

$$\mathbf{T}_{LDn} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad (2.26)$$

Se observa que los ángulos de interés se encuentran contenidos en $\mathbf{T}_{1519}$, es decir están en el lado izquierdo de la ecuación (2.25). Se observa que además de los tres ángulos de la junta esférica (θ_{1615} , θ_{1716} y θ_{1817}) lo demás es conocido debido a que es dato o ya se calculó, por lo que se procederá a dejar las transformaciones asociadas a los ángulos desconocidos del lado izquierdo de la ecuación y todo lo demás del lado derecho.

$$\mathbf{T}_{z4}(\theta_{1615})\mathbf{T}_{z6}(\theta_{1716})\mathbf{T}_{z4}(\theta_{1817}) = \mathbf{T}_{315}^{-1}\mathbf{T}_{03}^{-1}\mathbf{T}_{LDn}\mathbf{T}_{z1}(-x_{1918}) \quad (2.27)$$

Renombrando ambos lados de la ecuación (2.27):

$$\mathbf{T}_{LI} = \mathbf{T}_{LDn} \quad (2.28)$$

Dónde:

$$\mathbf{T}_{LI} = \begin{bmatrix} c\theta_{1716} & -c\theta_{1817}s\theta_{1716} & s\theta_{1716}s\theta_{1817} & 0 \\ c\theta_{1615}s\theta_{1716} & c\theta_{1615}c\theta_{1716}c\theta_{1817} - s\theta_{1615}s\theta_{1817} & -c\theta_{1817}s\theta_{1615} - c\theta_{1615}c\theta_{1716}s\theta_{1817} & 0 \\ s\theta_{1615}s\theta_{1716} & c\theta_{1716}c\theta_{1817}s\theta_{1615} + c\theta_{1615}s\theta_{1817} & c\theta_{1615}c\theta_{1817} - c\theta_{1716}s\theta_{1615}s\theta_{1817} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.29)$$

$$\mathbf{T}_{LDn2} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \quad (2.30)$$

Igualando los elementos (1,2) y (1,3) de las ecuaciones (2.29) y (2.30):

$$\begin{aligned} -c\theta_{18171}s\theta_{17161} &= b_{12} \\ s\theta_{17161}s\theta_{18171} &= b_{13} \end{aligned} \quad (2.31)$$

Despejando $c\theta_{1817}$ y $s\theta_{1817}$:

$$\begin{aligned} c\theta_{1817} &= -b_{12} \csc\theta_{1716} \\ s\theta_{1817} &= b_{13} \csc\theta_{1716} \end{aligned} \quad (2.32)$$

Formando la función tangente:

$$\tan\theta_{1817} = \frac{b_{13}}{-b_{12}} \quad (2.33)$$

Por lo tanto:

$$\theta_{1817} = \arctan\left(\frac{b_{13}}{-b_{12}}\right) \quad (2.34)$$

Dónde:

$$\begin{aligned} b_{12} &= s\theta_{54}(c\theta c\phi - s\theta s\phi s\psi) - c\theta_{54}(c\phi s\theta s(\theta_{21} + \theta_{43}) + s\phi(c(\theta_{21} + \theta_{43})c\psi \\ &\quad + c\theta s(\theta_{21} + \theta_{43})s\psi)) \\ b_{13} &= -c\theta c\theta_{54}c\psi s(\theta_{21} + \theta_{43}) - c\psi s\theta s\theta_{54} + c(\theta_{21} + \theta_{43})c\theta_{54}s\psi \end{aligned} \quad (2.35)$$

2.4.5 Solución del ángulo θ_{1716} .

De las ecuaciones (2.29) y (2.30) se igualan los elementos (1,1) y (1,3):

$$\begin{aligned} c\theta_{1716} &= b_{11} \\ s\theta_{1716}s\theta_{1817} &= b_{13} \end{aligned} \quad (2.36)$$

Despejando $c\theta_{1716}$ y $s\theta_{1716}$:

$$\begin{aligned} c\theta_{1716} &= b_{11} \\ s\theta_{1716} &= b_{13} \csc\theta_{1817} \end{aligned} \quad (2.37)$$

Formando la función tangente:

$$\tan\theta_{1716} = \frac{b_{13}\csc\theta_{18171}}{-b_{11}} \quad (2.38)$$

Por lo tanto:

$$\theta_{1716} = \arctan\left(\frac{b_{13}\csc\theta_{18171}}{-b_{11}}\right) \quad (2.39)$$

Dónde:

$$b_{11} = c(\theta_{21} + \theta_{43})c\theta_{54}c\phi c\psi + c\theta_{54}s(\theta_{21} + \theta_{43})(-s\theta s\phi + c\theta c\phi s\psi) + s\theta_{54}(c\theta s\phi + c\phi s\theta s\psi) \quad (2.40)$$

2.4.6 Solución del ángulo θ_{1615} .

De la ecuación (2.27) se despeja $T_{z4}(\theta_{1615})$:

$$\begin{aligned} T_{z4}(\theta_{1615}) &= T_{315}^{-1}T_{03}^{-1}T_{LDn}T_{z1}(-x_{1918})T_{z4}(-\theta_{1817})T_{z6}(-\theta_{1716}) \\ T_{z4}(\theta_{1615}) &= T_{LDn2}T_{z4}(-\theta_{1817})T_{z6}(-\theta_{1716}) \end{aligned} \quad (2.41)$$

Renombrando el lado derecho de la ecuación (2.41):

$$T_{z4}(\theta_{1615}) = T_{LDn3} \quad (2.42)$$

Dónde:

$$T_{z4}(\theta_{1615}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_{1615} & -s\theta_{16151} & 0 \\ 0 & s\theta_{16151} & c\theta_{16151} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.43)$$

$$T_{LDn3} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \quad (2.44)$$

De las ecuaciones (2.43) y (2.44) se igualan los elementos (2,2) y (3,2)

$$\begin{aligned} c\theta_{1615} &= c_{22} \\ s\theta_{1615} &= c_{32} \end{aligned} \quad (2.45)$$

Formando la función tangente:

$$\tan\theta_{1615} = \frac{c_{32}}{c_{22}} \quad (2.46)$$

Por lo tanto:

$$\theta_{1615} = \arctan\left(\frac{c_{32}}{c_{22}}\right) \quad (2.47)$$

Dónde:

$$\begin{aligned} c_{22} = & s\theta_{1716}(-c(\theta_{21} + \theta_{43})c\phi c\psi s\theta_{54} + s(\theta_{21} + \theta_{43})s\theta_{54}(s\theta s\phi - c\theta c\phi s\psi) \\ & + c\theta_{54}(c\theta s\phi + c\phi s\theta s\psi)) + c\theta_{1716}(s\theta_{1817}(c\theta_{54}c\psi s\theta \\ & + s\theta_{54}(-c\theta c\psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43})s\psi)) \\ & + c\theta_{1817}(c(\theta_{21} + \theta_{43})c\psi s\theta_{54}s\phi + s(\theta_{21} + \theta_{43})s\theta_{54}(c\phi s\theta \\ & + c\theta s\phi s\psi) + c\theta_{54}(c\theta c\phi - s\theta s\phi s\psi))) \\ c_{32} = & s\theta_{1716}(c\phi c\psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43})(s\theta s\phi - c\theta c\phi s\psi)) \\ & + c\theta_{1716}(-s\theta_{1817}(c\theta c(\theta_{21} + \theta_{43})c\psi + s(\theta_{21} + \theta_{43})s\psi) \\ & + c\theta_{1817}(-c\psi s(\theta_{21} + \theta_{43})s\phi + c(\theta_{21} + \theta_{43})(c\phi s\theta \\ & + c\theta s\phi s\psi))) \end{aligned} \quad (2.48)$$

2.5 Cálculo de velocidad para la cadena 0.

Para el cálculo de la velocidad se tiene como variables de entrada la velocidad del efector $(\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi})$ y las variables a calcular son las velocidades de los ángulos de la cadena 0 es decir $\dot{\theta}_{21}, \dot{\theta}_{43}, \dot{\theta}_{54}, \dot{\theta}_{1817}, \dot{\theta}_{1716}, \dot{\theta}_{1615}$. Para esta sección se asume que la posición de los ángulos es conocida debido a que se calculó en la sección anterior.

2.5.1 Solución de la velocidad para ángulo θ_{21} .

Derivando la ecuación (2.8) respecto al tiempo:

$$\dot{\theta}_{21} = \frac{-\dot{C}_{10} - \dot{A}_{10}c\theta_{21} - \dot{B}_{10}s\theta_{21}}{B_{10}c\theta_{21} - A_{10}s\theta_{21}} \quad (2.49)$$

Donde A_{10}, B_{10} y C_{10} se encuentran en las ecuaciones (2.9), además:

$$\begin{aligned} \dot{A}_{10} = & 2x_{32}(\dot{x} + \dot{\phi}x_{1918}c\psi s\phi + \dot{\psi}x_{1918}c\phi s\psi) \\ \dot{B}_{10} = & -2x_{32}(\dot{z} - x_{1918}c\phi s\theta(\dot{\phi} + \dot{\theta}s\psi) + x_{1918}c\theta(\dot{\psi}c\phi c\psi - s\phi(\dot{\theta} + \dot{\phi}s\psi))) \end{aligned} \quad (2.50)$$

$$\begin{aligned}\dot{C}_{10} = & 2(-\dot{x}x - \dot{y}y - \dot{z}z + \dot{z}z_{10} - \dot{\phi}xx_{1918}c\psi s\phi + \dot{z}x_{1918}s\theta s\phi - \dot{\theta}x_{1918}ys\theta s\phi \\ & - \dot{\phi}x_{1918}ys\theta s\phi s\psi + x_{1918}c\theta(c\phi(\dot{\phi}y - \dot{\psi}(z - z_{10})c\psi - \dot{z}s\psi \\ & + \dot{\theta}ys\psi) + s\phi(\dot{y} + \dot{\theta}z - \dot{\theta}z_{10} + \dot{\phi}(z - z_{10})s\psi)) \\ & + x_{1918}c\phi(c\psi(\dot{x} + \dot{\psi}ys\theta) - \dot{\psi}xs\psi + s\theta(\dot{\phi}(z - z_{10}) + (\dot{y} + \dot{\theta}z \\ & - \dot{\theta}z_{10})s\psi)))\end{aligned}$$

Sustituyendo las ecuaciones $A_{10}, B_{10}, C_{10}, \dot{A}_{10}, \dot{B}_{10}$ y $\dot{C}_{10}$ en (2.49) y al agrupar en $\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}$ se tiene:

$$\dot{\theta}_{21} = \frac{1}{V_1}(\dot{x}V_2 + \dot{y}V_3 + \dot{z}V_4 + \dot{\psi}V_5 + \dot{\theta}V_6 + \dot{\phi}V_7) \quad (2.51)$$

Donde los coeficientes se encuentran en el apéndice D.

2.5.2 Solución de la velocidad para ángulo θ_{43} .

Derivando respecto del tiempo la ecuación (2.19):

$$\dot{\theta}_{43} = \frac{-\dot{Z}c\theta_{43} - \dot{X}s\theta_{43}}{Xc\theta_{43} - Zs\theta_{43}} \quad (2.52)$$

Dónde:

$$\begin{aligned}\dot{X} &= \dot{x} + \theta_{21}x_{32}s\theta_{21} + \dot{\phi}x_{1918}c\psi s\phi + \dot{\psi}x_{1918}c\phi s\psi \\ \dot{Y} &= \dot{y} - x_{1918}(c\theta c\phi(\dot{\phi} + \dot{\theta}s\psi) + s\theta(\dot{\psi}c\phi c\psi - s\phi(\dot{\theta} + \dot{\phi}s\psi))) \\ \dot{Z} &= \dot{z} + \theta_{21}x_{32}c\theta_{21} - \dot{\phi}x_{1918}c\phi s\theta - \dot{\theta}x_{1918}c\phi s\theta s\psi + x_{1918}c\theta(\dot{\psi}c\phi c\psi \\ &\quad - s\phi(\dot{\theta} + \dot{\phi}s\psi))\end{aligned} \quad (2.53)$$

Sustituyendo X, Z que se encuentran en (2.20) así como $\dot{X}, \dot{Z}$ en (2.52), simplificando y agrupando:

$$\dot{\theta}_{43} = \frac{1}{V_8}(\dot{x}V_9 + \dot{z}V_{10} + \dot{\psi}V_{11} + \dot{\theta}V_{12} + \dot{\phi}V_{13} + \dot{\theta}_{21}V_{14}) \quad (2.54)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo la ecuación (2.51) en (2.54) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{43} = \frac{1}{V_8}(\dot{x}E_1 + \dot{y}E_2 + \dot{z}E_3 + \dot{\psi}E_4 + \dot{\theta}E_5 + \dot{\phi}E_6) \quad (2.55)$$

Dónde:

$$\begin{aligned}
E_1 &= \frac{V_{14}V_2}{V_2} + V_9 & E_4 &= V_{11} + \frac{V_{14}V_5}{V_1} \\
E_2 &= \frac{V_{14}V_3}{V_1} & E_5 &= V_{12} + \frac{V_{14}V_6}{V_1} \\
E_3 &= V_{10} + \frac{V_{14}V_4}{V_1} & E_6 &= V_{13} + \frac{V_{14}V_7}{V_1}
\end{aligned} \tag{2.56}$$

2.5.3 Solución de la velocidad, ángulo θ_{54} .

Derivando respecto del tiempo la ecuación (2.20):

$$\dot{\theta}_{54} = \frac{c\theta_{54}^2((\dot{Y}X - \dot{X}Y)c\theta_{43} - \dot{\theta}_{43}XYs\theta_{43})}{X^2} \tag{2.57}$$

Sustituyendo X, Y que se encuentran en (2.20) así como $\dot{X}, \dot{Y}$ en (2.57), simplificando y agrupando:

$$\dot{\theta}_{54} = \frac{1}{V_{15}} (\dot{x}V_{16} + \dot{y}V_{17} + \dot{\psi}V_{18} + \dot{\theta}V_{19} + \dot{\phi}V_{20} + \dot{\theta}_{21}V_{21} + \dot{\theta}_{43}V_{22}) \tag{2.58}$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55) en (2.58) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{54} = \frac{1}{V_{15}} (\dot{x}E_7 + \dot{y}E_8 + \dot{z}E_9 + \dot{\psi}E_{10} + \dot{\theta}E_{11} + \dot{\phi}E_{12}) \tag{2.59}$$

Dónde:

$$\begin{aligned}
E_7 &= V_{15} - \frac{V_2V_{20}}{V_1} + \frac{E_1V_{21}}{V_8} & E_{10} &= V_{17} - \frac{V_{20}V_5}{V_1} + \frac{E_4V_{21}}{V_8} \\
E_8 &= V_{16} - \frac{V_{20}V_3}{V_1} + \frac{E_2V_{21}}{V_8} & E_{11} &= V_{18} - \frac{V_{20}V_6}{V_1} + \frac{E_5V_{21}}{V_8} \\
E_9 &= -\frac{V_{20}V_4}{V_1} + \frac{E_3V_{21}}{V_8} & E_{12} &= V_{19} - \frac{V_{20}V_7}{V_1} + \frac{E_6V_{21}}{V_8}
\end{aligned} \tag{2.60}$$

2.5.4 Solución de la velocidad, ángulo θ_{1817} .

Derivando respecto del tiempo la ecuación (2.34):

$$\dot{\theta}_{1817} = \frac{(b_{13}\dot{b}_{12} - b_{12}\dot{b}_{13})c\theta_{1817}^2}{b_{12}^2} \tag{2.61}$$

Donde b_{12} y b_{13} se encuentran en (2.35), además:

$$\begin{aligned}\dot{b}_{12} = & \dot{\theta}_{54}c\theta_{54}(c\theta c\phi - s\theta s\phi s\psi) + \dot{\theta}_{54}s\theta_{54}(c\phi s\theta s(\theta_{21} + \theta_{43}) + s\phi(c(\theta_{21} \\ & + \theta_{43})c\psi + c\theta s(\theta_{21} + \theta_{43})s\psi)) - s\theta_{54}(c\phi s\theta(\theta + \phi s\psi) \\ & + s\phi(\psi c\psi s\theta + c\theta(\phi + \theta s\psi))) - c\theta_{54}(c(\theta_{21} + \theta_{43})(c\phi(\phi c\psi \\ & + (\dot{\theta}_{21} + \dot{\theta}_{43})s\theta) + (-\psi + (\dot{\theta}_{21} + \dot{\theta}_{43})c\theta)s\phi s\psi) + s(\theta_{21} \\ & + \theta_{43})(-s\phi((\theta_{21} + \theta_{43})c\psi + s\theta(\phi + \theta s\psi)) + c\theta(\psi c\psi s\phi \\ & + c\phi(\theta + \phi s\psi))))\end{aligned}\quad (2.62)$$

$$\begin{aligned}\dot{b}_{13} = & s\theta_{54}(-c\theta c\psi(\theta - \theta_{54}s(\theta_{21} + \theta_{43})) + \psi s\theta s\psi) - c\theta_{54}(c\psi s\theta(\dot{\theta}_{54} \\ & - \dot{\theta}s(\theta_{21} + \theta_{43})) + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi}c\theta)s(\theta_{21} + \theta_{43})s\psi) \\ & + c(\theta_{21} + \theta_{43})(\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43})c\theta)c\theta_{54}c\psi - \dot{\theta}_{54}s\theta_{54}s\psi)\end{aligned}$$

Sustituyendo las ecuaciones b_{12} , b_{13} , $\dot{b}_{12}$, $\dot{b}_{13}$ en (2.61), simplificando y agrupando:

$$\dot{\theta}_{1817} = \frac{1}{V_{23}}(\dot{\psi}V_{24} + \dot{\theta}V_{25} + \dot{\phi}V_{26} + \dot{\theta}_{21}V_{27} + \dot{\theta}_{43}V_{27} + \dot{\theta}_{54}V_{28}) \quad (2.63)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59) en (2.63) y al agrupar en $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, $\ddot{\psi}$, $\ddot{\theta}$, $\ddot{\phi}$ se tiene:

$$\dot{\theta}_{1817} = \frac{1}{V_{23}}(\dot{x}E_{13} + \dot{y}E_{14} + \dot{z}E_{15} + \dot{\psi}E_{16} + \dot{\theta}E_{17} + \dot{\phi}E_{18}) \quad (2.64)$$

Dónde:

$$\begin{aligned}E_{13} &= \frac{V_2V_{27}}{V_1} + \frac{E_7V_{28}}{V_{15}} + \frac{E_1V_{27}}{V_8} & E_{16} &= V_{24} + \frac{E_{10}V_{28}}{V_{15}} + \frac{V_{27}V_5}{V_1} + \frac{E_4V_{27}}{V_8} \\ E_{14} &= \frac{E_8V_{28}}{V_{15}} + \frac{V_{27}V_3}{V_1} + \frac{E_2V_{27}}{V_8} & E_{17} &= V_{25} + \frac{E_{11}V_{28}}{V_{15}} + \frac{V_{27}V_6}{V_1} + \frac{E_5V_{27}}{V_8} \\ E_{15} &= \frac{E_9V_{28}}{V_{15}} + \frac{V_{27}V_4}{V_1} + \frac{E_3V_{27}}{V_8} & E_{18} &= V_{26} + \frac{E_{12}V_{28}}{V_{15}} + \frac{V_{27}V_7}{V_1} + \frac{E_6V_{27}}{V_8}\end{aligned}\quad (2.65)$$

2.5.5 Solución de la velocidad, ángulo θ_{1716} .

Derivando respecto del tiempo la ecuación (2.39):

$$\dot{\theta}_{1716} = -\frac{c\theta_{1716}^2(b_{13}\dot{b}_{11}\csc\theta_{1817} - b_{11}\dot{b}_{13}\csc\theta_{1817} + b_{11}b_{13}\dot{\theta}_{1817}\cot\theta_{1817}\csc\theta_{1817})}{b_{11}^2} \quad (2.66)$$

Dónde:

$$\begin{aligned}
\dot{b}_{11} = & s\theta_{54}(-\dot{\theta}_{54}c(\theta_{21} + \theta_{43})c\phi c\psi + c\theta c\phi(\dot{\phi} + (\dot{\theta} - \dot{\theta}_{54}s(\theta_{21} + \theta_{43}))s\psi) \\
& + s\theta(\psi c\phi c\psi - s\phi(\dot{\theta} - \dot{\theta}_{54}s(\theta_{21} + \theta_{43}) + \dot{\phi}s\psi))) \\
& - c\theta_{54}(c\phi((\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi}c\theta)c\psi s(\theta_{21} + \theta_{43}) + (\dot{\psi} - (\dot{\theta}_{21} \\
& + \dot{\theta}_{43})c\theta)c(\theta_{21} + \theta_{43})s\psi + s\theta(-\dot{\theta}_{54}s\psi + s(\theta_{21} + \theta_{43})(\dot{\phi} \\
& + \dot{\theta}s\psi))) + s\phi(c(\theta_{21} + \theta_{43})(\dot{\phi}c\psi + (\dot{\theta}_{21} + \dot{\theta}_{43})s\theta) \\
& + c\theta(-\dot{\theta}_{54} + s(\theta_{21} + \theta_{43})(\dot{\theta} + \dot{\phi}s\psi))))
\end{aligned} \tag{2.67}$$

Sustituyendo las ecuaciones $b_{11}, b_{13}, \dot{b}_{11}, \dot{b}_{13}$ en (2.66), simplificando y agrupando:

$$\dot{\theta}_{1716} = \frac{1}{V_{29}}(\dot{\psi}V_{30} + \dot{\theta}V_{31} + \dot{\phi}V_{32} + \dot{\theta}_{21}V_{33} + \dot{\theta}_{43}V_{33} + \dot{\theta}_{54}V_{34} + \dot{\theta}_{1817}V_{35}) \tag{2.68}$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.64) en (2.68) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{1716} = \frac{1}{V_{29}}(\dot{x}E_{19} + \dot{y}E_{20} + \dot{z}E_{21} + \dot{\psi}E_{22} + \dot{\theta}E_{23} + \dot{\phi}E_{24}) \tag{2.69}$$

Dónde:

$$\begin{aligned}
E_{19} &= \frac{V_2V_{33}}{V_1} + \frac{E_7V_{34}}{V_{15}} + \frac{E_{13}V_{35}}{V_{23}} + \frac{E_1V_{33}}{V_8} \\
E_{20} &= \frac{V_3V_{33}}{V_1} + \frac{E_8V_{34}}{V_{15}} + \frac{E_{14}V_{35}}{V_{23}} + \frac{E_2V_{33}}{V_8} \\
E_{21} &= \frac{E_9V_{34}}{V_{15}} + \frac{E_{15}V_{35}}{V_{23}} + \frac{V_{33}V_4}{V_1} + \frac{E_3V_{33}}{V_8} \\
E_{22} &= V_{30} + \frac{E_{10}V_{34}}{V_{15}} + \frac{E_{16}V_{35}}{V_{23}} + \frac{V_{33}V_5}{V_1} + \frac{E_4V_{33}}{V_8} \\
E_{23} &= V_{31} + \frac{E_{11}V_{34}}{V_{15}} + \frac{E_{17}V_{35}}{V_{23}} + \frac{V_{33}V_6}{V_1} + \frac{E_5V_{33}}{V_8} \\
E_{24} &= V_{32} + \frac{E_{12}V_{34}}{V_{15}} + \frac{E_{18}V_{35}}{V_{23}} + \frac{V_{33}V_7}{V_1} + \frac{E_6V_{33}}{V_8}
\end{aligned} \tag{2.70}$$

2.5.6 Solución de la velocidad, ángulo θ_{1615} .

Derivando respecto del tiempo la ecuación (2.47):

$$\dot{\theta}_{1615} = -\frac{(c_{32}\dot{c}_{22} - c_{22}\dot{c}_{32})c\theta_{1615}^2}{c_{22}^2} \tag{2.71}$$

Al derivar la ecuación (2.48) se obtiene $\dot{c}_{22}$ y $\dot{c}_{32}$, que se encuentran en el apéndice C. Sustituyendo las ecuaciones (2.48) así como $\dot{c}_{22}$ y $\dot{c}_{32}$ en la ecuación (2.71), simplificando y agrupando:

$$\dot{\theta}_{1615} = \frac{1}{V_{36}} (\dot{\psi}V_{37} + \dot{\theta}V_{38} + \dot{\phi}V_{39} + \dot{\theta}_{21}V_{40} + \dot{\theta}_{43}V_{40} + \dot{\theta}_{54}V_{41} + \dot{\theta}_{1817}V_{42} + \dot{\theta}_{1716}V_{43}) \quad (2.72)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.64), (2.69) en (2.72) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{1615} = \frac{1}{V_{36}} (\dot{x}E_{25} + \dot{y}E_{26} + \dot{z}E_{27} + \dot{\psi}E_{28} + \dot{\theta}E_{29} + \dot{\phi}E_{30}) \quad (2.73)$$

Dónde:

$$\begin{aligned} E_{25} &= \frac{V_2V_{40}}{V_1} + \frac{E_7V_{41}}{V_{15}} + \frac{E_{13}V_{42}}{V_{23}} + \frac{E_{19}V_{43}}{V_{29}} + \frac{E_1V_{40}}{V_8} \\ E_{26} &= \frac{V_3V_{40}}{V_1} + \frac{E_8V_{41}}{V_{15}} + \frac{E_{14}V_{42}}{V_{23}} + \frac{E_{20}V_{43}}{V_{29}} + \frac{E_2V_{40}}{V_8} \\ E_{27} &= \frac{V_4V_{40}}{V_1} + \frac{E_9V_{41}}{V_{15}} + \frac{E_{15}V_{42}}{V_{23}} + \frac{E_{21}V_{43}}{V_{29}} + \frac{E_3V_{40}}{V_8} \\ E_{28} &= V_{37} + \frac{E_{10}V_{41}}{V_{15}} + \frac{E_{16}V_{42}}{V_{23}} + \frac{E_{22}V_{43}}{V_{29}} + \frac{V_{40}V_5}{V_1} + \frac{E_4V_{40}}{V_8} \\ E_{29} &= V_{38} + \frac{E_{11}V_{41}}{V_{15}} + \frac{E_{17}V_{42}}{V_{23}} + \frac{E_{23}V_{43}}{V_{29}} + \frac{V_{40}V_6}{V_1} + \frac{E_5V_{40}}{V_8} \\ E_{30} &= V_{39} + \frac{E_{12}V_{41}}{V_{15}} + \frac{E_{18}V_{42}}{V_{23}} + \frac{E_{24}V_{43}}{V_{29}} + \frac{V_{40}V_7}{V_1} + \frac{E_6V_{40}}{V_8} \end{aligned} \quad (2.74)$$

2.6 Calculo de la aceleración para la cadena 0.

Para el cálculo de la aceleración se tiene como variables de entrada la aceleración del efector ($\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$) y las variables a calcular son las aceleraciones de los ángulos de la cadena 0 es decir $\ddot{\theta}_{21}, \ddot{\theta}_{43}, \ddot{\theta}_{54}, \ddot{\theta}_{1817}, \ddot{\theta}_{1716}, \ddot{\theta}_{1615}$. Para esta sección se asume que la posición y la velocidad de los ángulos son conocidas debido a que se calcularon en secciones anteriores.

2.6.1 Solución de la aceleración, ángulo θ_{21} .

Derivando la ecuación (2.49) respecto al tiempo:

$$\ddot{\theta}_{21} = \frac{-\ddot{C}_{10} - \ddot{A}_{10}c\theta_{21} - 2\dot{B}_{10}\dot{\theta}_{21}c\theta_{21} + A_{10}\dot{\theta}_{21}^2 c\theta_{21} - \ddot{B}_{10}s\theta_{21} + 2\dot{A}_{10}\dot{\theta}_{21}s\theta_{21} + B_{10}\dot{\theta}_{21}^2 s\theta_{21}}{B_{10}c\theta_{21} - A_{10}s\theta_{21}} \quad (2.75)$$

Donde A_{10}, B_{10}, C_{10} se encuentran en (2.9) y $\dot{A}_{10}, \dot{B}_{10}, \dot{C}_{10}$ en (2.50), además:

$$\begin{aligned}
\ddot{A}_{10} &= 2x_{32}(\ddot{x} + \ddot{\phi}x_{1918}c\psi s\phi - 2\dot{\phi}\dot{\psi}x_{1918}s\phi s\psi + x_{1918}c\phi((\dot{\phi}^2 + \dot{\psi}^2)c\psi \\
&\quad + \dot{\psi}s\psi)) \\
\ddot{B}_{10} &= -2x_{32}(\ddot{z} + \dot{\theta}^2x_{1918}s\theta s\phi + \dot{\phi}^2x_{1918}s\theta s\phi + 2\dot{\theta}\dot{\phi}x_{1918}s\theta s\phi s\psi \\
&\quad - x_{1918}c\phi s\theta(\dot{\phi} + 2\dot{\theta}\dot{\psi}c\psi + \dot{\theta}s\psi) - x_{1918}c\theta(s\phi(\dot{\theta} + 2\dot{\phi}\dot{\psi}c\psi \\
&\quad + \dot{\phi}s\psi) + c\phi(2\dot{\theta}\dot{\phi} - \dot{\psi}c\psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2)s\psi))) \quad (2.76) \\
\ddot{C}_{10} &= 2(-\dot{x}^2 - \dot{y}^2 - \dot{z}^2 - \ddot{x}x - \ddot{y}y - \ddot{z}z + \ddot{z}z_{10} - x_{1918}y(\dot{\theta}^2c\theta + \dot{\theta}s\theta)s\phi \\
&\quad + x_{1918}z(\dot{\theta}c\theta - \dot{\theta}^2s\theta)s\phi - 2\dot{\theta}x_{1918}s\theta(\dot{\phi}yc\phi + \dot{y}s\phi) \\
&\quad + 2\dot{\theta}x_{1918}c\theta(\dot{\phi}zc\phi + \dot{z}s\phi) + x_{1918}c\theta((2\dot{y}\dot{\phi} + \dot{\phi}y)c\phi + (\dot{y} \\
&\quad - \dot{\phi}^2y)s\phi) + x_{1918}s\theta((2\dot{z}\dot{\phi} + \dot{\phi}z)c\phi + (\dot{z} - \dot{\phi}^2z)s\phi) \\
&\quad + x_{1918}z_{10}(-c\theta(2\dot{\theta}\dot{\phi}c\phi + \dot{\theta}s\phi) + s\theta(-\dot{\phi}c\phi + (\dot{\theta}^2 + \dot{\phi}^2)s\phi)) \\
&\quad + x_{1918}((- \dot{\phi}^2c\phi - \dot{\phi}s\phi)(xc\psi + ((-z + z_{10})c\theta + ys\theta)s\psi) \\
&\quad - 2\dot{\phi}s\phi(\dot{x}c\psi + \dot{\psi}c\psi((-z + z_{10})c\theta + ys\theta) - \dot{\psi}xs\psi + ((-\dot{z} \\
&\quad + \dot{\theta}y)c\theta + (\dot{y} + \dot{\theta}(z - z_{10}))s\theta)s\psi) + c\phi(\ddot{x}c\psi + 2\dot{\psi}c\psi((- \dot{z} \\
&\quad + \dot{\theta}y)c\theta + (\dot{y} + \dot{\theta}(z - z_{10}))s\theta) - 2\dot{x}\dot{\psi}s\psi + ((-\ddot{z} + 2\dot{y}\dot{\theta} + \dot{\theta}y \\
&\quad + \dot{\theta}^2z - \dot{\theta}^2z_{10})c\theta + (\ddot{y} + 2\dot{z}\dot{\theta} - \dot{\theta}^2y + \dot{\theta}z - \dot{\theta}z_{10})s\theta)s\psi \\
&\quad - x(\dot{\psi}^2c\psi + \dot{\psi}s\psi) + ((-z + z_{10})c\theta + ys\theta)(\dot{\psi}c\psi - \dot{\psi}^2s\psi))))
\end{aligned}$$

Sustituyendo $A_{10}, B_{10}, C_{10}, \dot{A}_{10}, \dot{B}_{10}, \dot{C}_{10}, \ddot{A}_{10}, \ddot{B}_{10}, \ddot{C}_{10}$ en la ecuación (2.75) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{21} = \frac{1}{V_1}(\ddot{x}G_1 + \ddot{y}G_2 + \ddot{z}G_3 + \ddot{\psi}G_4 + \ddot{\theta}G_5 + \ddot{\phi}G_6 + G_7) \quad (2.77)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.6.2 Solución de la aceleración, ángulo θ_{43} .

Derivando respecto del tiempo la ecuación (2.52):

$$\ddot{\theta}_{43} = \frac{-\ddot{Z}c\theta_{43} - 2\dot{X}\dot{\theta}_{43}c\theta_{43} + \dot{\theta}_{43}^2 Zc\theta_{43} - \ddot{X}s\theta_{43} + 2\dot{Z}\dot{\theta}_{43}s\theta_{43} + \dot{\theta}_{43}^2 Xs\theta_{43}}{Xc\theta_{43} - Zs\theta_{43}} \quad (2.78)$$

Donde X, Z se encuentran en (2.20) y $\dot{X}, \dot{Z}$ en 2.53, además:

$$\begin{aligned}
\ddot{X} &= \ddot{x} + \dot{\theta}_{21}^2 x_{32} c \theta_{21} + \ddot{\theta}_{21} x_{32} s \theta_{21} + \ddot{\phi} x_{1918} c \psi s \phi - 2 \dot{\phi} \dot{\psi} x_{1918} s \phi s \psi \\
&\quad + x_{1918} c \phi ((\dot{\phi}^2 + \dot{\psi}^2) c \psi + \dot{\psi} s \psi) \\
\ddot{Y} &= \ddot{y} + x_{1918} (c \theta (-c \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + s \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) \\
&\quad + s \theta (s \phi (\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi) + c \phi (2 \dot{\theta} \dot{\phi} - \dot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 \\
&\quad + \dot{\psi}^2) s \psi))) \\
\ddot{Z} &= \ddot{z} + \ddot{\theta}_{21} x_{32} c \theta_{21} - \ddot{\phi} x_{1918} c \phi s \theta - 2 \dot{\theta} \dot{\psi} x_{1918} c \phi c \psi s \theta - \dot{\theta}_{21}^2 x_{32} s \theta_{21} \\
&\quad + \dot{\theta}^2 x_{1918} s \theta s \phi + \dot{\phi}^2 x_{1918} s \theta s \phi - \ddot{\theta} x_{1918} c \phi s \theta s \psi \\
&\quad + 2 \dot{\theta} \dot{\phi} x_{1918} s \theta s \phi s \psi - x_{1918} c \theta (s \phi (\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi) \\
&\quad + c \phi (2 \dot{\theta} \dot{\phi} - \dot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))
\end{aligned} \tag{2.79}$$

Sustituyendo $X, Z, \dot{X}, \dot{Z}, \ddot{X}, \ddot{Z}$ en la ecuación (2.78) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{43} = \frac{1}{V_8} (\ddot{x} G_8 + \ddot{y} G_9 + \ddot{z} G_{10} + \ddot{\psi} G_{11} + \ddot{\theta} G_{12} + \ddot{\phi} G_{13} + G_{14}) \tag{2.80}$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.6.3 Solución de la aceleración, ángulo θ_{54} .

Derivando respecto del tiempo la ecuación (2.57):

$$\begin{aligned}
\ddot{\theta}_{54} &= \frac{1}{X^3} c \theta_{54}^2 (-2 \dot{X} \dot{Y} X c \theta_{43} + \ddot{Y} X^2 c \theta_{43} + 2 \dot{X}^2 Y c \theta_{43} - \ddot{X} X Y c \theta_{43} \\
&\quad - \dot{\theta}_{43}^2 X^2 Y c \theta_{43} - 2 \dot{Y} \dot{\theta}_{43} X^2 s \theta_{43} + 2 \dot{X} \dot{\theta}_{43} X Y s \theta_{43} - \ddot{\theta}_{43} X^2 Y s \theta_{43} \\
&\quad - 2 \dot{\theta}_{54}^2 X^3 \sec \theta_{54}^2 \tan \theta_{54})
\end{aligned} \tag{2.81}$$

Sustituyendo $X, Y, \dot{X}, \dot{Y}, \ddot{X}, \ddot{Y}$ en la ecuación (2.81) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{54} = \frac{1}{G_{15}} (\ddot{x} G_{16} + \ddot{y} G_{17} + \ddot{z} G_{18} + \ddot{\psi} G_{19} + \ddot{\theta} G_{20} + \ddot{\phi} G_{21} + G_{22}) \tag{2.82}$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.6.4 Solución de la aceleración, ángulo θ_{1817} .

Derivando respecto del tiempo la ecuación (2.61):

$$\ddot{\theta}_{1817} = -\frac{1}{b_{12}^3} c \theta_{1817}^2 (2b_{13} \dot{b}_{12}^2 - 2b_{12} \dot{b}_{12} \dot{b}_{13} - b_{12} b_{13} \ddot{b}_{12} + b_{12}^2 \ddot{b}_{13} + 2b_{12}^3 \dot{\theta}_{1817}^2 \sec \theta_{1817}^2 \tan \theta_{1817}) \quad (2.83)$$

Al derivar las ecuaciones (2.62) con respecto al tiempo se obtiene $\ddot{b}_{12}$ y $\ddot{b}_{13}$ que se encuentran en el apéndice C. De esta manera al sustituir $b_{12}, b_{13}, \dot{b}_{12}, \dot{b}_{13}, \ddot{b}_{12}, \ddot{b}_{13}$ en la ecuación (2.83) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{1817} = \frac{1}{G_{23}} (\ddot{x} G_{24} + \ddot{y} G_{25} + \ddot{z} G_{26} + \ddot{\psi} G_{27} + \ddot{\theta} G_{28} + \ddot{\phi} G_{29} + G_{30}) \quad (2.84)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.6.5 Solución de la aceleración, ángulo θ_{1716} .

Derivando respecto del tiempo la ecuación (2.66):

$$\ddot{\theta}_{1716} = \frac{1}{b_{11}^3} c \theta_{1716}^2 ((2b_{13} \dot{b}_{11}^2 - 2b_{11} \dot{b}_{11} \dot{b}_{13} - b_{11} b_{13} \ddot{b}_{11} + b_{11}^2 \ddot{b}_{13} - b_{11} (b_{11} b_{13} \ddot{\theta}_{1817} - 2b_{13} \dot{b}_{11} \dot{\theta}_{1817} + 2b_{11} \dot{b}_{13} \dot{\theta}_{1817}) \cot \theta_{1817} + b_{11}^2 b_{13} \dot{\theta}_{1817}^2 \cot \theta_{1817}^2) \csc \theta_{1817} + b_{11}^2 b_{13} \dot{\theta}_{1817}^2 \csc \theta_{1817}^3 - 2b_{11}^3 \dot{\theta}_{1716}^2 \sec \theta_{1716}^2 \tan \theta_{1716}) \quad (2.85)$$

Al derivar la ecuación (2.67) con respecto al tiempo se obtiene $\ddot{b}_{11}$ que se encuentran en el apéndice C. De esta manera al sustituir $b_{11}, b_{13}, \dot{b}_{11}, \dot{b}_{13}, \ddot{b}_{11}, \ddot{b}_{13}$ en la ecuación (2.85) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{1716} = \frac{1}{G_{31}} (\ddot{x} G_{32} + \ddot{y} G_{33} + \ddot{z} G_{34} + \ddot{\psi} G_{35} + \ddot{\theta} G_{36} + \ddot{\phi} G_{37} + G_{38}) \quad (2.86)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.6.6 Solución de la aceleración, ángulo θ_{1615} .

Derivando respecto del tiempo la ecuación (2.71):

$$\ddot{\theta}_{1615} = \frac{1}{c_{22}^3} c \theta_{1615}^2 (2c_{32} \dot{c}_{22}^2 - 2c_{22} \dot{c}_{22} \dot{c}_{32} - c_{22} c_{32} \ddot{c}_{22} + c_{22}^2 \ddot{c}_{32} - 2c_{22}^3 \dot{\theta}_{1615}^2 \sec \theta_{1615}^2 \tan \theta_{1615}) \quad (2.87)$$

Al derivar las ecuaciones (2.62) con respecto al tiempo se obtiene $\ddot{c}_{22}$ y $\ddot{c}_{32}$ que se encuentran en el apéndice C. De esta manera al sustituir $c_{22}, c_{32}, \dot{c}_{22}, \dot{c}_{32}, \ddot{c}_{22}, \ddot{c}_{32}$ en la ecuación (2.87) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{1615} = \frac{1}{G_{39}} (\ddot{x}G_{40} + \ddot{y}G_{41} + \ddot{z}G_{42} + \ddot{\psi}G_{43} + \ddot{\theta}G_{44} + \ddot{\phi}G_{45} + G_{46}) \quad (2.88)$$

Donde los coeficientes de la ecuación (2.88) se encuentran en el apéndice E.

2.7 Cálculo de posición para las cadenas i.

Para el cálculo de la posición de las cadenas i se tiene como variables a calcular los ángulos $\theta_{21i}, \theta_{65i}, \theta_{76i}, \theta_{1110i}, \theta_{109i}, \theta_{98i}$ y la posición $x, y, z, \psi, \theta, \phi$ del efector está dada.

2.7.1 Solución del ángulo θ_{21i} de las cadenas i.

A continuación se muestran las bases y los vectores a partir de los cuales se construye la ecuación de lazo para las cadenas 1 y 2. Al ser simétricas las cadenas 1 y 2 se lleva a cabo el análisis sobre una de ellas y asigna un iterador i. Así mismo se hace uso de una variable auxiliar llamada "signo" para diferenciar algunas distancias iguales en magnitud pero contrarias en signo.

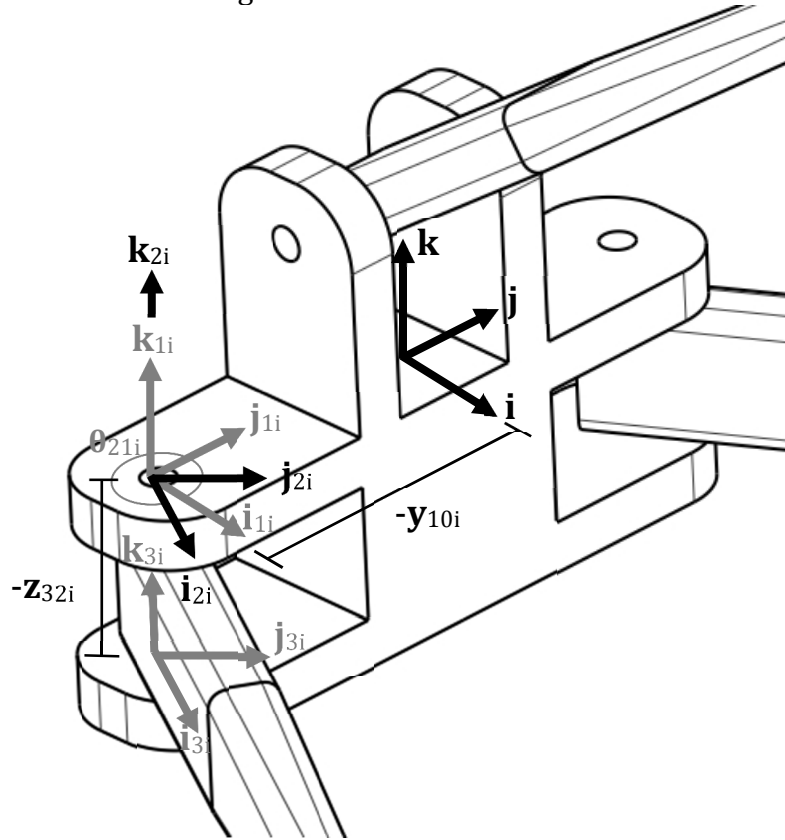


Figura 2-7 Bases locales de la 1i a la 3i para las cadenas i.

Partiendo de la base inercial se forma la base 1i por medio de una traslación sobre el eje Y. Posteriormente por tratarse de una junta rotacional se emplea una rotación sobre el eje Z para llegar a la base 2 y una traslación sobre Z para llegar a la base 3i como se observa en la figura (2-7).

En la figura (2-8) se aprecia que para formar la base 4i se traslada la base 3i sobre el eje X. Después para generar la base 5i se define un ángulo constante β_{54i} que le otorga una inclinación constante a la junta universal.

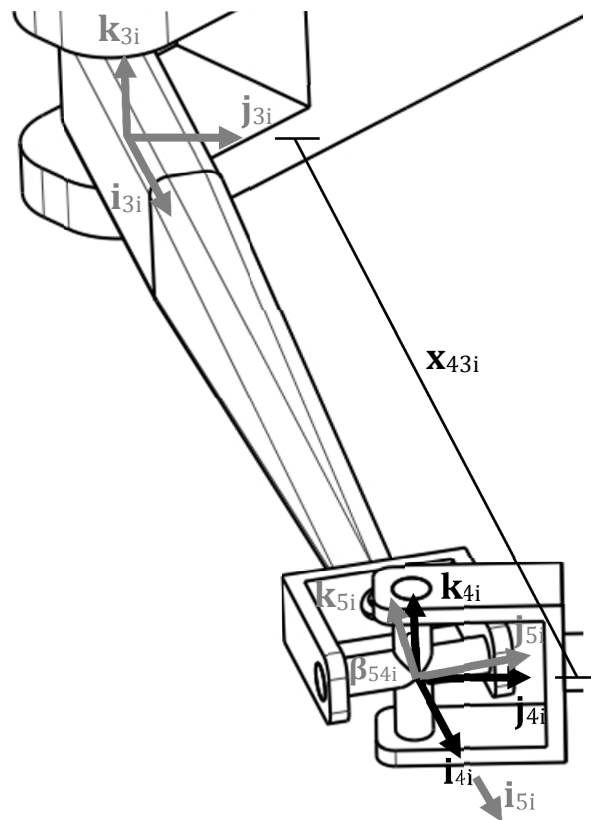


Figura 2-8 Bases locales 4i y 5i.

Por la naturaleza de la junta universal se necesita definir 2 rotaciones, lo cual se observa en la figura (2-9) generando así las bases 6i y 7i respectivamente. Posteriormente se crea la base 8i por medio de una traslación en el eje X, con lo cual se llega al centro de la junta esférica.

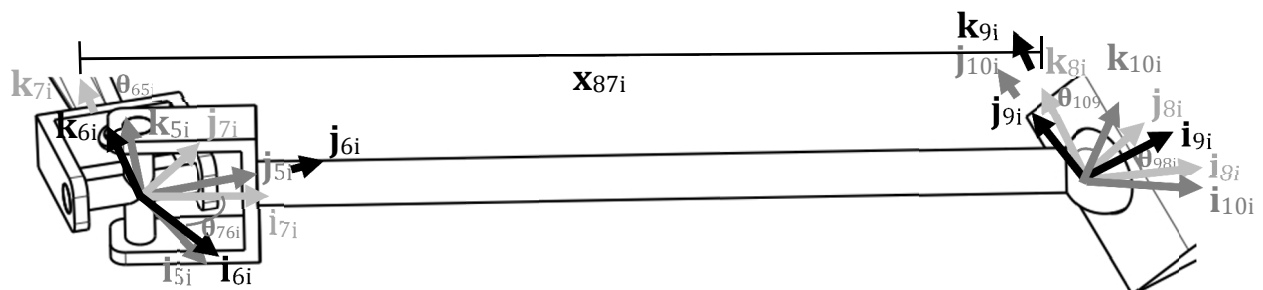


Figura 2-9 Bases locales de la 7i a la 10i.

Para la junta esférica es necesario definir tres rotaciones creando así las bases $9i$ y $10i$, que se observan en la figura (2-9) y $11i$ que se observa en la figura (2-10)

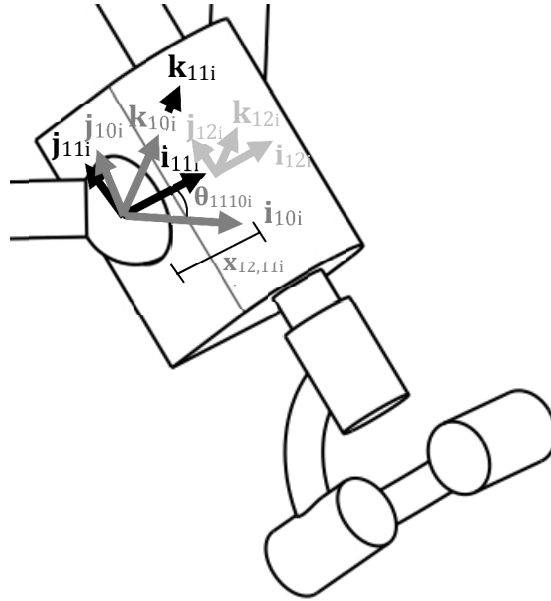


Figura 2-10 Bases locales de la $10i$ a la $12i$.

Por último se busca cerrar el lazo vectorial con algunos de los vectores ya definidos para la cadena 0. Para esto se generan las bases $12i$ y $13i$ mediante dos traslaciones, como se aprecia en las figuras (2-10) y (2-11).

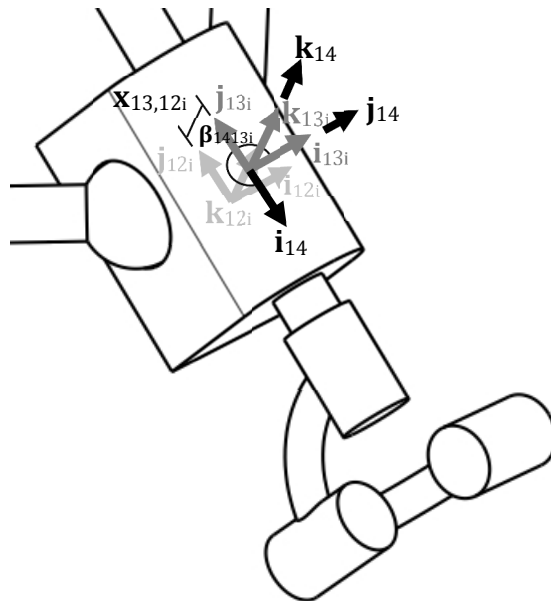


Figura 2-11 Bases locales de la $12i$ a la $14i$.

Finalmente se alinean las bases 13i y 14 mediante un ángulo β_{1413i} constante, cerrando de esta manera el lazo vectorial para las cadenas i.

En la figura (2-12) se muestra el lazo vectorial para las cadenas i.

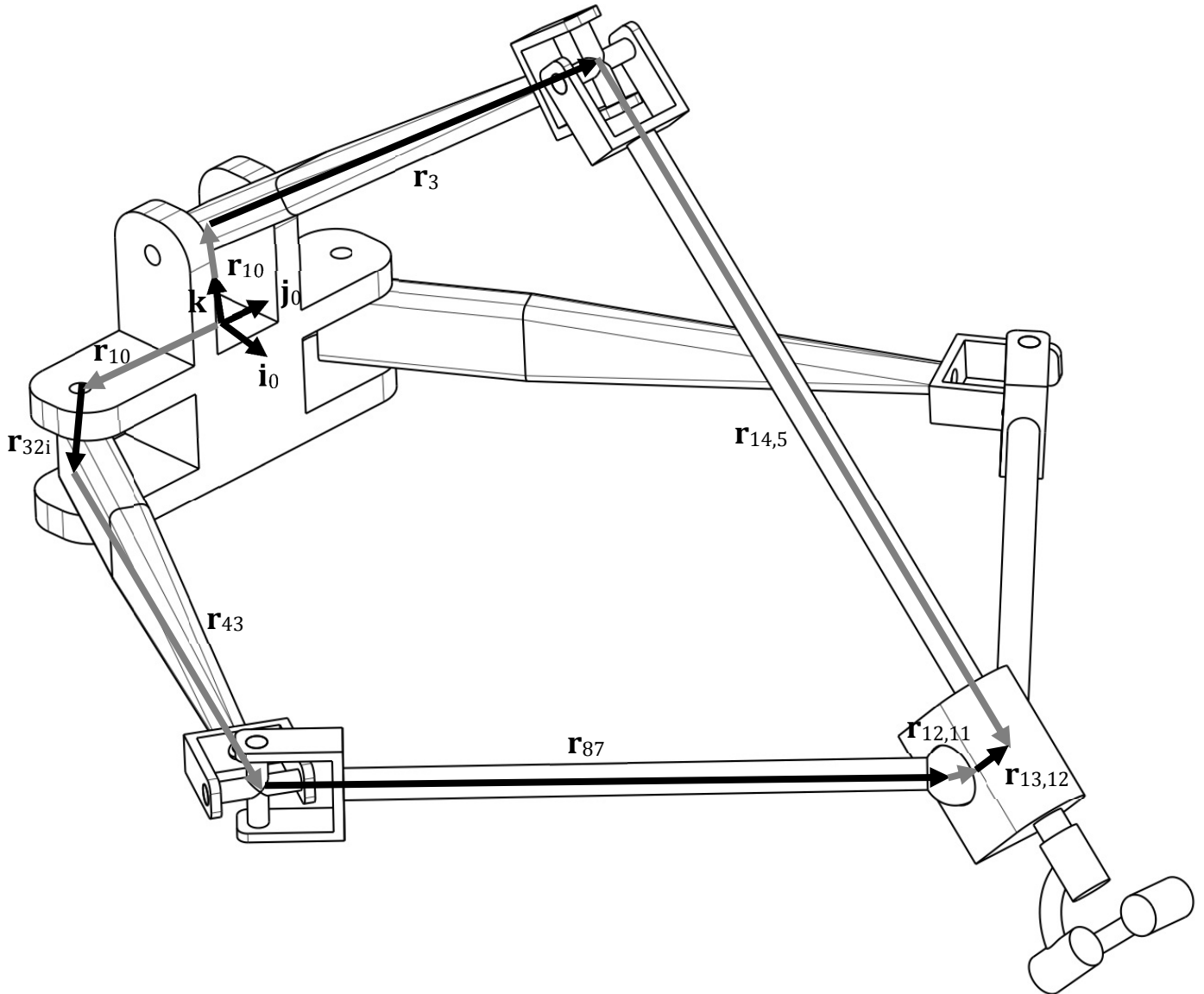


Figura 2-12 Lazo vectorial para las cadenas i.

A partir de la figura se forma la ecuación de lazo:

$$\mathbf{r}_{10i}^0 + \mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{87i}^0 = \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 - \mathbf{r}_{13,12i}^0 - \mathbf{r}_{12,11i}^0$$

Al expresar la ecuación con transformaciones homogéneas se obtiene lo siguiente:

$$\mathbf{T}_{02i} \mathbf{T}_{24i} \mathbf{T}_{48i} = \mathbf{T}_{03} \mathbf{T}_{311i} \quad (2.89)$$

Dónde:

$$\begin{aligned}
T_{02i} &= T_{z2}(\text{signo } y_{10i})T_{z6}(\theta_{21i}) \\
T_{24i} &= T_{z3}(-z_{32i})T_{z1}(x_{43i}) \\
T_{48i} &= T_{z4}(\beta_{54i})T_{z5}(\theta_{65i})T_{z6}(\theta_{76i})T_{z1}(x_{87i}) \\
T_{03} &= T_{z3}(z_{10})T_{z5}(\theta_{21})T_{z1}(x_{32}) \\
T_{311i} &= T_{z5}(\theta_{43})T_{z6}(\theta_{54})T_{z1}(x_{65})T_{z6}(-\beta_{1413i})T_{z3}(-z_{1312i})T_{z1}(-x_{1211i})
\end{aligned}$$

Se despeja T_{48i} y se posmultiplica ambos lados de la ecuación por un vector $\mathbf{n}$ con el fin de obtener la última columna de la matriz de transformación, con lo cual tenemos:

$$T_{48i}\mathbf{n} = T_{24i}^{-1}T_{02i}^{-1}T_{03}T_{311i}\mathbf{n} \quad (2.90)$$

Dónde:

$$\mathbf{n} = [0, 0, 0, 1]^T$$

Si renombramos el lado izquierdo de la ecuación (2.90) como $\mathbf{Ec}_3$ y el lado derecho como $\mathbf{Ec}_4$, se obtiene:

$$\mathbf{Ec}_3 = \mathbf{Ec}_4 \quad (2.91)$$

En esta ecuación se tienen como incógnitas a θ_{21i} , θ_{65i} y θ_{76i} . Como nos interesa despejar a θ_{21i} hay que eliminar las otras dos, lo cual se logra al obtener la magnitud de ambos lados de la ecuación:

$$\|\mathbf{Ec}_3\| = \|\mathbf{Ec}_4\| \quad (2.92)$$

$$\sqrt{\mathbf{Ec}_3^T \mathbf{Ec}_3} = \sqrt{\mathbf{Ec}_4^T \mathbf{Ec}_4} \quad (2.93)$$

Simplificado:

$$\mathbf{Ec}_3^T \mathbf{Ec}_3 - \mathbf{Ec}_4^T \mathbf{Ec}_4 = 0 \quad (2.94)$$

Al factorizar esta última ecuación en términos de $s\theta_{21i}$ y $c\theta_{21i}$ se obtiene una expresión de la forma:

$$C_{1i} + A_{1i}c\theta_{21i} + B_{1i}s\theta_{21i} = 0 \quad (2.95)$$

Dónde:

$$\begin{aligned}
A_{1i} &= x_{43i}(2 x_{32}c\theta_{21} - x_{1211i} c(\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54}) \\
&\quad + x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) - x_{1211i} c(\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) \\
&\quad + x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) - 2 z_{1312i} s(\theta_{21} + \theta_{43})) \\
B_{1i} &= 2x_{43i}(-signo y_{10i} + x_{1211i}s(\beta_{1413i} - \theta_{54}) + x_{65}s(\theta_{54})) \\
C_{1i} &= -x_{1211i}^2 - x_{32}^2 - x_{43i}^2 - x_{65}^2 + x_{87i}^2 - signo^2 y_{10i}^2 - z_{10}^2 - z_{1312i}^2 \\
&\quad - 2z_{10}z_{32i} - z_{32i}^2 + 2x_{1211i}x_{65}c(\beta_{1413i}) + 2z_{1312i}(z_{10} \\
&\quad + z_{32i})c(\theta_{21} + \theta_{43}) + x_{1211i}x_{32}c(\beta_{1413i} - \theta_{43} - \theta_{54}) \\
&\quad - x_{32}x_{65}c(\theta_{43} - \theta_{54}) + x_{1211i}x_{32}c(\beta_{1413i} + \theta_{43} - \theta_{54}) \\
&\quad - x_{32}x_{65}c(\theta_{43} + \theta_{54}) + 2x_{32}z_{10}s(\theta_{21}) + 2x_{32}z_{32i}s(\theta_{21}) \\
&\quad + 2x_{32}z_{1312i}s(\theta_{43}) + 2signox_{1211i}y_{10i}s(\beta_{1413i} - \theta_{54}) \\
&\quad + x_{1211i}z_{10}s(\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54}) + x_{1211i}z_{32i}s(\beta_{1413i} \\
&\quad - \theta_{21} - \theta_{43} - \theta_{54}) + x_{65}z_{10}s(\theta_{21} + \theta_{43} - \theta_{54}) + x_{65}z_{32i}s(\theta_{21} \\
&\quad + \theta_{43} - \theta_{54}) - x_{1211i}z_{10}s(\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) \\
&\quad - x_{1211i}z_{32i}s(\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) + 2signox_{65}y_{10i}s(\theta_{54}) \\
&\quad + x_{65}z_{10}s(\theta_{21} + \theta_{43} + \theta_{54}) + x_{65}z_{32i}s(\theta_{21} + \theta_{43} + \theta_{54})
\end{aligned} \tag{2.96}$$

De esta manera en la forma en la que se encuentra la ecuación (2.95) se puede despejar a θ_{21i} como:

$$\theta_{21i} = 2 \arctan \left(\frac{B_{1i} \pm \sqrt{A_{1i}^2 + B_{1i}^2 - C_{1i}^2}}{A_{1i} - C_{1i}} \right) \tag{2.97}$$

2.7.2 Solución del ángulo θ_{65i} de las cadenas i.

Del desarrollo anterior se puede obtener la forma matricial de ambos lados de la ecuación (2.91) como sigue:

$$\mathbf{Ec}_3 = \begin{bmatrix} x_{87i}c\theta_{65i}c\theta_{76i} \\ x_{87i}s\theta_{76i} \\ -x_{87i}c\theta_{76i}s\theta_{65i} \\ 1 \end{bmatrix} \tag{2.98}$$

Se observa que $\mathbf{Ec}_4$ esta formada por elementos constantes y por θ_{21i} que es el ángulo que se ya se despejó por lo que para un manejo más cómodo se renombraran sus elementos como X_i, Y_i, Z_i es decir:

$$\mathbf{Ec}_4 = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \quad (2.99)$$

Dónde:

$$\begin{aligned} X_i &= -x_{43i} - \text{signoy}_{10i}s\theta_{21i} + c\theta_{21}c\theta_{21i}(x_{32} + c\theta_{43}(-x_{1211i}c(\beta_{1413i} - \theta_{54}) \\ &\quad + x_{65}c\theta_{54}) - z_{1312i}s\theta_{43}) - c\theta_{21i}s\theta_{21}(z_{1312i}c\theta_{43} \\ &\quad - x_{1211i}c(\beta_{1413i} - \theta_{54})s\theta_{43} + x_{65}c\theta_{54}s\theta_{43}) \\ &\quad + x_{1211i}s\theta_{21i}s(\beta_{1413i} - \theta_{54}) + x_{65}s\theta_{21i}s\theta_{54} \\ Y_i &= -\text{signoy}_{10i}c\beta_{54i}c\theta_{21i} + z_{32i}s\beta_{54i} + s\beta_{54i}(z_{10} - x_{32}s\theta_{21}) \\ &\quad - x_{32}c\beta_{54i}c\theta_{21}s\theta_{21i} + (s\beta_{54i}s\theta_{21} \\ &\quad + c\beta_{54i}c\theta_{21}s\theta_{21i})(c\theta_{43}(x_{1211i}c(\beta_{1413i} - \theta_{54}) - x_{65}c\theta_{54}) \\ &\quad + z_{1312i}s\theta_{43}) - (c\theta_{21}s\beta_{54i} - c\beta_{54i}s\theta_{21}s\theta_{21i})(z_{1312i}c\theta_{43} \\ &\quad + (-x_{1211i}c(\beta_{1413i} - \theta_{54}) + x_{65}c\theta_{54})s\theta_{43}) \\ &\quad + c\beta_{54i}c\theta_{21i}(x_{1211i}s(\beta_{1413i} - \theta_{54}) + x_{65}s\theta_{54}) \\ Z_i &= z_{32i}c\beta_{54i} + \text{signoy}_{10i}c\theta_{21i}s\beta_{54i} + c\beta_{54i}(z_{10} - x_{32}s\theta_{21}) \\ &\quad + x_{32}c\theta_{21}s\beta_{54i}s\theta_{21i} + (c\beta_{54i}s\theta_{21} \\ &\quad - c\theta_{21}s\beta_{54i}s\theta_{21i})(c\theta_{43}(x_{1211i}c(\beta_{1413i} - \theta_{54}) - x_{65}c\theta_{54}) \\ &\quad + z_{1312i}s\theta_{43}) - (c\beta_{54i}c\theta_{21} + s\beta_{54i}s\theta_{21}s\theta_{21i})(z_{1312i}c\theta_{43} \\ &\quad + (-x_{1211i}c(\beta_{1413i} - \theta_{54}) + x_{65}c\theta_{54})s\theta_{43}) \\ &\quad - c\theta_{21i}s\beta_{54i}(x_{1211i}s(\beta_{1413i} - \theta_{54}) + x_{65}s\theta_{54}) \end{aligned} \quad (2.100)$$

De esta manera se manipulará más fácilmente el lado izquierdo de la ecuación, es decir $\mathbf{Ec}_3$, la cual contiene el ángulo que nos interesa despejar.

Se tiene el sistema de ecuaciones siguiente:

$$\begin{aligned} x_{87i}c\theta_{65i}c\theta_{76i} &= X_i \\ x_{87i}s\theta_{76i} &= Y_i \\ -x_{87i}c\theta_{76i}s\theta_{65i} &= Z_i \end{aligned} \quad (2.101)$$

De las ecuaciones (2.101) se obtiene $\tan\theta_{65i}$:

$$\tan\theta_{65i} = \frac{-Z_i}{X_i} \quad (2.102)$$

De esta manera:

$$\theta_{65i} = \arctan\left(\frac{-Z_i}{X_i}\right) \quad (2.103)$$

2.7.3 Solución del ángulo θ_{76i} las cadenas i.

De las ecuaciones (2.101) se tiene que:

$$\tan \theta_{76i} = \frac{Y_i c \theta_{65i}}{X_i} \quad (2.104)$$

Y por lo tanto:

$$\theta_{76i} = \arctan\left(\frac{Y_i c \theta_{65i}}{X_i}\right) \quad (2.105)$$

2.7.4 Solución del ángulo θ_{1110i} de las cadenas i.

Para el cálculo de los ángulos de la junta esférica que conecta la cadena i con el órgano terminal se formarán las ecuaciones de lazo pero esta vez considerando tanto distancias como ángulos.

Al expresarse en transformaciones homogéneas:

$$\mathbf{T}_{02i} \mathbf{T}_{24i} \mathbf{T}_{48i} \mathbf{T}_{z6}(\theta_{98i}) \mathbf{T}_{z5}(\theta_{109i}) \mathbf{T}_{z6}(\theta_{1110i}) = \mathbf{T}_{03} \mathbf{T}_{311i} \quad (2.106)$$

Dónde:

$$\begin{aligned} \mathbf{T}_{02i} &= \mathbf{T}_{z2}(\text{signo } y_{10i}) \mathbf{T}_{z6}(\theta_{21i}) \\ \mathbf{T}_{24i} &= \mathbf{T}_{z3}(-z_{32i}) \mathbf{T}_{z1}(x_{43i}) \\ \mathbf{T}_{48i} &= \mathbf{T}_{z4}(\beta_{54i}) \mathbf{T}_{z5}(\theta_{65i}) \mathbf{T}_{z6}(\theta_{76i}) \mathbf{T}_{z1}(x_{87i}) \\ \mathbf{T}_{03} &= \mathbf{T}_{z3}(z_{10}) \mathbf{T}_{z5}(\theta_{21}) \mathbf{T}_{z1}(x_{32}) \\ \mathbf{T}_{311i} &= \mathbf{T}_{z5}(\theta_{43}) \mathbf{T}_{z6}(\theta_{54}) \mathbf{T}_{z1}(x_{65}) \mathbf{T}_{z6}(-\beta_{1413i}) \mathbf{T}_{z3}(-z_{1312i}) \mathbf{T}_{z1}(-x_{1211i}) \end{aligned}$$

Se observa que además de los tres ángulos de la junta esférica (θ_{98i} , θ_{109i} y θ_{1110i}) lo demás es conocido debido a que es dato o ya se calculó, por lo que se procederá a dejar las transformaciones asociadas a los ángulos desconocidos del lado izquierdo de la ecuación y todo lo demás del lado derecho.

$$\mathbf{T}_{z6}(\theta_{98i}) \mathbf{T}_{z5}(\theta_{109i}) \mathbf{T}_{z6}(\theta_{1110i}) = \mathbf{T}_{48i}^{-1} \mathbf{T}_{24i}^{-1} \mathbf{T}_{02i}^{-1} \mathbf{T}_{03} \mathbf{T}_{311i} \quad (2.107)$$

Renombrando ambos lados de la ecuación (2.107):

$$\mathbf{T}_{Lli} = \mathbf{T}_{LDi} \quad (2.108)$$

Dónde:

$$\mathbf{T}_{Lli} = \begin{bmatrix} c\theta_{109i}c\theta_{1110i}c\theta_{98i} - s\theta_{1110i}s\theta_{98i} & -c\theta_{109i}c\theta_{98i}s\theta_{1110i} - c\theta_{1110i}s\theta_{98i} & c\theta_{98i}s\theta_{109i} & 0 \\ c\theta_{98i}s\theta_{1110i} + c\theta_{109i}c\theta_{1110i}s\theta_{98i} & c\theta_{1110i}c\theta_{98i} - c\theta_{109i}s\theta_{1110i}s\theta_{98i} & s\theta_{109i}s\theta_{98i} & 0 \\ -c\theta_{1110i}\theta_{109i} & s\theta_{109i}s\theta_{1110i} & c\theta_{109i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.109)$$

$$\mathbf{T}_{LDi} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \quad (2.110)$$

Igualando los elementos (3,1) y (3,2) de las ecuaciones (2.109) y (2.110):

$$\begin{aligned} -c\theta_{1110i}\theta_{109i} &= d_{31} \\ s\theta_{109i}s\theta_{1110i} &= d_{32} \end{aligned} \quad (2.111)$$

Despejando $c\theta_{1110i}$ y $s\theta_{1110i}$:

$$\begin{aligned} c\theta_{1110i} &= -d_{31}csc\theta_{109i} \\ s\theta_{1110i} &= d_{32}csc\theta_{109i} \end{aligned} \quad (2.112)$$

Formando la función tangente:

$$\tan\theta_{1110i} = \frac{d_{32}}{-d_{31}} \quad (2.113)$$

Por lo tanto:

$$\theta_{1110i} = \arctan\left(\frac{d_{32}}{-d_{31}}\right) \quad (2.114)$$

Dónde:

$$\begin{aligned} d_{31} &= s(\beta_{1413i} - \theta_{54})(c\theta_{21i}c\theta_{65i}s\beta_{54i} - s\theta_{21i}s\theta_{65i}) + c\theta_{43}c(\beta_{1413i} \\ &\quad - \theta_{54})(-c\beta_{54i}c\theta_{65i}s\theta_{21i} + c\theta_{21i}(c\theta_{65i}s\beta_{54i}s\theta_{21i} \\ &\quad + c\theta_{21i}s\theta_{65i})) - c(\beta_{1413i} - \theta_{54})s\theta_{43}(c\beta_{54i}c\theta_{21i}c\theta_{65i} \\ &\quad + s\theta_{21i}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})) \\ d_{32} &= c(\beta_{1413i} - \theta_{54})(-c\theta_{21i}c\theta_{65i}s\beta_{54i} + s\theta_{21i}s\theta_{65i}) + c\theta_{43}s(\beta_{1413i} \\ &\quad - \theta_{54})(-c\beta_{54i}c\theta_{65i}s\theta_{21i} + c\theta_{21i}(c\theta_{65i}s\beta_{54i}s\theta_{21i} \\ &\quad + c\theta_{21i}s\theta_{65i})) - s\theta_{43}s(\beta_{1413i} - \theta_{54})(c\beta_{54i}c\theta_{21i}c\theta_{65i} \\ &\quad + s\theta_{21i}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})) \end{aligned} \quad (2.115)$$

2.7.5 Solución del ángulo θ_{109i} de las cadenas i.

Despejando $\mathbf{T}_{z6}(\theta_{98i})\mathbf{T}_{z5}(\theta_{109i})$ de la ecuación (2.107):

$$\mathbf{T}_{z6}(\theta_{98i})\mathbf{T}_{z5}(\theta_{109i}) = \mathbf{T}_{48i}^{-1}\mathbf{T}_{24i}^{-1}\mathbf{T}_{02i}^{-1}\mathbf{T}_{03}\mathbf{T}_{311i}\mathbf{T}_{z6}^{-1}(\theta_{1110i}) \quad (2.116)$$

Renombrando el lado derecho de la ecuación (2.116) como:

$$\mathbf{T}_{z6}(\theta_{98i})\mathbf{T}_{z5}(\theta_{109i}) = \mathbf{T}_{LDi2} \quad (2.117)$$

Dónde:

$$\mathbf{T}_{z6}(\theta_{98i})\mathbf{T}_{z5}(\theta_{109i}) = \begin{bmatrix} c\theta_{109i}c\theta_{98i} & -s\theta_{98i} & c\theta_{98i}s\theta_{109i} & 0 \\ c\theta_{109i}s\theta_{98i} & c\theta_{98i} & s\theta_{109i}s\theta_{98i} & 0 \\ -s\theta_{109i} & 0 & c\theta_{109i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.118)$$

$$\mathbf{T}_{LDi2} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.119)$$

De las ecuaciones (2.118) y (2.119) se igualan los elementos (3,1) y (3,3):

$$\begin{aligned} -s\theta_{109i} &= e_{31} \\ c\theta_{109i} &= e_{33} \end{aligned} \quad (2.120)$$

Despejando $c\theta_{109i}$ y $s\theta_{109i}$ de las ecuaciones (2.120):

$$\begin{aligned} c\theta_{109i} &= e_{33} \\ s\theta_{109i} &= -e_{31} \end{aligned} \quad (2.121)$$

Formando la función tangente a partir de las ecuaciones (2.121):

$$\tan\theta_{109i} = \frac{-e_{31}}{e_{33}} \quad (2.122)$$

Por lo tanto:

$$\theta_{109i} = \arctan\left(\frac{-e_{31}}{e_{33}}\right) \quad (2.123)$$

Dónde:

$$\begin{aligned}
e_{31} = & -c\beta_{54i}c(\beta_{1413i} + \theta_{1110i} - \theta_{54})c\theta_{65i}s(\theta_{21} + \theta_{43}) + c\theta_{21i}(c(\beta_{1413i} \\
& - \theta_{54})(c\theta_{65i}s\beta_{54i}s\theta_{1110i} + c\theta_{1110i}c(\theta_{21} + \theta_{43})s\theta_{65i}) \\
& + s(\beta_{1413i} - \theta_{54})(c\theta_{1110i}c\theta_{65i}s\beta_{54i} - c(\theta_{21} \\
& + \theta_{43})s\theta_{1110i}s\theta_{65i})) \\
& + s\theta_{21i}(-s\theta_{1110i}(c\theta_{21}c\theta_{43}c\theta_{65i}s\beta_{54i}s(\beta_{1413i} - \theta_{54}) \\
& - c\theta_{65i}s\beta_{54i}s\theta_{21}s\theta_{43}s(\beta_{1413i} - \theta_{54}) + c(\beta_{1413i} - \theta_{54})s\theta_{65i}) \\
& + c\theta_{1110i}(c\theta_{21}c\theta_{43}c(\beta_{1413i} - \theta_{54})c\theta_{65i}s\beta_{54i} - c(\beta_{1413i} \\
& - \theta_{54})c\theta_{65i}s\beta_{54i}s\theta_{21}s\theta_{43} - s(\beta_{1413i} - \theta_{54})s\theta_{65i})) \\
e_{33} = & c\beta_{54i}c(\theta_{21} + \theta_{43})c\theta_{65i} + s(\theta_{21} + \theta_{43})(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})
\end{aligned} \tag{2.124}$$

2.7.6 Solución del ángulo θ_{98i} de las cadenas i.

Despejando $T_{z6}(\theta_{98i})$ de la ecuación (2.107):

$$T_{z6}(\theta_{98i}) = T_{48i}^{-1}T_{24i}^{-1}T_{02i}^{-1}T_{03}T_{311i}T_{z6}^{-1}(\theta_{1110i})T_{z5}^{-1}(\theta_{109i}) \tag{2.125}$$

Renombrando el lado derecho de la ecuación como:

$$T_{z6}(\theta_{98i}) = T_{LDi3} \tag{2.126}$$

Dónde:

$$T_{z6}(\theta_{98i}) = \begin{bmatrix} c\theta_{98i} & -s\theta_{98i} & 0 & 0 \\ s\theta_{98i} & c\theta_{98i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.127}$$

$$T_{LDi3} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.128}$$

De las ecuaciones (2.127) y (2.128) se igualan los elementos (1,1) y (2,3)

$$\begin{aligned}
s\theta_{98i} &= f_{21} \\
c\theta_{98i} &= f_{11}
\end{aligned} \tag{2.129}$$

Formando la función tangente a partir de las ecuaciones (2.129):

$$\tan\theta_{98i} = \frac{f_{21}}{f_{11}} \tag{2.130}$$

Por lo tanto:

$$\theta_{98i} = \arctan\left(\frac{f_{21}}{f_{11}}\right) \quad (2.131)$$

Donde f_{11} y f_{21} se encuentran en el apéndice C.

2.8 Cálculo de velocidad de las cadenas i.

Para el cálculo de la velocidad se tiene como variables de entrada la velocidad del efector $(\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi})$ y las variables a calcular son las velocidades de los ángulos de las cadenas i es decir $\dot{\theta}_{21i}, \dot{\theta}_{65i}, \dot{\theta}_{76i}, \dot{\theta}_{1110i}, \dot{\theta}_{109i}, \dot{\theta}_{98i}$. Para esta sección se asume que la posición de los ángulos es conocida debido a que se calculó en la sección anterior.

2.8.1 Solución de la velocidad ángulo θ_{21i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.95) y se despeja $\dot{\theta}_{21i}$:

$$\dot{\theta}_{21i} = \frac{-\dot{C}_{1i} - \dot{A}_{1i}c\theta_{21i} - \dot{B}_{1i}s\theta_{21i}}{B_{1i}c\theta_{21i} - A_{1i}s\theta_{21i}} \quad (2.132)$$

Dónde:

$$\begin{aligned} \dot{A}_{1i} &= x_{43i}(-2(\dot{\theta}_{21} + \dot{\theta}_{43})z_{1312i}c\theta_{21} + \dot{\theta}_{43} - 2\dot{\theta}_{21}x_{32}s\theta_{21} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{1211i}s\beta_{1413i} - \dot{\theta}_{21} - \dot{\theta}_{43} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{65}s\theta_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})x_{1211i}s\beta_{1413i} \\ &\quad + \dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{65}s\theta_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) \\ \dot{B}_{1i} &= -2\dot{\theta}_{54}x_{43i}(x_{1211i}c\beta_{1413i} - \theta_{54} - x_{65}c\theta_{54}) \\ \dot{C}_{1i} &= 2\dot{\theta}_{21}x_{32}z_{10}c\theta_{21} + 2\dot{\theta}_{21}x_{32}z_{32i}c\theta_{21} + 2\dot{\theta}_{43}x_{32}z_{1312i}c\theta_{43} \\ &\quad - 2\dot{\theta}_{54}\text{signox}_{1211i}y_{10i}c\beta_{1413i} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{1211i}z_{10}c\beta_{1413i} - \dot{\theta}_{21} - \dot{\theta}_{43} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{1211i}z_{32i}c\beta_{1413i} - \dot{\theta}_{21} - \dot{\theta}_{43} - \dot{\theta}_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{65}z_{10}c\theta_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{65}z_{32i}c\theta_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{1211i}z_{10}c\beta_{1413i} + \dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{1211i}z_{32i}c\beta_{1413i} + \dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} \\ &\quad + 2\dot{\theta}_{54}\text{signox}_{65}y_{10i}c\theta_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{65}z_{10}c\theta_{21} \\ &\quad + \dot{\theta}_{43} + \dot{\theta}_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{65}z_{32i}c\theta_{21} + \dot{\theta}_{43} + \dot{\theta}_{54} \\ &\quad - 2(\dot{\theta}_{21} + \dot{\theta}_{43})z_{1312i}(z_{10} + z_{32i})s\theta_{21} + \dot{\theta}_{43} + (\dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{1211i}x_{32}s\beta_{1413i} - \dot{\theta}_{43} - \dot{\theta}_{54} + (\dot{\theta}_{43} - \dot{\theta}_{54})x_{32}x_{65}s\theta_{43} \\ &\quad - \dot{\theta}_{54} - (\dot{\theta}_{43} - \dot{\theta}_{54})x_{1211i}x_{32}s\beta_{1413i} + \dot{\theta}_{43} - \dot{\theta}_{54} + (\dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{32}x_{65}s\theta_{43} + \dot{\theta}_{54} \end{aligned} \quad (2.133)$$

Sustituyendo $A_{1i}, B_{1i}, C_{1i}, \dot{A}_{1i}, \dot{B}_{1i}, \dot{C}_{1i}$ en (2.132), agrupando y simplificando:

$$\dot{\theta}_{21i} = \frac{1}{V_{44}} (\dot{\theta}_{21}V_{45} + \dot{\theta}_{43}V_{46} + \dot{\theta}_{54}V_{47}) \quad (2.134)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59) en (2.134) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{21i} = \frac{1}{V_{44}} (\dot{x}E_{31} + \dot{y}E_{32} + \dot{z}E_{33} + \dot{\psi}E_{34} + \dot{\theta}E_{35} + \dot{\phi}E_{36}) \quad (2.135)$$

Dónde:

$$\begin{aligned} E_{31} &= \frac{V_2V_{45}}{V_1} + \frac{E_7V_{47}}{V_{15}} + \frac{E_1V_{46}}{V_8} & E_{34} &= \frac{E_{10}V_{47}}{V_{15}} + \frac{V_{45}V_5}{V_1} + \frac{E_4V_{46}}{V_8} \\ E_{32} &= \frac{V_3V_{45}}{V_1} + \frac{E_8V_{47}}{V_{15}} + \frac{E_2V_{46}}{V_8} & E_{35} &= \frac{E_{11}V_{47}}{V_{15}} + \frac{V_{45}V_6}{V_1} + \frac{E_5V_{46}}{V_8} \\ E_{33} &= \frac{V_4V_{45}}{V_1} + \frac{E_9V_{47}}{V_{15}} + \frac{E_3V_{46}}{V_8} & E_{36} &= \frac{E_{12}V_{47}}{V_{15}} + \frac{V_{45}V_7}{V_1} + \frac{E_6V_{46}}{V_8} \end{aligned} \quad (2.136)$$

2.8.2 Solución de la velocidad ángulo θ_{65i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.103):

$$\dot{\theta}_{65i} = \frac{-\dot{Z}_i c \theta_{65i} - \dot{X}_i s \theta_{65i}}{X_i c \theta_{65i} - Z_i s \theta_{65i}} \quad (2.137)$$

Dónde:

$$\begin{aligned} \dot{X}_i &= \frac{1}{2} (-2(\dot{\theta}_{21} + \dot{\theta}_{43})z_{1312i}c\theta_{21} + \theta_{43} - 2\dot{\theta}_{21}x_{32}s\theta_{21} + 2\dot{\theta}_{21i}x_{43i}s\theta_{21i} \\ &\quad - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{1211i}s\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54} - (\dot{\theta}_{21} \\ &\quad + \dot{\theta}_{43} - \dot{\theta}_{54})x_{65}s\theta_{21} + \theta_{43} - \theta_{54} + (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad - \dot{\theta}_{54})x_{1211i}s\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} \\ &\quad + \dot{\theta}_{54})x_{65}s\theta_{21} + \theta_{43} + \theta_{54}) \\ \dot{Y}_i &= -c\beta_{54i}(\dot{\theta}_{21i}x_{43i}c\theta_{21i} + \dot{\theta}_{54}x_{1211i}c\beta_{1413i} - \theta_{54} - \dot{\theta}_{54}x_{65}c\theta_{54}) \\ &\quad + \frac{1}{2}s\beta_{54i}(-2\dot{\theta}_{21}x_{32}c\theta_{21} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{1211i}c\beta_{1413i} \\ &\quad - \theta_{21} - \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})x_{65}c\theta_{21} + \theta_{43} - \theta_{54} \\ &\quad + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})x_{1211i}c\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} \\ &\quad + \dot{\theta}_{43} + \dot{\theta}_{54})x_{65}c\theta_{21} + \theta_{43} + \theta_{54} + 2(\dot{\theta}_{21} + \dot{\theta}_{43})z_{1312i}s\theta_{21} \\ &\quad + \theta_{43}) \end{aligned} \quad (2.138)$$

$$\begin{aligned}\dot{Z}_i = & (\dot{\theta}_{21i}x_{43i}c\theta_{21i} + \dot{\theta}_{54}x_{1211i}c\beta_{1413i} - \theta_{54} - \dot{\theta}_{54}x_{65}c\theta_{54})s\beta_{54i} \\ & + \frac{1}{2}c\beta_{54i}(-2\dot{\theta}_{21}x_{32}c\theta_{21} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})x_{1211i}c\beta_{1413i} \\ & - \theta_{21} - \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})x_{65}c\theta_{21} + \theta_{43} - \theta_{54} \\ & + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})x_{1211i}c\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} \\ & + \dot{\theta}_{43} + \dot{\theta}_{54})x_{65}c\theta_{21} + \theta_{43} + \theta_{54} + 2(\dot{\theta}_{21} + \dot{\theta}_{43})z_{1312i}s\theta_{21} \\ & + \theta_{43})\end{aligned}$$

Sustituyendo $X_i, Z_i, \dot{X}_i, \dot{Z}_i$ en (2.137), agrupando y simplificando:

$$\dot{\theta}_{65i} = \frac{1}{V_{48}}(\dot{\theta}_{21}V_{49} + \dot{\theta}_{43}V_{50} + \dot{\theta}_{54}V_{51} + \dot{\theta}_{21i}V_{52}) \quad (2.139)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.135) en (2.139) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{65i} = \frac{1}{V_{48}}(\dot{x}E_{37} + \dot{y}E_{38} + \dot{z}E_{39} + \dot{\psi}E_{40} + \dot{\theta}E_{41} + \dot{\phi}E_{42}) \quad (2.140)$$

Dónde:

$$\begin{aligned}E_{37} &= \frac{V_2V_{49}}{V_1} + \frac{E_7V_{51}}{V_{15}} + \frac{E_{31}V_{52}}{V_{44}} + \frac{E_1V_{50}}{V_8} \\ E_{38} &= \frac{V_3V_{49}}{V_1} + \frac{E_8V_{51}}{V_{15}} + \frac{E_{32}V_{52}}{V_{44}} + \frac{E_2V_{50}}{V_8} \\ E_{39} &= \frac{V_4V_{49}}{V_1} + \frac{E_9V_{51}}{V_{15}} + \frac{E_{33}V_{52}}{V_{44}} + \frac{E_3V_{50}}{V_8} \\ E_{40} &= \frac{V_{49}V_5}{V_1} + \frac{E_{10}V_{51}}{V_{15}} + \frac{E_{34}V_{52}}{V_{44}} + \frac{E_4V_{50}}{V_8} \\ E_{41} &= \frac{E_{11}V_{51}}{V_{15}} + \frac{E_{35}V_{52}}{V_{44}} + \frac{V_{49}V_6}{V_1} + \frac{E_5V_{50}}{V_8} \\ E_{42} &= \frac{E_{12}V_{51}}{V_{15}} + \frac{E_{36}V_{52}}{V_{44}} + \frac{V_{49}V_7}{V_1} + \frac{E_6V_{50}}{V_8}\end{aligned} \quad (2.141)$$

2.8.3 Solución de la velocidad ángulo θ_{76i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.105):

$$\dot{\theta}_{76i} = \frac{c\theta_{76i}^2((\dot{Y}_iX_i - \dot{X}_iY_i)c\theta_{65i} - \dot{\theta}_{65i}X_iY_is\theta_{65i})}{X_i^2} \quad (2.142)$$

Donde X_i, Y_i se encuentran en las ecuaciones (2.100) y $\dot{X}_i, \dot{Y}_i$ en las ecuaciones (2.138). Sustituyendo $X_i, Y_i, \dot{X}_i, \dot{Y}_i$ en (2.142), agrupando y simplificando:

$$\dot{\theta}_{76i} = \frac{1}{V_{53}} (\dot{\theta}_{21}V_{54} + \dot{\theta}_{43}V_{55} + \dot{\theta}_{54}V_{56} + \dot{\theta}_{21i}V_{57} + \dot{\theta}_{65i}V_{58}) \quad (2.143)$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.135), (2.140) en (2.143) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{76i} = \frac{1}{V_{53}} (\dot{x}E_{43} + \dot{y}E_{44} + \dot{z}E_{45} + \dot{\psi}E_{46} + \dot{\theta}E_{47} + \dot{\phi}E_{48}) \quad (2.144)$$

Dónde:

$$\begin{aligned} E_{43} &= \frac{V_2V_{54}}{V_1} + \frac{E_7V_{56}}{V_{15}} + \frac{E_{31}V_{57}}{V_{44}} + \frac{E_{37}V_{58}}{V_{48}} + \frac{E_1V_{55}}{V_8} \\ E_{44} &= \frac{V_3V_{54}}{V_1} + \frac{E_8V_{56}}{V_{15}} + \frac{E_{32}V_{57}}{V_{44}} + \frac{E_{38}V_{58}}{V_{48}} + \frac{E_2V_{55}}{V_8} \\ E_{45} &= \frac{V_4V_{54}}{V_1} + \frac{E_9V_{56}}{V_{15}} + \frac{E_{33}V_{57}}{V_{44}} + \frac{E_{39}V_{58}}{V_{48}} + \frac{E_3V_{55}}{V_8} \\ E_{46} &= \frac{V_5V_{54}}{V_1} + \frac{E_{10}V_{56}}{V_{15}} + \frac{E_{34}V_{57}}{V_{44}} + \frac{E_{40}V_{58}}{V_{48}} + \frac{E_4V_{55}}{V_8} \\ E_{47} &= \frac{E_{11}V_{56}}{V_{15}} + \frac{E_{35}V_{57}}{V_{44}} + \frac{E_{41}V_{58}}{V_{48}} + \frac{V_{54}V_6}{V_1} + \frac{E_5V_{55}}{V_8} \\ E_{48} &= \frac{E_{12}V_{56}}{V_{15}} + \frac{E_{36}V_{57}}{V_{44}} + \frac{E_{42}V_{58}}{V_{48}} + \frac{V_{54}V_7}{V_1} + \frac{E_6V_{55}}{V_8} \end{aligned} \quad (2.145)$$

2.8.4 Solución de la velocidad ángulo θ_{1110i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.114):

$$\dot{\theta}_{1110i} = \frac{(d_{32}\dot{d}_{31} - d_{31}\dot{d}_{32})c\theta_{1110i}^2}{d_{31}^2} \quad (2.146)$$

Donde d_{31} y d_{32} se encuentran en las ecuaciones (2.115) y además:

$$\begin{aligned}
\dot{d}_{31} = & -s(\beta_{1413i} - \theta_{54})(c\theta_{65i}(\dot{\theta}_{65i} + \dot{\theta}_{21i}s\beta_{54i})s\theta_{21i} + c\theta_{21i}(\dot{\theta}_{21i} \\
& + \dot{\theta}_{65i}s\beta_{54i})s\theta_{65i}) - \dot{\theta}_{54}c(\beta_{1413i} - \theta_{54})(c\theta_{21i}c\theta_{65i}s\beta_{54i} \\
& - s\theta_{21i}s\theta_{65i}) + \dot{\theta}_{43}c(\beta_{1413i} - \theta_{54})s\theta_{43}(c\beta_{54i}c\theta_{65i}s\theta_{21} \\
& - c\theta_{21}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})) + \dot{\theta}_{54}c\theta_{43}s(\beta_{1413i} \\
& - \theta_{54})(-c\beta_{54i}c\theta_{65i}s\theta_{21} + c\theta_{21}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})) \\
& - \dot{\theta}_{43}c\theta_{43}c(\beta_{1413i} - \theta_{54})(c\beta_{54i}c\theta_{21}c\theta_{65i} + s\theta_{21}(c\theta_{65i}s\beta_{54i}s\theta_{21i} \\
& + c\theta_{21i}s\theta_{65i})) - \dot{\theta}_{54}s\theta_{43}s(\beta_{1413i} - \theta_{54})(c\beta_{54i}c\theta_{21}c\theta_{65i} \\
& + s\theta_{21}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i})) - c(\beta_{1413i} \\
& - \theta_{54})s\theta_{43}(c\theta_{21i}(c\theta_{65i}(\dot{\theta}_{65i} + \dot{\theta}_{21i}s\beta_{54i})s\theta_{21} + \dot{\theta}_{21}c\theta_{21}s\theta_{65i}) \\
& - c\beta_{54i}(\dot{\theta}_{21}c\theta_{65i}s\theta_{21} + \dot{\theta}_{65i}c\theta_{21}s\theta_{65i}) + s\theta_{21i}(\dot{\theta}_{21}c\theta_{21}c\theta_{65i}s\beta_{54i} \\
& - (\dot{\theta}_{21i} + \dot{\theta}_{65i}s\beta_{54i})s\theta_{21}s\theta_{65i})) + c\theta_{43}c(\beta_{1413i} \\
& - \theta_{54})(-\dot{\theta}_{21}s\theta_{21}(c\theta_{65i}s\beta_{54i}s\theta_{21i} + c\theta_{21i}s\theta_{65i}) \\
& + c\beta_{54i}(-\dot{\theta}_{21}c\theta_{21}c\theta_{65i} + \dot{\theta}_{65i}s\theta_{21}s\theta_{65i}) + c\theta_{21}(c\theta_{21i}c\theta_{65i}(\dot{\theta}_{65i} \\
& + \dot{\theta}_{21i}s\beta_{54i}) - (\dot{\theta}_{21i} + \dot{\theta}_{65i}s\beta_{54i})s\theta_{21i}s\theta_{65i}))
\end{aligned} \tag{2.147}$$

$$\begin{aligned}
\dot{d}_{32} = & c(\beta_{1413i} - \theta_{54})(c\theta_{65i}(\dot{\theta}_{65i} + (\dot{\theta}_{21i} - \dot{\theta}_{54}c(\theta_{21} + \theta_{43}))s\beta_{54i})s\theta_{21i} \\
& + c\theta_{21i}(\dot{\theta}_{21i} - \dot{\theta}_{54}c(\theta_{21} + \theta_{43}) + \dot{\theta}_{65i}s\beta_{54i})s\theta_{65i}) + s(\beta_{1413i} \\
& - \theta_{54})(-c\theta_{21i}(c\theta_{65i}(\dot{\theta}_{65i}s\theta_{21}s\theta_{43} + s\beta_{54i}(\dot{\theta}_{54} + \dot{\theta}_{21i}s\theta_{21}s\theta_{43})) \\
& + (\dot{\theta}_{21} + \dot{\theta}_{43})c\theta_{43}s\theta_{21}s\theta_{65i}) + s\theta_{21i}(-(\dot{\theta}_{21} \\
& + \dot{\theta}_{43})c\theta_{43}c\theta_{65i}s\beta_{54i}s\theta_{21} + (\dot{\theta}_{54} + (\dot{\theta}_{21i} \\
& + \dot{\theta}_{65i}s\beta_{54i})s\theta_{21}s\theta_{43})s\theta_{65i}) + c\theta_{21}(-s\theta_{21i}((\dot{\theta}_{21} \\
& + \dot{\theta}_{43})c\theta_{65i}s\beta_{54i}s\theta_{43} + c\theta_{43}(\dot{\theta}_{21i} + \dot{\theta}_{65i}s\beta_{54i})s\theta_{65i}) \\
& + c\theta_{21i}(c\theta_{43}c\theta_{65i}(\dot{\theta}_{65i} + \dot{\theta}_{21i}s\beta_{54i}) - (\dot{\theta}_{21} + \dot{\theta}_{43})s\theta_{43}s\theta_{65i}))) \\
& + c\beta_{54i}(c\theta_{43}(\dot{\theta}_{54}c(\beta_{1413i} - \theta_{54})c\theta_{65i}s\theta_{21} - s(\beta_{1413i} - \theta_{54})(\dot{\theta}_{21i} \\
& + \dot{\theta}_{43})c\theta_{21}c\theta_{65i} - \dot{\theta}_{65i}s\theta_{21}s\theta_{65i})) + s\theta_{43}((\dot{\theta}_{21} \\
& + \dot{\theta}_{43})c\theta_{65i}s\theta_{21}s(\beta_{1413i} - \theta_{54}) + c\theta_{21}(\dot{\theta}_{54}c(\beta_{1413i} - \theta_{54})c\theta_{65i} \\
& + \dot{\theta}_{65i}s(\beta_{1413i} - \theta_{54})s\theta_{65i})))
\end{aligned}$$

Sustituyendo las ecuaciones $\dot{d}_{31}$, $\dot{d}_{32}$, $\dot{d}_{31}$, $\dot{d}_{32}$ en (2.146), agrupando y simplificando:

$$\dot{\theta}_{1110i} = \frac{1}{V_{59}} (\dot{\theta}_{21}V_{60} + \dot{\theta}_{43}V_{61} + \dot{\theta}_{54}V_{62} + \dot{\theta}_{21i}V_{63} + \dot{\theta}_{65i}V_{64}) \tag{2.148}$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.135), (2.140), (2.144) en (2.148) y al agrupar en $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, $\ddot{\psi}$, $\ddot{\theta}$, $\ddot{\phi}$ se tiene:

$$\dot{\theta}_{1110i} = \frac{1}{V_{59}} (\dot{x}E_{49} + \dot{y}E_{50} + \dot{z}E_{51} + \dot{\psi}E_{52} + \dot{\theta}E_{53} + \dot{\phi}E_{54}) \tag{2.149}$$

Dónde:

$$\begin{aligned}
E_{49} &= \frac{V_2 V_{60}}{V_1} + \frac{E_7 V_{62}}{V_{15}} + \frac{E_{31} V_{63}}{V_{44}} + \frac{E_{37} V_{64}}{V_{48}} + \frac{E_1 V_{61}}{V_8} \\
E_{50} &= \frac{V_3 V_{60}}{V_1} + \frac{E_8 V_{62}}{V_{15}} + \frac{E_{32} V_{63}}{V_{44}} + \frac{E_{38} V_{64}}{V_{48}} + \frac{E_2 V_{61}}{V_8} \\
E_{51} &= \frac{V_4 V_{60}}{V_1} + \frac{E_9 V_{62}}{V_{15}} + \frac{E_{33} V_{63}}{V_{44}} + \frac{E_{39} V_{64}}{V_{48}} + \frac{E_3 V_{61}}{V_8} \\
E_{52} &= \frac{V_5 V_{60}}{V_1} + \frac{E_{10} V_{62}}{V_{15}} + \frac{E_{34} V_{63}}{V_{44}} + \frac{E_{40} V_{64}}{V_{48}} + \frac{E_4 V_{61}}{V_8} \\
E_{53} &= \frac{V_6 V_{60}}{V_1} + \frac{E_{11} V_{62}}{V_{15}} + \frac{E_{35} V_{63}}{V_{44}} + \frac{E_{41} V_{64}}{V_{48}} + \frac{E_5 V_{61}}{V_8} \\
E_{54} &= \frac{E_{12} V_{62}}{V_{15}} + \frac{E_{36} V_{63}}{V_{44}} + \frac{E_{42} V_{64}}{V_{48}} + \frac{V_{60} V_7}{V_1} + \frac{E_6 V_{61}}{V_8}
\end{aligned} \tag{2.150}$$

2.8.5 Solución de la velocidad ángulo θ_{109i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.123):

$$\dot{\theta}_{109i} = \frac{(\dot{e}_{33} e_{31} - \dot{e}_{31} e_{33}) c \theta_{109i}^2}{e_{33}^2} \tag{2.151}$$

Al derivar la ecuación (2.124) se obtiene $\dot{e}_{31}$ y $\dot{e}_{33}$, que se encuentran en el apéndice C. Sustituyendo las ecuaciones (2.124) así como $\dot{e}_{31}$ y $\dot{e}_{33}$ en la ecuación (2.151), simplificando y agrupando:

$$\begin{aligned}
\dot{\theta}_{109i} &= \frac{1}{V_{65}} \left(\dot{\theta}_{21} V_{66} + \dot{\theta}_{43} V_{66} + \dot{\theta}_{54} V_{67} + \dot{\theta}_{21i} V_{68} + \dot{\theta}_{65i} V_{69} \right. \\
&\quad \left. + \dot{\theta}_{1110i} V_{70} \right)
\end{aligned} \tag{2.152}$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.135), (2.140), (2.144), (2.149) en (2.152) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{109i} = \frac{1}{V_{65}} (\dot{x} E_{55} + \dot{y} E_{56} + \dot{z} E_{57} + \dot{\psi} E_{58} + \dot{\theta} E_{59} + \dot{\phi} E_{60}) \tag{2.153}$$

Dónde:

$$\begin{aligned}
E_{55} &= \frac{V_2 V_{66}}{V_1} + \frac{E_7 V_{67}}{V_{15}} + \frac{E_{31} V_{68}}{V_{44}} + \frac{E_{37} V_{69}}{V_{48}} + \frac{E_{49} V_{70}}{V_{59}} + \frac{E_1 V_{66}}{V_8} \\
E_{56} &= \frac{V_3 V_{66}}{V_1} + \frac{E_8 V_{67}}{V_{15}} + \frac{E_{32} V_{68}}{V_{44}} + \frac{E_{38} V_{69}}{V_{48}} + \frac{E_{50} V_{70}}{V_{59}} + \frac{E_2 V_{66}}{V_8} \\
E_{57} &= \frac{V_4 V_{66}}{V_1} + \frac{E_9 V_{67}}{V_{15}} + \frac{E_{33} V_{68}}{V_{44}} + \frac{E_{39} V_{69}}{V_{48}} + \frac{E_{51} V_{70}}{V_{59}} + \frac{E_3 V_{66}}{V_8} \\
E_{58} &= \frac{V_5 V_{66}}{V_1} + \frac{E_{10} V_{67}}{V_{15}} + \frac{E_{34} V_{68}}{V_{44}} + \frac{E_{40} V_{69}}{V_{48}} + \frac{E_{52} V_{70}}{V_{59}} + \frac{E_4 V_{66}}{V_8} \\
E_{59} &= \frac{V_6 V_{66}}{V_1} + \frac{E_{11} V_{67}}{V_{15}} + \frac{E_{35} V_{68}}{V_{44}} + \frac{E_{41} V_{69}}{V_{48}} + \frac{E_{53} V_{70}}{V_{59}} + \frac{E_5 V_{66}}{V_8} \\
E_{60} &= \frac{E_{12} V_{67}}{V_{15}} + \frac{E_{36} V_{68}}{V_{44}} + \frac{E_{42} V_{69}}{V_{48}} + \frac{V_{66} V_7}{V_1} + \frac{E_{54} V_{70}}{V_{59}} + \frac{E_6 V_{66}}{V_8}
\end{aligned} \tag{2.154}$$

2.8.6 Solución de la velocidad ángulo θ_{98i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.131):

$$\dot{\theta}_{98i} = - \frac{(-\dot{f}_{21} f_{11} + \dot{f}_{11} f_{21}) c \theta_{98i}^2}{f_{11}^2} \tag{2.155}$$

Al derivar f_{11} y f_{21} se obtiene $\dot{f}_{11}$ y $\dot{f}_{21}$, que se encuentran en el apéndice C. Sustituyendo las ecuaciones f_{11} y f_{21} así como $\dot{f}_{11}$ y $\dot{f}_{21}$ en la ecuación (2.155), simplificando y agrupando:

$$\begin{aligned}
\dot{\theta}_{98i} &= \frac{1}{V_{71}} (\dot{\theta}_{21} V_{72} + \dot{\theta}_{43} V_{73} + \dot{\theta}_{54} V_{74} + \dot{\theta}_{21i} V_{75} + \dot{\theta}_{65i} V_{76} + \dot{\theta}_{76i} V_{77} \\
&\quad + \dot{\theta}_{1110i} V_{78} + \dot{\theta}_{109i} V_{79})
\end{aligned} \tag{2.156}$$

Donde los coeficientes se encuentran en el apéndice D. Finalmente sustituyendo las ecuaciones (2.51), (2.55), (2.59), (2.135), (2.140), (2.144), (2.149), (2.153) en (2.156) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\dot{\theta}_{98i} = \frac{1}{V_{71}} (\dot{x} E_{61} + \dot{y} E_{62} + \dot{z} E_{63} + \dot{\psi} E_{64} + \dot{\theta} E_{65} + \dot{\phi} E_{66}) \tag{2.157}$$

Dónde:

$$\begin{aligned}
E_{61} &= \frac{V_2 V_{72}}{V_1} + \frac{E_7 V_{74}}{V_{15}} + \frac{E_{31} V_{75}}{V_{44}} + \frac{E_{37} V_{76}}{V_{48}} + \frac{E_{43} V_{77}}{V_{53}} + \frac{E_{49} V_{78}}{V_{59}} + \frac{E_{55} V_{79}}{V_{65}} + \frac{E_1 V_{73}}{V_8} \\
E_{62} &= \frac{V_3 V_{72}}{V_1} + \frac{E_8 V_{74}}{V_{15}} + \frac{E_{32} V_{75}}{V_{44}} + \frac{E_{38} V_{76}}{V_{48}} + \frac{E_{44} V_{77}}{V_{53}} + \frac{E_{50} V_{78}}{V_{59}} + \frac{E_{56} V_{79}}{V_{65}} + \frac{E_2 V_{73}}{V_8} \\
E_{63} &= \frac{V_4 V_{72}}{V_1} + \frac{E_9 V_{74}}{V_{15}} + \frac{E_{33} V_{75}}{V_{44}} + \frac{E_{39} V_{76}}{V_{48}} + \frac{E_{45} V_{77}}{V_{53}} + \frac{E_{51} V_{78}}{V_{59}} + \frac{E_{57} V_{79}}{V_{65}} + \frac{E_3 V_{73}}{V_8} \\
E_{64} &= \frac{V_5 V_{72}}{V_1} + \frac{E_{10} V_{74}}{V_{15}} + \frac{E_{34} V_{75}}{V_{44}} + \frac{E_{40} V_{76}}{V_{48}} + \frac{E_{46} V_{77}}{V_{53}} + \frac{E_{52} V_{78}}{V_{59}} + \frac{E_{58} V_{79}}{V_{65}} + \frac{E_4 V_{73}}{V_8} \\
E_{65} &= \frac{V_6 V_{72}}{V_1} + \frac{E_{11} V_{74}}{V_{15}} + \frac{E_{35} V_{75}}{V_{44}} + \frac{E_{41} V_{76}}{V_{48}} + \frac{E_{47} V_{77}}{V_{53}} + \frac{E_{53} V_{78}}{V_{59}} + \frac{E_{59} V_{79}}{V_{65}} + \frac{E_5 V_{73}}{V_8} \\
E_{66} &= \frac{V_7 V_{72}}{V_1} + \frac{E_{12} V_{74}}{V_{15}} + \frac{E_{36} V_{75}}{V_{44}} + \frac{E_{42} V_{76}}{V_{48}} + \frac{E_{48} V_{77}}{V_{53}} + \frac{E_{54} V_{78}}{V_{59}} + \frac{E_{60} V_{79}}{V_{65}} + \frac{E_6 V_{73}}{V_8}
\end{aligned} \tag{2.158}$$

2.9 Calculo de la aceleración para las cadenas i.

Para el cálculo de la aceleración se tiene como variables de entrada la aceleración del efector ($\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$) y las variables a calcular son las aceleraciones de los ángulos de las cadenas i es decir $\ddot{\theta}_{21i}, \ddot{\theta}_{65i}, \ddot{\theta}_{76i}, \ddot{\theta}_{1110i}, \ddot{\theta}_{109i}, \ddot{\theta}_{98i}$. Para esta sección se asume que la posición y la velocidad de los ángulos son conocidas debido a que se calcularon en secciones anteriores.

2.9.1 Solución de la aceleración ángulo θ_{21i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.132):

$$\ddot{\theta}_{21i} = \frac{(-\ddot{C}_{1i} - \ddot{A}_{1i} c \theta_{21i} - 2\dot{B}_{1i} \dot{\theta}_{21i} c \theta_{21i} + A_{1i} \dot{\theta}_{21i}^2 c \theta_{21i} - \ddot{B}_{1i} s \theta_{21i} + 2\dot{A}_{1i} \dot{\theta}_{21i} s \theta_{21i} + B_{1i} \dot{\theta}_{21i}^2 s \theta_{21i})}{(B_{1i} c \theta_{21i} - A_{1i} s \theta_{21i})} \tag{2.159}$$

Dónde:

$$\begin{aligned}
\ddot{A}_{1i} &= x_{43i} (-2x_{32} (\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) - 2z_{1312i} ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta_{21} + \ddot{\theta}_{43} \\
&\quad - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s \theta_{21} + \ddot{\theta}_{43}) \\
&\quad - x_{1211i} (-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c \beta_{1413i} - \ddot{\theta}_{21} - \ddot{\theta}_{43} - \ddot{\theta}_{54} + (\ddot{\theta}_{21} \\
&\quad + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s \beta_{1413i} - \ddot{\theta}_{21} - \ddot{\theta}_{43} - \ddot{\theta}_{54}) \\
&\quad + x_{65} (-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c \theta_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54} - (\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad - \ddot{\theta}_{54}) s \theta_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) - x_{1211i} (-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c \beta_{1413i} \\
&\quad + \ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54} - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) s \beta_{1413i} + \ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad - \ddot{\theta}_{54}) + x_{65} (-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c \theta_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54} - (\ddot{\theta}_{21} \\
&\quad + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s \theta_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54})) \\
\ddot{B}_{1i} &= -2x_{43i} (\ddot{\theta}_{54} x_{1211i} c \beta_{1413i} - \ddot{\theta}_{54} - \ddot{\theta}_{54} x_{65} c \theta_{54} + \dot{\theta}_{54}^2 (x_{1211i} s \beta_{1413i} \\
&\quad - \ddot{\theta}_{54} + x_{65} s \theta_{54}))
\end{aligned} \tag{2.160}$$

$$\begin{aligned}
\ddot{C}_{1i} = & 2x_{32}z_{10}(\ddot{\theta}_{21}c\theta_{21} - \dot{\theta}_{21}^2 s\theta_{21}) + 2x_{32}z_{32i}(\ddot{\theta}_{21}c\theta_{21} - \dot{\theta}_{21}^2 s\theta_{21}) \\
& + 2x_{32}z_{1312i}(\ddot{\theta}_{43}c\theta_{43} - \dot{\theta}_{43}^2 s\theta_{43}) + 2z_{1312i}(z_{10} \\
& + z_{32i})(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c\theta_{21} + \theta_{43} - (\ddot{\theta}_{21} + \ddot{\theta}_{43})s\theta_{21} + \theta_{43}) \\
& - 2\text{signox}_{1211i}y_{10i}(\ddot{\theta}_{54}c\beta_{1413i} - \theta_{54} + \dot{\theta}_{54}^2 s\beta_{1413i} - \theta_{54}) \\
& + x_{1211i}x_{32}(-(\dot{\theta}_{43} + \dot{\theta}_{54})^2 c\beta_{1413i} - \theta_{43} - \theta_{54} + (\ddot{\theta}_{43} \\
& + \ddot{\theta}_{54})s\beta_{1413i} - \theta_{43} - \theta_{54}) + x_{1211i}z_{10}(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
& + \ddot{\theta}_{54})c\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\beta_{1413i} \\
& - \theta_{21} - \theta_{43} - \theta_{54}) + x_{1211i}z_{32i}(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54})c\beta_{1413i} \\
& - \theta_{21} - \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\beta_{1413i} - \theta_{21} - \theta_{43} \\
& - \theta_{54}) - x_{32}x_{65}(-(\dot{\theta}_{43} - \dot{\theta}_{54})^2 c\theta_{43} - \theta_{54} + (-\ddot{\theta}_{43} + \ddot{\theta}_{54})s\theta_{43} \\
& - \theta_{54}) + x_{1211i}x_{32}(-(\dot{\theta}_{43} - \dot{\theta}_{54})^2 c\beta_{1413i} + \theta_{43} - \theta_{54} + (-\ddot{\theta}_{43} \\
& + \ddot{\theta}_{54})s\beta_{1413i} + \theta_{43} - \theta_{54}) + x_{65}z_{10}((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})c\theta_{21} \\
& + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} - \theta_{54}) \\
& + x_{65}z_{32i}((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})c\theta_{21} + \theta_{43} - \theta_{54} \\
& - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} - \theta_{54}) - x_{1211i}z_{10}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
& - \ddot{\theta}_{54})c\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\beta_{1413i} \\
& + \theta_{21} + \theta_{43} - \theta_{54}) - x_{1211i}z_{32i}((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})c\beta_{1413i} \\
& + \theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\beta_{1413i} + \theta_{21} + \theta_{43} \\
& - \theta_{54}) + 2\text{signox}_{65}y_{10i}(\ddot{\theta}_{54}c\theta_{54} - \dot{\theta}_{54}^2 s\theta_{54}) \\
& - x_{32}x_{65}(-(\dot{\theta}_{43} + \dot{\theta}_{54})^2 c\theta_{43} + \theta_{54} - (\ddot{\theta}_{43} + \ddot{\theta}_{54})s\theta_{43} + \theta_{54}) \\
& + x_{65}z_{10}((\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54})c\theta_{21} + \theta_{43} + \theta_{54} \\
& - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} + \theta_{54}) + x_{65}z_{32i}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
& + \ddot{\theta}_{54})c\theta_{21} + \theta_{43} + \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} \\
& + \theta_{54})
\end{aligned}$$

Sustituyendo $A_{1i}, B_{1i}, C_{1i}, \dot{A}_{1i}, \dot{B}_{1i}, \dot{C}_{1i}, \ddot{A}_{1i}, \ddot{B}_{1i}, \ddot{C}_{1i}$ en la ecuación (2.160) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{21i} = \frac{1}{V_{44}} (\ddot{x}G_{47} + \ddot{y}G_{48} + \ddot{z}G_{49} + \ddot{\psi}G_{50} + \ddot{\theta}G_{51} + \ddot{\phi}G_{52} + G_{53}) \quad (2.161)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.9.2 Solución de la aceleración ángulo θ_{65i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.137):

$$\ddot{\theta}_{65i} = \frac{-\ddot{Z}_i c\theta_{65i} - 2\dot{X}_i \dot{\theta}_{65i} c\theta_{65i} + \dot{\theta}_{65i}^2 Z_i c\theta_{65i} - \ddot{X}_i s\theta_{65i} + 2\dot{Z}_i \dot{\theta}_{65i} s\theta_{65i} + \dot{\theta}_{65i}^2 X_i s\theta_{65i}}{X_i c\theta_{65i} - Z_i s\theta_{65i}} \quad (2.162)$$

Dónde:

$$\begin{aligned}
\ddot{X}_i &= \frac{1}{2}(-2x_{32}(\dot{\theta}_{21}^2 c\theta_{21} + \ddot{\theta}_{21}s\theta_{21}) + 2x_{43i}(\dot{\theta}_{21i}^2 c\theta_{21i} + \ddot{\theta}_{21i}s\theta_{21i}) \\
&\quad - 2z_{1312i}((\ddot{\theta}_{21} + \ddot{\theta}_{43})c\theta_{21} + \theta_{43} - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s\theta_{21} + \theta_{43}) \\
&\quad - x_{1211i}(-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54} + (\ddot{\theta}_{21} \\
&\quad + \ddot{\theta}_{43} + \ddot{\theta}_{54})s\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54}) \\
&\quad + x_{65}(-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c\theta_{21} + \theta_{43} - \theta_{54} - (\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad - \ddot{\theta}_{54})s\theta_{21} + \theta_{43} - \theta_{54}) - x_{1211i}(-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c\beta_{1413i} \\
&\quad + \theta_{21} + \theta_{43} - \theta_{54} - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})s\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) \\
&\quad + x_{65}(-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c\theta_{21} + \theta_{43} + \theta_{54} - (\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad + \ddot{\theta}_{54})s\theta_{21} + \theta_{43} + \theta_{54})) \\
\ddot{Y}_i &= -c\beta_{54i}(\ddot{\theta}_{21i}x_{43i}c\theta_{21i} + \ddot{\theta}_{54}x_{1211i}c\beta_{1413i} - \theta_{54} - \ddot{\theta}_{54}x_{65}c\theta_{54} \\
&\quad - \dot{\theta}_{21i}^2 x_{43i}s\theta_{21i} + \dot{\theta}_{54}^2 x_{1211i}s\beta_{1413i} - \theta_{54} + \dot{\theta}_{54}^2 x_{65}s\theta_{54}) \\
&\quad + \frac{1}{2}s\beta_{54i}(-2\ddot{\theta}_{21}x_{32}c\theta_{21} + 2\dot{\theta}_{21}^2 x_{32}s\theta_{21} \\
&\quad - 2z_{1312i}(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c\theta_{21} + \theta_{43} - (\ddot{\theta}_{21} + \ddot{\theta}_{43})s\theta_{21} + \theta_{43}) \\
&\quad - x_{1211i}(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54})c\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54} \\
&\quad - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54}) - x_{65}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad - \ddot{\theta}_{54})c\theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} - \theta_{54}) \\
&\quad + x_{1211i}((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})c\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} \\
&\quad - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) - x_{65}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad + \ddot{\theta}_{54})c\theta_{21} + \theta_{43} + \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} + \theta_{54})) \\
\ddot{Z}_i &= s\beta_{54i}(\ddot{\theta}_{21i}x_{43i}c\theta_{21i} + \ddot{\theta}_{54}x_{1211i}c\beta_{1413i} - \theta_{54} - \ddot{\theta}_{54}x_{65}c\theta_{54} \\
&\quad - \dot{\theta}_{21i}^2 x_{43i}s\theta_{21i} + \dot{\theta}_{54}^2 x_{1211i}s\beta_{1413i} - \theta_{54} + \dot{\theta}_{54}^2 x_{65}s\theta_{54}) \\
&\quad + \frac{1}{2}c\beta_{54i}(-2\ddot{\theta}_{21}x_{32}c\theta_{21} + 2\dot{\theta}_{21}^2 x_{32}s\theta_{21} \\
&\quad - 2z_{1312i}(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c\theta_{21} + \theta_{43} - (\ddot{\theta}_{21} + \ddot{\theta}_{43})s\theta_{21} + \theta_{43}) \\
&\quad - x_{1211i}(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54})c\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54} \\
&\quad - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\beta_{1413i} - \theta_{21} - \theta_{43} - \theta_{54}) - x_{65}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad - \ddot{\theta}_{54})c\theta_{21} + \theta_{43} - \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} - \theta_{54}) \\
&\quad + x_{1211i}((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54})c\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54} \\
&\quad - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s\beta_{1413i} + \theta_{21} + \theta_{43} - \theta_{54}) - x_{65}((\ddot{\theta}_{21} + \ddot{\theta}_{43} \\
&\quad + \ddot{\theta}_{54})c\theta_{21} + \theta_{43} + \theta_{54} - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s\theta_{21} + \theta_{43} + \theta_{54}))
\end{aligned} \tag{2.163}$$

Sustituyendo $X_i, Z_i, \dot{X}_i, \dot{Z}_i, \ddot{X}_i, \ddot{Z}_i$ en la ecuación (2.162) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{65i} = \frac{1}{V_{48}} (\ddot{x}G_{54} + \ddot{y}G_{55} + \ddot{z}G_{56} + \ddot{\psi}G_{57} + \ddot{\theta}G_{58} + \ddot{\phi}G_{59} + G_{60}) \quad (2.164)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.9.3 Solución de la aceleración ángulo θ_{76i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.142):

$$\begin{aligned} \ddot{\theta}_{76i} = -\frac{1}{X^3} c\theta_{76i}^2 ((2\dot{X}\dot{Y}X - 2\dot{X}^2Y + X(-\ddot{Y}X + \ddot{X}Y + \dot{\theta}_{65i}^2 XY))c\theta_{65i} \\ + X((2\dot{Y}\dot{\theta}_{65i}X - 2\dot{X}\dot{\theta}_{65i}Y + \ddot{\theta}_{65i}XY)s\theta_{65i} \\ + 2\dot{\theta}_{76i}^2 X^2 \sec\theta_{76i}^2 \tan\theta_{76i})) \end{aligned} \quad (2.165)$$

Sustituyendo $X_i, Y_i, \dot{X}_i, \dot{Y}_i, \ddot{X}_i, \ddot{Y}_i$ en la ecuación (2.165) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{76i} = \frac{1}{G_{61}} (\ddot{x}G_{62} + \ddot{y}G_{63} + \ddot{z}G_{64} + \ddot{\psi}G_{65} + \ddot{\theta}G_{66} + \ddot{\phi}G_{67} + G_{68}) \quad (2.166)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.9.4 Solución de la aceleración ángulo θ_{1110i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.146):

$$\begin{aligned} \ddot{\theta}_{1110i} = -\frac{1}{d_{31}^3} c\theta_{1110i}^2 (2d_{32}\dot{d}_{31}^2 - 2d_{31}\dot{d}_{31}\dot{d}_{32} - d_{31}d_{32}\ddot{d}_{31} + d_{31}^2\ddot{d}_{32} \\ + 2d_{31}^3\dot{\theta}_{1110i}^2 \sec\theta_{1110i}^2 \tan\theta_{1110i}) \end{aligned} \quad (2.167)$$

Al derivar las ecuaciones (2.147) se obtienen $\ddot{d}_{31}$ y $\ddot{d}_{32}$, que se encuentran en el apéndice C. Sustituyendo $d_{31}, d_{32}, \dot{d}_{31}, \dot{d}_{32}, \ddot{d}_{31}, \ddot{d}_{32}$ en la ecuación (2.167) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{1110i} = \frac{1}{G_{69}} (\ddot{x}G_{70} + \ddot{y}G_{71} + \ddot{z}G_{72} + \ddot{\psi}G_{73} + \ddot{\theta}G_{74} + \ddot{\phi}G_{75} + G_{76}) \quad (2.168)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.9.5 Solución de la aceleración ángulo θ_{109i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.151):

$$\ddot{\theta}_{109i} = -\frac{1}{e_{33}^3} c \theta_{109i}^2 (2 \dot{e}_{33}^2 e_{31} - 2 \dot{e}_{31} \dot{e}_{33} e_{33} + e_{33} (-\ddot{e}_{33} e_{31} + \ddot{e}_{31} e_{33}) + 2 \dot{\theta}_{109i}^2 e_{33}^3 \sec \theta_{109i}^2 \tan \theta_{109i}) \quad (2.169)$$

Donde $\ddot{e}_{31}$ y $\ddot{e}_{33}$ se obtienen al derivar $\dot{e}_{31}$ y $\dot{e}_{33}$ y se encuentran en el apéndice C. Sustituyendo $e_{31}, e_{33}, \dot{e}_{31}, \dot{e}_{33}, \ddot{e}_{31}, \ddot{e}_{33}$ en la ecuación (2.169) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{109i} = \frac{1}{G_{77}} (\ddot{x} G_{78} + \ddot{y} G_{79} + \ddot{z} G_{80} + \ddot{\psi} G_{81} + \ddot{\theta} G_{82} + \ddot{\phi} G_{83} + G_{84}) \quad (2.170)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

2.9.6 Solución de la aceleración ángulo θ_{109i} de las cadenas i.

Se deriva con respecto al tiempo la ecuación (2.155):

$$\ddot{\theta}_{98i} = \frac{c \theta_{98i}^2 (-2 \dot{f}_{11} \dot{f}_{21} f_{11} + \ddot{f}_{21} f_{11}^2 + 2 \dot{f}_{11}^2 f_{21} - \ddot{f}_{11} f_{11} f_{21} - 2 \dot{\theta}_{98i}^2 f_{11}^3 \sec \theta_{98i}^2 \tan \theta_{98i})}{f_{11}^3} \quad (2.171)$$

Donde $\ddot{f}_{11}$ y $\ddot{f}_{21}$ se obtienen al derivar $\dot{f}_{11}$ y $\dot{f}_{21}$ y se encuentran en el apéndice C. Sustituyendo $f_{11}, f_{21}, \dot{f}_{11}, \dot{f}_{21}, \ddot{f}_{11}, \ddot{f}_{21}$ en la ecuación (2.169) y al agrupar en $\ddot{x}, \ddot{y}, \ddot{z}, \ddot{\psi}, \ddot{\theta}, \ddot{\phi}$ se tiene:

$$\ddot{\theta}_{98i} = \frac{1}{G_{85}} (\ddot{x} G_{86} + \ddot{y} G_{87} + \ddot{z} G_{88} + \ddot{\psi} G_{89} + \ddot{\theta} G_{90} + \ddot{\phi} G_{91} + G_{92}) \quad (2.172)$$

Donde los coeficientes de la ecuación se encuentran en el apéndice E.

Capítulo 3

Análisis Dinámico Formulación Euler-Lagrange

3.1 Introducción

En este capítulo, se considera la dinámica del robot, con el fin de poder determinar los torques aplicados por los actuadores en los eslabones de entrada para que el efector final alcance una trayectoria dada.

El método de Euler-Lagrange formula ecuaciones de movimiento usando un conjunto de coordenadas generalizadas (Spong et al., 1989). Esto permite eliminar todas o algunas de las fuerzas de restricción y permite manejar desplazamientos tanto desplazamientos lineales como angulares con un solo tipo de coordenadas. Con el entendimiento de la dinámica del manipulador, es posible diseñar un controlador con mejores características de ejecución que las realizadas con los típicos encontrados usando métodos heurísticos después de que el manipulador ha sido construido.

En este capítulo se empleará la siguiente notación:

La función Lagrangiana es definida como la diferencia entre la energía cinética y la energía potencial de un sistema como:

$$L = K - U \quad 3.1$$

Donde K es la energía cinética definida como:

$$K = \frac{1}{2} m \mathbf{v}^T \mathbf{v} + \boldsymbol{\omega}^T \mathbf{I} \boldsymbol{\omega} \quad 3.2$$

Y la energía potencial como:

$$U = -m \mathbf{g}^T \mathbf{r}_G \quad 3.3$$

La energía cinética depende de la velocidad de los eslabones del manipulador, mientras que la energía potencial depende únicamente de la localización de los eslabones. La ecuación de Lagrange de movimiento es formulada en términos de la función Lagrangiana como:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j \quad 3.4$$

El término Q_j es conocido como el vector de fuerzas generalizadas y se obtendrá a partir de expresiones, que involucren los torques y las coordenadas generalizadas.

3.2 Velocidad de Centros de Gravedad y Velocidad Angular para cadena 0

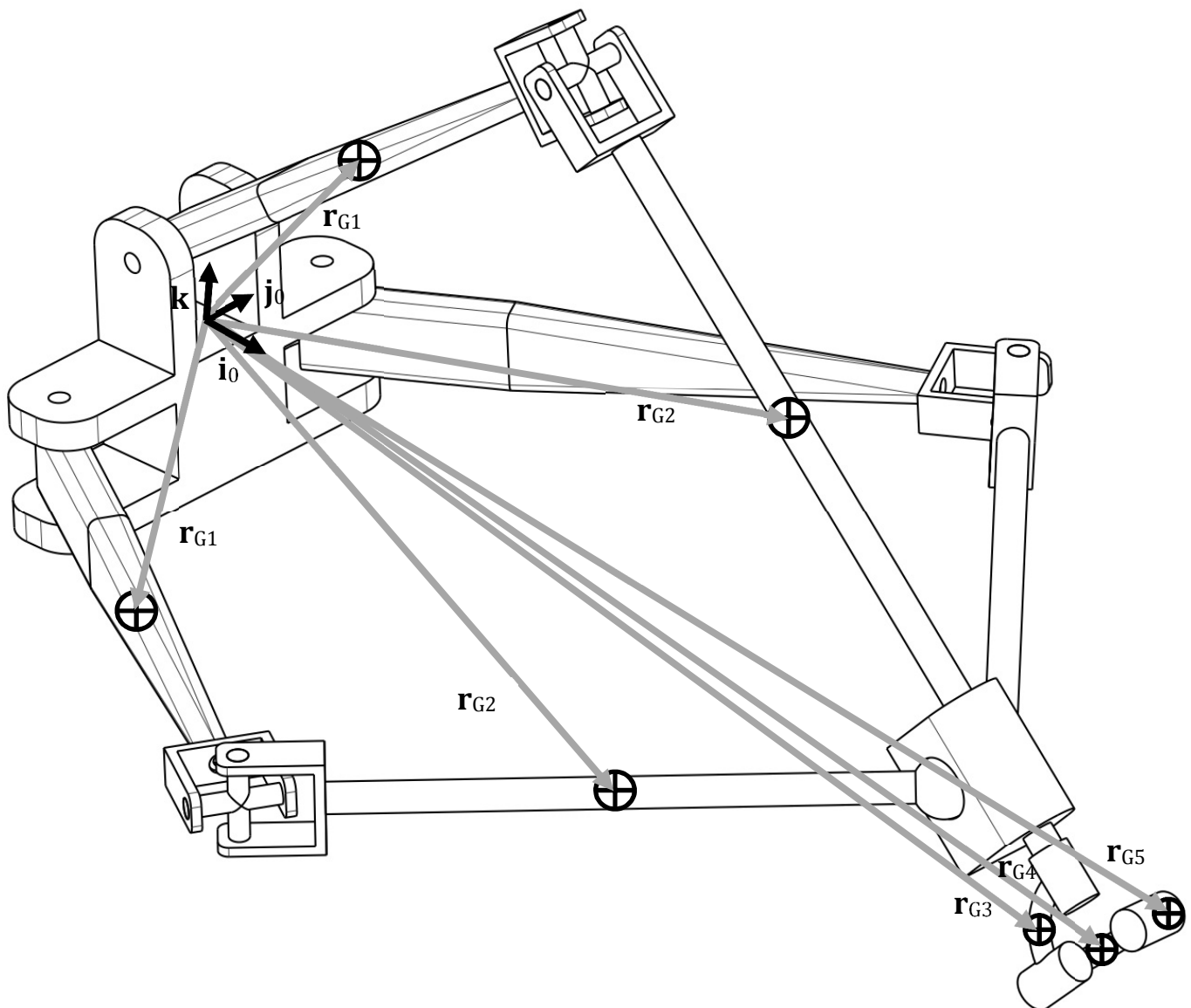


Figura 3-1 Centros de gravedad a partir de la base inercial.

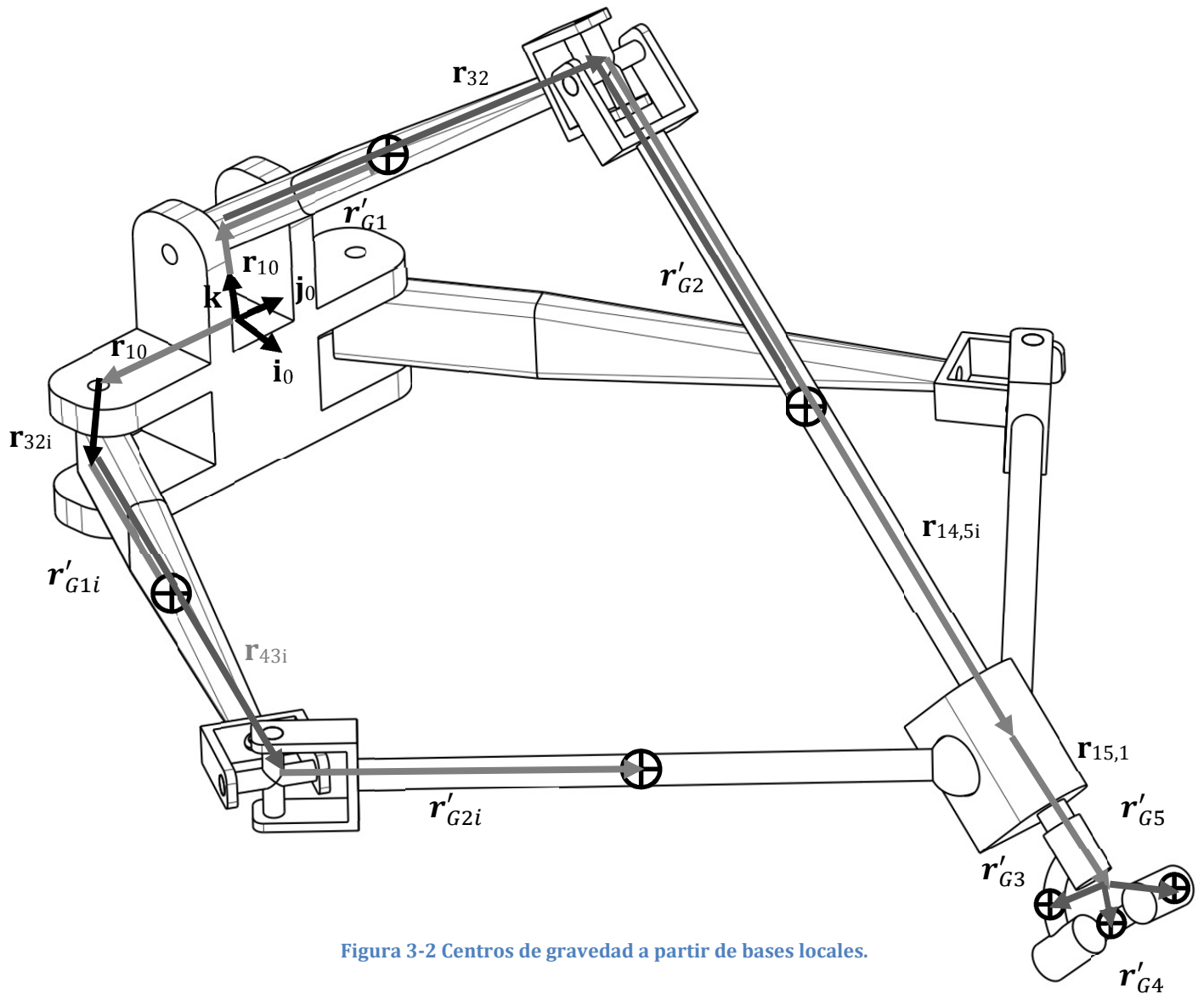


Figura 3-2 Centros de gravedad a partir de bases locales.

Con base en las figuras 3-1 y 3-2 se construye las ecuaciones vectoriales para la posición de los centros de gravedad de la cadena 0:

$$\begin{aligned}
 \mathbf{r}_{G1}^0 &= \mathbf{r}_{10}^0 + \mathbf{r}_{G1'}^0 \\
 \mathbf{r}_{G2}^0 &= \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0 \\
 \mathbf{r}_{G3}^0 &= \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0 \\
 \mathbf{r}_{G4}^0 &= \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0 \\
 \mathbf{r}_{G5}^0 &= \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{r}_{10}^0 &= z_{10} \mathbf{k}_0 \\
 \mathbf{r}_{32}^0 &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 \\
 \mathbf{r}_{14,5i}^0 &= \mathbf{R}_5^0 \mathbf{r}_{14,5i}^5
 \end{aligned}$$

$$\mathbf{r}_{15,14i}^0 = \mathbf{R}_{14}^0 \mathbf{r}_{15,14i}^{14}$$

$$\begin{aligned}\mathbf{r}_{G1'}^0 &= \mathbf{R}_2^0 \mathbf{r}_{G1'}^2 \\ \mathbf{r}_{G2'}^0 &= \mathbf{R}_5^0 \mathbf{r}_{G2'}^5 \\ \mathbf{r}_{G3'}^0 &= \mathbf{R}_{16}^0 \mathbf{r}_{G3'}^{16} \\ \mathbf{r}_{G4'}^0 &= \mathbf{R}_{17}^0 \mathbf{r}_{G4'}^{17} \\ \mathbf{r}_{G5'}^0 &= \mathbf{R}_{18}^0 \mathbf{r}_{G5'}^{18}\end{aligned}$$

$$\begin{aligned}\mathbf{R}_2^0 &= \mathbf{R}_y(\theta_{21}) \\ \mathbf{R}_5^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \\ \mathbf{R}_{16}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \\ \mathbf{R}_{17}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \\ \mathbf{R}_{18}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17})\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{32}^2 &= [x_{32}, 0, 0]^T \\ \mathbf{r}_{14,5i}^5 &= [x_{14,5i}, 0, 0]^T \\ \mathbf{r}_{15,14i}^{14} &= [x_{15,14i}, 0, 0]^T\end{aligned}$$

$$\begin{aligned}\mathbf{r}_{G1'}^2 &= [x_{G1}, y_{G1}, z_{G1}]^T \\ \mathbf{r}_{G2'}^5 &= [x_{G2}, y_{G2}, z_{G2}]^T \\ \mathbf{r}_{G3'}^{16} &= [x_{G3}, y_{G3}, z_{G3}]^T \\ \mathbf{r}_{G4'}^{17} &= [x_{G4}, y_{G4}, z_{G4}]^T \\ \mathbf{r}_{G5'}^{18} &= [x_{G5}, y_{G5}, z_{G5}]^T\end{aligned}$$

Ahora se definen las ecuaciones para la velocidad de los centros de gravedad para la cadena 0:

$$\begin{aligned}\mathbf{v}_{G1}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{G1'}^0 \\ \mathbf{v}_{G2}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{G2'}^0 \\ \mathbf{v}_{G3}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G3'}^0 \\ \mathbf{v}_{G4}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G4'}^0 \\ \mathbf{v}_{G5}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G5'}^0\end{aligned}$$

$$\begin{aligned}\mathbf{v}_{10}^0 &= 0 \\ \mathbf{v}_{32}^0 &= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 \\ \mathbf{v}_{14,5i}^0 &= \boldsymbol{\omega}_5^0 \times \mathbf{r}_{14,5i}^0 \\ \mathbf{v}_{15,14i}^0 &= \boldsymbol{\omega}_5^0 \times \mathbf{r}_{15,14i}^0\end{aligned}$$

$$\begin{aligned}\mathbf{v}_{G1'}^0 &= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{G1'}^0 \\ \mathbf{v}_{G2'}^0 &= \boldsymbol{\omega}_5^0 \times \mathbf{r}_{G2'}^0 \\ \mathbf{v}_{G3'}^0 &= \boldsymbol{\omega}_{16}^0 \times \mathbf{r}_{G3'}^0\end{aligned}$$

$$\begin{aligned} \mathbf{v}_{G4'}^0 &= \boldsymbol{\omega}_{17}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{v}_{G5'}^0 &= \boldsymbol{\omega}_{18}^0 \times \mathbf{r}_{G5'}^0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}_2^0 &= \boldsymbol{\omega}_{21}^0 \\ \boldsymbol{\omega}_5^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 \\ \boldsymbol{\omega}_{16}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 \\ \boldsymbol{\omega}_{17}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 \\ \boldsymbol{\omega}_{18}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 + \boldsymbol{\omega}_{18,17}^0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}_{21}^0 &= \dot{\theta}_{21} \mathbf{j}_1^0 \\ \boldsymbol{\omega}_{43}^0 &= \dot{\theta}_{43} \mathbf{j}_3^0 \\ \boldsymbol{\omega}_{54}^0 &= \dot{\theta}_{54} \mathbf{k}_4^0 \\ \boldsymbol{\omega}_{16,15}^0 &= \dot{\theta}_{16,15} \mathbf{i}_{15}^0 \\ \boldsymbol{\omega}_{17,16}^0 &= \dot{\theta}_{17,16} \mathbf{k}_{16}^0 \\ \boldsymbol{\omega}_{18,17}^0 &= \dot{\theta}_{18,17} \mathbf{i}_{17}^0 \end{aligned}$$

$$\begin{aligned} \mathbf{j}_1^0 &= \mathbf{j}_0 \\ \mathbf{j}_3^0 &= \mathbf{j}_0 \\ \mathbf{k}_4^0 &= \mathbf{R}_4^0 \mathbf{k}_4^4 \\ \mathbf{i}_{15}^0 &= \mathbf{R}_{15}^0 \mathbf{i}_{15}^{15} \\ \mathbf{k}_{16}^0 &= \mathbf{R}_{16}^0 \mathbf{k}_{16}^{16} \\ \mathbf{i}_{17}^0 &= \mathbf{R}_{17}^0 \mathbf{i}_{17}^{17} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_4^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \\ \mathbf{R}_{15}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \end{aligned}$$

3.2.1 Velocidad de Centro de Gravedad del Cuerpo 1

$$\begin{aligned} \mathbf{v}_{G1}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{G1'}^0 \\ &= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{G1'}^0 \\ &= \boldsymbol{\omega}_{21}^0 \times \mathbf{r}_{G1'}^0 \\ &= (\dot{\theta}_{21} \mathbf{j}_1^0) \times \mathbf{r}_{G1'}^0 \\ &= (\mathbf{j}_1^0 \times \mathbf{r}_{G1'}^0) \dot{\theta}_{21} \\ \mathbf{v}_{G1}^0 &= \mathbf{k}_1 \dot{\theta}_{21} \end{aligned} \tag{3.5}$$

Dónde:

$$\mathbf{k}_1 = \mathbf{j}_1^0 \times \mathbf{r}_{G1'}^0 \tag{3.6}$$

De la ec. (2.51):

$$\dot{\theta}_{21} = \frac{1}{V_1} (V_2 \dot{x} + V_3 \dot{y} + V_4 \dot{z} + V_5 \dot{\psi} + V_6 \dot{\theta} + V_7 \dot{\phi})$$

Renombrando:

$$\dot{\theta}_{21} = \mathbf{k}_2^T \dot{\mathbf{q}} \quad 3.7$$

Dónde:

$$\begin{aligned} \mathbf{k}_2^T &= \frac{1}{V_1} [V_2, V_3, V_4, V_5, V_6, V_7] \\ \dot{\mathbf{q}} &= [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]^T \end{aligned} \quad 3.8$$

Sustituyendo la ec. (3.7) en la ec. (3.5) con el fin de poner a esta última en función de las coordenadas generalizadas se tiene:

$$\begin{aligned} \mathbf{v}_{G1}^0 &= \mathbf{k}_1 (\mathbf{k}_2^T \dot{\mathbf{q}}) \\ \mathbf{v}_{G1}^0 &= (\mathbf{k}_1 \mathbf{k}_2^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\mathbf{v}_{G1}^0 = \mathbf{M}_1 \dot{\mathbf{q}} \quad 3.9$$

Dónde:

$$\mathbf{M}_1 = \mathbf{k}_1 \mathbf{k}_2^T \quad 3.10$$

3.2.2 Velocidad Angular del Cuerpo 1

$$\begin{aligned} \boldsymbol{\omega}_2^0 &= \boldsymbol{\omega}_{21}^0 \\ \boldsymbol{\omega}_{21}^0 &= \mathbf{j}_1^0 \dot{\theta}_{21} \end{aligned}$$

Sustituyendo (3.7) en la ec. anterior:

$$\begin{aligned} \boldsymbol{\omega}_{21}^0 &= \mathbf{j}_1^0 (\mathbf{k}_2^T \dot{\mathbf{q}}) \\ \boldsymbol{\omega}_{21}^0 &= (\mathbf{j}_1^0 \mathbf{k}_2^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_{21}^0 = \mathbf{M}_2 \dot{\mathbf{q}} \quad 3.11$$

Dónde:

$$\mathbf{M}_2 = \mathbf{j}_1^0 \mathbf{k}_2^T \quad 3.12$$

3.2.3 Velocidad de Centro de Gravedad del Cuerpo 2

$$\begin{aligned} \mathbf{v}_{G2}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{G2'}^0 \\ &= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{G2'}^0 \\ &= \boldsymbol{\omega}_{21}^0 \times \mathbf{r}_{32}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{G2'}^0 \\ &= \boldsymbol{\omega}_{21}^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0) + (\boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{G2'}^0 \\ &= (\dot{\theta}_{21} \mathbf{j}_1^0) \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0) + (\dot{\theta}_{43} \mathbf{j}_3^0 + \dot{\theta}_{54} \mathbf{k}_4^0) \times \mathbf{r}_{G2'}^0 \\ &= (\mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0)) \dot{\theta}_{21} + (\mathbf{j}_3^0 \times \mathbf{r}_{G2'}^0) \dot{\theta}_{43} + (\mathbf{k}_4^0 \times \mathbf{r}_{G2'}^0) \dot{\theta}_{54} \\ \mathbf{v}_{G2}^0 &= \mathbf{k}_3 \dot{\theta}_{21} + \mathbf{k}_4 \dot{\theta}_{43} + \mathbf{k}_5 \dot{\theta}_{54} \end{aligned} \quad 3.13$$

Dónde:

$$\begin{aligned} \mathbf{k}_3 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0) \\ \mathbf{k}_4 &= \mathbf{j}_3^0 \times \mathbf{r}_{G2'}^0 \\ \mathbf{k}_5 &= \mathbf{k}_4^0 \times \mathbf{r}_{G2'}^0 \end{aligned} \quad 3.14$$

De las ecs. (2.55) y (2.59):

$$\begin{aligned} \dot{\theta}_{43} &= \frac{1}{V_8} (E_1 \dot{x} + E_2 \dot{y} + E_3 \dot{z} + E_4 \dot{\psi} + E_5 \dot{\theta} + E_6 \dot{\phi}) \\ \dot{\theta}_{54} &= \frac{1}{V_{15}} (E_7 \dot{x} + E_8 \dot{y} + E_9 \dot{z} + E_{10} \dot{\psi} + E_{11} \dot{\theta} + E_{12} \dot{\phi}) \end{aligned}$$

Renombrando:

$$\dot{\theta}_{43} = \mathbf{k}_6^T \dot{\mathbf{q}} \quad 3.15$$

$$\dot{\theta}_{54} = \mathbf{k}_7^T \dot{\mathbf{q}} \quad 3.16$$

Dónde:

$$\begin{aligned} \mathbf{k}_6^T &= \frac{1}{V_8} [E_1, E_2, E_3, E_4, E_5, E_6] \\ \mathbf{k}_7^T &= \frac{1}{V_{15}} [E_7, E_8, E_9, E_{10}, E_{11}, E_{12}] \end{aligned} \quad 3.17$$

Sustituyendo la ec. (3.7), (3.15) y (3.16) en la ec. (3.13):

$$\begin{aligned} \mathbf{v}_{G2}^0 &= \mathbf{k}_3 \dot{\theta}_{21} + \mathbf{k}_4 \dot{\theta}_{43} + \mathbf{k}_5 \dot{\theta}_{54} \\ &= (\mathbf{k}_3 \mathbf{k}_2^T) \dot{\mathbf{q}} + (\mathbf{k}_4 \mathbf{k}_6^T) \dot{\mathbf{q}} + (\mathbf{k}_5 \mathbf{k}_7^T) \dot{\mathbf{q}} \\ \mathbf{v}_{G2}^0 &= (\mathbf{k}_3 \mathbf{k}_2^T + \mathbf{k}_4 \mathbf{k}_6^T + \mathbf{k}_5 \mathbf{k}_7^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\mathbf{v}_{G2}^0 = \mathbf{M}_3 \dot{\mathbf{q}} \quad 3.18$$

Dónde:

$$\mathbf{M}_3 = \mathbf{k}_3 \mathbf{k}_2^T + \mathbf{k}_4 \mathbf{k}_6^T + \mathbf{k}_5 \mathbf{k}_7^T \quad 3.19$$

3.2.4 Velocidad Angular del Cuerpo 2

$$\begin{aligned} \boldsymbol{\omega}_5^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 \\ &= \mathbf{j}_1^0 \dot{\theta}_{21} + \mathbf{j}_3^0 \dot{\theta}_{43} + \mathbf{k}_4^0 \dot{\theta}_{54} \\ &= \mathbf{j}_1^0 \mathbf{k}_2^T \dot{\mathbf{q}} + \mathbf{j}_3^0 \mathbf{k}_6^T \dot{\mathbf{q}} + \mathbf{k}_4^0 \mathbf{k}_7^T \dot{\mathbf{q}} \\ \boldsymbol{\omega}_5^0 &= (\mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_5^0 = \mathbf{M}_4 \dot{\mathbf{q}} \quad 3.20$$

Dónde:

$$\mathbf{M}_4 = \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T \quad 3.21$$

3.2.5 Velocidad de Centro de Gravedad del Cuerpo 3

$$\begin{aligned}
\mathbf{v}_{G3}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G3'}^0 \\
&= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{14,5i}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{15,14i}^0 + \boldsymbol{\omega}_{16}^0 \times \mathbf{r}_{G3'}^0 \\
&= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 + \boldsymbol{\omega}_5^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) + \boldsymbol{\omega}_{16}^0 \times \mathbf{r}_{G3'}^0 \\
&= \boldsymbol{\omega}_{21}^0 \times \mathbf{r}_{32}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 \\
&\quad + \boldsymbol{\omega}_{16,15}^0) \times \mathbf{r}_{G3'}^0 \\
&= \boldsymbol{\omega}_{21}^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) + (\boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
&\quad + \boldsymbol{\omega}_{16,15}^0 \times \mathbf{r}_{G3'}^0 \\
&= (\dot{\theta}_{21} \mathbf{j}_1^0) \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) + (\dot{\theta}_{43} \mathbf{j}_3^0 + \dot{\theta}_{54} \mathbf{k}_4^0) \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 \\
&\quad + \mathbf{r}_{G3'}^0) + (\dot{\theta}_{1615} \mathbf{i}_{15}^0) \times \mathbf{r}_{G3'}^0 \\
&= (\mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0)) \dot{\theta}_{21} + (\mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0)) \dot{\theta}_{43} \\
&\quad + (\mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0)) \dot{\theta}_{54} + (\mathbf{i}_{15}^0 \times \mathbf{r}_{G3'}^0) \dot{\theta}_{1615} \\
\mathbf{v}_{G3}^0 &= \mathbf{k}_8 \dot{\theta}_{21} + \mathbf{k}_9 \dot{\theta}_{43} + \mathbf{k}_{10} \dot{\theta}_{54} + \mathbf{k}_{11} \dot{\theta}_{1615} \quad 3.22
\end{aligned}$$

Dónde:

$$\begin{aligned}
\mathbf{k}_8 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) \\
\mathbf{k}_9 &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{10} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{11} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G3'}^0 \quad 3.23
\end{aligned}$$

De la ec. (2.73):

$$\dot{\theta}_{1615} = \frac{1}{V_{36}} (E_{25} \dot{x} + E_{26} \dot{y} + E_{27} \dot{z} + E_{28} \dot{\psi} + E_{29} \dot{\theta} + E_{30} \dot{\phi})$$

Renombrando:

$$\dot{\theta}_{1615} = \mathbf{k}_{12}^T \dot{\mathbf{q}} \quad 3.24$$

Dónde:

$$\mathbf{k}_{12}^T = \frac{1}{V_{36}} [E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}] \quad 3.25$$

Sustituyendo la ec. (3.7), (3.15), (3.16) y (3.24) en la ec. (3.22):

$$\begin{aligned}
\mathbf{v}_{G3}^0 &= \mathbf{k}_8 \dot{\theta}_{21} + \mathbf{k}_9 \dot{\theta}_{43} + \mathbf{k}_{10} \dot{\theta}_{54} + \mathbf{k}_{11} \dot{\theta}_{1615} \\
&= (\mathbf{k}_8 \mathbf{k}_2^T) \dot{\mathbf{q}} + (\mathbf{k}_9 \mathbf{k}_6^T) \dot{\mathbf{q}} + (\mathbf{k}_{10} \mathbf{k}_7^T) \dot{\mathbf{q}} + (\mathbf{k}_{11} \mathbf{k}_{12}^T) \dot{\mathbf{q}}
\end{aligned}$$

$$\mathbf{v}_{G3}^0 = (\mathbf{k}_8 \mathbf{k}_2^T + \mathbf{k}_9 \mathbf{k}_6^T + \mathbf{k}_{10} \mathbf{k}_7^T + \mathbf{k}_{11} \mathbf{k}_{12}^T) \dot{q}$$

Renombrando:

$$\mathbf{v}_{G3}^0 = \mathbf{M}_5 \dot{q} \quad 3.26$$

Dónde:

$$\mathbf{M}_5 = \mathbf{k}_8 \mathbf{k}_2^T + \mathbf{k}_9 \mathbf{k}_6^T + \mathbf{k}_{10} \mathbf{k}_7^T + \mathbf{k}_{11} \mathbf{k}_{12}^T \quad 3.27$$

3.2.6 Velocidad Angular del Cuerpo 3

$$\begin{aligned} \boldsymbol{\omega}_{16}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 \\ &= \mathbf{j}_1^0 \dot{\theta}_{21} + \mathbf{j}_3^0 \dot{\theta}_{43} + \mathbf{k}_4^0 \dot{\theta}_{54} + \mathbf{i}_{15}^0 \dot{\theta}_{16,15} \\ &= \mathbf{j}_1^0 \mathbf{k}_2^T \dot{q} + \mathbf{j}_3^0 \mathbf{k}_6^T \dot{q} + \mathbf{k}_4^0 \mathbf{k}_7^T \dot{q} + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T \dot{q} \\ \boldsymbol{\omega}_{16}^0 &= (\mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T) \dot{q} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_{16}^0 = \mathbf{M}_6 \dot{q} \quad 3.28$$

Dónde:

$$\mathbf{M}_6 = \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T \quad 3.29$$

3.2.7 Velocidad de Centro de Gravedad del Cuerpo 4

$$\begin{aligned} \mathbf{v}_{G4}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G4'}^0 \\ &= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{14,5i}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{15,14i}^0 + \boldsymbol{\omega}_{17}^0 \times \mathbf{r}_{G4'}^0 \\ &= \boldsymbol{\omega}_{21}^0 \times \mathbf{r}_{32}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{14,5i}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{15,14i}^0 \\ &\quad + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0) \times \mathbf{r}_{G4'}^0 \\ &= \boldsymbol{\omega}_{21}^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + (\boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 \\ &\quad + \mathbf{r}_{G4'}^0) + (\boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0) \times \mathbf{r}_{G4'}^0 \\ &= (\dot{\theta}_{21} \mathbf{j}_1^0) \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + (\dot{\theta}_{43} \mathbf{j}_3^0 + \dot{\theta}_{54} \mathbf{k}_4^0) \times (\mathbf{r}_{14,5i}^0 \\ &\quad + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + (\dot{\theta}_{16,15} \mathbf{i}_{15}^0 + \dot{\theta}_{17,16} \mathbf{k}_{16}^0) \times \mathbf{r}_{G4'}^0 \\ &= (\mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0)) \dot{\theta}_{21} + (\mathbf{j}_3^0 \\ &\quad \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0)) \dot{\theta}_{43} + (\mathbf{k}_4^0 \\ &\quad \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0)) \dot{\theta}_{54} + (\mathbf{i}_{15}^0 \times \mathbf{r}_{G4'}^0) \dot{\theta}_{16,15} + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{r}_{G4'}^0) \dot{\theta}_{17,16} \\ \mathbf{v}_{G4}^0 &= \mathbf{k}_{13} \dot{\theta}_{21} + \mathbf{k}_{14} \dot{\theta}_{43} + \mathbf{k}_{15} \dot{\theta}_{54} + \mathbf{k}_{16} \dot{\theta}_{16,15} + \mathbf{k}_{17} \dot{\theta}_{17,16} \end{aligned} \quad 3.30$$

Dónde:

$$\begin{aligned} \mathbf{k}_{13} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{14} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{15} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{16} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{k}_{17} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G4'}^0 \end{aligned} \quad 3.31$$

De la ec. (2.69):

$$\dot{\theta}_{1716} = \frac{1}{V_{29}} (E_{19}\dot{x} + E_{20}\dot{y} + E_{21}\dot{z} + E_{22}\dot{\psi} + E_{23}\dot{\theta} + E_{24}\dot{\phi})$$

Renombrando:

$$\dot{\theta}_{1716} = \mathbf{k}_{18}^T \dot{\mathbf{q}} \quad 3.32$$

Dónde:

$$\mathbf{k}_{18}^T = \frac{1}{V_{29}} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] \quad 3.33$$

Sustituyendo las ecs. (3.7), (3.15), (3.16) ,(3.24) y (3.32) en la ec. (3.30):

$$\begin{aligned} \mathbf{v}_{G4}^0 &= \mathbf{k}_{13}\dot{\theta}_{21} + \mathbf{k}_{14}\dot{\theta}_{43} + \mathbf{k}_{15}\dot{\theta}_{54} + \mathbf{k}_{16}\dot{\theta}_{1615} + \mathbf{k}_{17}\dot{\theta}_{1716} \\ &= (\mathbf{k}_{13}\mathbf{k}_2^T)\dot{\mathbf{q}} + (\mathbf{k}_{14}\mathbf{k}_6^T)\dot{\mathbf{q}} + (\mathbf{k}_{15}\mathbf{k}_7^T)\dot{\mathbf{q}} + (\mathbf{k}_{16}\mathbf{k}_{12}^T)\dot{\mathbf{q}} + (\mathbf{k}_{17}\mathbf{k}_{18}^T)\dot{\mathbf{q}} \\ \mathbf{v}_{G4}^0 &= (\mathbf{k}_{13}\mathbf{k}_2^T + \mathbf{k}_{14}\mathbf{k}_6^T + \mathbf{k}_{15}\mathbf{k}_7^T + \mathbf{k}_{16}\mathbf{k}_{12}^T + \mathbf{k}_{17}\mathbf{k}_{18}^T)\dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\mathbf{v}_{G4}^0 = \mathbf{M}_7 \dot{\mathbf{q}} \quad 3.34$$

Dónde:

$$\mathbf{M}_7 = \mathbf{k}_{13}\mathbf{k}_2^T + \mathbf{k}_{14}\mathbf{k}_6^T + \mathbf{k}_{15}\mathbf{k}_7^T + \mathbf{k}_{16}\mathbf{k}_{12}^T + \mathbf{k}_{17}\mathbf{k}_{18}^T \quad 3.35$$

3.2.8 Velocidad Angular del Cuerpo 4

$$\begin{aligned} \boldsymbol{\omega}_{17}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 \\ &= \mathbf{j}_1^0\dot{\theta}_{21} + \mathbf{j}_3^0\dot{\theta}_{43} + \mathbf{k}_4^0\dot{\theta}_{54} + \mathbf{i}_{15}^0\dot{\theta}_{1615} + \mathbf{k}_{16}^0\dot{\theta}_{1716} \\ &= \mathbf{j}_1^0\mathbf{k}_2^T\dot{\mathbf{q}} + \mathbf{j}_3^0\mathbf{k}_6^T\dot{\mathbf{q}} + \mathbf{k}_4^0\mathbf{k}_7^T\dot{\mathbf{q}} + \mathbf{i}_{15}^0\mathbf{k}_{12}^T\dot{\mathbf{q}} + \mathbf{k}_{16}^0\mathbf{k}_{18}^T\dot{\mathbf{q}} \\ &= (\mathbf{j}_1^0\mathbf{k}_2^T + \mathbf{j}_3^0\mathbf{k}_6^T + \mathbf{k}_4^0\mathbf{k}_7^T + \mathbf{i}_{15}^0\mathbf{k}_{12}^T + \mathbf{k}_{16}^0\mathbf{k}_{18}^T)\dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_{17}^0 = \mathbf{M}_8 \dot{\mathbf{q}} \quad 3.36$$

Dónde:

$$\mathbf{M}_8 = \mathbf{j}_1^0\mathbf{k}_2^T + \mathbf{j}_3^0\mathbf{k}_6^T + \mathbf{k}_4^0\mathbf{k}_7^T + \mathbf{i}_{15}^0\mathbf{k}_{12}^T + \mathbf{k}_{16}^0\mathbf{k}_{18}^T \quad 3.37$$

3.2.9 Velocidad de Centro de Gravedad del Cuerpo 5

$$\begin{aligned}
\mathbf{v}_{G5}^0 &= \mathbf{v}_{10}^0 + \mathbf{v}_{32}^0 + \mathbf{v}_{14,5i}^0 + \mathbf{v}_{15,14i}^0 + \mathbf{v}_{G5'}^0 \\
&= \boldsymbol{\omega}_2^0 \times \mathbf{r}_{32}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{14,5i}^0 + \boldsymbol{\omega}_5^0 \times \mathbf{r}_{15,14i}^0 + \boldsymbol{\omega}_{18}^0 \times \mathbf{r}_{G5'}^0 \\
&= \boldsymbol{\omega}_{21}^0 \times \mathbf{r}_{32}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{14,5i}^0 + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times \mathbf{r}_{15,14i}^0 \\
&\quad + (\boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 + \boldsymbol{\omega}_{18,17}^0) \times \mathbf{r}_{G5'}^0 \\
&= \boldsymbol{\omega}_{21}^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + (\boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0) \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 \\
&\quad + \mathbf{r}_{G5'}^0) + (\boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 + \boldsymbol{\omega}_{18,17}^0) \times \mathbf{r}_{G5'}^0 \\
&= (\dot{\theta}_{21} \mathbf{j}_1^0) \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + (\dot{\theta}_{43} \mathbf{j}_3^0 + \dot{\theta}_{54} \mathbf{k}_4^0) \times (\mathbf{r}_{14,5i}^0 \\
&\quad + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + (\dot{\theta}_{1615} \mathbf{i}_{15}^0 + \dot{\theta}_{1716} \mathbf{k}_{16}^0 + \dot{\theta}_{1817} \mathbf{i}_{17}^0) \times \mathbf{r}_{G5'}^0 \\
&= (\mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0)) \dot{\theta}_{21} + (\mathbf{j}_3^0 \\
&\quad \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0)) \dot{\theta}_{43} + (\mathbf{k}_4^0 \\
&\quad \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0)) \dot{\theta}_{54} + (\mathbf{i}_{15}^0 \times \mathbf{r}_{G5'}^0) \dot{\theta}_{1615} + (\mathbf{k}_{16}^0 \\
&\quad \times \mathbf{r}_{G5'}^0) \dot{\theta}_{1716} + (\mathbf{i}_{17}^0 \times \mathbf{r}_{G5'}^0) \dot{\theta}_{1817} \\
\mathbf{v}_{G5}^0 &= \mathbf{k}_{19} \dot{\theta}_{21} + \mathbf{k}_{20} \dot{\theta}_{43} + \mathbf{k}_{21} \dot{\theta}_{54} + \mathbf{k}_{22} \dot{\theta}_{1615} + \mathbf{k}_{23} \dot{\theta}_{1716} + \mathbf{k}_{24} \dot{\theta}_{1817}
\end{aligned} \tag{3.38}$$

Dónde:

$$\begin{aligned}
\mathbf{k}_{19} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{20} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{21} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{22} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G5'}^0 \\
\mathbf{k}_{23} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G5'}^0 \\
\mathbf{k}_{24} &= \mathbf{i}_{17}^0 \times \mathbf{r}_{G5'}^0
\end{aligned} \tag{3.39}$$

De la ec. (2.64):

$$\dot{\theta}_{1817} = \frac{1}{V_{23}} (E_{13} \dot{x} + E_{14} \dot{y} + E_{15} \dot{z} + E_{16} \dot{\psi} + E_{17} \dot{\theta} + E_{18} \dot{\phi})$$

Renombrando:

$$\dot{\theta}_{1817} = \mathbf{k}_{25}^T \dot{\mathbf{q}} \tag{3.40}$$

Dónde:

$$\mathbf{k}_{25}^T = \frac{1}{V_{23}} [E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_{18}] \tag{3.41}$$

Sustituyendo las ecs. (3.7), (3.15), (3.16), (3.24), (3.32) y (3.40) en la ec. (3.38):

$$\begin{aligned}
\mathbf{v}_{G5}^0 &= \mathbf{k}_{19} \dot{\theta}_{21} + \mathbf{k}_{20} \dot{\theta}_{43} + \mathbf{k}_{21} \dot{\theta}_{54} + \mathbf{k}_{22} \dot{\theta}_{1615} + \mathbf{k}_{23} \dot{\theta}_{1716} + \mathbf{k}_{24} \dot{\theta}_{1817} \\
&= (\mathbf{k}_{19} \mathbf{k}_{25}^T) \dot{\mathbf{q}} + (\mathbf{k}_{20} \mathbf{k}_{25}^T) \dot{\mathbf{q}} + (\mathbf{k}_{21} \mathbf{k}_{25}^T) \dot{\mathbf{q}} + (\mathbf{k}_{22} \mathbf{k}_{25}^T) \dot{\mathbf{q}} + (\mathbf{k}_{23} \mathbf{k}_{25}^T) \dot{\mathbf{q}} + (\mathbf{k}_{24} \mathbf{k}_{25}^T) \dot{\mathbf{q}}
\end{aligned}$$

$$\mathbf{v}_{G5}^0 = (\mathbf{k}_{19}\mathbf{k}_2^T + \mathbf{k}_{20}\mathbf{k}_6^T + \mathbf{k}_{21}\mathbf{k}_7^T + \mathbf{k}_{22}\mathbf{k}_{12}^T + \mathbf{k}_{23}\mathbf{k}_{18}^T + \mathbf{k}_{24}\mathbf{k}_{25}^T)\dot{\mathbf{q}}$$

Renombrando:

$$\mathbf{v}_{G5}^0 = \mathbf{M}_9\dot{\mathbf{q}} \quad 3.42$$

Dónde:

$$\mathbf{M}_9 = \mathbf{k}_{19}\mathbf{k}_2^T + \mathbf{k}_{20}\mathbf{k}_6^T + \mathbf{k}_{21}\mathbf{k}_7^T + \mathbf{k}_{22}\mathbf{k}_{12}^T + \mathbf{k}_{23}\mathbf{k}_{18}^T + \mathbf{k}_{24}\mathbf{k}_{25}^T \quad 3.43$$

3.2.10 Velocidad Angular del Cuerpo 5

$$\begin{aligned} \boldsymbol{\omega}_{18}^0 &= \boldsymbol{\omega}_{21}^0 + \boldsymbol{\omega}_{43}^0 + \boldsymbol{\omega}_{54}^0 + \boldsymbol{\omega}_{16,15}^0 + \boldsymbol{\omega}_{17,16}^0 + \boldsymbol{\omega}_{18,17}^0 \\ &= \mathbf{j}_1^0\dot{\theta}_{21} + \mathbf{j}_3^0\dot{\theta}_{43} + \mathbf{k}_4^0\dot{\theta}_{54} + \mathbf{i}_{15}^0\dot{\theta}_{16,15} + \mathbf{k}_{16}^0\dot{\theta}_{17,16} + \mathbf{i}_{17}^0\dot{\theta}_{18,17} \\ &= \mathbf{j}_1^0\mathbf{k}_2^T\dot{\mathbf{q}} + \mathbf{j}_3^0\mathbf{k}_6^T\dot{\mathbf{q}} + \mathbf{k}_4^0\mathbf{k}_7^T\dot{\mathbf{q}} + \mathbf{i}_{15}^0\mathbf{k}_{12}^T\dot{\mathbf{q}} + \mathbf{k}_{16}^0\mathbf{k}_{18}^T\dot{\mathbf{q}} + \mathbf{i}_{17}^0\mathbf{k}_{25}^T\dot{\mathbf{q}} \\ \boldsymbol{\omega}_{18}^0 &= (\mathbf{j}_1^0\mathbf{k}_2^T + \mathbf{j}_3^0\mathbf{k}_6^T + \mathbf{k}_4^0\mathbf{k}_7^T + \mathbf{i}_{15}^0\mathbf{k}_{12}^T + \mathbf{k}_{16}^0\mathbf{k}_{18}^T + \mathbf{i}_{17}^0\mathbf{k}_{25}^T)\dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_{18}^0 = \mathbf{M}_{10}\dot{\mathbf{q}} \quad 3.44$$

Dónde:

$$\mathbf{M}_{10} = \mathbf{j}_1^0\mathbf{k}_2^T + \mathbf{j}_3^0\mathbf{k}_6^T + \mathbf{k}_4^0\mathbf{k}_7^T + \mathbf{i}_{15}^0\mathbf{k}_{12}^T + \mathbf{k}_{16}^0\mathbf{k}_{18}^T + \mathbf{i}_{17}^0\mathbf{k}_{25}^T \quad 3.45$$

Matrices de inercia medidas en la base inercial:

$$\begin{aligned} \mathbf{J}_1^0 &= \mathbf{R}_2^0 \mathbf{J}_1^2 \mathbf{R}_2^{0T} \\ \mathbf{J}_2^0 &= \mathbf{R}_5^0 \mathbf{J}_2^5 \mathbf{R}_5^{0T} \\ \mathbf{J}_3^0 &= \mathbf{R}_{16}^0 \mathbf{J}_3^{16} \mathbf{R}_{16}^{0T} \\ \mathbf{J}_4^0 &= \mathbf{R}_{17}^0 \mathbf{J}_4^{17} \mathbf{R}_{17}^{0T} \\ \mathbf{J}_5^0 &= \mathbf{R}_{18}^0 \mathbf{J}_5^{18} \mathbf{R}_{18}^{0T} \end{aligned}$$

3.3 Función Lagrangiana

Aplicando la ec. (3.1) se consigue de manera general la siguiente expresión:

$$L = \sum_{h=1}^5 (K_h - U_h) + \sum_{i=1}^2 \left(\sum_{k=1}^2 (K_{ki} - U_{ki}) \right) \quad 3.46$$

h =número de cuerpos en la cadena 0.

i =número de la cadena 1 o 2.

k =número de cuerpos en la cadena h .

Expandiendo los términos del primer paréntesis:

$$L = \sum_{h=1}^5 (K_h - U_h) + \sum_{i=1}^2 ((K_{1i} - U_{1i}) + (K_{2i} - U_{2i})) \quad 3.47$$

$$L = \sum_{h=1}^5 L_h + \sum_{i=1}^2 (L_{1i} + L_{2i}) \quad 3.48$$

Dónde:

$$\begin{aligned} L_1 &= \frac{1}{2} (m_1 \mathbf{v}_{G1}^0{}^T \mathbf{v}_{G1}^0 + \boldsymbol{\omega}_2^0{}^T J_1 \boldsymbol{\omega}_2^0) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\ L_2 &= \frac{1}{2} (m_2 \mathbf{v}_{G2}^0{}^T \mathbf{v}_{G2}^0 + \boldsymbol{\omega}_5^0{}^T J_2 \boldsymbol{\omega}_5^0) + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0 \\ L_3 &= \frac{1}{2} (m_3 \mathbf{v}_{G3}^0{}^T \mathbf{v}_{G3}^0 + \boldsymbol{\omega}_{16}^0{}^T J_3 \boldsymbol{\omega}_{16}^0) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0 \\ L_4 &= \frac{1}{2} (m_4 \mathbf{v}_{G4}^0{}^T \mathbf{v}_{G4}^0 + \boldsymbol{\omega}_{17}^0{}^T J_4 \boldsymbol{\omega}_{17}^0) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0 \\ L_5 &= \frac{1}{2} (m_5 \mathbf{v}_{G5}^0{}^T \mathbf{v}_{G5}^0 + \boldsymbol{\omega}_{18}^0{}^T J_5 \boldsymbol{\omega}_{18}^0) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0 \\ L_{1i} &= \frac{1}{2} (m_{1h} \mathbf{v}_{G1i}^0{}^T \mathbf{v}_{G1i}^0 + \boldsymbol{\omega}_{2i}^0{}^T J_{1i} \boldsymbol{\omega}_{2i}^0) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \\ L_{2i} &= \frac{1}{2} (m_{2h} \mathbf{v}_{G2i}^0{}^T \mathbf{v}_{G2i}^0 + \boldsymbol{\omega}_{7i}^0{}^T J_{2i} \boldsymbol{\omega}_{7i}^0) + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0 \end{aligned}$$

Donde $\mathbf{g} = [0, 0, -9.81]^T$

3.3.1 Desarrollo del Primer Término de la Ecuación de Lagrange para la cadena 0.

A partir de la ec. (3.4):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$

Desarrollando el primer término de la ecuación anterior a partir de la ec. (3.47):

$$\frac{\partial L}{\partial \dot{q}_j} = \sum_{h=1}^5 \frac{\partial L_h}{\partial \dot{q}_j} + \sum_{i=1}^2 \left(\frac{\partial L_{1i}}{\partial \dot{q}_j} + \frac{\partial L_{2i}}{\partial \dot{q}_j} \right) \quad 3.49$$

Para $j = 1, 2, 3, 4, 5, 6$, donde:

$$\begin{aligned} \dot{q}_1 &= \dot{x} & \dot{q}_2 &= \dot{y} & \dot{q}_3 &= \dot{z} \\ \dot{q}_4 &= \dot{\psi} & \dot{q}_5 &= \dot{\theta} & \dot{q}_6 &= \dot{\phi} \end{aligned}$$

Desarrollando $\frac{\partial L_1}{\partial \dot{q}_j}$:

Se tiene:

$$L_1 = \frac{1}{2} \left(m_1 \mathbf{v}_{G1}^0{}^T \mathbf{v}_{G1}^0 + \boldsymbol{\omega}_2^{0T} \mathbf{J}_1 \boldsymbol{\omega}_2^0 \right) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0$$

Sustituyendo ec.(3.9) y ec.(3.11):

$$\begin{aligned} L_1 &= \frac{1}{2} (m_1 (\mathbf{M}_1 \dot{\mathbf{q}})^T (\mathbf{M}_1 \dot{\mathbf{q}}) + (\mathbf{M}_2 \dot{\mathbf{q}})^T \mathbf{J}_1 (\mathbf{M}_2 \dot{\mathbf{q}})) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\ &= \frac{1}{2} (m_1 \dot{\mathbf{q}}^T \mathbf{M}_1^T (\mathbf{M}_1 \dot{\mathbf{q}}) + \dot{\mathbf{q}}^T \mathbf{M}_2^T \mathbf{J}_1 (\mathbf{M}_2 \dot{\mathbf{q}})) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\ &= \frac{1}{2} (m_1 \dot{\mathbf{q}}^T (\mathbf{M}_1^T \mathbf{M}_1) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2) \dot{\mathbf{q}}) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\ &= \frac{1}{2} \dot{\mathbf{q}}^T (m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2) \dot{\mathbf{q}} + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\ L_1 &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_1 \dot{\mathbf{q}} + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \end{aligned} \quad 3.50$$

Dónde:

$$\mathbf{N}_1 = m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2$$

Efectuando la derivada:

$$\begin{aligned} \frac{\partial L_1}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_1 \dot{\mathbf{q}}) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \right) \\ \frac{\partial L_1}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_1 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \end{aligned} \quad 3.51$$

Donde se cumple:

$$\begin{aligned} \mathbf{N}_1^T &= (m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2)^T \\ &= (m_1 \mathbf{M}_1^T \mathbf{M}_1)^T + (\mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2)^T \\ &= m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1^T \mathbf{M}_2 \\ &= m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2 \\ \mathbf{N}_1^T &= \mathbf{N}_1 \end{aligned} \quad 3.52$$

Tal que $\mathbf{J}_1 = \mathbf{J}_1^T$, ya que la matriz de inercia es simétrica. Además:

$$\begin{aligned} \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} &= (\mathbf{N}_1 \dot{\mathbf{q}})^T \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \\ &= \dot{\mathbf{q}}^T \mathbf{N}_1^T \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \\ \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} &= \dot{\mathbf{q}}^T \mathbf{N}_1 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \end{aligned}$$

Estas dos propiedades anteriores se ocupan para los demás cuerpos.

Simplificando:

$$\begin{aligned}
\frac{\partial L_1}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_1 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\
&= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} \right) \\
\frac{\partial L_1}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}}
\end{aligned} \tag{3.53}$$

Desarrollando $\frac{\partial L_2}{\partial \dot{q}_j}$:

Se tiene:

$$L_2 = \frac{1}{2} \left(m_2 \mathbf{v}_{G2}^{0T} \mathbf{v}_{G2}^0 + \boldsymbol{\omega}_5^{0T} \mathbf{J}_2 \boldsymbol{\omega}_5^0 \right) + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0$$

Sustituyendo ec.(3.18) y ec.(3.20):

$$\begin{aligned}
L_2 &= \frac{1}{2} (m_2 (\mathbf{M}_3 \dot{\mathbf{q}})^T (\mathbf{M}_3 \dot{\mathbf{q}}) + (\mathbf{M}_4 \dot{\mathbf{q}})^T \mathbf{J}_1 (\mathbf{M}_4 \dot{\mathbf{q}})) + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0 \\
&= \frac{1}{2} \dot{\mathbf{q}}^T (m_2 \mathbf{M}_3^T \mathbf{M}_3 + \mathbf{M}_4^T \mathbf{J}_2 \mathbf{M}_4) \dot{\mathbf{q}} + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0 \\
L_2 &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_2 \dot{\mathbf{q}} + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0
\end{aligned} \tag{3.54}$$

Dónde:

$$\mathbf{N}_2 = m_2 \mathbf{M}_3^T \mathbf{M}_3 + \mathbf{M}_4^T \mathbf{J}_2 \mathbf{M}_4$$

Efectuando la derivada:

$$\begin{aligned}
\frac{\partial L_2}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_2 \dot{\mathbf{q}}) + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0 \right) \\
\frac{\partial L_2}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_2 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right)
\end{aligned} \tag{3.55}$$

Simplificando:

$$\begin{aligned}
\frac{\partial L_2}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_2 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\
&= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}} \right) \\
\frac{\partial L_2}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}}
\end{aligned} \tag{3.56}$$

Desarrollando $\frac{\partial L_3}{\partial \dot{q}_j}$:

Se tiene:

$$L_3 = \frac{1}{2} \left(m_3 \mathbf{v}_{G3}^0{}^T \mathbf{v}_{G3}^0 + \boldsymbol{\omega}_{16}^0{}^T J_3 \boldsymbol{\omega}_{16}^0 \right) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0$$

Sustituyendo ec.(3.26) y ec.(3.28):

$$\begin{aligned} L_3 &= \frac{1}{2} (m_3 (\mathbf{M}_5 \dot{\mathbf{q}})^T (\mathbf{M}_5 \dot{\mathbf{q}}) + (\mathbf{M}_6 \dot{\mathbf{q}})^T J_3 (\mathbf{M}_6 \dot{\mathbf{q}})) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0 \\ &= \frac{1}{2} (\dot{\mathbf{q}}^T (m_3 \mathbf{M}_5^T \mathbf{M}_5 + \mathbf{M}_6^T J_3 \mathbf{M}_6) \dot{\mathbf{q}}) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0 \\ L_3 &= \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_3 \dot{\mathbf{q}}) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0 \end{aligned} \quad 3.57$$

Dónde:

$$\mathbf{N}_3 = m_3 \mathbf{M}_5^T \mathbf{M}_5 + \mathbf{M}_6^T J_3 \mathbf{M}_6$$

Efectuando la derivada:

$$\begin{aligned} \frac{\partial L_3}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_3 \dot{\mathbf{q}}) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0 \right) \\ \frac{\partial L_3}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_3 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_3 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \end{aligned} \quad 3.58$$

Simplificando:

$$\begin{aligned} \frac{\partial L_3}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_3 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_3 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\ &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_3 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_3 \dot{\mathbf{q}} \right) \\ \frac{\partial L_3}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_3 \dot{\mathbf{q}} \end{aligned} \quad 3.59$$

Desarrollando $\frac{\partial L_4}{\partial \dot{q}_j}$:

Se tiene:

$$L_4 = \frac{1}{2} \left(m_4 \mathbf{v}_{G4}^0{}^T \mathbf{v}_{G4}^0 + \boldsymbol{\omega}_{17}^0{}^T J_4 \boldsymbol{\omega}_{17}^0 \right) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0$$

Sustituyendo ec.(3.34) y ec.(3.36):

$$\begin{aligned} L_4 &= \frac{1}{2} (m_4 (\mathbf{M}_7 \dot{\mathbf{q}})^T (\mathbf{M}_7 \dot{\mathbf{q}}) + (\mathbf{M}_8 \dot{\mathbf{q}})^T J_4 (\mathbf{M}_8 \dot{\mathbf{q}})) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0 \\ &= \frac{1}{2} (\dot{\mathbf{q}}^T (m_4 \mathbf{M}_7^T \mathbf{M}_7 + \mathbf{M}_8^T J_4 \mathbf{M}_8) \dot{\mathbf{q}}) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0 \\ L_4 &= \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_4 \dot{\mathbf{q}}) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0 \end{aligned} \quad 3.60$$

Dónde:

$$\mathbf{N}_4 = m_4 \mathbf{M}_7^T \mathbf{M}_7 + \mathbf{M}_8^T \mathbf{J}_4 \mathbf{M}_8$$

Efectuando la derivada:

$$\begin{aligned} \frac{\partial L_4}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_4 \dot{\mathbf{q}}) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0 \right) \\ \frac{\partial L_4}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_4 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \end{aligned} \quad 3.61$$

Simplificando:

$$\begin{aligned} \frac{\partial L_4}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_4 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\ &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} \right) \\ \frac{\partial L_4}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} \end{aligned} \quad 3.62$$

Desarrollando $\frac{\partial L_5}{\partial \dot{q}_j}$:

Se tiene:

$$L_5 = \frac{1}{2} \left(m_5 \mathbf{v}_{G5}^0{}^T \mathbf{v}_{G5}^0 + \boldsymbol{\omega}_{18}^0{}^T \mathbf{J}_5 \boldsymbol{\omega}_{18}^0 \right) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0$$

Sustituyendo ec.(3.42) y ec.(3.44):

$$\begin{aligned} L_5 &= \frac{1}{2} (m_5 (\mathbf{M}_9 \dot{\mathbf{q}})^T (\mathbf{M}_9 \dot{\mathbf{q}}) + (\mathbf{M}_{10} \dot{\mathbf{q}})^T \mathbf{J}_5 (\mathbf{M}_{10} \dot{\mathbf{q}})) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0 \\ &= \frac{1}{2} (\dot{\mathbf{q}}^T (m_5 \mathbf{M}_9^T \mathbf{M}_9 + \mathbf{M}_{10}^T \mathbf{J}_5 \mathbf{M}_{10}) \dot{\mathbf{q}}) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0 \\ L_5 &= \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_5 \dot{\mathbf{q}}) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0 \end{aligned} \quad 3.63$$

Dónde:

$$\mathbf{N}_5 = m_5 \mathbf{M}_9^T \mathbf{M}_9 + \mathbf{M}_{10}^T \mathbf{J}_5 \mathbf{M}_{10}$$

Efectuando la derivada:

$$\begin{aligned}\frac{\partial L_5}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_5 \dot{\mathbf{q}}) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0 \right) \\ \frac{\partial L_5}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_5 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right)\end{aligned}\tag{3.64}$$

Simplificando:

$$\begin{aligned}\frac{\partial L_5}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_5 \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\ &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \dot{\mathbf{q}} \right) \\ \frac{\partial L_5}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \dot{\mathbf{q}}\end{aligned}\tag{3.65}$$

Al evaluar el término $\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}$, dependerá que valor tome j , de tal manera que se tienen los siguientes resultados para diferente valor del iterador j . De esta forma, para:

$j=1$

$$\begin{aligned}\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_1} &= \frac{\partial}{\partial \dot{q}_1} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{x}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] \\ \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_1} &= [1, 0, 0, 0, 0, 0]\end{aligned}$$

$j=2$

$$\begin{aligned}\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_2} &= \frac{\partial}{\partial \dot{q}_2} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{y}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] \\ \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_2} &= [0, 1, 0, 0, 0, 0]\end{aligned}$$

$j=3$

$$\begin{aligned}\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_3} &= \frac{\partial}{\partial \dot{q}_3} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{z}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] \\ \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_3} &= [0, 0, 1, 0, 0, 0]\end{aligned}$$

$j=4$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_4} = \frac{\partial}{\partial \dot{q}_4} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{\psi}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_4} = [0, 0, 0, 1, 0, 0]$$

$j=5$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_5} = \frac{\partial}{\partial \dot{q}_5} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{\theta}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_5} = [0, 0, 0, 0, 1, 0]$$

$j=6$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_6} = \frac{\partial}{\partial \dot{q}_6} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}] = \frac{\partial}{\partial \dot{\phi}} [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_6} = [0, 0, 0, 0, 0, 1]$$

Por lo tanto, se hace notar que al derivar el término $\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}$ respecto al tiempo se tiene:

$$\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) = 0$$

Ahora, tomando la ecuación (3.48) y derivando respecto al tiempo se tiene:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = \frac{d}{dt} \left(\sum_{h=1}^5 \frac{\partial L_h}{\partial \dot{q}_j} + \sum_{i=1}^2 \left(\frac{\partial L_{1i}}{\partial \dot{q}_j} + \frac{\partial L_{2i}}{\partial \dot{q}_j} \right) \right) \quad 3.66$$

El desarrollo de la derivada con respecto al tiempo de cada elemento se muestra a continuación:

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_1}{\partial \dot{q}_j} \right)$:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L_1}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \dot{\mathbf{q}} \right) \\ &= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) \mathbf{N}_1 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \frac{d\mathbf{N}_1}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \frac{d\dot{\mathbf{q}}}{dt} \right) \\ &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_1 \dot{\mathbf{q}} \\ \frac{d}{dt} \left(\frac{\partial L_1}{\partial \dot{q}_j} \right) &= \mathbb{D}_{1j} \ddot{\mathbf{q}} + \mathbb{V}_{1j} \dot{\mathbf{q}} \end{aligned} \quad 3.67$$

Dónde:

$$\begin{aligned}\mathbb{D}_{1j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_1 \\ \mathbb{V}_{1j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_1\end{aligned}\tag{3.68}$$

Además:

$$\begin{aligned}\mathbf{N}_1 &= m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2 \\ \mathbf{M}_1 &= \mathbf{k}_1 \mathbf{k}_2^T \\ \mathbf{M}_2 &= \mathbf{j}_1^0 \mathbf{k}_2^T \\ \mathbf{J}_1^0 &= \mathbf{R}_2^0 \mathbf{J}_1^2 \mathbf{R}_2^{0T} \\ \mathbf{k}_1 &= \mathbf{j}_1^0 \times \mathbf{r}_{G1'}^0 \\ \mathbf{k}_2^T &= \frac{1}{V_1} [V_2, V_3, V_4, V_5, V_6, V_7] \\ \mathbf{r}_{G1'}^0 &= \mathbf{R}_2^0 \mathbf{r}_{G1'}^2 \\ \mathbf{R}_2^0 &= \mathbf{R}_y(\theta_{21}) \\ \mathbf{j}_1^0 &= \mathbf{j}_0 \\ \mathbf{r}_{G1'}^2 &= [x_{G1}, y_{G1}, z_{G1}]^T\end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_1 = m_1 \left(\dot{\mathbf{M}}_1^T \mathbf{M}_1 + \mathbf{M}_1^T \dot{\mathbf{M}}_1 \right) + (\dot{\mathbf{M}}_2^T \mathbf{J}_1 \mathbf{M}_2 + \mathbf{M}_2^T \dot{\mathbf{J}}_1 \mathbf{M}_2 + \mathbf{M}_2^T \mathbf{J}_1 \dot{\mathbf{M}}_2)$$

A su vez:

$$\begin{aligned}\dot{\mathbf{M}}_1 &= \dot{\mathbf{k}}_1 \mathbf{k}_2^T + \mathbf{k}_1 \dot{\mathbf{k}}_2^T \\ \dot{\mathbf{M}}_2 &= \mathbf{j}_1^0 \dot{\mathbf{k}}_2^T \\ \dot{\mathbf{J}}_1^0 &= \dot{\mathbf{R}}_2^0 \mathbf{J}_1^2 \mathbf{R}_2^{0T} + \mathbf{R}_2^0 \mathbf{J}_1^2 \dot{\mathbf{R}}_2^{0T} \\ \dot{\mathbf{k}}_1 &= \mathbf{j}_1^0 \times \dot{\mathbf{r}}_{G1'}^0 \\ \dot{\mathbf{k}}_2^T &= \frac{d}{dt} (V_1^{-1} [V_2, V_3, V_4, V_5, V_6, V_7]) \\ &= -V_1^{-2} \dot{V}_1 [V_2, V_3, V_4, V_5, V_6, V_7] + V_1^{-1} [\dot{V}_2, \dot{V}_3, \dot{V}_4, \dot{V}_5, \dot{V}_6, \dot{V}_7] \\ \dot{\mathbf{k}}_2^T &= -\frac{1}{V_1^2} \dot{V}_1 [V_2, V_3, V_4, V_5, V_6, V_7] + \frac{1}{V_1} [\dot{V}_2, \dot{V}_3, \dot{V}_4, \dot{V}_5, \dot{V}_6, \dot{V}_7] \\ \dot{\mathbf{r}}_{G1'}^0 &= \dot{\mathbf{R}}_2^0 \mathbf{r}_{G1'}^2 \\ \dot{\mathbf{R}}_2^0 &= \mathbf{B}_y(\theta_{21}) \dot{\theta}_{21} \\ \mathbf{B}_y(\theta_{21}) &= \frac{d\mathbf{R}_y(\theta_{21})}{d\theta_{21}}\end{aligned}$$

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_2}{\partial \dot{q}_j} \right)$:

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial L_2}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \dot{\mathbf{q}} \right) \\
 &= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) \mathbf{N}_2 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \frac{d\mathbf{N}_2}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \frac{d\dot{\mathbf{q}}}{dt} \right) \\
 &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_2 \dot{\mathbf{q}} \\
 \frac{d}{dt} \left(\frac{\partial L_2}{\partial \dot{q}_j} \right) &= \mathbb{D}_{2j} \ddot{\mathbf{q}} + \mathbb{V}_{2j} \dot{\mathbf{q}}
 \end{aligned} \tag{3.69}$$

Dónde:

$$\begin{aligned}
 \mathbb{D}_{2j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_2 \\
 \mathbb{V}_{2j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_2
 \end{aligned} \tag{3.70}$$

Además:

$$\begin{aligned}
 \mathbf{N}_2 &= m_2 \mathbf{M}_3^T \mathbf{M}_3 + \mathbf{M}_4^T \mathbf{J}_2 \mathbf{M}_4 \\
 \mathbf{M}_3 &= \mathbf{k}_3 \mathbf{k}_2^T + \mathbf{k}_4 \mathbf{k}_6^T + \mathbf{k}_5 \mathbf{k}_7^T \\
 \mathbf{M}_4 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T \\
 \mathbf{J}_2^0 &= \mathbf{R}_5^0 \mathbf{J}_2^5 \mathbf{R}_5^{0T} \\
 \mathbf{k}_3 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0) \\
 \mathbf{k}_4 &= \mathbf{j}_3^0 \times \mathbf{r}_{G2'}^0 \\
 \mathbf{k}_5 &= \mathbf{k}_4^0 \times \mathbf{r}_{G2'}^0 \\
 \mathbf{k}_6^T &= \frac{1}{V_8} [E_1, E_2, E_3, E_4, E_5, E_6] \\
 \mathbf{k}_7^T &= \frac{1}{V_{15}} [E_7, E_8, E_9, E_{10}, E_{11}, E_{12}] \\
 \mathbf{r}_{G2'}^0 &= \mathbf{R}_5^0 \mathbf{r}_{G2'}^5 \\
 \mathbf{k}_4^0 &= \mathbf{R}_4^0 \mathbf{k}_4^4 \\
 \mathbf{r}_{32}^0 &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 \\
 \mathbf{R}_4^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \\
 \mathbf{R}_5^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \\
 \mathbf{j}_3^0 &= \mathbf{j}_0
 \end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_2 = m_2 \left(\dot{\mathbf{M}}_3^T \mathbf{M}_3 + \mathbf{M}_3^T \dot{\mathbf{M}}_3 \right) + \left(\dot{\mathbf{M}}_4^T \mathbf{J}_2 \mathbf{M}_4 + \mathbf{M}_4^T \dot{\mathbf{J}}_2 \mathbf{M}_4 + \mathbf{M}_4^T \mathbf{J}_2 \dot{\mathbf{M}}_4 \right)$$

A su vez:

$$\begin{aligned}
 \dot{\mathbf{M}}_3 &= \dot{\mathbf{k}}_3 \mathbf{k}_2^T + \mathbf{k}_3 \dot{\mathbf{k}}_2^T + \dot{\mathbf{k}}_4 \mathbf{k}_6^T + \mathbf{k}_4 \dot{\mathbf{k}}_6^T + \dot{\mathbf{k}}_5 \mathbf{k}_7^T + \mathbf{k}_5 \dot{\mathbf{k}}_7^T \\
 \dot{\mathbf{M}}_4 &= \mathbf{j}_1^0 \dot{\mathbf{k}}_2^T + \dot{\mathbf{j}}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \dot{\mathbf{k}}_7^T + \dot{\mathbf{k}}_4^0 \mathbf{k}_7^T
 \end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{j}}_2^0 &= \dot{\mathbf{R}}_5^0 \mathbf{J}_2^5 \mathbf{R}_5^{0T} + \mathbf{R}_5^0 \mathbf{J}_2^5 \dot{\mathbf{R}}_5^{0T} \\
\dot{\mathbf{k}}_3 &= \dot{\mathbf{j}}_1^0 \times (\dot{\mathbf{r}}_{32}^0 + \dot{\mathbf{r}}_{G2'}^0) \\
\dot{\mathbf{k}}_4 &= \dot{\mathbf{j}}_3^0 \times \dot{\mathbf{r}}_{G2'}^0 \\
\dot{\mathbf{k}}_5 &= \dot{\mathbf{k}}_4^0 \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \times \dot{\mathbf{r}}_{G2'}^0 \\
\dot{\mathbf{k}}_6^T &= -\frac{1}{V_8^2} \dot{V}_8 [E_1, E_2, E_3, E_4, E_5, E_6] + \frac{1}{V_8} [\dot{E}_1, \dot{E}_2, \dot{E}_3, \dot{E}_4, \dot{E}_5, \dot{E}_6] \\
\dot{\mathbf{k}}_7^T &= -\frac{1}{V_{15}^2} \dot{V}_{15} [E_7, E_8, E_9, E_{10}, E_{11}, E_{12}] + \frac{1}{V_{15}} [\dot{E}_7, \dot{E}_8, \dot{E}_9, \dot{E}_{10}, \dot{E}_{11}, \dot{E}_{12}] \\
\dot{\mathbf{r}}_{32}^0 &= \dot{\mathbf{R}}_2^0 \mathbf{r}_{32}^2 \\
\dot{\mathbf{r}}_{G2'}^0 &= \dot{\mathbf{R}}_5^0 \mathbf{r}_{G2'}^5 \\
\dot{\mathbf{k}}_4^0 &= \dot{\mathbf{R}}_4^0 \mathbf{k}_4^4 \\
\dot{\mathbf{R}}_5^0 &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \dot{\theta}_{21} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \dot{\theta}_{43} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \dot{\theta}_{54} \\
\mathbf{B}_y(\theta_{43}) &= \frac{d\mathbf{R}_y(\theta_{43})}{d\theta_{43}} \\
\mathbf{B}_z(\theta_{54}) &= \frac{d\mathbf{R}_z(\theta_{54})}{d\theta_{54}} \\
\dot{\mathbf{R}}_4^0 &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \dot{\theta}_{21} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \dot{\theta}_{43}
\end{aligned}$$

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_3}{\partial \dot{\mathbf{q}}_j} \right)$:

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_3}{\partial \dot{\mathbf{q}}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \mathbf{N}_3 \dot{\mathbf{q}} \right) \\
&= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \right) \mathbf{N}_3 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \frac{d\mathbf{N}_3}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \mathbf{N}_3 \frac{d\dot{\mathbf{q}}}{dt} \right) \\
&= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \mathbf{N}_3 \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \dot{\mathbf{N}}_3 \dot{\mathbf{q}} \\
\frac{d}{dt} \left(\frac{\partial L_3}{\partial \dot{\mathbf{q}}_j} \right) &= \mathbb{D}_{3j} \ddot{\mathbf{q}} + \mathbb{V}_{3j} \dot{\mathbf{q}}
\end{aligned} \tag{3.71}$$

Dónde:

$$\begin{aligned}
\mathbb{D}_{3j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \mathbf{N}_3 \\
\mathbb{V}_{3j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{\mathbf{q}}_j} \dot{\mathbf{N}}_3
\end{aligned} \tag{3.72}$$

Además:

$$\begin{aligned}
\mathbf{N}_3 &= m_3 \mathbf{M}_5^T \mathbf{M}_5 + \mathbf{M}_6^T \mathbf{J}_3 \mathbf{M}_6 \\
\mathbf{M}_5 &= \mathbf{k}_8 \mathbf{k}_2^T + \mathbf{k}_9 \mathbf{k}_6^T + \mathbf{k}_{10} \mathbf{k}_7^T + \mathbf{k}_{11} \mathbf{k}_{12}^T
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_6 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T \\
\mathbf{J}_3^0 &= \mathbf{R}_{16}^0 \mathbf{J}_3^{16} \mathbf{R}_{16}^{0^T} \\
\mathbf{k}_8 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) \\
\mathbf{k}_9 &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{10} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{11} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G3'}^0 \\
\mathbf{k}_{12}^T &= \frac{1}{V_{36}} [E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}] \\
\mathbf{i}_{15}^0 &= \mathbf{R}_{15}^0 \mathbf{i}_{15}^{15} \\
\mathbf{r}_{14,5i}^0 &= \mathbf{R}_5^0 \mathbf{r}_{14,5i}^5 \\
\mathbf{r}_{15,14i}^0 &= \mathbf{R}_{14}^0 \mathbf{r}_{15,14i}^{14} \\
\mathbf{r}_{G3'}^0 &= \mathbf{R}_{16}^0 \mathbf{r}_{G3'}^{16} \\
\mathbf{R}_{14}^0 &= \mathbf{R}_5^0 \\
\mathbf{R}_{15}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \\
\mathbf{R}_{16}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15})
\end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_3 = m_3 (\dot{\mathbf{M}}_5^T \mathbf{M}_5 + \mathbf{M}_5^T \dot{\mathbf{M}}_5) + (\dot{\mathbf{M}}_6^T \mathbf{J}_3 \mathbf{M}_6 + \mathbf{M}_6^T \dot{\mathbf{J}}_3 \mathbf{M}_6 + \mathbf{M}_6^T \mathbf{J}_3 \dot{\mathbf{M}}_6)$$

A su vez:

$$\begin{aligned}
\dot{\mathbf{M}}_5 &= \dot{\mathbf{k}}_8 \mathbf{k}_2^T + \mathbf{k}_8 \dot{\mathbf{k}}_2^T + \dot{\mathbf{k}}_9 \mathbf{k}_6^T + \mathbf{k}_9 \dot{\mathbf{k}}_6^T + \dot{\mathbf{k}}_{10} \mathbf{k}_7^T + \mathbf{k}_{10} \dot{\mathbf{k}}_7^T + \dot{\mathbf{k}}_{11} \mathbf{k}_{12}^T + \mathbf{k}_{11} \dot{\mathbf{k}}_{12}^T \\
\dot{\mathbf{M}}_6 &= \mathbf{j}_1^0 \dot{\mathbf{k}}_2^T + \mathbf{j}_3^0 \dot{\mathbf{k}}_6^T + \mathbf{k}_4^0 \dot{\mathbf{k}}_7^T + \mathbf{k}_4^0 \dot{\mathbf{k}}_7^T + \mathbf{i}_{15}^0 \dot{\mathbf{k}}_{12}^T + \mathbf{i}_{15}^0 \dot{\mathbf{k}}_{12}^T \\
\dot{\mathbf{J}}_3^0 &= \dot{\mathbf{R}}_{16}^0 \mathbf{J}_3^{16} \mathbf{R}_{16}^{0^T} + \mathbf{R}_{16}^0 \mathbf{J}_3^{16} \dot{\mathbf{R}}_{16}^{0^T} \\
\dot{\mathbf{k}}_8 &= \mathbf{j}_1^0 \times (\dot{\mathbf{r}}_{32}^0 + \dot{\mathbf{r}}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0) \\
\dot{\mathbf{k}}_9 &= \mathbf{j}_3^0 \times (\dot{\mathbf{r}}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0 + \dot{\mathbf{r}}_{G3'}^0) \\
\dot{\mathbf{k}}_{10} &= \dot{\mathbf{k}}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) + \mathbf{k}_4^0 \times (\dot{\mathbf{r}}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0 + \dot{\mathbf{r}}_{G3'}^0) \\
\dot{\mathbf{k}}_{11} &= \dot{\mathbf{i}}_{15}^0 \times \mathbf{r}_{G3'}^0 + \mathbf{i}_{15}^0 \times \dot{\mathbf{r}}_{G3'}^0 \\
\dot{\mathbf{k}}_{12}^T &= -\frac{1}{V_{36}^2} \dot{V}_{36} [E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}] + \frac{1}{V_{36}} [\dot{E}_{25}, \dot{E}_{26}, \dot{E}_{27}, \dot{E}_{28}, \dot{E}_{29}, \dot{E}_{30}] \\
\dot{\mathbf{r}}_{14,5i}^0 &= \dot{\mathbf{R}}_5^0 \mathbf{r}_{14,5i}^5 \\
\dot{\mathbf{r}}_{15,14i}^0 &= \dot{\mathbf{R}}_{14}^0 \mathbf{r}_{15,14i}^{14} \\
\dot{\mathbf{r}}_{G3'}^0 &= \dot{\mathbf{R}}_{16}^0 \mathbf{r}_{G3'}^{16} \\
\dot{\mathbf{i}}_{15}^0 &= \dot{\mathbf{R}}_{15}^0 \mathbf{i}_{15}^{15} \\
\dot{\mathbf{R}}_{15}^0 &= \dot{\mathbf{R}}_5^0 \\
\dot{\mathbf{R}}_{16}^0 &= \mathbf{B}_y(\theta_{21}) \dot{\theta}_{21} \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \dot{\theta}_{43} \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \dot{\theta}_{54} \mathbf{R}_x(\theta_{16,15}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \dot{\theta}_{16,15}
\end{aligned}$$

$$\mathbf{B}_x(\theta_{16,15}) = \frac{d\mathbf{R}_x(\theta_{16,15})}{d\theta_{16,15}}$$

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_4}{\partial \dot{q}_j} \right)$:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L_4}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \dot{\mathbf{q}} \right) \\ &= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) \mathbf{N}_4 \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \frac{d\mathbf{N}_4}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \frac{d\dot{\mathbf{q}}}{dt} \right) \\ &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_4 \dot{\mathbf{q}} \\ \frac{d}{dt} \left(\frac{\partial L_4}{\partial \dot{q}_j} \right) &= \mathbb{D}_{4j} \ddot{\mathbf{q}} + \mathbb{V}_{4j} \dot{\mathbf{q}} \end{aligned} \tag{3.73}$$

Dónde:

$$\begin{aligned} \mathbb{D}_{4j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_4 \\ \mathbb{V}_{4j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_4 \end{aligned} \tag{3.74}$$

Además:

$$\begin{aligned} \mathbf{N}_4 &= m_4 \mathbf{M}_7^T \mathbf{M}_7 + \mathbf{M}_8^T \mathbf{J}_4 \mathbf{M}_8 \\ \mathbf{M}_7 &= \mathbf{k}_{13} \mathbf{k}_2^T + \mathbf{k}_{14} \mathbf{k}_6^T + \mathbf{k}_{15} \mathbf{k}_7^T + \mathbf{k}_{16} \mathbf{k}_{12}^T + \mathbf{k}_{17} \mathbf{k}_{18}^T \\ \mathbf{M}_8 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T + \mathbf{k}_{16}^0 \mathbf{k}_{18}^T \\ \mathbf{J}_4^0 &= \mathbf{R}_{17}^0 \mathbf{J}_4^{17} \mathbf{R}_{17}^{0^T} \\ \mathbf{k}_{13} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{14} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{15} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{16} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{k}_{17} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{k}_{18}^T &= \frac{1}{V_{29}} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] \\ \mathbf{r}_{G4'}^0 &= \mathbf{R}_{17}^0 \mathbf{r}_{G4'}^{17}, \\ \mathbf{k}_{16}^0 &= \mathbf{R}_{16}^0 \mathbf{k}_{16}^{16} \\ \mathbf{R}_{17}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \end{aligned}$$

Derivando:

$$\dot{N}_4 = m_4 \left(\dot{M}_7^T M_7 + M_7^T \dot{M}_7 \right) + (\dot{M}_8^T J_4 M_8 + M_8^T J_4 \dot{M}_8 + M_8^T J_4 \dot{M}_8)$$

A su vez:

$$\begin{aligned} \dot{M}_7 &= \dot{k}_{13} k_2^T + k_{13} \dot{k}_2^T + \dot{k}_{14} k_6^T + k_{14} \dot{k}_6^T + \dot{k}_{15} k_7^T + k_{15} \dot{k}_7^T + \dot{k}_{16} k_{12}^T + k_{16} \dot{k}_{12}^T \\ &\quad + \dot{k}_{17} k_{18}^T + k_{17} \dot{k}_{18}^T \\ \dot{M}_8 &= j_1^0 \dot{k}_2^T + j_3^0 \dot{k}_6^T + \dot{k}_4^0 k_7^T + k_4^0 \dot{k}_7^T + i_{15}^0 k_{12}^T + i_{15}^0 \dot{k}_{12}^T + \dot{k}_{16}^0 k_{18}^T + k_{16}^0 \dot{k}_{18}^T \\ j_4^0 &= \dot{R}_{17}^0 J_4^{17} R_{17}^{0T} + R_{17}^0 J_4^{17} \dot{R}_{17}^{0T} \\ \dot{k}_{13} &= j_1^0 \times (\dot{r}_{32}^0 + \dot{r}_{14,5i}^0 + \dot{r}_{15,14i}^0 + \dot{r}_{G4'}^0) \\ \dot{k}_{14} &= j_3^0 \times (\dot{r}_{14,5i}^0 + \dot{r}_{15,14i}^0 + \dot{r}_{G4'}^0) \\ \dot{k}_{15} &= \dot{k}_4^0 \times (r_{14,5i}^0 + r_{15,14i}^0 + r_{G4'}^0) + k_4^0 \times (\dot{r}_{14,5i}^0 + \dot{r}_{15,14i}^0 + \dot{r}_{G4'}^0) \\ \dot{k}_{16} &= i_{15}^0 \times r_{G4'}^0 + i_{15}^0 \times \dot{r}_{G4'}^0 \\ \dot{k}_{17} &= \dot{k}_{16}^0 \times r_{G4'}^0 + k_{16}^0 \times \dot{r}_{G4'}^0 \\ \dot{k}_{16}^0 &= \dot{R}_{16}^0 k_{16}^{16} \\ \dot{k}_{18}^T &= -\frac{1}{V_{29}^2} \dot{V}_{29} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] + \frac{1}{V_{29}} [\dot{E}_{19}, \dot{E}_{20}, \dot{E}_{21}, \dot{E}_{22}, \dot{E}_{23}, \dot{E}_{24}] \\ \dot{r}_{G4'}^0 &= \dot{R}_{17}^0 r_{G4'}^{17} \\ \dot{R}_{17}^0 &= B_y(\theta_{21}) \dot{\theta}_{21} R_y(\theta_{43}) R_z(\theta_{54}) R_x(\theta_{16,15}) R_z(\theta_{17,16}) \\ &\quad + R_y(\theta_{21}) B_y(\theta_{43}) \dot{\theta}_{43} R_z(\theta_{54}) R_x(\theta_{16,15}) R_z(\theta_{17,16}) \\ &\quad + R_y(\theta_{21}) R_y(\theta_{43}) B_z(\theta_{54}) \dot{\theta}_{54} R_x(\theta_{16,15}) R_z(\theta_{17,16}) \\ &\quad + R_y(\theta_{21}) R_y(\theta_{43}) R_z(\theta_{54}) B_x(\theta_{16,15}) \dot{\theta}_{16,15} R_z(\theta_{17,16}) \\ &\quad + R_y(\theta_{21}) R_y(\theta_{43}) R_z(\theta_{54}) R_x(\theta_{16,15}) B_z(\theta_{17,16}) \dot{\theta}_{17,16} \\ B_z(\theta_{17,16}) &= \frac{dR_z(\theta_{17,16})}{d\theta_{17,16}} \end{aligned}$$

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_5}{\partial \dot{q}_j} \right)$:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L_5}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{q}^T}{\partial \dot{q}_j} N_5 \dot{q} \right) \\ &= \left(\frac{d}{dt} \left(\frac{\partial \dot{q}^T}{\partial \dot{q}_j} \right) N_5 \dot{q} + \frac{\partial \dot{q}^T}{\partial \dot{q}_j} \frac{dN_5}{dt} \dot{q} + \frac{\partial \dot{q}^T}{\partial \dot{q}_j} N_5 \frac{d\dot{q}}{dt} \right) \\ &= \frac{\partial \dot{q}^T}{\partial \dot{q}_j} N_5 \ddot{q} + \frac{\partial \dot{q}^T}{\partial \dot{q}_j} \dot{N}_5 \dot{q} \\ \frac{d}{dt} \left(\frac{\partial L_5}{\partial \dot{q}_j} \right) &= \mathbb{D}_{5j} \ddot{q} + \mathbb{V}_{5j} \dot{q} \end{aligned} \tag{3.75}$$

Dónde:

$$\begin{aligned}\mathbb{D}_{5j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_5 \\ \mathbb{V}_{5j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_5\end{aligned}\tag{3.76}$$

Además:

$$\begin{aligned}\mathbf{N}_5 &= m_5 \mathbf{M}_9^T \mathbf{M}_9 + \mathbf{M}_{10}^T \mathbf{J}_5 \mathbf{M}_{10} \\ \mathbf{M}_9 &= \mathbf{k}_{19} \mathbf{k}_2^T + \mathbf{k}_{20} \mathbf{k}_6^T + \mathbf{k}_{21} \mathbf{k}_7^T + \mathbf{k}_{22} \mathbf{k}_{12}^T + \mathbf{k}_{23} \mathbf{k}_{18}^T + \mathbf{k}_{24} \mathbf{k}_{25}^T \\ \mathbf{M}_{10} &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T + \mathbf{k}_{16}^0 \mathbf{k}_{18}^T + \mathbf{i}_{17}^0 \mathbf{k}_{25}^T \\ \mathbf{J}_5^0 &= \mathbf{R}_{18}^0 \mathbf{J}_5^{18} \mathbf{R}_{18}^{0^T} \\ \mathbf{k}_{19} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\ \mathbf{k}_{20} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\ \mathbf{k}_{21} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\ \mathbf{k}_{22} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G5'}^0 \\ \mathbf{k}_{23} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G5'}^0 \\ \mathbf{k}_{24} &= \mathbf{i}_{17}^0 \times \mathbf{r}_{G5'}^0 \\ \mathbf{k}_{25}^T &= \frac{1}{V_{23}} [E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_{18}] \\ \mathbf{i}_{17}^0 &= \mathbf{R}_{17}^0 \mathbf{i}_{17}^{17} \\ \mathbf{r}_{G5'}^0 &= \mathbf{R}_{18}^0 \mathbf{r}_{G5'}^{18} \\ \mathbf{R}_{18}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \\ \mathbf{r}_{G5'}^{18} &= [x_{G5}, y_{G5}, z_{G5}]^T\end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_5 = m_5 \left(\dot{\mathbf{M}}_9^T \mathbf{M}_9 + \mathbf{M}_9^T \dot{\mathbf{M}}_9 \right) + (\dot{\mathbf{M}}_{10}^T \mathbf{J}_5 \mathbf{M}_{10} + \mathbf{M}_{10}^T \dot{\mathbf{J}}_5 \mathbf{M}_{10} + \mathbf{M}_{10}^T \mathbf{J}_5 \dot{\mathbf{M}}_{10})$$

A su vez:

$$\begin{aligned}\dot{\mathbf{M}}_9 &= \dot{\mathbf{k}}_{19} \mathbf{k}_2^T + \mathbf{k}_{19} \dot{\mathbf{k}}_2^T + \dot{\mathbf{k}}_{20} \mathbf{k}_6^T + \mathbf{k}_{20} \dot{\mathbf{k}}_6^T + \dot{\mathbf{k}}_{21} \mathbf{k}_7^T + \mathbf{k}_{21} \dot{\mathbf{k}}_7^T + \dot{\mathbf{k}}_{22} \mathbf{k}_{12}^T + \mathbf{k}_{22} \dot{\mathbf{k}}_{12}^T + \dot{\mathbf{k}}_{23} \mathbf{k}_{18}^T \\ &\quad + \mathbf{k}_{23} \dot{\mathbf{k}}_{18}^T + \dot{\mathbf{k}}_{24} \mathbf{k}_{25}^T + \mathbf{k}_{24} \dot{\mathbf{k}}_{25}^T \\ \dot{\mathbf{M}}_{10} &= \mathbf{j}_1^0 \dot{\mathbf{k}}_2^T + \mathbf{j}_3^0 \dot{\mathbf{k}}_6^T + \dot{\mathbf{k}}_4^0 \mathbf{k}_7^T + \mathbf{k}_4^0 \dot{\mathbf{k}}_7^T + \mathbf{i}_{15}^0 \dot{\mathbf{k}}_{12}^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T + \dot{\mathbf{k}}_{16}^0 \mathbf{k}_{18}^T + \mathbf{k}_{16}^0 \dot{\mathbf{k}}_{18}^T + \mathbf{i}_{17}^0 \dot{\mathbf{k}}_{25}^T \\ &\quad + \mathbf{i}_{17}^0 \mathbf{k}_{25}^T \\ \dot{\mathbf{J}}_5^0 &= \dot{\mathbf{R}}_{18}^0 \mathbf{J}_5^{18} \mathbf{R}_{18}^{0^T} + \mathbf{R}_{18}^0 \mathbf{J}_5^{18} \dot{\mathbf{R}}_{18}^{0^T} \\ \dot{\mathbf{k}}_{19} &= \dot{\mathbf{j}}_1^0 \times (\mathbf{r}_{32}^0 + \dot{\mathbf{r}}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0 + \dot{\mathbf{r}}_{G5'}^0) \\ \dot{\mathbf{k}}_{20} &= \dot{\mathbf{j}}_3^0 \times (\mathbf{r}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0 + \dot{\mathbf{r}}_{G5'}^0) \\ \dot{\mathbf{k}}_{21} &= \dot{\mathbf{k}}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + \mathbf{k}_4^0 \times (\dot{\mathbf{r}}_{14,5i}^0 + \dot{\mathbf{r}}_{15,14i}^0 + \dot{\mathbf{r}}_{G5'}^0) \\ \dot{\mathbf{k}}_{22} &= \dot{\mathbf{i}}_{15}^0 \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{15}^0 \times \dot{\mathbf{r}}_{G5'}^0 \\ \dot{\mathbf{k}}_{23} &= \dot{\mathbf{k}}_{16}^0 \times \mathbf{r}_{G5'}^0 + \mathbf{k}_{16}^0 \times \dot{\mathbf{r}}_{G5'}^0 \\ \dot{\mathbf{k}}_{24} &= \dot{\mathbf{i}}_{17}^0 \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{17}^0 \times \dot{\mathbf{r}}_{G5'}^0\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{k}}_{25}^T &= -\frac{1}{V_{23}^2} \dot{V}_{23} [E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_{18}] + \frac{1}{V_{23}} [\dot{E}_{13}, \dot{E}_{14}, \dot{E}_{15}, \dot{E}_{16}, \dot{E}_{17}, \dot{E}_{18}] \\
\dot{\mathbf{i}}_{17}^0 &= \dot{\mathbf{R}}_{17}^0 \mathbf{i}_{17}^{17} \\
\dot{\mathbf{r}}_{G5'}^0 &= \dot{\mathbf{R}}_{18}^0 \mathbf{r}_{G5'}^{18}, \\
\dot{\mathbf{R}}_{18}^0 &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \dot{\theta}_{21} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \dot{\theta}_{43} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \dot{\theta}_{54} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \dot{\theta}_{16,15} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \dot{\theta}_{17,16} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \dot{\theta}_{18,17}
\end{aligned}$$

3.3.2. Desarrollo del Segundo Término de la Ecuación de Lagrange para la cadena 0

Tomando la ec. (3.47) y aplicando la derivada parcial con respecto a la variable de coordenadas cartesianas:

$$\frac{\partial L}{\partial q_j} = \sum_{h=1}^5 \frac{\partial L_h}{\partial q_j} + \sum_{i=1}^2 \left(\frac{\partial L_{1i}}{\partial q_j} + \frac{\partial L_{2i}}{\partial q_j} \right)$$

Desarrollando $\frac{\partial L_1}{\partial q_j}$

$$L_1 = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_1 \dot{\mathbf{q}} + m_1 \mathbf{g}^T \mathbf{r}_{G1}^0$$

Dónde:

$$\begin{aligned}
\mathbf{N}_1 &= m_1 \mathbf{M}_1^T \mathbf{M}_1 + \mathbf{M}_2^T \mathbf{J}_1 \mathbf{M}_2 \\
\mathbf{M}_1 &= \mathbf{k}_1 \mathbf{k}_2^T \\
\mathbf{M}_2 &= \mathbf{j}_1^0 \mathbf{k}_2^T \\
\mathbf{J}_1^0 &= \mathbf{R}_2^0 \mathbf{J}_1^2 \mathbf{R}_2^{0T} \\
\mathbf{k}_1 &= \mathbf{j}_1^0 \times \mathbf{r}_{G1'}^0 \\
\mathbf{k}_2^T &= \frac{1}{V_1} [V_2, V_3, V_4, V_5, V_6, V_7] \\
\mathbf{r}_{G1'}^0 &= \mathbf{R}_2^0 \mathbf{r}_{G1'}^2 \\
\mathbf{R}_2^0 &= \mathbf{R}_y(\theta_{21}) \\
\mathbf{j}_1^0 &= \mathbf{j}_0 \\
\mathbf{r}_{G1'}^2 &= [x_{G1}, y_{G1}, z_{G1}]^T
\end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned}
\frac{\partial L_1}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_1}{\partial q_j} \dot{\mathbf{q}} + m_1 \mathbf{g}^T \frac{\partial \mathbf{r}_{G1}^0}{\partial q_j} \\
\frac{\partial L_1}{\partial q_j} &= \mathbb{V}'_{1j}{}^T \dot{\mathbf{q}} + \mathbb{C}_{1j}
\end{aligned} \tag{3.77}$$

Dónde:

$$\begin{aligned}
\mathbb{V}'_{1j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_1}{\partial q_j} \\
\mathbb{C}_{1j} &= m_1 \mathbf{g}^T \frac{\partial \mathbf{r}_{G1}^0}{\partial q_j}
\end{aligned} \tag{3.78}$$

A su vez:

$$\begin{aligned}
\frac{\partial \mathbf{N}_1}{\partial q_j} &= m_1 \left(\frac{\partial \mathbf{M}_1^T}{\partial q_j} \mathbf{M}_1 + \mathbf{M}_1^T \frac{\partial \mathbf{M}_1}{\partial q_j} \right) + \frac{\partial \mathbf{M}_2^T}{\partial q_j} \mathbf{J}_1 \mathbf{M}_2 + \mathbf{M}_2^T \frac{\partial \mathbf{J}_1}{\partial q_j} \mathbf{M}_2 + \mathbf{M}_2^T \mathbf{J}_1 \frac{\partial \mathbf{M}_2}{\partial q_j} \\
\frac{\partial \mathbf{M}_1}{\partial q_j} &= \frac{\partial \mathbf{k}_1}{\partial q_j} \mathbf{k}_2^T + \mathbf{k}_1 \frac{\partial \mathbf{k}_2^T}{\partial q_j} \\
\frac{\partial \mathbf{M}_2}{\partial q_j} &= \mathbf{j}_1^0 \frac{\partial \mathbf{k}_2^T}{\partial q_j} \\
\frac{\partial \mathbf{J}_1^0}{\partial q_j} &= \frac{\partial \mathbf{R}_2^0}{\partial q_j} \mathbf{J}_1^2 \mathbf{R}_2^{0T} + \mathbf{R}_2^0 \mathbf{J}_1^2 \frac{\partial \mathbf{R}_2^{0T}}{\partial q_j} \\
\frac{\partial \mathbf{k}_1}{\partial q_j} &= \mathbf{j}_1^0 \times \frac{\partial \mathbf{r}_{G1'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_2^T}{\partial q_j} &= -\frac{1}{V_1^2} \frac{\partial V_1}{\partial q_j} [V_2, V_3, V_4, V_5, V_6, V_7] + \frac{1}{V_1} \left[\frac{\partial V_2}{\partial q_j}, \frac{\partial V_3}{\partial q_j}, \frac{\partial V_4}{\partial q_j}, \frac{\partial V_5}{\partial q_j}, \frac{\partial V_6}{\partial q_j}, \frac{\partial V_7}{\partial q_j} \right] \\
\frac{\partial \mathbf{r}_{10}^0}{\partial q_j} &= 0 \\
\frac{\partial \mathbf{r}_{G1'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_2^0}{\partial q_j} \mathbf{r}_{G1'}^2 \\
\frac{\partial \mathbf{r}_{G1}^0}{\partial q_j} &= \frac{\partial \mathbf{r}_{10}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1'}^0}{\partial q_j} \\
\frac{\partial \mathbf{R}_2^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} = \mathbf{B}_y(\theta_{21}) \frac{\partial \theta_{21}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_1}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_1}{\partial q_j} &= \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \frac{\partial \theta_{21}}{\partial q_j} \mathbf{r}_{G1'}^2 = (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{G1'}^2) \frac{\partial \theta_{21}}{\partial q_j} \\
\frac{\partial \mathbf{k}_1}{\partial q_j} &= \mathbf{J}_1 \frac{\partial \theta}{\partial q_j}
\end{aligned} \tag{3.79}$$

Dónde:

$$J_1 = [(j_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{G1'}^2) \quad \mathbf{0} \quad \mathbf{0}]$$

Desarrollando $\frac{\partial \mathbf{k}_2^T}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_2^T}{\partial q_j} &= \frac{\partial \mathbf{k}_2^T}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial \mathbf{k}_2'^T}{\partial q_j} \\ \frac{\partial \mathbf{k}_2^T}{\partial q_j} &= J_2 \frac{\partial \theta}{\partial q_j} + \frac{\partial \mathbf{k}_2'^T}{\partial q_j} \end{aligned} \quad 3.80$$

Dónde:

$$J_2 = \left[\frac{\partial \mathbf{k}_2^T}{\partial \theta_{21}} \quad \mathbf{0} \quad \mathbf{0} \right]$$

De manera general cada V es función de las coordenadas generalizadas $\mathbf{q}$ y de los ángulos de la cadena, es decir:

$$V = V(\theta_{21}, \theta_{43}, \theta_{54}, \theta_{1615}, \theta_{1716}, \theta_{1817}, x, y, z, \psi, \theta, \phi)$$

Por lo tanto:

$$\begin{aligned} \frac{\partial V}{\partial q_j} &= \frac{\partial V}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial V}{\partial \theta_{43}} \frac{\partial \theta_{43}}{\partial q_j} + \frac{\partial V}{\partial \theta_{54}} \frac{\partial \theta_{54}}{\partial q_j} + \frac{\partial V}{\partial \theta_{1615}} \frac{\partial \theta_{1615}}{\partial q_j} + \frac{\partial V}{\partial \theta_{1716}} \frac{\partial \theta_{1716}}{\partial q_j} \\ &\quad + \frac{\partial V}{\partial \theta_{1817}} \frac{\partial \theta_{1817}}{\partial q_j} + \frac{\partial V}{\partial x} \frac{\partial x}{\partial q_j} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial q_j} + \frac{\partial V}{\partial z} \frac{\partial z}{\partial q_j} + \frac{\partial V}{\partial \psi} \frac{\partial \psi}{\partial q_j} + \frac{\partial V}{\partial \theta} \frac{\partial \theta}{\partial q_j} \\ &\quad + \frac{\partial V}{\partial \phi} \frac{\partial \phi}{\partial q_j} \end{aligned}$$

Los términos $(\frac{\partial \theta_{21}}{\partial q_j}, \frac{\partial \theta_{43}}{\partial q_j}, \frac{\partial \theta_{54}}{\partial q_j}, \frac{\partial \theta_{1615}}{\partial q_j}, \frac{\partial \theta_{1716}}{\partial q_j}, \frac{\partial \theta_{1817}}{\partial q_j})$ se determinan en el **apéndice A**

Desarrollando $\frac{\partial L_2}{\partial q_j}$

$$L_2 = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_2 \dot{\mathbf{q}} + m_2 \mathbf{g}^T \mathbf{r}_{G2}^0$$

Dónde:

$$\begin{aligned} \mathbf{N}_2 &= m_2 \mathbf{M}_3^T \mathbf{M}_3 + \mathbf{M}_4^T J_2 \mathbf{M}_4 \\ \mathbf{M}_3 &= \mathbf{k}_3 \mathbf{k}_2^T + \mathbf{k}_4 \mathbf{k}_6^T + \mathbf{k}_5 \mathbf{k}_7^T \end{aligned}$$

$$\begin{aligned}
\mathbf{M}_4 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T \\
\mathbf{J}_2^0 &= \mathbf{R}_5^0 \mathbf{J}_2^5 \mathbf{R}_5^{0T} \\
\mathbf{k}_3 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{G2'}^0) \\
\mathbf{k}_4 &= \mathbf{j}_3^0 \times \mathbf{r}_{G2'}^0 \\
\mathbf{k}_5 &= \mathbf{k}_4^0 \times \mathbf{r}_{G2'}^0 \\
\mathbf{k}_6^T &= \frac{1}{V_8} [E_1, E_2, E_3, E_4, E_5, E_6] \\
\mathbf{k}_7^T &= \frac{1}{V_{15}} [E_7, E_8, E_9, E_{10}, E_{11}, E_{12}] \\
\mathbf{k}_4^0 &= \mathbf{R}_4^0 \mathbf{k}_4^4 \\
\mathbf{r}_{32}^0 &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 \\
\mathbf{r}_{G2'}^0 &= \mathbf{R}_5^0 \mathbf{r}_{G2'}^5 \\
\mathbf{R}_4^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \\
\mathbf{R}_5^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \\
\mathbf{j}_3^0 &= \mathbf{j}_0 \\
\mathbf{r}_{32}^2 &= [x_{32}, 0, 0]^T \\
\mathbf{r}_{G2'}^5 &= [x_{G2}, y_{G2}, z_{G2}]^T
\end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned}
\frac{\partial L_2}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_2}{\partial q_j} \dot{\mathbf{q}} + m_2 \mathbf{g}^T \frac{\partial \mathbf{r}_{G2}^0}{\partial q_j} \\
\frac{\partial L_2}{\partial q_j} &= \mathbb{V}'_{2j} \dot{\mathbf{q}} + \mathbb{C}_{2j}
\end{aligned} \tag{3.81}$$

Dónde:

$$\begin{aligned}
\mathbb{V}'_{2j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_2}{\partial q_j} \\
\mathbb{C}_{2j} &= m_1 \mathbf{g}^T \frac{\partial \mathbf{r}_{G2}^0}{\partial q_j}
\end{aligned} \tag{3.82}$$

A su vez:

$$\begin{aligned}
\frac{\partial \mathbf{N}_2}{\partial q_j} &= m_2 \left(\frac{\partial \mathbf{M}_3^T}{\partial q_j} \mathbf{M}_3 + \mathbf{M}_3^T \frac{\partial \mathbf{M}_3}{\partial q_j} \right) + \frac{\partial \mathbf{M}_4^T}{\partial q_j} \mathbf{J}_2 \mathbf{M}_4 + \mathbf{M}_4^T \frac{\partial \mathbf{J}_2}{\partial q_j} \mathbf{M}_4 + \mathbf{M}_4^T \mathbf{J}_2 \frac{\partial \mathbf{M}_4}{\partial q_j} \\
\frac{\partial \mathbf{M}_3}{\partial q_j} &= \frac{\partial \mathbf{k}_3}{\partial q_j} \mathbf{k}_2^T + \mathbf{k}_3 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \frac{\partial \mathbf{k}_4}{\partial q_j} \mathbf{k}_6^T + \mathbf{k}_4 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_5}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_5 \frac{\partial \mathbf{k}_7^T}{\partial q_j} \\
\frac{\partial \mathbf{M}_4}{\partial q_j} &= \mathbf{j}_1^0 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \mathbf{j}_3^0 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_4^0}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_4^0 \frac{\partial \mathbf{k}_7^T}{\partial q_j} \\
\frac{\partial \mathbf{J}_2^0}{\partial q_j} &= \frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{J}_2^5 \mathbf{R}_5^{0T} + \mathbf{R}_5^0 \mathbf{J}_2^5 \frac{\partial \mathbf{R}_5^{0T}}{\partial q_j} \\
\frac{\partial \mathbf{k}_3}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2'}^0}{\partial q_j} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_4}{\partial q_j} &= \mathbf{j}_3^0 \times \frac{\partial \mathbf{r}_{G2'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_5}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \times \frac{\partial \mathbf{r}_{G2'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_6^T}{\partial q_j} &= -\frac{1}{V_8^2} \frac{\partial V_8}{\partial q_j} [E_1, E_2, E_3, E_4, E_5, E_6] + \frac{1}{V_8} \left[\frac{\partial E_1}{\partial q_j}, \frac{\partial E_2}{\partial q_j}, \frac{\partial E_3}{\partial q_j}, \frac{\partial E_4}{\partial q_j}, \frac{\partial E_5}{\partial q_j}, \frac{\partial E_6}{\partial q_j} \right] \\
\frac{\partial \mathbf{k}_7^T}{\partial q_j} &= -\frac{1}{V_{15}^2} \frac{\partial V_{15}}{\partial q_j} [E_7, E_8, E_9, E_{10}, E_{11}, E_{12}] + \frac{1}{V_8} \left[\frac{\partial E_7}{\partial q_j}, \frac{\partial E_8}{\partial q_j}, \frac{\partial E_9}{\partial q_j}, \frac{\partial E_{10}}{\partial q_j}, \frac{\partial E_{11}}{\partial q_j}, \frac{\partial E_{12}}{\partial q_j} \right] \\
\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_2^0}{\partial q_j} \mathbf{r}_{32}^2 \\
\frac{\partial \mathbf{r}_{G2'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{G2'}^5 \\
\frac{\partial \mathbf{k}_4^0}{\partial q_j} &= \frac{\partial \mathbf{R}_4^0}{\partial q_j} \mathbf{k}_4^4 \\
\frac{\partial \mathbf{R}_5^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) + \mathbf{R}_y(\theta_{21}) \frac{\partial \mathbf{R}_y(\theta_{43})}{\partial q_j} \mathbf{R}_z(\theta_{54}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \frac{\partial \mathbf{R}_z(\theta_{54})}{\partial q_j} \\
&= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{R}_4^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} \mathbf{R}_y(\theta_{43}) + \mathbf{R}_y(\theta_{21}) \frac{\partial \mathbf{R}_y(\theta_{43})}{\partial q_j} \\
\frac{\partial \mathbf{R}_4^0}{\partial q_j} &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \frac{\partial \theta_{43}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_3}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_3}{\partial q_j} &= \left(\mathbf{j}_1^0 \times \left(\mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{G2'}^5 \right) \right) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \left(\mathbf{j}_1^0 \times \left(\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{G2'}^5 \right) \right) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \left(\mathbf{j}_1^0 \times \left(\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{G2'}^5 \right) \right) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{k}_3}{\partial q_j} &= \mathbf{J}_3 \frac{\partial \theta}{\partial q_j}
\end{aligned}$$

3.83

Dónde:

$$J_3 = [J_{31} \quad J_{32} \quad J_{33}]$$

A su vez:

$$\begin{aligned} J_{31} &= \mathbf{j}_1^0 \times (\mathbf{B}_y(\theta_{21})\mathbf{r}_{32}^2 + \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \\ J_{32} &= \mathbf{j}_1^0 \times (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \\ J_{33} &= \mathbf{j}_1^0 \times (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{G2'}^5) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_4}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_4}{\partial q_j} &= (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{21}}{\partial q_j} \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{54}}{\partial q_j} \\ \frac{\partial \mathbf{k}_4}{\partial q_j} &= J_4 \frac{\partial \boldsymbol{\theta}}{\partial q_j} \end{aligned} \tag{3.84}$$

Dónde:

$$J_4 = [J_{41} \quad J_{42} \quad J_{43}]$$

A su vez:

$$\begin{aligned} J_{41} &= \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5, \\ J_{42} &= \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5, \\ J_{43} &= (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{G2'}^5) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_5}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_5}{\partial q_j} &= (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^4 \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{21}}{\partial q_j} \\ &\quad + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^4 \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{G2'}^5) \frac{\partial \theta_{54}}{\partial q_j} \\ \frac{\partial \mathbf{k}_5}{\partial q_j} &= J_5 \frac{\partial \boldsymbol{\theta}}{\partial q_j} \end{aligned} \tag{3.85}$$

Dónde:

$$J_5 = [J_{51} \quad J_{52} \quad J_{53}]$$

A su vez:

$$\begin{aligned} J_{51} &= \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^4 \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5, \\ J_{52} &= \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^4 \times \mathbf{r}_{G2'}^0 + \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{G2'}^5, \\ J_{53} &= \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{G2'}^5, \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_6^T}{\partial q_j}$:

$$\frac{\partial \mathbf{k}_6^T}{\partial q_j} = J_6 \frac{\partial \boldsymbol{\theta}}{\partial q_j} + \frac{\partial \mathbf{k}'_6^T}{\partial q_j} \quad 3.86$$

Dónde:

$$J_6 = \begin{bmatrix} \frac{\partial \mathbf{k}_6^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_6^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_6^T}{\partial \theta_{54}} \end{bmatrix}$$

Desarrollando $\frac{\partial \mathbf{k}_7^T}{\partial q_j}$:

$$\frac{\partial \mathbf{k}_7^T}{\partial q_j} = J_7 \frac{\partial \boldsymbol{\theta}}{\partial q_j} + \frac{\partial \mathbf{k}'_7^T}{\partial q_j} \quad 3.87$$

Dónde:

$$J_7 = \begin{bmatrix} \frac{\partial \mathbf{k}_7^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_7^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_7^T}{\partial \theta_{54}} \end{bmatrix}$$

Desarrollando $\frac{\partial L_3}{\partial q_j}$

$$L_3 = \frac{1}{2}(\dot{\mathbf{q}}^T \mathbf{N}_3 \dot{\mathbf{q}}) + m_3 \mathbf{g}^T \mathbf{r}_{G3}^0$$

Dónde:

$$\begin{aligned} \mathbf{N}_3 &= m_3 \mathbf{M}_5^T \mathbf{M}_5 + \mathbf{M}_6^T J_3 \mathbf{M}_6 \\ \mathbf{M}_5 &= \mathbf{k}_8 \mathbf{k}_2^T + \mathbf{k}_9 \mathbf{k}_6^T + \mathbf{k}_{10} \mathbf{k}_7^T + \mathbf{k}_{11} \mathbf{k}_{12}^T \\ \mathbf{M}_6 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T \end{aligned}$$

$$\begin{aligned}
\mathbf{k}_8 &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) \\
\mathbf{k}_9 &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{10} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) \\
\mathbf{k}_{11} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G3'}^0 \\
\mathbf{k}_{12}^T &= \frac{1}{V_{36}} [E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}] \\
\mathbf{i}_{15}^0 &= \mathbf{R}_{15}^0 \mathbf{i}_{15}^{15} \\
\mathbf{r}_{14,5i}^0 &= \mathbf{R}_5^0 \mathbf{r}_{14,5i}^5 \\
\mathbf{r}_{15,14i}^0 &= \mathbf{R}_{14}^0 \mathbf{r}_{15,14i}^{14} \\
\mathbf{r}_{G3'}^0 &= \mathbf{R}_{16}^0 \mathbf{r}_{G3'}^{16} \\
\mathbf{R}_{14}^0 &= \mathbf{R}_5^0 \\
\mathbf{R}_{15}^0 &= \mathbf{R}_5^0 \\
\mathbf{R}_{16}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \\
\mathbf{r}_{14,5i}^5 &= [x_{14,5i}, 0, 0]^T \\
\mathbf{r}_{15,14i}^{14} &= [x_{15,14i}, 0, 0]^T \\
\mathbf{r}_{G3'}^{16} &= [x_{G3}, y_{G3}, z_{G3}]^T \\
\mathbf{J}_3^0 &= \mathbf{R}_{16}^0 \mathbf{J}_3^{16} \mathbf{R}_{16}^{0T}
\end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned}
\frac{\partial L_3}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_3}{\partial q_j} \dot{\mathbf{q}} + m_3 \mathbf{g}^T \frac{\partial \mathbf{r}_{G3}^0}{\partial q_j} \\
\frac{\partial L_3}{\partial q_j} &= \mathbb{V}'_{3j} \dot{\mathbf{q}} + \mathbb{C}_{3j}
\end{aligned} \tag{3.88}$$

Dónde:

$$\begin{aligned}
\mathbb{V}'_{3j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_3}{\partial q_j} \\
\mathbb{C}_{3j} &= m_3 \mathbf{g}^T \frac{\partial \mathbf{r}_{G3}^0}{\partial q_j}
\end{aligned} \tag{3.89}$$

A su vez:

$$\begin{aligned}
\frac{\partial \mathbf{N}_3}{\partial q_j} &= m_3 \left(\frac{\partial \mathbf{M}_5^T}{\partial q_j} \mathbf{M}_5 + \mathbf{M}_5^T \frac{\partial \mathbf{M}_5}{\partial q_j} \right) + \frac{\partial \mathbf{M}_6^T}{\partial q_j} \mathbf{J}_3 \mathbf{M}_6 + \mathbf{M}_6^T \frac{\partial \mathbf{J}_3}{\partial q_j} \mathbf{M}_6 + \mathbf{M}_6^T \mathbf{J}_3 \frac{\partial \mathbf{M}_6}{\partial q_j} \\
\frac{\partial \mathbf{M}_5}{\partial q_j} &= \frac{\partial \mathbf{k}_8}{\partial q_j} \mathbf{k}_2^T + \mathbf{k}_8 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \frac{\partial \mathbf{k}_9}{\partial q_j} \mathbf{k}_6^T + \mathbf{k}_9 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_{10}}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_{10} \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{k}_{11}}{\partial q_j} \mathbf{k}_{12}^T \\
&\quad + \mathbf{k}_{11} \frac{\partial \mathbf{k}_{12}^T}{\partial q_j}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_8}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{14,5i}^5 \\
\frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{14}^0}{\partial q_j} \mathbf{r}_{15,14i}^{14} \\
\frac{\partial \mathbf{k}_9}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{r}_{G3'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{16}^0}{\partial q_j} \mathbf{r}_{G3'}^{16} \\
\frac{\partial \mathbf{R}_{16}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) + \mathbf{R}_y(\theta_{21}) \frac{\partial \mathbf{R}_y(\theta_{43})}{\partial q_j} \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \frac{\partial \mathbf{R}_z(\theta_{54})}{\partial q_j} \mathbf{R}_x(\theta_{16,15}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \mathbf{R}_x(\theta_{16,15})}{\partial q_j} \\
&= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
\frac{\partial \mathbf{k}_{10}}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G3'}^0) + \mathbf{k}_4^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{11}}{\partial q_j} &= \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \times \mathbf{r}_{G3'}^0 + \mathbf{i}_{15}^0 \times \frac{\partial \mathbf{r}_{G3'}^0}{\partial q_j} \\
\frac{\partial \mathbf{i}_{15}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{15}^0}{\partial q_j} \mathbf{i}_{15}^{15} \\
\frac{\partial \mathbf{k}_{12}^T}{\partial q_j} &= -\frac{1}{V_{36}^2} \frac{\partial V_{36}}{\partial q_j} [E_{25}, E_{26}, E_{27}, E_{28}, E_{29}, E_{30}] \\
&\quad + \frac{1}{V_{36}} \left[\frac{\partial E_{25}}{\partial q_j}, \frac{\partial E_{26}}{\partial q_j}, \frac{\partial E_{27}}{\partial q_j}, \frac{\partial E_{28}}{\partial q_j}, \frac{\partial E_{29}}{\partial q_j}, \frac{\partial E_{30}}{\partial q_j} \right] \\
\frac{\partial \mathbf{M}_6}{\partial q_j} &= \mathbf{j}_1^0 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \mathbf{j}_3^0 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_4^0}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_4^0 \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \mathbf{k}_{12}^T + \mathbf{i}_{15}^0 \frac{\partial \mathbf{k}_{12}^T}{\partial q_j} \\
\frac{\partial \mathbf{J}_3^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{16}^0}{\partial q_j} \mathbf{J}_3^{16} \mathbf{R}_{16}^{0^T} + \mathbf{R}_{16}^0 \mathbf{J}_3^{16} \frac{\partial \mathbf{R}_{16}^{0^T}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_8}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_8}{\partial q_j} = & (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{21}}{\partial q_j} \\
& + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \\
& \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{43}}{\partial q_j} \\
& + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \\
& \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{k}_8}{\partial q_j} = & \mathbf{J}_8 \frac{\partial \boldsymbol{\theta}}{\partial q_j}
\end{aligned} \tag{3.90}$$

Dónde:

$$\mathbf{J}_8 = [\mathbf{J}_{81} \quad \mathbf{J}_{82} \quad \mathbf{J}_{83}]$$

A su vez:

$$\begin{aligned}
\mathbf{J}_{81} = & \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
\mathbf{J}_{82} = & \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
\mathbf{J}_{83} = & \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_9}{\partial q_j}$:

$$\frac{\partial \mathbf{k}_9}{\partial q_j} = \mathbf{J}_9 \frac{\partial \boldsymbol{\theta}}{\partial q_j} + \mathbf{J}'_9 \frac{\partial \theta_{16,15}}{\partial q_j} \tag{3.91}$$

Dónde:

$$\begin{aligned}
\mathbf{J}_9 = & [\mathbf{J}_{91} \quad \mathbf{J}_{92} \quad \mathbf{J}_{93}] \\
\mathbf{J}'_9 = & (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{r}_{G3'}^{16})
\end{aligned}$$

A su vez:

$$\begin{aligned}
\mathbf{J}_{91} = & \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{r}_{G3'}^{16} \\
\mathbf{J}_{92} = & (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \\
& \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{r}_{G3'}^{16})
\end{aligned}$$

$$J_{93} = \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \\ \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{r}_{G3'}^{16},$$

Desarrollando $\frac{\partial L_4}{\partial q_j}$

$$L_4 = \frac{1}{2}(\dot{\mathbf{q}}^T \mathbf{N}_4 \dot{\mathbf{q}}) + m_4 \mathbf{g}^T \mathbf{r}_{G4}^0$$

Dónde:

$$\begin{aligned} \mathbf{N}_4 &= m_4 \mathbf{M}_7^T \mathbf{M}_7 + \mathbf{M}_8^T \mathbf{J}_4 \mathbf{M}_8 \\ \mathbf{M}_7 &= \mathbf{k}_{13} \mathbf{k}_2^T + \mathbf{k}_{14} \mathbf{k}_6^T + \mathbf{k}_{15} \mathbf{k}_7^T + \mathbf{k}_{16} \mathbf{k}_{12}^T + \mathbf{k}_{17} \mathbf{k}_{18}^T \\ \mathbf{M}_8 &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T + \mathbf{k}_{16}^0 \mathbf{k}_{18}^T \\ \mathbf{k}_{13} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{14} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{15} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) \\ \mathbf{k}_{16} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{k}_{17} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G4'}^0 \\ \mathbf{k}_{18}^T &= \frac{1}{V_{29}} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] \\ \mathbf{r}_{G4'}^0 &= \mathbf{R}_{17}^0 \mathbf{r}_{G4'}^{17} \\ \mathbf{k}_{16}^0 &= \mathbf{R}_{16}^0 \mathbf{k}_{16}^{16} \\ \mathbf{R}_{17}^0 &= \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \\ \mathbf{r}_{G4'}^{17} &= [x_{G4'}, y_{G4'}, z_{G4'}]^T \\ \mathbf{J}_4^0 &= \mathbf{R}_{17}^0 \mathbf{J}_4^{17} \mathbf{R}_{17}^{0T} \end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned} \frac{\partial L_4}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_4}{\partial q_j} \dot{\mathbf{q}} + m_4 \mathbf{g}^T \frac{\partial \mathbf{r}_{G4}^0}{\partial q_j} \\ \frac{\partial L_4}{\partial q_j} &= \mathbb{V}'_{4j} \dot{\mathbf{q}} + \mathbb{C}_{4j} \end{aligned} \tag{3.92}$$

Dónde:

$$\begin{aligned} \mathbb{V}'_{4j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_4}{\partial q_j} \\ \mathbb{C}_{4j} &= m_4 \mathbf{g}^T \frac{\partial \mathbf{r}_{G4}^0}{\partial q_j} \end{aligned} \tag{3.93}$$

A su vez:

$$\begin{aligned}
\frac{\partial N_4}{\partial q_j} &= m_4 \left(\frac{\partial \mathbf{M}_7^T}{\partial q_j} \mathbf{M}_7 + \mathbf{M}_7^T \frac{\partial \mathbf{M}_7}{\partial q_j} \right) + \frac{\partial \mathbf{M}_8^T}{\partial q_j} J_4 \mathbf{M}_8 + \mathbf{M}_8^T \frac{\partial J_4}{\partial q_j} \mathbf{M}_8 + \mathbf{M}_8^T J_4 \frac{\partial \mathbf{M}_8}{\partial q_j} \\
\frac{\partial \mathbf{M}_7}{\partial q_j} &= \frac{\partial \mathbf{k}_{13}}{\partial q_j} \mathbf{k}_2^T + \mathbf{k}_{13} \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \frac{\partial \mathbf{k}_{14}}{\partial q_j} \mathbf{k}_6^T + \mathbf{k}_{14} \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_{15}}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_{15} \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{k}_{16}}{\partial q_j} \mathbf{k}_{12}^T \\
&\quad + \mathbf{k}_{16} \frac{\partial \mathbf{k}_{12}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{17}}{\partial q_j} \mathbf{k}_{18}^T + \mathbf{k}_{17} \frac{\partial \mathbf{k}_{18}^T}{\partial q_j} \\
\frac{\partial \mathbf{k}_{13}}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{r}_{G4'}^{17} \\
\frac{\partial \mathbf{R}_{17}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \frac{\partial \mathbf{R}_y(\theta_{43})}{\partial q_j} \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \frac{\partial \mathbf{R}_z(\theta_{54})}{\partial q_j} \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \mathbf{R}_x(\theta_{16,15})}{\partial q_j} \mathbf{R}_z(\theta_{17,16}) \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \frac{\partial \mathbf{R}_z(\theta_{17,16})}{\partial q_j} \\
&= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \\
\frac{\partial \mathbf{k}_{14}}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{15}}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + \mathbf{k}_4^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{16}}{\partial q_j} &= \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \times \mathbf{r}_{G4'}^0 + \mathbf{i}_{15}^0 \times \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{17}}{\partial q_j} &= \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \times \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{18}^T}{\partial q_j} &= -\frac{1}{V_{29}^2} \frac{\partial V_{29}}{\partial q_j} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] \\
&\quad + \frac{1}{V_{29}} \left[\frac{\partial E_{19}}{\partial q_j}, \frac{\partial E_{20}}{\partial q_j}, \frac{\partial E_{21}}{\partial q_j}, \frac{\partial E_{22}}{\partial q_j}, \frac{\partial E_{23}}{\partial q_j}, \frac{\partial E_{24}}{\partial q_j} \right] \\
\frac{\partial \mathbf{k}_{16}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{16}^0}{\partial q_j} \mathbf{k}_{16}^{16} \\
\frac{\partial \mathbf{M}_8}{\partial q_j} &= \mathbf{j}_1^0 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \mathbf{j}_3^0 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_4^0}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_4^0 \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \mathbf{k}_{12}^T + \mathbf{i}_{15}^0 \frac{\partial \mathbf{k}_{12}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \mathbf{k}_{18}^T \\
&\quad + \mathbf{k}_{16}^0 \frac{\partial \mathbf{k}_{18}^T}{\partial q_j} \\
\frac{\partial \mathbf{J}_4^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{J}_4^{17} \mathbf{R}_{17}^{0T} + \mathbf{R}_{17}^0 \mathbf{J}_4^{17} \frac{\partial \mathbf{R}_{17}^{0T}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{13}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{13}}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{R}_2^0}{\partial q_j} \mathbf{r}_{32}^2 + \frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{14,5i}^5 + \frac{\partial \mathbf{R}_{14}^0}{\partial q_j} \mathbf{r}_{15,14i}^{14} + \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{r}_{G4'}^{17} \right) \\
\frac{\partial \mathbf{k}_{13}}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\mathbf{B}_y(\theta_{21}) \frac{\partial \theta_{21}}{\partial q_j} \mathbf{r}_{32}^2 \right. \\
&\quad + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \\
&\quad + \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{14,5i}^5 \\
&\quad + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \\
&\quad + \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{15,14i}^{14} \\
&\quad + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \right. \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad + \left. \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \mathbf{r}_{G4'}^{17} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{13}}{\partial q_j} = & \left(\left(\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \right. \right. \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_1^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \left. \right) \frac{\partial \theta_{21}}{\partial q_j} \\
& + \left(\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \right. \\
& + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \left. \right) \frac{\partial \theta_{43}}{\partial q_j} \\
& + \left(\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \right. \\
& + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \left. \right) \frac{\partial \theta_{54}}{\partial q_j} \\
& + \left(\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{16,15}}{\partial q_j} \\
& + \left(\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{17,16}}{\partial q_j} \left. \right) \\
\frac{\partial \mathbf{k}_{13}}{\partial q_j} = & J_{13} \frac{\partial \theta}{\partial q_j} + J'_{13} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{13} \frac{\partial \theta_{17,16}}{\partial q_j}
\end{aligned} \tag{3.94}$$

Dónde:

$$\begin{aligned}
J_{13} &= [J_{131} \quad J_{132} \quad J_{133}] \\
J'_{13} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \\
J''_{13} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17})
\end{aligned}$$

A su vez:

$$\begin{aligned}
J_{131} &= (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_1^0 \\
& \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \\
J_{132} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_1^0 \\
& \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \\
J_{133} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_1^0 \\
& \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17})
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{14}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{14}}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{14}}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{14,5i}^5 + \frac{\partial \mathbf{R}_{14}^0}{\partial q_j} \mathbf{r}_{15,14i}^{14} + \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{r}_{G4'}^{17} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{14}}{\partial q_j} = & \mathbf{j}_3^0 \times \left(\left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \right. \\
& + \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{14,5i}^5 \\
& + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \\
& + \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{15,14i}^{14} \\
& + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \right. \\
& + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \\
& + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \\
& + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
& + \left. \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \mathbf{r}_{G4'}^{17} \right) \\
\frac{\partial \mathbf{k}_{14}}{\partial q_j} = & \left(\left(\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \right. \right. \\
& \times \left. \left. \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{21}}{\partial q_j} \right. \\
& + \left(\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \right. \\
& + \left. \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{43}}{\partial q_j} \\
& + \left(\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \right. \\
& + \left. \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{54}}{\partial q_j} \\
& + \left(\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{16,15}}{\partial q_j} \\
& + \left. \left(\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{14}}{\partial q_j} = & J_{14} \frac{\partial \theta}{\partial q_j} + J'_{14} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{14} \frac{\partial \theta_{17,16}}{\partial q_j}
\end{aligned} \tag{3.95}$$

Dónde:

$$\begin{aligned}
J_{14} &= [J_{141} \quad J_{142} \quad J_{143}] \\
J'_{14} &= (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \\
J''_{14} &= (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17})
\end{aligned}$$

A su vez:

$$J_{141} = (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

$$J_{142} = (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

$$J_{143} = (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{15,14i}^{14} + \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

Desarrollando $\frac{\partial \mathbf{k}_{15}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{15}}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + \mathbf{k}_4^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \right) \\ &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + \mathbf{k}_4^0 \\ &\quad \times \left(\frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{14,5i}^5 + \frac{\partial \mathbf{R}_{14}^0}{\partial q_j} \mathbf{r}_{15,14i}^{14} + \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{r}_{G4'}^{17} \right) \\ \frac{\partial \mathbf{k}_{15}}{\partial q_j} &= \left((\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^0) \frac{\partial \theta_{21}}{\partial q_j} + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{k}_4^0) \frac{\partial \theta_{43}}{\partial q_j} \right) \mathbf{k}_4^0 \\ &\quad \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G4'}^0) + \mathbf{k}_4^0 \\ &\quad \times \left(\left(\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{14,5i}^5 \right. \\ &\quad \left. + \left(\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{r}_{15,14i}^{14} \right. \\ &\quad \left. + \left(\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \right. \right. \\ &\quad \left. \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \mathbf{r}_{G4'}^{17} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial \mathbf{k}_{15}}{\partial q_j} = & \left((\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \times \mathbf{r}_{14,5i}^0 + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \times \mathbf{r}_{15,14i}^0 + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \right. \\ & \times \mathbf{r}_{G4'}^0 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\ & \left. + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{21}}{\partial q_j} \\ & + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \times \mathbf{r}_{14,5i}^0 + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \times \mathbf{r}_{15,14i}^0 \right. \\ & + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4) \times \mathbf{r}_{G4'}^0 + \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \\ & \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{k}_4^0 \\ & \left. \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{43}}{\partial q_j} \\ & + \left(\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{k}_4^0 \right. \\ & \left. \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{54}}{\partial q_j} \\ & + \left(\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{16,15}}{\partial q_j} \\ & + \left(\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \frac{\partial \theta_{17,16}}{\partial q_j} \\ \\ \frac{\partial \mathbf{k}_{15}}{\partial q_j} = & J_{15} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{15} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{15} \frac{\partial \theta_{17,16}}{\partial q_j} \end{aligned} \quad 3.96$$

Dónde:

$$\begin{aligned}
J_{151} &= \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \times \mathbf{r}_{14,5i}^0 + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \times \mathbf{r}_{15,14i}^0 + \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \\
&\quad \times \mathbf{r}_{G4'}^0 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
&\quad + \mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}, \\
J_{152} &= \left(\left(\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \times \mathbf{r}_{14,5i}^0 + \left(\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \times \mathbf{r}_{15,14i}^0 \right. \\
&\quad + \left(\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \right) \times \mathbf{r}_{G4'}^0 + \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \\
&\quad \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{k}_4^0 \\
&\quad \times \left. \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right) \\
J_{153} &= \left(\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} + \mathbf{k}_4^0 \right. \\
&\quad \times \left. \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{16}}{\partial q_j} &= \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \times \mathbf{r}_{G4'}^0 + \mathbf{i}_{15}^0 \times \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{16}}{\partial q_j} &= \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \right. \\
&\quad \left. + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \right) \mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0 + \mathbf{i}_{15}^0 \\
&\quad \times \left(\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \right. \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad \left. + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \mathbf{r}_{G4'}^{17} \\
\frac{\partial \mathbf{k}_{16}}{\partial q_j} &= \left((\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \right. \\
&\quad \left. + (\mathbf{i}_{15}^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \right) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \right. \\
&\quad \left. + (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \right) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \right. \\
&\quad \left. + (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \right) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \left((\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \right) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad + \left((\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \right) \frac{\partial \theta_{17,16}}{\partial q_j} \\
\frac{\partial \mathbf{k}_{16}}{\partial q_j} &= J_{16} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{16} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{16} \frac{\partial \theta_{17,16}}{\partial q_j} \tag{3.97}
\end{aligned}$$

Dónde:

$$\begin{aligned}
J_{16} &= [J_{161} \quad J_{162} \quad J_{163}] \\
J'_{16} &= (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17}) \\
J''_{16} &= (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{r}_{G4'}^{17})
\end{aligned}$$

A su vez:

$$J_{161} = (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \\ + (\mathbf{i}_{15}^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

$$J_{162} = (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \\ + (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

$$J_{163} = (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G4'}^0) \\ + (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})$$

$$\begin{aligned} \frac{\partial \mathbf{k}_{17}}{\partial q_j} &= \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \times \frac{\partial \mathbf{r}_{G4'}^0}{\partial q_j} \\ \frac{\partial \mathbf{k}_{17}}{\partial q_j} &= \frac{\partial \mathbf{R}_{16}^0}{\partial q_j} \mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \times \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{r}_{G4'}^{17} \\ \frac{\partial \mathbf{k}_{17}}{\partial q_j} &= \left(\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \frac{\partial \theta_{21}}{\partial q_j} \right. \\ &\quad + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \frac{\partial \theta_{16,15}}{\partial q_j} \right) \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\ &\quad \times \left(\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{21}}{\partial q_j} \right. \\ &\quad + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16}) \frac{\partial \theta_{16,15}}{\partial q_j} \\ &\quad \left. + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16}) \frac{\partial \theta_{17,16}}{\partial q_j} \right) \mathbf{r}_{G4'}^{17} \end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{17}}{\partial q_j} = & (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
& \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \frac{\partial \theta_{21}}{\partial q_j} \\
& + (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \frac{\partial \theta_{43}}{\partial q_j} \\
& + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \frac{\partial \theta_{54}}{\partial q_j} \\
& + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
& + (\mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \frac{\partial \theta_{17,16}}{\partial q_j} \\
\\
\frac{\partial \mathbf{k}_{17}}{\partial q_j} = & J_{17} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{17} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{17} \frac{\partial \theta_{17,16}}{\partial q_j}
\end{aligned} \tag{3.98}$$

Dónde:

$$\begin{aligned}
J_{17} &= [J_{171} \quad J_{172} \quad J_{173}] \\
J'_{17} &= (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
&\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \\
J''_{17} &= (\mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})
\end{aligned}$$

A su vez:

$$\begin{aligned}
J_{171} &= (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
&\quad \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \\
J_{172} &= (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
&\quad \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17}) \\
J_{173} &= (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G4'}^0 + \mathbf{k}_{16}^0 \\
&\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{r}_{G4'}^{17})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{18}^T}{\partial q_j} &= -\frac{1}{V_{29}^2} \frac{\partial V_{29}}{\partial q_j} [E_{19}, E_{20}, E_{21}, E_{22}, E_{23}, E_{24}] \\
&\quad + \frac{1}{V_{29}} \left[\frac{\partial E_{19}}{\partial q_j}, \frac{\partial E_{20}}{\partial q_j}, \frac{\partial E_{21}}{\partial q_j}, \frac{\partial E_{22}}{\partial q_j}, \frac{\partial E_{23}}{\partial q_j}, \frac{\partial E_{24}}{\partial q_j} \right] \\
\frac{\partial \mathbf{k}_{18}^T}{\partial q_j} &= \mathbf{J}_{18} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + \frac{\partial \mathbf{k}_{18}'^T}{\partial q_j}
\end{aligned} \tag{3.99}$$

Dónde:

$$\mathbf{J}_{18} = \begin{bmatrix} \frac{\partial \mathbf{k}_{18}^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_{18}^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_{18}^T}{\partial \theta_{54}} \end{bmatrix}$$

Desarrollando $\frac{\partial L_5}{\partial q_j}$

$$L_5 = \frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_5 \dot{\mathbf{q}}) + m_5 \mathbf{g}^T \mathbf{r}_{G5}^0$$

Dónde:

$$\begin{aligned}
\mathbf{N}_5 &= m_5 \mathbf{M}_9^T \mathbf{M}_9 + \mathbf{M}_{10}^T \mathbf{J}_5 \mathbf{M}_{10} \\
\mathbf{M}_9 &= \mathbf{k}_{19} \mathbf{k}_2^T + \mathbf{k}_{20} \mathbf{k}_6^T + \mathbf{k}_{21} \mathbf{k}_7^T + \mathbf{k}_{22} \mathbf{k}_{12}^T + \mathbf{k}_{23} \mathbf{k}_{18}^T + \mathbf{k}_{24} \mathbf{k}_{25}^T \\
\mathbf{M}_{10} &= \mathbf{j}_1^0 \mathbf{k}_2^T + \mathbf{j}_3^0 \mathbf{k}_6^T + \mathbf{k}_4^0 \mathbf{k}_7^T + \mathbf{i}_{15}^0 \mathbf{k}_{12}^T + \mathbf{k}_{16}^0 \mathbf{k}_{18}^T + \mathbf{i}_{17}^0 \mathbf{k}_{25}^T \\
\mathbf{k}_{19} &= \mathbf{j}_1^0 \times (\mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{20} &= \mathbf{j}_3^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{21} &= \mathbf{k}_4^0 \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) \\
\mathbf{k}_{22} &= \mathbf{i}_{15}^0 \times \mathbf{r}_{G5'}^0 \\
\mathbf{k}_{23} &= \mathbf{k}_{16}^0 \times \mathbf{r}_{G5'}^0 \\
\mathbf{k}_{24} &= \mathbf{i}_{17}^0 \times \mathbf{r}_{G5'}^0 \\
\mathbf{k}_{25}^T &= \frac{1}{V_{23}} [E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_{18}] \\
\mathbf{i}_{17}^0 &= \mathbf{R}_{17}^0 \mathbf{i}_{17}^{17} \\
\mathbf{r}_{G5'}^0 &= \mathbf{R}_{18}^0 \mathbf{r}_{G5'}^{18} \\
\mathbf{R}_{18}^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \\
\mathbf{r}_{G5'}^{18} &= [x_{G5}, y_{G5}, z_{G5}]^T \\
\mathbf{J}_5^0 &= \mathbf{R}_{18}^0 \mathbf{J}_5^{18} \mathbf{R}_{18}^{0^T}
\end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned}
\frac{\partial L_5}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_5}{\partial q_j} \dot{\mathbf{q}} + m_5 \mathbf{g}^T \frac{\partial \mathbf{r}_{G5}^0}{\partial q_j} \\
\frac{\partial L_5}{\partial q_j} &= \mathbb{V}'_{5j} \dot{\mathbf{q}} + \mathbb{C}_{5j}
\end{aligned} \tag{3.100}$$

Dónde:

$$\begin{aligned}
\mathbb{V}'_{5j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_5}{\partial q_j} \\
\mathbb{C}_{5j} &= m_5 \mathbf{g}^T \frac{\partial \mathbf{r}_{G5}^0}{\partial q_j}
\end{aligned} \tag{3.101}$$

Además:

$$\begin{aligned}
\frac{\partial \mathbf{N}_5}{\partial q_j} &= m_5 \left(\frac{\partial \mathbf{M}_9^T}{\partial q_j} \mathbf{M}_9 + \mathbf{M}_9^T \frac{\partial \mathbf{M}_9}{\partial q_j} \right) + \frac{\partial \mathbf{M}_{10}^T}{\partial q_j} \mathbf{J}_5 \mathbf{M}_{10} + \mathbf{M}_{10}^T \frac{\partial \mathbf{J}_5}{\partial q_j} \mathbf{M}_{10} + \mathbf{M}_{10}^T \mathbf{J}_5 \frac{\partial \mathbf{M}_{10}}{\partial q_j} \\
\frac{\partial \mathbf{M}_9}{\partial q_j} &= \frac{\partial \mathbf{k}_{19}}{\partial q_j} \mathbf{k}_2^T + \mathbf{k}_{19} \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \frac{\partial \mathbf{k}_{20}}{\partial q_j} \mathbf{k}_6^T + \mathbf{k}_{20} \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_{21}}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_{21} \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{k}_{22}}{\partial q_j} \mathbf{k}_{12}^T \\
&\quad + \mathbf{k}_{22} \frac{\partial \mathbf{k}_{12}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{23}}{\partial q_j} \mathbf{k}_{18}^T + \mathbf{k}_{23} \frac{\partial \mathbf{k}_{18}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{24}}{\partial q_j} \mathbf{k}_{25}^T + \mathbf{k}_{24} \frac{\partial \mathbf{k}_{25}^T}{\partial q_j} \\
\frac{\partial \mathbf{k}_{19}}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{18}^0}{\partial q_j} \mathbf{r}_{G5'}^{18} \\
\frac{\partial \mathbf{R}_{18}^0}{\partial q_j} &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \frac{\partial \theta_{17,16}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \frac{\partial \theta_{18,17}}{\partial q_j} \\
\frac{\partial \mathbf{k}_{20}}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{21}}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + \mathbf{k}_4^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{22}}{\partial q_j} &= \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{15}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{23}}{\partial q_j} &= \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{k}_{16}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{24}}{\partial q_j} &= \frac{\partial \mathbf{i}_{17}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{17}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \\
\frac{\partial \mathbf{i}_{17}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{17}^0}{\partial q_j} \mathbf{i}_{17}^{17} \\
\frac{\partial \mathbf{k}_{25}^T}{\partial q_j} &= -\frac{1}{V_{23}^2} \frac{\partial V_{23}}{\partial q_j} [E_{13}, E_{14}, E_{15}, E_{16}, E_{17}, E_{18}] \\
&\quad + \frac{1}{V_{23}} \left[\frac{\partial E_{13}}{\partial q_j}, \frac{\partial E_{14}}{\partial q_j}, \frac{\partial E_{15}}{\partial q_j}, \frac{\partial E_{16}}{\partial q_j}, \frac{\partial E_{17}}{\partial q_j}, \frac{\partial E_{18}}{\partial q_j} \right] \\
\frac{\partial \mathbf{M}_{10}}{\partial q_j} &= \mathbf{j}_1^0 \frac{\partial \mathbf{k}_2^T}{\partial q_j} + \mathbf{j}_3^0 \frac{\partial \mathbf{k}_6^T}{\partial q_j} + \frac{\partial \mathbf{k}_4^0}{\partial q_j} \mathbf{k}_7^T + \mathbf{k}_4^0 \frac{\partial \mathbf{k}_7^T}{\partial q_j} + \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \mathbf{k}_{12}^T + \mathbf{i}_{15}^0 \frac{\partial \mathbf{k}_{12}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \mathbf{k}_{18}^T \\
&\quad + \mathbf{k}_{16}^0 \frac{\partial \mathbf{k}_{18}^T}{\partial q_j} + \frac{\partial \mathbf{i}_{17}^0}{\partial q_j} \mathbf{k}_{25}^T + \mathbf{i}_{17}^0 \frac{\partial \mathbf{k}_{25}^T}{\partial q_j} \\
\frac{\partial \mathbf{J}_5^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{18}^0}{\partial q_j} \mathbf{J}_5^{18} \mathbf{R}_{18}^{0T} + \mathbf{R}_{18}^0 \mathbf{J}_5^{18} \frac{\partial \mathbf{R}_{18}^{0T}}{\partial q_j} \\
\frac{\partial \mathbf{k}_{19}}{\partial q_j} &= \mathbf{j}_1^0 \times \left(\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right)
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{19}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{19}}{\partial q_j} &= \left((\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \left((\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \left((\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{16,15}}{\partial q_j} \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{17,16}}{\partial q_j} \\
&\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{18,17}}{\partial q_j}
\end{aligned}$$

$$\frac{\partial \mathbf{k}_{19}}{\partial q_j} = J_{19} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{19} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{19} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{19} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.102$$

Dónde:

$$\begin{aligned} J_{19} &= [J_{191} \quad J_{192} \quad J_{193}] \\ J'_{19} &= \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \\ J''_{19} &= \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \\ J'''_{19} &= \mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \end{aligned}$$

A su vez:

$$\begin{aligned} J_{191} &= (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \\ &\quad + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_1^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{192} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{193} &= (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_1^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{20}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{20}}{\partial q_j} &= \mathbf{j}_3^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right) \\ \frac{\partial \mathbf{k}_{20}}{\partial q_j} &= \left(\left((\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \right. \right. \\ &\quad \left. \left. + (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \right) \frac{\partial \theta_{21}}{\partial q_j} \right. \\ &\quad + \left((\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\ &\quad \left. + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \right. \\ &\quad \left. + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \right) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \left((\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\ &\quad \left. + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \right. \\ &\quad \left. + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \right) \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{16,15}}{\partial q_j} \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{17,16}}{\partial q_j} \\ &\quad \left. + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{18,17}}{\partial q_j} \right) \end{aligned}$$

$$\frac{\partial \mathbf{k}_{20}}{\partial q_j} = J_{20} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{20} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{20} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{20} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.103$$

Dónde:

$$\begin{aligned} J_{20} &= [J_{201} \quad J_{202} \quad J_{203}] \\ J'_{20} &= \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \\ J''_{20} &= \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \\ J'''_{20} &= \mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \end{aligned}$$

A su vez:

$$\begin{aligned} J_{201} &= (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{202} &= (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{203} &= (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{j}_3^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{21}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{21}}{\partial q_j} &= \frac{\partial \mathbf{k}_4^0}{\partial q_j} \times (\mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 + \mathbf{r}_{G5'}^0) + \mathbf{k}_4^0 \times \left(\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \right) \\ \frac{\partial \mathbf{k}_{21}}{\partial q_j} &= \left((\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{14,5i}^0) + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{15,14i}^0) \right. \\ &\quad + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{G5'}^0) + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{21}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{14,5i}^0) + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{15,14i}^0) \right. \\ &\quad + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^0 \times \mathbf{r}_{G5'}^0) + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \left((\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{16,15}}{\partial q_j} \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{17,16}}{\partial q_j} \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \frac{\partial \theta_{18,17}}{\partial q_j} \end{aligned}$$

$$\frac{\partial \mathbf{k}_{21}}{\partial q_j} = J_{21} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{21} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{21} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{21} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.104$$

Dónde:

$$\begin{aligned} J_{21} &= [J_{211} \quad J_{212} \quad J_{213}] \\ J'_{21} &= \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}, \\ J''_{21} &= \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}, \\ J'''_{21} &= \mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18} \end{aligned}$$

A su vez:

$$\begin{aligned} J_{211} &= (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{14,5i}^0) + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{15,14i}^0) \\ &\quad + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{G5'}^0) + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{212} &= (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{14,5i}^0) + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{15,14i}^0) \\ &\quad + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{k}_4^4 \times \mathbf{r}_{G5'}^0) + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{213} &= (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \\ &\quad + (\mathbf{k}_4^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{22}}{\partial q_j}$:

$$\frac{\partial \mathbf{k}_{22}}{\partial q_j} = \frac{\partial \mathbf{i}_{15}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{15}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_{22}}{\partial q_j} = & \left((\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \right. \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{21}}{\partial q_j} \\
& + \left((\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \right. \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{43}}{\partial q_j} \\
& + \left((\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \right. \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{54}}{\partial q_j} \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{16,15}}{\partial q_j} \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{17,16}}{\partial q_j} \\
& + (\mathbf{i}_{15}^0 \\
& \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{B}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \left. \right) \frac{\partial \theta_{18,17}}{\partial q_j}
\end{aligned}$$

$$\frac{\partial \mathbf{k}_{22}}{\partial q_j} = J_{22} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{22} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{22} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{22} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.105$$

Dónde:

$$\begin{aligned}
J_{22} &= [J_{221} \quad J_{222} \quad J_{223}] \\
J'_{22} &= \mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}, \\
J''_{22} &= \mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}, \\
J'''_{22} &= \mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{B}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}
\end{aligned}$$

A su vez:

$$\begin{aligned}
J_{221} &= (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \\
&+ (\mathbf{i}_{15}^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \\
J_{222} &= (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \\
&+ (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})
\end{aligned}$$

$$J_{223} = (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{i}_{15}^{15} \times \mathbf{r}_{G5'}^0) \\ + (\mathbf{i}_{15}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})$$

Desarrollando $\frac{\partial \mathbf{k}_{23}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{23}}{\partial q_j} &= \frac{\partial \mathbf{k}_{16}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{k}_{16}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \\ \frac{\partial \mathbf{k}_{23}}{\partial q_j} &= \left((\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{21}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{16,15}}{\partial q_j} \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{17,16}}{\partial q_j} \\ &\quad + (\mathbf{k}_{16}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{B}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{18,17}}{\partial q_j} \\ \frac{\partial \mathbf{k}_{23}}{\partial q_j} &= J_{23} \frac{\partial \theta}{\partial q_j} + J'_{23} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{23} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{23} \frac{\partial \theta_{18,17}}{\partial q_j} \end{aligned} \quad 3.106$$

Dónde:

$$J_{22} = [J_{231} \quad J_{232} \quad J_{233}] \\ J'_{23} = (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \\ + (\mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{B}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18})$$

$$\begin{aligned} J''_{23} &= \mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{B}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18} \\ J'''_{23} &= \mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{B}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18} \end{aligned}$$

A su vez:

$$\begin{aligned} J_{231} &= (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{k}_{16}^0 \times \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \\ J_{232} &= (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \\ J_{233} &= (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{k}_{16}^{16} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{k}_{16}^0 \times \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_x(\theta_{16,15})\mathbf{R}_z(\theta_{17,16})\mathbf{R}_x(\theta_{18,17})\mathbf{r}_{G5'}^{18}) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{24}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{24}}{\partial q_j} &= \frac{\partial \mathbf{i}_{17}^0}{\partial q_j} \times \mathbf{r}_{G5'}^0 + \mathbf{i}_{17}^0 \times \frac{\partial \mathbf{r}_{G5'}^0}{\partial q_j} \\ \frac{\partial \mathbf{k}_{24}}{\partial q_j} &= \left((\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{i}_{17}^0 \\ &\quad \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{21}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{i}_{17}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{43}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{i}_{17}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{54}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{i}_{17}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{16,15}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \right. \\ &\quad + (\mathbf{i}_{17}^0 \\ &\quad \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18})) \frac{\partial \theta_{17,16}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \right. \\ &\quad \times \mathbf{r}_{G5'}^{18} \left. \right) \frac{\partial \theta_{18,17}}{\partial q_j} \end{aligned}$$

$$\frac{\partial \mathbf{k}_{24}}{\partial q_j} = J_{24} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{24} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{24} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{24} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.107$$

Dónde:

$$\begin{aligned} J_{24} &= [J_{241} \quad J_{242} \quad J_{243}] \\ J'_{24} &= (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{i}_{17}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{B}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J''_{24} &= (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{i}_{17}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{B}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J'''_{24} &= (\mathbf{i}_{17}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{B}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \end{aligned}$$

$$\begin{aligned} J_{241} &= (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{i}_{17}^0 \times \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{242} &= (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{i}_{17}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \\ J_{243} &= (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{i}_{17}^{17} \times \mathbf{r}_{G5'}^0) \\ &\quad + (\mathbf{i}_{17}^0 \times \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_x(\theta_{16,15}) \mathbf{R}_z(\theta_{17,16}) \mathbf{R}_x(\theta_{18,17}) \mathbf{r}_{G5'}^{18}) \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{25}^T}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{25}^T}{\partial q_j} &= \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{43}} \frac{\partial \theta_{43}}{\partial q_j} + \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{54}} \frac{\partial \theta_{54}}{\partial q_j} + \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{16,15}} \frac{\partial \theta_{16,15}}{\partial q_j} + \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{17,16}} \frac{\partial \theta_{17,16}}{\partial q_j} \\ &\quad + \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{18,17}} \frac{\partial \theta_{18,17}}{\partial q_j} \\ \frac{\partial \mathbf{k}_{25}^T}{\partial q_j} &= J_{25} \frac{\partial \boldsymbol{\theta}}{\partial q_j} + J'_{25} \frac{\partial \theta_{16,15}}{\partial q_j} + J''_{25} \frac{\partial \theta_{17,16}}{\partial q_j} + J'''_{25} \frac{\partial \theta_{18,17}}{\partial q_j} \quad 3.108 \end{aligned}$$

Dónde:

$$\begin{aligned} J_{25} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{54}} \end{bmatrix} \\ J'_{25} &= \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{16,15}} \\ J''_{25} &= \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{17,16}} \\ J'''_{25} &= \frac{\partial \mathbf{k}_{25}^T}{\partial \theta_{18,17}} \end{aligned}$$

3.4 Velocidad de Centros de Gravedad y Velocidad Angular para las cadenas i.

Al igual que con la cadena 0 se construyen las ecuaciones de lazo de los centros de gravedad on base en las figuras 3-1 y 3-2:

$$\begin{aligned} \mathbf{r}_{G1i}^0 &= \mathbf{r}_{10i}^0 + \mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0 \\ \mathbf{r}_{G2i}^0 &= \mathbf{r}_{10i}^0 + \mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0 \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{10i}^0 &= \text{signo } y_{10} \mathbf{j}_0 \\ \mathbf{r}_{32i}^0 &= \mathbf{R}_{2i}^0 \mathbf{r}_{32i}^{2i} \\ \mathbf{r}_{43i}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{43i}^{3i} \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{G1i'}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{G1i'}^{3i} \\ \mathbf{r}_{G2i'}^0 &= \mathbf{R}_{7i}^0 \mathbf{r}_{G2i'}^{7i} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{2i}^0 &= \mathbf{R}_z(\theta_{21i}) \\ \mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\ \mathbf{R}_{7i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{32i}^{2i} &= [-z_{32i}, 0, 0]^T \\ \mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{G1i'}^{3i} &= [x_{G1i'}, y_{G1i'}, z_{G1i'}]^T \\ \mathbf{r}_{G2i'}^{7i} &= [x_{G2i'}, y_{G2i'}, z_{G2i'}]^T \end{aligned}$$

$$\begin{aligned} \mathbf{v}_{G1i}^0 &= \mathbf{v}_{10i}^0 + \mathbf{v}_{32i}^0 + \mathbf{v}_{G1i'}^0 \\ \mathbf{v}_{G2i}^0 &= \mathbf{v}_{10i}^0 + \mathbf{v}_{32i}^0 + \mathbf{v}_{43i}^0 + \mathbf{v}_{G2i'}^0 \end{aligned}$$

$$\begin{aligned} \mathbf{v}_{10i}^0 &= 0 \\ \mathbf{v}_{32i}^0 &= \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{32i}^0 \\ \mathbf{v}_{43i}^0 &= \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{43i}^0 \end{aligned}$$

$$\begin{aligned} \mathbf{v}_{G1i'}^0 &= \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{G1i'}^0 \\ \mathbf{v}_{G2i'}^0 &= \boldsymbol{\omega}_{7i}^0 \times \mathbf{r}_{G2i'}^0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}_{2i}^0 &= \boldsymbol{\omega}_{21i}^0 \\ \boldsymbol{\omega}_{7i}^0 &= \boldsymbol{\omega}_{21i}^0 + \boldsymbol{\omega}_{65i}^0 + \boldsymbol{\omega}_{76i}^0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{\omega}_{21i}^0 &= \dot{\theta}_{21i} \mathbf{k}_{1i}^0 \\ \boldsymbol{\omega}_{65i}^0 &= \dot{\theta}_{65i} \mathbf{j}_{5i}^0 \\ \boldsymbol{\omega}_{76i}^0 &= \dot{\theta}_{76i} \mathbf{k}_{6i}^0 \end{aligned}$$

$$\begin{aligned} \mathbf{k}_{1i}^0 &= \mathbf{k}_0 \\ \mathbf{j}_{5i}^0 &= \mathbf{R}_{5i}^0 \mathbf{j}_{5i}^{5i} \\ \mathbf{k}_{6i}^0 &= \mathbf{R}_{6i}^0 \mathbf{k}_{6i}^{6i} \end{aligned}$$

$$\begin{aligned} \mathbf{R}_{5i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \\ \mathbf{R}_{6i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \end{aligned}$$

3.4.1 Velocidad de Centro de Gravedad del Cuerpo 1i

$$\begin{aligned} \mathbf{v}_{G1i}^0 &= \mathbf{v}_{10i}^0 + \mathbf{v}_{32i}^0 + \mathbf{v}_{G1i'}^0 \\ &= \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{32i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{G1i'}^0 \\ &= \boldsymbol{\omega}_{21i}^0 \times \mathbf{r}_{32i}^0 + \boldsymbol{\omega}_{21i}^0 \times \mathbf{r}_{G1i'}^0 \\ &= \boldsymbol{\omega}_{21i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0) \\ &= (\dot{\theta}_{21i} \mathbf{k}_{1i}^0) \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0) \\ &= (\mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0)) \dot{\theta}_{21i} \end{aligned}$$

$$\mathbf{v}_{G1i}^0 = \mathbf{k}_{1i} \dot{\theta}_{21i} \quad 3.109$$

Dónde:

$$\mathbf{k}_{1i} = \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0) \quad 3.110$$

De la ec. (2.40):

$$\dot{\theta}_{21i} = \frac{1}{V_{44i}} (E_{31i} \dot{x} + E_{32i} \dot{y} + E_{33i} \dot{z} + E_{34i} \dot{\psi} + E_{35i} \dot{\theta} + E_{36i} \dot{\phi})$$

Renombrando:

$$\dot{\theta}_{21i} = \mathbf{k}_{2i}^T \dot{\mathbf{q}} \quad 3.111$$

Dónde:

$$\begin{aligned} \mathbf{k}_{2i}^T &= \frac{1}{V_{44i}} [E_{31i}, E_{32i}, E_{33i}, E_{34i}, E_{35i}, E_{36i}] \\ \dot{\mathbf{q}} &= [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]^T \end{aligned} \quad 3.112$$

Sustituyendo la ec. (3.48) en la ec. (3.46) con el fin de poner a esta última en función de las coordenadas generalizadas se tiene:

$$\begin{aligned} \mathbf{v}_{G1i}^0 &= \mathbf{k}_{1i} \mathbf{k}_{2i}^T \dot{\mathbf{q}} \\ &= (\mathbf{k}_{1i} \mathbf{k}_{2i}^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\mathbf{v}_{G1i}^0 = \mathbf{M}_{1i} \dot{\mathbf{q}} \quad 3.113$$

Dónde:

$$\mathbf{M}_{1i} = \mathbf{k}_{1i} \mathbf{k}_{2i}^T \quad 3.114$$

3.4.2 Velocidad Angular del Cuerpo 1i

$$\begin{aligned}\omega_{2i}^0 &= \omega_{21i}^0 \\ \omega_{21i}^0 &= \mathbf{k}_{1i}^0 \dot{\theta}_{21i}\end{aligned}$$

Sustituyendo (3.48) en la ec. anterior:

$$\begin{aligned}\omega_{21i}^0 &= \mathbf{k}_{1i}^0 (\mathbf{k}_{2i}^T \dot{\mathbf{q}}) \\ &= (\mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T) \dot{\mathbf{q}}\end{aligned}$$

Renombrando:

$$\omega_{2i}^0 = \mathbf{M}_{2i} \dot{\mathbf{q}} \quad 3.115$$

Dónde:

$$\mathbf{M}_{2i} = \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T \quad 3.116$$

3.4.3 Velocidad de Centro de Gravedad del Cuerpo 2i

$$\begin{aligned}\mathbf{v}_{G2i}^0 &= \mathbf{v}_{10i}^0 + \mathbf{v}_{32i}^0 + \mathbf{v}_{43i}^0 + \mathbf{v}_{G2i'}^0 \\ &= \omega_{2i}^0 \times \mathbf{r}_{32i}^0 + \omega_{2i}^0 \times \mathbf{r}_{43i}^0 + \omega_{7i}^0 \times \mathbf{r}_{G2i'}^0 \\ &= \omega_{21i}^0 \times \mathbf{r}_{32i}^0 + \omega_{21i}^0 \times \mathbf{r}_{43i}^0 + (\omega_{21i}^0 + \omega_{65i}^0 + \omega_{76i}^0) \times \mathbf{r}_{G2i'}^0 \\ &= (\dot{\theta}_{21i} \mathbf{k}_{1i}^0) \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0) + (\dot{\theta}_{65i} \mathbf{j}_{5i}^0 + \dot{\theta}_{76i} \mathbf{k}_{6i}^0) \times \mathbf{r}_{G2i'}^0 \\ &= (\mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0)) \dot{\theta}_{21i} + (\mathbf{j}_{5i}^0 \times \mathbf{r}_{G2i'}^0) \dot{\theta}_{65i} + (\mathbf{k}_{6i}^0 \\ &\quad \times \mathbf{r}_{G2i'}^0) \dot{\theta}_{76i} \\ \mathbf{v}_{G2i}^0 &= \mathbf{k}_{3i} \dot{\theta}_{21i} + \mathbf{k}_{4i} \dot{\theta}_{65i} + \mathbf{k}_{5i} \dot{\theta}_{76i}\end{aligned} \quad 3.117$$

Dónde:

$$\begin{aligned}\mathbf{k}_{3i} &= \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0) \\ \mathbf{k}_{4i} &= \mathbf{j}_{5i}^0 \times \mathbf{r}_{G2i'}^0 \\ \mathbf{k}_{5i} &= \mathbf{k}_{6i}^0 \times \mathbf{r}_{G2i'}^0\end{aligned} \quad 3.118$$

De la ec. (2.40):

$$\begin{aligned}\dot{\theta}_{65i} &= \frac{1}{V_{48i}} (E_{37i} \dot{x} + E_{38i} \dot{y} + E_{39i} \dot{z} + E_{40i} \dot{\psi} + E_{41i} \dot{\theta} + E_{42i} \dot{\phi}) \\ \dot{\theta}_{76i} &= \frac{1}{V_{53i}} (E_{43i} \dot{x} + E_{44i} \dot{y} + E_{45i} \dot{z} + E_{46i} \dot{\psi} + E_{47i} \dot{\theta} + E_{48i} \dot{\phi})\end{aligned}$$

Renombrando:

$$\begin{aligned}\dot{\theta}_{65i} &= \mathbf{k}_{6i}^T \dot{\mathbf{q}} \\ \dot{\theta}_{76i} &= \mathbf{k}_{7i}^T \dot{\mathbf{q}}\end{aligned} \quad 3.119$$

Dónde:

$$\begin{aligned} \mathbf{k}_{6i}^T &= \frac{1}{V_{48i}} [E_{37i}, E_{38i}, E_{39i}, E_{40i}, E_{41i}, E_{42i}] \\ \mathbf{k}_{7i}^T &= \frac{1}{V_{53i}} [E_{43i}, E_{44i}, E_{45i}, E_{46i}, E_{47i}, E_{48i}] \\ \dot{\mathbf{q}} &= [\dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}]^T \end{aligned} \quad 3.120$$

Sustituyendo la ec.(3.48) y (3.56) en la ec. (3.54) con el fin de poner a esta última en función de las coordenadas generalizadas se tiene:

$$\begin{aligned} \mathbf{v}_{G2i}^0 &= \mathbf{k}_{3i} \mathbf{k}_{2i}^T \dot{\mathbf{q}} + \mathbf{k}_{4i} \mathbf{k}_{6i}^T \dot{\mathbf{q}} + \mathbf{k}_{5i} \mathbf{k}_{7i}^T \dot{\mathbf{q}} \\ \mathbf{v}_{G2i}^0 &= (\mathbf{k}_{3i} \mathbf{k}_{2i}^T + \mathbf{k}_{4i} \mathbf{k}_{6i}^T + \mathbf{k}_{5i} \mathbf{k}_{7i}^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\mathbf{v}_{G2i}^0 = \mathbf{M}_{3i} \dot{\mathbf{q}} \quad 3.121$$

Dónde:

$$\mathbf{M}_{3i} = \mathbf{k}_{3i} \mathbf{k}_{2i}^T + \mathbf{k}_{4i} \mathbf{k}_{6i}^T + \mathbf{k}_{5i} \mathbf{k}_{7i}^T \quad 3.122$$

3.4.4 Velocidad Angular del Cuerpo 2i

$$\begin{aligned} \boldsymbol{\omega}_{7i}^0 &= \boldsymbol{\omega}_{21i}^0 + \boldsymbol{\omega}_{65i}^0 + \boldsymbol{\omega}_{76i}^0 \\ \boldsymbol{\omega}_{7i}^0 &= \mathbf{k}_{1i}^0 \dot{\theta}_{21i} + \mathbf{j}_{5i}^0 \dot{\theta}_{65i} + \mathbf{k}_{6i}^0 \dot{\theta}_{76i} \end{aligned}$$

Sustituyendo (3.48) y (3.56) en la ec. anterior:

$$\begin{aligned} \boldsymbol{\omega}_{7i}^0 &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T \dot{\mathbf{q}} + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T \dot{\mathbf{q}} + \mathbf{k}_{6i}^0 \mathbf{k}_{7i}^T \dot{\mathbf{q}} \\ \boldsymbol{\omega}_{7i}^0 &= (\mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T + \mathbf{k}_{6i}^0 \mathbf{k}_{7i}^T) \dot{\mathbf{q}} \end{aligned}$$

Renombrando:

$$\boldsymbol{\omega}_{7i}^0 = \mathbf{M}_{4i} \dot{\mathbf{q}} \quad 3.123$$

Dónde:

$$\mathbf{M}_{4i} = \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T + \mathbf{k}_{6i}^0 \mathbf{k}_{7i}^T \quad 3.124$$

Matrices de inercia medidas en la base inercial:

$$\begin{aligned} \mathbf{J}_{1i}^0 &= \mathbf{R}_{2i}^0 \mathbf{J}_{1i}^{2i} \mathbf{R}_{2i}^{0T} \\ \mathbf{J}_{2i}^0 &= \mathbf{R}_{7i}^0 \mathbf{J}_{2i}^{7i} \mathbf{R}_{7i}^{0T} \end{aligned}$$

3.5 Función Lagrangiana para cadenas i.

3.5.1 Desarrollo del Primer Término de la Ecuación de Lagrange para las cadenas i.

Desarrollando $\frac{\partial L_{1i}}{\partial \dot{q}_j}$:

Se tiene:

$$L_{1i} = \frac{1}{2} \left(m_{1i} \mathbf{v}_{G1i}^0{}^T \mathbf{v}_{G1i}^0 + \boldsymbol{\omega}_{2i}^0{}^T \mathbf{J}_{1i} \boldsymbol{\omega}_{2i}^0 \right) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0$$

Sustituyendo ec.(3.113) y ec.(3.115):

$$\begin{aligned} L_{1i} &= \frac{1}{2} (m_{1i} (\mathbf{M}_{1i} \dot{\mathbf{q}})^T (\mathbf{M}_{1i} \dot{\mathbf{q}}) + (\mathbf{M}_{2i} \dot{\mathbf{q}})^T \mathbf{J}_{1i} (\mathbf{M}_{2i} \dot{\mathbf{q}})) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \\ &= \frac{1}{2} (m_{1i} \dot{\mathbf{q}}^T \mathbf{M}_{1i}^T (\mathbf{M}_{1i} \dot{\mathbf{q}}) + \dot{\mathbf{q}}^T \mathbf{M}_{2i}^T \mathbf{J}_{1i} (\mathbf{M}_{2i} \dot{\mathbf{q}})) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \\ &= \frac{1}{2} (m_{1i} \dot{\mathbf{q}}^T (\mathbf{M}_{1i}^T \mathbf{M}_{1i}) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T (\mathbf{M}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i}) \dot{\mathbf{q}}) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \\ &= \frac{1}{2} \dot{\mathbf{q}}^T (m_{1i} \mathbf{M}_{1i}^T \mathbf{M}_{1i} + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i}) \dot{\mathbf{q}} + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \\ L_{1i} &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_{1i} \dot{\mathbf{q}} + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \end{aligned} \quad 3.125$$

Dónde:

$$\mathbf{N}_{1i} = m_{1i} \mathbf{M}_{1i}^T \mathbf{M}_{1i} + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i}$$

Efectuando la derivada:

$$\begin{aligned} \frac{\partial L_{1i}}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_{1i} \dot{\mathbf{q}}) + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0 \right) \\ \frac{\partial L_{1i}}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_{1i} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \end{aligned} \quad 3.126$$

Simplificando:

$$\begin{aligned} \frac{\partial L_{1i}}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_{1i} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\ &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} \right) \\ \frac{\partial L_{1i}}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} \end{aligned} \quad 3.127$$

Desarrollando $\frac{\partial L_{2i}}{\partial \dot{q}_j}$:

Se tiene:

$$L_{2i} = \frac{1}{2} \left(m_{2i} \mathbf{v}_{G2i}^0{}^T \mathbf{v}_{G2i}^0 + \boldsymbol{\omega}_{7i}^0{}^T \mathbf{J}_{2i} \boldsymbol{\omega}_{7i}^0 \right) + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0$$

Sustituyendo ec.(3.121) y ec.(3.123):

$$\begin{aligned} L_{2i} &= \frac{1}{2} (m_{2i} (\mathbf{M}_{3i} \dot{\mathbf{q}})^T (\mathbf{M}_{3i} \dot{\mathbf{q}}) + (\mathbf{M}_{4i} \dot{\mathbf{q}})^T \mathbf{J}_{1i} (\mathbf{M}_{4i} \dot{\mathbf{q}})) + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0 \\ &= \frac{1}{2} \dot{\mathbf{q}}^T (m_{2i} \mathbf{M}_{3i}^T \mathbf{M}_{3i} + \mathbf{M}_{4i}^T \mathbf{J}_{2i} \mathbf{M}_{4i}) \dot{\mathbf{q}} + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0 \\ L_{2i} &= \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_{2i} \dot{\mathbf{q}} + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0 \end{aligned} \quad 3.128$$

Dónde:

$$\mathbf{N}_{2i} = m_{2i} \mathbf{M}_{3i}^T \mathbf{M}_{3i} + \mathbf{M}_{4i}^T \mathbf{J}_{2i} \mathbf{M}_{4i}$$

Efectuando la derivada:

$$\begin{aligned} \frac{\partial L_{2i}}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} (\dot{\mathbf{q}}^T \mathbf{N}_{2i} \dot{\mathbf{q}}) + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0 \right) \\ \frac{\partial L_{2i}}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_{2i} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \end{aligned} \quad 3.129$$

Simplificando:

$$\begin{aligned} \frac{\partial L_{2i}}{\partial \dot{q}_j} &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_{2i} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \\ &= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} \right) \\ \frac{\partial L_{2i}}{\partial \dot{q}_j} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} \end{aligned} \quad 3.130$$

El desarrollo de la derivada con respecto al tiempo de cada elemento se muestra a continuación:

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_{1i}}{\partial \dot{q}_j} \right)$:

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_{1i}}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \dot{\mathbf{q}} \right) \\
&= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) \mathbf{N}_{1i} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \frac{d\mathbf{N}_{1i}}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \frac{d\dot{\mathbf{q}}}{dt} \right) \\
&= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_{1i} \dot{\mathbf{q}} \\
\frac{d}{dt} \left(\frac{\partial L_{1i}}{\partial \dot{q}_j} \right) &= \mathbb{D}_{1ij} \ddot{\mathbf{q}} + \mathbb{V}_{1ij} \dot{\mathbf{q}}
\end{aligned} \tag{3.131}$$

Dónde:

$$\begin{aligned}
\mathbb{D}_{1ij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{1i} \\
\mathbb{V}_{1ij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_{1i}
\end{aligned} \tag{3.132}$$

Además:

$$\begin{aligned}
\mathbf{N}_{1i} &= m_{1i} \mathbf{M}_{1i}^T \mathbf{M}_{1i} + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i} \\
\mathbf{M}_{1i} &= \mathbf{k}_{1i} \mathbf{k}_{2i}^T \\
\mathbf{M}_{2i} &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T \\
\mathbf{k}_{1i} &= \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0) \\
\mathbf{k}_{2i}^T &= \frac{1}{V_{44}} [E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{36}] \\
\mathbf{k}_{1i}^0 &= \mathbf{k}_0 \\
\mathbf{r}_{32i}^0 &= \mathbf{R}_{2i}^0 \mathbf{r}_{32i}^{2i} \\
\mathbf{r}_{G1i'}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{G1i'}^{3i} \\
\mathbf{R}_{2i}^0 &= \mathbf{R}_z(\theta_{21i}) \\
\mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\
\mathbf{r}_{32i}^{2i} &= [-z_{32i}, 0, 0]^T \\
\mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \\
\mathbf{r}_{G1i'}^{3i} &= [x_{G1i}, y_{G1i}, z_{G1i}]^T \\
\mathbf{J}_{1i}^0 &= \mathbf{R}_{2i}^0 \mathbf{J}_{1i}^{2i} \mathbf{R}_{2i}^{0T}
\end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_{1i} = m_{1i} \left(\dot{\mathbf{M}}_{1i}^T \mathbf{M}_{1i} + \mathbf{M}_{1i}^T \dot{\mathbf{M}}_{1i} \right) + \left(\dot{\mathbf{M}}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i} + \mathbf{M}_{2i}^T \dot{\mathbf{J}}_{1i} \mathbf{M}_{2i} + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \dot{\mathbf{M}}_{2i} \right)$$

A su vez:

$$\begin{aligned}
\dot{\mathbf{M}}_{1i} &= \dot{\mathbf{k}}_{1i} \mathbf{k}_{2i}^T + \mathbf{k}_{1i} \dot{\mathbf{k}}_{2i}^T \\
\dot{\mathbf{k}}_{1i} &= \mathbf{k}_{1i}^0 \times (\dot{\mathbf{r}}_{32i}^0 + \dot{\mathbf{r}}_{G1i'}^0)
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{r}}_{32i}^0 &= \dot{\mathbf{R}}_{2i}^0 \mathbf{r}_{32i}^{2i} \\
\dot{\mathbf{R}}_{2i}^0 &= \mathbf{B}_z(\theta_{21i}) \dot{\theta}_{21i} \\
\mathbf{B}_z(\theta_{21i}) &= \frac{d\mathbf{R}_z(\theta_{21i})}{d\theta_{21i}} \\
\dot{\mathbf{r}}_{G1i'}^0 &= \dot{\mathbf{R}}_{3i}^0 \mathbf{r}_{G1i'}^{3i} \\
\dot{\mathbf{R}}_{3i}^0 &= \dot{\mathbf{R}}_{2i}^0 \\
\mathbf{k}_{2i}^T &= -\frac{1}{V_{44}^2} \dot{V}_{44} [E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{36}] + \frac{1}{V_{44}} [\dot{E}_{31}, \dot{E}_{32}, \dot{E}_{33}, \dot{E}_{34}, \dot{E}_{35}, \dot{E}_{36}] \\
\dot{\mathbf{M}}_{2i} &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T \\
\dot{\mathbf{J}}_{1i}^0 &= \dot{\mathbf{R}}_{2i}^0 \mathbf{J}_{1i}^{2i} \mathbf{R}_{2i}^{0T} + \mathbf{R}_{2i}^0 \mathbf{J}_{1i}^{2i} \dot{\mathbf{R}}_{2i}^{0T}
\end{aligned}$$

Desarrollando $\frac{d}{dt} \left(\frac{\partial L_{2i}}{\partial \dot{q}_j} \right)$:

$$\begin{aligned}
\frac{d}{dt} \left(\frac{\partial L_{2i}}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \dot{\mathbf{q}} \right) \\
&= \left(\frac{d}{dt} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \right) \right) \mathbf{N}_{2i} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \frac{d\mathbf{N}_{2i}}{dt} \dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \frac{d\dot{\mathbf{q}}}{dt} \\
&= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_{2i} \dot{\mathbf{q}} \\
\frac{d}{dt} \left(\frac{\partial L_{2i}}{\partial \dot{q}_j} \right) &= \mathbb{D}_{2ij} \ddot{\mathbf{q}} + \mathbb{V}_{2ij} \dot{\mathbf{q}}
\end{aligned} \tag{3.133}$$

Dónde:

$$\begin{aligned}
\mathbb{D}_{2ij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{2i} \\
\mathbb{V}_{2ij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \dot{\mathbf{N}}_{2i}
\end{aligned} \tag{3.134}$$

Además:

$$\begin{aligned}
\mathbf{N}_{2i} &= m_{2i} \mathbf{M}_{3i}^T \mathbf{M}_{3i} + \mathbf{M}_{4i}^T \mathbf{J}_{2i} \mathbf{M}_{4i} \\
\mathbf{M}_{3i} &= \mathbf{k}_{3i} \mathbf{k}_{2i}^T + \mathbf{k}_{4i} \mathbf{k}_{6i}^T + \mathbf{k}_{5i} \mathbf{k}_{7i}^T \\
\mathbf{M}_{4i} &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T + \mathbf{k}_{6i}^0 \mathbf{k}_{7i}^T \\
\mathbf{k}_{3i} &= \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0) \\
\mathbf{k}_{4i} &= \mathbf{j}_{5i}^0 \times \mathbf{r}_{G2i'}^0 \\
\mathbf{k}_{5i} &= \mathbf{k}_{6i}^0 \times \mathbf{r}_{G2i'}^0 \\
\mathbf{k}_{6i}^T &= \frac{1}{V_{48}} [E_{37}, E_{38}, E_{39}, E_{40}, E_{41}, E_{42}] \\
\mathbf{k}_{7i}^T &= \frac{1}{V_{53}} [E_{43}, E_{44}, E_{45}, E_{46}, E_{47}, E_{48}] \\
\mathbf{j}_{5i}^0 &= \mathbf{R}_{5i}^0 \mathbf{j}_{5i}^{5i}
\end{aligned}$$

$$\begin{aligned}
\mathbf{k}_{6i}^0 &= \mathbf{R}_{6i}^0 \mathbf{k}_{6i}^{6i} \\
\mathbf{R}_{5i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \\
\mathbf{R}_{6i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \\
\mathbf{r}_{43i}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{43i}^{3i} \\
\mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\
\mathbf{r}_{G2i'}^0 &= \mathbf{R}_{7i}^0 \mathbf{r}_{G2i'}^{7i} \\
\mathbf{R}_{7i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \\
\mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \\
\mathbf{r}_{G2i'}^{7i} &= [x_{G2i}, y_{G2i}, z_{G2i}]^T \\
\mathbf{J}_{2i}^0 &= \mathbf{R}_{7i}^0 \mathbf{J}_{2i}^{7i} \mathbf{R}_{7i}^{0T}
\end{aligned}$$

Derivando:

$$\dot{\mathbf{N}}_{2i} = m_{2i} \left(\dot{\mathbf{M}}_{3i}^T \mathbf{M}_{3i} + \mathbf{M}_{3i}^T \dot{\mathbf{M}}_{3i} \right) + \left(\dot{\mathbf{M}}_{4i}^T \mathbf{J}_{2i} \mathbf{M}_{4i} + \mathbf{M}_{4i}^T \dot{\mathbf{J}}_{2i} \mathbf{M}_{4i} + \mathbf{M}_{4i}^T \mathbf{J}_{2i} \dot{\mathbf{M}}_{4i} \right)$$

A su vez:

$$\begin{aligned}
\dot{\mathbf{M}}_{3i} &= \dot{\mathbf{k}}_{3i} \mathbf{k}_{2i}^T + \mathbf{k}_{3i} \dot{\mathbf{k}}_{2i}^T + \dot{\mathbf{k}}_{4i} \mathbf{k}_{6i}^T + \mathbf{k}_{4i} \dot{\mathbf{k}}_{6i}^T + \dot{\mathbf{k}}_{5i} \mathbf{k}_{7i}^T + \mathbf{k}_{5i} \dot{\mathbf{k}}_{7i}^T \\
\dot{\mathbf{M}}_{4i} &= \dot{\mathbf{k}}_{1i}^0 \mathbf{k}_{2i}^T + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T + \dot{\mathbf{j}}_{5i}^0 \mathbf{k}_{6i}^T + \dot{\mathbf{k}}_{6i}^0 \mathbf{k}_{7i}^T + \mathbf{k}_{6i}^0 \dot{\mathbf{k}}_{7i}^T \\
\dot{\mathbf{k}}_{3i} &= \dot{\mathbf{k}}_{1i}^0 \times (\dot{\mathbf{r}}_{32i}^0 + \dot{\mathbf{r}}_{43i}^0 + \dot{\mathbf{r}}_{G2i'}^0) \\
\dot{\mathbf{r}}_{43i}^0 &= \dot{\mathbf{R}}_{3i}^0 \mathbf{r}_{43i}^{3i} \\
\dot{\mathbf{r}}_{G2i'}^0 &= \dot{\mathbf{R}}_{7i}^0 \mathbf{r}_{G2i'}^{7i} \\
\dot{\mathbf{R}}_{7i}^0 &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \dot{\theta}_{21i} \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \dot{\theta}_{65i} \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \dot{\theta}_{76i} \\
\mathbf{B}_y(\theta_{65i}) &= \frac{d\mathbf{R}_y(\theta_{65i})}{d\theta_{65i}} \\
\mathbf{B}_z(\theta_{76i}) &= \frac{d\mathbf{R}_z(\theta_{76i})}{d\theta_{76i}} \\
\dot{\mathbf{k}}_{4i} &= \mathbf{j}_{5i}^0 \times \mathbf{r}_{G2i'}^0 + \mathbf{j}_{5i}^0 \times \dot{\mathbf{r}}_{G2i'}^0 \\
\dot{\mathbf{j}}_{5i}^0 &= \dot{\mathbf{R}}_{5i}^0 \mathbf{j}_{5i}^{5i} \\
\dot{\mathbf{R}}_{5i}^0 &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \dot{\theta}_{21i} \\
\dot{\mathbf{k}}_{5i} &= \dot{\mathbf{k}}_{6i}^0 \times \mathbf{r}_{G2i'}^0 + \mathbf{k}_{6i}^0 \times \dot{\mathbf{r}}_{G2i'}^0 \\
\dot{\mathbf{k}}_{6i}^0 &= \dot{\mathbf{R}}_{6i}^0 \mathbf{k}_{6i}^{6i} \\
\dot{\mathbf{R}}_{6i}^0 &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \dot{\theta}_{21i} + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \dot{\theta}_{65i} \\
\dot{\mathbf{k}}_{6i}^T &= -\frac{1}{V_{48}^2} \dot{V}_{48} [E_{37}, E_{38}, E_{39}, E_{40}, E_{41}, E_{42}] + \frac{1}{V_{48}} [\dot{E}_{37}, \dot{E}_{38}, \dot{E}_{39}, \dot{E}_{40}, \dot{E}_{41}, \dot{E}_{42}] \\
\dot{\mathbf{k}}_{7i}^T &= -\frac{1}{V_{53}^2} \dot{V}_{53} [E_{43}, E_{44}, E_{45}, E_{46}, E_{47}, E_{48}] + \frac{1}{V_{53}} [\dot{E}_{43}, \dot{E}_{44}, \dot{E}_{45}, \dot{E}_{46}, \dot{E}_{47}, \dot{E}_{48}] \\
\dot{\mathbf{j}}_{2i}^0 &= \dot{\mathbf{R}}_{7i}^0 \mathbf{J}_{2i}^{7i} \mathbf{R}_{7i}^{0T} + \mathbf{R}_{7i}^0 \mathbf{J}_{2i}^{7i} \dot{\mathbf{R}}_{7i}^{0T}
\end{aligned}$$

3.5.2. Desarrollo del Segundo Término de la Ecuación de Lagrange para las cadenas i.

Desarrollando $\frac{\partial L_{1i}}{\partial q_j}$

$$L_{1i} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_{1i} \dot{\mathbf{q}} + m_{1i} \mathbf{g}^T \mathbf{r}_{G1i}^0$$

Dónde:

$$\begin{aligned} \mathbf{N}_{1i} &= m_{1i} \mathbf{M}_{1i}^T \mathbf{M}_{1i} + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \mathbf{M}_{2i} \\ \mathbf{M}_{1i} &= \mathbf{k}_{1i} \mathbf{k}_{2i}^T \\ \mathbf{M}_{2i} &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T \\ \mathbf{k}_{1i} &= \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{G1i'}^0) \\ \mathbf{k}_{2i}^T &= \frac{1}{V_{44}} [E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{36}] \\ \mathbf{k}_{1i}^0 &= \mathbf{k}_0 \\ \mathbf{r}_{32i}^0 &= \mathbf{R}_{2i}^0 \mathbf{r}_{32i}^{2i} \\ \mathbf{r}_{G1i'}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{G1i'}^{3i} \\ \mathbf{R}_{2i}^0 &= \mathbf{R}_z(\theta_{21i}) \\ \mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\ \mathbf{r}_{32i}^{2i} &= [-z_{32i}, 0, 0]^T \\ \mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \\ \mathbf{r}_{G1i'}^{3i} &= [x_{G1i}, y_{G1i}, z_{G1i}]^T \\ \mathbf{J}_{1i}^0 &= \mathbf{R}_{2i}^0 \mathbf{J}_{1i}^{2i} \mathbf{R}_{2i}^{0T} \end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned} \frac{\partial L_{1i}}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_{1i}}{\partial q_j} \dot{\mathbf{q}} + m_{1i} \mathbf{g}^T \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j} \\ \frac{\partial L_{1i}}{\partial q_j} &= \mathbb{V}'_{1ij} \dot{\mathbf{q}} + \mathbb{C}_{1ij} \end{aligned} \quad 3.135$$

Dónde:

$$\begin{aligned} \mathbb{V}'_{1ij} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_{1i}}{\partial q_j} \\ \mathbb{C}_{1ij} &= m_{1i} \mathbf{g}^T \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j} \end{aligned} \quad 3.136$$

A su vez:

$$\begin{aligned} \frac{\partial \mathbf{N}_{1i}}{\partial q_j} &= m_{1i} \left(\frac{\partial \mathbf{M}_{1i}^T}{\partial q_j} \mathbf{M}_{1i} + \mathbf{M}_{1i}^T \frac{\partial \mathbf{M}_{1i}}{\partial q_j} \right) + \frac{\partial \mathbf{M}_{2i}^T}{\partial q_j} \mathbf{J}_{1i} \mathbf{M}_{2i} + \mathbf{M}_{2i}^T \frac{\partial \mathbf{J}_{1i}}{\partial q_j} \mathbf{M}_{2i} \\ &\quad + \mathbf{M}_{2i}^T \mathbf{J}_{1i} \frac{\partial \mathbf{M}_{2i}}{\partial q_j} \end{aligned}$$

Además:

$$\begin{aligned}
\frac{\partial \mathbf{M}_{1i}}{\partial q_j} &= \frac{\partial \mathbf{k}_{1i}}{\partial q_j} \mathbf{k}_{2i}^T + \mathbf{k}_{1i} \frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} \\
\frac{\partial \mathbf{M}_{2i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} \\
\frac{\partial \mathbf{k}_{1i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} &= -\frac{1}{V_{44}^2} \frac{\partial V_{44}}{\partial q_j} [E_{31}, E_{32}, E_{33}, E_{34}, E_{35}, E_{36}] \\
&\quad + \frac{1}{V_{44}} \left[\frac{\partial E_{31}}{\partial q_j}, \frac{\partial E_{32}}{\partial q_j}, \frac{\partial E_{33}}{\partial q_j}, \frac{\partial E_{34}}{\partial q_j}, \frac{\partial E_{35}}{\partial q_j}, \frac{\partial E_{36}}{\partial q_j} \right] \\
\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \mathbf{r}_{32i}^{2i} \\
\frac{\partial \mathbf{r}_{G1i'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{r}_{G1i'}^{3i} \\
\frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} &= \mathbf{B}_z(\theta_{21i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \\
\frac{\partial \mathbf{J}_{1i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \mathbf{J}_{1i}^{2i} \mathbf{R}_{2i}^{0T} + \mathbf{R}_{2i}^0 \mathbf{J}_{1i}^{2i} \frac{\partial \mathbf{R}_{2i}^{0T}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{1i}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{1i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{1i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \mathbf{r}_{32i}^{2i} + \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{r}_{G1i'}^{3i} \right) \\
\frac{\partial \mathbf{k}_{1i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\mathbf{B}_z(\theta_{21i}) \frac{\partial \theta_{21i}}{\partial q_j} \mathbf{r}_{32i}^{2i} + \mathbf{B}_z(\theta_{21i}) \frac{\partial \theta_{21i}}{\partial q_j} \mathbf{r}_{G1i'}^{3i} \right) \\
\frac{\partial \mathbf{k}_{1i}}{\partial q_j} &= (\mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i} + \mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{G1i'}^{3i}) \frac{\partial \theta_{21i}}{\partial q_j}
\end{aligned}$$

$$\frac{\partial \mathbf{k}_{1i}}{\partial q_j} = \mathbf{J}_{1i} \frac{\partial \beta}{\partial q_j}$$

3.137

Dónde:

$$\mathbf{J}_{1i} = [\mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i} + \mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{G1i'}^{3i} \quad \mathbf{0} \quad \mathbf{0}]$$

Desarrollando $\frac{\partial \mathbf{k}_{2i}^T}{\partial q_j}$:

$$\begin{aligned}\frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} &= \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{21i}} \frac{\partial \theta_{21i}}{\partial q_j} + \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{65i}} \frac{\partial \theta_{65i}}{\partial q_j} + \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{76i}} \frac{\partial \theta_{76i}}{\partial q_j} + \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{43}} \frac{\partial \theta_{43}}{\partial q_j} + \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{54}} \frac{\partial \theta_{54}}{\partial q_j} \\ \frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} &= J_{2i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} + J'_{2i} \frac{\partial \boldsymbol{\theta}}{\partial q_j}\end{aligned}\tag{3.138}$$

Dónde:

$$\begin{aligned}J_{2i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{21i}} & \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{65i}} & \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{76i}} \end{bmatrix} \\ J'_{2i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_{2i}^T}{\partial \theta_{54}} \end{bmatrix}\end{aligned}$$

Desarrollando $\frac{\partial L_{2i}}{\partial q_j}$

$$L_{2i} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_{2i} \dot{\mathbf{q}} + m_{2i} \mathbf{g}^T \mathbf{r}_{G2i}^0$$

Dónde:

$$\begin{aligned}N_{2i} &= m_{2i} \mathbf{M}_{3i}^T \mathbf{M}_{3i} + \mathbf{M}_{4i}^T J_{2i} \mathbf{M}_{4i} \\ \mathbf{M}_{3i} &= \mathbf{k}_{3i} \mathbf{k}_{2i}^T + \mathbf{k}_{4i} \mathbf{k}_{6i}^T + \mathbf{k}_{5i} \mathbf{k}_{7i}^T \\ \mathbf{M}_{4i} &= \mathbf{k}_{1i}^0 \mathbf{k}_{2i}^T + \mathbf{j}_{5i}^0 \mathbf{k}_{6i}^T + \mathbf{k}_{6i}^0 \mathbf{k}_{7i}^T \\ \mathbf{k}_{3i} &= \mathbf{k}_{1i}^0 \times (\mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{G2i'}^0) \\ \mathbf{k}_{4i} &= \mathbf{j}_{5i}^0 \times \mathbf{r}_{G2i'}^0 \\ \mathbf{k}_{5i} &= \mathbf{k}_{6i}^0 \times \mathbf{r}_{G2i'}^0 \\ \mathbf{k}_{6i}^T &= \frac{1}{V_{48}} [E_{37}, E_{38}, E_{39}, E_{40}, E_{41}, E_{42}] \\ \mathbf{k}_{7i}^T &= \frac{1}{V_{53}} [E_{43}, E_{44}, E_{45}, E_{46}, E_{47}, E_{48}] \\ \mathbf{j}_{5i}^0 &= \mathbf{R}_{5i}^0 \mathbf{j}_{5i}^{5i} \\ \mathbf{k}_{6i}^0 &= \mathbf{R}_{6i}^0 \mathbf{k}_{6i}^{6i} \\ \mathbf{R}_{5i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \\ \mathbf{R}_{6i}^0 &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \\ \mathbf{r}_{43i}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{43i}^{3i} \\ \mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\ \mathbf{r}_{G2i'}^0 &= \mathbf{R}_{7i}^0 \mathbf{r}_{G2i'}^{7i}\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{7i}^0 &= \mathbf{R}_z(\theta_{21i})\mathbf{R}_x(\beta_{54i})\mathbf{R}_y(\theta_{65i})\mathbf{R}_z(\theta_{76i}) \\
\mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \\
\mathbf{r}_{G2i'}^{7i} &= [x_{G2i}, y_{G2i}, z_{G2i}]^T \\
\mathbf{J}_{2i}^0 &= \mathbf{R}_{7i}^0 \mathbf{J}_{2i}^{7i} \mathbf{R}_{7i}^{0T}
\end{aligned}$$

Derivando con respecto a q_j :

$$\begin{aligned}
\frac{\partial L_2}{\partial q_j} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_{2i}}{\partial q_j} \dot{\mathbf{q}} + m_{2i} \mathbf{g}^T \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j} \\
\frac{\partial L_{2i}}{\partial q_j} &= \mathbb{V}'_{2ij} \dot{\mathbf{q}} + \mathbb{C}_{2ij}
\end{aligned} \tag{3.139}$$

Dónde:

$$\begin{aligned}
\mathbb{V}'_{2ij} &= \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{N}_{2i}}{\partial q_j} \\
\mathbb{C}_{2ij} &= m_{2i} \mathbf{g}^T \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}
\end{aligned} \tag{3.140}$$

Además:

$$\begin{aligned}
\frac{\partial \mathbf{N}_{2i}}{\partial q_j} &= m_{2i} \left(\frac{\partial \mathbf{M}_{3i}^T}{\partial q_j} \mathbf{M}_{3i} + \mathbf{M}_{3i}^T \frac{\partial \mathbf{M}_{3i}}{\partial q_j} \right) + \frac{\partial \mathbf{M}_{4i}^T}{\partial q_j} \mathbf{J}_{2i} \mathbf{M}_{4i} + \mathbf{M}_{4i}^T \frac{\partial \mathbf{J}_{2i}}{\partial q_j} \mathbf{M}_{4i} \\
&\quad + \mathbf{M}_{4i}^T \mathbf{J}_{2i} \frac{\partial \mathbf{M}_{4i}}{\partial q_j}
\end{aligned}$$

A su vez:

$$\begin{aligned}
\frac{\partial \mathbf{M}_{3i}}{\partial q_j} &= \frac{\partial \mathbf{k}_{3i}}{\partial q_j} \mathbf{k}_{2i}^T + \mathbf{k}_{3i} \frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{4i}}{\partial q_j} \mathbf{k}_{6i}^T + \mathbf{k}_{4i} \frac{\partial \mathbf{k}_{6i}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{5i}}{\partial q_j} \mathbf{k}_{7i}^T + \mathbf{k}_{5i} \frac{\partial \mathbf{k}_{7i}^T}{\partial q_j} \\
\frac{\partial \mathbf{M}_{4i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \frac{\partial \mathbf{k}_{2i}^T}{\partial q_j} + \frac{\partial \mathbf{j}_{5i}^0}{\partial q_j} \mathbf{k}_{6i}^T + \mathbf{j}_{5i}^0 \frac{\partial \mathbf{k}_{6i}^T}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \mathbf{k}_{7i}^T + \mathbf{k}_{6i}^0 \frac{\partial \mathbf{k}_{7i}^T}{\partial q_j} \\
\frac{\partial \mathbf{k}_{3i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{43i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{4i}}{\partial q_j} &= \frac{\partial \mathbf{j}_{5i}^0}{\partial q_j} \times \mathbf{r}_{G2i'}^0 + \mathbf{j}_{5i}^0 \times \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{5i}}{\partial q_j} &= \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \times \mathbf{r}_{G2i'}^0 + \mathbf{k}_{6i}^0 \times \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{6i}^0}{\partial q_j} \mathbf{k}_{6i}^0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{j}_{5i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{j}_{5i}^{5i} \\
\frac{\partial \mathbf{k}_{6i}^T}{\partial q_j} &= -\frac{1}{V_{48}^2} \frac{\partial V_{48}}{\partial q_j} [E_{37}, E_{38}, E_{39}, E_{40}, E_{41}, E_{42}] \\
&\quad + \frac{1}{V_{48}} \left[\frac{\partial E_{37}}{\partial q_j}, \frac{\partial E_{38}}{\partial q_j}, \frac{\partial E_{39}}{\partial q_j}, \frac{\partial E_{40}}{\partial q_j}, \frac{\partial E_{41}}{\partial q_j}, \frac{\partial E_{42}}{\partial q_j} \right] \\
\frac{\partial \mathbf{k}_{7i}^T}{\partial q_j} &= -\frac{1}{V_{53}^2} \frac{\partial V_{53}}{\partial q_j} [E_{43}, E_{44}, E_{45}, E_{46}, E_{47}, E_{48}] \\
&\quad + \frac{1}{V_{53}} \left[\frac{\partial E_{43}}{\partial q_j}, \frac{\partial E_{44}}{\partial q_j}, \frac{\partial E_{45}}{\partial q_j}, \frac{\partial E_{46}}{\partial q_j}, \frac{\partial E_{47}}{\partial q_j}, \frac{\partial E_{48}}{\partial q_j} \right] \\
\frac{\partial \mathbf{r}_{43i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{r}_{43i}^{3i} \\
\frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{r}_{G2i'}^{7i} \\
\frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_z(\theta_{21i})}{\partial q_j} \mathbf{R}_x(\beta_{54i}) + \mathbf{R}_z(\theta_{21i}) \frac{\partial \mathbf{R}_x(\beta_{54i})}{\partial q_j} \\
\frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{6i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_z(\theta_{21i})}{\partial q_j} \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) + \mathbf{R}_z(\theta_{21i}) \frac{\partial \mathbf{R}_x(\beta_{54i})}{\partial q_j} \mathbf{R}_y(\theta_{65i}) \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \frac{\partial \mathbf{R}_y(\theta_{65i})}{\partial q_j} \\
\frac{\partial \mathbf{R}_{6i}^0}{\partial q_j} &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \frac{\partial \theta_{21i}}{\partial q_j} + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_z(\theta_{21i})}{\partial q_j} \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) + \mathbf{R}_z(\theta_{21i}) \frac{\partial \mathbf{R}_x(\beta_{54i})}{\partial q_j} \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \frac{\partial \mathbf{R}_y(\theta_{65i})}{\partial q_j} \mathbf{R}_z(\theta_{76i}) \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \frac{\partial \mathbf{R}_z(\theta_{76i})}{\partial q_j} \\
\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} &= \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
&\quad + \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \frac{\partial \theta_{76i}}{\partial q_j} \\
\frac{\partial \mathbf{J}_{2i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{J}_{2i}^{7i} \mathbf{R}_{7i}^{0T} + \mathbf{R}_{7i}^0 \mathbf{J}_{2i}^{7i} \frac{\partial \mathbf{R}_{7i}^{0T}}{\partial q_j}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{3i}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{3i}}{\partial q_j} &= \mathbf{k}_{1i}^0 \times \left(\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{43i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{3i}}{\partial q_j} &= \left((\mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i} + \mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{43i}^{3i} + \mathbf{k}_{1i}^0 \right. \\
&\quad \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
&\quad + (\mathbf{k}_{1i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
&\quad \left. + (\mathbf{k}_{1i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{76i}}{\partial q_j} \right) \\
\frac{\partial \mathbf{k}_{3i}}{\partial q_j} &= J_{3i} \frac{\partial \boldsymbol{\beta}}{\partial q_j}
\end{aligned} \tag{3.141}$$

Dónde:

$$J_{3i} = [J_{31i} \quad J_{32i} \quad J_{33i}]$$

A su vez:

$$\begin{aligned}
J_{31i} &= \mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i} + \mathbf{k}_{1i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{r}_{43i}^{3i} + \mathbf{k}_{1i}^0 \\
&\quad \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i} \\
J_{32i} &= \mathbf{k}_{1i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i} \\
J_{33i} &= \mathbf{k}_{1i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}
\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{4i}}{\partial q_j}$:

$$\begin{aligned}
\frac{\partial \mathbf{k}_{4i}}{\partial q_j} &= \frac{\partial \mathbf{j}_{5i}^0}{\partial q_j} \times \mathbf{r}_{G2i'}^0 + \mathbf{j}_{5i}^0 \times \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \\
\frac{\partial \mathbf{k}_{4i}}{\partial q_j} &= \left((\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{j}_{5i}^{5i} \times \mathbf{r}_{G2i'}^0) \right. \\
&\quad + (\mathbf{j}_{5i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
&\quad + (\mathbf{j}_{5i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
&\quad \left. + (\mathbf{j}_{5i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{76i}}{\partial q_j} \right)
\end{aligned}$$

$$\frac{\partial \mathbf{k}_{4i}}{\partial q_j} = J_{4i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} \quad 3.142$$

Dónde:

$$J_{4i} = [J_{41i} \quad J_{42i} \quad J_{43i}]$$

A su vez:

$$\begin{aligned} J_{41i} &= (\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{j}_{5i}^5 \times \mathbf{r}_{G2i'}^0) + (\mathbf{j}_{5i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \\ J_{42i} &= \mathbf{j}_{5i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i} \\ J_{43i} &= \mathbf{j}_{5i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i} \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{5i}}{\partial q_j}$:

$$\begin{aligned} \frac{\partial \mathbf{k}_{5i}}{\partial q_j} &= \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \times \mathbf{r}_{G2i'}^0 + \mathbf{k}_{6i}^0 \times \frac{\partial \mathbf{r}_{G2i'}^0}{\partial q_j} \\ \frac{\partial \mathbf{k}_{5i}}{\partial q_j} &= \left((\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{k}_{6i}^{6i} \times \mathbf{r}_{G2i'}^0) \right. \\ &\quad \left. + (\mathbf{k}_{6i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \right) \frac{\partial \theta_{21i}}{\partial q_j} \\ &\quad + \left((\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{k}_{6i}^{6i} \times \mathbf{r}_{G2i'}^0) \right. \\ &\quad \left. + (\mathbf{k}_{6i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \right) \frac{\partial \theta_{65i}}{\partial q_j} \\ &\quad + (\mathbf{k}_{6i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \frac{\partial \theta_{76i}}{\partial q_j} \end{aligned}$$

$$\frac{\partial \mathbf{k}_{5i}}{\partial q_j} = J_{5i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} \quad 3.143$$

Dónde:

$$J_{5i} = [J_{51i} \quad J_{52i} \quad J_{53i}]$$

A su vez:

$$\begin{aligned} J_{51i} &= (\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{k}_{6i}^{6i} \times \mathbf{r}_{G2i'}^0) \\ &\quad + (\mathbf{k}_{6i}^0 \times \mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \\ J_{52i} &= (\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{k}_{6i}^{6i} \times \mathbf{r}_{G2i'}^0) \\ &\quad + (\mathbf{k}_{6i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i}) \\ J_{53i} &= \mathbf{k}_{6i}^0 \times \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{G2i'}^{7i} \end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{6i}^T}{\partial q_j}$:

$$\begin{aligned}\frac{\partial \mathbf{k}_{6i}^T}{\partial q_j} &= \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{21i}} \frac{\partial \theta_{21i}}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{65i}} \frac{\partial \theta_{65i}}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{76i}} \frac{\partial \theta_{76i}}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{43}} \frac{\partial \theta_{43}}{\partial q_j} + \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{54}} \frac{\partial \theta_{54}}{\partial q_j} \\ \frac{\partial \mathbf{k}_{6i}^T}{\partial q_j} &= J_{6i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} + J'_{6i} \frac{\partial \boldsymbol{\theta}}{\partial q_j}\end{aligned}\quad 3.144$$

Dónde:

$$\begin{aligned}J_{6i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{21i}} & \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{65i}} & \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{76i}} \end{bmatrix} \\ J'_{6i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_{6i}^T}{\partial \theta_{54}} \end{bmatrix}\end{aligned}$$

Desarrollando $\frac{\partial \mathbf{k}_{7i}^T}{\partial q_j}$:

$$\begin{aligned}\frac{\partial \mathbf{k}_{7i}^T}{\partial q_j} &= \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{21i}} \frac{\partial \theta_{21i}}{\partial q_j} + \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{65i}} \frac{\partial \theta_{65i}}{\partial q_j} + \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{76i}} \frac{\partial \theta_{76i}}{\partial q_j} + \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{21}} \frac{\partial \theta_{21}}{\partial q_j} + \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{43}} \frac{\partial \theta_{43}}{\partial q_j} + \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{54}} \frac{\partial \theta_{54}}{\partial q_j} \\ \frac{\partial \mathbf{k}_{7i}^T}{\partial q_j} &= J_{7i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} + J'_{7i} \frac{\partial \boldsymbol{\theta}}{\partial q_j}\end{aligned}\quad 3.145$$

Dónde:

$$\begin{aligned}J_{7i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{21i}} & \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{65i}} & \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{76i}} \end{bmatrix} \\ J'_{7i} &= \begin{bmatrix} \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{21}} & \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{43}} & \frac{\partial \mathbf{k}_{7i}^T}{\partial \theta_{54}} \end{bmatrix}\end{aligned}$$

3.6 Fuerzas Generalizadas.

La formulación de Lagrange considera el uso de fuerzas generalizadas es decir fuerzas y torques aplicadas externamente por los actuadores, de modo que es necesario desarrollar expresiones que sean compatibles con el Lagrangiano, esto se obtiene a partir de la expresión de trabajo virtual. Considerando los torques ejercidos por los actuadores en la base del robot:

$$\begin{aligned}
\delta W &= \sum_{i=1}^3 \mathbf{F}_i^T \delta \mathbf{R}_i + \mathbf{M}_i^T \delta \mathbf{Q}_i \\
&= \boldsymbol{\tau}_1^T \delta \mathbf{Q}_1 + \boldsymbol{\tau}_2^T \delta \mathbf{Q}_2 + \boldsymbol{\tau}_3^T \delta \mathbf{Q}_3
\end{aligned} \tag{3.146}$$

Obteniendo los términos $\boldsymbol{\tau}_i$:

$$\begin{aligned}
\boldsymbol{\tau}_1 &= \tau_1 \mathbf{j}_0 \\
\boldsymbol{\tau}_2 &= \tau_2 \mathbf{k}_0 \\
\boldsymbol{\tau}_3 &= \tau_3 \mathbf{k}_0
\end{aligned}$$

Las velocidades angulares se relacionan con los desplazamientos virtuales, esto es:

$$\delta \mathbf{Q} = \frac{\partial \boldsymbol{\omega}}{\partial \dot{\theta}} \delta \theta$$

$$\begin{aligned}
\delta \mathbf{Q}_1 &= \frac{\partial \boldsymbol{\omega}_2^0}{\partial \dot{\theta}_{21}} \delta \theta_{21} \\
&= \frac{\partial \boldsymbol{\omega}_{21}^0}{\partial \dot{\theta}_{21}} \delta \theta_{21} = \frac{\partial \dot{\theta}_{21} \mathbf{j}_1^0}{\partial \dot{\theta}_{21}} \delta \theta_{21} \\
\delta \mathbf{Q}_1 &= \mathbf{j}_0 \delta \theta_{21}
\end{aligned} \tag{3.147}$$

$$\begin{aligned}
\delta \mathbf{Q}_2 &= \frac{\partial \boldsymbol{\omega}_{21}^0}{\partial \dot{\theta}_{211}} \delta \theta_{211} \\
&= \frac{\partial \boldsymbol{\omega}_{211}^0}{\partial \dot{\theta}_{211}} \delta \theta_{211} = \frac{\partial \dot{\theta}_{211} \mathbf{k}_{11}^0}{\partial \dot{\theta}_{211}} \delta \theta_{211} \\
\delta \mathbf{Q}_2 &= \mathbf{k}_0 \delta \theta_{211}
\end{aligned} \tag{3.148}$$

$$\begin{aligned}
\delta \mathbf{Q}_3 &= \frac{\partial \boldsymbol{\omega}_{22}^0}{\partial \dot{\theta}_{212}} \delta \theta_{212} \\
&= \frac{\partial \boldsymbol{\omega}_{212}^0}{\partial \dot{\theta}_{212}} \delta \theta_{212} = \frac{\partial \dot{\theta}_{212} \mathbf{k}_{12}^0}{\partial \dot{\theta}_{212}} \delta \theta_{212} \\
\delta \mathbf{Q}_3 &= \mathbf{k}_0 \delta \theta_{212}
\end{aligned} \tag{3.149}$$

$$\begin{aligned}
\delta W &= \boldsymbol{\tau}_1^T \delta \mathbf{Q}_{21} + \boldsymbol{\tau}_2^T \delta \mathbf{Q}_{211} + \boldsymbol{\tau}_3^T \delta \mathbf{Q}_{212} \\
&= (\tau_1 \mathbf{j}_0)^T (\mathbf{j}_0 \delta \theta_{21}) + (\tau_2 \mathbf{k}_0)^T (\mathbf{k}_0 \delta \theta_{211}) + (\tau_3 \mathbf{k}_0)^T (\mathbf{k}_0 \delta \theta_{212}) \\
&= \tau_1 \delta \theta_{21} + \tau_2 \delta \theta_{211} + \tau_3 \delta \theta_{212} \\
\delta W &= \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \end{bmatrix} \begin{bmatrix} \delta \theta_{21} \\ \delta \theta_{211} \\ \delta \theta_{212} \end{bmatrix}
\end{aligned} \tag{3.150}$$

Usando:

$$\begin{aligned}\delta\theta_{21} &= \mathbf{k}_2^T \delta\mathbf{q} \\ \delta\theta_{21i} &= \mathbf{k}_{2i}^T \delta\mathbf{q}\end{aligned}$$

Sustituyendo:

$$\begin{aligned}\delta W &= [\tau_1 \ \tau_2 \ \tau_3] \begin{bmatrix} \mathbf{k}_2^T \\ \mathbf{k}_{21}^T \\ \mathbf{k}_{22}^T \end{bmatrix} \delta\mathbf{q} \\ &= \boldsymbol{\tau}^T \mathbf{J}_\tau \delta\mathbf{q} \\ \delta W &= \mathbf{Q}^T \delta\mathbf{q}\end{aligned}\tag{3.151}$$

Dónde:

$$\begin{aligned}\boldsymbol{\tau}^T &= [\tau_1 \ \tau_2 \ \tau_3] \\ \mathbf{J}_\tau &= [\mathbf{k}_2^T \ \mathbf{k}_{21}^T \ \mathbf{k}_{22}^T]^T \\ \mathbf{k}_2^T &= \frac{1}{V_1} [V_2, V_3, V_4, V_5, V_6, V_7] \\ \mathbf{k}_{2i}^T &= \frac{1}{V_{44i}} [V_{31i}, V_{32i}, V_{33i}, V_{34i}, V_{35i}, V_{36i}] \\ \delta\mathbf{q} &= [\delta x, \delta y, \delta z, \delta\psi, \delta\theta, \delta\phi]^T \\ \mathbf{Q}^T &= \boldsymbol{\tau}^T \mathbf{J}_\tau \\ \mathbf{Q} &= \mathbf{J}_\tau^T \boldsymbol{\tau}\end{aligned}\tag{3.152}$$

3.7 Sustitución de los términos de la ecuación de Lagrange.

Por Ultimo se tiene la ecuación (3.4):

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j$$

Dónde:

$$\begin{aligned}\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) &= \frac{d}{dt} \left(\sum_{h=1}^5 L_h + \sum_{i=1}^2 (L_{1i} + L_{2i}) \right) \\ &= \mathbb{D}_{1j} \ddot{\mathbf{q}} + \mathbb{V}_{1j} \dot{\mathbf{q}} + \mathbb{D}_{2j} \ddot{\mathbf{q}} + \mathbb{V}_{2j} \dot{\mathbf{q}} + \mathbb{D}_{3j} \ddot{\mathbf{q}} + \mathbb{V}_{3j} \dot{\mathbf{q}} + \mathbb{D}_{4j} \ddot{\mathbf{q}} + \mathbb{V}_{4j} \dot{\mathbf{q}} + \mathbb{D}_{5j} \ddot{\mathbf{q}} \\ &\quad + \mathbb{V}_{5j} \dot{\mathbf{q}} + \sum_{i=1}^2 (\mathbb{D}_{1ij} \ddot{\mathbf{q}} + \mathbb{V}_{1ij} \dot{\mathbf{q}} + \mathbb{D}_{2ij} \ddot{\mathbf{q}} + \mathbb{V}_{2ij} \dot{\mathbf{q}}) \\ &= \left[\mathbb{D}_{1j} + \mathbb{D}_{2j} + \mathbb{D}_{3j} + \mathbb{D}_{4j} + \mathbb{D}_{5j} + \sum_{i=1}^2 (\mathbb{D}_{1ij} + \mathbb{D}_{2ij}) \right] \ddot{\mathbf{q}} \\ &\quad + \left[\mathbb{V}_{1j} + \mathbb{V}_{2j} + \mathbb{V}_{3j} + \mathbb{V}_{4j} + \mathbb{V}_{5j} + \sum_{i=1}^2 (\mathbb{V}_{1ij} + \mathbb{V}_{2ij}) \right] \dot{\mathbf{q}}\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial q_j} &= \sum_{h=1}^5 \frac{\partial L_h}{\partial q_j} + \sum_{i=1}^2 \left(\frac{\partial L_{1i}}{\partial q_j} + \frac{\partial L_{2i}}{\partial q_j} \right) \\
\frac{\partial L}{\partial q_j} &= \mathbb{V}'_{1j} \dot{\mathbf{q}} + \mathbb{C}_{1j} + \mathbb{V}'_{2j} \dot{\mathbf{q}} + \mathbb{C}_{2j} + \mathbb{V}'_{3j} \dot{\mathbf{q}} + \mathbb{C}_{3j} + \mathbb{V}'_{4j} \dot{\mathbf{q}} + \mathbb{C}_{4j} + \mathbb{V}'_{5j} \dot{\mathbf{q}} + \mathbb{C}_{5j} \\
&\quad + \sum_{i=1}^2 (\mathbb{V}'_{1ij} \dot{\mathbf{q}} + \mathbb{C}_{1ij} + \mathbb{V}'_{2ij} \dot{\mathbf{q}} + \mathbb{C}_{2ij}) \\
&= \left[\mathbb{V}'_{1j} + \mathbb{V}'_{2j} + \mathbb{V}'_{3j} + \mathbb{V}'_{4j} + \mathbb{V}'_{5j} + \sum_{i=1}^2 (\mathbb{V}'_{1ij} + \mathbb{V}'_{2ij}) \right] \dot{\mathbf{q}} \\
&\quad + \left[\mathbb{C}_{1j} + \mathbb{C}_{2j} + \mathbb{C}_{3j} + \mathbb{C}_{4j} + \mathbb{C}_{5j} + \sum_{i=1}^2 (\mathbb{C}_{1ij} + \mathbb{C}_{2ij}) \right]
\end{aligned}$$

Sustituyendo en la ecuación (3.4):

$$\begin{aligned}
&\left[\mathbb{D}_{1j} + \mathbb{D}_{2j} + \mathbb{D}_{3j} + \mathbb{D}_{4j} + \mathbb{D}_{5j} + \sum_{i=1}^2 (\mathbb{D}_{1ij} + \mathbb{D}_{2ij}) \right] \ddot{\mathbf{q}} \\
&\quad + \left[\mathbb{V}_{1j} + \mathbb{V}_{2j} + \mathbb{V}_{3j} + \mathbb{V}_{4j} + \mathbb{V}_{5j} + \sum_{i=1}^2 (\mathbb{V}_{1ij} + \mathbb{V}_{2ij}) \right] \dot{\mathbf{q}} \\
&\quad - \left[\mathbb{V}'_{1j} + \mathbb{V}'_{2j} + \mathbb{V}'_{3j} + \mathbb{V}'_{4j} + \mathbb{V}'_{5j} + \sum_{i=1}^2 (\mathbb{V}'_{1ij} + \mathbb{V}'_{2ij}) \right] \dot{\mathbf{q}} \\
&\quad - \left[\mathbb{C}_{1j} + \mathbb{C}_{2j} + \mathbb{C}_{3j} + \mathbb{C}_{4j} + \mathbb{C}_{5j} + \sum_{i=1}^2 (\mathbb{C}_{1ij} + \mathbb{C}_{2ij}) \right] = \mathbf{Q}_j
\end{aligned}$$

$$\begin{aligned}
&\left[\mathbb{D}_{1j} + \mathbb{D}_{2j} + \mathbb{D}_{3j} + \mathbb{D}_{4j} + \mathbb{D}_{5j} + \sum_{i=1}^2 (\mathbb{D}_{1ij} + \mathbb{D}_{2ij}) \right] \ddot{\mathbf{q}} \\
&\quad + \left[\mathbb{V}_{1j} + \mathbb{V}_{2j} + \mathbb{V}_{3j} + \mathbb{V}_{4j} + \mathbb{V}_{5j} + \sum_{i=1}^2 (\mathbb{V}_{1ij} + \mathbb{V}_{2ij}) - \mathbb{V}'_{1j} - \mathbb{V}'_{1j} - \mathbb{V}'_{3j} \right. \\
&\quad \left. - \mathbb{V}'_{4j} - \mathbb{V}'_{5j} - \sum_{i=1}^2 (\mathbb{V}'_{1ij} + \mathbb{V}'_{2ij}) \right] \dot{\mathbf{q}} \\
&\quad - \left[\mathbb{C}_{1j} + \mathbb{C}_{2j} + \mathbb{C}_{3j} + \mathbb{C}_{4j} + \mathbb{C}_{5j} + \sum_{i=1}^2 (\mathbb{C}_{1ij} + \mathbb{C}_{2ij}) \right] = \mathbf{Q}_j
\end{aligned}$$

Finalmente:

$$D_j \ddot{\mathbf{q}} + V_j \dot{\mathbf{q}} + C_j = \mathbf{Q}_j \quad 3.153$$

Dónde:

$$\begin{aligned}
 D_j &= \mathbb{D}_{1j} + \mathbb{D}_{2j} + \mathbb{D}_{3j} + \mathbb{D}_{4j} + \mathbb{D}_{5j} + \sum_{i=1}^2 (\mathbb{D}_{1ij} + \mathbb{D}_{2ij}) \\
 V_j^T &= \mathbb{V}_{1j} + \mathbb{V}_{2j} + \mathbb{V}_{3j} + \mathbb{V}_{4j} + \mathbb{V}_{5j} + \sum_{i=1}^2 (\mathbb{V}_{1ij} + \mathbb{V}_{2ij}) - \mathbb{V}'_{1j} - \mathbb{V}'_{1j} - \mathbb{V}'_{3j} - \mathbb{V}'_{4j} - \mathbb{V}'_{5j} \\
 &\quad - \sum_{i=1}^2 (\mathbb{V}'_{1ij} + \mathbb{V}'_{2ij}) \\
 C_j &= -\mathbb{C}_{1j} - \mathbb{C}_{2j} - \mathbb{C}_{3j} - \mathbb{C}_{4j} - \mathbb{C}_{5j} - \sum_{i=1}^2 (-\mathbb{C}_{1ij} - \mathbb{C}_{2ij})
 \end{aligned}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}$$

$$D_j'^T \ddot{\mathbf{q}} + V_j'^T \dot{\mathbf{q}} + C_j' = \mathbf{Q}$$

$$D_j'^T \ddot{\mathbf{q}} + V_j'^T \dot{\mathbf{q}} + C_j' = \mathbf{J}_\tau^T \boldsymbol{\tau}$$

$$\mathbf{J}_\tau^{-T} D_j'^T \ddot{\mathbf{q}} + \mathbf{J}_\tau^{-T} V_j'^T \dot{\mathbf{q}} + \mathbf{J}_\tau^{-T} C_j' = \boldsymbol{\tau}$$

Finalmente:

$$D_j^T \ddot{\mathbf{q}} + V_j^T \dot{\mathbf{q}} + C_j = \boldsymbol{\tau} \quad 3.154$$

Dónde:

$$D_j^T = \mathbf{J}_\tau^{-T} D_j'^T$$

$$V_j = \mathbf{J}_\tau^{-T} V_j'^T$$

$$C_j = \mathbf{J}_\tau^{-T} C_j'$$

Capítulo 4

Resultados

A continuación, se muestran los resultados del análisis cinemático y dinámico del robot paralelo para la trayectoria descrita en el apéndice B.

4.1 Gráficas de Posición

Los parámetros geométricos relacionados con la estructura del robot son los siguientes:

Cadena 0:

$$z_{10} = .05 \text{ m}$$

$$x_{32} = .33 \text{ m}$$

$$x_{65} = .37 \text{ m}$$

Cadena 1:

$$y_{101} = .085 \text{ m}$$

$$z_{321} = .035 \text{ m}$$

$$x_{431} = .33 \text{ m}$$

$$\beta_{541} = 0^\circ$$

$$x_{871} = .325 \text{ m}$$

$$x_{12111} = .015 \text{ m}$$

$$z_{13121} = .02 \text{ m}$$

$$\beta_{14131} = 270^\circ$$

$$x_{15141} = .1 \text{ m}$$

$$x_{19181} = .06 \text{ m}$$

Cadena 2:

$$y_{102} = .085 \text{ m}$$

$$z_{322} = .035 \text{ m}$$

$$x_{432} = .33 \text{ m}$$

$$\beta_{542} = 0^\circ$$

$$x_{872} = .325 \text{ m}$$

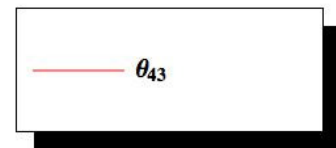
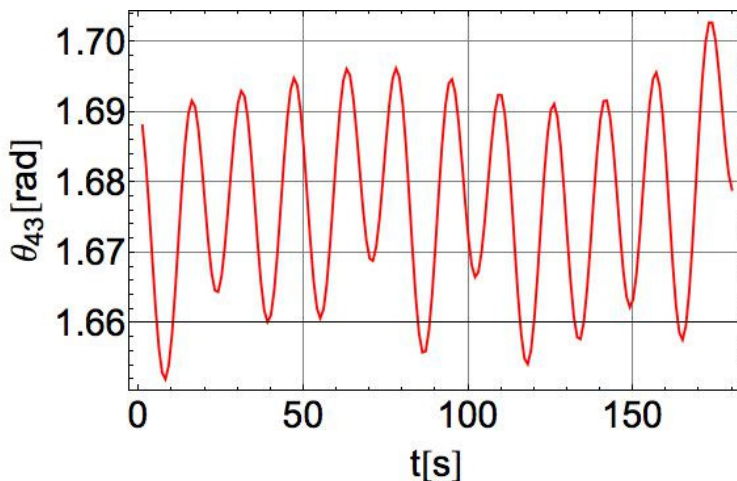
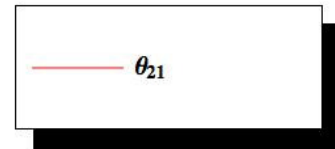
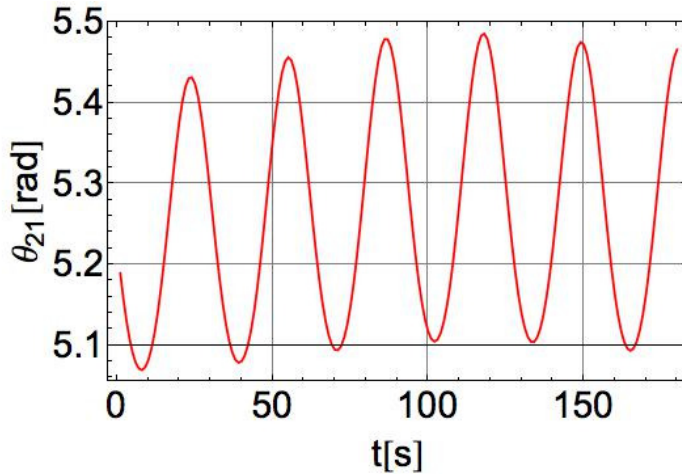
$$x_{12112} = .015 \text{ m}$$

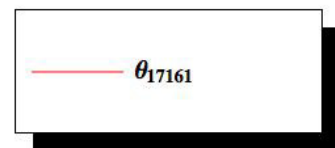
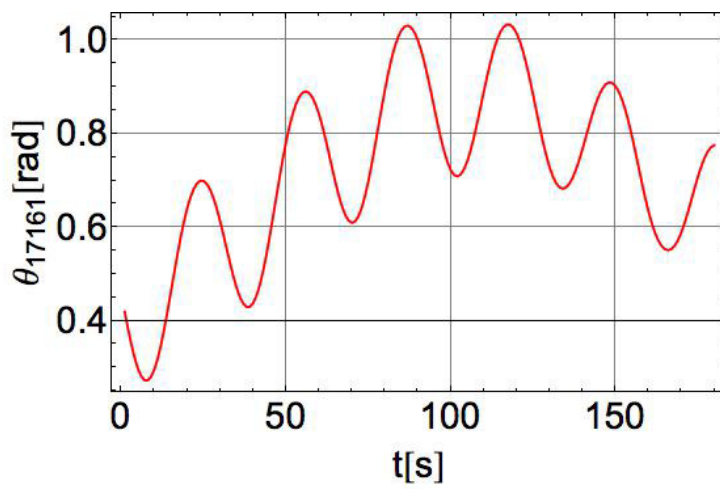
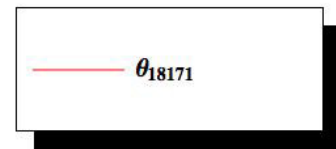
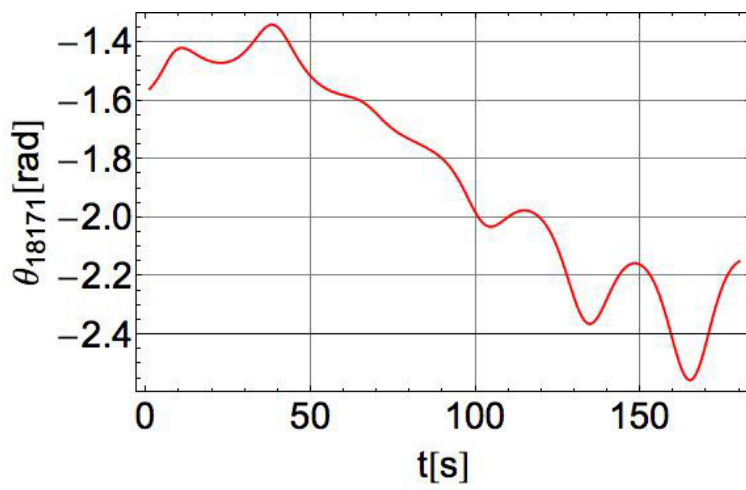
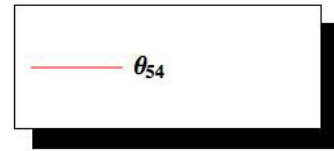
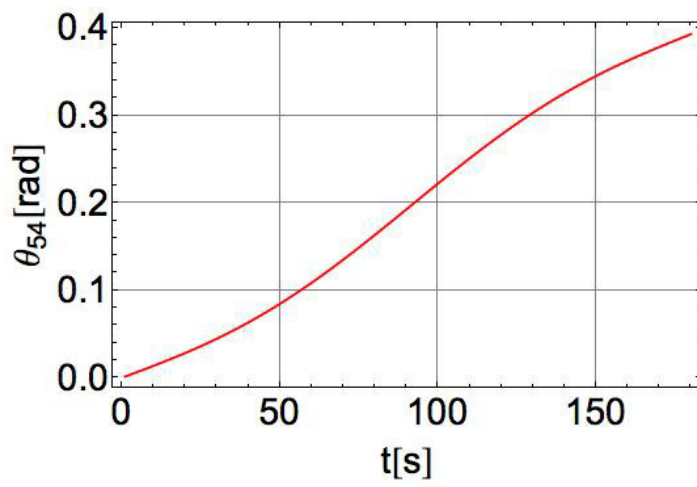
$$z_{13122} = .02 \text{ m}$$

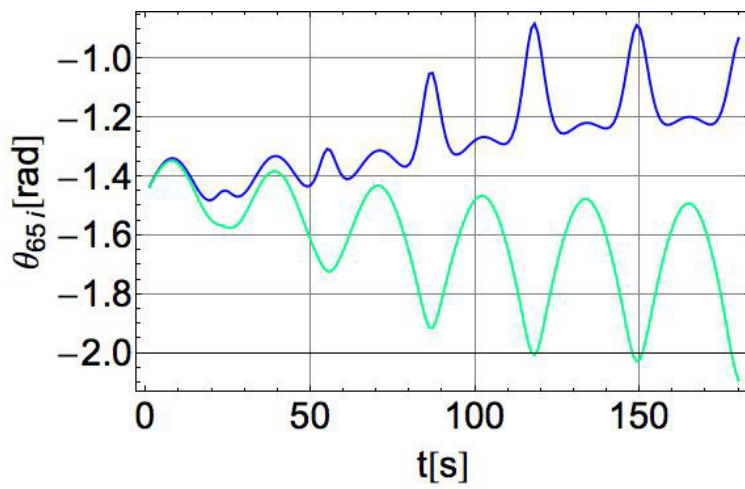
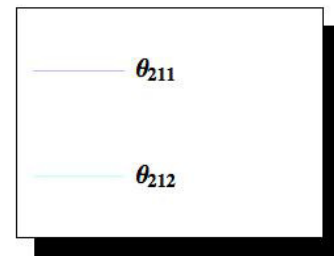
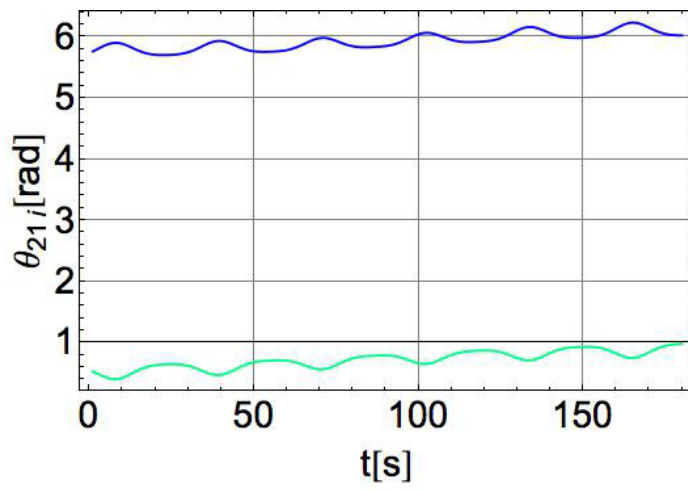
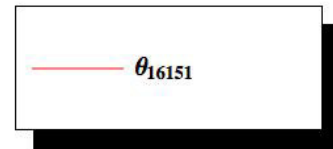
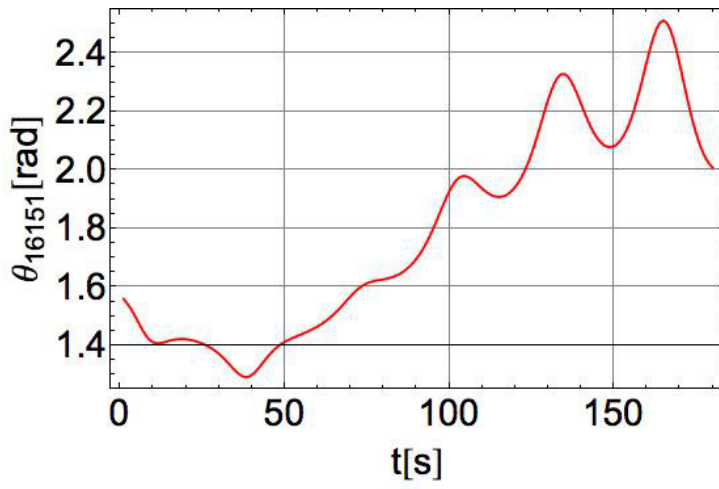
$$\beta_{14132} = 90^\circ$$

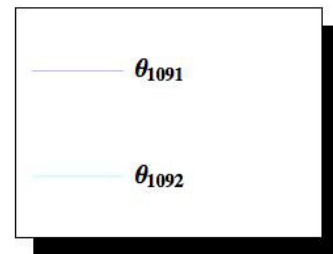
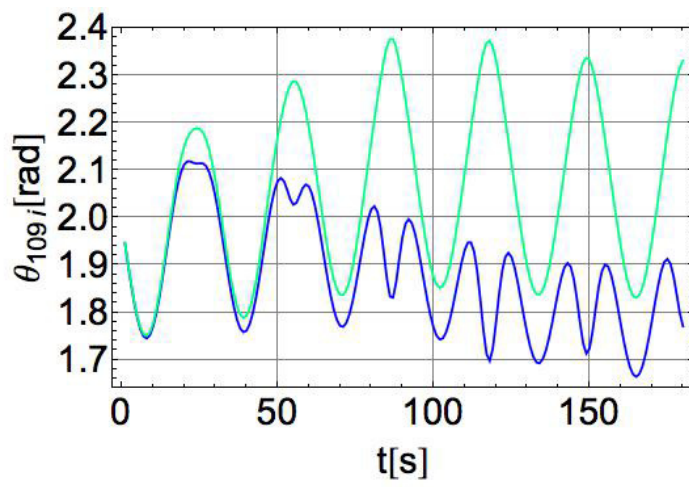
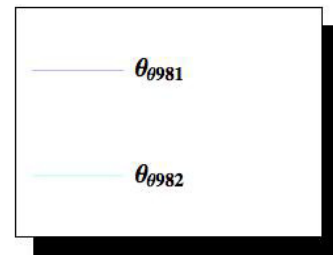
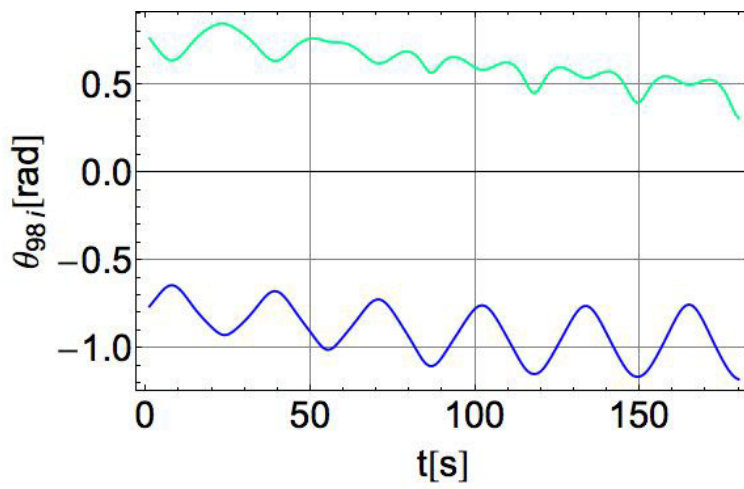
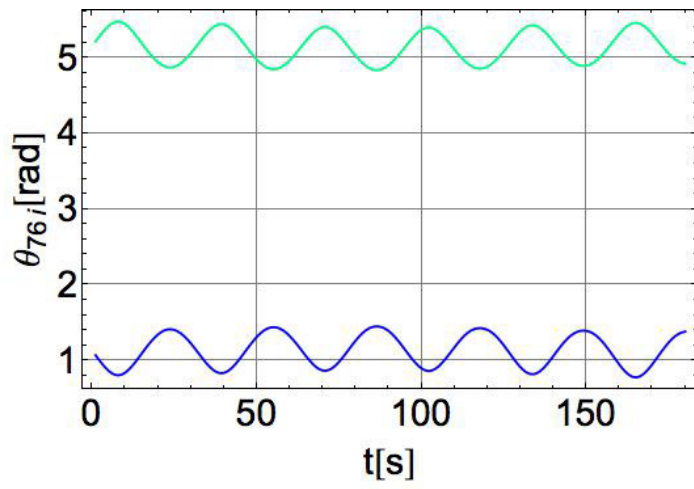
$$x_{15142} = .1 \text{ m}$$

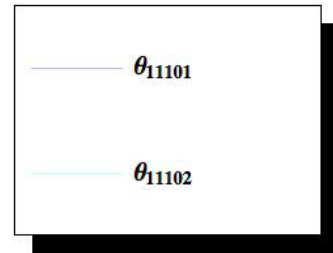
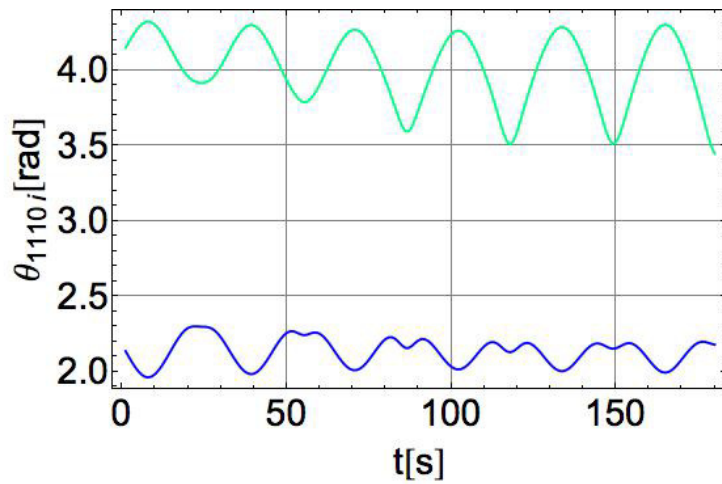
$$x_{19182} = .06 \text{ m}$$



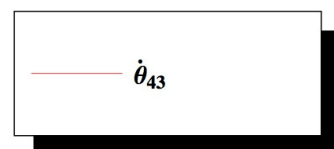
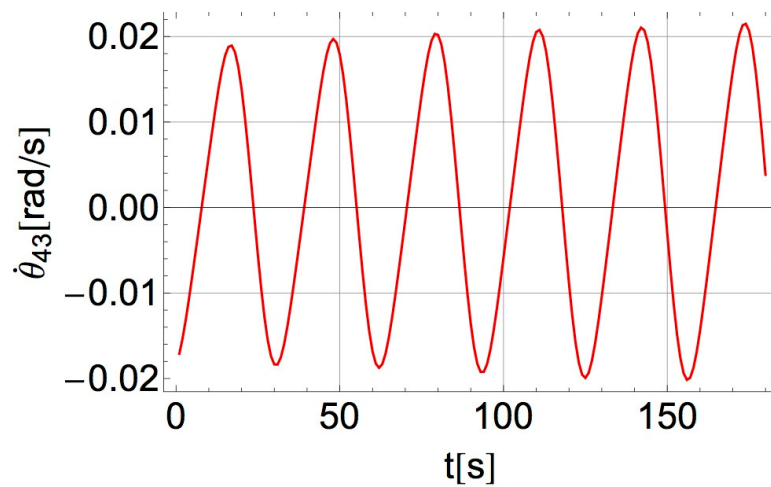
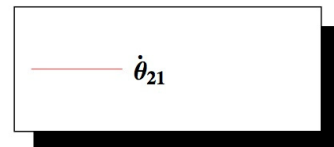
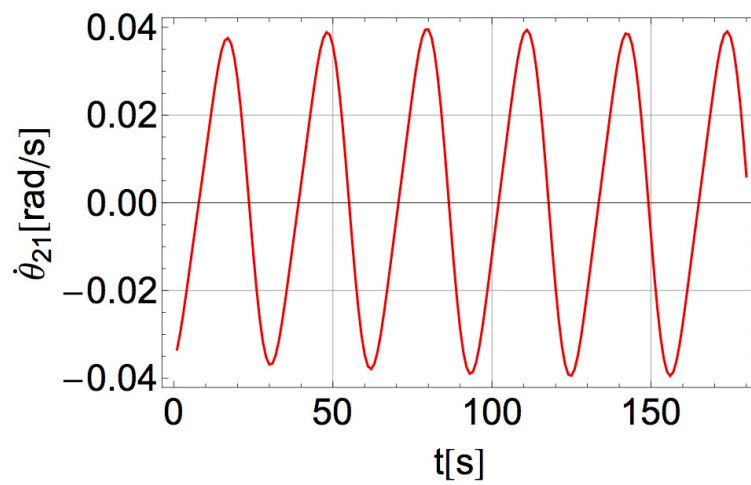


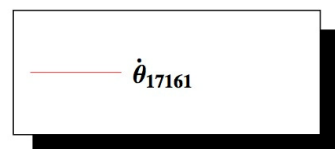
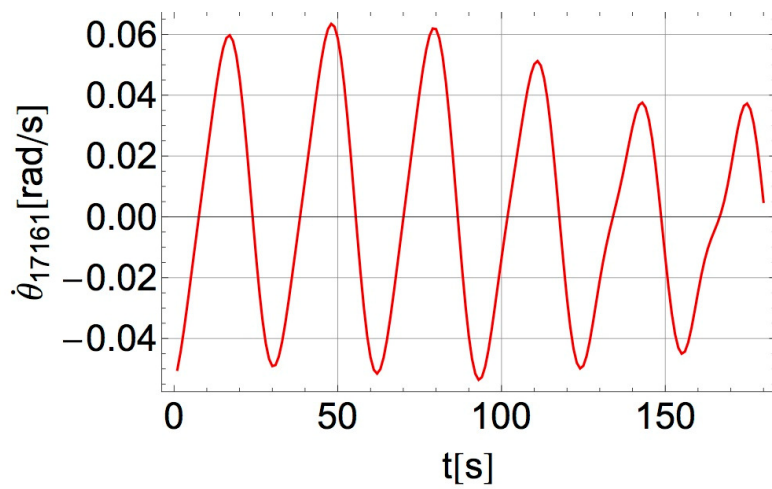
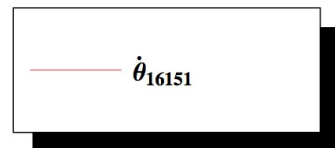
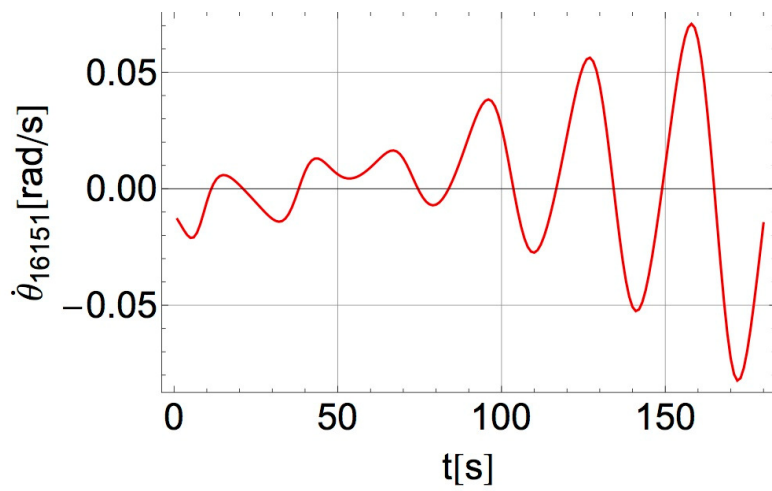
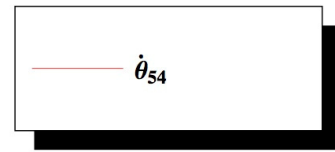
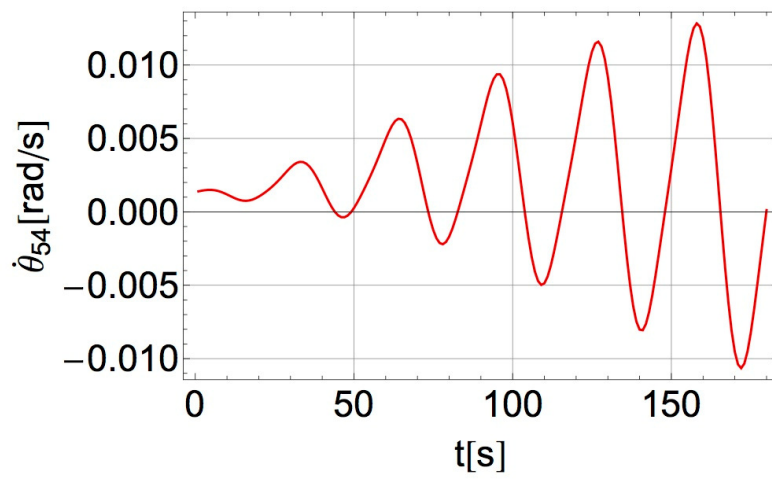


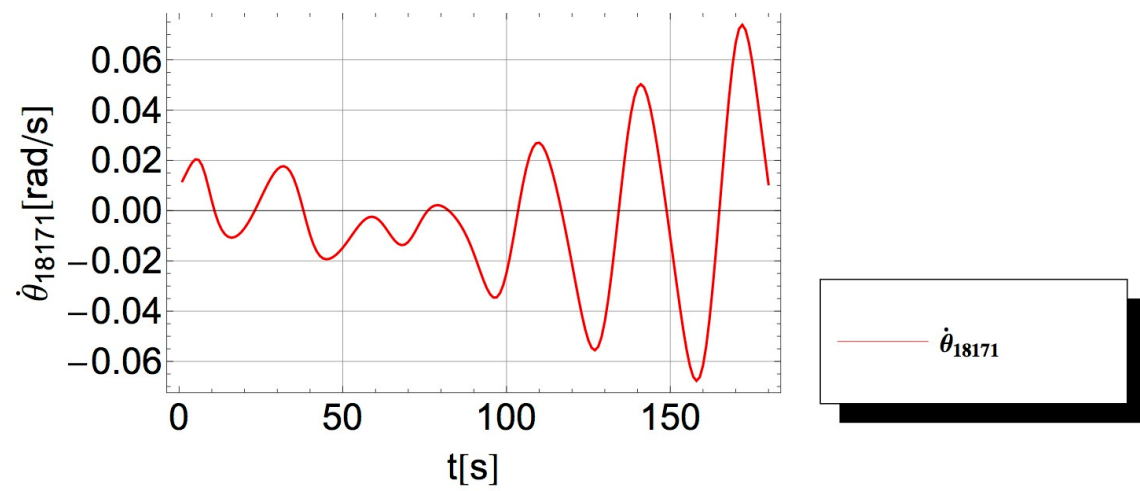




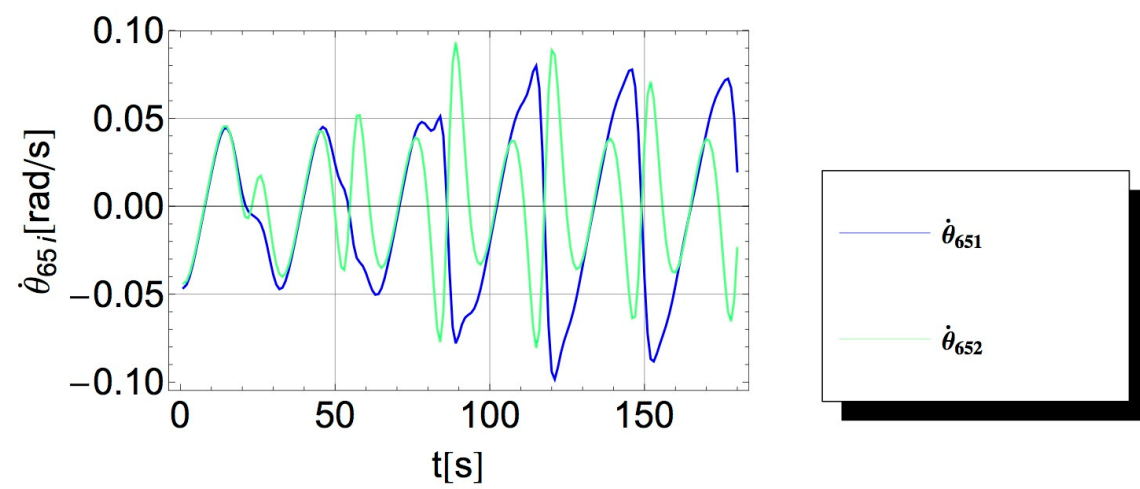
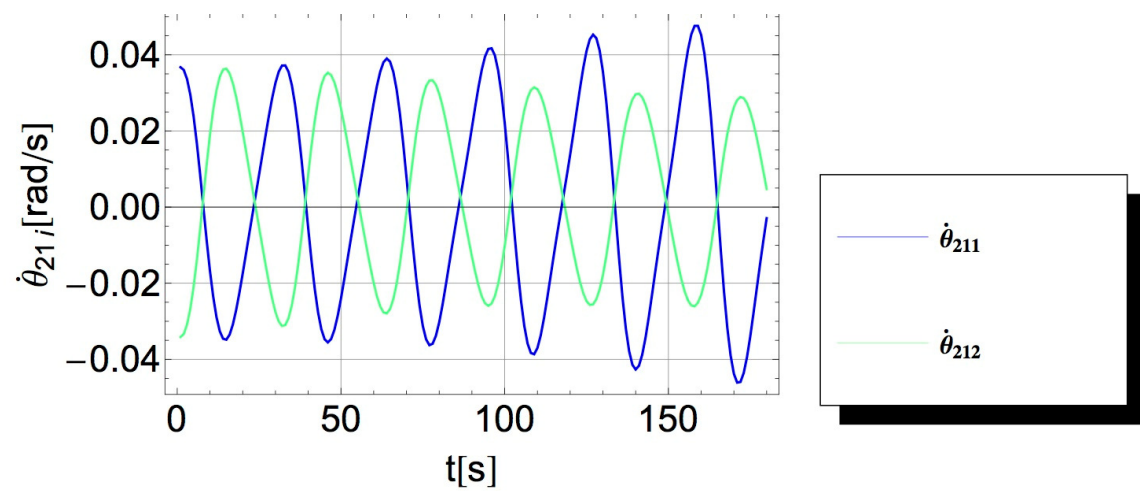
4.2 Gráficas de Velocidad

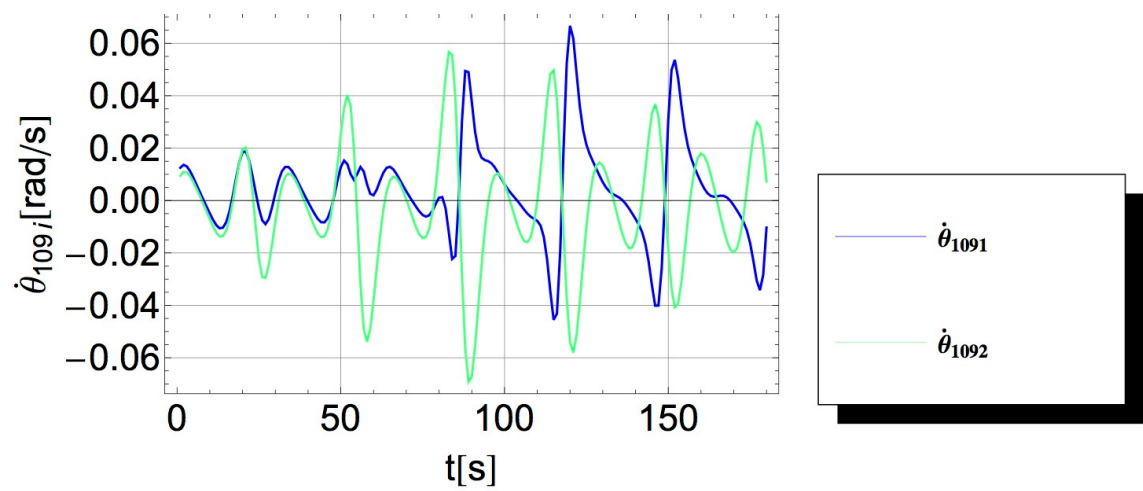
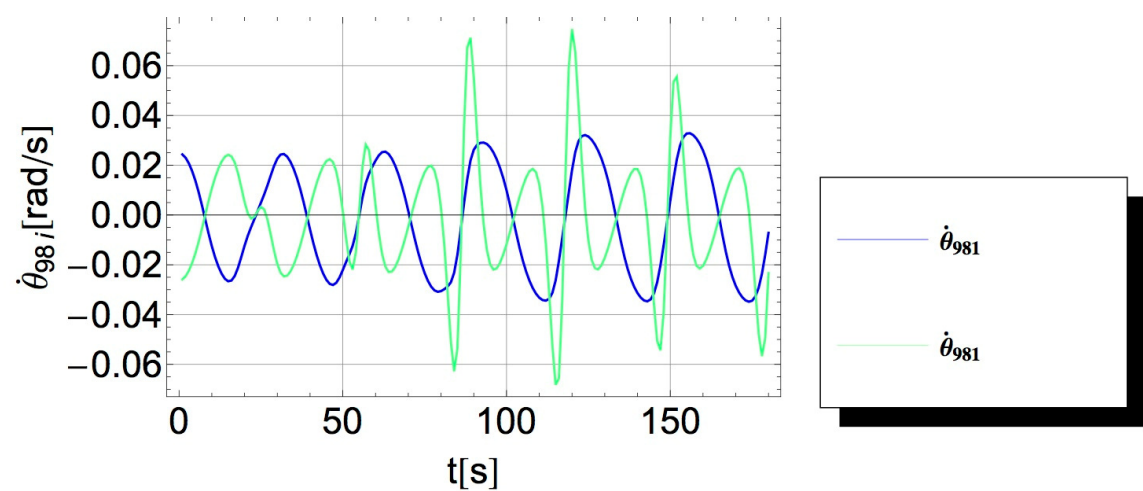
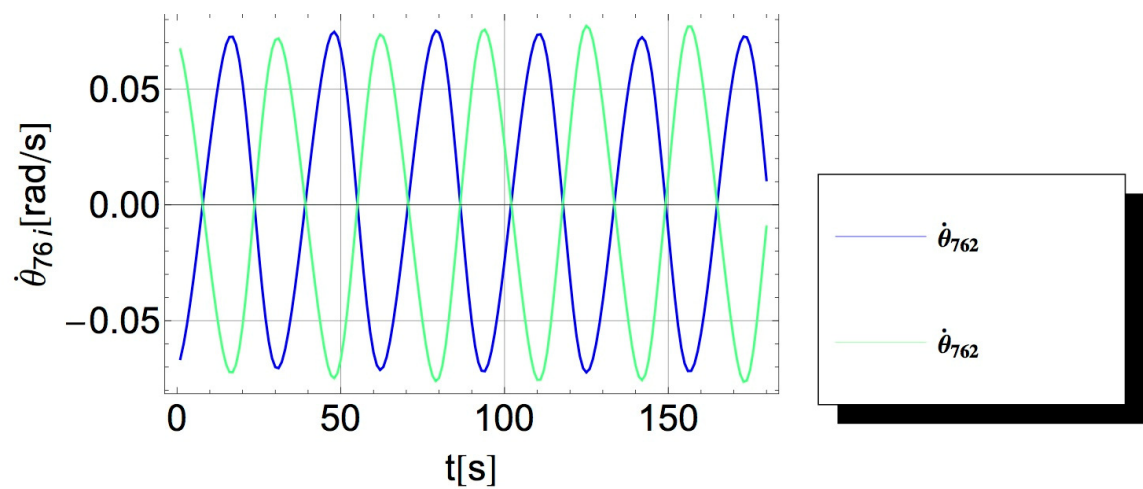


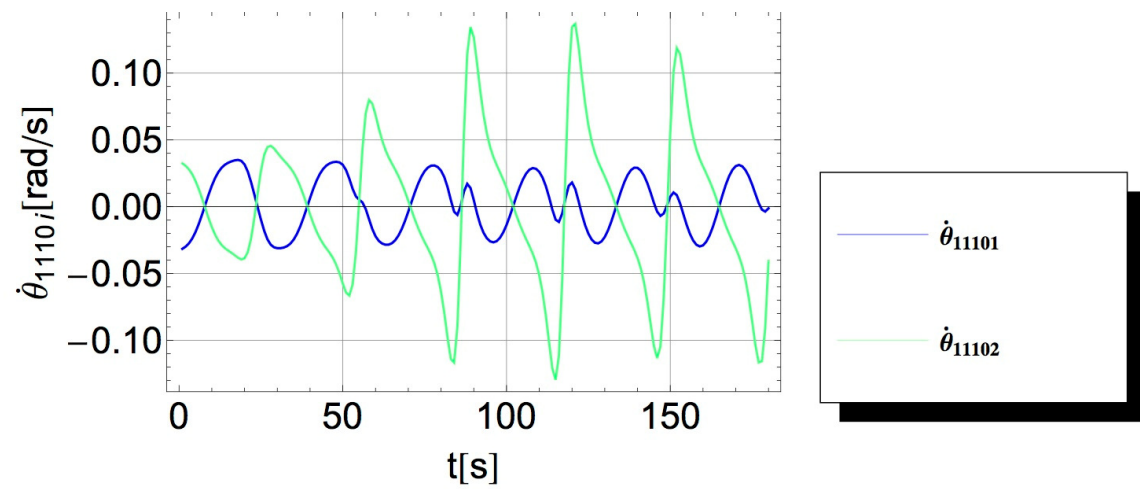




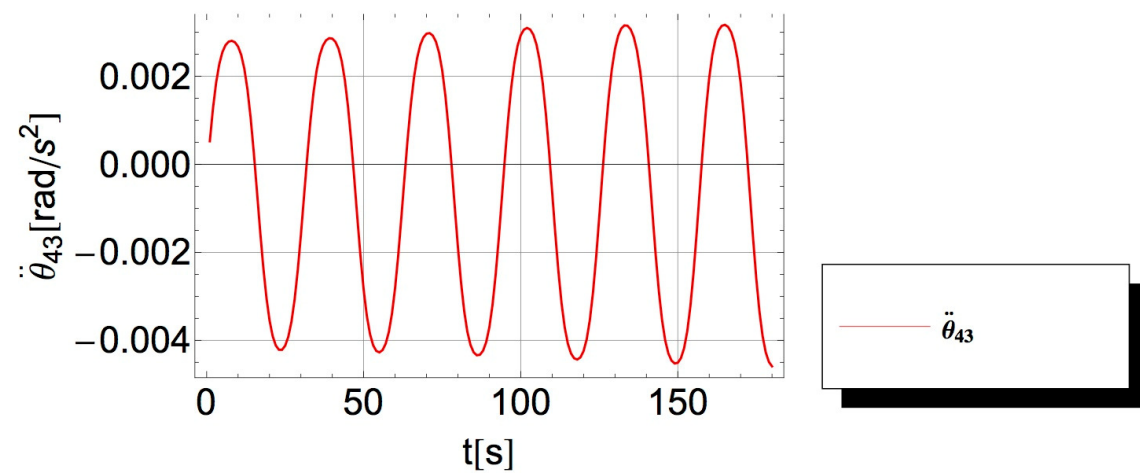
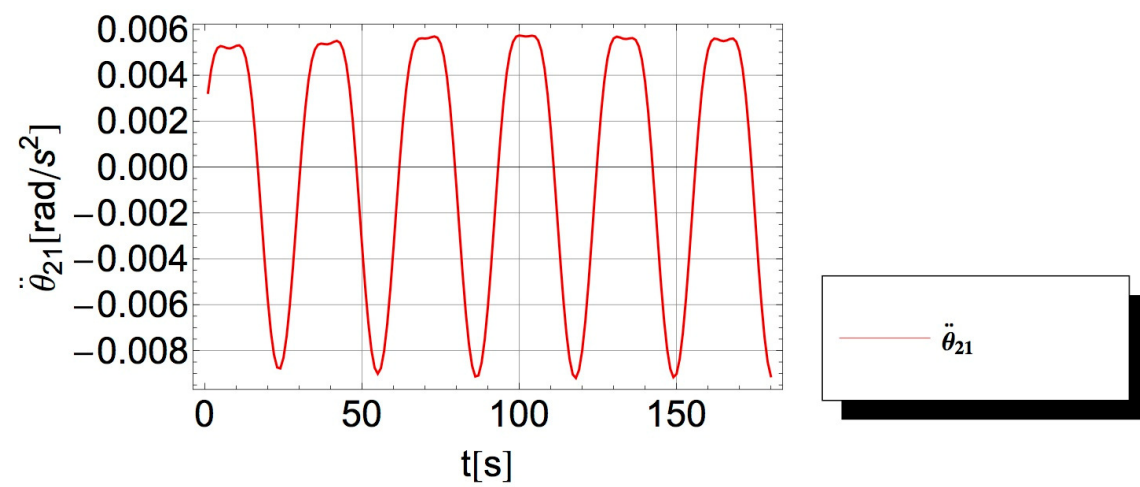
Cadena i

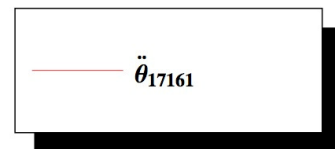
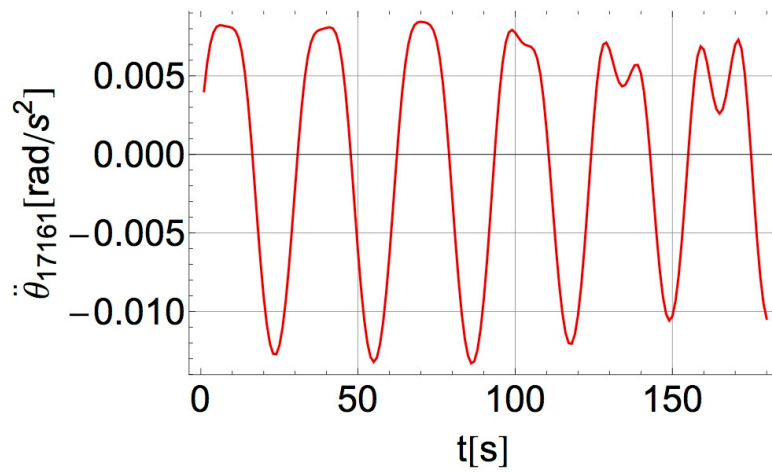
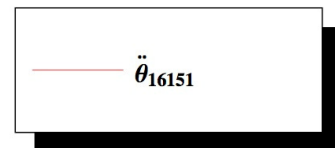
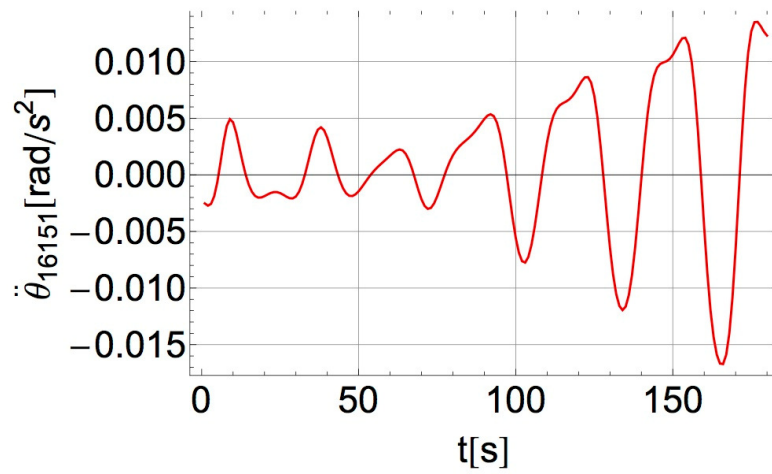
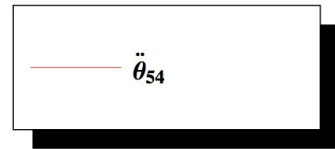
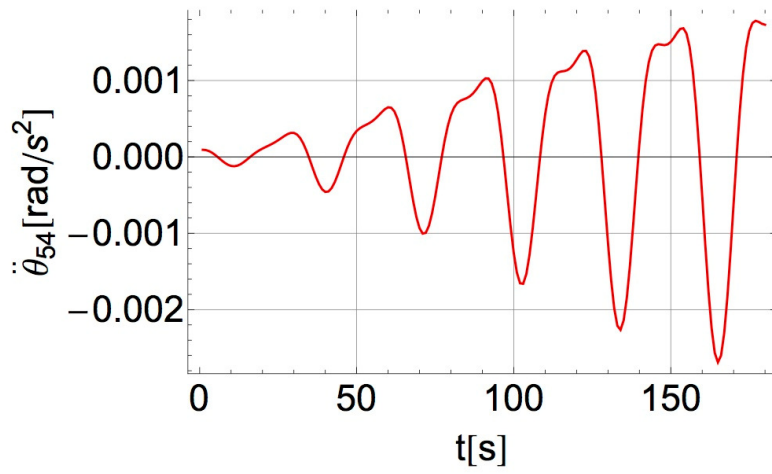


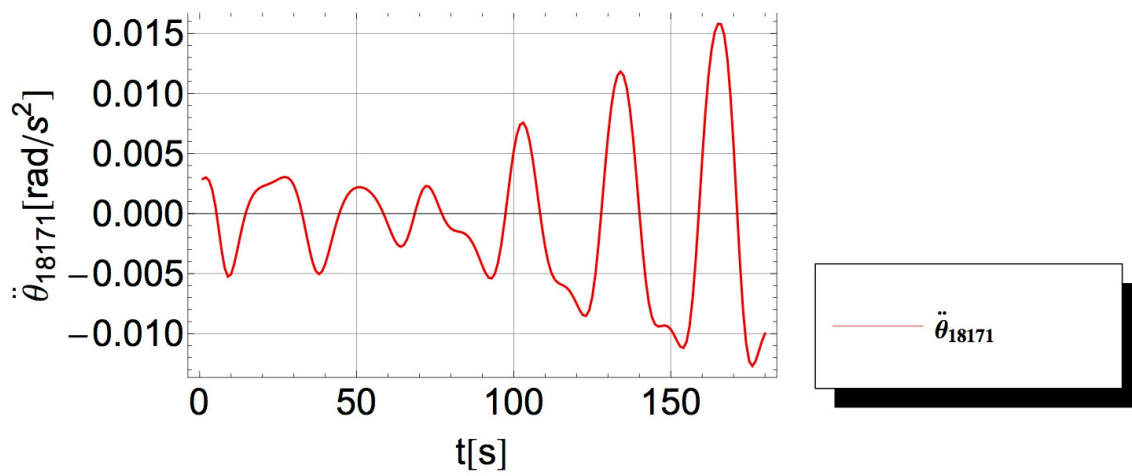




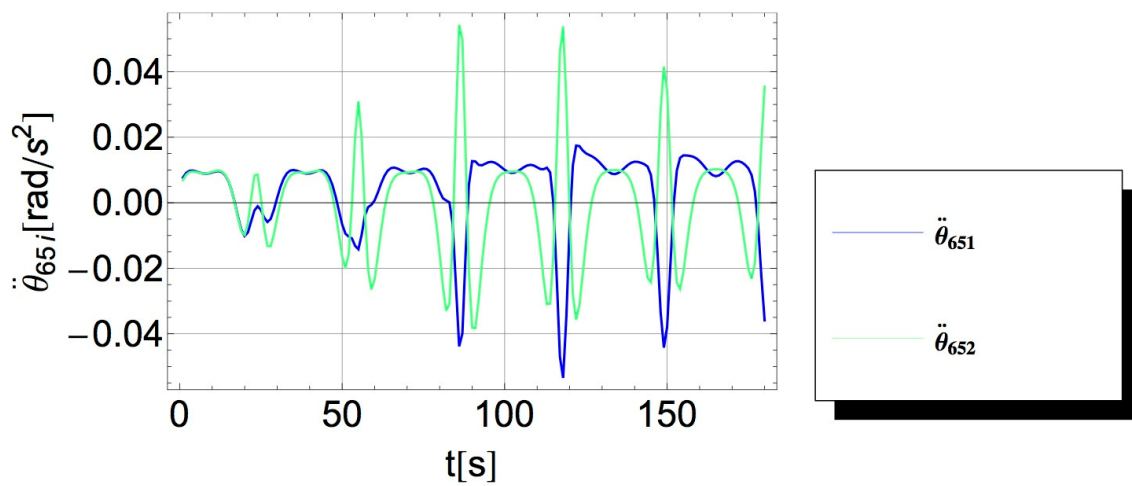
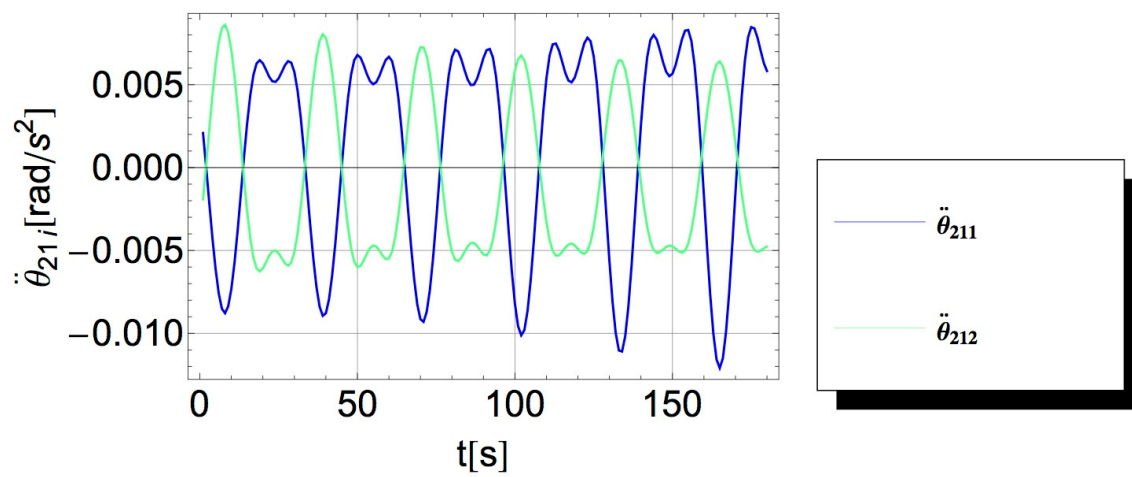
4.3 Gráficas de Aceleración

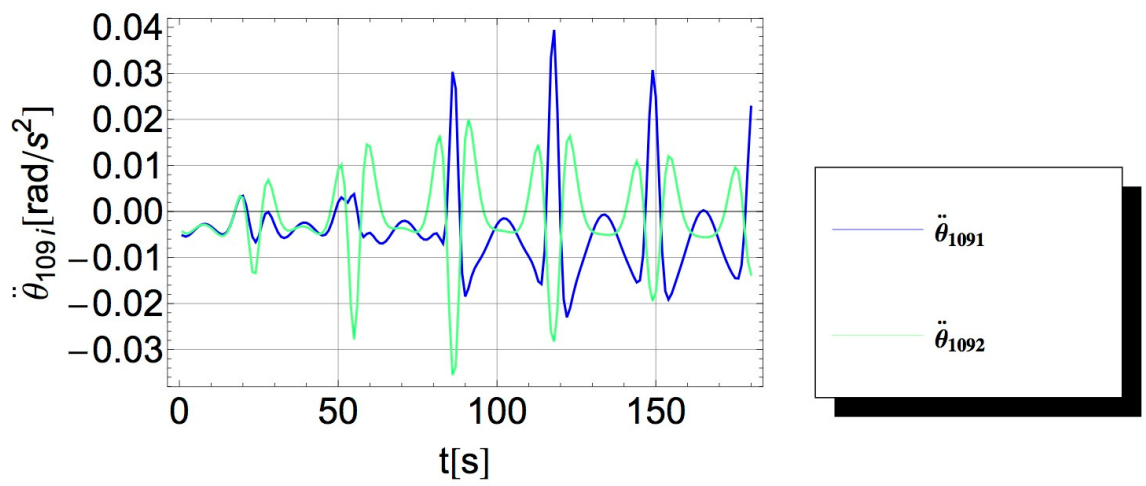
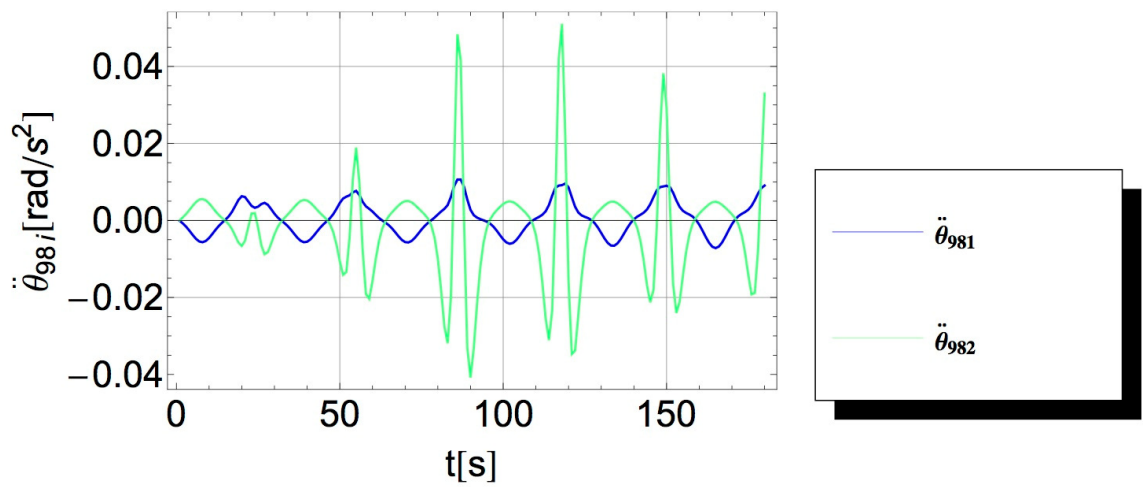
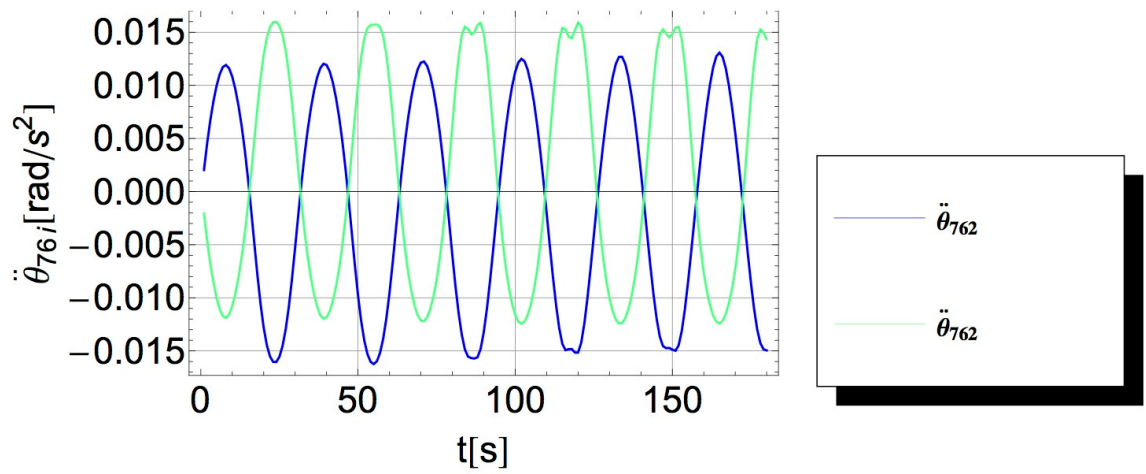


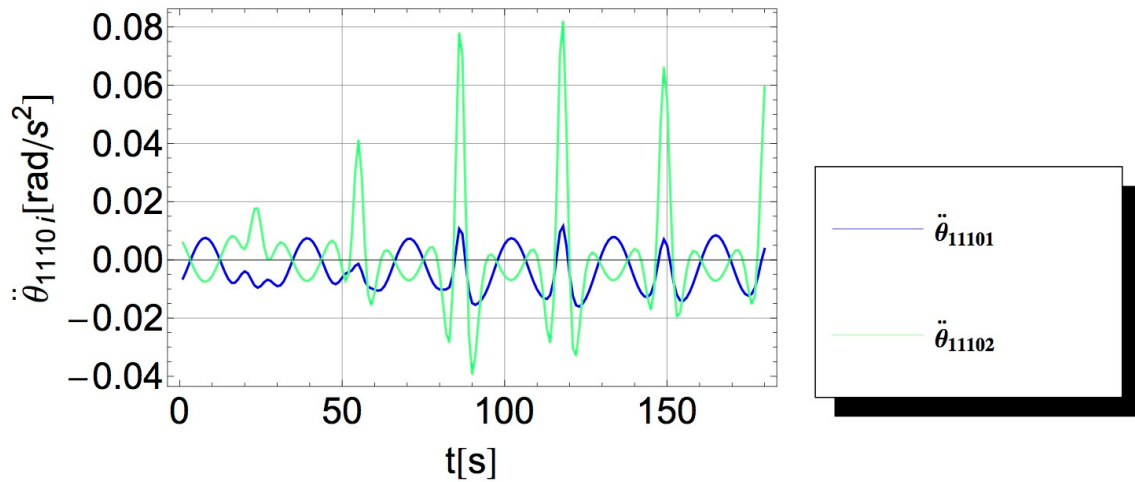




Cadena i







4.4 Solución del método Euler-Lagrange

Para la solución del método de Euler-Lagrange se evaluó la ecuación (3.154) con cada uno de los términos que la componen. Para la obtención de los datos de masa e inercia se usó un software de CAD y la simulación de un material. Dichos datos se presentan a continuación:

Cadena 0:

$$J_1^2 = \begin{bmatrix} 20.988 & 0 & 0 \\ 0 & 3027.487 & 0 \\ 0 & 0 & 3016.058 \end{bmatrix} kg * mm^2$$

$$J_2^5 = \begin{bmatrix} 199.423 & -5.176 & 1381.812 \\ -5.176 & 45670 & 0.314 \\ 1381.812 & 0.314 & 45670 \end{bmatrix} kg * mm^2$$

$$J_3^{16} = \begin{bmatrix} 1.681 & 0 & 0 \\ 0 & 95.822 & 0 \\ 0 & 0 & 95.822 \end{bmatrix} kg * mm^2$$

$$J_4^{17} = \begin{bmatrix} 1.681 & 0 & 0 \\ 0 & 95.822 & 0 \\ 0 & 0 & 95.822 \end{bmatrix} kg * mm^2$$

$$J_5^{18} = \begin{bmatrix} 1.681 & 0 & 0 \\ 0 & 95.822 & 0 \\ 0 & 0 & 95.822 \end{bmatrix} kg * mm^2$$

$$m_1 = 0.166 kg$$

$$m_2 = 0.188 kg$$

$$m_3 = 0.04 kg$$

$$m_4 = 0.035 kg$$

$$m_5 = 0.035 kg$$

Cadenas i:

$$J_1^{2i} = \begin{bmatrix} 20.988 & 0 & 0 \\ 0 & 3027.487 & 0 \\ 0 & 0 & 3016.058 \end{bmatrix} kg * mm^2$$

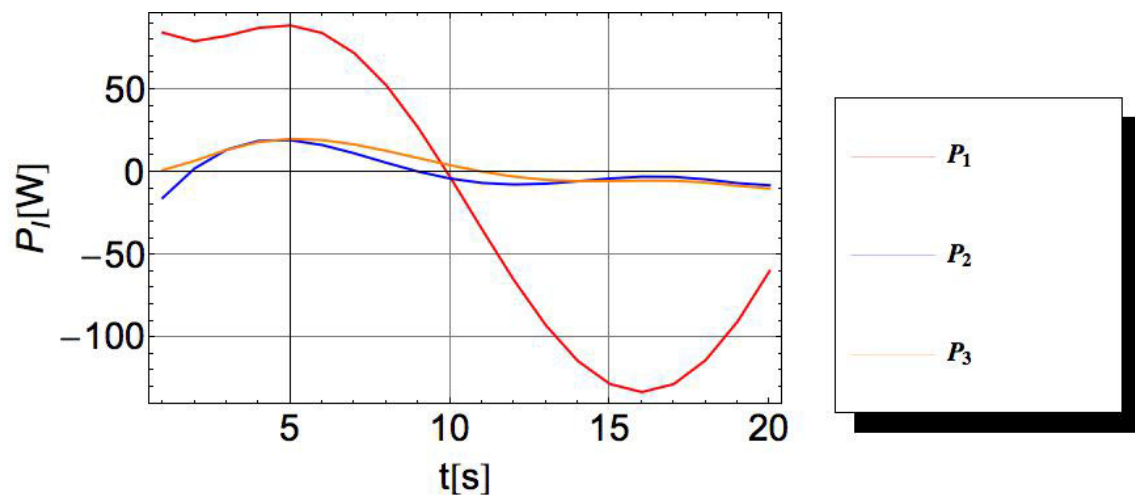
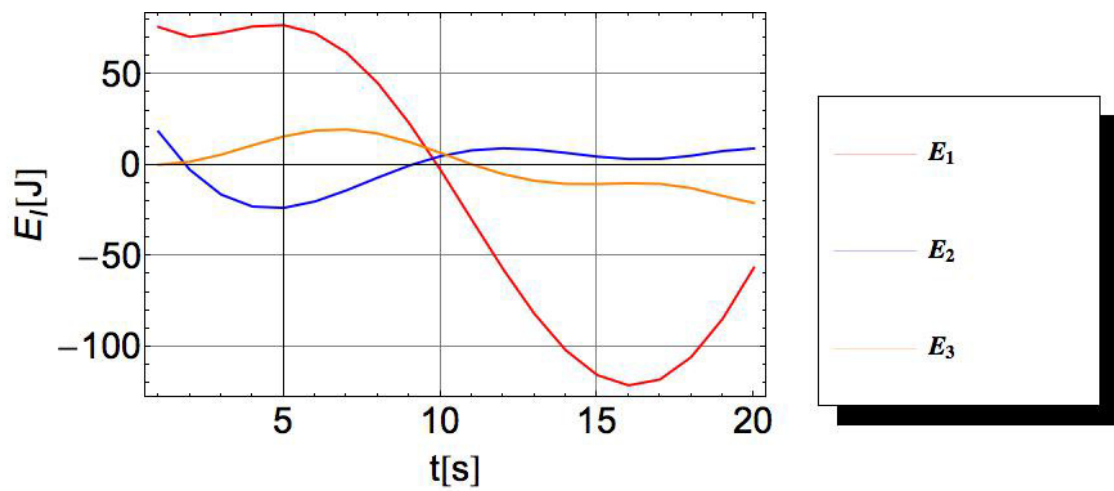
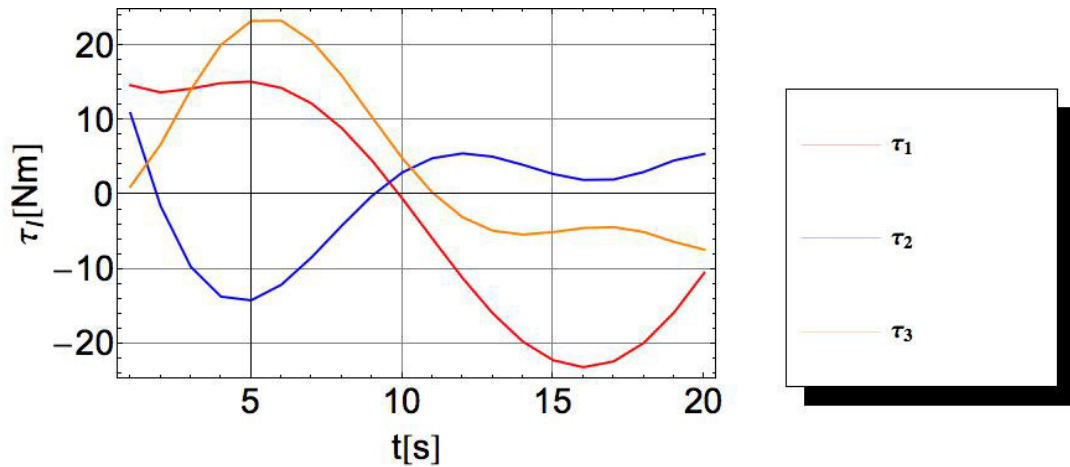
$$J_2^{7i} = \begin{bmatrix} 2.62 & 0 & 0 \\ 0 & 2322.307 & 0 \\ 0 & 0 & 2322.307 \end{bmatrix} kg * mm^2$$

$$m_{1i} = 0.166 kg$$

$$m_{2i} = 0.062 kg$$

Para i=1,2

Con base en los datos antes mencionados y en las ecuaciones obtenidas se obtuvieron las gráficas correspondientes al comportamiento dinámico de los actuadores presentes en la base del robot ya que estos son los que soportan la carga para mover el mismo. Las gráficas obtenidas corresponden al torque, trabajo y potencia para la trayectoria descrita en el apéndice A.



Capítulo 5

Conclusiones

Con el presente trabajo fue posible aplicar métodos para el análisis cinemático y dinámico de una configuración geométrica espacial. Dichos métodos permitieron la obtención de variables importantes para evaluar el funcionamiento del robot y de esta manera evaluar si es viable su construcción para realizar el propósito para el que fue diseñado.

En el caso de la cinemática se empleó el método de transformaciones homogéneas, el cual es muy versátil para analizar configuraciones geométricas compuestas por diferentes tipos de juntas. El método permite la descomposición de dichas juntas en traslaciones y rotaciones en el espacio lo cual es útil para el análisis de cada uno de los ángulos por separado. Sin embargo una de las desventajas del método es que las ecuaciones obtenidas fueron muy complejas lo cual derivó en dificultad para su manejo y en la gran capacidad de procesamiento necesaria para la realización de los cálculos.

Dado que el propósito original del robot fue el de pintar partes en una línea industrial, la trayectoria se eligió para simular el movimiento que tendría al realizar esta tarea. De esta manera fue posible observar el comportamiento detallado de posición, velocidad y aceleración de cada ángulo al trazar la trayectoria descrita. Con base en las gráficas cinemáticas obtenidas se observa que los resultados son coherentes con la trayectoria senoidal propuesta ya que se observa un comportamiento cíclico.

Para la dinámica el método empleado fue el de Euler-Lagrange el cual es un método que no contempla las fuerzas y momentos de reacción en cada uno de los eslabones de las cadenas cinemáticas sino solo las fuerzas requeridas para que el sistema realice un movimiento específico. Al observar las gráficas dinámicas se pueden apreciar los límites superiores e inferiores lo cual es útil para la selección de los actuadores. Para la aplicación del método fue necesario conocer los parámetros físicos involucrados en la dinámica como son las masas los momentos de inercia y las distancias a los centros de gravedad los cuales se obtuvieron con el uso de un software CAD.

Por lo anterior se puede concluir que el método de Euler-Lagrange se puede usar cuando se desea conocer las especificaciones de los actuadores y cuando no es crítico el conocimiento de las fuerzas involucradas en los elementos de unión como son los pernos y rodamientos. Debido a esto y a la selección de la trayectoria se puede decir que los resultados dinámicos obtenidos se pueden usar para la selección real de los actuadores. Así mismo el modelo dinámico obtenido se puede usar en un trabajo posterior para la formulación de un esquema de control.

Bibliografía

- [1] Angeles, Jorge. 1997. "Fundamentals of Robotic Mechanical Systems". Montreal Quebec. Springer, 1997.
- [2] Arai, T, et al. 1991. "Development of Parallel Link Manipulator for Underground Excavation Task". Proc. 1991 International Symposium on Advanced Robot Technology. 1991. pp. 541-548.
- [3] Barrientos A., Peñin L., Balaguer, C., Aracil, R., 2007. "Fundamentos de Robótica", McGraw-hill.
- [4] Cappel, K.L., "Motion simulator," US Patent No. 3,295,224, Enero 3, 1967.
- [5] Clavel, R. 1988. "A Fast Robot with Parallel Geometry". Proc. 18th International Symposium on Industrial Robots. 1988. pp. 91-100.
- [6] Clavel, R., "Device for the Movement and Positioning of an Element in Space," US Patent No. 4,976,582, Diciembre 11, 1990.
- [7] Gallardo, Jaime. 2016. "Fundamentals of Robotic Mechanical Systems". Celaya México Springer, 2016.
- [8] Geng Z. y Haynes L.S. Six-degree-of-freedom active vibration isolation using a Stewart platform mechanism. J. of Robotic Systems, 10(5):725-744, Julio 1993.
- [9] Gosselin C. M. y Hamel J. F., "The Agile Eye: a high-performance three-degree-of-freedom camera-orienting device," in Proceedings of the IEEE International Conference on Robotics and Automation, vol. 1, pp. 781-786, San Diego, Calif, USA, May 1994.
- [10] Gough, V.E. and Whitehall, S.G., "Universal tyre test machine," Proceedings of the FISITA Ninth International Technical Congress, pp. 117-137, May, 1962.
- [11] Gwinnett, J.E., "Amusement devices", US Patent No. 1,789,680, Enero 20 1931.
- [12] Kane, T. R. and Levinson, D. A., 1985, "Dynamics: Theory and Application", McGraw-Hill, New York.
- [13] Kim, H. S., & Tsai, L.-W. 2003. "Design optimization of a Cartesian parallel manipulator". ASME Journal of Mechanical Design, 125(1), 43-51.

[14] Merlet, Jean Pierre. 2000. "Parallel Robots". s.l. : Kluwer Academic Publishers, 2000.

[15] Nevatia, Ramakant & Dunn, Lewis A & Harris, William R. (William Robert), 1941- & Blum, Ronald Evan & Dudzinsky, J. S et al. (1982). "Machine perception". Prentice-Hall, Englewood Cliffs, N.J.

[16] Nguyen C.C. et al. "Adaptive control of a Stewart platformbased manipulator". J. of Robotic Systems, 10(5):657–687, July 1993.

[17] Pierrot, F., Fournier, A. and Dauchez, P. 1991. "Toward a Fully Parallel 6-DOF Robot for High-Speed Applications". Proc. 1991 IEEE International Conference on Robotics and Automation. 1991. pp. 1288-1293.

[18] Pollard, W.L.G., "Spray painting machine", US Patent No. 2,213,108, Agosto 26 1940.

[19] Pollard, W.L.V., "Position controlling apparatus", US Patent No. 2,286,571, Junio 16 1942.

[20] Roselund, H.A., "Means for moving spray guns or other devices through predetermined paths", US Patent No. 2,344,108, Marzo 14 1944.

[21] Spong, Mark W. and Vidyasagar, M. 1989. "Robot Dynamics and Control". S.l. : John Wiley & Sons, 1989. pp. 129-133.

[22] Stejskal, Vladimir and Valásek, Michael. 1996, "Kinematics and Dynamics of Machinery". s.l. : Marcel Dekker, Inc., 1996.

[23] Stewart, D. 1965. "A platform with 6 degrees of freedom". Proc. Institution of Mechanical Engineers. 1965. Vol. 180, pp. 371-386.

[24] Tsai, Lun Wen.. 1999. "Robot Analysis" The mechanics of serial and parallel manipulators. s.l. : John Wiley & Sons, 1999.

[25] Yangmin, L. & Qingsong, X. "Dynamic Analysis of a Modified Delta Parallel Robot for Cardiopulmonary Resuscitation", Macau China, 2005.

[26] Yun, Y. & Li, Y. "Design and analysis of a novel 6-DOF redundant actuated parallel robot with compliant hinges for high precision positioning", 2010.

Apéndice A.

Desarrollo de la ecuación de lazo.

Con el fin de obtener el término $\frac{\partial \theta}{\partial q_j}$ se procederá a generar una ecuación de lazo que nos permitirá relacionar el vector de derivadas parciales respecto a q_j de los ángulos $\theta_{21}, \theta_{43}, \theta_{54}$ con la derivada parcial respecto a q_j del vector de coordenadas generalizadas, esto es:

$$J_{\theta i} \frac{\partial \theta}{\partial q_j} = J_{qi} \frac{\partial q}{\partial q_j}$$

A partir de la figura 2-6 se tiene la ecuación de lazo:

$$\mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0 = \mathbf{r}_p^0 - \mathbf{r}_{1918}^0 \quad (\text{A.1})$$

Dónde:

$$\begin{aligned} \mathbf{r}_{10}^0 &= z_{10} \mathbf{k}_0 \\ \mathbf{r}_{32}^0 &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 \\ \mathbf{r}_{14,5i}^0 &= \mathbf{R}_5^0 \mathbf{r}_{14,5i}^5 \\ \mathbf{r}_{15,14i}^0 &= \mathbf{R}_{14}^0 \mathbf{r}_{15,14i}^{14} \\ \mathbf{r}_p^0 &= [x, y, z]^T \\ \mathbf{r}_{1918}^0 &= \mathbf{R}_p^0 \mathbf{r}_{1918}^p \end{aligned}$$

$$\begin{aligned} \mathbf{R}_2^0 &= \mathbf{R}_y(\theta_{21}) \\ \mathbf{R}_5^0 &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \\ \mathbf{R}_{14}^0 &= \mathbf{R}_5^0 \\ \mathbf{R}_p^0 &= \mathbf{R}_x(\theta) \mathbf{R}_y(\psi) \mathbf{R}_z(\phi) \end{aligned}$$

$$\begin{aligned} \mathbf{r}_{32}^2 &= [x_{32}, 0, 0]^T \\ \mathbf{r}_{14,5i}^5 &= [x_{14,5i}, 0, 0]^T \\ \mathbf{r}_{15,14i}^{14} &= [x_{15,14i}, 0, 0]^T \\ \mathbf{r}_{1918}^p &= [-x_{1918}, 0, 0]^T \end{aligned}$$

Derivando la ecuación (A.1) con respecto a q_j :

$$\begin{aligned} \frac{\partial}{\partial q_j} (\mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 + \mathbf{r}_{15,14i}^0) &= \frac{\partial}{\partial q_j} (\mathbf{r}_p^0 - \mathbf{r}_{1918}^0) \\ \frac{\partial \mathbf{r}_{10}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{32}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} &= \frac{\partial \mathbf{r}_p^0}{\partial q_j} - \frac{\partial \mathbf{r}_{1918}^0}{\partial q_j} \end{aligned} \quad (\text{A.2})$$

Obteniendo cada uno de los componentes de la ec. anterior, primero escribimos las parciales de las matrices previamente obtenidas:

$$\begin{aligned}
\frac{\partial \mathbf{R}_2^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} = \mathbf{B}_y(\theta_{21}) \frac{\partial \theta_{21}}{\partial q_j} \\
\frac{\partial \mathbf{R}_5^0}{\partial q_j} &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{14}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_5^0}{\partial q_j} \\
\frac{\partial \mathbf{R}_p^0}{\partial q_j} &= \frac{\partial \mathbf{R}_x(\theta)}{\partial q_j} \mathbf{R}_y(\psi) \mathbf{R}_z(\phi) + \mathbf{R}_x(\theta) \frac{\partial \mathbf{R}_y(\psi)}{\partial q_j} \mathbf{R}_y(\psi) \mathbf{R}_z(\phi) + \mathbf{R}_x(\theta) \mathbf{R}_y(\psi) \frac{\partial \mathbf{R}_z(\phi)}{\partial q_j} \\
&= \mathbf{B}_x(\theta) \mathbf{R}_y(\psi) \mathbf{R}_z(\phi) \frac{\partial \theta}{\partial q_j} + \mathbf{R}_x(\theta) \mathbf{B}_y(\psi) \mathbf{R}_z(\phi) \frac{\partial \psi}{\partial q_j} + \mathbf{R}_x(\theta) \mathbf{R}_y(\psi) \mathbf{B}_z(\phi) \frac{\partial \phi}{\partial q_j} \\
\frac{\partial \mathbf{r}_{10}^0}{\partial q_j} &= 0 \\
\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 = (\mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) \frac{\partial \theta_{21}}{\partial q_j} \\
\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} &= (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{21}}{\partial q_j} + (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{r}_{15,14i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{14}^0}{\partial q_j} \mathbf{r}_{15,14i}^{14} \\
&= (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{r}_p^0}{\partial q_j} &= \frac{\partial \mathbf{r}_p^0}{\partial x} \frac{\partial x}{\partial q_j} + \frac{\partial \mathbf{r}_p^0}{\partial y} \frac{\partial y}{\partial q_j} + \frac{\partial \mathbf{r}_p^0}{\partial z} \frac{\partial z}{\partial q_j} \\
\frac{\partial \mathbf{r}_{1918}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_p^0}{\partial q_j} \mathbf{r}_{1918}^p
\end{aligned}$$

$$\begin{aligned}
&= (\mathbf{B}_x(\theta)\mathbf{R}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \theta}{\partial q_j} + (\mathbf{R}_x(\theta)\mathbf{B}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \psi}{\partial q_j} \\
&\quad + (\mathbf{R}_x(\theta)\mathbf{R}_y(\psi)\mathbf{B}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \phi}{\partial q_j}
\end{aligned}$$

Sustituyendo los términos anteriores en la ec. (A.1):

$$\begin{aligned}
&(\mathbf{B}_y(\theta_{21})\mathbf{r}_{32}^2) \frac{\partial \theta_{21}}{\partial q_j} + (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{14,5i}^5) \frac{\partial \theta_{54}}{\partial q_j} \\
&\quad + (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{54}}{\partial q_j} \\
&= \frac{\partial \mathbf{r}_p^0}{\partial x} \frac{\partial x}{\partial q_j} + \frac{\partial \mathbf{r}_p^0}{\partial y} \frac{\partial y}{\partial q_j} + \frac{\partial \mathbf{r}_p^0}{\partial z} \frac{\partial z}{\partial q_j} - (\mathbf{B}_x(\theta)\mathbf{R}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \theta}{\partial q_j} \\
&\quad - (\mathbf{R}_x(\theta)\mathbf{B}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \psi}{\partial q_j} - (\mathbf{R}_x(\theta)\mathbf{R}_y(\psi)\mathbf{B}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \phi}{\partial q_j}
\end{aligned}$$

Simplificando y agrupando los términos semejantes:

$$\begin{aligned}
&(\mathbf{B}_y(\theta_{21})\mathbf{r}_{32}^2 + \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{14,5i}^5 + \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{15,14i}^{14}) \frac{\partial \theta_{54}}{\partial q_j} \\
&= \frac{\partial \mathbf{r}_p^0}{\partial q_j} - (\mathbf{B}_x(\theta)\mathbf{R}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \theta}{\partial q_j} - (\mathbf{R}_x(\theta)\mathbf{B}_y(\psi)\mathbf{R}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \psi}{\partial q_j} \\
&\quad - (\mathbf{R}_x(\theta)\mathbf{R}_y(\psi)\mathbf{B}_z(\phi)\mathbf{r}_{1918}^p) \frac{\partial \phi}{\partial q_j}
\end{aligned}$$

Renombrando y acomodando matricialmente:

$$\mathbf{r}_{Ai} \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{r}_{Bi} \frac{\partial \theta_{43}}{\partial q_j} + \mathbf{r}_{Ci} \frac{\partial \theta_{54}}{\partial q_j} = \mathbf{I}_{3 \times 3} \frac{\partial \mathbf{r}_p^0}{\partial q_j} + \mathbf{r}_{Di} \frac{\partial \theta}{\partial q_j} + \mathbf{r}_{Ei} \frac{\partial \psi}{\partial q_j} + \mathbf{r}_{Fi} \frac{\partial \phi}{\partial q_j}$$

$$\begin{bmatrix} \mathbf{r}_{Ai} & \mathbf{r}_{Bi} & \mathbf{r}_{Ci} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_{21}}{\partial q_j} \\ \frac{\partial \theta_{43}}{\partial q_j} \\ \frac{\partial \theta_{54}}{\partial q_j} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{r}_{Di} & \mathbf{r}_{Ei} & \mathbf{r}_{Fi} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{r}_p^0}{\partial q_j} \\ \frac{\partial \theta}{\partial q_j} \\ \frac{\partial \psi}{\partial q_j} \\ \frac{\partial \phi}{\partial q_j} \end{bmatrix}$$

Finalmente se tiene:

$$J_{\theta i} \frac{\partial \theta}{\partial q_j} = J_{qi} \frac{\partial q}{\partial q_j} \quad (\text{A.3})$$

Donde:

$$\begin{aligned}
 \mathbf{r}_{Ai} &= \mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2 + \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
 \mathbf{r}_{Bi} &= \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
 \mathbf{r}_{Ci} &= \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5 + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{15,14i}^{14} \\
 \mathbf{r}_{Di} &= -(\mathbf{B}_x(\theta) \mathbf{R}_y(\psi) \mathbf{R}_z(\phi) \mathbf{r}_{1918}^p) \\
 \mathbf{r}_{Ei} &= -(\mathbf{R}_x(\theta) \mathbf{B}_y(\psi) \mathbf{R}_z(\phi) \mathbf{r}_{1918}^p) \\
 \mathbf{r}_{Fi} &= -(\mathbf{R}_x(\theta) \mathbf{R}_y(\psi) \mathbf{B}_z(\phi) \mathbf{r}_{1918}^p)
 \end{aligned}$$

Ahora se genera una ecuación de lazo que nos permitirá relacionar el vector de derivadas parciales respecto a q_j de los ángulos $\theta_{21i}, \theta_{65i}, \theta_{76i}$ con la derivada parcial respecto a q_j .

Con base en la figura 2-12 se tiene la ecuación de lazo siguiente:

$$\mathbf{r}_{10i}^0 + \mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{87i}^0 = \mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 - \mathbf{r}_{13,12i}^0 - \mathbf{r}_{12,11i}^0 \quad (\text{A.4})$$

Dónde:

$$\begin{aligned}
 \mathbf{r}_{10i}^0 &= \text{signo } y_{10} \mathbf{j}_0 \\
 \mathbf{r}_{32i}^0 &= \mathbf{R}_{2i}^0 \mathbf{r}_{32i}^{2i} \\
 \mathbf{r}_{43i}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{43i}^{3i} \\
 \mathbf{r}_{87i}^0 &= \mathbf{R}_{7i}^0 \mathbf{r}_{87i}^{7i} \\
 \mathbf{r}_{10}^0 &= z_{10} \mathbf{k}_0 \\
 \mathbf{r}_{32}^0 &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 \\
 \mathbf{r}_{14,5i}^0 &= \mathbf{R}_5^0 \mathbf{r}_{14,5i}^5 \\
 \mathbf{r}_{13,12i}^0 &= \mathbf{R}_{12i}^0 \mathbf{r}_{13,12i}^{12i} \\
 \mathbf{r}_{12,11i}^0 &= \mathbf{R}_{11i}^0 \mathbf{r}_{12,11i}^{11i}
 \end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{2i}^0 &= \mathbf{R}_z(\theta_{21i}) \\
\mathbf{R}_{3i}^0 &= \mathbf{R}_{2i}^0 \\
\mathbf{R}_{7i}^0 &= \mathbf{R}_z(\theta_{21i})\mathbf{R}_x(\beta_{54i})\mathbf{R}_y(\theta_{65i})\mathbf{R}_z(\theta_{76i}) \\
\mathbf{R}_2^0 &= \mathbf{R}_y(\theta_{21}) \\
\mathbf{R}_5^0 &= \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54}) \\
\mathbf{R}_{11i}^0 &= \mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i}) \\
\mathbf{R}_{12i}^0 &= \mathbf{R}_{11i}^0
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_{32i}^{2i} &= [-z_{32i}, 0, 0]^T \\
\mathbf{r}_{43i}^{3i} &= [x_{43i}, 0, 0]^T \\
\mathbf{r}_{87i}^{7i} &= [x_{87i}, 0, 0]^T \\
\mathbf{r}_{32}^2 &= [x_{32}, 0, 0]^T \\
\mathbf{r}_{14,5i}^5 &= [x_{14,5i}, 0, 0]^T \\
\mathbf{r}_{12,11i}^0 &= [-x_{12,11i}, 0, 0]^T \\
\mathbf{r}_{13,12i}^0 &= [-z_{13,12i}, 0, 0]^T
\end{aligned}$$

Derivando la ecuación (A.4) con respecto a q_j

$$\begin{aligned}
\frac{\partial}{\partial q_j}(\mathbf{r}_{10i}^0 + \mathbf{r}_{32i}^0 + \mathbf{r}_{43i}^0 + \mathbf{r}_{87i}^0) &= \frac{\partial}{\partial q_j}(\mathbf{r}_{10}^0 + \mathbf{r}_{32}^0 + \mathbf{r}_{14,5i}^0 - \mathbf{r}_{13,12i}^0 - \mathbf{r}_{12,11i}^0) \\
\frac{\partial}{\partial q_j}\mathbf{r}_{10i}^0 + \frac{\partial}{\partial q_j}\mathbf{r}_{32i}^0 + \frac{\partial}{\partial q_j}\mathbf{r}_{43i}^0 + \frac{\partial}{\partial q_j}\mathbf{r}_{87i}^0 &= \frac{\partial}{\partial q_j}\mathbf{r}_{10}^0 + \frac{\partial}{\partial q_j}\mathbf{r}_{32}^0 + \frac{\partial}{\partial q_j}\mathbf{r}_{14,5i}^0 - \frac{\partial}{\partial q_j}\mathbf{r}_{13,12i}^0 - \frac{\partial}{\partial q_j}\mathbf{r}_{12,11i}^0 \quad (A.5)
\end{aligned}$$

Obteniendo cada uno de los componentes de la ec. anterior, primero escribimos las parciales de las matrices previamente obtenidas:

$$\begin{aligned}
\frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} &= \mathbf{B}_z(\theta_{21i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \\
\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_z(\theta_{21i})}{\partial q_j} \mathbf{R}_x(\beta_{54i})\mathbf{R}_y(\theta_{65i})\mathbf{R}_z(\theta_{76i}) + \mathbf{R}_z(\theta_{21i}) \frac{\partial \mathbf{R}_x(\beta_{54i})}{\partial q_j} \mathbf{R}_y(\theta_{65i})\mathbf{R}_z(\theta_{76i}) \\
&\quad + \mathbf{R}_z(\theta_{21i})\mathbf{R}_x(\beta_{54i}) \frac{\partial \mathbf{R}_y(\theta_{65i})}{\partial q_j} \mathbf{R}_z(\theta_{76i}) \\
&\quad + \mathbf{R}_z(\theta_{21i})\mathbf{R}_x(\beta_{54i})\mathbf{R}_y(\theta_{65i}) \frac{\partial \mathbf{R}_z(\theta_{76i})}{\partial q_j}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathbf{R}_2^0}{\partial q_j} &= \frac{\partial \mathbf{R}_y(\theta_{21})}{\partial q_j} = \mathbf{B}_y(\theta_{21}) \frac{\partial \theta_{21}}{\partial q_j} \\
\frac{\partial \mathbf{R}_5^0}{\partial q_j} &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} &= \mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \frac{\partial \theta_{21}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \frac{\partial \theta_{43}}{\partial q_j} \\
&\quad + \mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \frac{\partial \theta_{54}}{\partial q_j} \\
\frac{\partial \mathbf{R}_{12i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} \\
\frac{\partial \mathbf{r}_{10i}^0}{\partial q_j} &= 0 \\
\frac{\partial \mathbf{r}_{32i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{2i}^0}{\partial q_j} \mathbf{r}_{32i}^{2i} \\
\frac{\partial \mathbf{r}_{43i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{r}_{43i}^{3i} \\
\frac{\partial \mathbf{r}_{87i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{r}_{87i}^{7i} \\
\frac{\partial \mathbf{r}_{10}^0}{\partial q_j} &= 0 \\
\frac{\partial \mathbf{r}_{32}^0}{\partial q_j} &= \mathbf{R}_2^0 \mathbf{r}_{32}^2 = (\mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) \frac{\partial \theta_{21}}{\partial q_j} \\
\frac{\partial \mathbf{r}_{14,5i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_5^0}{\partial q_j} \mathbf{r}_{14,5i}^5 \\
\frac{\partial \mathbf{r}_{12,11i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} \mathbf{r}_{12,11i}^5 \\
\frac{\partial \mathbf{r}_{13,12i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{12i}^0}{\partial q_j} \mathbf{r}_{13,12i}^5
\end{aligned}$$

Sustituyendo los términos anteriores en la ec. (A.4):

$$\begin{aligned}
& (\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i}) \frac{\partial \theta_{21i}}{\partial q_j} + (\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{43i}^{3i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
& + (\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \frac{\partial \theta_{21i}}{\partial q_j} \\
& + (\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
& + (\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \frac{\partial \theta_{76i}}{\partial q_j} \\
& = (\mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) \frac{\partial \theta_{21}}{\partial q_j} + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{21}}{\partial q_j} \\
& + (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{43}}{\partial q_j} \\
& + (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \frac{\partial \theta_{54}}{\partial q_j} \\
& - (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \frac{\partial \theta_{21}}{\partial q_j} \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \frac{\partial \theta_{43}}{\partial q_j} \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \frac{\partial \theta_{54}}{\partial q_j} \\
& - (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \frac{\partial \theta_{21}}{\partial q_j} \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \frac{\partial \theta_{43}}{\partial q_j} \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \frac{\partial \theta_{54}}{\partial q_j}
\end{aligned}$$

Simplificando y agrupando los términos semejantes:

$$\begin{aligned}
& \left((\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i}) + (\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{43i}^{3i}) + (\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \right) \frac{\partial \theta_{21i}}{\partial q_j} \\
& + (\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \frac{\partial \theta_{65i}}{\partial q_j} \\
& + (\mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{B}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \frac{\partial \theta_{76i}}{\partial q_j} \\
& = \left((\mathbf{B}_y(\theta_{21}) \mathbf{r}_{32}^2) + (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
& - (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \\
& - (\mathbf{B}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \left. \right) \frac{\partial \theta_{21}}{\partial q_j} \\
& + \left((\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{B}_y(\theta_{43}) \mathbf{R}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \left. \right) \frac{\partial \theta_{43}}{\partial q_j} \\
& + \left((\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{r}_{14,5i}^5) \right. \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{13,12i}^5) \\
& - (\mathbf{R}_y(\theta_{21}) \mathbf{R}_y(\theta_{43}) \mathbf{B}_z(\theta_{54}) \mathbf{R}_z(\beta_{14,13i}) \mathbf{r}_{12,11i}^5) \left. \right) \frac{\partial \theta_{54}}{\partial q_j}
\end{aligned}$$

Renombrando y acomodando matricialmente:

$$\begin{aligned}
& \mathbf{r}_{Gi} \frac{\partial \theta_{21i}}{\partial q_j} + \mathbf{r}_{Hi} \frac{\partial \theta_{65i}}{\partial q_j} + \mathbf{r}_{Li} \frac{\partial \theta_{76i}}{\partial q_j} = \mathbf{r}_{Ji} \frac{\partial \theta_{21}}{\partial q_j} + \mathbf{r}_{Ki} \frac{\partial \theta_{43}}{\partial q_j} + \mathbf{r}_{Li} \frac{\partial \theta_{54}}{\partial q_j} \\
& \begin{bmatrix} \mathbf{r}_{Gi} & \mathbf{r}_{Hi} & \mathbf{r}_{Li} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_{21i}}{\partial q_j} \\ \frac{\partial \theta_{65i}}{\partial q_j} \\ \frac{\partial \theta_{76i}}{\partial q_j} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{Ji} & \mathbf{r}_{Ki} & \mathbf{r}_{Li} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta_{21}}{\partial q_j} \\ \frac{\partial \theta_{43}}{\partial q_j} \\ \frac{\partial \theta_{54}}{\partial q_j} \end{bmatrix}
\end{aligned}$$

Finalmente se tiene:

$$\mathbf{J}_{\beta i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} = \mathbf{J}_{ki} \frac{\partial \boldsymbol{\theta}}{\partial q_j} \quad (\text{A.6})$$

Donde:

$$\begin{aligned}
\mathbf{r}_{Gi} &= (\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{32i}^{2i}) + (\mathbf{B}_z(\theta_{21i}) \mathbf{r}_{43i}^{3i}) + (\mathbf{B}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{R}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}) \\
\mathbf{r}_{Hi} &= \mathbf{R}_z(\theta_{21i}) \mathbf{R}_x(\beta_{54i}) \mathbf{B}_y(\theta_{65i}) \mathbf{R}_z(\theta_{76i}) \mathbf{r}_{87i}^{7i}
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_{li} &= \mathbf{R}_z(\theta_{21i})\mathbf{R}_x(\beta_{54i})\mathbf{R}_y(\theta_{65i})\mathbf{B}_z(\theta_{76i})\mathbf{r}_{87i}^{7i} \\
\mathbf{r}_{ji} &= (\mathbf{B}_y(\theta_{21})\mathbf{r}_{32}^2) + (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5) \\
&\quad - (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{13,12i}^5) \\
&\quad - (\mathbf{B}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{12,11i}^5) \\
\mathbf{r}_{Ki} &= (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{r}_{14,5i}^5) - (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{13,12i}^5) \\
&\quad - (\mathbf{R}_y(\theta_{21})\mathbf{B}_y(\theta_{43})\mathbf{R}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{12,11i}^5) \\
\mathbf{r}_{Li} &= (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{r}_{14,5i}^5) - (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{13,12i}^5) \\
&\quad - (\mathbf{R}_y(\theta_{21})\mathbf{R}_y(\theta_{43})\mathbf{B}_z(\theta_{54})\mathbf{R}_z(\beta_{14,13i})\mathbf{r}_{12,11i}^5)
\end{aligned}$$

Despejando y sustituyendo:

$$\begin{aligned}
\frac{\partial \boldsymbol{\theta}}{\partial q_j} &= \mathbf{J}_{\theta i}^{-1} \mathbf{J}_{qi} \frac{\partial \mathbf{q}}{\partial q_j} \\
\mathbf{J}_{\beta i} \frac{\partial \boldsymbol{\beta}}{\partial q_j} &= \mathbf{J}_{ki} \mathbf{J}_{\theta i}^{-1} \mathbf{J}_{qi} \frac{\partial \mathbf{q}}{\partial q_j}
\end{aligned} \tag{A.7}$$

Apéndice B

Trayectoria

La trayectoria que recorre el efector final del robot se describe a continuación. Está compuesta por una función senoidal y una parábola por lo que tiene las siguientes ecuaciones paramétricas:

$$\begin{aligned}x &= -0.001t^2 \\y &= t \\z &= 100 \operatorname{sen}(0.3t)\end{aligned}$$

Para t de $(-60\pi, 60\pi)$

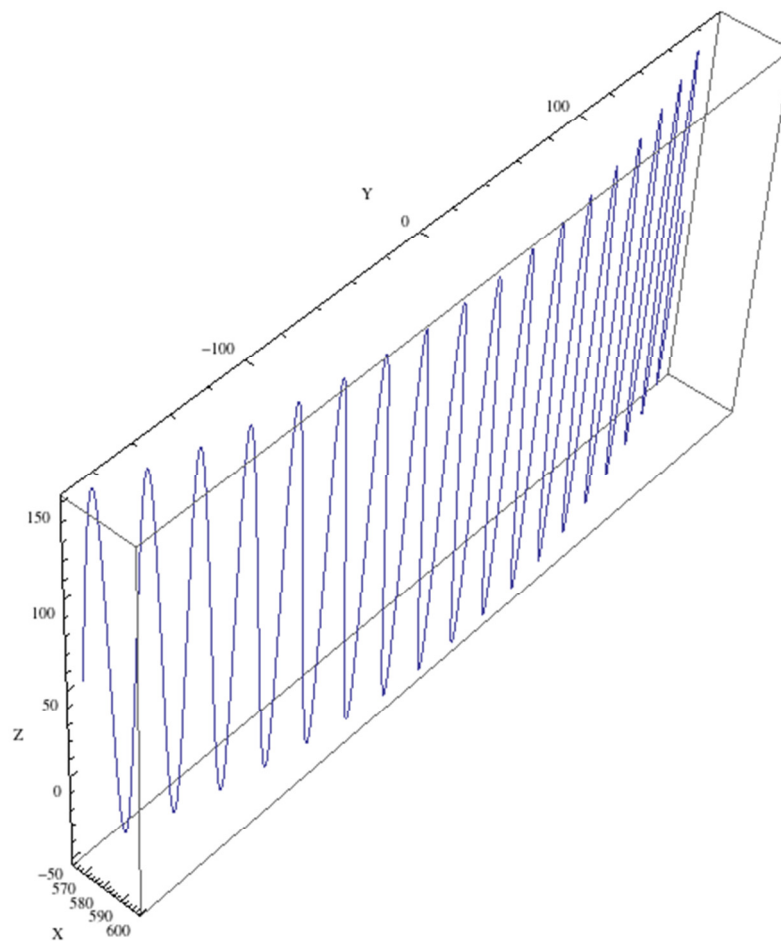


Figura B1 Trayectoria trazada por el efector final.

Apéndice C

Coeficientes de la ecuación (2.71)

$$\begin{aligned}
 \dot{c}_{22} = & \dot{\theta}_{1716} c \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\
 & \dot{\theta}_{1716} s \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + \\
 & c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))) + \\
 & s \theta_{1716} (-\dot{\theta}_{54} c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \phi c \psi s(\theta_{21} + \theta_{43}) s \theta_{54} + \dot{\phi} c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + \\
 & \dot{\psi} c(\theta_{21} + \theta_{43}) c \phi s \theta_{54} s \psi + \dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54} \\
 & (s \theta s \phi - c \theta c \phi s \psi) - \dot{\theta}_{54} s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi) + c \theta_{54} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
 & s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (-\dot{\psi} c \phi c \psi + s \phi (\dot{\theta} + \dot{\phi} s \psi)))) + \\
 & c \theta_{1716} (\dot{\theta}_{1817} c \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) - \\
 & \dot{\theta}_{1817} s \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi)) + \\
 & s \theta_{1817} (\dot{\theta} c \theta c \theta_{54} c \psi - \dot{\theta}_{54} c \psi s \theta s \theta_{54} - \dot{\psi} c \theta_{54} s \theta s \psi + \dot{\theta}_{54} c \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi) + \\
 & s \theta_{54} ((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) (\dot{\theta} c \psi s \theta - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi))) + c \theta_{1817} \\
 & (\dot{\phi} c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + \dot{\theta}_{54} c(\theta_{21} + \theta_{43}) c \theta_{54} c \psi s \phi - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi - \dot{\psi} c(\theta_{21} + \theta_{43}) \\
 & s \theta_{54} s \phi s \psi + \dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) - \\
 & \dot{\theta}_{54} s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \phi (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
 & s(\theta_{21} + \theta_{43}) s \theta_{54} (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi)))))) \\
 \\
 \dot{c}_{22} = & \dot{\theta}_{1716} c \theta_{1716} (c \phi c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi)) - \dot{\theta}_{1716} s \theta_{1716} \\
 & (-s \theta_{1817} (c \theta c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) s \psi) + c \theta_{1817} (-c \psi s(\theta_{21} + \theta_{43}) s \phi + c(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi))) + \\
 & s \theta_{1716} (-s(\theta_{21} + \theta_{43}) (\dot{\phi} c \psi s \phi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta s \phi + (\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c \phi s \psi) + \\
 & c(\theta_{21} + \theta_{43}) (c \theta s \phi (\dot{\theta} + \dot{\phi} s \psi) + c \phi ((\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) c \psi + s \theta (\dot{\phi} + \dot{\theta} s \psi)))) + \\
 & c \theta_{1716} (-c(\theta_{21} + \theta_{43}) (\dot{\theta}_{1817} c \phi s \theta s \theta_{1817} + \dot{\phi} c \theta_{1817} s \theta s \phi + c \psi (-\dot{\theta} s \theta s \theta_{1817} + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1817} s \phi) + \\
 & \dot{\theta}_{21} s \theta_{1817} s \psi + \dot{\theta}_{43} s \theta_{1817} s \psi + \dot{\theta} c \theta_{1817} s \theta s \phi s \psi) - \\
 & s(\theta_{21} + \theta_{43}) (c \psi s \theta_{1817} (\dot{\psi} - \dot{\theta}_{1817} s \phi) + c \theta_{1817} (c \phi (\dot{\phi} c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) + (\dot{\theta}_{1817} - \dot{\psi} s \phi) s \psi)) + \\
 & c \theta (s \theta_{1817} ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (\dot{\psi} - \dot{\theta}_{1817} s \phi) s \psi) + \\
 & c \theta_{1817} (- (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \phi s \psi + c(\theta_{21} + \theta_{43}) (c \psi (-\dot{\theta}_{1817} + \dot{\psi} s \phi) + c \phi (\dot{\theta} + \dot{\phi} s \psi))))))
 \end{aligned}$$

Coeficientes de la ecuación (2.83)

$$\begin{aligned}
 \ddot{b}_{12} = & (\ddot{\theta}_{54} c \theta_{54} - \dot{\theta}_{54}^2 s \theta_{54}) (c \theta c \phi - s \theta s \phi s \psi) + \\
 & (\dot{\theta}_{54}^2 c \theta_{54} + \ddot{\theta}_{54} s \theta_{54}) (c \phi s \theta s(\theta_{21} + \theta_{43}) + s \phi (c(\theta_{21} + \theta_{43}) c \psi + c \theta s(\theta_{21} + \theta_{43}) s \psi)) - \\
 & 2 \dot{\theta}_{54} c \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \phi (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
 & s \theta_{54} (-c \theta (s \phi (\ddot{\theta} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + c \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \\
 & s \theta (-c \phi (\ddot{\theta} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + s \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))) - c \theta_{54} \\
 & (c \phi s \theta ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c(\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s(\theta_{21} + \theta_{43})) + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (\dot{\theta} c \theta c \phi - \dot{\phi} s \theta s \phi) + s(\theta_{21} + \theta_{43}) \\
 & (-s \theta ((\dot{\theta}^2 + \dot{\phi}^2) c \phi + \ddot{\theta} s \phi) + c \theta (\ddot{\theta} c \phi - 2 \dot{\theta} \dot{\phi} s \phi))) + (\ddot{\theta} c \phi - \dot{\phi}^2 s \phi) (c(\theta_{21} + \theta_{43}) c \psi + c \theta s(\theta_{21} + \theta_{43}) s \psi) +
 \end{aligned}$$

$$\begin{aligned}
& 2 \dot{\psi} c \phi \left(-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) c \psi s(\theta_{21} + \theta_{43}) + ((-\dot{\psi} + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c(\theta_{21} + \theta_{43}) - \dot{\theta} s \theta s(\theta_{21} + \theta_{43})) s \psi \right) + \\
& s \phi \left(-s(\theta_{21} + \theta_{43}) \left(c \psi \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\psi} c \theta + 2 \dot{\theta} \dot{\psi} s \theta \right) + \right. \right. \\
& \quad \left. \left(-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} + \left(\dot{\theta}^2 + \dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 \right) c \theta + \ddot{\theta} s \theta \right) s \psi \right) - c(\theta_{21} + \theta_{43}) \\
& \quad \left. \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} c \theta \right) c \psi + \left(\ddot{\psi} - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta + 2 \dot{\theta} (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta \right) s \psi \right) \right) + \\
& 2 \dot{\theta}_{54} s \theta_{54} (c(\theta_{21} + \theta_{43}) (c \phi (\dot{\phi} c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) + (-\dot{\psi} + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) s \phi s \psi) + \\
& s(\theta_{21} + \theta_{43}) (-s \phi ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi + s \theta (\dot{\phi} + \dot{\theta} s \psi)) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi))))
\end{aligned}$$

$$\begin{aligned}
\ddot{b}_{13} = & 2 \dot{\psi} c \psi \left(-(\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{54} s(\theta_{21} + \theta_{43}) - \dot{\theta}_{54} c(\theta_{21} + \theta_{43}) s \theta_{54} \right) + \\
& c \psi s(\theta_{21} + \theta_{43}) \left(c \theta \left(\left(\dot{\theta}^2 + \dot{\theta}_{54}^2 \right) c \theta_{54} + \ddot{\theta}_{54} s \theta_{54} \right) + s \theta \left(\ddot{\theta} c \theta_{54} - 2 \dot{\theta} \dot{\theta}_{54} s \theta_{54} \right) \right) + \\
& c \psi \left(-c \theta \left(2 \dot{\theta} \dot{\theta}_{54} c \theta_{54} + \ddot{\theta} s \theta_{54} \right) + s \theta \left(-\ddot{\theta}_{54} c \theta_{54} + \left(\dot{\theta}^2 + \dot{\theta}_{54}^2 \right) s \theta_{54} \right) \right) + 2 \dot{\psi} \left(\dot{\theta}_{54} c \theta_{54} s \theta + \dot{\theta} c \theta s \theta_{54} \right) s \psi + \\
& \left(-c(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 \right) c \theta_{54} + \ddot{\theta}_{54} s \theta_{54} \right) - s(\theta_{21} + \theta_{43}) \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{54} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} s \theta_{54} \right) \right) \\
& s \psi + s \theta s \theta_{54} \left(\dot{\psi}^2 c \psi + \ddot{\psi} s \psi \right) + c(\theta_{21} + \theta_{43}) c \theta_{54} \left(\ddot{\psi} c \psi - \dot{\psi}^2 s \psi \right) + \\
& 2 \left(\dot{\theta} c \theta_{54} s \theta + \dot{\theta}_{54} c \theta s \theta_{54} \right) \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c \psi - \dot{\psi} s(\theta_{21} + \theta_{43}) s \psi \right) - \\
& c \theta c \theta_{54} \left(-s(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 \right) c \psi + \ddot{\psi} s \psi \right) + c(\theta_{21} + \theta_{43}) \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \psi - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} s \psi \right) \right)
\end{aligned}$$

Coeficientes de la ecuación (2.85)

$$\begin{aligned}
\ddot{b}_{11} = & c \phi c \psi \left(-c(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 \right) c \theta_{54} + \ddot{\theta}_{54} s \theta_{54} \right) - \right. \\
& s(\theta_{21} + \theta_{43}) \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{54} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} s \theta_{54} \right) \left. \right) + \\
& 2 \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{54} s(\theta_{21} + \theta_{43}) + \dot{\theta}_{54} c(\theta_{21} + \theta_{43}) s \theta_{54} \right) (\dot{\phi} c \psi s \phi + \dot{\psi} c \phi s \psi) + \\
& \left(-s(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 \right) c \theta_{54} + \ddot{\theta}_{54} s \theta_{54} \right) + c(\theta_{21} + \theta_{43}) \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{54} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} s \theta_{54} \right) \right) \\
& (-s \theta s \phi + c \theta c \phi s \psi) + \left(\ddot{\theta}_{54} c \theta_{54} - \dot{\theta}_{54}^2 s \theta_{54} \right) (c \theta s \phi + c \phi s \theta s \psi) - \\
& c(\theta_{21} + \theta_{43}) c \theta_{54} \left(c \phi \left(\left(\dot{\phi}^2 + \dot{\psi}^2 \right) c \psi + \ddot{\psi} s \psi \right) + s \phi \left(\ddot{\phi} c \psi - 2 \dot{\phi} \dot{\psi} s \psi \right) \right) + \\
& 2 \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c \theta_{54} - \dot{\theta}_{54} s(\theta_{21} + \theta_{43}) s \theta_{54} \right) (-c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& 2 \dot{\theta}_{54} c \theta_{54} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) - \\
& c \theta_{54} s(\theta_{21} + \theta_{43}) \left(s \theta \left(c \phi \left(\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi \right) - s \phi \left(\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi \right) \right) + \right. \\
& \quad \left. c \theta \left(s \phi \left(\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi \right) + c \phi \left(2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 \right) s \psi \right) \right) \right) + \\
& s \theta_{54} \left(c \theta \left(c \phi \left(\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi \right) - s \phi \left(\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi \right) \right) - \right. \\
& \quad \left. s \theta \left(s \phi \left(\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi \right) + c \phi \left(2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 \right) s \psi \right) \right) \right)
\end{aligned}$$

Coeficientes de la ecuación (2.87)

$$\begin{aligned}
\ddot{c}_{22} = & \left(\ddot{\theta}_{1716} c \theta_{1716} - \dot{\theta}_{1716}^2 s \theta_{1716} \right) \\
& (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) +
\end{aligned}$$

$$\begin{aligned}
& \left(-\dot{\theta}_{1716}^2 c \theta_{1716} - \ddot{\theta}_{1716} s \theta_{1716} \right) (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi)) + \\
& \quad c \theta_{1817} (c (\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s (\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))) + \\
& 2 \dot{\theta}_{1716} c \theta_{1716} \left(-\dot{\theta}_{54} c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \phi c \psi s (\theta_{21} + \theta_{43}) s \theta_{54} + \dot{\phi} c (\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + \right. \\
& \quad \dot{\psi} c (\theta_{21} + \theta_{43}) c \phi s \theta_{54} s \psi + \dot{\theta}_{54} c \theta_{54} s (\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c (\theta_{21} + \theta_{43}) s \theta_{54} \\
& \quad (s \theta s \phi - c \theta c \phi s \psi) - \dot{\theta}_{54} s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi) + c \theta_{54} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& \quad s (\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (-\dot{\psi} c \phi c \psi + s \phi (\dot{\theta} + \dot{\phi} s \psi))) + s \theta_{1716} \\
& \quad \left(c \psi s \theta_{54} (c (\theta_{21} + \theta_{43}) ((\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\phi}^2) c \phi + \ddot{\phi} s \phi) + s (\theta_{21} + \theta_{43}) ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \phi - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\phi} s \phi)) + \right. \\
& \quad \left(c (\theta_{21} + \theta_{43}) (2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} c \theta_{54} + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s \theta_{54}) + \right. \\
& \quad \left. s (\theta_{21} + \theta_{43}) (\ddot{\theta}_{54} c \theta_{54} - (\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2) s \theta_{54}) \right) (s \theta s \phi - c \theta c \phi s \psi) - \\
& \quad (\dot{\theta}_{54}^2 c \theta_{54} + \ddot{\theta}_{54} s \theta_{54}) (c \theta s \phi + c \phi s \theta s \psi) + 2 ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \phi s (\theta_{21} + \theta_{43}) + \dot{\phi} c (\theta_{21} + \theta_{43}) s \phi) \\
& \quad (\dot{\theta}_{54} c \theta_{54} c \psi - \dot{\psi} s \theta_{54} s \psi) - c (\theta_{21} + \theta_{43}) c \phi (-s \theta_{54} ((\dot{\theta}_{54}^2 + \dot{\psi}^2) c \psi + \ddot{\psi} s \psi) + c \theta_{54} (\ddot{\theta}_{54} c \psi - 2 \dot{\theta}_{54} \dot{\psi} s \psi)) - \\
& 2 \dot{\theta}_{54} s \theta_{54} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& 2 (\dot{\theta}_{54} c \theta_{54} s (\theta_{21} + \theta_{43}) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c (\theta_{21} + \theta_{43}) s \theta_{54}) (c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (-\dot{\psi} c \phi c \psi + s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& s (\theta_{21} + \theta_{43}) s \theta_{54} (s \theta (c \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) - s \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \\
& \quad c \theta (s \phi (\ddot{\theta} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\phi} s \psi) + c \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))) + \\
& c \theta_{54} (c \theta (c \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) - s \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) - \\
& \quad s \theta (s \phi (\ddot{\theta} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\phi} s \psi) + c \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))) - \\
& 2 \dot{\theta}_{1716} s \theta_{1716} (\dot{\theta}_{1817} c \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi)) - \\
& \quad \dot{\theta}_{1817} s \theta_{1817} (c (\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s (\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi)) + \\
& \quad s \theta_{1817} (\dot{\theta} c \theta c \theta_{54} c \psi - \dot{\theta}_{54} c \psi s \theta s \theta_{54} - \dot{\psi} c \theta_{54} s \theta s \psi + \dot{\theta}_{54} c \theta_{54} (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi) + \\
& \quad s \theta_{54} ((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c (\theta_{21} + \theta_{43}) c \psi + s (\theta_{21} + \theta_{43}) (\dot{\theta} c \psi s \theta - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi))) + c \theta_{1817} \\
& \quad (\dot{\phi} c (\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + \dot{\theta}_{54} c (\theta_{21} + \theta_{43}) c \theta_{54} c \psi s \phi - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s (\theta_{21} + \theta_{43}) s \theta_{54} s \phi - \dot{\psi} c (\theta_{21} + \theta_{43}) \\
& \quad s \theta_{54} s \phi s \psi + \dot{\theta}_{54} c \theta_{54} s (\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c (\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) - \\
& \quad \dot{\theta}_{54} s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \theta (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
& \quad s (\theta_{21} + \theta_{43}) s \theta_{54} (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi)))) + \\
& c \theta_{1716} ((\ddot{\theta}_{1817} c \theta_{1817} - \dot{\theta}_{1817}^2 s \theta_{1817}) (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi)) + \\
& \quad (-\dot{\theta}_{1817}^2 c \theta_{1817} - \ddot{\theta}_{1817} s \theta_{1817}) \\
& \quad (c (\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s (\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi)) + \\
& 2 \dot{\theta}_{1817} c \theta_{1817} (\dot{\theta} c \theta c \theta_{54} c \psi - \dot{\theta}_{54} c \psi s \theta s \theta_{54} - \dot{\psi} c \theta_{54} s \theta s \psi + \dot{\theta}_{54} c \theta_{54} (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi) + \\
& \quad s \theta_{54} ((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c (\theta_{21} + \theta_{43}) c \psi + s (\theta_{21} + \theta_{43}) (\dot{\theta} c \psi s \theta - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi))) + \\
& s \theta_{1817} (c \theta_{54} c \psi (\ddot{\theta} c \theta - \dot{\theta}^2 s \theta) + (\ddot{\theta}_{54} c \theta_{54} - \dot{\theta}_{54}^2 s \theta_{54}) (-c \theta c \psi s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \psi) - \\
& \quad 2 \dot{\theta} c \theta (\dot{\theta}_{54} c \psi s \theta_{54} + \dot{\psi} c \theta_{54} s \psi) - s \theta (c \theta_{54} ((\dot{\theta}_{54}^2 + \dot{\psi}^2) c \psi + \ddot{\psi} s \psi) + s \theta_{54} (\ddot{\theta}_{54} c \psi - 2 \dot{\theta}_{54} \dot{\psi} s \psi)) + \\
& \quad 2 \dot{\theta}_{54} c \theta_{54} ((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c (\theta_{21} + \theta_{43}) c \psi + s (\theta_{21} + \theta_{43}) (\dot{\theta} c \psi s \theta - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi)) + \\
& \quad s \theta_{54} (c (\theta_{21} + \theta_{43}) (c \psi (\ddot{\psi} - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta + 2 \dot{\theta} (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) -
\end{aligned}$$

$$\begin{aligned}
& \left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} c \theta \right) s \psi + s(\theta_{21} + \theta_{43}) \left(c \psi \left(-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} + \right. \right. \\
& \left. \left. (\dot{\theta}^2 + \dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2) c \theta + \ddot{\theta} s \theta \right) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\psi} c \theta + 2 \dot{\theta} \dot{\psi} s \theta) s \psi \right) \Big) - \\
& 2 \dot{\theta}_{1817} s \theta_{1817} \left(\dot{\phi} c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + \dot{\theta}_{54} c(\theta_{21} + \theta_{43}) c \theta_{54} c \psi s \phi - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi - \right. \\
& \left. \dot{\psi} c(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi + \dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi) + \right. \\
& \left. (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) - \dot{\theta}_{54} s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - \right. \\
& \left. c \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \phi (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \right. \\
& \left. s(\theta_{21} + \theta_{43}) s \theta_{54} (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi))) \right) + \\
& c \theta_{1817} \left(c(\theta_{21} + \theta_{43}) c \psi \left(c \theta_{54} (2 \dot{\theta}_{54} \dot{\phi} c \phi + \ddot{\theta}_{54} s \phi) + s \theta_{54} (\ddot{\phi} c \phi - (\dot{\theta}_{54}^2 + \dot{\phi}^2) s \phi) \right) + 2 (\dot{\phi} c \phi s \theta_{54} + \dot{\theta}_{54} c \theta_{54} s \phi) \right. \\
& \left. (- (\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) - \dot{\psi} c(\theta_{21} + \theta_{43}) s \psi) + (c(\theta_{21} + \theta_{43}) (2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} c \theta_{54} + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s \theta_{54}) + \right. \\
& \left. s(\theta_{21} + \theta_{43}) (\ddot{\theta}_{54} c \theta_{54} - (\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2) s \theta_{54})) (c \phi s \theta + c \theta s \phi s \psi) - \right. \\
& \left. (\dot{\theta}_{54}^2 c \theta_{54} + \ddot{\theta}_{54} s \theta_{54}) (c \theta c \phi - s \theta s \phi s \psi) + s \theta_{54} s \phi (-c(\theta_{21} + \theta_{43}) \right. \\
& \left. ((\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2) c \psi + \ddot{\psi} s \psi) - s(\theta_{21} + \theta_{43}) ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \psi - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} s \psi)) \right) + \\
& 2 \dot{\theta}_{54} s \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \phi (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
& 2 (\dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54}) (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& s(\theta_{21} + \theta_{43}) s \theta_{54} (-s \theta (s \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + c \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \\
& c \theta (c \phi (\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi) - s \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))) + \\
& c \theta_{54} (-c \theta (s \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + c \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \\
& s \theta (-c \phi (\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi) + s \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi))))))
\end{aligned}$$

$$\begin{aligned}
\ddot{c}_{32} = & (\ddot{\theta}_{1716} c \theta_{1716} - \dot{\theta}_{1716}^2 s \theta_{1716}) (c \phi c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi)) + (\dot{\theta}_{1716}^2 c \theta_{1716} + \ddot{\theta}_{1716} s \theta_{1716}) \\
& (s \theta_{1817} (c \theta c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) s \psi) - c \theta_{1817} (-c \psi s(\theta_{21} + \theta_{43}) s \phi + c(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi))) + \\
& 2 \dot{\theta}_{1716} c \theta_{1716} (-s(\theta_{21} + \theta_{43}) (\dot{\phi} c \psi s \phi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta s \phi + (\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c \phi s \psi) + \\
& c(\theta_{21} + \theta_{43}) (c \theta s \phi (\dot{\theta} + \dot{\phi} s \psi) + c \phi ((\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) c \psi + s \theta (\dot{\phi} + \dot{\theta} s \psi)))) + \\
& s \theta_{1716} (c \phi c \psi ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c(\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s(\theta_{21} + \theta_{43})) - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (\dot{\phi} c \psi s \phi + \dot{\psi} c \phi s \psi) + \\
& (- (\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43})) (s \theta s \phi - c \theta c \phi s \psi) - \\
& s(\theta_{21} + \theta_{43}) (c \phi ((\dot{\phi}^2 + \dot{\psi}^2) c \psi + \ddot{\psi} s \psi) + s \phi (\ddot{\phi} c \psi - 2 \dot{\phi} \dot{\psi} s \psi)) - \\
& 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) (c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (-\dot{\psi} c \phi c \psi + s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& c(\theta_{21} + \theta_{43}) (s \theta (c \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) - s \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \\
& c \theta (s \phi (\ddot{\theta} + 2 \dot{\phi} \dot{\psi} c \psi + \ddot{\phi} s \psi) + c \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi)))) - \\
& 2 \dot{\theta}_{1716} s \theta_{1716} (-c(\theta_{21} + \theta_{43}) (\dot{\theta}_{1817} c \phi s \theta s \theta_{1817} + \dot{\phi} c \theta_{1817} s \theta s \phi + c \psi (-\dot{\theta} s \theta s \theta_{1817} + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1817} s \phi) + \\
& \dot{\theta}_{21} s \theta_{1817} s \psi + \dot{\theta}_{43} s \theta_{1817} s \psi + \dot{\theta} c \theta_{1817} s \theta s \phi s \psi) - \\
& s(\theta_{21} + \theta_{43}) (c \psi s \theta_{1817} (\dot{\psi} - \dot{\theta}_{1817} s \phi) + c \theta_{1817} (c \phi (\dot{\phi} c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) + (\dot{\theta}_{1817} - \dot{\psi} s \phi) s \psi)) + \\
& c \theta (s \theta_{1817} ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (\dot{\psi} - \dot{\theta}_{1817} s \phi) s \psi) + \\
& c \theta_{1817} (- (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \phi s \psi + c(\theta_{21} + \theta_{43}) (c \psi (-\dot{\theta}_{1817} + \dot{\psi} s \phi) + c \phi (\dot{\theta} + \dot{\phi} s \psi)))))) +
\end{aligned}$$

$$\begin{aligned}
& c \theta_{1716} \left(\left(-\ddot{\theta}_{1817} c \theta_{1817} + \dot{\theta}_{1817}^2 s \theta_{1817} \right) (c \theta c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) s \psi) - \right. \\
& 2 \dot{\theta}_{1817} c \theta_{1817} \left((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (-\dot{\theta} c \psi s \theta + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi) \right) - \\
& s \theta_{1817} \left(s(\theta_{21} + \theta_{43}) \left(c \psi (\ddot{\psi} - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta + 2 \dot{\theta} (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) - \right. \right. \\
& \quad \left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} c \theta \right) s \psi \Big) - c(\theta_{21} + \theta_{43}) \\
& \quad \left. \left(c \psi \left(-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} + (\dot{\theta}^2 + \dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2) c \theta + \ddot{\theta} s \theta \right) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\psi} c \theta + 2 \dot{\theta} \dot{\psi} s \theta) s \psi \right) \right) + \\
& \left(\dot{\theta}_{1817}^2 c \theta_{1817} + \ddot{\theta}_{1817} s \theta_{1817} \right) (c \psi s(\theta_{21} + \theta_{43}) s \phi - c(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi)) - \\
& 2 \dot{\theta}_{1817} s \theta_{1817} \left(-c \phi (\dot{\phi} c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) s(\theta_{21} + \theta_{43}) - \right. \\
& \quad s \phi (-\dot{\psi} s(\theta_{21} + \theta_{43}) s \psi + c(\theta_{21} + \theta_{43}) ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi + s \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
& \quad c \theta (- (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \phi s \psi + c(\theta_{21} + \theta_{43}) (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi))) + c \theta_{1817} \\
& \quad \left. (-c \psi (c(\theta_{21} + \theta_{43}) (2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\phi} c \phi + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s \phi) + s(\theta_{21} + \theta_{43}) (\ddot{\phi} c \phi - (\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\phi}^2) s \phi)) \right) + \\
& \quad 2 \dot{\psi} (\dot{\phi} c \phi s(\theta_{21} + \theta_{43}) + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \phi) s \psi + s(\theta_{21} + \theta_{43}) s \phi (\dot{\psi}^2 c \psi + \ddot{\psi} s \psi) + \\
& \quad \left(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) (c \phi s \theta + c \theta s \phi s \psi) - \\
& \quad 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& \quad c(\theta_{21} + \theta_{43}) \left(-s \theta (s \phi (\ddot{\phi} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\theta} s \psi) + c \phi (\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi)) + \right. \\
& \quad \left. c \theta (c \phi (\ddot{\theta} + 2 \dot{\theta} \dot{\psi} c \psi + \ddot{\phi} s \psi) - s \phi (2 \dot{\theta} \dot{\phi} - \ddot{\psi} c \psi + (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) s \psi)) \right) \Big)
\end{aligned}$$

Coefficientes de la ecuación (2.131)

$$\begin{aligned}
f_{11} = & s \theta_{109 \text{ i}} (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} s(\theta_{21} + \theta_{43}) - \\
& s \beta_{54 \text{ i}} (c \theta_{43} c \theta_{76 \text{ i}} s \theta_{21} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}} + c \theta_{21} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} s \theta_{43} s \theta_{65 \text{ i}} - c \theta_{21} c \theta_{43} s \theta_{76 \text{ i}} + s \theta_{21} s \theta_{43} s \theta_{76 \text{ i}}) - \\
& c \beta_{54 \text{ i}} (c \theta_{21} c \theta_{43} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} - c \theta_{76 \text{ i}} s \theta_{21} s \theta_{43} s \theta_{65 \text{ i}} + c \theta_{43} s \theta_{21} s \theta_{21 \text{ i}} s \theta_{76 \text{ i}} + c \theta_{21} s \theta_{21 \text{ i}} s \theta_{43} s \theta_{76 \text{ i}}) + \\
& c \theta_{109 \text{ i}} (c \theta_{1110 \text{ i}} (-s(\beta_{1413 \text{ i}} - \theta_{54}) (c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} + c \theta_{21 \text{ i}} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{76 \text{ i}})) + \\
& \quad c(\beta_{1413 \text{ i}} - \theta_{54}) s \theta_{43} (-c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21} + s \beta_{54 \text{ i}} (c \theta_{76 \text{ i}} s \theta_{21} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}} - c \theta_{21} s \theta_{76 \text{ i}}) + \\
& \quad c \beta_{54 \text{ i}} (c \theta_{21} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} + s \theta_{21} s \theta_{21 \text{ i}} s \theta_{76 \text{ i}})) + c \theta_{43} c(\beta_{1413 \text{ i}} - \theta_{54}) (s \theta_{21} (c \beta_{54 \text{ i}} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} - \\
& \quad s \beta_{54 \text{ i}} s \theta_{76 \text{ i}}) + c \theta_{21} (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} - s \theta_{21 \text{ i}} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{76 \text{ i}})))) - \\
& s \theta_{1110 \text{ i}} (c(\beta_{1413 \text{ i}} - \theta_{54}) (c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} + c \theta_{21 \text{ i}} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{76 \text{ i}})) + \\
& \quad s(\beta_{1413 \text{ i}} - \theta_{54}) (c \theta_{21} (c \theta_{21 \text{ i}} c \theta_{43} c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} - c \theta_{43} s \theta_{21 \text{ i}} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{76 \text{ i}}) + \\
& \quad s \theta_{43} (c \beta_{54 \text{ i}} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{76 \text{ i}})) + s \theta_{21} (-c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} c \theta_{76 \text{ i}} s \theta_{43} + s \beta_{54 \text{ i}} \\
& \quad (c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} s \theta_{43} s \theta_{65 \text{ i}} - c \theta_{43} s \theta_{76 \text{ i}}) + c \beta_{54 \text{ i}} (c \theta_{43} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} + s \theta_{21 \text{ i}} s \theta_{43} s \theta_{76 \text{ i}})))))) \\
f_{21} = & s \theta_{109 \text{ i}} (-s \theta_{43} \\
& (c \theta_{76 \text{ i}} (s \beta_{54 \text{ i}} s \theta_{21} + c \beta_{54 \text{ i}} c \theta_{21} s \theta_{21 \text{ i}}) + (c \beta_{54 \text{ i}} s \theta_{21} s \theta_{65 \text{ i}} + c \theta_{21} (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}})) s \theta_{76 \text{ i}}) + \\
& c \theta_{43} (c \theta_{21} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{65 \text{ i}} s \theta_{76 \text{ i}}) - \\
& \quad s \theta_{21} (c \beta_{54 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} + (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}})) + \\
& c \theta_{109 \text{ i}} (c \theta_{1110 \text{ i}} (s(\beta_{1413 \text{ i}} - \theta_{54}) (-c \beta_{54 \text{ i}} c \theta_{21 \text{ i}} c \theta_{76 \text{ i}} + (c \theta_{65 \text{ i}} s \theta_{21 \text{ i}} + c \theta_{21 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}}) - c \theta_{43} c(\beta_{1413 \text{ i}} - \theta_{54}) \\
& \quad (c \theta_{76 \text{ i}} (s \beta_{54 \text{ i}} s \theta_{21} + c \beta_{54 \text{ i}} c \theta_{21} s \theta_{21 \text{ i}}) + (c \beta_{54 \text{ i}} s \theta_{21} s \theta_{65 \text{ i}} + c \theta_{21} (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}})) \\
& \quad s \theta_{76 \text{ i}}) - c(\beta_{1413 \text{ i}} - \theta_{54}) s \theta_{43} (c \theta_{21} (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{65 \text{ i}} s \theta_{76 \text{ i}}) - \\
& \quad s \theta_{21} (c \beta_{54 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} + (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}})) - \\
& s \theta_{1110 \text{ i}} (c(\beta_{1413 \text{ i}} - \theta_{54}) (c \beta_{54 \text{ i}} c \theta_{21 \text{ i}} c \theta_{76 \text{ i}} - (c \theta_{65 \text{ i}} s \theta_{21 \text{ i}} + c \theta_{21 \text{ i}} s \beta_{54 \text{ i}} s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}}) - \\
& \quad c \theta_{43} s(\beta_{1413 \text{ i}} - \theta_{54}) (c \theta_{76 \text{ i}} (s \beta_{54 \text{ i}} s \theta_{21} + c \beta_{54 \text{ i}} c \theta_{21} s \theta_{21 \text{ i}}) +
\end{aligned}$$

$$\begin{aligned} & (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\ & s \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\ & s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)))) \end{aligned}$$

Coeficientes de la ecuación (2.151)

$$\begin{aligned} \dot{e}_{31} = & -c \beta_{54} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) c \theta_{65} i - \\ & s(\theta_{21} + \theta_{43}) ((\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{65} i s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) + \dot{\theta}_{65} i c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s \theta_{65} i)) + \\ & c \theta_{1110} i (-s \theta_{21} i (c \theta_{65} i (\dot{\theta}_{65} i + (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43})) s \beta_{54} i) s(\beta_{1413} i - \theta_{54}) + \\ & c(\beta_{1413} i - \theta_{54}) ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \beta_{54} i s(\theta_{21} + \theta_{43}) + \\ & (\dot{\theta}_{1110} i - \dot{\theta}_{54} + \dot{\theta}_{21} i c(\theta_{21} + \theta_{43}) + \dot{\theta}_{65} i c \theta_{21} c \theta_{43} s \beta_{54} i - \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i)) + \\ & c \theta_{21} i (-(\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i + c(\beta_{1413} i - \theta_{54}) \\ & ((\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{65} i s \beta_{54} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\ & c(\theta_{21} + \theta_{43}) (c(\beta_{1413} i - \theta_{54}) c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (-\dot{\theta}_{1110} i + \dot{\theta}_{54}) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i))) + \\ & s \theta_{1110} i (-c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i (\dot{\theta}_{65} i + (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43})) s \beta_{54} i) s \theta_{21} i + \\ & c \theta_{21} i (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43}) + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i) + s(\beta_{1413} i - \theta_{54}) \\ & (c \theta_{21} i (-(\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\ & s \theta_{21} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{43} + (\dot{\theta}_{1110} i - \dot{\theta}_{54} + \dot{\theta}_{21} i c(\theta_{21} + \theta_{43}) - \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i + \\ & c \theta_{43} s \beta_{54} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i)))))) \end{aligned}$$

$$\begin{aligned} \dot{e}_{33} = & -(\dot{\theta}_{21} + \dot{\theta}_{43}) c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) - \\ & \dot{\theta}_{65} i c \beta_{54} i c(\theta_{21} + \theta_{43}) s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \\ & s(\theta_{21} + \theta_{43}) (c \theta_{21} i c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{65} i) \end{aligned}$$

Coeficientes de la ecuación (2.155)

$$\begin{aligned} \dot{f}_{11} = & \dot{\theta}_{109} i c \theta_{109} i (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - \\ & s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{65} i + c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\ & c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\ & s \theta_{109} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} i c(\theta_{21} + \theta_{43}) c \theta_{65} i c \theta_{76} i - \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s(\theta_{21} + \theta_{43}) - \\ & \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s(\theta_{21} + \theta_{43}) s \theta_{65} i - \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s \theta_{76} i - \\ & c \beta_{54} i (c \theta_{21} (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\ & c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\ & s \theta_{21} (c \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) - \\ & s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) - \\ & s \beta_{54} i (s \theta_{21} (c \theta_{76} i s \theta_{43} (\dot{\theta}_{76} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i) + c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + \\ & (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + c \theta_{21} (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i) + \\ & s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)))) - \\ & \dot{\theta}_{109} i s \theta_{109} i (c \theta_{1110} i (-s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\ & c(\beta_{1413} i - \theta_{54}) s \theta_{43} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} + s \beta_{54} i (c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i - c \theta_{21} s \theta_{76} i) + \\ & c \beta_{54} i (c \theta_{21} c \theta_{76} i s \theta_{65} i + s \theta_{21} s \theta_{21} i s \theta_{76} i)) + \end{aligned}$$

[illegible]

$$\begin{aligned}
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) - \\
& c \theta_{43} s(\beta_{1413} i - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + \\
& c \theta_{21} (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))))))
\end{aligned}$$

Coeficientes de la ecuación (2.167)

$$\begin{aligned}
\ddot{d}_{31} = & 2 \dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i) - \\
& (\ddot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413} i - \theta_{54})) (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\
& (-c \theta_{43} ((\dot{\theta}_{43}^2 + \dot{\theta}_{54}^2) c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) - s \theta_{43} (\ddot{\theta}_{43} c(\beta_{1413} i - \theta_{54}) + 2 \dot{\theta}_{43} \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}))) \\
& (-c \beta_{54} i c \theta_{65} i s \theta_{21} i + c \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) + \\
& s \theta_{43} (\dot{\theta}_{54}^2 c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) (c \beta_{54} i c \theta_{21} c \theta_{65} i + s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) - \\
& s(\beta_{1413} i - \theta_{54}) (c \theta_{21} i (c \theta_{65} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) s \beta_{54} i) + (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i) + \\
& s \theta_{21} i (c \theta_{65} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i)) + \\
& 2 (-\dot{\theta}_{43} c(\beta_{1413} i - \theta_{54}) s \theta_{43} + \dot{\theta}_{54} c \theta_{43} s(\beta_{1413} i - \theta_{54})) \\
& (-\dot{\theta}_{21} s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + c \beta_{54} i (-\dot{\theta}_{21} c \theta_{21} c \theta_{65} i + \dot{\theta}_{65} i s \theta_{21} s \theta_{65} i) + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{65} i)) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (c \beta_{54} i (s \theta_{21} ((\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + c \theta_{21} (-\ddot{\theta}_{21} c \theta_{65} i + 2 \dot{\theta}_{21} \dot{\theta}_{65} i s \theta_{65} i)) - \\
& s \theta_{21} (c \theta_{21} i (2 \dot{\theta}_{21} c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + \ddot{\theta}_{21} s \theta_{65} i) + \\
& s \theta_{21} i (\ddot{\theta}_{21} c \theta_{65} i s \beta_{54} i - 2 \dot{\theta}_{21} (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i)) + \\
& c \theta_{21} (-s \theta_{21} i (c \theta_{65} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) s \beta_{54} i) + (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i) + \\
& c \theta_{21} i (c \theta_{65} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21}^2 + \dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i))) - \\
& 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (\dot{\theta}_{43} c \theta_{43} (c \beta_{54} i c \theta_{21} c \theta_{65} i + s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) + \\
& s \theta_{43} (c \theta_{21} i (c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{21} c \theta_{21} s \theta_{65} i) - c \beta_{54} i (\dot{\theta}_{21} c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i) + \\
& s \theta_{21} i (\dot{\theta}_{21} c \theta_{21} c \theta_{65} i s \beta_{54} i - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} s \theta_{65} i))) - \\
& c(\beta_{1413} i - \theta_{54}) ((\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43}) (c \beta_{54} i c \theta_{21} c \theta_{65} i + s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) + \\
& 2 \dot{\theta}_{43} c \theta_{43} (c \theta_{21} i (c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{21} c \theta_{21} s \theta_{65} i) - c \beta_{54} i (\dot{\theta}_{21} c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i) + \\
& s \theta_{21} i (\dot{\theta}_{21} c \theta_{21} c \theta_{65} i s \beta_{54} i - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} s \theta_{65} i)) + \\
& s \theta_{43} (-c \beta_{54} i (c \theta_{21} ((\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + s \theta_{21} (\ddot{\theta}_{21} c \theta_{65} i - 2 \dot{\theta}_{21} \dot{\theta}_{65} i s \theta_{65} i)) + \\
& c \theta_{21} (c \theta_{21} i (2 \dot{\theta}_{21} c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + \ddot{\theta}_{21} s \theta_{65} i) + \\
& s \theta_{21} i (\ddot{\theta}_{21} c \theta_{65} i s \beta_{54} i - 2 \dot{\theta}_{21} (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i)) +
\end{aligned}$$

$$\begin{aligned}
& c \beta_{54} i c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) \left(-s(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{65}^2 \right) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + \right. \\
& \quad c(\theta_{21} + \theta_{43}) \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{65} i - 2 \left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) \dot{\theta}_{65} i s \theta_{65} i \right) \left. + \right. \\
& \quad \left(-\dot{\theta}_{21}^2 c \theta_{21} i - \ddot{\theta}_{21} i s \theta_{21} i \right) (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i s \beta_{54} i s \theta_{1110} i + c \theta_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\
& \quad s(\beta_{1413} i - \theta_{54}) (c \theta_{1110} i c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i)) + \left(\ddot{\theta}_{21} i c \theta_{21} i - \dot{\theta}_{21}^2 s \theta_{21} i \right) \\
& \quad (-s \theta_{1110} i (c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{54} i s(\beta_{1413} i - \theta_{54}) - c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s(\beta_{1413} i - \theta_{54}) + c(\beta_{1413} i - \theta_{54}) s \theta_{65} i) + \\
& \quad c \theta_{1110} i (c \theta_{21} c \theta_{43} c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i - c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - s(\beta_{1413} i - \theta_{54}) s \theta_{65} i)) - \\
& \quad 2 \dot{\theta}_{21} i s \theta_{21} i (\dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i s \beta_{54} i s \theta_{1110} i + c \theta_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{65} i) - \\
& \quad \dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) (c \theta_{1110} i c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i) + \\
& \quad c(\beta_{1413} i - \theta_{54}) (\dot{\theta}_{65} i c \theta_{1110} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + \dot{\theta}_{1110} i c \theta_{1110} i c \theta_{65} i s \beta_{54} i - \\
& \quad \dot{\theta}_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i - \dot{\theta}_{65} i s \beta_{54} i s \theta_{1110} i s \theta_{65} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110} i s(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\
& \quad s(\beta_{1413} i - \theta_{54}) (-\dot{\theta}_{1110} i c \theta_{65} i s \beta_{54} i s \theta_{1110} i + (-\dot{\theta}_{65} i c \theta_{1110} i s \beta_{54} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{1110} i s(\theta_{21} + \theta_{43})) s \theta_{65} i - \\
& \quad c(\theta_{21} + \theta_{43}) (\dot{\theta}_{65} i c \theta_{65} i s \theta_{1110} i + \dot{\theta}_{1110} i c \theta_{1110} i s \theta_{65} i))) + \\
& \quad c \theta_{21} i \left((-\dot{\theta}_{54}^2 c(\beta_{1413} i - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) (c \theta_{65} i s \beta_{54} i s \theta_{1110} i + c \theta_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{65} i) - \right. \\
& \quad \left(\ddot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413} i - \theta_{54}) \right) (c \theta_{1110} i c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i) + \\
& \quad 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (\dot{\theta}_{65} i c \theta_{1110} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + \dot{\theta}_{1110} i c \theta_{1110} i c \theta_{65} i s \beta_{54} i - \\
& \quad \dot{\theta}_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i - \dot{\theta}_{65} i s \beta_{54} i s \theta_{1110} i s \theta_{65} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110} i s(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\
& \quad c(\beta_{1413} i - \theta_{54}) \left(c \theta_{65} i s \beta_{54} i \left(\ddot{\theta}_{1110} i c \theta_{1110} i - \dot{\theta}_{1110}^2 s \theta_{1110} i \right) + 2 \dot{\theta}_{65} i c \theta_{65} i \right. \\
& \quad \left. (-\dot{\theta}_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{1110} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110} i s(\theta_{21} + \theta_{43})) - 2 \dot{\theta}_{1110} i \dot{\theta}_{65} i c \theta_{1110} i s \beta_{54} i s \theta_{65} i + \right. \\
& \quad \left. (-c \theta_{1110} i \left((\dot{\theta}_{1110} i^2 + (\dot{\theta}_{21} + \dot{\theta}_{43})^2 \right) c(\theta_{21} + \theta_{43}) + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) + \\
& \quad s \theta_{1110} i \left(-\ddot{\theta}_{1110} i c(\theta_{21} + \theta_{43}) + 2 \dot{\theta}_{1110} i (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) \left. \right) s \theta_{65} i - \\
& \quad s \beta_{54} i s \theta_{1110} i \left(\dot{\theta}_{65}^2 c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + c \theta_{1110} i c(\theta_{21} + \theta_{43}) \left(\ddot{\theta}_{65} i c \theta_{65} i - \dot{\theta}_{65}^2 s \theta_{65} i \right) \left. \right) - \\
& \quad 2 \dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) (-\dot{\theta}_{1110} i c \theta_{65} i s \beta_{54} i s \theta_{1110} i + (-\dot{\theta}_{65} i c \theta_{1110} i s \beta_{54} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{1110} i s(\theta_{21} + \theta_{43})) \\
& \quad s \theta_{65} i - c(\theta_{21} + \theta_{43}) (\dot{\theta}_{65} i c \theta_{65} i s \theta_{1110} i + \dot{\theta}_{1110} i c \theta_{1110} i s \theta_{65} i)) + \\
& \quad s(\beta_{1413} i - \theta_{54}) \left(s \theta_{1110} i \left((\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) s \theta_{65} i + \right. \\
& \quad 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) (\dot{\theta}_{65} i c \theta_{65} i s \theta_{1110} i + \dot{\theta}_{1110} i c \theta_{1110} i s \theta_{65} i) - s \beta_{54} i \left(c \theta_{1110} i \right. \\
& \quad \left. \left((\dot{\theta}_{1110} i^2 + \dot{\theta}_{65} i^2 \right) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + s \theta_{1110} i \left(\ddot{\theta}_{1110} i c \theta_{65} i - 2 \dot{\theta}_{1110} i \dot{\theta}_{65} i s \theta_{65} i \right) \left. \right) + c(\theta_{21} + \theta_{43}) \\
& \quad \left. \left(-c \theta_{1110} i \left(2 \dot{\theta}_{1110} i \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{1110} i s \theta_{65} i \right) + s \theta_{1110} i \left(-\ddot{\theta}_{65} i c \theta_{65} i + (\dot{\theta}_{1110} i^2 + \dot{\theta}_{65} i^2) s \theta_{65} i \right) \right) \right) \left. \right) + \\
& \quad 2 \dot{\theta}_{21} i c \theta_{21} i (-c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i (s \theta_{1110} i (\dot{\theta}_{65} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{21} c \theta_{43} s \beta_{54} i - (\dot{\theta}_{1110} i - \dot{\theta}_{54}) s \beta_{54} i s \theta_{21} s \theta_{43}) + \\
& \quad (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110} i s \beta_{54} i s(\theta_{21} + \theta_{43})) + \\
& \quad c \theta_{1110} i (\dot{\theta}_{1110} i - \dot{\theta}_{54} + \dot{\theta}_{65} i c \theta_{21} c \theta_{43} s \beta_{54} i - \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i) + \\
& \quad s(\beta_{1413} i - \theta_{54}) (-c \theta_{1110} i c \theta_{65} i (\dot{\theta}_{65} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{21} c \theta_{43} s \beta_{54} i - (\dot{\theta}_{1110} i - \dot{\theta}_{54}) s \beta_{54} i s \theta_{21} s \theta_{43}) + \\
& \quad s \theta_{1110} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{43} + (\dot{\theta}_{1110} i - \dot{\theta}_{54} - \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i + \\
& \quad c \theta_{43} s \beta_{54} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i)))) + \\
& \quad s \theta_{21} i \left((-\ddot{\$$

$$\begin{aligned}
& c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s (\beta_{1413} i - \theta_{54}) + c (\beta_{1413} i - \theta_{54}) s \theta_{65} i - \left(\dot{\theta}_{1110} i^2 c \theta_{1110} i + \ddot{\theta}_{1110} i s \theta_{1110} i \right) \\
& (c \theta_{21} c \theta_{43} c (\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i - c (\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - s (\beta_{1413} i - \theta_{54}) s \theta_{65} i) + \\
& c \theta_{1110} i \left(-s (\beta_{1413} i - \theta_{54}) \left(c \theta_{65} i \left(\ddot{\theta}_{65} i + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} c \theta_{43} s \beta_{54} i s \theta_{21} + \right. \right. \right. \\
& \quad \ddot{\theta}_{54} s \beta_{54} i s \theta_{21} s \theta_{43} - c \theta_{21} s \beta_{54} i \left(\ddot{\theta}_{54} c \theta_{43} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} s \theta_{43} \right) \Big) - \\
& \quad \left(\dot{\theta}_{54}^2 + \dot{\theta}_{65} i^2 - 2 \dot{\theta}_{54} \dot{\theta}_{65} i c \theta_{21} c \theta_{43} s \beta_{54} i + 2 \dot{\theta}_{54} \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} \right) s \theta_{65} i \Big) + \\
& c (\beta_{1413} i - \theta_{54}) \left(-c \theta_{65} i \left(-2 \dot{\theta}_{54} \dot{\theta}_{65} i + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{43} s \beta_{54} i s \theta_{21} - \dot{\theta}_{21}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \right. \\
& \quad 2 \dot{\theta}_{21} \dot{\theta}_{43} s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{43}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{54}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{65} i^2 s \beta_{54} i s \\
& \quad \left. \left. \theta_{21} s \theta_{43} + c \theta_{21} s \beta_{54} i \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 + \dot{\theta}_{65} i^2 \right) c \theta_{43} + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) s \theta_{43} \right) \right) \right) + \\
& \quad \left(\ddot{\theta}_{54} + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i c \theta_{43} s \beta_{54} i s \theta_{21} + \ddot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \\
& \quad \left. c \theta_{21} s \beta_{54} i \left(\ddot{\theta}_{65} i c \theta_{43} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i s \theta_{43} \right) \right) s \theta_{65} i \Big) - \\
& s \theta_{1110} i \left(c (\beta_{1413} i - \theta_{54}) \left(c \theta_{65} i \left(\ddot{\theta}_{65} i + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} c \theta_{43} s \beta_{54} i s \theta_{21} + \ddot{\theta}_{54} s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \right. \right. \\
& \quad \left. \left. c \theta_{21} s \beta_{54} i \left(\ddot{\theta}_{54} c \theta_{43} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{54} s \theta_{43} \right) \right) - \right. \\
& \quad \left(\dot{\theta}_{54}^2 + \dot{\theta}_{65} i^2 - 2 \dot{\theta}_{54} \dot{\theta}_{65} i c \theta_{21} c \theta_{43} s \beta_{54} i + 2 \dot{\theta}_{54} \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} \right) s \theta_{65} i \Big) + \\
& s (\beta_{1413} i - \theta_{54}) \left(-c \theta_{65} i \left(-2 \dot{\theta}_{54} \dot{\theta}_{65} i + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{43} s \beta_{54} i s \theta_{21} - \dot{\theta}_{21}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \right. \\
& \quad 2 \dot{\theta}_{21} \dot{\theta}_{43} s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{43}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{54}^2 s \beta_{54} i s \theta_{21} s \theta_{43} - \dot{\theta}_{65} i^2 s \beta_{54} i s \\
& \quad \left. \left. \theta_{21} s \theta_{43} + c \theta_{21} s \beta_{54} i \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 + \dot{\theta}_{65} i^2 \right) c \theta_{43} + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) s \theta_{43} \right) \right) \right) + \\
& \quad \left(\ddot{\theta}_{54} + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i c \theta_{43} s \beta_{54} i s \theta_{21} + \ddot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \\
& \quad \left. c \theta_{21} s \beta_{54} i \left(\ddot{\theta}_{65} i c \theta_{43} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i s \theta_{43} \right) \right) s \theta_{65} i \Big) + \\
& 2 \dot{\theta}_{1110} i s \theta_{1110} i \left(\left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) c \theta_{21} c (\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i s \theta_{43} + \dot{\theta}_{65} i c \theta_{65} i s (\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54} c \theta_{65} i \right. \\
& \quad s \beta_{54} i s \theta_{21} s \theta_{43} s (\beta_{1413} i - \theta_{54}) - \dot{\theta}_{54} c (\beta_{1413} i - \theta_{54}) s \theta_{65} i - \dot{\theta}_{65} i c (\beta_{1413} i - \theta_{54}) s \beta_{54} i s \theta_{21} s \theta_{43} s \theta_{65} i + \\
& \quad \left. c \theta_{43} s \beta_{54} i \left(-\dot{\theta}_{54} c \theta_{21} c \theta_{65} i s (\beta_{1413} i - \theta_{54}) + c (\beta_{1413} i - \theta_{54}) \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i \right) \right) \right) - \\
& 2 \dot{\theta}_{1110} i c \theta_{1110} i \left(c (\beta_{1413} i - \theta_{54}) c \theta_{65} i \left(\dot{\theta}_{65} i - \dot{\theta}_{54} c \theta_{21} c \theta_{43} s \beta_{54} i + \dot{\theta}_{54} s \beta_{54} i s \theta_{21} s \theta_{43} \right) - \right. \\
& \quad s (\beta_{1413} i - \theta_{54}) \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{43} - (\dot{\theta}_{54} + \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i + \right. \\
& \quad \left. \left. c \theta_{43} s \beta_{54} i \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i \right) \right) \right) \Big)
\end{aligned}$$

$$\begin{aligned}
\ddot{e}_{33} = & c \beta_{54} i c \theta_{65} i \left(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c (\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s (\theta_{21} + \theta_{43}) \right) + \\
& 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i c \beta_{54} i s (\theta_{21} + \theta_{43}) s \theta_{65} i - c \beta_{54} i c (\theta_{21} + \theta_{43}) \left(\dot{\theta}_{65} i^2 c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + \\
& \left((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c (\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s (\theta_{21} + \theta_{43}) \right) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \\
& 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) c (\theta_{21} + \theta_{43}) (c \theta_{21} i c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{65} i) - \\
& s (\theta_{21} + \theta_{43}) \left(s \theta_{21} i \left(c \theta_{65} i \left(2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \beta_{54} i \right) + (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i \right) + \right. \\
& \quad \left. c \theta_{21} i \left(-c \theta_{65} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i \right) \right)
\end{aligned}$$

Coeficientes de la ecuación (2.171)

$$\begin{aligned}
\ddot{f}_{11} = & \left(\ddot{\theta}_{109} i c \theta_{109} i - \dot{\theta}_{109} i^2 s \theta_{109} i \right) (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i + c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& 2 \dot{\theta}_{109} i c \theta_{109} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} i c(\theta_{21} + \theta_{43}) c \theta_{65} i c \theta_{76} i - \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s(\theta_{21} + \theta_{43}) - \\
& \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s(\theta_{21} + \theta_{43}) s \theta_{65} i - \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s \theta_{76} i - \\
& c \beta_{54} i (c \theta_{21} (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\
& s \theta_{21} (c \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) - \\
& s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) - \\
& s \beta_{54} i (s \theta_{21} (c \theta_{76} i s \theta_{43} (\dot{\theta}_{76} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i) + c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + c \theta_{21} (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i) + \\
& s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + \\
& s \theta_{109} i (-c \theta_{76} i s(\theta_{21} + \theta_{43}) (c \theta_{21} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i)) + \\
& 2 (-\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i - \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i) ((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c \theta_{76} i - \dot{\theta}_{76} i s(\theta_{21} + \theta_{43}) s \theta_{76} i) + \\
& c \theta_{21} i c \theta_{65} i (-s(\theta_{21} + \theta_{43}) ((\dot{\theta}_{21} i^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& c(\theta_{21} + \theta_{43}) ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta_{76} i - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{76} i s \theta_{76} i)) - \\
& c \beta_{54} i (-c \theta_{76} i (c \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{43} i^2) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) + s \theta_{21} (\ddot{\theta}_{21} c \theta_{43} - 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{43})) s \theta_{65} i + \\
& c \theta_{76} i s \theta_{21} (-c \theta_{43} (2 \dot{\theta}_{43} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{43} s \theta_{65} i) + s \theta_{43} (-\ddot{\theta}_{65} i c \theta_{65} i + (\dot{\theta}_{43} i^2 + \dot{\theta}_{65} i^2) s \theta_{65} i)) - \\
& (s \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i) + c \theta_{21} (-\ddot{\theta}_{21} i c \theta_{21} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{21} i^2) s \theta_{21} i)) s \theta_{43} s \theta_{76} i + \\
& s \theta_{21} i (-s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{43} i^2) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) + c \theta_{21} (\ddot{\theta}_{21} c \theta_{43} - 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{43})) s \theta_{76} i + \\
& 2 (\dot{\theta}_{21} c \theta_{21} c \theta_{43} - \dot{\theta}_{43} s \theta_{21} s \theta_{43}) (\dot{\theta}_{76} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& 2 (\dot{\theta}_{21} i c \theta_{21} c \theta_{21} i - \dot{\theta}_{21} s \theta_{21} s \theta_{21} i) (\dot{\theta}_{76} i c \theta_{76} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{76} i) + \\
& 2 (\dot{\theta}_{65} i c \theta_{65} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{65} i) (-\dot{\theta}_{21} c \theta_{21} c \theta_{76} i + \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) - \\
& 2 (\dot{\theta}_{21} c \theta_{43} s \theta_{21} + \dot{\theta}_{43} c \theta_{21} s \theta_{43}) (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i - \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i) + \\
& s \theta_{43} s \theta_{65} i (s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} (-\ddot{\theta}_{21} c \theta_{76} i + 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i)) + \\
& c \theta_{21} c \theta_{43} (-s \theta_{65} i ((\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{65} i (\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i)) + \\
& c \theta_{43} s \theta_{21} (c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i) + s \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i)) + \\
& c \theta_{21} s \theta_{21} i (c \theta_{43} (2 \dot{\theta}_{43} \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{43} s \theta_{76} i) + s \theta_{43} (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{43} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i))) - \\
& s \beta_{54} i (c \theta_{21} (-s \theta_{43} (c \theta_{76} i (-2 \dot{\theta}_{21} \dot{\theta}_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i - \ddot{\theta}_{65} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} i^2 s \theta_{21} i s \theta_{65} i + \dot{\theta}_{21} i^2 s \theta_{21} i s \theta_{65} i + \\
& 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i + \dot{\theta}_{43} i^2 s \theta_{21} i s \theta_{65} i + \dot{\theta}_{65} i^2 s \theta_{21} i s \theta_{65} i + \dot{\theta}_{76} i^2 s \theta_{21} i s \theta_{65} i - \\
& c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} i s \theta_{65} i)) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i c \theta_{65} i s \theta_{21} i - 2 \dot{\theta}_{21} i \\
& \dot{\theta}_{76} i c \theta_{21} i s \theta_{65} i - \ddot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + c \theta_{43} (c \theta_{76} i (-\ddot{\theta}_{76} i + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i \\
& c \theta_{65} i s \theta_{21} i + 2 \dot{\theta}_{21} i (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} i s \theta_{65} i + \ddot{\theta}_{21} s \theta_{21} i s \theta_{65} i + \ddot{\theta}_{43} s \theta_{21} i s \theta_{65} i) + \\
& (\dot{\theta}_{21} i^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43} i^2 + \dot{\theta}_{76} i^2 - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) -
\end{aligned}$$

$$\begin{aligned}
& (-c \theta_{21} c \theta_{76} i (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) + \\
& s \theta_{21} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i + \\
& c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{21} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& c \theta_{21} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{21} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + c(\beta_{1413} i - \theta_{54}) s \theta_{43} \\
& (c \theta_{76} i s \theta_{21} (c \theta_{21} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i)) + \\
& 2 (\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i) (\dot{\theta}_{21} c \theta_{21} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) - \\
& c \theta_{21} i c \theta_{65} i (-s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} (\ddot{\theta}_{21} c \theta_{76} i - 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i)) + \\
& s \beta_{54} i (2 \dot{\theta}_{21} \dot{\theta}_{76} i c \theta_{76} i s \theta_{21} + c \theta_{76} i s \theta_{21} (c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} i s \theta_{65} i) + \\
& s \theta_{21} i (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \theta_{65} i)) + \\
& (\dot{\theta}_{21} i^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) s \theta_{76} i + c \theta_{21} (-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i) + \\
& 2 (\dot{\theta}_{65} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{65} i) (\dot{\theta}_{21} c \theta_{21} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) + \\
& s \theta_{21} i s \theta_{65} i (-s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} (\ddot{\theta}_{21} c \theta_{76} i - 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i))) + \\
& c \beta_{54} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{21} i c \theta_{21} i s \theta_{21} + \dot{\theta}_{21} c \theta_{21} s \theta_{21} i) + c \theta_{21} c \theta_{76} i (\ddot{\theta}_{65} i c \theta_{65} i - \dot{\theta}_{65} i^2 s \theta_{65} i) + \\
& (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i) + s \theta_{21} (\ddot{\theta}_{21} i c \theta_{21} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{21} i^2) s \theta_{21} i)) s \theta_{76} i + \\
& s \theta_{21} s \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76} i^2 s \theta_{76} i) - 2 \dot{\theta}_{65} i c \theta_{65} i (\dot{\theta}_{21} c \theta_{76} i s \theta_{21} + \dot{\theta}_{76} i c \theta_{21} s \theta_{76} i) - \\
& s \theta_{65} i (c \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + s \theta_{21} (\ddot{\theta}_{21} c \theta_{76} i - 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i))) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) ((\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21} i^2 s \theta_{21}) (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) - \\
& (\dot{\theta}_{21} i^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& 2 \dot{\theta}_{21} s \theta_{21} (c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \\
& \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i) + s \theta_{65} i (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - \dot{\theta}_{76} i s \beta_{54} i s \theta_{21} i s \theta_{76} i)) - \\
& 2 \dot{\theta}_{21} c \theta_{21} (\dot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + c \beta_{54} i (-\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i)) + \\
& s \theta_{21} (s \beta_{54} i (-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i) + \\
& c \beta_{54} i (-s \theta_{65} i ((\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{65} i (\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i))) + \\
& c \theta_{21} (c \beta_{54} i (-c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i) + s \theta_{21} i (-\ddot{\theta}_{76} i c \theta_{76} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i)) - \\
& c \theta_{21} i (c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& s \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i)) + \\
& s \theta_{21} i (s \theta_{65} i (c \theta_{76} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \beta_{54} i) + \ddot{\theta}_{76} i s \beta_{54} i s \theta_{76} i) - \\
& c \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i)))) - \\
& 2 \dot{\theta}_{1110} i c \theta_{1110} i (\dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& c(\beta_{1413} i - \theta_{54}) (-s \theta_{21} i (c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + \dot{\theta}_{76} i c \theta_{65} i s \theta_{76} i) + \\
& c \beta_{54} i (\dot{\theta}_{76} i c \theta_{21} i c \theta_{76} i - \dot{\theta}_{21} i s \theta_{21} i s \theta_{76} i) + \\
& c \theta_{21} i (c \theta_{65} i c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) - \dot{\theta}_{76} i s \beta_{54} i s \theta_{65} i s \theta_{76} i)) -
\end{aligned}$$

$$\begin{aligned}
& \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s \theta_{43} s \theta_{76} i + c \beta_{54} i (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i) + s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \\
& \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + s(\beta_{1413} i - \theta_{54}) \\
& ((-\dot{\theta}_{21}^2 c \theta_{21} - \ddot{\theta}_{21} s \theta_{21}) (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + \\
& s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + \\
& s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i)) - \\
& 2 \dot{\theta}_{21} s \theta_{21} (-c \theta_{76} i s \theta_{43} (\dot{\theta}_{43} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i)) - \\
& c \theta_{43} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i + \\
& c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) + \\
& c \beta_{54} i (-c \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) + 2 \dot{\theta}_{21} c \theta_{21} \\
& (-\dot{\theta}_{43} c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i + \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s \theta_{43} + \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s \theta_{43} s \theta_{65} i + \\
& \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s \theta_{43} s \theta_{76} i + c \beta_{54} i (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i) + s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \\
& \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + s \theta_{21} \\
& (c \theta_{76} i s \theta_{43} (c \theta_{21} i ((\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i))) + \\
& 2 (\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i) (\dot{\theta}_{43} c \theta_{43} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{43} s \theta_{76} i) - \\
& c \theta_{21} i c \theta_{65} i (-s \theta_{43} ((\dot{\theta}_{43}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{43} (\ddot{\theta}_{43} c \theta_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i s \theta_{76} i)) + \\
& s \beta_{54} i (2 \dot{\theta}_{43} \dot{\theta}_{76} i c \theta_{76} i s \theta_{43} + c \theta_{76} i s \theta_{21} i (c \theta_{43} (2 \dot{\theta}_{43} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{43} s \theta_{65} i) + \\
& s \theta_{43} (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{43}^2 + \dot{\theta}_{65}^2) s \theta_{65} i)) + (\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) s \theta_{76} i + \\
& c \theta_{43} (-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76}^2 s \theta_{76} i) + 2 (\dot{\theta}_{65} i c \theta_{65} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{65} i) \\
& (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} i s \theta_{76} i) + s \theta_{43} s \theta_{65} i (-s \theta_{21} i ((\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \\
& \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} i (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i))) + c \beta_{54} i \\
& (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{43} c \theta_{43} s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{43}) + c \theta_{43} c \theta_{76} i (\ddot{\theta}_{65} i c \theta_{65} i - \dot{\theta}_{65}^2 s \theta_{65} i) + \\
& (c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{43} c \theta_{43} + \ddot{\theta}_{21} i s \theta_{43}) + s \theta_{21} i (\ddot{\theta}_{43} c \theta_{43} - (\dot{\theta}_{21}^2 + \dot{\theta}_{43}^2) s \theta_{43})) s \theta_{76} i + \\
& s \theta_{21} i s \theta_{43} (\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76}^2 s \theta_{76} i) - 2 \dot{\theta}_{65} i c \theta_{65} i (\dot{\theta}_{43} c \theta_{76} i s \theta_{43} + \dot{\theta}_{76} i c \theta_{43} s \theta_{76} i) - \\
& s \theta_{65} i (c \theta_{43} ((\dot{\theta}_{43}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + s \theta_{43} (\ddot{\theta}_{43} c \theta_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i s \theta_{76} i))) + \\
& c \theta_{21} (-c \theta_{65} i c \theta_{76} i (c \theta_{21} i ((\dot{\theta}_{21}^2 + \dot{\theta}_{43}^2) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) + s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{43} - 2 \dot{\theta}_{21} i \dot{\theta}_{43} s \theta_{43})) + \\
& s \theta_{21} i (\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + \\
& 2 (\dot{\theta}_{21} i c \theta_{43} s \theta_{21} i + \dot{\theta}_{43} c \theta_{21} i s \theta_{43}) (\dot{\theta}_{65} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{76} i c \theta_{65} i s \theta_{76} i) + \\
& (\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43}) (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) - c \theta_{21} i c \theta_{43} \\
& (c \theta_{65} i ((\dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + s \theta_{65} i (\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i)) -
\end{aligned}$$

$$\begin{aligned}
& 2 \dot{\theta}_{43} c \theta_{43} (\dot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + c \beta_{54} i (-\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i)) + \\
& s \theta_{43} (s \beta_{54} i (-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i) + c \beta_{54} i (-s \theta_{65} i ((\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& c \theta_{65} i (\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i))) + \\
& 2 \dot{\theta}_{43} s \theta_{43} (\dot{\theta}_{21} i c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + s \theta_{21} i \\
& (\dot{\theta}_{76} i c \beta_{54} i c \theta_{76} i + s \beta_{54} i (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i - \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i))) - c \theta_{43} \\
& (c \beta_{54} i (c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i) + s \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i)) - \\
& s \beta_{54} i (-c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + s \theta_{65} i (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i)) + \\
& s \theta_{21} i (s \theta_{65} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& c \theta_{65} i (-\ddot{\theta}_{65} i c \theta_{76} i + 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i))))))
\end{aligned}$$

$$\begin{aligned}
\ddot{f}_{21} = & (\ddot{\theta}_{109} i c \theta_{109} i - \dot{\theta}_{109} i^2 s \theta_{109} i) (-s \theta_{43} \\
& (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) + \\
& c \theta_{43} (c \theta_{21} i (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} i (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + 2 \dot{\theta}_{109} i c \theta_{109} i (-\dot{\theta}_{43} c \theta_{43} \\
& (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& \dot{\theta}_{43} s \theta_{43} (c \theta_{21} i (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} i (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + \\
& c \theta_{43} (-\dot{\theta}_{21} s \theta_{21} i (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} i (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} i (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} i (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) - s \theta_{43} \\
& (-s \theta_{21} i (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - \\
& c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i + c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} i (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + \\
& c \theta_{21} i (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))) + (-\dot{\theta}_{109} i^2 c \theta_{109} i - \ddot{\theta}_{109} i s \theta_{109} i) \\
& (c \theta_{1110} i (s(\beta_{1413} i - \theta_{54}) (-c \beta_{54} i c \theta_{21} i c \theta_{76} i + (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (c \theta_{21} i (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} i (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \beta_{54} i c \theta_{21} i c \theta_{76} i - (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& s \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{21} i (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} i (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)))) + \\
& s \theta_{109} i ((-\ddot{\theta}_{43} c \theta_{43} + \dot{\theta}_{43}^2 s \theta_{43}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) -
\end{aligned}$$

184

[illegible]

$$\begin{aligned}
& 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) \left(\dot{\theta}_{43} c \theta_{76} i (c \theta_{76} i + c \beta_{54} i + c \beta_{54} i s \theta_{65} i) + \right. \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& s \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \\
& \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) + \\
& 2 \dot{\theta}_{43} s \theta_{43} (\dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c(\beta_{1413} i - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} i - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + \\
& c \theta_{21} (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))) - \\
& c \theta_{43} \left((-\dot{\theta}_{54}^2 c(\beta_{1413} i - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) + \right. \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i (\dot{\theta}_{21} c \theta_{21} i - \dot{\theta}_{21}^2 s \theta_{21} i) + c \beta_{54} i (-s \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i) + \\
& c \theta_{21} (\ddot{\theta}_{21} i c \theta_{21} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2) s \theta_{21} i))) + 2 \dot{\theta}_{76} i c \theta_{76} i \\
& (\dot{\theta}_{65} i c \beta_{54} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} c \beta_{54} i c \theta_{21} i s \theta_{65} i - c \theta_{21} (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \\
& c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) - \dot{\theta}_{21} s \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) - \\
& 2 \dot{\theta}_{76} i (\dot{\theta}_{21} c \theta_{21} s \beta_{54} i + c \beta_{54} i (\dot{\theta}_{21} i c \theta_{21} i c \theta_{21} i - \dot{\theta}_{21} s \theta_{21} s \theta_{21} i)) s \theta_{76} i + \\
& (c \beta_{54} i (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} s \theta_{65} i) + s \theta_{21} (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{65} i^2) s \theta_{65} i)) + s \theta_{21} \\
& (s \theta_{21} i (2 \dot{\theta}_{21} c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{21} s \beta_{54} i s \theta_{65} i) + c \theta_{21} i (-\ddot{\theta}_{21} c \theta_{65} i + 2 \dot{\theta}_{21} \\
& (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i)) - c \theta_{21} (c \theta_{21} i (c \theta_{65} i (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \\
& 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) + s \theta_{21} i (c \theta_{65} i \\
& (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \beta_{54} i) s \theta_{65} i))) - \\
& s \theta_{76} i - (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} i s \theta_{21} i) (\dot{\theta}_{76} i^2 c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& (c \beta_{54} i s \theta_{21} i s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) (\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76} i^2 s \theta_{76} i)) + \\
& 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} i - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + \\
& c \theta_{21} (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))) - \\
& c(\beta_{1413} i - \theta_{54}) \left((\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \right. \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& 2 \dot{\theta}_{43} c \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i +
\end{aligned}$$

$$\begin{aligned}
& \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) + \\
& s \theta_{43} \left(-(\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - 2 \dot{\theta}_{21} c \theta_{21} \right. \\
& \quad \left(\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i \right. \\
& \quad \left. s \theta_{76} i - (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) + \right. \\
& \quad \left. (-\ddot{\theta}_{21} c \theta_{21} + \dot{\theta}_{21}^2 s \theta_{21}) (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) - \right. \\
& \quad \left. 2 \dot{\theta}_{21} s \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i)) + \right. \\
& \quad \left. c \theta_{21} (-s \beta_{54} i (\dot{\theta}_{76}^2 c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \beta_{54} i (c \theta_{65} i (2 \dot{\theta}_{65} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{65} i s \theta_{76} i) + \right. \\
& \quad \left. s \theta_{65} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) s \theta_{76} i)) \right) - s \theta_{21} (c \beta_{54} i (-s \theta_{21} i \\
& \quad ((\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} i (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i)) + \\
& \quad c \theta_{21} i (-s \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i) + \\
& \quad c \theta_{65} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i)) + \\
& \quad s \theta_{21} i (-c \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i) + s \theta_{65} i \\
& \quad (-\ddot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) s \beta_{54} i) s \theta_{76} i)))) - \\
& s \theta_{1110} i \left((-\dot{\theta}_{54}^2 c(\beta_{1413} i - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) (c \beta_{54} i c \theta_{21} i c \theta_{76} i - \right. \\
& \quad (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) + \\
& \quad (\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) s(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& \quad (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& \quad 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (c \beta_{54} i (\dot{\theta}_{21} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i) + \\
& \quad c \theta_{65} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i) + \\
& \quad s \theta_{65} i (\dot{\theta}_{76} i c \theta_{21} i c \theta_{76} i s \beta_{54} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i)) + \\
& \quad (-\ddot{\theta}_{43} c \theta_{43} + \dot{\theta}_{43}^2 s \theta_{43}) s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& \quad s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + c(\beta_{1413} i - \theta_{54}) \\
& \quad (-c \beta_{54} i (c \theta_{21} i ((\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i)) + \\
& \quad s \theta_{21} i (s \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i) + \\
& \quad c \theta_{65} i (-\ddot{\theta}_{76} i c \theta_{76} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i)) + \\
& \quad c \theta_{21} i (-c \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i) + \\
& \quad s \theta_{65} i (-\ddot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) s \beta_{54} i) s \theta_{76} i)))) - \\
& 2 \dot{\theta}_{43} c \theta_{43} (-\dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& \quad s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + \\
& \quad s(\beta_{1413} i - \theta_{54}) (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \\
& \quad \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& \quad (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) -
\end{aligned}$$

$$\begin{aligned}
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}) + \\
& c \theta_{21} (-\dot{\theta}_{76 i} s \beta_{54 i} s \theta_{76 i} + c \beta_{54 i} (\dot{\theta}_{76 i} c \theta_{76 i} s \theta_{65 i} + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i})) - \\
c \theta_{43} & \left(-(\ddot{\theta}_{54} c(\beta_{1413 i} - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413 i} - \theta_{54})) (c \theta_{76 i} (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) + \right. \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}) + \\
& s(\beta_{1413 i} - \theta_{54}) (c \theta_{76 i} (s \beta_{54 i} (\dot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) + c \beta_{54 i} (-s \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21 i} c \theta_{21 i} + \ddot{\theta}_{21} s \theta_{21 i}) + \\
& c \theta_{21} (\ddot{\theta}_{21 i} c \theta_{21 i} - (\dot{\theta}_{21}^2 + \dot{\theta}_{21 i}^2) s \theta_{21 i}))) + 2 \dot{\theta}_{76 i} c \theta_{76 i} \\
& (\dot{\theta}_{65 i} c \beta_{54 i} c \theta_{65 i} s \theta_{21} + \dot{\theta}_{21} c \beta_{54 i} c \theta_{21} s \theta_{65 i} - c \theta_{21} (c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} + \\
& c \theta_{21 i} (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i}) - \dot{\theta}_{21} s \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) - \\
& 2 \dot{\theta}_{76 i} (\dot{\theta}_{21} c \theta_{21} s \beta_{54 i} + c \beta_{54 i} (\dot{\theta}_{21 i} c \theta_{21} c \theta_{21 i} - \dot{\theta}_{21} s \theta_{21} s \theta_{21 i})) s \theta_{76 i} + \\
& (c \beta_{54 i} (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{65 i} c \theta_{65 i} + \ddot{\theta}_{21} s \theta_{65 i}) + s \theta_{21} (\ddot{\theta}_{65 i} c \theta_{65 i} - (\dot{\theta}_{21}^2 + \dot{\theta}_{65 i}^2) s \theta_{65 i})) + s \theta_{21} \\
& (s \theta_{21 i} (2 \dot{\theta}_{21} c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) + \ddot{\theta}_{21} s \beta_{54 i} s \theta_{65 i}) + c \theta_{21 i} (-\ddot{\theta}_{21} c \theta_{65 i} + 2 \dot{\theta}_{21} \\
& (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i})) - c \theta_{21} (c \theta_{21 i} (c \theta_{65 i} (\dot{\theta}_{21}^2 + \dot{\theta}_{21 i}^2 + \dot{\theta}_{65 i}^2 + \\
& 2 \dot{\theta}_{21 i} \dot{\theta}_{65 i} s \beta_{54 i}) + (\ddot{\theta}_{65 i} + \ddot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i}) + s \theta_{21 i} (c \theta_{65 i} \\
& (\ddot{\theta}_{21 i} + \ddot{\theta}_{65 i} s \beta_{54 i}) - (2 \dot{\theta}_{21 i} \dot{\theta}_{65 i} + (\dot{\theta}_{21}^2 + \dot{\theta}_{21 i}^2 + \dot{\theta}_{65 i}^2) s \beta_{54 i}) s \theta_{65 i})) \\
& s \theta_{76 i} - (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) (\dot{\theta}_{76 i}^2 c \theta_{76 i} + \ddot{\theta}_{76 i} s \theta_{76 i}) + \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) (\ddot{\theta}_{76 i} c \theta_{76 i} - \dot{\theta}_{76 i}^2 s \theta_{76 i})) - \\
2 \dot{\theta}_{54} & c(\beta_{1413 i} - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21 i} c \theta_{65 i} + s \beta_{54 i} (\dot{\theta}_{76 i} - \dot{\theta}_{21} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i} + \\
& c \theta_{21} (c \theta_{76 i} s \beta_{54 i} (\dot{\theta}_{21} - \dot{\theta}_{76 i} s \theta_{21 i} s \theta_{65 i}) - c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} s \theta_{76 i} + \\
& c \theta_{21 i} (\dot{\theta}_{76 i} c \theta_{65 i} c \theta_{76 i} - (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i} s \theta_{76 i})) + \\
& c \beta_{54 i} (s \theta_{21} (c \theta_{76 i} (-\dot{\theta}_{21} s \theta_{21 i} + \dot{\theta}_{76 i} s \theta_{65 i}) + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}) + \\
& c \theta_{21} (\dot{\theta}_{21 i} c \theta_{21 i} c \theta_{76 i} + (-\dot{\theta}_{76 i} s \theta_{21 i} + \dot{\theta}_{21} s \theta_{65 i}) s \theta_{76 i}))) + \\
2 \dot{\theta}_{43} & s \theta_{43} (-\dot{\theta}_{54} c(\beta_{1413 i} - \theta_{54}) (c \theta_{76 i} (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) + \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}) + \\
& s(\beta_{1413 i} - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21 i} c \theta_{65 i} + s \beta_{54 i} (\dot{\theta}_{76 i} - \dot{\theta}_{21} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i} + \\
& c \theta_{21} (c \theta_{76 i} s \beta_{54 i} (\dot{\theta}_{21} - \dot{\theta}_{76 i} s \theta_{21 i} s \theta_{65 i}) - c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} s \theta_{76 i} + \\
& c \theta_{21 i} (\dot{\theta}_{76 i} c \theta_{65 i} c \theta_{76 i} - (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i} s \theta_{76 i})) + \\
& c \beta_{54 i} (s \theta_{21} (c \theta_{76 i} (-\dot{\theta}_{21} s \theta_{21 i} + \dot{\theta}_{76 i} s \theta_{65 i}) + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}) + \\
& c \theta_{21} (\dot{\theta}_{21 i} c \theta_{21 i} c \theta_{76 i} + (-\dot{\theta}_{76 i} s \theta_{21 i} + \dot{\theta}_{21} s \theta_{65 i}) s \theta_{76 i}))) - \\
s \theta_{43} & \left(-(\ddot{\theta}_{54} c(\beta_{1413 i} - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413 i} - \theta_{54})) (c \theta_{21} (c \theta_{76 i} s \beta_{54 i} + c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}) - \right. \\
& s \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i})) - \\
2 \dot{\theta}_{54} & c(\beta_{1413 i} - \theta_{54}) (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76 i} s \beta_{54 i} + c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}) - s \theta_{21} \\
& (\dot{\theta}_{21 i} c \beta_{54 i} c \theta_{21 i} c \theta_{76 i} + \dot{\theta}_{76 i} c \theta_{76 i} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) - \dot{\theta}_{76 i} c \beta_{54 i} s \theta_{21 i} \\
& s \theta_{76 i} - (c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} + c \theta_{21 i} (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i}) s \theta_{76 i}) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}) + \\
& c \theta_{21} (-\dot{\theta}_{76 i} s \beta_{54 i} s \theta_{76 i} + c \beta_{54 i} (\dot{\theta}_{76 i} c \theta_{76 i} s \theta_{65 i} + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}))) +
\end{aligned}$$

$$\begin{aligned}
& s(\beta_{1413 \text{ i}} - \theta_{54}) \left(-\left(\dot{\theta}_{21 \text{ i}}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21} \right) (c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} + c \beta_{54 \text{ i}} s \theta_{65 \text{ i}} s \theta_{76 \text{ i}}) - 2 \dot{\theta}_{21} c \theta_{21} \right. \\
& \quad \left(\dot{\theta}_{21 \text{ i}} c \beta_{54 \text{ i}} c \theta_{21 \text{ i}} c \theta_{76 \text{ i}} + \dot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}}) - \dot{\theta}_{76 \text{ i}} c \beta_{54 \text{ i}} s \theta_{21 \text{ i}} \right. \\
& \quad \left. s \theta_{76 \text{ i}} - (c \theta_{65 \text{ i}} (\dot{\theta}_{21 \text{ i}} + \dot{\theta}_{65 \text{ i}} s \beta_{54 \text{ i}}) s \theta_{21 \text{ i}} + c \theta_{21 \text{ i}} (\dot{\theta}_{65 \text{ i}} + \dot{\theta}_{21 \text{ i}} s \beta_{54 \text{ i}}) s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}} \right) + \\
& \quad \left(-\ddot{\theta}_{21} c \theta_{21} + \dot{\theta}_{21}^2 s \theta_{21} \right) (c \beta_{54 \text{ i}} c \theta_{76 \text{ i}} s \theta_{21 \text{ i}} + (c \theta_{21 \text{ i}} c \theta_{65 \text{ i}} - s \beta_{54 \text{ i}} s \theta_{21 \text{ i}} s \theta_{65 \text{ i}}) s \theta_{76 \text{ i}}) - \\
& \quad 2 \dot{\theta}_{21} s \theta_{21} (-\dot{\theta}_{76 \text{ i}} s \beta_{54 \text{ i}} s \theta_{76 \text{ i}} + c \beta_{54 \text{ i}} (\dot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} s \theta_{65 \text{ i}} + \dot{\theta}_{65 \text{ i}} c \theta_{65 \text{ i}} s \theta_{76 \text{ i}})) + \\
& \quad c \theta_{21} \left(-s \beta_{54 \text{ i}} (\dot{\theta}_{76 \text{ i}}^2 c \theta_{76 \text{ i}} + \ddot{\theta}_{76 \text{ i}} s \theta_{76 \text{ i}}) + c \beta_{54 \text{ i}} (c \theta_{65 \text{ i}} (2 \dot{\theta}_{65 \text{ i}} \dot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} + \ddot{\theta}_{65 \text{ i}} s \theta_{76 \text{ i}}) + \right. \\
& \quad \left. s \theta_{65 \text{ i}} (\ddot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} - (\dot{\theta}_{65 \text{ i}}^2 + \dot{\theta}_{76 \text{ i}}^2) s \theta_{76 \text{ i}})) \right) - s \theta_{21} \left(c \beta_{54 \text{ i}} (-s \theta_{21 \text{ i}} \right. \\
& \quad \left. ((\dot{\theta}_{21 \text{ i}}^2 + \dot{\theta}_{76 \text{ i}}^2) c \theta_{76 \text{ i}} + \ddot{\theta}_{76 \text{ i}} s \theta_{76 \text{ i}}) + c \theta_{21 \text{ i}} (\ddot{\theta}_{21 \text{ i}} c \theta_{76 \text{ i}} - 2 \dot{\theta}_{21 \text{ i}} \dot{\theta}_{76 \text{ i}} s \theta_{76 \text{ i}})) \right) + \\
& \quad c \theta_{21 \text{ i}} \left(-s \theta_{65 \text{ i}} (2 \dot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} (\dot{\theta}_{65 \text{ i}} + \dot{\theta}_{21 \text{ i}} s \beta_{54 \text{ i}}) + (\ddot{\theta}_{65 \text{ i}} + \ddot{\theta}_{21 \text{ i}} s \beta_{54 \text{ i}}) s \theta_{76 \text{ i}}) + \right. \\
& \quad \left. c \theta_{65 \text{ i}} (\ddot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} - (\dot{\theta}_{21 \text{ i}}^2 + \dot{\theta}_{65 \text{ i}}^2 + \dot{\theta}_{76 \text{ i}}^2 + 2 \dot{\theta}_{21 \text{ i}} \dot{\theta}_{65 \text{ i}} s \beta_{54 \text{ i}}) s \theta_{76 \text{ i}}) \right) + \\
& \quad s \theta_{21 \text{ i}} \left(-c \theta_{65 \text{ i}} (2 \dot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} (\dot{\theta}_{21 \text{ i}} + \dot{\theta}_{65 \text{ i}} s \beta_{54 \text{ i}}) + (\ddot{\theta}_{21 \text{ i}} + \ddot{\theta}_{65 \text{ i}} s \beta_{54 \text{ i}}) s \theta_{76 \text{ i}}) + s \theta_{65 \text{ i}} \right. \\
& \quad \left. (-\ddot{\theta}_{76 \text{ i}} c \theta_{76 \text{ i}} s \beta_{54 \text{ i}} + (2 \dot{\theta}_{21 \text{ i}} \dot{\theta}_{65 \text{ i}} + (\dot{\theta}_{21 \text{ i}}^2 + \dot{\theta}_{65 \text{ i}}^2 + \dot{\theta}_{76 \text{ i}}^2) s \beta_{54 \text{ i}}) s \theta_{76 \text{ i}})) \right) \Big)
\end{aligned}$$

Apéndice D

Coeficientes de la ecuación (2.51)

$$V_1 = -2 x_{32} (x - x_{1918} c \phi c \psi) s \theta_{21} - 2 x_{32} c \theta_{21} (z - z_{10} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi)$$

$$V_2 = 2 (x - x_{32} c \theta_{21} - x_{1918} c \phi c \psi)$$

$$V_3 = 2 (y - x_{1918} c \theta s \phi - x_{1918} c \phi s \theta s \psi)$$

$$V_4 = 2 (z - z_{10} + x_{32} s \theta_{21} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi)$$

$$V_5 = 2 x_{1918} c \phi (-y c \psi s \theta + c \theta c \psi (z - z_{10} + x_{32} s \theta_{21}) + (x - x_{32} c \theta_{21}) s \psi)$$

$$V_6 = -2 x_{1918} (c \theta ((z - z_{10} + x_{32} s \theta_{21}) s \phi + y c \phi s \psi) + s \theta (-y s \phi + c \phi (z - z_{10} + x_{32} s \theta_{21}) s \psi))$$

$$V_7 = 2 x_{1918} (-c \phi s \theta (z - z_{10} + x_{32} s \theta_{21}) + s \phi ((x - x_{32} c \theta_{21}) c \psi + y s \theta s \psi) - c \theta (y c \phi + (z - z_{10} + x_{32} s \theta_{21}) s \phi s \psi))$$

Coeficientes de la ecuación (2.54)

$$V_8 = c \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) - s \theta_{43} ((x - x_{1918} c \phi c \psi) s \theta_{21} + c \theta_{21} (z - z_{10} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi))$$

$$V_9 = -s \theta_{43}$$

$$V_{10} = -c \theta_{43}$$

$$V_{11} = -x_{1918} c \phi (c \theta c \theta_{43} c \psi + s \theta_{43} s \psi)$$

$$V_{12} = x_{1918} c \theta_{43} (c \theta s \phi + c \phi s \theta s \psi)$$

$$V_{13} = x_{1918} (-c \psi s \theta_{43} s \phi + c \theta_{43} (c \phi s \theta + c \theta s \phi s \psi))$$

$$V_{14} = -x_{32} c(\theta_{21} - \theta_{43})$$

Coeficientes de la ecuación (2.58)

$$V_{15} = (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2$$

$$V_{16} = -c \theta_{43} c \theta_{54}^2 (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi))$$

$$V_{17} = c \theta_{43} c \theta_{54}^2 (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))$$

$$V_{18} = x_{1918} c \theta_{43} c \theta_{54}^2 c \phi (s \psi (-y + x_{1918} c \theta s \phi + x_{1918} c \phi s \theta s \psi) - c \psi s \theta (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)))$$

$$V_{19} = x_{1918} c \theta_{43} c \theta_{54}^2 (s \theta s \phi - c \theta c \phi s \psi) (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))$$

$$V_{20} = x_{1918} c \theta_{43} c \theta_{54}^2 (c \psi s \phi (-y + x_{1918} c \theta s \phi + x_{1918} c \phi s \theta s \psi) - \\ c \theta c \phi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) + \\ s \theta s \phi s \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)))$$

$$V_{21} = -x_{32} c \theta_{43} c \theta_{54}^2 s \theta_{21} (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi))$$

$$V_{22} = \\ c \theta_{54}^2 s \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi))$$

Coeficientes de la ecuación (2.63)

$$V_{23} = (s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi)))^2$$

$$V_{24} = \\ c \theta_{1817}^2 (c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi (c \theta_{54} s \theta s (\theta_{21} + \theta_{43}) - c \theta s \theta_{54}) + c (\theta_{21} + \theta_{43})^2 c \theta_{54}^2 s \phi + (c \theta c \theta_{54} s (\theta_{21} + \theta_{43}) + s \theta s \theta_{54}) \\ (s \theta (s \theta_{54} s \phi + c \theta_{54} c \phi s (\theta_{21} + \theta_{43}) s \psi) + c \theta (c \theta_{43} c \theta_{54} s \theta_{21} s \phi + c \theta_{21} c \theta_{54} s \theta_{43} s \phi - c \phi s \theta_{54} s \psi)))$$

$$V_{25} = c \theta_{1817}^2 (c \theta^2 c \phi c \psi (c \theta_{54}^2 s (\theta_{21} + \theta_{43})^2 + s \theta_{54}^2) - c \theta c (\theta_{21} + \theta_{43}) c \theta_{54} (s \theta_{54} s \phi + c \theta_{54} c \phi s (\theta_{21} + \theta_{43}) s \psi) + \\ s \theta (c \phi c \psi s \theta s \theta_{54}^2 + c \theta_{54}^2 s (\theta_{21} + \theta_{43}) (c \phi c \psi s \theta s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \phi) - c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi s \theta_{54} s \psi)))$$

$$V_{26} = c \theta_{1817}^2 (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s (\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi))$$

$$V_{27} = -c \theta_{1817}^2 c \theta_{54} ((c \theta c \theta_{54} + c \theta_{43} s \theta s \theta_{21} s \theta_{54}) s \phi + \\ c \phi (c \psi s \theta_{21} s \theta_{43} s \theta_{54} + c \theta_{54} s \theta s \psi - c \theta c \theta_{43} s \theta_{21} s \theta_{54} s \psi) - c \theta_{21} s \theta_{54} (c \theta_{43} c \phi c \psi + s \theta_{43} (-s \theta s \phi + c \theta c \phi s \psi)))$$

$$V_{28} = -c \theta_{1817}^2 (c \theta_{43} (c \theta_{21} s \theta s \phi + c \phi (c \psi s \theta_{21} - c \theta c \theta_{21} s \psi)) + s \theta_{43} (c \theta_{21} c \phi c \psi + s \theta_{21} (-s \theta s \phi + c \theta c \phi s \psi)))$$

Coeficientes de la ecuación (2.68)

$$V_{29} = (c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s (\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi))^2$$

$$V_{30} = c \theta_{1716}^2 \csc(\theta_{1817}) \\ (c (\theta_{21} + \theta_{43})^2 c \theta_{54}^2 c \phi + c (\theta_{21} + \theta_{43}) c \theta_{54} c \psi (-c \theta_{54} s \theta s (\theta_{21} + \theta_{43}) + c \theta s \theta_{54}) s \phi + (c \theta c \theta_{54} s (\theta_{21} + \theta_{43}) + s \theta s \theta_{54}) \\ (s \theta (c \phi s \theta_{54} - c \theta_{54} s (\theta_{21} + \theta_{43}) s \phi s \psi) + c \theta (c \theta_{43} c \theta_{54} c \phi s \theta_{21} + c \theta_{21} c \theta_{54} c \phi s \theta_{43} + s \theta_{54} s \phi s \psi)))$$

$$V_{31} = -c \theta_{1716}^2 \csc(\theta_{1817}) \\ \left(-\frac{1}{8} (-6 + 2 c (2 (\theta_{21} + \theta_{43})) + c (2 (\theta_{21} + \theta_{43} - \theta_{54})) + 2 c (2 \theta_{54}) + c (2 (\theta_{21} + \theta_{43} + \theta_{54}))) c \psi s \phi - c (\theta_{21} + \theta_{43}) c \theta_{54} \right. \\ \left. (c \theta_{43} c \theta_{54} s \theta_{21} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{21} c \theta_{54} s \theta_{43} (c \phi s \theta + c \theta s \phi s \psi) + s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi)) \right)$$

$$V_{32} = -c \theta_{1716}^2 \csc(\theta_{1817}) (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi)$$

$$(s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi) + c \theta_{54} (c \phi s \theta s(\theta_{21} + \theta_{43}) + s \phi (c(\theta_{21} + \theta_{43}) c \psi + c \theta s(\theta_{21} + \theta_{43}) s \psi)))$$

$$V_{33} = -c \theta_{1716}^2 c \theta_{54} \csc(\theta_{1817}) (c \theta_{21} c \phi s \theta s \theta_{43} s \theta_{54} - c \psi s \theta_{21} s \theta_{43} s \theta_{54} s \phi + c \theta_{43} s \theta_{54} (c \phi s \theta s \theta_{21} + c \theta_{21} c \psi s \phi) - c \theta_{54} s \theta s \phi s \psi + c \theta (c \theta_{54} c \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi))$$

$$V_{34} = c \theta_{1716}^2 \csc(\theta_{1817}) (s \theta_{21} (c \phi s \theta s \theta_{43} + s \phi (c \theta_{43} c \psi + c \theta s \theta_{43} s \psi)) + c \theta_{21} (c \psi s \theta_{43} s \phi - c \theta_{43} (c \phi s \theta + c \theta s \phi s \psi)))$$

$$V_{35} = c \theta_{1716}^2 \cot(\theta_{1817}) \csc(\theta_{1817}) (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi))$$

Coeficientes de la ecuación (2.72)

$$V_{36} = (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) + c \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))))^2$$

$$V_{37} = c \theta_{1615}^2 (c \theta^2 c(\theta_{21} + \theta_{43}) c \theta_{54} (c \theta_{1716} c \theta_{1817} c \phi + s \theta_{1716} s \phi) (-c \phi c \psi s \theta_{1716} + c \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi)) + c \theta (c \phi s \theta_{1716}^2 (c \phi s \theta_{54} - c \theta_{54} s(\theta_{21} + \theta_{43}) s \phi s \psi) - c \theta_{1716} s \theta_{1716} (c \theta_{1817} s \theta_{54} s(2 \phi) + c \theta_{43} c \theta_{54} s \theta_{21} (c \psi s \theta_{1817} s \phi + c \theta_{1817} c(2 \phi) s \psi) + c \theta_{1716}^2 (-c \theta_{1817} c \theta_{54} c \phi c \psi s \theta_{1817} s(\theta_{21} + \theta_{43}) + s \theta_{1817}^2 s \theta_{54} + c \theta_{1817}^2 s \phi (s \theta_{54} s \phi + c \theta_{54} c \phi s(\theta_{21} + \theta_{43}) s \psi))) + s \theta (-c \phi s \theta_{1716}^2 (c \theta_{21} c \theta_{54} c \phi s \theta_{43} + c \theta_{43} c \theta_{54} (c \phi s \theta_{21} + c \theta_{21} c \psi s \theta s \phi) + s \phi (-c \theta_{54} c \psi s \theta s \theta_{21} s \theta_{43} + s \theta_{54} s \psi)) + c \theta_{1716} s \theta_{1716} (s \theta_{1817} s \phi (-c \psi s \theta_{54} + c(\theta_{21} + \theta_{43}) c \theta_{54} s \theta s \psi) + c \theta_{1817} (c \theta_{54} s \theta_{21} (c \phi^2 c \psi s \theta s \theta_{43} - c \psi s \theta s \theta_{43} s \phi^2 + c \theta_{43} s(2 \phi)) + c \theta_{21} c \theta_{54} (-c \theta_{43} c(2 \phi) c \psi s \theta + s \theta_{43} s(2 \phi)) - c(2 \phi) s \theta_{54} s \psi)) - c \theta_{1716}^2 (c \theta_{1817} c \phi s \theta_{54} (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi) + c \theta_{54} (-c \theta_{1817}^2 c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta s \phi + c \psi^2 s(\theta_{21} + \theta_{43}) (s \theta_{1817}^2 + c \theta_{1817}^2 s \phi^2) + s \psi (-c \theta_{1817} c(\theta_{21} + \theta_{43}) c \phi s \theta s \theta_{1817} + s \theta_{1817}^2 s(\theta_{21} + \theta_{43}) s \psi + c \theta_{1817}^2 s(\theta_{21} + \theta_{43}) s \phi^2 s \psi))))))$$

$$V_{38} = c \theta_{1615}^2 (c \theta (-c \phi c \psi s \theta_{1716} + c \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi)) (s \theta_{1716} (s \theta_{54} s \phi + c \theta_{54} c \phi s(\theta_{21} + \theta_{43}) s \psi) + c \theta_{1716} (c \theta_{1817} c \phi s \theta_{54} + c \theta_{54} s(\theta_{21} + \theta_{43}) (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi))) + c \theta^2 c(\theta_{21} + \theta_{43}) c \theta_{54} (2 c \theta_{1716} c \phi c \psi s \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi) + s \theta_{1716}^2 (s \phi^2 + c \phi^2 s \psi^2) + c \theta_{1716}^2 (c \psi^2 s \theta_{1817}^2 - 2 c \theta_{1817} c \psi s \theta_{1817} s \phi s \psi + c \theta_{1817}^2 (c \phi^2 + s \phi^2 s \psi^2))) + s \theta (s \theta_{1716}^2 (-c \phi^2 c \psi s \theta_{54} s \psi + c \theta_{54} (c \phi c \psi s(\theta_{21} + \theta_{43}) s \phi + c(\theta_{21} + \theta_{43}) s \theta s \phi^2 + c(\theta_{21} + \theta_{43}) c \phi^2 s \theta s \psi^2)) + c \theta_{1716} s \theta_{1716} (c \theta_{1817} c \psi (c \theta_{54} (c \phi^2 s(\theta_{21} + \theta_{43}) - s(\theta_{21} + \theta_{43}) s \phi^2 + c(\theta_{21} + \theta_{43}) c \psi s \theta s(2 \phi)) + s \theta_{54} s(2 \phi) s \psi) - s \theta_{1817} (c \theta_{54} s(\theta_{21} + \theta_{43}) s \phi s \psi + c \phi (c \psi^2 s \theta_{54} - 2 c \theta_{21} c \theta_{43} c \theta_{54} c \psi s \theta s \psi - s \theta_{54} s \psi^2 + c \theta_{54} s \theta s \theta_{21} s \theta_{43} s(2 \psi)))) - c \theta_{1716}^2 (-c \psi s \theta_{1817}^2 (c \theta_{21} c \theta_{43} c \theta_{54} c \psi s \theta - c \theta_{54} c \psi s \theta s \theta_{21} s \theta_{43} + s \theta_{54} s \psi) + c \theta_{1817}^2 (c(\theta_{21} + \theta_{43})^2 c \psi s \theta_{54} s \phi^2 s \psi + c \psi s(\theta_{21} + \theta_{43}) s \phi (c \theta_{54} c \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi) - c(\theta_{21} + \theta_{43}) c \theta_{54} s \theta (c \phi^2 + s \phi^2 s \psi^2)) + c \theta_{1817} s \theta_{1817} (-c \psi^2 s \theta_{54} s \phi + c \theta_{21} c \theta_{54} c \phi s \theta_{43} s \psi + s \theta_{54} s \phi s \psi^2 - c \theta_{54} s \theta s \theta_{21} s \theta_{43} s \phi s(2 \psi) + c \theta_{43} c \theta_{54} (c \phi s \theta_{21} s \psi + c \theta_{21} s \theta s \phi s(2 \psi))))))$$

$$V_{39} = c \theta_{1615}^2 (c \theta^2 c(\theta_{21} + \theta_{43}) c \theta_{54} (c \theta_{1716} c \phi c \psi s \theta_{1716} s \theta_{1817} + s \theta_{1716}^2 s \psi + c \theta_{1716}^2 c \theta_{1817} (-c \psi s \theta_{1817} s \phi + c \theta_{1817} s \psi)) +$$

$$\begin{aligned}
& c\theta(-c\theta_{54}c\psi s\theta_{1716}^2 s(\theta_{21}+\theta_{43})+c\theta_{1716}s\theta_{1716}s\theta_{1817}(s\theta_{54}s\phi+c\theta_{54}c\phi s(\theta_{21}+\theta_{43})s\psi)- \\
& c\theta_{1716}^2 c\theta_{1817}(c\theta_{1817}c\theta_{54}c\psi s(\theta_{21}+\theta_{43})+s\theta_{1817}(-c\phi s\theta_{54}+c\theta_{54}s(\theta_{21}+\theta_{43})s\phi s\psi)))- \\
& s\theta(s\theta_{1716}^2(c\psi s\theta_{54}-c(\theta_{21}+\theta_{43})c\theta_{54}s\theta s\psi)+c\theta_{1716}s\theta_{1716}s\theta_{1817} \\
& (c\theta_{54}s\theta_{21}(c\phi c\psi s\theta s\theta_{43}+c\theta_{43}s\phi)+c\theta_{21}c\theta_{54}(-c\theta_{43}c\phi c\psi s\theta+s\theta_{43}s\phi)-c\phi s\theta_{54}s\psi)+c\theta_{1716}^2 \\
& c\theta_{1817}(c\theta_{21}c\theta_{54}c\phi s\theta_{1817}s\theta_{43}+c\theta_{1817}c\psi s\theta_{54}-c\theta_{54}c\psi s\theta s\theta_{1817}s\theta_{21}s\theta_{43}s\phi+c\theta_{1817}c\theta_{54}s\theta s\theta_{21} \\
& s\theta_{43}s\psi+s\theta_{1817}s\theta_{54}s\phi s\psi+c\theta_{43}c\theta_{54}(c\phi s\theta_{1817}s\theta_{21}+c\theta_{21}s\theta(c\psi s\theta_{1817}s\phi-c\theta_{1817}s\psi))))))
\end{aligned}$$

$$\begin{aligned}
V_{40} = & -c\theta_{1615}^2 \left(s\theta_{1716}^2 (s\theta^2 s\theta_{54}s\phi^2 + c\theta_{54}c\phi s\theta^2 s(\theta_{21}+\theta_{43})s\phi s\psi + c\phi^2 c\psi(c\psi s\theta_{54}-c(\theta_{21}+\theta_{43})c\theta_{54}s\theta s\psi)) + \right. \\
& c\theta^2(c\phi s\theta_{1716}s\psi + c\theta_{1716}(c\psi s\theta_{1817}-c\theta_{1817}s\phi s\psi)) \\
& (-s\theta_{1716}(c\theta_{43}c\theta_{54}s\theta_{21}s\phi + c\theta_{21}c\theta_{54}s\theta_{43}s\phi - c\phi s\theta_{54}s\psi) - \\
& c\theta_{1716}(-c\psi s\theta_{1817}s\theta_{54} + c\theta_{1817}(c\theta_{43}c\theta_{54}c\phi s\theta_{21} + c\theta_{21}c\theta_{54}c\phi s\theta_{43} + s\theta_{54}s\phi s\psi))) + \\
& c\theta_{1716}^2 (s\theta_{1817}^2 s\psi(c\theta_{21}c\theta_{43}c\theta_{54}c\psi s\theta - c\theta_{54}c\psi s\theta s\theta_{21}s\theta_{43} + s\theta_{54}s\psi) + \\
& c\theta_{1817}^2 (c\phi^2 s\theta^2 s\theta_{54} - c\theta_{54}c\phi s\theta^2 s(\theta_{21}+\theta_{43})s\phi s\psi + c\psi s\phi^2(c\psi s\theta_{54}-c(\theta_{21}+\theta_{43})c\theta_{54}s\theta s\psi)) + \\
& c\theta_{1817}s\theta_{1817}(c\theta_{54}s\theta(c\phi c\psi s\theta s(\theta_{21}+\theta_{43}) + c(\theta_{21}+\theta_{43})c(2\psi)s\phi) + s\theta_{54}s\phi s(2\psi))) + \\
& c\theta_{1716}s\theta_{1716} \left(-c(\theta_{21}+\theta_{43})^2 s\theta_{54} \left(\frac{1}{2} c\theta_{1817}(c(2\theta) + c(2\psi))s(2\phi) + c\phi s\theta_{1817}s(2\psi) \right) + c(\theta_{21}+\theta_{43})c\theta_{54}c\phi s\theta \right. \\
& \left. (-c\psi^2 s\theta_{1817} + s\theta_{1817}s\psi^2 + c\theta_{1817}s\phi s(2\psi)) + s(\theta_{21}+\theta_{43}) \left(c\theta_{54}s\theta^2(c\psi s\theta_{1817}s\phi + c\theta_{1817}c(2\phi)s\psi) - \right. \right. \\
& \left. \left. s(\theta_{21}+\theta_{43})s\theta_{54} \left(\frac{1}{2} c\theta_{1817}(c(2\theta) + c(2\psi))s(2\phi) + c\phi s\theta_{1817}s(2\psi) \right) \right) \right) \Bigg) + \\
& c\theta(-s\theta_{1716}(c\theta_{54}s\theta_{1716}s\theta_{21}(-c\phi c\psi s\theta_{43}s\phi - c\theta_{43}s\theta s\phi^2 + c\theta_{43}c\phi^2 s\theta s\psi^2) + \\
& s\theta s\theta_{54}(s\theta_{1716}s(2\phi)s\psi + 2c\theta_{1716}(c\psi s\theta_{1817}s\phi + c\theta_{1817}c(2\phi)s\psi)) + \\
& c\theta_{21}c\theta_{54}s\theta_{1716}(c\theta_{43}c\phi c\psi s\phi + s\theta s\theta_{43}(-s\phi^2 + c\phi^2 s\psi^2))) + \\
& c\theta_{1716}^2 (-c\psi s\theta s\theta_{1817}(c\theta_{43}c\theta_{54}c\psi s\theta_{1817}s\theta_{21} + c\theta_{21}c\theta_{54}c\psi s\theta_{1817}s\theta_{43} + 2c\theta_{1817}c\phi s\theta_{54}) + \\
& c\theta_{1817}c\theta_{54}s\theta_{1817}(c(\theta_{21}+\theta_{43})c\phi + 2c\psi s\theta s(\theta_{21}+\theta_{43})s\phi)s\psi + c\theta_{1817}^2 \\
& (s\theta s\theta_{54}s(2\phi)s\psi + c\theta_{54}(c\phi^2 s\theta s(\theta_{21}+\theta_{43}) + c(\theta_{21}+\theta_{43})c\phi c\psi s\phi - s\theta s(\theta_{21}+\theta_{43})s\phi^2 s\psi^2))) + \\
& c\theta_{1716}c\theta_{54}s\theta_{1716}(c\theta_{1817}(-c(\theta_{21}+\theta_{43})c(2\phi)c\psi + s\theta s(\theta_{21}+\theta_{43})s(2\phi)(1+s\psi^2)) + \\
& s\theta_{1817}(c(\theta_{21}+\theta_{43})s\phi s\psi - c\phi s\theta s(\theta_{21}+\theta_{43})s(2\psi)))) \Bigg)
\end{aligned}$$

$$\begin{aligned}
V_{41} = & -c\theta_{1615}^2 (s\theta_{1716}(c\phi c\psi s(\theta_{21}+\theta_{43}) + c(\theta_{21}+\theta_{43})(s\theta s\phi - c\theta c\phi s\psi)) + c\theta_{1716} \\
& (-s\theta_{1817}(c\theta c(\theta_{21}+\theta_{43})c\psi + s(\theta_{21}+\theta_{43})s\psi) + c\theta_{1817}(-c\psi s(\theta_{21}+\theta_{43})s\phi + c(\theta_{21}+\theta_{43})(c\phi s\theta + c\theta s\phi s\psi)))) \\
& (s\theta_{1716}(c\theta_{54}s(\theta_{21}+\theta_{43})(s\theta s\phi - c\theta c\phi s\psi) - s\theta_{54}(c\theta s\phi + c\phi s\theta s\psi)) + c(\theta_{21}+\theta_{43})c\theta_{54} \\
& (-c\phi c\psi s\theta_{1716} + c\theta_{1716}(c\theta_{1817}c\psi s\phi + s\theta_{1817}s\psi)) + c\theta_{1716}(-c\psi s\theta_{1817}(c\theta c\theta_{54}s(\theta_{21}+\theta_{43}) + s\theta s\theta_{54}) + \\
& c\theta_{1817}(c\theta_{54}s(\theta_{21}+\theta_{43})(c\phi s\theta + c\theta s\phi s\psi) + s\theta_{54}(-c\theta c\phi + s\theta s\phi s\psi))))
\end{aligned}$$

$$\begin{aligned}
V_{42} = & -\frac{1}{2} c\theta_{1615}^2 \\
& (-c\theta_{54}c\phi c\psi s\theta_{21}s\theta_{43} + c\theta_{54}c\phi c\psi s\theta_{1716}^2 s\theta_{21}s\theta_{43} - c\theta_{43}c\theta_{54}s\theta s\theta_{21}s\phi + c\theta_{43}c\theta_{54}s\theta s\theta_{1716}^2 s\theta_{21}s\phi + \\
& c\theta s\theta_{54}s\phi - c\theta s\theta_{1716}^2 s\theta_{54}s\phi + c\theta_{21}c\theta_{43}c\theta_{54}s(2\theta_{1716})s\theta_{1817}s\psi + c\theta c\theta_{43}c\theta_{54}c\phi s\theta_{21}s\psi - \\
& c\theta c\theta_{43}c\theta_{54}c\phi s\theta_{1716}^2 s\theta_{21}s\psi + c\phi s\theta s\theta_{54}s\psi - c\phi s\theta s\theta_{1716}^2 s\theta_{54}s\psi - \\
& 2c\theta_{1716}s\theta_{1716}(c\theta(c\theta_{43}c\theta_{54}c\psi s\theta_{1817}s\theta_{21} + c\theta_{21}c\theta_{54}c\psi s\theta_{1817}s\theta_{43} + c\theta_{1817}c\phi s\theta_{54}) +
\end{aligned}$$

$$\begin{aligned}
& c \psi (s \theta s \theta_{1817} s \theta_{54} + c \theta_{1817} c \theta_{54} s \theta_{21} s \theta_{43} s \phi) + c \theta_{54} s \theta_{1817} s \theta_{21} s \theta_{43} s \psi) + \\
& c \theta_{1716}^2 (-c \theta_{54} s \theta_{21} (c \phi c \psi s \theta_{43} + c \theta_{43} s \theta s \phi - c \theta c \theta_{43} c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi) + \\
& 2 c \theta_{21} c \theta_{54} (c \theta_{43} c \phi c \psi + s \theta_{43} (-s \theta s \phi + c \theta c \phi s \psi))) + c \theta_{1817} s (2 \theta_{1716}) \\
& (s \theta s \theta_{54} s \phi s \psi + c \theta_{21} c \theta_{54} s \theta_{43} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{43} c \theta_{54} (c \phi s \theta s \theta_{21} + s \phi (c \theta_{21} c \psi + c \theta s \theta_{21} s \psi))))
\end{aligned}$$

$$\begin{aligned}
V_{43} = & c \theta_{1615}^2 (c \theta_{21} c \theta_{54} c \phi s \theta s \theta_{1817} s \theta_{43} + c \theta_{1817} c \psi s \theta s \theta_{54} - c \theta_{54} c \psi s \theta_{1817} s \theta_{21} s \theta_{43} s \phi + c \theta_{1817} c \theta_{54} s \theta_{21} s \theta_{43} s \psi + \\
& s \theta s \theta_{1817} s \theta_{54} s \phi s \psi + c \theta_{43} c \theta_{54} (c \phi s \theta s \theta_{1817} s \theta_{21} + c \theta_{21} (c \psi s \theta_{1817} s \phi - c \theta_{1817} s \psi)) + \\
& c \theta (c \theta_{1817} c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + s \theta_{1817} (-c \phi s \theta_{54} + c \theta_{54} s (\theta_{21} + \theta_{43}) s \phi s \psi)))
\end{aligned}$$

Coeficientes de la ecuación (2.134)

$$\begin{aligned}
V_{44} = & -x_{43} i s \theta_{21} i (2 x_{32} c \theta_{21} - x_{1211} i c (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + x_{65} c (\theta_{21} + \theta_{43} - \theta_{54}) - x_{1211} i c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) + \\
& x_{65} c (\theta_{21} + \theta_{43} + \theta_{54}) - 2 z_{1312} i s (\theta_{21} + \theta_{43})) + 2 x_{43} i c \theta_{21} i (-\text{signo } y_{10} i + x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})
\end{aligned}$$

$$\begin{aligned}
V_{45} = & -2 x_{32} (z_{10} + z_{32} i) c \theta_{21} + (z_{10} + z_{32} i) (x_{1211} i c (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - x_{65} c (\theta_{21} + \theta_{43} - \theta_{54}) + \\
& x_{1211} i c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - x_{65} c (\theta_{21} + \theta_{43} + \theta_{54}) + 2 z_{1312} i s (\theta_{21} + \theta_{43})) + \\
& 2 x_{43} i c \theta_{21} i (z_{1312} i c (\theta_{21} + \theta_{43}) + x_{32} s \theta_{21} + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s (\theta_{21} + \theta_{43}))
\end{aligned}$$

$$\begin{aligned}
V_{46} = & -2 x_{32} z_{1312} i c \theta_{43} + x_{1211} i z_{10} c (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + \\
& x_{1211} i z_{32} i c (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - x_{65} z_{10} c (\theta_{21} + \theta_{43} - \theta_{54}) - x_{65} z_{32} i c (\theta_{21} + \theta_{43} - \theta_{54}) + \\
& x_{1211} i z_{10} c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) + x_{1211} i z_{32} i c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - x_{65} z_{10} c (\theta_{21} + \theta_{43} + \theta_{54}) - \\
& x_{65} z_{32} i c (\theta_{21} + \theta_{43} + \theta_{54}) + 2 z_{10} z_{1312} i s (\theta_{21} + \theta_{43}) + 2 z_{1312} i z_{32} i s (\theta_{21} + \theta_{43}) + \\
& 2 x_{43} i c \theta_{21} i (z_{1312} i c (\theta_{21} + \theta_{43}) + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s (\theta_{21} + \theta_{43})) - \\
& x_{1211} i x_{32} s (\beta_{1413} i - \theta_{43} - \theta_{54}) - x_{32} x_{65} s (\theta_{43} - \theta_{54}) + x_{1211} i x_{32} s (\beta_{1413} i + \theta_{43} - \theta_{54}) - x_{32} x_{65} s (\theta_{43} + \theta_{54})
\end{aligned}$$

$$\begin{aligned}
V_{47} = & x_{1211} i (z_{10} + z_{32} i) c (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + \\
& x_{65} z_{10} c (\theta_{21} + \theta_{43} - \theta_{54}) + x_{65} z_{32} i c (\theta_{21} + \theta_{43} - \theta_{54}) - x_{1211} i z_{10} c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - \\
& x_{1211} i z_{32} i c (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - 2 \text{signo } x_{65} y_{10} i c \theta_{54} - x_{65} z_{10} c (\theta_{21} + \theta_{43} + \theta_{54}) - \\
& x_{65} z_{32} i c (\theta_{21} + \theta_{43} + \theta_{54}) - 2 x_{43} i x_{65} c \theta_{54} s \theta_{21} i + 2 x_{1211} i c (\beta_{1413} i - \theta_{54}) (\text{signo } y_{10} i + x_{43} i s \theta_{21} i) - \\
& x_{1211} i x_{32} s (\beta_{1413} i - \theta_{43} - \theta_{54}) + x_{1211} i x_{43} i c \theta_{21} i s (\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + \\
& x_{32} x_{65} s (\theta_{43} - \theta_{54}) - x_{1211} i x_{32} s (\beta_{1413} i + \theta_{43} - \theta_{54}) - x_{43} i x_{65} c \theta_{21} i s (\theta_{21} + \theta_{43} - \theta_{54}) + \\
& x_{1211} i x_{43} i c \theta_{21} i s (\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - x_{32} x_{65} s (\theta_{43} + \theta_{54}) + x_{43} i x_{65} c \theta_{21} i s (\theta_{21} + \theta_{43} + \theta_{54})
\end{aligned}$$

Coeficientes de la ecuación (2.139)

$$\begin{aligned}
V_{48} = & c \theta_{65} i \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - c \theta_{21} i s \theta_{21} \\
& (z_{1312} i c \theta_{43} - x_{1211} i c (\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + x_{1211} i s \theta_{21} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) - \\
& (z_{32} i c \beta_{54} i + \text{signo } y_{10} i c \theta_{21} i s \beta_{54} i + c \beta_{54} i (z_{10} - x_{32} s \theta_{21}) + x_{32} c \theta_{21} s \beta_{54} i s \theta_{21} i + \\
& (c \beta_{54} i s \theta_{21} - c \theta_{21} s \beta_{54} i s \theta_{21} i) (c \theta_{43} (x_{1211} i c (\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \beta_{54} i c \theta_{21} + s \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) - \\
& c \theta_{21} i s \beta_{54} i (x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) s \theta_{65} i
\end{aligned}$$

$$V_{49} = \frac{1}{2} (c \beta_{54} i c \theta_{65} i (2 x_{32} c \theta_{21} - x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) - \\ x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) + x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) - 2 z_{1312} i s(\theta_{21} + \theta_{43})) + \\ 2 (z_{1312} i c(\theta_{21} + \theta_{43}) + x_{32} s \theta_{21} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s(\theta_{21} + \theta_{43})) s \theta_{65} i)$$

$$V_{50} = \frac{1}{2} (-c \beta_{54} i c \theta_{65} i (x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) + \\ x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) + 2 z_{1312} i s(\theta_{21} + \theta_{43})) + \\ 2 (z_{1312} i c(\theta_{21} + \theta_{43}) + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s(\theta_{21} + \theta_{43})) s \theta_{65} i)$$

$$V_{51} = \frac{1}{2} (-2 x_{1211} i c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i + 2 x_{65} c \theta_{54} c \theta_{65} i s \beta_{54} i - \\ 2 (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54}) (c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) - c(\theta_{21} + \theta_{43}) s \theta_{65} i))$$

$$V_{52} = -x_{43} i (c \theta_{21} i c \theta_{65} i s \beta_{54} i + s \theta_{21} i s \theta_{65} i)$$

Coeficientes de la ecuación (2.143)

$$V_{53} = (x_{43} i + \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (-x_{32} + c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) + c \theta_{21} i s \theta_{21} \\ (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) - x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) - x_{65} s \theta_{21} i s \theta_{54})^2$$

$$V_{54} = \frac{1}{2} c \theta_{65} i c \theta_{76} i^2 (-2 x_{32} c \theta_{21} s \beta_{54} i \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) + x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) s \beta_{54} i \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) - x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) s \beta_{54} i \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) + x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) s \beta_{54} i \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) - x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) s \beta_{54} i \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) + 2 z_{1312} i s \beta_{54} i s(\theta_{21} + \theta_{43}) \\ (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\ c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\ x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) - \\ 2 z_{1312} i c(\theta_{21} + \theta_{43}) (\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i - z_{32} i s \beta_{54} i - s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) + x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i - \\ (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) + \\ (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) - \\ c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) -$$

$$\begin{aligned}
& c \beta_{54} i c \theta_{21} i (x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) - \\
& x_{65} (\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i - z_{32} i s \beta_{54} i - s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) + x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i - \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c (\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) + \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) - \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) s (\theta_{21} + \theta_{43} + \theta_{54}))
\end{aligned}$$

$$\begin{aligned}
V_{57} = & x_{43} i c \theta_{65} i c \theta_{76} i^2 (-c \beta_{54} i c \theta_{21} i \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c (\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) + \\
& s \theta_{21} i (\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i - z_{32} i s \beta_{54} i - s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) + x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i - \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c (\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) + \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) - \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})))
\end{aligned}$$

$$\begin{aligned}
V_{58} = & c \theta_{76} i^2 \\
& (x_{43} i + \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (-x_{32} + c \theta_{43} (x_{1211} i c (\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) + c \theta_{21} i s \theta_{21} \\
& (z_{1312} i c \theta_{43} - x_{1211} i c (\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) - x_{1211} i s \theta_{21} i s (\beta_{1413} i - \theta_{54}) - x_{65} s \theta_{21} i s \theta_{54}) \\
& (-\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i + z_{32} i s \beta_{54} i + s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i + \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c (\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c (\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) + \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s (\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) s \theta_{65} i
\end{aligned}$$

Coeficientes de la ecuación (2.148)

$$\begin{aligned}
V_{59} = & (s (\beta_{1413} i - \theta_{54}) (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\
& c \theta_{43} c (\beta_{1413} i - \theta_{54}) (-c \beta_{54} i c \theta_{65} i s \theta_{21} + c \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) - \\
& c (\beta_{1413} i - \theta_{54}) s \theta_{43} (c \beta_{54} i c \theta_{21} c \theta_{65} i + s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)))^2
\end{aligned}$$

$$\begin{aligned}
V_{60} = & \\
& c \theta_{1110} i^2 (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) (c \beta_{54} i c (\theta_{21} + \theta_{43}) c \theta_{65} i + s (\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i))
\end{aligned}$$

$$\begin{aligned}
V_{61} = & \\
& c \theta_{1110} i^2 (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) (c \beta_{54} i c (\theta_{21} + \theta_{43}) c \theta_{65} i + s (\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i))
\end{aligned}$$

$$\begin{aligned}
V_{62} = & c \theta_{1110} i^2 \\
& \left(-\frac{1}{8} c \theta_{21} i^2 (-4 + 2 c (2 \beta_{54} i) - 2 c (2 (\theta_{21} + \theta_{43}))) + c (2 (\beta_{54} i - \theta_{65} i)) + c (2 (\theta_{21} + \theta_{43} - \theta_{65} i)) + c (2 (\beta_{54} i + \theta_{65} i)) + \right. \\
& c (2 (\theta_{21} + \theta_{43} + \theta_{65} i))) + c \beta_{54} i^2 c \theta_{65} i^2 s (\theta_{21} + \theta_{43})^2 - c \beta_{54} i c \theta_{65} i^2 s \beta_{54} i s \theta_{21} i s (2 (\theta_{21} + \theta_{43})) + \\
& s \theta_{21} i^2 (c \theta_{21}^2 c \theta_{43}^2 c \theta_{65} i^2 s \beta_{54} i^2 - 2 c \theta_{21} c \theta_{43} c \theta_{65} i^2 s \beta_{54} i^2 s \theta_{21} s \theta_{43} + c \theta_{65} i^2 s \beta_{54} i^2 s \theta_{21}^2 s \theta_{43}^2 + s \theta_{65} i^2) - \\
& \left. c \theta_{21} i s (\theta_{21} + \theta_{43}) (c \beta_{54} i c (\theta_{21} + \theta_{43}) + s \beta_{54} i s \theta_{21} i s (\theta_{21} + \theta_{43})) s (2 \theta_{65} i) \right)
\end{aligned}$$

$$V_{63} = \frac{1}{8} c \theta_{1110} i^2 (s \theta_{21} (-(-6 + 2 c (2 \beta_{54} i) + c (2 (\beta_{54} i - \theta_{65} i)) + 2 c (2 \theta_{65} i) + c (2 (\beta_{54} i + \theta_{65} i))) s \theta_{43} +$$

$$2 c \theta_{43} \left((1 + c \theta_{65} i)^2 s(2 \beta_{54} i) s \theta_{21} i - 2 c \beta_{54} i s \theta_{65} i (-2 c \theta_{21} i c \theta_{65} i + s \beta_{54} i s \theta_{21} i s \theta_{65} i) \right) + \\ c \theta_{21} (c \theta_{43} (-6 + 2 c(2 \beta_{54} i) + c(2 (\beta_{54} i - \theta_{65} i)) + 2 c(2 \theta_{65} i) + c(2 (\beta_{54} i + \theta_{65} i))) + \\ 2 s \theta_{43} ((1 + c \theta_{65} i)^2 s(2 \beta_{54} i) s \theta_{21} i - 2 c \beta_{54} i s \theta_{65} i (-2 c \theta_{21} i c \theta_{65} i + s \beta_{54} i s \theta_{21} i s \theta_{65} i)))$$

$$V_{64} = c \theta_{1110} i^2 (s \theta_{21} (c \beta_{54} i c \theta_{43} s \theta_{21} i + s \beta_{54} i s \theta_{43}) + c \theta_{21} (-c \theta_{43} s \beta_{54} i + c \beta_{54} i s \theta_{21} i s \theta_{43}))$$

Coeficientes de la ecuación (2.152)

$$V_{65} = (c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i))^2$$

$$V_{66} = c \theta_{109} i^2 \left(c \beta_{54} i^2 c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) c \theta_{65} i^2 - \right. \\ c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\ (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) \left(c \theta_{21} i (c(\theta_{21} + \theta_{43}) c \theta_{65} i s \beta_{54} i s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) + \right. \\ c(\theta_{21} + \theta_{43})^2 c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s \theta_{65} i + c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s(\theta_{21} + \theta_{43})^2 s \theta_{65} i) + \\ \left. \frac{1}{4} s \theta_{21} i (-c(\beta_{1413} i + \theta_{1110} i - \theta_{21} - \theta_{43} - \theta_{54} - \theta_{65} i) - c(\beta_{1413} i + \theta_{1110} i + \theta_{21} + \theta_{43} - \theta_{54} - \theta_{65} i) + \right. \\ c(\beta_{1413} i + \theta_{1110} i - \theta_{21} - \theta_{43} - \theta_{54} + \theta_{65} i) + c(\beta_{1413} i + \theta_{1110} i + \theta_{21} + \theta_{43} - \theta_{54} + \theta_{65} i) - \\ s(\beta_{1413} i - \beta_{54} i + \theta_{1110} i - \theta_{54} - \theta_{65} i) + s(\beta_{1413} i + \beta_{54} i + \theta_{1110} i - \theta_{54} - \theta_{65} i) - \\ \left. \left. s(\beta_{1413} i - \beta_{54} i + \theta_{1110} i - \theta_{54} + \theta_{65} i) + s(\beta_{1413} i + \beta_{54} i + \theta_{1110} i - \theta_{54} + \theta_{65} i) \right) \right) \right)$$

$$V_{67} = c \theta_{109} i^2 (-c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i - s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) \\ (c \theta_{21} i c \theta_{65} i s \beta_{54} i s \theta_{1110} i s(\beta_{1413} i - \theta_{54}) - c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) - \\ s \theta_{1110} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) s \theta_{65} i + c(\theta_{21} + \theta_{43}) c(\beta_{1413} i - \theta_{54}) s \theta_{1110} i (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \\ c \theta_{1110} i (s \theta_{21} i (c(\theta_{21} + \theta_{43}) c \theta_{65} i s \beta_{54} i s(\beta_{1413} i - \theta_{54}) + c(\beta_{1413} i - \theta_{54}) s \theta_{65} i) + \\ c \theta_{21} i (-c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i + c(\theta_{21} + \theta_{43}) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i)))$$

$$V_{68} =$$

$$c \theta_{109} i^2 \left(-\frac{1}{8} (-6 + 2 c(2 \beta_{54} i) + c(2 (\beta_{54} i - \theta_{65} i)) + 2 c(2 \theta_{65} i) + c(2 (\beta_{54} i + \theta_{65} i))) s(\theta_{21} + \theta_{43}) s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) + \right. \\ c \beta_{54} i c \theta_{65} i (c(\theta_{21} + \theta_{43}) c(\beta_{1413} i - \theta_{54}) s \theta_{1110} i (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \\ c(\theta_{21} + \theta_{43})^2 s \theta_{1110} i s(\beta_{1413} i - \theta_{54}) (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\ c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s(\theta_{21} + \theta_{43})^2 (-c \theta_{21} i c \theta_{65} i s \beta_{54} i + s \theta_{21} i s \theta_{65} i) + \\ c \theta_{1110} i c(\theta_{21} + \theta_{43}) (s \theta_{21} i (c \theta_{65} i s \beta_{54} i s(\beta_{1413} i - \theta_{54}) + c(\theta_{21} + \theta_{43}) c(\beta_{1413} i - \theta_{54}) s \theta_{65} i) + \\ \left. \left. c \theta_{21} i (-c(\theta_{21} + \theta_{43}) c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i + s(\beta_{1413} i - \theta_{54}) s \theta_{65} i)) \right) \right)$$

$$V_{69} =$$

$$c \theta_{109} i^2 (s \beta_{1413} i (c(\theta_{1110} i - \theta_{54}) s \beta_{54} i s(\theta_{21} + \theta_{43}) + c \beta_{54} i (c(\theta_{21} + \theta_{43}) c(\theta_{1110} i - \theta_{54}) s \theta_{21} i + c \theta_{21} i s(\theta_{1110} i - \theta_{54}))) - \\ c \beta_{1413} i (-s \beta_{54} i s(\theta_{21} + \theta_{43}) s(\theta_{1110} i - \theta_{54}) + c \beta_{54} i (s \theta_{1110} i (-c \theta_{21} i c \theta_{43} c \theta_{54} s \theta_{21} i + \\ c \theta_{54} s \theta_{21} i s \theta_{21} i s \theta_{43} + c \theta_{21} i s \theta_{54}) + c \theta_{1110} i (c \theta_{21} i c \theta_{54} + c(\theta_{21} + \theta_{43}) s \theta_{21} i s \theta_{54}))))$$

$$\begin{aligned}
V_{70} = & -c \theta_{109} i^2 (c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) \\
& (-c \theta_{21} i c \theta_{65} i s \beta_{54} i s \theta_{1110} i s(\beta_{1413} i - \theta_{54}) + c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) + \\
& s \theta_{1110} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) s \theta_{65} i - c(\theta_{21} + \theta_{43}) c(\beta_{1413} i - \theta_{54}) s \theta_{1110} i (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) - \\
& c \theta_{1110} i (s \theta_{21} i (c(\theta_{21} + \theta_{43}) c \theta_{65} i s \beta_{54} i s(\beta_{1413} i - \theta_{54}) + c(\beta_{1413} i - \theta_{54}) s \theta_{65} i) + \\
& c \theta_{21} i (-c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i + c(\theta_{21} + \theta_{43}) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i)))
\end{aligned}$$

Coefficientes de la ecuación (2.156)

$$\begin{aligned}
V_{71} = & (s \theta_{109} i (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i + c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& c \theta_{109} i (c \theta_{1110} i (-s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + s \beta_{54} i (c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i - c \theta_{21} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{21} c \theta_{76} i s \theta_{65} i + s \theta_{21} s \theta_{21} i s \theta_{76} i)) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + \\
& s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& s \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i))))))^2
\end{aligned}$$

$$\begin{aligned}
V_{72} = & \frac{1}{128} c \theta_{98} i^2 (8 c(2 \theta_{109} i - \theta_{21} i - \theta_{65} i) + 48 c(\theta_{21} i - \theta_{65} i) - \\
& 8 c(2 \theta_{109} i + \theta_{21} i - \theta_{65} i) - 8 c(2 \beta_{1413} i + 2 \theta_{1110} i - \theta_{21} i - 2 \theta_{54} - \theta_{65} i) - \\
& 4 c(2 \beta_{1413} i - 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - 2 \theta_{54} - \theta_{65} i) - 4 c(2 \beta_{1413} i + 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - 2 \theta_{54} - \theta_{65} i) + \\
& 8 c(2 \beta_{1413} i + 2 \theta_{1110} i + \theta_{21} i - 2 \theta_{54} - \theta_{65} i) + 4 c(2 \beta_{1413} i - 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i - 2 \theta_{54} - \theta_{65} i) + \\
& 4 c(2 \beta_{1413} i + 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i - 2 \theta_{54} - \theta_{65} i) - 4 c(2 \beta_{1413} i - \beta_{54} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) - \\
& 4 c(2 \beta_{1413} i + \beta_{54} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) - \\
& 2 c(2 \beta_{1413} i - \beta_{54} i - 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) - \\
& 2 c(2 \beta_{1413} i + \beta_{54} i - 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) - \\
& 2 c(2 \beta_{1413} i - \beta_{54} i + 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) - \\
& 2 c(2 \beta_{1413} i + \beta_{54} i + 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 4 c(2 \beta_{1413} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i - 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i + 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i - \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 4 c(2 \beta_{1413} i + 2 \theta_{1110} i - \theta_{21} i + \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i - 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i + \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i + 2 \theta_{109} i + 2 \theta_{1110} i - \theta_{21} i + \theta_{21} i - \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 4 c(2 \beta_{1413} i - \beta_{54} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 4 c(2 \beta_{1413} i + \beta_{54} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i - \beta_{54} i - 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i + \beta_{54} i - 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i - \beta_{54} i + 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) + \\
& 2 c(2 \beta_{1413} i + \beta_{54} i + 2 \theta_{109} i + 2 \theta_{1110} i + \theta_{21} i + \theta_{43} - 2 \theta_{54} - \theta_{65} i) +
\end{aligned}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

214

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& s \theta_{21} s \theta_{43} s \theta_{54} + c \theta_{54} s \theta_{65} i) - c \theta_{21} i (c \theta_{54} c \theta_{65} i s \beta_{54} i + c(\theta_{21} + \theta_{43}) s \theta_{54} s \theta_{65} i))) + \\
& s \theta_{109} i^2 \left(s \theta_{21}^2 (c \theta_{21} i^2 c \theta_{43}^2 c \theta_{65} i^2 + c \beta_{54} i c \theta_{65} i^2 s \beta_{54} i s \theta_{21} i s(2 \theta_{43}) + 2 c \theta_{21} i c \theta_{43} c \theta_{65} i \right. \\
& \quad (-c \theta_{43} s \beta_{54} i s \theta_{21} i + c \beta_{54} i s \theta_{43}) s \theta_{65} i + \\
& \quad s \beta_{54} i^2 (s \theta_{43}^2 + c \theta_{43}^2 s \theta_{21} i^2 s \theta_{65} i^2) + c \beta_{54} i^2 (c \theta_{43}^2 s \theta_{21} i^2 + s \theta_{43}^2 s \theta_{65} i^2)) + \\
& \quad c \theta_{21}^2 (-2 c \beta_{54} i c \theta_{43} c \theta_{65} i s \theta_{43} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + c \theta_{43}^2 (s \beta_{54} i^2 + c \beta_{54} i^2 s \theta_{65} i^2) + \\
& \quad s \theta_{43}^2 (c \theta_{21} i^2 c \theta_{65} i^2 - 2 c \theta_{21} i c \theta_{65} i s \beta_{54} i s \theta_{21} i s \theta_{65} i + s \theta_{21} i^2 (c \beta_{54} i^2 + s \beta_{54} i^2 s \theta_{65} i^2))) + 2 c \theta_{21} \\
& \quad s \theta_{21} \left(-\frac{1}{4} c \beta_{54} i^2 (c(2 \theta_{21} i) - c(2 \theta_{65} i)) s(2 \theta_{43}) - c \beta_{54} i c(2 \theta_{43}) c \theta_{65} i (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \right. \\
& \quad \left. c \theta_{43} s \theta_{43} (c \theta_{21} i^2 c \theta_{65} i^2 - 2 c \theta_{21} i c \theta_{65} i s \beta_{54} i s \theta_{21} i s \theta_{65} i + s \beta_{54} i^2 (-1 + s \theta_{21} i^2 s \theta_{65} i^2)) \right) \Big) + \\
& \frac{1}{16} c \theta_{109} i^2 (4 c \beta_{54} i^2 (c \theta_{21} c \theta_{43} (c(2 \beta_{1413} i - \theta_{1110} i - 2 \theta_{54}) + 3 c(2 \beta_{1413} i + \theta_{1110} i - 2 \theta_{54})) s \theta_{1110} i s(2 \theta_{21} i) - \\
& \quad 2 c(\beta_{1413} i - \theta_{54})^2 s(2 \theta_{1110} i) s \theta_{21} s(2 \theta_{21} i) s \theta_{43} + 2 s(2 \theta_{1110} i) s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s(\beta_{1413} i - \theta_{54})^2 + \\
& \quad 4 s \theta_{1110} i^2 s \theta_{21}^2 s \theta_{21} i^2 s \theta_{43}^2 s(\beta_{1413} i - \theta_{54})^2 - 2 s \theta_{1110} i^2 s(2 \theta_{21}) s \theta_{21} i^2 s(2 \theta_{43}) s(\beta_{1413} i - \theta_{54})^2 + \\
& \quad 2 s \theta_{1110} i^2 s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s(2(\beta_{1413} i - \theta_{54})) - 2 s(2 \theta_{1110} i) s \theta_{21}^2 s \theta_{21} i^2 s \theta_{43}^2 s(2(\beta_{1413} i - \theta_{54})) + \\
& \quad s(2 \theta_{1110} i) s(2 \theta_{21}) s \theta_{21} i^2 s(2 \theta_{43}) s(2(\beta_{1413} i - \theta_{54})) + \\
& \quad 2 c \theta_{21} i^2 (2 c(\beta_{1413} i - \theta_{54})^2 s \theta_{1110} i^2 + s(2 \theta_{1110} i) s(2(\beta_{1413} i - \theta_{54}))) + \\
& \quad 4 c \theta_{43}^2 s \theta_{1110} i^2 s \theta_{21}^2 s(\beta_{1413} i - \theta_{54})^2 s \theta_{65} i^2 + 2 s \theta_{1110} i^2 s(2 \theta_{21}) s(2 \theta_{43}) s(\beta_{1413} i - \theta_{54})^2 s \theta_{65} i^2 - \\
& \quad 2 c \theta_{43}^2 s(2 \theta_{1110} i) s \theta_{21}^2 s(2(\beta_{1413} i - \theta_{54})) s \theta_{65} i^2 - \\
& \quad s(2 \theta_{1110} i) s(2 \theta_{21}) s(2 \theta_{43}) s(2(\beta_{1413} i - \theta_{54})) s \theta_{65} i^2 + \\
& \quad 2 c \theta_{21}^2 (2 s \theta_{1110} i^2 s(\beta_{1413} i - \theta_{54})^2 - s(2 \theta_{1110} i) s(2(\beta_{1413} i - \theta_{54}))) (c \theta_{43}^2 s \theta_{21} i^2 + s \theta_{43}^2 s \theta_{65} i^2)) + \\
& \quad c \beta_{54} i s \theta_{1110} i s(\theta_{21} + \theta_{43}) (2 s(\theta_{1110} i - \theta_{21} - \theta_{21} i - \theta_{43}) + 2 s(\theta_{1110} i - \theta_{21} + \theta_{21} i - \theta_{43}) + \\
& \quad 2 s(\theta_{1110} i + \theta_{21} - \theta_{21} i + \theta_{43}) + 2 s(\theta_{1110} i + \theta_{21} + \theta_{21} i + \theta_{43}) + 2 s(2 \beta_{1413} i - \theta_{1110} i - \theta_{21} i - 2 \theta_{54}) + \\
& \quad 6 s(2 \beta_{1413} i + \theta_{1110} i - \theta_{21} i - 2 \theta_{54}) - 2 s(2 \beta_{1413} i - \theta_{1110} i + \theta_{21} i - 2 \theta_{54}) - \\
& \quad 6 s(2 \beta_{1413} i + \theta_{1110} i + \theta_{21} i - 2 \theta_{54}) - s(2 \beta_{1413} i - \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54}) - \\
& \quad 3 s(2 \beta_{1413} i + \theta_{1110} i - \theta_{21} i - \theta_{43} - 2 \theta_{54}) - s(2 \beta_{1413} i - \theta_{1110} i - \theta_{21} i + \theta_{21} i - \theta_{43} - 2 \theta_{54}) - \\
& \quad 3 s(2 \beta_{1413} i + \theta_{1110} i - \theta_{21} i + \theta_{21} i - \theta_{43} - 2 \theta_{54}) - s(2 \beta_{1413} i - \theta_{1110} i + \theta_{21} i - \theta_{21} i + \theta_{43} - 2 \theta_{54}) - \\
& \quad 3 s(2 \beta_{1413} i + \theta_{1110} i + \theta_{21} i - \theta_{21} i + \theta_{43} - 2 \theta_{54}) - s(2 \beta_{1413} i - \theta_{1110} i + \theta_{21} i + \theta_{21} i + \theta_{43} - 2 \theta_{54}) - \\
& \quad 3 s(2 \beta_{1413} i + \theta_{1110} i + \theta_{21} i + \theta_{21} i + \theta_{43} - 2 \theta_{54})) s(2 \theta_{65} i) + \\
& \quad 4 c \theta_{1110} i^2 (4 c \theta_{21} i^2 c(\beta_{1413} i - \theta_{54})^2 c \theta_{65} i^2 s \theta_{21}^2 s \theta_{43}^2 - 2 c(\beta_{1413} i - \theta_{54})^2 s(2 \beta_{54} i) s(2 \theta_{21}) s \theta_{21} i s \theta_{43}^2 + \\
& \quad 4 c \beta_{54} i^2 c(\beta_{1413} i - \theta_{54})^2 s \theta_{21}^2 s \theta_{21} i^2 s \theta_{43}^2 - 2 c \theta_{21} i^2 c(\beta_{1413} i - \theta_{54})^2 c \theta_{65} i^2 s(2 \theta_{21}) s(2 \theta_{43}) + \\
& \quad 2 c(\beta_{1413} i - \theta_{54})^2 s \beta_{54} i^2 s(2 \theta_{21}) s(2 \theta_{43}) - 2 c(\beta_{1413} i - \theta_{54})^2 s(2 \beta_{54} i) s \theta_{21}^2 s \theta_{21} i s(2 \theta_{43}) - \\
& \quad 2 c \beta_{54} i^2 c(\beta_{1413} i - \theta_{54})^2 s(2 \theta_{21}) s \theta_{21} i^2 s(2 \theta_{43}) + 4 c \beta_{54} i^2 c \theta_{21} i^2 s(\beta_{1413} i - \theta_{54})^2 + \\
& \quad 4 c \theta_{65} i^2 s \theta_{21} i^2 s(\beta_{1413} i - \theta_{54})^2 - 2 c \beta_{54} i^2 s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s(2(\beta_{1413} i - \theta_{54})) + \\
& \quad 2 c \theta_{65} i^2 s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s(2(\beta_{1413} i - \theta_{54})) + 2 c(\beta_{1413} i - \theta_{54})^2 s(2 \beta_{54} i) s(2 \theta_{21}) \\
& \quad s \theta_{21} i s \theta_{43}^2 s \theta_{65} i^2 + 4 c(\beta_{1413} i - \theta_{54})^2 s \beta_{54} i^2 s \theta_{21}^2 s \theta_{21} i^2 s \theta_{43}^2 s \theta_{65} i^2 + \\
& \quad 2 c \beta_{54} i^2 c(\beta_{1413} i - \theta_{54})^2 s(2 \theta_{21}) s(2 \theta_{43}) s \theta_{65} i^2 + 2 c(\beta_{1413} i - \theta_{54})^2 s(2 \beta_{54} i) s \theta_{21}^2 \\
& \quad s \theta_{21} i s(2 \theta_{43}) s \theta_{65} i^2 - 2 c(\beta_{1413} i - \theta_{54})^2 s \beta_{54} i^2 s(2 \theta_{21}) s \theta_{21} i^2 s(2 \theta_{43}) s \theta_{65} i^2 + \\
& \quad 4 c \theta_{21} i^2 s \beta_{54} i^2 s(\beta_{1413} i - \theta_{54})^2 s \theta_{65} i^2 - 2 s \beta_{54} i^2 s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s(2(\beta_{1413} i - \theta_{54})) s \theta_{65} i^2 - \\
& \quad 2 c \beta_{54} i c \theta_{21} i c(\beta_{1413} i - \theta_{54})^2 s(2 \theta_{21}) s \theta_{43}^2 s(2 \theta_{65} i) - 2 c(\beta_{1413} i - \theta_{54})^2 s \beta_{54} i s \theta_{21}^2 \\
& \quad s(2 \theta_{21} i) s \theta_{43}^2 s(2 \theta_{65} i) - 2 c \beta_{54} i c \theta_{21} i c(\beta_{1413} i - \theta_{54})^2 s \theta_{21}^2 s(2 \theta_{43}) s(2 \theta_{65} i) + \\
& \quad c(\beta_{1413} i - \theta_{54})^2 s \beta_{54} i s(2 \theta_{21}) s(2 \theta_{21} i) s(2 \theta_{43}) s(2 \theta_{65} i) + 2 s \beta_{54} i s(2 \theta_{21} i) s(\beta_{1413} i - \theta_{54})^2 s(2 \theta_{65} i) +
\end{aligned}$$

$$\begin{aligned}
& 2c\theta_{21}i^2s\beta_{54}is\theta_{21}s\theta_{43}s(2(\beta_{1413}i-\theta_{54}))s(2\theta_{65}i)-2s\beta_{54}is\theta_{21}s\theta_{21}i^2s\theta_{43}s(2(\beta_{1413}i-\theta_{54})) \\
& s(2\theta_{65}i)+2c\theta_{43}s\theta_{21}s(2(\beta_{1413}i-\theta_{54}))(c\theta_{21}ic\theta_{65}i^2s(2\beta_{54}i)-c\beta_{54}is\theta_{21}is(2\theta_{65}i))+ \\
& 2c\theta_{21}^2c(\beta_{1413}i-\theta_{54})^2(2c\theta_{21}i^2c\theta_{43}^2c\theta_{65}i^2+2s\beta_{54}i^2s\theta_{43}^2+s(2\beta_{54}i)s\theta_{21}is(2\theta_{43})+ \\
& 2c\theta_{43}^2s\beta_{54}i^2s\theta_{21}i^2s\theta_{65}i^2-s(2\beta_{54}i)s\theta_{21}is(2\theta_{43})s\theta_{65}i^2+2c\beta_{54}i^2(c\theta_{43}^2s\theta_{21}i^2+ \\
& s\theta_{43}^2s\theta_{65}i^2)-c\theta_{43}^2s\beta_{54}is(2\theta_{21}i)s(2\theta_{65}i)+c\beta_{54}ic\theta_{21}is(2\theta_{43})s(2\theta_{65}i))+ \\
& 2c\theta_{21}s(2(\beta_{1413}i-\theta_{54}))(c\beta_{54}i^2c\theta_{43}s(2\theta_{21}i)+c\theta_{21}ic\theta_{65}i^2s(2\beta_{54}i)s\theta_{43}-c\beta_{54}is\theta_{21}i \\
& s\theta_{43}s(2\theta_{65}i)-c\theta_{43}(c\theta_{65}i^2s(2\theta_{21}i)-s\beta_{54}i^2s(2\theta_{21}i)s\theta_{65}i^2+c(2\theta_{21}i)s\beta_{54}is(2\theta_{65}i)))+ \\
& 2c\theta_{43}^2c(\beta_{1413}i-\theta_{54})^2(2s\beta_{54}i^2s\theta_{21}^2+c\theta_{65}i^2s(2\beta_{54}i)s(2\theta_{21})s\theta_{21}i+ \\
& c\beta_{54}i(2c\beta_{54}is\theta_{21}^2s\theta_{65}i^2+c\theta_{21}is(2\theta_{21})s(2\theta_{65}i))))+ \\
& 2(2c\theta_{21}i(c(2\beta_{1413}i-\theta_{1110}i-2\theta_{54})+3c(2\beta_{1413}i+\theta_{1110}i-2\theta_{54}))c\theta_{65}i^2s(2\beta_{54}i)s\theta_{1110}is(\theta_{21}+\theta_{43})+ \\
& 8c\theta_{43}^2s\beta_{54}i^2s\theta_{1110}i^2s\theta_{21}^2s(\beta_{1413}i-\theta_{54})^2+4c\theta_{43}^2s(2\beta_{54}i)s\theta_{1110}i^2s(2\theta_{21}) \\
& s\theta_{21}is(\beta_{1413}i-\theta_{54})^2+4c\theta_{21}c\theta_{43}c\theta_{65}i^2s(2\theta_{1110}i)s(2\theta_{21}i)s(\beta_{1413}i-\theta_{54})^2- \\
& 4c\theta_{65}i^2s(2\theta_{1110}i)s\theta_{21}s(2\theta_{21}i)s\theta_{43}s(\beta_{1413}i-\theta_{54})^2+8c\theta_{21}^2s\beta_{54}i^2s\theta_{1110}i^2 \\
& s\theta_{43}^2s(\beta_{1413}i-\theta_{54})^2-4s(2\beta_{54}i)s\theta_{1110}i^2s(2\theta_{21})s\theta_{21}is\theta_{43}^2s(\beta_{1413}i-\theta_{54})^2+ \\
& 4s\beta_{54}i^2s\theta_{1110}i^2s(2\theta_{21})s(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2+4c\theta_{21}^2s(2\beta_{54}i)s\theta_{1110}i^2s\theta_{21}i \\
& s(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2-4s(2\beta_{54}i)s\theta_{1110}i^2s\theta_{21}^2s\theta_{21}is(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2- \\
& 4c\theta_{43}^2s\beta_{54}i^2s(2\theta_{1110}i)s\theta_{21}^2s(2(\beta_{1413}i-\theta_{54}))-2c\theta_{43}^2s(2\beta_{54}i)s(2\theta_{1110}i) \\
& s(2\theta_{21})s\theta_{21}is(2(\beta_{1413}i-\theta_{54}))+4c\theta_{65}i^2s(2\theta_{1110}i)s\theta_{21}i^2s(2(\beta_{1413}i-\theta_{54}))+ \\
& 4c\theta_{21}c\theta_{43}c\theta_{65}i^2s\theta_{1110}i^2s(2\theta_{21}i)s(2(\beta_{1413}i-\theta_{54}))-4c\theta_{65}i^2s\theta_{1110}i^2s\theta_{21} \\
& s(2\theta_{21}i)s\theta_{43}s(2(\beta_{1413}i-\theta_{54}))-4c\theta_{21}^2s\beta_{54}i^2s(2\theta_{1110}i)s\theta_{43}^2s(2(\beta_{1413}i-\theta_{54}))+ \\
& 2s(2\beta_{54}i)s(2\theta_{1110}i)s(2\theta_{21})s\theta_{21}is\theta_{43}^2s(2(\beta_{1413}i-\theta_{54}))-2s\beta_{54}i^2s(2\theta_{1110}i)s(2\theta_{21}) \\
& s(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))-2c\theta_{21}^2s(2\beta_{54}i)s(2\theta_{1110}i)s\theta_{21}is(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))+ \\
& 2s(2\beta_{54}i)s(2\theta_{1110}i)s\theta_{21}^2s\theta_{21}is(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))-4c\theta_{43}^2s(2\beta_{54}i)s\theta_{1110}i^2s(2\theta_{21}) \\
& s\theta_{21}is(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+8c\theta_{21}^2c\theta_{43}^2s\beta_{54}i^2s\theta_{1110}i^2s\theta_{21}i^2s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2- \\
& 4c\theta_{21}c\theta_{43}s\beta_{54}i^2s(2\theta_{1110}i)s(2\theta_{21}i)s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+4s\beta_{54}i^2s(2\theta_{1110}i)s\theta_{21}s(2\theta_{21}i) \\
& s\theta_{43}s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+4s(2\beta_{54}i)s\theta_{1110}i^2s(2\theta_{21})s\theta_{21}is\theta_{43}^2s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+ \\
& 8s\beta_{54}i^2s\theta_{1110}i^2s\theta_{21}^2s\theta_{21}i^2s\theta_{43}^2s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2-4c\theta_{21}^2s(2\beta_{54}i)s\theta_{1110}i^2s\theta_{21}i \\
& s(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+4s(2\beta_{54}i)s\theta_{1110}i^2s\theta_{21}^2s\theta_{21}is(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2- \\
& 4s\beta_{54}i^2s\theta_{1110}i^2s(2\theta_{21})s\theta_{21}i^2s(2\theta_{43})s(\beta_{1413}i-\theta_{54})^2s\theta_{65}i^2+ \\
& 2c\theta_{43}^2s(2\beta_{54}i)s(2\theta_{1110}i)s(2\theta_{21})s\theta_{21}is(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 4c\theta_{21}^2c\theta_{43}^2s\beta_{54}i^2s(2\theta_{1110}i)s\theta_{21}i^2s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 4c\theta_{21}c\theta_{43}s\beta_{54}i^2s\theta_{1110}i^2s(2\theta_{21}i)s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2+ \\
& 4s\beta_{54}i^2s\theta_{1110}i^2s\theta_{21}s(2\theta_{21}i)s\theta_{43}s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 2s(2\beta_{54}i)s(2\theta_{1110}i)s(2\theta_{21})s\theta_{21}is\theta_{43}^2s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 4s\beta_{54}i^2s(2\theta_{1110}i)s\theta_{21}^2s\theta_{21}i^2s\theta_{43}^2s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2+ \\
& 2c\theta_{21}^2s(2\beta_{54}i)s(2\theta_{1110}i)s\theta_{21}is(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 2s(2\beta_{54}i)s(2\theta_{1110}i)s\theta_{21}^2s\theta_{21}is(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2+ \\
& 2s\beta_{54}i^2s(2\theta_{1110}i)s(2\theta_{21})s\theta_{21}i^2s(2\theta_{43})s(2(\beta_{1413}i-\theta_{54}))s\theta_{65}i^2- \\
& 4c\theta_{21}c\theta_{43}s\beta_{54}is(2\theta_{1110}i)s\theta_{21}i^2s(\beta_{1413}i-\theta_{54})^2s(2\theta_{65}i)- \\
& 4c\theta_{21}^2c\theta_{43}^2s\beta_{54}is\theta_{1110}i^2s(2\theta_{21}i)s(\beta_{1413}i-\theta_{54})^2s(2\theta_{65}i)+ \\
& 4s\beta_{54}is(2\theta_{1110}i)s\theta_{21}s\theta_{21}i^2s\theta_{43}s(\beta_{1413}i-\theta_{54})^2s(2\theta_{65}i)- \\
& 4s\beta_{54}is\theta_{1110}i^2s\theta_{21}^2s(2\theta_{21}i)s\theta_{43}^2s(\beta_{1413}i-\theta_{54})^2s(2\theta_{65}i)+ \\
&$$

$$\begin{aligned}
& 2 s \beta_{54} i s(2 \theta_{1110} i) s(2 \theta_{21} i) s(2 (\beta_{1413} i - \theta_{54})) s(2 \theta_{65} i) + \\
& 2 c \theta_{21}^2 c \theta_{43}^2 s \beta_{54} i s(2 \theta_{1110} i) s(2 \theta_{21} i) s(2 (\beta_{1413} i - \theta_{54})) s(2 \theta_{65} i) + \\
& 4 s \beta_{54} i s \theta_{1110} i^2 s \theta_{21} s \theta_{21} i^2 s \theta_{43} s(2 (\beta_{1413} i - \theta_{54})) s(2 \theta_{65} i) + \\
& 2 s \beta_{54} i s(2 \theta_{1110} i) s \theta_{21}^2 s(2 \theta_{21} i) s \theta_{43}^2 s(2 (\beta_{1413} i - \theta_{54})) s(2 \theta_{65} i) - \\
& s \beta_{54} i s(2 \theta_{1110} i) s(2 \theta_{21} i) s(2 \theta_{21} i) s(2 \theta_{43} i) s(2 (\beta_{1413} i - \theta_{54})) s(2 \theta_{65} i) + \\
& 2 c \theta_{21} i^2 ((c(\theta_{21} - \theta_{43}) + 3 c(\theta_{21} + \theta_{43}))(c(\beta_{1413} i - \theta_{1110} i - \theta_{54}) + 3 c(\beta_{1413} i + \theta_{1110} i - \theta_{54})) \\
& \quad c \theta_{65} i^2 s \theta_{1110} i s \theta_{21} s \theta_{43} s(\beta_{1413} i - \theta_{54}) + \\
& \quad 2 c \theta_{21}^2 c \theta_{43}^2 c \theta_{65} i^2 (2 s \theta_{1110} i^2 s(\beta_{1413} i - \theta_{54})^2 - s(2 \theta_{1110} i) s(2 (\beta_{1413} i - \theta_{54}))) - \\
& \quad c \theta_{21} c \theta_{43} (c(2 \beta_{1413} i - \theta_{1110} i - 2 \theta_{54}) + 3 c(2 \beta_{1413} i + \theta_{1110} i - 2 \theta_{54})) s \beta_{54} i s \theta_{1110} i s(2 \theta_{65} i) + \\
& \quad 2 s \beta_{54} i (s \beta_{54} i s(2 \theta_{1110} i) s(2 (\beta_{1413} i - \theta_{54})) s \theta_{65} i^2 - \\
& \quad \quad 2 s \theta_{1110} i s \theta_{21} s \theta_{43} s(\beta_{1413} i - \theta_{54}) s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s(2 \theta_{65} i))) + \\
& 4 c(\beta_{1413} i - \theta_{54})^2 (c \theta_{65} i^2 (2 s \theta_{1110} i^2 s \theta_{21} i^2 - c(\theta_{21} + \theta_{43}) s(2 \theta_{1110} i) s(2 \theta_{21} i)) + s \beta_{54} i \\
& \quad (-s \beta_{54} i s(2 \theta_{1110} i) s \theta_{21} s(2 \theta_{21} i) s \theta_{43} s \theta_{65} i^2 + s \theta_{1110} i^2 s(2 \theta_{21} i) s(2 \theta_{65} i) - s(2 \theta_{1110} i) s \theta_{21} \\
& \quad \quad s \theta_{21} i^2 s \theta_{43} s(2 \theta_{65} i) + c \theta_{21} c \theta_{43} s(2 \theta_{1110} i) (s \beta_{54} i s(2 \theta_{21} i) s \theta_{65} i^2 + s \theta_{21} i^2 s(2 \theta_{65} i)) + \\
& \quad \quad c \theta_{21} i^2 (2 s \beta_{54} i s \theta_{1110} i^2 s \theta_{65} i^2 + s(2 \theta_{1110} i) s \theta_{21} s \theta_{43} s(2 \theta_{65} i))))))
\end{aligned}$$

$$\begin{aligned}
V_{78} = & -\frac{1}{2} c \theta_{98} i^2 (c \theta_{21} c \theta_{43} c \theta_{54} c \theta_{65} i s \beta_{1413} i s \beta_{54} i s(2 \theta_{109} i) s \theta_{1110} i s \theta_{21} i + \\
& c \theta_{43} c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{21} i - c \theta_{43} c \theta_{65} i s \beta_{54} i s \theta_{109} i^2 s \theta_{21} s \theta_{21} i + c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{21} i s \theta_{43} - \\
& c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{109} i^2 s \theta_{21} i s \theta_{43} + c \beta_{1413} i c \theta_{1110} i c \theta_{54} c \theta_{65} i s \beta_{54} i s(2 \theta_{109} i) s \theta_{21} s \theta_{21} i s \theta_{43} + \\
& c \beta_{54} i c \theta_{65} i (2 c \theta_{109} i^2 c(\theta_{21} + \theta_{43}) + c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s(2 \theta_{109} i) s(\theta_{21} + \theta_{43})) + \\
& c \beta_{1413} i c \theta_{1110} i c \theta_{21} i c \theta_{65} i s \beta_{54} i s(2 \theta_{109} i) s \theta_{54} + c \theta_{1110} i c \theta_{65} i s \beta_{1413} i s \beta_{54} i s(2 \theta_{109} i) s \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{54} + \\
& c \beta_{1413} i c \theta_{65} i s \beta_{54} i s(2 \theta_{109} i) s \theta_{1110} i s \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{54} + \\
& c \theta_{21} c \theta_{21} i c \theta_{43} c \theta_{54} s \beta_{1413} i s(2 \theta_{109} i) s \theta_{1110} i s \theta_{65} i + c \theta_{21} i c \theta_{43} s \theta_{21} s \theta_{65} i - \\
& c \theta_{21} i c \theta_{43} s \theta_{109} i^2 s \theta_{21} s \theta_{65} i + c \theta_{1110} i c \theta_{54} s \beta_{1413} i s(2 \theta_{109} i) s \theta_{21} i s \theta_{65} i + \\
& c \beta_{1413} i c \theta_{54} s(2 \theta_{109} i) s \theta_{1110} i s \theta_{21} i s \theta_{65} i + c \theta_{21} c \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{21} i s \theta_{109} i^2 s \theta_{43} s \theta_{65} i + \\
& c \beta_{1413} i c \theta_{1110} i c \theta_{21} i c \theta_{54} s(2 \theta_{109} i) s \theta_{21} s \theta_{43} s \theta_{65} i + s \beta_{1413} i s(2 \theta_{109} i) s \theta_{1110} i s \theta_{21} i s \theta_{54} s \theta_{65} i + \\
& c \theta_{1110} i c \theta_{21} i s \beta_{1413} i s(2 \theta_{109} i) s \theta_{21} s \theta_{43} s \theta_{54} s \theta_{65} i + c \beta_{1413} i c \theta_{21} i s(2 \theta_{109} i) s \theta_{1110} i s \theta_{21} s \theta_{43} s \theta_{54} s \theta_{65} i + \\
& c \theta_{109} i^2 s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) - 2 c \theta_{109} i s \theta_{109} i (c \theta_{1110} i \\
& \quad (s \theta_{21} i (c \beta_{1413} i c \theta_{21} c \theta_{43} c \theta_{54} c \theta_{65} i s \beta_{54} i + c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{1413} i s \beta_{54} i s \theta_{54} + c \beta_{1413} i s \theta_{54} s \theta_{65} i) + \\
& \quad \quad c \theta_{21} i (c \theta_{54} c \theta_{65} i s \beta_{1413} i s \beta_{54} i + c \beta_{1413} i c \theta_{21} c \theta_{43} c \theta_{54} s \theta_{65} i + c \theta_{21} c \theta_{43} s \beta_{1413} i s \theta_{54} s \theta_{65} i)) + \\
& \quad s \theta_{1110} i (s \beta_{1413} i (c \theta_{21} i c \theta_{65} i s \beta_{54} i s \theta_{54} + c \theta_{54} s \theta_{21} s \theta_{43} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) + \\
& \quad \quad c \beta_{1413} i (c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{54} i s \theta_{21} i s \theta_{54} + c \theta_{21} i (c \theta_{54} c \theta_{65} i s \beta_{54} i + c \theta_{21} c \theta_{43} s \theta_{54} s \theta_{65} i))))))
\end{aligned}$$

$$\begin{aligned}
V_{79} = & c \theta_{98} i^2 (s \beta_{1413} i (c \beta_{54} i c(\theta_{1110} i - \theta_{54}) c \theta_{65} i s(\theta_{21} + \theta_{43}) - c \theta_{21} i (c \theta_{54} (c \theta_{65} i s \beta_{54} i s \theta_{1110} i + c \theta_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\
& \quad s \theta_{54} (-c \theta_{1110} i c \theta_{65} i s \beta_{54} i + c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i)) + \\
& \quad s \theta_{21} i (s \theta_{1110} i (-c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{54} i s \theta_{54} + c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s \theta_{54} + c \theta_{54} s \theta_{65} i) - \\
& \quad \quad c \theta_{1110} i (c \theta_{21} c \theta_{43} c \theta_{54} c \theta_{65} i s \beta_{54} i - c \theta_{54} c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} + s \theta_{54} s \theta_{65} i))) + \\
& c \beta_{1413} i (s \theta_{1110} i (-c \theta_{21} c \theta_{43} c \theta_{54} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + s \theta_{54} (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\
& \quad \quad c \theta_{54} (c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{21} i s \theta_{43} + c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) + c \theta_{21} i s \theta_{21} s \theta_{43} s \theta_{65} i)) - \\
& \quad c \theta_{1110} i (c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) s \theta_{54} + s \theta_{21} i (-c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{54} i s \theta_{54} + \\
& \quad \quad c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s \theta_{54} + c \theta_{54} s \theta_{65} i) - c \theta_{21} i (c \theta_{54} c \theta_{65} i s \beta_{54} i + c(\theta_{21} + \theta_{43}) s \theta_{54} s \theta_{65} i))))
\end{aligned}$$

Apéndice E

Coeficientes de la ecuación (2.77)

$$G_1 = 2 (x - x_{32} c \theta_{21} - x_{1918} c \phi c \psi)$$

$$G_2 = 2 (y - x_{1918} c \theta s \phi - x_{1918} c \phi s \theta s \psi)$$

$$G_3 = 2 (z - z_{10} + x_{32} s \theta_{21} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi)$$

$$G_4 = 2 x_{1918} c \phi (-y c \psi s \theta + c \theta c \psi (z - z_{10} + x_{32} s \theta_{21}) + (x - x_{32} c \theta_{21}) s \psi)$$

$$G_5 = -2 x_{1918} (c \theta ((z - z_{10} + x_{32} s \theta_{21}) s \phi + y c \phi s \psi) + s \theta (-y s \phi + c \phi (z - z_{10} + x_{32} s \theta_{21}) s \psi))$$

$$G_6 = 2 x_{1918} (-c \phi s \theta (z - z_{10} + x_{32} s \theta_{21}) + s \phi ((x - x_{32} c \theta_{21}) c \psi + y s \theta s \psi) - c \theta (y c \phi + (z - z_{10} + x_{32} s \theta_{21}) s \phi s \psi))$$

$$\begin{aligned} G_7 = & 2 \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2 \dot{y} \dot{\phi} x_{1918} c \theta c \phi + 2 \dot{z} \dot{\phi} x_{1918} z_{10} c \theta c \phi + \dot{\psi}^2 x_{1918} c \phi c \psi - (\dot{\phi}^2 + \dot{\psi}^2) x_{1918} x_{32} c \theta_{21} c \phi c \psi + \right. \\ & \dot{\theta}_{21}^2 x_{32} c \theta_{21} (x - x_{1918} c \phi c \psi) - 2 \dot{z} \dot{\phi} x_{1918} c \phi s \theta - 2 \dot{\psi} x_{1918} c \phi c \psi ((-\dot{z} + \dot{\theta} y) c \theta + (\dot{y} + \dot{\theta} (z - z_{10})) s \theta) - \\ & 2 \dot{\theta} \dot{\phi} x_{1918} x_{32} c \theta c \phi s \theta_{21} - 2 \dot{\theta} \dot{\psi} x_{1918} x_{32} c \phi c \psi s \theta s \theta_{21} + \dot{\theta}^2 x_{1918} y c \theta s \phi + \dot{\phi}^2 x_{1918} y c \theta s \phi + \\ & \dot{\theta}^2 x_{1918} z s \theta s \phi + \dot{\phi}^2 x_{1918} z s \theta s \phi - (\dot{\theta}^2 + \dot{\phi}^2) x_{1918} z_{10} s \theta s \phi - 2 \dot{\phi} \dot{\psi} x_{1918} x_{32} c \theta c \psi s \theta_{21} s \phi + \\ & \dot{\theta}^2 x_{1918} x_{32} s \theta s \theta_{21} s \phi + \dot{\phi}^2 x_{1918} x_{32} s \theta s \theta_{21} s \phi + 2 \dot{\theta} x_{1918} s \theta (\dot{\phi} y c \phi + \dot{y} s \phi) - 2 \dot{\theta} x_{1918} c \theta (\dot{\phi} z c \phi + \dot{z} s \phi) + \\ & 2 \dot{x} \dot{\psi} x_{1918} c \phi s \psi - 2 \dot{y} \dot{\theta} x_{1918} c \theta c \phi s \psi - \dot{\theta}^2 x_{1918} z c \theta c \phi s \psi + \dot{\theta}^2 x_{1918} z_{10} c \theta c \phi s \psi - 2 \dot{z} \dot{\theta} x_{1918} c \phi s \theta s \psi + \\ & \dot{\theta}^2 x_{1918} y c \phi s \theta s \psi + \dot{\psi}^2 x_{1918} c \phi ((-z + z_{10}) c \theta + y s \theta) s \psi - (\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2) x_{1918} x_{32} c \theta c \phi s \theta_{21} s \psi + \\ & 2 \dot{\phi} \dot{\psi} x_{1918} x_{32} c \theta_{21} s \phi s \psi + 2 \dot{\theta} \dot{\phi} x_{1918} x_{32} s \theta s \theta_{21} s \phi s \psi + 2 \dot{\theta}_{21} x_{32} s \theta_{21} (\dot{x} + \dot{\phi} x_{1918} c \psi s \phi + \dot{\psi} x_{1918} c \phi s \psi) - \\ & \dot{\theta}_{21}^2 x_{32} s \theta_{21} (z - z_{10} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi) + \dot{\phi}^2 x_{1918} c \phi (x c \psi + ((-z + z_{10}) c \theta + y s \theta) s \psi) + \\ & 2 \dot{\phi} x_{1918} s \phi (\dot{x} c \psi + \dot{\psi} c \psi ((-z + z_{10}) c \theta + y s \theta) - \dot{\psi} x s \psi + ((-\dot{z} + \dot{\theta} y) c \theta + (\dot{y} + \dot{\theta} (z - z_{10})) s \theta) s \psi) + \\ & 2 \dot{\theta}_{21} x_{32} c \theta_{21} (\dot{z} - x_{1918} c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + x_{1918} c \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) \end{aligned}$$

Coeficientes de la ecuación (2.80)

$$G_8 = -s \theta_{43}$$

$$G_9 = -c \theta_{43}$$

$$G_{10} = 0$$

$$G_{11} = -x_{1918} c \phi (c \theta c \theta_{43} c \psi + s \theta_{43} s \psi)$$

$$G_{12} = x_{1918} c \theta_{43} (c \theta s \phi + c \phi s \theta s \psi)$$

$$G_{13} = x_{1918} (-c \psi s \theta_{43} s \phi + c \theta_{43} (c \phi s \theta + c \theta s \phi s \psi))$$

$$\begin{aligned}
G_{14} = & -\ddot{\theta}_{21} x_{32} c \theta_{21} c \theta_{43} + 2 \dot{\theta} \dot{\phi} x_{1918} c \theta c \theta_{43} c \phi + 2 \dot{\theta} \dot{\psi} x_{1918} c \theta_{43} c \phi c \psi s \theta + \dot{\theta}_{21}^2 x_{32} c \theta_{43} s \theta_{21} - \\
& \dot{\theta}_{21}^2 x_{32} c \theta_{21} s \theta_{43} - \left(\dot{\phi}^2 + \dot{\psi}^2 \right) x_{1918} c \phi c \psi s \theta_{43} - \ddot{\theta}_{21} x_{32} s \theta_{21} s \theta_{43} + 2 \dot{\phi} \dot{\psi} x_{1918} c \theta c \theta_{43} c \psi s \phi - \\
& \dot{\theta}^2 x_{1918} c \theta_{43} s \theta s \phi - \dot{\phi}^2 x_{1918} c \theta_{43} s \theta s \phi + \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 \right) x_{1918} c \theta c \theta_{43} c \phi s \psi - 2 \dot{\theta} \dot{\phi} x_{1918} c \theta_{43} s \theta s \phi s \psi + \\
& 2 \dot{\phi} \dot{\psi} x_{1918} s \theta_{43} s \phi s \psi - 2 \dot{\theta}_{43} c \theta_{43} \left(\dot{x} + \dot{\theta}_{21} x_{32} s \theta_{21} + \dot{\phi} x_{1918} c \psi s \phi + \dot{\psi} x_{1918} c \phi s \psi \right) + \\
& \dot{\theta}_{43}^2 s \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) + \\
& \dot{\theta}_{43}^2 c \theta_{43} ((x - x_{1918} c \phi c \psi) s \theta_{21} + c \theta_{21} (z - z_{10} - x_{1918} s \theta s \phi + x_{1918} c \theta c \phi s \psi)) + \\
& 2 \dot{\theta}_{43} s \theta_{43} (\dot{z} + \dot{\theta}_{21} x_{32} c \theta_{21} - \dot{\phi} x_{1918} c \phi s \theta - \dot{\theta} x_{1918} c \phi s \theta s \psi + x_{1918} c \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi)))
\end{aligned}$$

Coefficientes de la ecuación (2.82)

$$G_{15} = (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^3$$

$$\begin{aligned}
G_{16} = & -c \theta_{43} c \theta_{54}^2 \\
& (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi))
\end{aligned}$$

$$G_{17} = c \theta_{43} c \theta_{54}^2 (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2$$

$$G_{18} = 0$$

$$\begin{aligned}
G_{19} = & x_{1918} c \theta_{43} c \theta_{54}^2 c \phi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) \\
& (s \psi (-y + x_{1918} c \theta s \phi + x_{1918} c \phi s \theta s \psi) - \\
& c \psi s \theta (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)))
\end{aligned}$$

$$G_{20} = x_{1918} c \theta_{43} c \theta_{54}^2 (s \theta s \phi - c \theta c \phi s \psi) (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2$$

$$\begin{aligned}
G_{21} = & x_{1918} c \theta_{43} c \theta_{54}^2 (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) \\
& (c \psi s \phi (-y + x_{1918} c \theta s \phi + x_{1918} c \phi s \theta s \psi) - \\
& c \theta c \phi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) + \\
& s \theta s \phi s \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)))
\end{aligned}$$

$$G_{22} =$$

$$\begin{aligned}
& c \theta_{54}^2 \left(-2 \dot{\theta} \dot{\psi} x_{1918} c \theta c \theta_{43} c \phi c \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 + 2 \dot{\theta} \dot{\phi} \right. \\
& x_{1918} c \theta_{43} c \phi s \theta (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 + 2 \dot{\phi} \dot{\psi} x_{1918} \\
& c \theta_{43} c \psi s \theta s \phi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 + \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 \right) \\
& x_{1918} c \theta_{43} c \phi s \theta s \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 + x_{1918} c \theta \\
& c \theta_{43} s \phi \left(\dot{\theta}^2 + \dot{\phi}^2 + 2 \dot{\theta} \dot{\phi} s \psi \right) (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 + \\
& 2 c \theta_{43} \left(\dot{x} + \dot{\theta}_{21} x_{32} s \theta_{21} + \dot{\phi} x_{1918} c \psi s \phi + \dot{\psi} x_{1918} c \phi s \psi \right)^2 (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - \dot{\theta}_{21}^2 x_{32} c \theta_{21} c \theta_{43} \\
& (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - \\
& \left(\dot{\phi}^2 + \dot{\psi}^2 \right) x_{1918} c \theta_{43} c \phi c \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) \\
& (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - \ddot{\theta}_{21} x_{32} c \theta_{43} s \theta_{21} \\
& (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) +
\end{aligned}$$

$$\begin{aligned}
& 2 \dot{\phi} \dot{\psi} x_{1918} c \theta_{43} s \phi s \psi (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) \\
& (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) + 2 \dot{\theta}_{43} s \theta_{43} (\dot{x} + \dot{\theta}_{21} x_{32} s \theta_{21} + \dot{\phi} x_{1918} c \psi s \phi + \dot{\psi} x_{1918} c \phi s \psi) \\
& (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - \\
& \dot{\theta}_{43}^2 c \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 \\
& (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - \\
& \ddot{\theta}_{43} s \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 \\
& (y - x_{1918} (c \theta s \phi + c \phi s \theta s \psi)) - 2 c \theta_{43} (\dot{x} + \dot{\theta}_{21} x_{32} s \theta_{21} + \dot{\phi} x_{1918} c \psi s \phi + \dot{\psi} x_{1918} c \phi s \psi) \\
& (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi)) \\
& (\dot{y} - x_{1918} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi)))) - \\
& 2 \dot{\theta}_{43} s \theta_{43} (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^2 \\
& (\dot{y} - x_{1918} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi)))) - \\
& 2 \dot{\theta}_{54}^2 \sec(\theta_{54})^2 (-x_{32} + c \theta_{21} (x - x_{1918} c \phi c \psi) + s \theta_{21} (-z + z_{10} + x_{1918} s \theta s \phi - x_{1918} c \theta c \phi s \psi))^3 \tan(\theta_{54})
\end{aligned}$$

Coeficientes de la ecuación (2.82)

$$G_{23} = (s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi)))^3$$

$$G_{24} = 0$$

$$G_{25} = 0$$

$$G_{26} = 0$$

$$\begin{aligned}
G_{27} = & -c \theta_{1817}^2 (s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi) + c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi))) \\
& (c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi (c \theta_{54} s \theta s (\theta_{21} + \theta_{43}) - c \theta s \theta_{54}) + c (\theta_{21} + \theta_{43})^2 c \theta_{54}^2 s \phi + (c \theta c \theta_{54} s (\theta_{21} + \theta_{43}) + s \theta s \theta_{54}) \\
& (s \theta (s \theta_{54} s \phi + c \theta_{54} c \phi s (\theta_{21} + \theta_{43}) s \psi) + c \theta (c \theta_{43} c \theta_{54} s \theta_{21} s \phi + c \theta_{21} c \theta_{54} s \theta_{43} s \phi - c \phi s \theta_{54} s \psi)))
\end{aligned}$$

$$\begin{aligned}
G_{28} = & -c \theta_{1817}^2 (c \theta^2 c \phi c \psi (c \theta_{54}^2 s (\theta_{21} + \theta_{43})^2 + s \theta_{54}^2) - c \theta c (\theta_{21} + \theta_{43}) c \theta_{54} (s \theta_{54} s \phi + c \theta_{54} c \phi s (\theta_{21} + \theta_{43}) s \psi) + \\
& s \theta (c \phi c \psi s \theta s \theta_{54}^2 + c \theta_{54}^2 s (\theta_{21} + \theta_{43}) (c \phi c \psi s \theta s (\theta_{21} + \theta_{43}) + c (\theta_{21} + \theta_{43}) s \phi) - c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi s \theta_{54} s \psi)) \\
& (s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi) + c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi)))
\end{aligned}$$

$$\begin{aligned}
G_{29} = & -c \theta_{1817}^2 (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\
& (c (\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s (\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) \\
& (s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi) + c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi)))
\end{aligned}$$

$$\begin{aligned}
G_{30} = & -c \theta_{1817}^2 \left((\dot{\theta}^2 + \dot{\phi}^2) c \theta_{54} c \phi s \theta s (\theta_{21} + \theta_{43}) (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \right. \\
& (s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi))) - c \theta_{54} c \phi s \theta \\
& \left((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c (\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s (\theta_{21} + \theta_{43}) \right) (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\
& (s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi))) + \\
& \left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\psi}^2 - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\psi} c \theta \right) c (\theta_{21} + \theta_{43}) c \theta_{54} c \psi s \phi \\
& (c \theta c \theta_{54} c \psi s (\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c (\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\
& (s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s (\theta_{21} + \theta_{43}) + s \phi (c (\theta_{21} + \theta_{43}) c \psi + c \theta s (\theta_{21} + \theta_{43}) s \psi))) +
\end{aligned}$$

225

226

$$(s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - c \theta_{54} (c \phi s \theta s(\theta_{21} + \theta_{43}) + s \phi (c(\theta_{21} + \theta_{43}) c \psi + c \theta s(\theta_{21} + \theta_{43}) s \psi)))^3 \tan(\theta_{1817}))$$

Coefficientes de la ecuación (2.86)

$$G_{31} = (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi))^3$$

$$G_{32} = 0$$

$$G_{33} = 0$$

$$G_{34} = 0$$

$$G_{35} = c \theta_{1716}^2 \csc(\theta_{1817}) (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) \\ (c(\theta_{21} + \theta_{43})^2 c \theta_{54}^2 c \phi + c(\theta_{21} + \theta_{43}) c \theta_{54} c \psi (-c \theta_{54} s \theta s(\theta_{21} + \theta_{43}) + c \theta s \theta_{54}) s \phi + (c \theta c \theta_{54} s(\theta_{21} + \theta_{43}) + s \theta s \theta_{54}) \\ (s \theta (c \phi s \theta_{54} - c \theta_{54} s(\theta_{21} + \theta_{43}) s \phi s \psi) + c \theta (c \theta_{43} c \theta_{54} c \phi s \theta_{21} + c \theta_{21} c \theta_{54} c \phi s \theta_{43} + s \theta_{54} s \phi s \psi)))$$

$$G_{36} = c \theta_{1716}^2 \csc(\theta_{1817}) (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) \\ \left(\frac{1}{8} (-6 + 2 c(2(\theta_{21} + \theta_{43})) + c(2(\theta_{21} + \theta_{43} - \theta_{54})) + 2 c(2 \theta_{54}) + c(2(\theta_{21} + \theta_{43} + \theta_{54}))) c \psi s \phi + c(\theta_{21} + \theta_{43}) c \theta_{54} \right. \\ \left. (c \theta_{43} c \theta_{54} s \theta_{21} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{21} c \theta_{54} s \theta_{43} (c \phi s \theta + c \theta s \phi s \psi) + s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi)) \right)$$

$$G_{37} = -c \theta_{1716}^2 \csc(\theta_{1817}) (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) \\ (s \theta_{54} (-c \theta c \phi + s \theta s \phi s \psi) + c \theta_{54} (c \phi s \theta s(\theta_{21} + \theta_{43}) + s \phi (c(\theta_{21} + \theta_{43}) c \psi + c \theta s(\theta_{21} + \theta_{43}) s \psi)))$$

$$G_{38} = c \theta_{1716}^2 \left(-2 \dot{\theta} \dot{\phi} c \theta c \theta_{54} c \phi \csc(\theta_{1817}) s(\theta_{21} + \theta_{43}) (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \right. \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\ 2 \dot{\theta} \dot{\psi} c \theta_{54} c \phi c \psi \csc(\theta_{1817}) s \theta s(\theta_{21} + \theta_{43}) (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) + \\ 2 \dot{\theta} \dot{\psi} c \theta c \phi c \psi \csc(\theta_{1817}) s \theta_{54} (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\ 2 \dot{\theta} \dot{\phi} c \phi \csc(\theta_{1817}) s \theta s \theta_{54} (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) + \\ c \phi c \psi \csc(\theta_{1817}) \left(-c(\theta_{21} + \theta_{43}) \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 \right) c \theta_{54} + \ddot{\theta}_{54} s \theta_{54} \right) - s(\theta_{21} + \theta_{43}) \right. \\ \left. \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c \theta_{54} - 2 \left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) \dot{\theta}_{54} s \theta_{54} \right) \right) (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\ 2 \dot{\theta} \dot{\psi} c \theta c \theta_{54} c \psi \csc(\theta_{1817}) s(\theta_{21} + \theta_{43}) s \phi (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\ 2 \dot{\theta} \dot{\psi} c \psi \csc(\theta_{1817}) s \theta s \theta_{54} s \phi (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\ \left(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 \right) c \theta c \theta_{54} c \phi \csc(\theta_{1817}) s(\theta_{21} + \theta_{43}) s \psi (c \theta c \theta_{54} c \psi s(\theta_{21} + \theta_{43}) + c \psi s \theta s \theta_{54} - c(\theta_{21} + \theta_{43}) c \theta_{54} s \psi) \\ (c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + c \theta_{54} s(\theta_{21} + \theta_{43}) (-s \theta s \phi + c \theta c \phi s \psi) + s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) -$$

[illegible]

[illegible]

Coeficientes de la ecuación (2.88)

$$G_{39} = (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) + \\ c \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + \\ c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))))^3$$

$$G_{40} = 0$$

$$G_{41} = 0$$

$$G_{42} = 0$$

$$G_{43} = c \theta_{1615}^2 (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s \theta (s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi + c \theta_{54} c \phi s \psi)) + \\ c \theta_{1716} (s \theta_{54} (c \theta_{1817} c \phi s \theta s(\theta_{21} + \theta_{43}) + c \theta_{1817} c(\theta_{21} + \theta_{43}) c \psi s \phi + c(\theta_{21} + \theta_{43}) s \theta_{1817} s \psi) + \\ c \theta_{54} s \theta (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi)) + c \theta (s \theta_{1716} (c \theta_{54} s \phi - c \phi s(\theta_{21} + \theta_{43}) s \theta_{54} s \psi) + \\ c \theta_{1716} (c \theta_{1817} c \theta_{54} c \phi - c \psi s \theta_{1817} s(\theta_{21} + \theta_{43}) s \theta_{54} + c \theta_{1817} s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi))) \\ (c \theta^2 c(\theta_{21} + \theta_{43}) c \theta_{54} (c \theta_{1716} c \theta_{1817} c \phi + s \theta_{1716} s \phi) (-c \phi c \psi s \theta_{1716} + c \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi)) + \\ c \theta (c \phi s \theta_{1716}^2 (c \phi s \theta_{54} - c \theta_{54} s(\theta_{21} + \theta_{43}) s \phi s \psi) - c \theta_{1716} s \theta_{1716} (c \theta_{1817} s \theta_{54} s(2 \phi) + c \theta_{43} c \theta_{54} s \theta_{21} \\ (c \psi s \theta_{1817} s \phi + c \theta_{1817} c(2 \phi) s \psi) + c \theta_{21} c \theta_{54} s \theta_{43} (c \psi s \theta_{1817} s \phi + c \theta_{1817} c(2 \phi) s \psi)) + c \theta_{1716}^2 \\ (-c \theta_{1817} c \theta_{54} c \phi c \psi s \theta_{1817} s(\theta_{21} + \theta_{43}) + s \theta_{1817}^2 s \theta_{54} + c \theta_{1817}^2 s \phi (s \theta_{54} s \phi + c \theta_{54} c \phi s(\theta_{21} + \theta_{43}) s \psi))) + \\ s \theta (-c \phi s \theta_{1716}^2 (c \theta_{21} c \theta_{54} c \phi s \theta_{43} + c \theta_{43} c \theta_{54} (c \phi s \theta_{21} + c \theta_{21} c \psi s \theta s \phi) + s \phi \\ (-c \theta_{54} c \psi s \theta s \theta_{21} s \theta_{43} + s \theta_{54} s \psi)) + c \theta_{1716} s \theta_{1716} (s \theta_{1817} s \phi (-c \psi s \theta_{54} + c(\theta_{21} + \theta_{43}) c \theta_{54} s \theta s \psi) + \\ c \theta_{1817} (c \theta_{54} s \theta_{21} (c \phi^2 c \psi s \theta s \theta_{43} - c \psi s \theta s \theta_{43} s \phi^2 + c \theta_{43} s(2 \phi)) + c \theta_{21} c \theta_{54} (-c \theta_{43} c(2 \phi) c \psi s \theta + \\ s \theta_{43} s(2 \phi)) - c(2 \phi) s \theta_{54} s \psi)) - c \theta_{1716}^2 (c \theta_{1817} c \phi s \theta_{54} (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi) + \\ c \theta_{54} (-c \theta_{1817}^2 c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta s \phi + c \psi^2 s(\theta_{21} + \theta_{43}) (s \theta_{1817}^2 + c \theta_{1817}^2 s \phi^2) + \\ s \psi (-c \theta_{1817} c(\theta_{21} + \theta_{43}) c \phi s \theta s \theta_{1817} + s \theta_{1817}^2 s(\theta_{21} + \theta_{43}) s \psi + c \theta_{1817}^2 s(\theta_{21} + \theta_{43}) s \phi^2 s \psi))))))$$

$$G_{44} = c \theta_{1615}^2 (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s \theta (s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi + c \theta_{54} c \phi s \psi)) + \\ c \theta_{1716} (s \theta_{54} (c \theta_{1817} c \phi s \theta s(\theta_{21} + \theta_{43}) + c \theta_{1817} c(\theta_{21} + \theta_{43}) c \psi s \phi + c(\theta_{21} + \theta_{43}) s \theta_{1817} s \psi) + \\ c \theta_{54} s \theta (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi)) + c \theta (s \theta_{1716} (c \theta_{54} s \phi - c \phi s(\theta_{21} + \theta_{43}) s \theta_{54} s \psi) + \\ c \theta_{1716} (c \theta_{1817} c \theta_{54} c \phi - c \psi s \theta_{1817} s(\theta_{21} + \theta_{43}) s \theta_{54} + c \theta_{1817} s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi))) \\ (c \theta (-c \phi c \psi s \theta_{1716} + c \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi)) (s \theta_{1716} (s \theta_{54} s \phi + c \theta_{54} c \phi s(\theta_{21} + \theta_{43}) s \psi) + \\ c \theta_{1716} (c \theta_{1817} c \phi s \theta_{54} + c \theta_{54} s(\theta_{21} + \theta_{43}) (c \psi s \theta_{1817} - c \theta_{1817} s \phi s \psi))) + \\ c \theta^2 c(\theta_{21} + \theta_{43}) c \theta_{54} (2 c \theta_{1716} c \phi c \psi s \theta_{1716} (c \theta_{1817} c \psi s \phi + s \theta_{1817} s \psi) + s \theta_{1716}^2 (s \phi^2 + c \phi^2 s \psi^2) + \\ c \theta_{1716}^2 (c \psi^2 s \theta_{1817}^2 - 2 c \theta_{1817} c \psi s \theta_{1817} s \phi s \psi + c \theta_{1817}^2 (c \phi^2 + s \phi^2 s \psi^2))) + \\ s \theta (s \theta_{1716}^2 (-c \phi^2 c \psi s \theta_{54} s \psi + c \theta_{54} (c \phi c \psi s(\theta_{21} + \theta_{43}) s \phi + c(\theta_{21} + \theta_{43}) s \theta s \phi^2 + c(\theta_{21} + \theta_{43}) c \phi^2 s \theta s \psi^2)) + \\ c \theta_{1716} s \theta_{1716} (c \theta_{1817} c \psi (c \theta_{54} (c \phi^2 s(\theta_{21} + \theta_{43}) - s(\theta_{21} + \theta_{43}) s \phi^2 + c(\theta_{21} + \theta_{43}) c \psi s \theta s(2 \phi)) + \\ s \theta_{54} s(2 \phi) s \psi) - s \theta_{1817} (c \theta_{54} s(\theta_{21} + \theta_{43}) s \phi s \psi + \\ c \phi (c \psi^2 s \theta_{54} - 2 c \theta_{21} c \theta_{43} c \theta_{54} c \psi s \theta s \psi - s \theta_{54} s \psi^2 + c \theta_{54} s \theta s \theta_{21} s \theta_{43} s(2 \psi)))) - \\ c \theta_{1716}^2 (-c \psi s \theta_{1817}^2 (c \theta_{21} c \theta_{43} c \theta_{54} c \psi s \theta - c \theta_{54} c \psi s \theta s \theta_{21} s \theta_{43} + s \theta_{54} s \psi) + \\ c \theta_{1817}^2 (c(\theta_{21} + \theta_{43})^2 c \psi s \theta_{54} s \phi^2 s \psi + c \psi s(\theta_{21} + \theta_{43}) s \phi (c \theta_{54} c \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi) - \\ c(\theta_{21} + \theta_{43}) c \theta_{54} s \theta (c \phi^2 + s \phi^2 s \psi^2)) + c \theta_{1817} s \theta_{1817} (-c \psi^2 s \theta_{54} s \phi + c \theta_{21} c \theta_{54} c \phi s \theta_{43} s \psi + \\ s \theta_{54} s \phi s \psi^2 - c \theta_{54} s \theta s \theta_{21} s \theta_{43} s \phi s(2 \psi) + c \theta_{43} c \theta_{54} (c \phi s \theta_{21} s \psi + c \theta_{21} s \theta s \phi s(2 \psi))))))$$

$$G_{45} = c \theta_{1615}^2 (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s \theta (s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi + c \theta_{54} c \phi s \psi)) +$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

240

241

[illegible]

[illegible]

$$\begin{aligned}
& (s\theta_{1716}(-c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+s(\theta_{21}+\theta_{43})s\theta_{54}(s\theta s\phi-c\theta c\phi s\psi))+ \\
& c\theta_{1716}(s\theta_{1817}(c\theta_{54}c\psi s\theta+s\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi))+ \\
& c\theta_{1817}(c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)+c\theta_{54}(c\theta c\phi-s\theta s\phi s\psi)))) \\
& (\dot{\theta}_{1817}c\theta_{1817}(c\theta_{54}c\psi s\theta+s\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi))- \\
& \dot{\theta}_{1817}s\theta_{1817}(c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)+c\theta_{54}(c\theta c\phi-s\theta s\phi s\psi))+ \\
& s\theta_{1817}(\dot{\theta}c\theta c\theta_{54}c\psi-\dot{\theta}_{54}c\psi s\theta s\theta_{54}-\dot{\psi}c\theta_{54}s\theta s\psi+\dot{\theta}_{54}c\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi)+ \\
& s\theta_{54}((\dot{\psi}-(\dot{\theta}_{21}+\dot{\theta}_{43})c\theta)c(\theta_{21}+\theta_{43})c\psi+s(\theta_{21}+\theta_{43})(\dot{\theta}c\psi s\theta-(\dot{\theta}_{21}+\dot{\theta}_{43}-\dot{\psi}c\theta)s\psi))) + \\
& c\theta_{1817}(\dot{\phi}c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+\dot{\theta}_{54}c(\theta_{21}+\theta_{43})c\theta_{54}c\psi s\phi-(\dot{\theta}_{21}+\dot{\theta}_{43})c\psi s(\theta_{21}+\theta_{43})s\theta_{54}s\phi- \\
& \dot{\psi}c(\theta_{21}+\theta_{43})s\theta_{54}s\phi s\psi+\dot{\theta}_{54}c\theta_{54}s(\theta_{21}+\theta_{43})(c\phi s\theta+c\theta s\phi s\psi)+ \\
& (\dot{\theta}_{21}+\dot{\theta}_{43})c(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)-\dot{\theta}_{54}s\theta_{54}(c\theta c\phi-s\theta s\phi s\psi)- \\
& c\theta_{54}(c\phi s\theta(\dot{\theta}+\dot{\phi}s\psi)+s\phi(\dot{\psi}c\psi s\theta+c\theta(\dot{\phi}+\dot{\theta}s\psi)))) + \\
& s(\theta_{21}+\theta_{43})s\theta_{54}(-s\theta s\phi(\dot{\phi}+\dot{\theta}s\psi)+c\theta(\dot{\psi}c\psi s\phi+c\phi(\dot{\theta}+\dot{\phi}s\psi)))))) - \\
& 2\dot{\theta}_{1716}s\theta_{1716}(s\theta_{1716}(-c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+s(\theta_{21}+\theta_{43})s\theta_{54}(s\theta s\phi-c\theta c\phi s\psi))+c\theta_{54}(c\theta s\phi+c\phi s\theta s\psi))+ \\
& c\theta_{1716}(s\theta_{1817}(c\theta_{54}c\psi s\theta+s\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi))+c\theta_{1817} \\
& (c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)+c\theta_{54}(c\theta c\phi-s\theta s\phi s\psi))))^2 \\
& (-c(\theta_{21}+\theta_{43})(\dot{\theta}_{1817}c\phi s\theta s\theta_{1817}+\dot{\phi}c\theta_{1817}s\theta s\phi+c\psi(-\dot{\theta}s\theta s\theta_{1817}+(\dot{\theta}_{21}+\dot{\theta}_{43})c\theta_{1817}s\phi)+ \\
& \dot{\theta}_{21}s\theta_{1817}s\psi+\dot{\theta}_{43}s\theta_{1817}s\psi+\dot{\theta}c\theta_{1817}s\theta s\phi s\psi)- \\
& s(\theta_{21}+\theta_{43})(c\psi s\theta_{1817}(\dot{\psi}-\dot{\theta}_{1817}s\phi)+c\theta_{1817}(c\phi(\dot{\phi}c\psi+(\dot{\theta}_{21}+\dot{\theta}_{43})s\theta)+(\dot{\theta}_{1817}-\dot{\psi}s\phi)s\psi))+ \\
& c\theta(s\theta_{1817}((\dot{\theta}_{21}+\dot{\theta}_{43})c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})(\dot{\psi}-\dot{\theta}_{1817}s\phi)s\psi)+ \\
& c\theta_{1817}(-(\dot{\theta}_{21}+\dot{\theta}_{43})s(\theta_{21}+\theta_{43})s\phi s\psi+c(\theta_{21}+\theta_{43})(c\psi(-\dot{\theta}_{1817}+\dot{\psi}s\phi)+c\phi(\dot{\theta}+\dot{\phi}s\psi)))))) + \\
& 2(s\theta_{1716}(c\phi c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})(s\theta s\phi-c\theta c\phi s\psi))+ \\
& c\theta_{1716}(-s\theta_{1817}(c\theta c(\theta_{21}+\theta_{43})c\psi+s(\theta_{21}+\theta_{43})s\psi)+ \\
& c\theta_{1817}(-c\psi s(\theta_{21}+\theta_{43})s\phi+c(\theta_{21}+\theta_{43})(c\phi s\theta+c\theta s\phi s\psi)))) \\
& (\dot{\theta}_{1716}c\theta_{1716}(-c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+s(\theta_{21}+\theta_{43})s\theta_{54}(s\theta s\phi-c\theta c\phi s\psi))+c\theta_{54}(c\theta s\phi+c\phi s\theta s\psi))- \\
& \dot{\theta}_{1716}s\theta_{1716}(s\theta_{1817}(c\theta_{54}c\psi s\theta+s\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi))+c\theta_{1817} \\
& (c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)+c\theta_{54}(c\theta c\phi-s\theta s\phi s\psi)))) + \\
& s\theta_{1716}(-\dot{\theta}_{54}c(\theta_{21}+\theta_{43})c\theta_{54}c\phi c\psi+(\dot{\theta}_{21}+\dot{\theta}_{43})c\phi c\psi s(\theta_{21}+\theta_{43})s\theta_{54}+ \\
& \dot{\phi}c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+\dot{\psi}c(\theta_{21}+\theta_{43})c\phi s\theta_{54}s\psi+ \\
& \dot{\theta}_{54}c\theta_{54}s(\theta_{21}+\theta_{43})(s\theta s\phi-c\theta c\phi s\psi)+(\dot{\theta}_{21}+\dot{\theta}_{43})c(\theta_{21}+\theta_{43})s\theta_{54}(s\theta s\phi-c\theta c\phi s\psi)- \\
& \dot{\theta}_{54}s\theta_{54}(c\theta s\phi+c\phi s\theta s\psi)+c\theta_{54}(c\theta c\phi(\dot{\phi}+\dot{\theta}s\psi)+s\theta(\dot{\psi}c\phi c\psi-s\phi(\dot{\theta}+\dot{\phi}s\psi)))+ \\
& s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta(\dot{\phi}+\dot{\theta}s\psi)+c\theta(-\dot{\psi}c\phi c\psi+s\phi(\dot{\theta}+\dot{\phi}s\psi)))) + \\
& c\theta_{1716}(\dot{\theta}_{1817}c\theta_{1817}(c\theta_{54}c\psi s\theta+s\theta_{54}(-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi))-\dot{\theta}_{1817}s\theta_{1817} \\
& (c(\theta_{21}+\theta_{43})c\psi s\theta_{54}s\phi+s(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)+c\theta_{54}(c\theta c\phi-s\theta s\phi s\psi))+ \\
& s\theta_{1817}(\dot{\theta}c\theta c\theta_{54}c\psi-\dot{\theta}_{54}c\psi s\theta s\theta_{54}-\dot{\psi}c\theta_{54}s\theta s\psi+\dot{\theta}_{54}c\theta_{54} \\
& (-c\theta c\psi s(\theta_{21}+\theta_{43})+c(\theta_{21}+\theta_{43})s\psi)+s\theta_{54}((\dot{\psi}-(\dot{\theta}_{21}+\dot{\theta}_{43})c\theta)c(\theta_{21}+\theta_{43})c\psi+ \\
& s(\theta_{21}+\theta_{43})(\dot{\theta}c\psi s\theta-(\dot{\theta}_{21}+\dot{\theta}_{43}-\dot{\psi}c\theta)s\psi)))+c\theta_{1817}(\dot{\phi}c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+ \\
& \dot{\theta}_{54}c(\theta_{21}+\theta_{43})c\theta_{54}c\psi s\phi-(\dot{\theta}_{21}+\dot{\theta}_{43})c\psi s(\theta_{21}+\theta_{43})s\theta_{54}s\phi-\dot{\psi}c(\theta_{21}+\theta_{43})s\theta_{54}s\phi s\psi+ \\
& \dot{\theta}_{54}c\theta_{54}s(\theta_{21}+\theta_{43})(c\phi s\theta+c\theta s\phi s\psi)+(\dot{\theta}_{21}+\dot{\theta}_{43})c(\theta_{21}+\theta_{43})s\theta_{54}(c\phi s\theta+c\theta s\phi s\psi)- \\
& \dot{\theta}_{54}s\theta_{54}(c\theta c\phi-s\theta s\phi s\psi)-c\theta_{54}(c\phi s\theta(\dot{\theta}+\dot{\phi}s\psi)+s\phi(\dot{\psi}c\psi s\theta+c\theta(\dot{\phi}+\dot{\theta}s\psi)))+ \\
& s(\theta_{21}+\theta_{43})s\theta_{54}(-s\theta s\phi(\dot{\phi}+\dot{\theta}s\psi)+c\theta(\dot{\psi}c\psi s\phi+c\phi(\dot{\theta}+\dot{\phi}s\psi))))^2 - \\
& 2(s\theta_{1716}(-c(\theta_{21}+\theta_{43})c\phi c\psi s\theta_{54}+s(\theta_{21}+\theta_{43})s\theta_{54}(s\theta s\phi-c\theta c\phi s\psi))+c\theta_{54}(c\theta s\phi+c\phi s\theta s\psi))+
\end{aligned}$$

$$\begin{aligned}
& c \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + \\
& c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))) \\
& (\dot{\theta}_{1716} c \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) - \\
& \dot{\theta}_{1716} s \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + \\
& c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))) + \\
& s \theta_{1716} (-\dot{\theta}_{54} c(\theta_{21} + \theta_{43}) c \theta_{54} c \phi c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \phi c \psi s(\theta_{21} + \theta_{43}) s \theta_{54} + \dot{\phi} c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + \\
& \dot{\psi} c(\theta_{21} + \theta_{43}) c \phi s \theta_{54} s \psi + \dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi) + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) - \dot{\theta}_{54} s \theta_{54} (c \theta s \phi + c \phi s \theta s \psi) + \\
& c \theta_{54} (c \theta c \phi (\dot{\phi} + \dot{\theta} s \psi) + s \theta (\dot{\psi} c \phi c \psi - s \phi (\dot{\theta} + \dot{\phi} s \psi))) + \\
& s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta (\dot{\phi} + \dot{\theta} s \psi) + c \theta (-\dot{\psi} c \phi c \psi + s \phi (\dot{\theta} + \dot{\phi} s \psi)))) + \\
& c \theta_{1716} (\dot{\theta}_{1817} c \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) - \dot{\theta}_{1817} s \theta_{1817} \\
& (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi)) + s \theta_{1817} \\
& (\dot{\theta} c \theta c \theta_{54} c \psi - \dot{\theta}_{54} c \psi s \theta s \theta_{54} - \dot{\psi} c \theta_{54} s \theta s \psi + \dot{\theta}_{54} c \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi) + \\
& s \theta_{54} ((\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) (\dot{\theta} c \psi s \theta - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) s \psi))) + \\
& c \theta_{1817} (\dot{\phi} c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + \dot{\theta}_{54} c(\theta_{21} + \theta_{43}) c \theta_{54} c \psi s \phi - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) s \theta_{54} s \phi - \\
& \dot{\psi} c(\theta_{21} + \theta_{43}) s \theta_{54} s \phi s \psi + \dot{\theta}_{54} c \theta_{54} s(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi) + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) - \dot{\theta}_{54} s \theta_{54} (c \theta c \phi - s \theta s \phi s \psi) - \\
& c \theta_{54} (c \phi s \theta (\dot{\theta} + \dot{\phi} s \psi) + s \phi (\dot{\psi} c \psi s \theta + c \theta (\dot{\phi} + \dot{\theta} s \psi))) + \\
& s(\theta_{21} + \theta_{43}) s \theta_{54} (-s \theta s \phi (\dot{\phi} + \dot{\theta} s \psi) + c \theta (\dot{\psi} c \psi s \phi + c \phi (\dot{\theta} + \dot{\phi} s \psi)))))) \\
& (\dot{\theta}_{1716} c \theta_{1716} (c \phi c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (s \theta s \phi - c \theta c \phi s \psi)) - \dot{\theta}_{1716} s \theta_{1716} \\
& (-s \theta_{1817} (c \theta c(\theta_{21} + \theta_{43}) c \psi + s(\theta_{21} + \theta_{43}) s \psi) + \\
& c \theta_{1817} (-c \psi s(\theta_{21} + \theta_{43}) s \phi + c(\theta_{21} + \theta_{43}) (c \phi s \theta + c \theta s \phi s \psi))) + \\
& s \theta_{1716} (-s(\theta_{21} + \theta_{43}) (\dot{\phi} c \psi s \phi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta s \phi + (\dot{\psi} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta) c \phi s \psi) + \\
& c(\theta_{21} + \theta_{43}) (c \theta s \phi (\dot{\theta} + \dot{\phi} s \psi) + c \phi ((\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\psi} c \theta) c \psi + s \theta (\dot{\phi} + \dot{\theta} s \psi)))) + \\
& c \theta_{1716} (-c(\theta_{21} + \theta_{43}) (\dot{\theta}_{1817} c \phi s \theta s \theta_{1817} + \dot{\phi} c \theta_{1817} s \theta s \phi + c \psi (-\dot{\theta} s \theta s \theta_{1817} + (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1817} s \phi) + \\
& \dot{\theta}_{21} s \theta_{1817} s \psi + \dot{\theta}_{43} s \theta_{1817} s \psi + \dot{\theta} c \theta_{1817} s \theta s \phi s \psi) - \\
& s(\theta_{21} + \theta_{43}) (c \psi s \theta_{1817} (\dot{\psi} - \dot{\theta}_{1817} s \phi) + c \theta_{1817} (c \phi (\dot{\phi} c \psi + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta) + (\dot{\theta}_{1817} - \dot{\psi} s \phi) s \psi)) + \\
& c \theta (s \theta_{1817} ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) (\dot{\psi} - \dot{\theta}_{1817} s \phi) s \psi) + c \theta_{1817} (- (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \\
& s \phi s \psi + c(\theta_{21} + \theta_{43}) (c \psi (-\dot{\theta}_{1817} + \dot{\psi} s \phi) + c \phi (\dot{\theta} + \dot{\phi} s \psi)))))) - 2 \dot{\theta}_{1615}^2 \sec(\theta_{1615})^2 \\
& (s \theta_{1716} (-c(\theta_{21} + \theta_{43}) c \phi c \psi s \theta_{54} + s(\theta_{21} + \theta_{43}) s \theta_{54} (s \theta s \phi - c \theta c \phi s \psi) + c \theta_{54} (c \theta s \phi + c \phi s \theta s \psi)) + \\
& c \theta_{1716} (s \theta_{1817} (c \theta_{54} c \psi s \theta + s \theta_{54} (-c \theta c \psi s(\theta_{21} + \theta_{43}) + c(\theta_{21} + \theta_{43}) s \psi)) + c \theta_{1817} (c(\theta_{21} + \theta_{43}) c \psi s \theta_{54} \\
& s \phi + s(\theta_{21} + \theta_{43}) s \theta_{54} (c \phi s \theta + c \theta s \phi s \psi) + c \theta_{54} (c \theta c \phi - s \theta s \phi s \psi))))^3 \tan(\theta_{1615}))
\end{aligned}$$

Coeficientes de la ecuación (2.161)

$$G_{47} = 0$$

$$G_{48} = 0$$

$$G_{49} = 0$$

$$G_{50} = 0$$

$$G_{51} = 0$$

$$G_{52} = 0$$

$$\begin{aligned}
G_{53} = & 4 \dot{\theta}_{21} \dot{\theta}_{54} x_{43} i c \theta_{21} i (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) - \\
& 2 x_{32} z_{10} (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) - 2 x_{32} z_{32} i (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) - 2 x_{32} z_{1312} i (\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43}) - \\
& 2 z_{1312} i (z_{10} + z_{32} i) \left(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) + \\
& \dot{\theta}_{21}^2 x_{43} i c \theta_{21} i (2 x_{32} c \theta_{21} - x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) - \\
& x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) + x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) - 2 z_{1312} i s(\theta_{21} + \theta_{43})) + \\
& 2 \text{signo } x_{1211} i y_{10} i (\ddot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413} i - \theta_{54})) - \\
& x_{1211} i x_{32} \left(-(\dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\beta_{1413} i - \theta_{43} - \theta_{54}) + (\ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\beta_{1413} i - \theta_{43} - \theta_{54}) \right) - \\
& x_{1211} i z_{10} \left(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) \right) - \\
& x_{1211} i z_{32} i \left(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) \right) + \\
& x_{32} x_{65} \left(-(\dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\theta_{43} - \theta_{54}) + (-\ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\theta_{43} - \theta_{54}) \right) - \\
& x_{1211} i x_{32} \left(-(\dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\beta_{1413} i + \theta_{43} - \theta_{54}) + (-\ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\beta_{1413} i + \theta_{43} - \theta_{54}) \right) - \\
& x_{65} z_{10} \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} - \theta_{54}) \right) - \\
& x_{65} z_{32} i \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} - \theta_{54}) \right) + \\
& x_{1211} i z_{10} \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) \right) + \\
& x_{1211} i z_{32} i \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) \right) - \\
& 2 \text{signo } x_{65} y_{10} i (\ddot{\theta}_{54} c \theta_{54} - \dot{\theta}_{54}^2 s \theta_{54}) + 2 \dot{\theta}_{21}^2 x_{43} i s \theta_{21} i (-\text{signo } y_{10} i + x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54}) + \\
& 2 x_{43} i s \theta_{21} i (\ddot{\theta}_{54} x_{1211} i c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} x_{65} c \theta_{54} + \dot{\theta}_{54}^2 (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) + \\
& x_{32} x_{65} \left(-(\dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\theta_{43} + \theta_{54}) - (\ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\theta_{43} + \theta_{54}) \right) - \\
& x_{65} z_{10} \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} + \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} + \theta_{54}) \right) - \\
& x_{65} z_{32} i \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} + \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} + \theta_{54}) \right) + \\
& 2 \dot{\theta}_{21} i x_{43} i s \theta_{21} i (-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i c(\theta_{21} + \theta_{43}) - 2 \dot{\theta}_{21} x_{32} s \theta_{21} - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} - \theta_{54}) + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} + \theta_{54})) - \\
& x_{43} i c \theta_{21} i (-2 x_{32} (\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) - 2 z_{1312} i ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c(\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s(\theta_{21} + \theta_{43})) - \\
& x_{1211} i \left(-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + (\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) \right) + \\
& x_{65} \left(-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\theta_{21} + \theta_{43} - \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) s(\theta_{21} + \theta_{43} - \theta_{54}) \right) - \\
& x_{1211} i \left(-(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) \right) + \\
& x_{65} \left(-(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\theta_{21} + \theta_{43} + \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\theta_{21} + \theta_{43} + \theta_{54}) \right)
\end{aligned}$$

Coeficientes de la ecuación (2.164)

$$G_{54} = 0$$

$$G_{55} = 0$$

$$G_{56} = 0$$

$$G_{57} = 0$$

$$G_{58} = 0$$

$$G_{59} = 0$$

$$\begin{aligned}
 G_{60} = & \dot{\theta}_{65}^2 \left(c \theta_{65} \left(z_{32} i c \beta_{54} i + \text{signo } y_{10} i c \theta_{21} i s \beta_{54} i + c \beta_{54} i (z_{10} - x_{32} s \theta_{21}) + x_{32} c \theta_{21} s \beta_{54} i s \theta_{21} i + \right. \right. \\
 & (c \beta_{54} i s \theta_{21} - c \theta_{21} s \beta_{54} i s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
 & (c \beta_{54} i c \theta_{21} + s \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) - \\
 & \left. c \theta_{21} i s \beta_{54} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54}) \right) - \\
 & \dot{\theta}_{65} i c \theta_{65} i \left(-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i c(\theta_{21} + \theta_{43}) - 2 \dot{\theta}_{21} x_{32} s \theta_{21} + 2 \dot{\theta}_{21} i x_{43} i s \theta_{21} i - \right. \\
 & (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} - \theta_{54}) + \\
 & \left. (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} + \theta_{54}) \right) + \\
 & c \theta_{65} i \left(-s \beta_{54} i \left(\ddot{\theta}_{21} i x_{43} i c \theta_{21} i + \ddot{\theta}_{54} x_{1211} i c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} x_{65} c \theta_{54} - \right. \right. \\
 & \left. \dot{\theta}_{21}^2 x_{43} i s \theta_{21} i + \dot{\theta}_{54}^2 x_{1211} i s(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54}^2 x_{65} s \theta_{54} \right) - \\
 & \frac{1}{2} c \beta_{54} i \left(-2 \ddot{\theta}_{21} x_{32} c \theta_{21} + 2 \dot{\theta}_{21}^2 x_{32} s \theta_{21} - 2 z_{1312} i \left(-(\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43}) \right) - \right. \\
 & x_{1211} i \left(-(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) \right) - \\
 & x_{65} \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} - \theta_{54}) \right) + \\
 & x_{1211} i \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) \right) - \\
 & \left. x_{65} \left((\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} + \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} + \theta_{54}) \right) \right) \Bigg) + \\
 & 2 \dot{\theta}_{65} i \left((\dot{\theta}_{21} i x_{43} i c \theta_{21} i + \dot{\theta}_{54} x_{1211} i c(\beta_{1413} i - \theta_{54}) - \dot{\theta}_{54} x_{65} c \theta_{54}) s \beta_{54} i + \right. \\
 & \frac{1}{2} c \beta_{54} i \left(-2 \dot{\theta}_{21} x_{32} c \theta_{21} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - \right. \\
 & (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - \\
 & \left. \left. (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i s(\theta_{21} + \theta_{43}) \right) \right) s \theta_{65} i + \\
 & \dot{\theta}_{65}^2 \left(-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \right. \\
 & \left. c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \right.
 \end{aligned}$$

$$\begin{aligned}
& x_{1211} \text{ i } s \theta_{21} \text{ i } s(\beta_{1413} \text{ i } - \theta_{54}) + x_{65} s \theta_{21} \text{ i } s \theta_{54}) s \theta_{65} \text{ i } - \\
& \frac{1}{2} \left(-2 x_{32} \left(\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21} \right) + 2 x_{43} \text{ i } \left(\dot{\theta}_{21}^2 c \theta_{21} \text{ i } + \ddot{\theta}_{21} \text{ i } s \theta_{21} \text{ i } \right) - \right. \\
& 2 z_{1312} \text{ i } \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) c(\theta_{21} + \theta_{43}) - \left(\dot{\theta}_{21} + \dot{\theta}_{43} \right)^2 s(\theta_{21} + \theta_{43}) \right) - \\
& x_{1211} \text{ i } \left(-\left(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54} \right)^2 c(\beta_{1413} \text{ i } - \theta_{21} - \theta_{43} - \theta_{54}) + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54} \right) s(\beta_{1413} \text{ i } - \theta_{21} - \theta_{43} - \theta_{54}) \right) + \\
& x_{65} \left(-\left(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} \right)^2 c(\theta_{21} + \theta_{43} - \theta_{54}) - \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54} \right) s(\theta_{21} + \theta_{43} - \theta_{54}) \right) - \\
& x_{1211} \text{ i } \left(-\left(\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54} \right)^2 c(\beta_{1413} \text{ i } + \theta_{21} + \theta_{43} - \theta_{54}) - \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54} \right) s(\beta_{1413} \text{ i } + \theta_{21} + \theta_{43} - \theta_{54}) \right) + \\
& x_{65} \left(-\left(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54} \right)^2 c(\theta_{21} + \theta_{43} + \theta_{54}) - \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54} \right) s(\theta_{21} + \theta_{43} + \theta_{54}) \right) \left. \right) s \theta_{65} \text{ i}
\end{aligned}$$

Coeficientes de la ecuación (2.166)

$$\begin{aligned}
G_{61} = & (-x_{43} \text{ i } - \text{signo } y_{10} \text{ i } s \theta_{21} \text{ i } + c \theta_{21} c \theta_{21} \text{ i } (x_{32} + c \theta_{43} (-x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} \text{ i } s \theta_{43}) - c \theta_{21} \text{ i } s \theta_{21} \\
& (z_{1312} \text{ i } c \theta_{43} - x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + x_{1211} \text{ i } s \theta_{21} \text{ i } s(\beta_{1413} \text{ i } - \theta_{54}) + x_{65} s \theta_{21} \text{ i } s \theta_{54})^3
\end{aligned}$$

$$G_{62} = 0$$

$$G_{63} = 0$$

$$G_{64} = 0$$

$$G_{65} = 0$$

$$G_{66} = 0$$

$$G_{67} = 0$$

$$G_{68} =$$

$$\begin{aligned}
& -c \theta_{76} \text{ i }^2 \left(c \theta_{65} \text{ i } \left(\left(-c \beta_{54} \text{ i } (\dot{\theta}_{21} \text{ i } x_{43} \text{ i } c \theta_{21} \text{ i } + \dot{\theta}_{54} x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) - \dot{\theta}_{54} x_{65} c \theta_{54}) + \frac{1}{2} s \beta_{54} \text{ i } (-2 \dot{\theta}_{21} x_{32} c \theta_{21} + (\dot{\theta}_{21} + \right. \right. \\
& \quad \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) + \\
& \quad (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} \text{ i } c(\beta_{1413} \text{ i } + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} \\
& \quad \left. \left. c(\theta_{21} + \theta_{43} + \theta_{54}) + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} \text{ i } s(\theta_{21} + \theta_{43}) \right) \right) \right) \\
& (-x_{43} \text{ i } - \text{signo } y_{10} \text{ i } s \theta_{21} \text{ i } + c \theta_{21} c \theta_{21} \text{ i } (x_{32} + c \theta_{43} (-x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} \text{ i } s \theta_{43}) - \\
& c \theta_{21} \text{ i } s \theta_{21} (z_{1312} \text{ i } c \theta_{43} - x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + x_{1211} \text{ i } s \theta_{21} \text{ i } s(\beta_{1413} \text{ i } - \theta_{54}) + \\
& x_{65} s \theta_{21} \text{ i } s \theta_{54}) (-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} \text{ i } c(\theta_{21} + \theta_{43}) - 2 \dot{\theta}_{21} x_{32} s \theta_{21} + 2 \dot{\theta}_{21} \text{ i } x_{43} \text{ i } s \theta_{21} \text{ i } - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} \text{ i } s(\beta_{1413} \text{ i } - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} - \theta_{54}) + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} \text{ i } s(\beta_{1413} \text{ i } + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} + \theta_{54})) - \\
& \frac{1}{2} (-\text{signo } y_{10} \text{ i } c \beta_{54} \text{ i } c \theta_{21} \text{ i } + z_{32} \text{ i } s \beta_{54} \text{ i } + s \beta_{54} \text{ i } (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} \text{ i } c \theta_{21} s \theta_{21} \text{ i } + \\
& (s \beta_{54} \text{ i } s \theta_{21} + c \beta_{54} \text{ i } c \theta_{21} s \theta_{21} \text{ i}) (c \theta_{43} (x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} \text{ i } s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} \text{ i } - c \beta_{54} \text{ i } s \theta_{21} s \theta_{21} \text{ i}) (z_{1312} \text{ i } c \theta_{43} + (-x_{1211} \text{ i } c(\beta_{1413} \text{ i } - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) +
\end{aligned}$$

$$\begin{aligned}
& c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) \\
& (-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i c(\theta_{21} + \theta_{43}) - 2 \dot{\theta}_{21} x_{32} s \theta_{21} + 2 \dot{\theta}_{21} i x_{43} i s \theta_{21} i - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} - \theta_{54}) + \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} + \theta_{54}))^2 + \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54})) \\
& \left(\dot{\theta}_{65} i^2 (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - \right. \\
& z_{1312} i s \theta_{43}) - c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + \\
& x_{65} c \theta_{54} s \theta_{43}) + x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54})) \\
& (-\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i + z_{32} i s \beta_{54} i + s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i + \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) + \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) + \\
& \frac{1}{2} (-\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i + z_{32} i s \beta_{54} i + s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i + \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) + \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) \\
& (-2 x_{32} (\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) + 2 x_{43} i (\dot{\theta}_{21}^2 c \theta_{21} i + \ddot{\theta}_{21} i s \theta_{21} i) - \\
& 2 z_{1312} i ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c(\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s(\theta_{21} + \theta_{43})) - x_{1211} i \\
& (-\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) + (\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54})) + \\
& x_{65} (-\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\theta_{21} + \theta_{43} - \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) s(\theta_{21} + \theta_{43} - \theta_{54})) - x_{1211} i \\
& (-\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54})) + \\
& x_{65} (-\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 c(\theta_{21} + \theta_{43} + \theta_{54}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54}) s(\theta_{21} + \theta_{43} + \theta_{54})) - \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54})) \\
& \left(-c \beta_{54} i (\ddot{\theta}_{21} i x_{43} i c \theta_{21} i + \ddot{\theta}_{54} x_{1211} i c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} x_{65} c \theta_{54} - \dot{\theta}_{21} i^2 x_{43} i s \theta_{21} i + \right. \\
& \dot{\theta}_{54}^2 x_{1211} i s(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54}^2 x_{65} s \theta_{54})) + \frac{1}{2} s \beta_{54} i (-2 \ddot{\theta}_{21} x_{32} c \theta_{21} + 2 \dot{\theta}_{21}^2 x_{32} s \theta_{21} - \\
& 2 z_{1312} i (-\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43})) - x_{1211} i ((-\ddot{\theta}_{21} - \ddot{\theta}_{43} - \ddot{\theta}_{54}) \\
& c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54})^2 s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54})) - \\
& x_{65} ((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\theta_{21} + \theta_{43} - \theta_{54}) - (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\theta_{21} + \theta_{43} - \theta_{54})) + \\
& x_{1211} i ((\ddot{\theta}_{21} + \ddot{\theta}_{43} - \ddot{\theta}_{54}) c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54})^2 s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54})) -
\end{aligned}$$

$$\begin{aligned}
& x_{65} \left(\left(\ddot{\theta}_{21} + \ddot{\theta}_{43} + \ddot{\theta}_{54} \right) c(\theta_{21} + \theta_{43} + \theta_{54}) - \left(\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54} \right)^2 s(\theta_{21} + \theta_{43} + \theta_{54}) \right) \Big) \Big) \Big) + \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} \\
& (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) \\
& \left(\left(2 \dot{\theta}_{65} i \left(-c \beta_{54} i (\dot{\theta}_{21} i x_{43} i c \theta_{21} i + \dot{\theta}_{54} x_{1211} i c(\beta_{1413} i - \theta_{54}) - \dot{\theta}_{54} x_{65} c \theta_{54}) + \right. \right. \right. \\
& \frac{1}{2} s \beta_{54} i (-2 \dot{\theta}_{21} x_{32} c \theta_{21} + (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i c(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} c(\theta_{21} + \theta_{43} - \theta_{54}) + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i c(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - \\
& \left. \left. (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} c(\theta_{21} + \theta_{43} + \theta_{54}) + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i s(\theta_{21} + \theta_{43}) \right) \right) \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - \\
& z_{1312} i s \theta_{43}) - c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) + \ddot{\theta}_{65} i (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + \\
& c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54}) \\
& (-\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i + z_{32} i s \beta_{54} i + s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i + \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) + \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) - \\
& \dot{\theta}_{65} i (-\text{signo } y_{10} i c \beta_{54} i c \theta_{21} i + z_{32} i s \beta_{54} i + s \beta_{54} i (z_{10} - x_{32} s \theta_{21}) - x_{32} c \beta_{54} i c \theta_{21} s \theta_{21} i + \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (c \theta_{43} (x_{1211} i c(\beta_{1413} i - \theta_{54}) - x_{65} c \theta_{54}) + z_{1312} i s \theta_{43}) - \\
& (c \theta_{21} s \beta_{54} i - c \beta_{54} i s \theta_{21} s \theta_{21} i) (z_{1312} i c \theta_{43} + (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) s \theta_{43}) + \\
& c \beta_{54} i c \theta_{21} i (x_{1211} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{54})) (-2 (\dot{\theta}_{21} + \dot{\theta}_{43}) z_{1312} i c(\theta_{21} + \theta_{43}) - \\
& 2 \dot{\theta}_{21} x_{32} s \theta_{21} + 2 \dot{\theta}_{21} i x_{43} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i - \theta_{21} - \theta_{43} - \theta_{54}) - \\
& (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} - \theta_{54}) + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{54}) x_{1211} i s(\beta_{1413} i + \theta_{21} + \theta_{43} - \theta_{54}) - \\
& \left. (\dot{\theta}_{21} + \dot{\theta}_{43} + \dot{\theta}_{54}) x_{65} s(\theta_{21} + \theta_{43} + \theta_{54}) \right) s \theta_{65} i + 2 \dot{\theta}_{76} i^2 \sec(\theta_{76} i)^2 \\
& (-x_{43} i - \text{signo } y_{10} i s \theta_{21} i + c \theta_{21} c \theta_{21} i (x_{32} + c \theta_{43} (-x_{1211} i c(\beta_{1413} i - \theta_{54}) + x_{65} c \theta_{54}) - z_{1312} i s \theta_{43}) - \\
& c \theta_{21} i s \theta_{21} (z_{1312} i c \theta_{43} - x_{1211} i c(\beta_{1413} i - \theta_{54}) s \theta_{43} + x_{65} c \theta_{54} s \theta_{43}) + \\
& x_{1211} i s \theta_{21} i s(\beta_{1413} i - \theta_{54}) + x_{65} s \theta_{21} i s \theta_{54})^2 \tan(\theta_{76} i) \Big) \Big) \Big)
\end{aligned}$$

Coeficientes de la ecuación (2.168)

$$\begin{aligned}
G_{69} = & (s(\beta_{1413} i - \theta_{54}) (c \theta_{21} i c \theta_{65} i s \beta_{54} i - s \theta_{21} i s \theta_{65} i) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (-c \beta_{54} i c \theta_{65} i s \theta_{21} + c \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) - \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (c \beta_{54} i c \theta_{21} c \theta_{65} i + s \theta_{21} (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)))^3
\end{aligned}$$

$$G_{70} = 0$$

[illegible]

[illegible]

Coeficientes de la ecuación (2.170)

$$G_{77} = (c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i))^3$$

$$G_{78} = 0$$

$$G_{79} = 0$$

$$G_{80} = 0$$

$$G_{81} = 0$$

$$G_{82} = 0$$

$$G_{83} = 0$$

$$\begin{aligned} G_{84} = & -c \theta_{109} i^2 \\ & \left(2 \left((-\dot{\theta}_{21} - \dot{\theta}_{43}) c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) - \dot{\theta}_{65} i c \beta_{54} i c(\theta_{21} + \theta_{43}) s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + \right. \right. \\ & \quad \left. \left. c \theta_{21} i s \theta_{65} i) + s(\theta_{21} + \theta_{43}) (c \theta_{21} i c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{65} i) \right)^2 \right. \\ & \quad (-c \beta_{54} i c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) c \theta_{65} i s(\theta_{21} + \theta_{43}) + c \theta_{21} i (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i s \beta_{54} i s \theta_{1110} i + \\ & \quad c \theta_{1110} i c(\theta_{21} + \theta_{43}) s \theta_{65} i) + s(\beta_{1413} i - \theta_{54}) (c \theta_{1110} i c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) s \theta_{1110} i s \theta_{65} i)) + \\ & \quad s \theta_{21} i (-s \theta_{1110} i (c \theta_{21} c \theta_{43} c \theta_{65} i s \beta_{54} i s(\beta_{1413} i - \theta_{54}) - c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s(\beta_{1413} i - \theta_{54}) + \\ & \quad c(\beta_{1413} i - \theta_{54}) s \theta_{65} i) + c \theta_{1110} i (c \theta_{21} c \theta_{43} c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i - \\ & \quad c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - s(\beta_{1413} i - \theta_{54}) s \theta_{65} i)) - \\ & \quad 2 (c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i)) \\ & \quad \left((-\dot{\theta}_{21} - \dot{\theta}_{43}) c \beta_{54} i c \theta_{65} i s(\theta_{21} + \theta_{43}) - \dot{\theta}_{65} i c \beta_{54} i c(\theta_{21} + \theta_{43}) s \theta_{65} i + \right. \\ & \quad \left. (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i) + \right. \\ & \quad \left. s(\theta_{21} + \theta_{43}) (c \theta_{21} i c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{65} i) \right) \\ & \quad (-c \beta_{54} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) c \theta_{65} i - \\ & \quad s(\theta_{21} + \theta_{43}) ((\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{65} i s(\beta_{1413} i + \theta_{1110} i - \theta_{54}) + \dot{\theta}_{65} i c(\beta_{1413} i + \theta_{1110} i - \theta_{54}) s \theta_{65} i)) + \\ & \quad c \theta_{1110} i (-s \theta_{21} i (c \theta_{65} i (\dot{\theta}_{65} i + (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43})) s \beta_{54} i) s(\beta_{1413} i - \theta_{54}) + c(\beta_{1413} i - \theta_{54}) \\ & \quad ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \beta_{54} i s(\theta_{21} + \theta_{43}) + (\dot{\theta}_{1110} i - \dot{\theta}_{54} + \dot{\theta}_{21} i c(\theta_{21} + \theta_{43}) + \dot{\theta}_{65} i c \theta_{21} c \theta_{43} s \beta_{54} i - \\ & \quad \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i)) + c \theta_{21} i ((-\dot{\theta}_{21} i - \dot{\theta}_{65} i s \beta_{54} i) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i + \\ & \quad c(\beta_{1413} i - \theta_{54}) ((\dot{\theta}_{1110} i - \dot{\theta}_{54}) c \theta_{65} i s \beta_{54} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \theta_{65} i) + \\ & \quad c(\theta_{21} + \theta_{43}) (c(\beta_{1413} i - \theta_{54}) c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (-\dot{\theta}_{1110} i + \dot{\theta}_{54}) s(\beta_{1413} i - \theta_{54}) s \theta_{65} i))) + \\ & \quad s \theta_{1110} i (-c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i (\dot{\theta}_{65} i + (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43})) s \beta_{54} i) s \theta_{21} i + \\ & \quad c \theta_{21} i (\dot{\theta}_{21} i + (\dot{\theta}_{1110} i - \dot{\theta}_{54}) c(\theta_{21} + \theta_{43}) + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{65} i) + \\ & \quad s(\beta_{1413} i - \theta_{54}) (c \theta_{21} i ((-\dot{\theta}_{1110} i + \dot{\theta}_{54}) c \theta_{65} i s \beta_{54} i - c(\theta_{21} + \theta_{43}) c \theta_{65} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + \\ & \quad (\dot{\theta}_{21} + \dot{\theta}_{43}) s(\theta_{21} + \theta_{43}) s \theta_{65} i) + s \theta_{21} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{43} + \\ & \quad (\dot{\theta}_{1110} i - \dot{\theta}_{54} + \dot{\theta}_{21} i c(\theta_{21} + \theta_{43}) - \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43}) s \theta_{65} i + \\ & \quad c \theta_{43} s \beta_{54} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i)))))) + \end{aligned}$$

$$\begin{aligned}
& (c \beta_{54 \ i} c(\theta_{21} + \theta_{43}) c \theta_{65 \ i} + s(\theta_{21} + \theta_{43}) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} + c \theta_{21 \ i} s \theta_{65 \ i})) \\
& \left((-c \beta_{54 \ i} c \theta_{65 \ i} (-\dot{\theta}_{21} + \dot{\theta}_{43})^2 c(\theta_{21} + \theta_{43}) - (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43})) - \right. \\
& \quad 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65 \ i} c \beta_{54 \ i} s(\theta_{21} + \theta_{43}) s \theta_{65 \ i} + c \beta_{54 \ i} c(\theta_{21} + \theta_{43}) (\dot{\theta}_{65 \ i}^2 c \theta_{65 \ i} + \ddot{\theta}_{65 \ i} s \theta_{65 \ i}) - \\
& \quad \left((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c(\theta_{21} + \theta_{43}) - (\dot{\theta}_{21} + \dot{\theta}_{43})^2 s(\theta_{21} + \theta_{43}) \right) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} + c \theta_{21 \ i} s \theta_{65 \ i}) - \\
& \quad 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) (c \theta_{21 \ i} c \theta_{65 \ i} (\dot{\theta}_{65 \ i} + \dot{\theta}_{21 \ i} s \beta_{54 \ i}) - (\dot{\theta}_{21 \ i} + \dot{\theta}_{65 \ i} s \beta_{54 \ i}) s \theta_{21 \ i} s \theta_{65 \ i}) + \\
& \quad s(\theta_{21} + \theta_{43}) (s \theta_{21 \ i} (c \theta_{65 \ i} (2 \dot{\theta}_{21 \ i} \dot{\theta}_{65 \ i} + (\dot{\theta}_{21 \ i}^2 + \dot{\theta}_{65 \ i}^2) s \beta_{54 \ i}) + (\ddot{\theta}_{21 \ i} + \ddot{\theta}_{65 \ i} s \beta_{54 \ i}) s \theta_{65 \ i}) + \\
& \quad c \theta_{21 \ i} (-c \theta_{65 \ i} (\ddot{\theta}_{65 \ i} + \ddot{\theta}_{21 \ i} s \beta_{54 \ i}) + (\dot{\theta}_{21 \ i}^2 + \dot{\theta}_{65 \ i}^2 + 2 \dot{\theta}_{21 \ i} \dot{\theta}_{65 \ i} s \beta_{54 \ i}) s \theta_{65 \ i})) \Big) \\
& (-c \beta_{54 \ i} c(\beta_{1413 \ i} + \theta_{1110 \ i} - \theta_{54}) c \theta_{65 \ i} s(\theta_{21} + \theta_{43}) + c \theta_{21 \ i} (c(\beta_{1413 \ i} - \theta_{54}) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} + c \theta_{1110 \ i} \\
& \quad c(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) + s(\beta_{1413 \ i} - \theta_{54}) (c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} s \theta_{65 \ i})) + \\
& \quad s \theta_{21 \ i} (-s \theta_{1110 \ i} (c \theta_{21 \ i} c \theta_{43} c \theta_{65 \ i} s \beta_{54 \ i} s(\beta_{1413 \ i} - \theta_{54}) - c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} s \theta_{43} s(\beta_{1413 \ i} - \theta_{54}) + \\
& \quad c(\beta_{1413 \ i} - \theta_{54}) s \theta_{65 \ i}) + c \theta_{1110 \ i} (c \theta_{21 \ i} c \theta_{43} c(\beta_{1413 \ i} - \theta_{54}) c \theta_{65 \ i} s \beta_{54 \ i} - \\
& \quad c(\beta_{1413 \ i} - \theta_{54}) c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} s \theta_{43} - s(\beta_{1413 \ i} - \theta_{54}) s \theta_{65 \ i})) + \\
& (c \beta_{54 \ i} c(\theta_{21} + \theta_{43}) c \theta_{65 \ i} + s(\theta_{21} + \theta_{43}) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} + c \theta_{21 \ i} s \theta_{65 \ i})) \Big(-c \beta_{54 \ i} c \theta_{65 \ i} \\
& \quad s(\theta_{21} + \theta_{43}) \Big(-(\dot{\theta}_{1110 \ i} - \dot{\theta}_{54})^2 c(\beta_{1413 \ i} + \theta_{1110 \ i} - \theta_{54}) + (-\ddot{\theta}_{1110 \ i} + \ddot{\theta}_{54}) s(\beta_{1413 \ i} + \theta_{1110 \ i} - \theta_{54}) \Big) + \\
& \quad 2 (\dot{\theta}_{1110 \ i} - \dot{\theta}_{54}) c \beta_{54 \ i} s(\beta_{1413 \ i} + \theta_{1110 \ i} - \theta_{54}) ((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c \theta_{65 \ i} - \dot{\theta}_{65 \ i} s(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) - \\
& \quad c \beta_{54 \ i} c(\beta_{1413 \ i} + \theta_{1110 \ i} - \theta_{54}) \Big(-s(\theta_{21} + \theta_{43}) ((\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{65 \ i}^2) c \theta_{65 \ i} + \ddot{\theta}_{65 \ i} s \theta_{65 \ i}) + \\
& \quad c(\theta_{21} + \theta_{43}) ((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta_{65 \ i} - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65 \ i} s \theta_{65 \ i}) \Big) + \\
& \quad \Big(-\dot{\theta}_{21 \ i}^2 c \theta_{21 \ i} - \ddot{\theta}_{21 \ i} s \theta_{21 \ i} \Big) (c(\beta_{1413 \ i} - \theta_{54}) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} + c \theta_{1110 \ i} c(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) + \\
& \quad s(\beta_{1413 \ i} - \theta_{54}) (c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} s \theta_{65 \ i})) + \\
& \quad \Big(\ddot{\theta}_{21 \ i} c \theta_{21 \ i} - \dot{\theta}_{21 \ i}^2 s \theta_{21 \ i} \Big) (-s \theta_{1110 \ i} (c \theta_{21 \ i} c \theta_{43} c \theta_{65 \ i} s \beta_{54 \ i} s(\beta_{1413 \ i} - \theta_{54}) - c \theta_{65 \ i} s \beta_{54 \ i} \\
& \quad s \theta_{21 \ i} s \theta_{43} s(\beta_{1413 \ i} - \theta_{54}) + c(\beta_{1413 \ i} - \theta_{54}) s \theta_{65 \ i}) + c \theta_{1110 \ i} (c \theta_{21 \ i} c \theta_{43} c(\beta_{1413 \ i} - \theta_{54}) \\
& \quad c \theta_{65 \ i} s \beta_{54 \ i} - c(\beta_{1413 \ i} - \theta_{54}) c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{21 \ i} s \theta_{43} - s(\beta_{1413 \ i} - \theta_{54}) s \theta_{65 \ i})) - \\
& \quad 2 \dot{\theta}_{21 \ i} s \theta_{21 \ i} (\dot{\theta}_{54} s(\beta_{1413 \ i} - \theta_{54}) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} + c \theta_{1110 \ i} c(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) - \\
& \quad \dot{\theta}_{54} c(\beta_{1413 \ i} - \theta_{54}) (c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} s \theta_{65 \ i}) + \\
& \quad c(\beta_{1413 \ i} - \theta_{54}) (\dot{\theta}_{65 \ i} c \theta_{1110 \ i} c(\theta_{21} + \theta_{43}) c \theta_{65 \ i} + \dot{\theta}_{1110 \ i} c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - \dot{\theta}_{1110 \ i} c(\theta_{21} + \theta_{43}) \\
& \quad s \theta_{1110 \ i} s \theta_{65 \ i} - \dot{\theta}_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} s \theta_{65 \ i} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110 \ i} s(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) + \\
& \quad s(\beta_{1413 \ i} - \theta_{54}) (-\dot{\theta}_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} + (-\dot{\theta}_{65 \ i} c \theta_{1110 \ i} s \beta_{54 \ i} + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{1110 \ i} \\
& \quad s(\theta_{21} + \theta_{43})) s \theta_{65 \ i} - c(\theta_{21} + \theta_{43}) (\dot{\theta}_{65 \ i} c \theta_{65 \ i} s \theta_{1110 \ i} + \dot{\theta}_{1110 \ i} c \theta_{1110 \ i} s \theta_{65 \ i})) \Big) + \\
& \quad c \theta_{21 \ i} \Big((-\dot{\theta}_{54}^2 c(\beta_{1413 \ i} - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413 \ i} - \theta_{54})) (c \theta_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} + c \theta_{1110 \ i} c(\theta_{21} + \theta_{43}) s \theta_{65 \ i}) - \\
& \quad (\ddot{\theta}_{54} c(\beta_{1413 \ i} - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413 \ i} - \theta_{54})) (c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} s \theta_{65 \ i}) + \\
& \quad 2 \dot{\theta}_{54} s(\beta_{1413 \ i} - \theta_{54}) (\dot{\theta}_{65 \ i} c \theta_{1110 \ i} c(\theta_{21} + \theta_{43}) c \theta_{65 \ i} + \dot{\theta}_{1110 \ i} c \theta_{1110 \ i} c \theta_{65 \ i} s \beta_{54 \ i} - \dot{\theta}_{1110 \ i} \\
& \quad c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} s \theta_{65 \ i} - \dot{\theta}_{65 \ i} s \beta_{54 \ i} s \theta_{1110 \ i} s \theta_{65 \ i} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110 \ i} s(\theta_{21} + \theta_{43}) \\
& \quad s \theta_{65 \ i}) + c(\beta_{1413 \ i} - \theta_{54}) (c \theta_{65 \ i} s \beta_{54 \ i} (\dot{\theta}_{1110 \ i} c \theta_{1110 \ i} - \dot{\theta}_{1110 \ i}^2 s \theta_{1110 \ i}) + 2 \dot{\theta}_{65 \ i} c \theta_{65 \ i} \\
& \quad (-\dot{\theta}_{1110 \ i} c(\theta_{21} + \theta_{43}) s \theta_{1110 \ i} - (\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{1110 \ i} s(\theta_{21} + \theta_{43})) - 2 \dot{\theta}_{1110 \ i} \dot{\theta}_{65 \ i} c \theta_{1110 \ i} \\
& \quad s \beta_{54 \ i} s \theta_{65 \ i} + (-c \theta_{1110 \ i} ((\dot{\theta}_{1110 \ i}^2 + (\dot{\theta}_{21} + \dot{\theta}_{43})^2) c(\theta_{21} + \theta_{43}) + (\ddot{\theta}_{21} + \ddot{\theta}_{43}) s(\theta_{21} + \theta_{43})) +
\end{aligned}$$

[illegible]

$$\begin{aligned}
& c \theta_{21} s \beta_{54} i \left(\left(\dot{\theta}_{21}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 + \dot{\theta}_{65}^2 \right) c \theta_{43} + \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} \right) s \theta_{43} \right) + \\
& \left(\ddot{\theta}_{54} + 2 \left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) \dot{\theta}_{65} i c \theta_{43} s \beta_{54} i s \theta_{21} + \ddot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} - \right. \\
& \left. c \theta_{21} s \beta_{54} i \left(\ddot{\theta}_{65} i c \theta_{43} - 2 \left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) \dot{\theta}_{65} i s \theta_{43} \right) s \theta_{65} i \right) + \\
& 2 \dot{\theta}_{1110} i s \theta_{1110} i \left(\left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) c \theta_{21} c(\beta_{1413} i - \theta_{54}) c \theta_{65} i s \beta_{54} i s \theta_{43} + \dot{\theta}_{65} i c \theta_{65} i \right. \\
& \left. s(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54} c \theta_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} s(\beta_{1413} i - \theta_{54}) - \dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) s \theta_{65} i - \right. \\
& \left. \dot{\theta}_{65} i c(\beta_{1413} i - \theta_{54}) s \beta_{54} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \beta_{54} i \left(-\dot{\theta}_{54} c \theta_{21} c \theta_{65} i s(\beta_{1413} i - \theta_{54}) + \right. \right. \\
& \left. \left. c(\beta_{1413} i - \theta_{54}) \left(\left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i \right) \right) \right) - \\
& 2 \dot{\theta}_{1110} i c \theta_{1110} i \left(c(\beta_{1413} i - \theta_{54}) c \theta_{65} i \left(\dot{\theta}_{65} i - \dot{\theta}_{54} c \theta_{21} c \theta_{43} s \beta_{54} i + \dot{\theta}_{54} s \beta_{54} i s \theta_{21} s \theta_{43} \right) - \right. \\
& \left. s(\beta_{1413} i - \theta_{54}) \left(\left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) c \theta_{21} c \theta_{65} i s \beta_{54} i s \theta_{43} - \left(\dot{\theta}_{54} + \dot{\theta}_{65} i s \beta_{54} i s \theta_{21} s \theta_{43} \right) s \theta_{65} i + \right. \right. \\
& \left. \left. c \theta_{43} s \beta_{54} i \left(\left(\dot{\theta}_{21} + \dot{\theta}_{43} \right) c \theta_{65} i s \theta_{21} + \dot{\theta}_{65} i c \theta_{21} s \theta_{65} i \right) \right) \right) \right) + \\
& 2 \dot{\theta}_{109} i^2 \sec(\theta_{109} i)^2 (c \beta_{54} i c(\theta_{21} + \theta_{43}) c \theta_{65} i + s(\theta_{21} + \theta_{43}) (c \theta_{65} i s \beta_{54} i s \theta_{21} i + c \theta_{21} i s \theta_{65} i))^3 \\
& \tan(\theta_{109} i)
\end{aligned}$$

Coeficientes de la ecuación (2.172)

$$\begin{aligned}
G_{85} = & (s \theta_{109} i (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i + c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& c \theta_{109} i (c \theta_{1110} i (-s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} + s \beta_{54} i (c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i - c \theta_{21} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{21} c \theta_{76} i s \theta_{65} i + s \theta_{21} s \theta_{21} i s \theta_{76} i)) + c \theta_{43} c(\beta_{1413} i - \theta_{54}) (s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - \\
& s \beta_{54} i s \theta_{76} i) + c \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + \\
& s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& s \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i))))))^3
\end{aligned}$$

$$G_{86} = 0$$

$$G_{87} = 0$$

$$G_{88} = 0$$

$$G_{89} = 0$$

$$G_{90} = 0$$

$$G_{91} = 0$$

$$G_{92} =$$

$$\begin{aligned} & c \theta_{98 i}^2 \left(-2(s \theta_{109 i}(c \theta_{21 i} c \theta_{65 i} s \theta_{76 i} s(\theta_{21} + \theta_{43}) - s \beta_{54 i}(c \theta_{43} c \theta_{76 i} s \theta_{21} s \theta_{21 i} s \theta_{65 i} + c \theta_{21} c \theta_{76 i} s \theta_{21 i} s \theta_{43} s \theta_{65 i} - \right. \\ & \quad c \theta_{21} c \theta_{43} s \theta_{76 i} + s \theta_{21} s \theta_{43} s \theta_{76 i})) - c \beta_{54 i} \\ & \quad (c \theta_{21} c \theta_{43} c \theta_{76 i} s \theta_{65 i} - c \theta_{76 i} s \theta_{21} s \theta_{43} s \theta_{65 i} + c \theta_{43} s \theta_{21} s \theta_{21 i} s \theta_{76 i} + c \theta_{21} s \theta_{21 i} s \theta_{43} s \theta_{76 i})) + \\ & \quad c \theta_{109 i}(c \theta_{1110 i}(-s(\beta_{1413 i}-\theta_{54})(c \theta_{65 i} c \theta_{76 i} s \theta_{21 i}+c \theta_{21 i}(c \theta_{76 i} s \beta_{54 i} s \theta_{65 i}+c \beta_{54 i} s \theta_{76 i}))+ \\ & \quad c(\beta_{1413 i}-\theta_{54}) s \theta_{43}(-c \theta_{21 i} c \theta_{65 i} c \theta_{76 i} s \theta_{21}+s \beta_{54 i}(c \theta_{76 i} s \theta_{21} s \theta_{21 i} s \theta_{65 i}-c \theta_{21} s \theta_{76 i}))+ \\ & \quad c \beta_{54 i}(c \theta_{21} c \theta_{76 i} s \theta_{65 i}+s \theta_{21} s \theta_{21 i} s \theta_{76 i}))+ \\ & \quad c \theta_{43} c(\beta_{1413 i}-\theta_{54})(s \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{65 i}-s \beta_{54 i} s \theta_{76 i}))+ \\ & \quad c \theta_{21}(c \theta_{21 i} c \theta_{65 i} c \theta_{76 i}-s \theta_{21 i}(c \theta_{76 i} s \beta_{54 i} s \theta_{65 i}+c \beta_{54 i} s \theta_{76 i}))))- \\ & \quad s \theta_{1110 i}(c(\beta_{1413 i}-\theta_{54})(c \theta_{65 i} c \theta_{76 i} s \theta_{21 i}+c \theta_{21 i}(c \theta_{76 i} s \beta_{54 i} s \theta_{65 i}+c \beta_{54 i} s \theta_{76 i}))+ \\ & \quad s(\beta_{1413 i}-\theta_{54})(c \theta_{21}(c \theta_{21 i} c \theta_{43} c \theta_{65 i} c \theta_{76 i}-c \theta_{43} s \theta_{21 i}(c \theta_{76 i} s \beta_{54 i} s \theta_{65 i}+c \beta_{54 i} s \theta_{76 i}))+ \\ & \quad s \theta_{43}(c \beta_{54 i} c \theta_{76 i} s \theta_{65 i}-s \beta_{54 i} s \theta_{76 i}))+ \\ & \quad s \theta_{21}(-c \theta_{21 i} c \theta_{65 i} c \theta_{76 i} s \theta_{43}+s \beta_{54 i}(c \theta_{76 i} s \theta_{21 i} s \theta_{43} s \theta_{65 i}-c \theta_{43} s \theta_{76 i}))+ \\ & \quad c \beta_{54 i}(c \theta_{43} c \theta_{76 i} s \theta_{65 i}+s \theta_{21 i} s \theta_{43} s \theta_{76 i})))))) \\ & (\dot{\theta}_{109 i} c \theta_{109 i}(-s \theta_{43}(c \theta_{76 i}(s \beta_{54 i} s \theta_{21}+c \beta_{54 i} c \theta_{21} s \theta_{21 i})+(c \beta_{54 i} s \theta_{21} s \theta_{65 i}+c \theta_{21}(c \theta_{21 i} c \theta_{65 i}- \\ & \quad s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}))+c \theta_{43}(c \theta_{21}(c \theta_{76 i} s \beta_{54 i}+c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}))- \\ & \quad s \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{21 i}+(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}))) + \\ & s \theta_{109 i}\left(-\dot{\theta}_{43} c \theta_{43}(c \theta_{76 i}(s \beta_{54 i} s \theta_{21}+c \beta_{54 i} c \theta_{21} s \theta_{21 i})+(c \beta_{54 i} s \theta_{21} s \theta_{65 i}+ \right. \\ & \quad c \theta_{21}(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i})- \\ & \quad \dot{\theta}_{43} s \theta_{43}(c \theta_{21}(c \theta_{76 i} s \beta_{54 i}+c \beta_{54 i} s \theta_{65 i} s \theta_{76 i})-s \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{21 i}+ \\ & \quad (c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}))+ \\ & \quad c \theta_{43}\left(-\dot{\theta}_{21} s \theta_{21}(c \theta_{76 i} s \beta_{54 i}+c \beta_{54 i} s \theta_{65 i} s \theta_{76 i})-s \theta_{21}\left(\dot{\theta}_{21 i} c \beta_{54 i} c \theta_{21 i} c \theta_{76 i}+ \right. \\ & \quad \dot{\theta}_{76 i} c \theta_{76 i}(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})-\dot{\theta}_{76 i} c \beta_{54 i} s \theta_{21 i} s \theta_{76 i}- \\ & \quad (c \theta_{65 i}(\dot{\theta}_{21 i}+\dot{\theta}_{65 i} s \beta_{54 i})) s \theta_{21 i}+c \theta_{21 i}(\dot{\theta}_{65 i}+\dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i}) s \theta_{76 i})- \\ & \quad \dot{\theta}_{21} c \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{21 i}+(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}))+ \\ & \quad c \theta_{21}\left(-\dot{\theta}_{76 i} s \beta_{54 i} s \theta_{76 i}+c \beta_{54 i}(\dot{\theta}_{76 i} c \theta_{76 i} s \theta_{65 i}+\dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}))\right)- \\ & \quad s \theta_{43}\left(-s \theta_{21}\left(\dot{\theta}_{21} c \theta_{21 i} c \theta_{65 i}+s \beta_{54 i}\left(\dot{\theta}_{76 i}-\dot{\theta}_{21} s \theta_{21 i} s \theta_{65 i}\right)\right) s \theta_{76 i}+ \right. \\ & \quad c \theta_{21}\left(c \theta_{76 i} s \beta_{54 i}\left(\dot{\theta}_{21}-\dot{\theta}_{76 i} s \theta_{21 i} s \theta_{65 i}\right)-c \theta_{65 i}\left(\dot{\theta}_{21 i}+\dot{\theta}_{65 i} s \beta_{54 i}\right) s \theta_{21 i} s \theta_{76 i}+ \right. \\ & \quad c \theta_{21 i}\left(\dot{\theta}_{76 i} c \theta_{65 i} c \theta_{76 i}-\left(\dot{\theta}_{65 i}+\dot{\theta}_{21 i} s \beta_{54 i}\right) s \theta_{65 i} s \theta_{76 i}\right))+ \\ & \quad c \beta_{54 i}\left(s \theta_{21}\left(c \theta_{76 i}\left(-\dot{\theta}_{21} s \theta_{21 i}+\dot{\theta}_{76 i} s \theta_{65 i}\right)+\dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}\right)+ \right. \\ & \quad c \theta_{21}\left(\dot{\theta}_{21 i} c \theta_{21 i} c \theta_{76 i}+\left(-\dot{\theta}_{76 i} s \theta_{21 i}+\dot{\theta}_{21} s \theta_{65 i}\right) s \theta_{76 i}\right))))- \\ & \quad \dot{\theta}_{109 i} s \theta_{109 i}(c \theta_{1110 i}(s(\beta_{1413 i}-\theta_{54})(-c \beta_{54 i} c \theta_{21 i} c \theta_{76 i}+(c \theta_{65 i} s \theta_{21 i}+c \theta_{21 i} s \beta_{54 i} s \theta_{65 i}) s \theta_{76 i}))- \\ & \quad c \theta_{43} c(\beta_{1413 i}-\theta_{54})(c \theta_{76 i}(s \beta_{54 i} s \theta_{21}+c \beta_{54 i} c \theta_{21} s \theta_{21 i}))+ \\ & \quad (c \beta_{54 i} s \theta_{21} s \theta_{65 i}+c \theta_{21}(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}))- \\ & \quad c(\beta_{1413 i}-\theta_{54}) s \theta_{43}(c \theta_{21}(c \theta_{76 i} s \beta_{54 i}+c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}))- \\ & \quad s \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{21 i}+(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}))- \\ & \quad s \theta_{1110 i}(c(\beta_{1413 i}-\theta_{54})(c \beta_{54 i} c \theta_{21 i} c \theta_{76 i}-(c \theta_{65 i} s \theta_{21 i}+c \theta_{21 i} s \beta_{54 i} s \theta_{65 i}) s \theta_{76 i}))- \\ & \quad c \theta_{43} s(\beta_{1413 i}-\theta_{54})(c \theta_{76 i}(s \beta_{54 i} s \theta_{21}+c \beta_{54 i} c \theta_{21} s \theta_{21 i}))+ \\ & \quad (c \beta_{54 i} s \theta_{21} s \theta_{65 i}+c \theta_{21}(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}))- \\ & \quad s \theta_{43} s(\beta_{1413 i}-\theta_{54})(c \theta_{21}(c \theta_{76 i} s \beta_{54 i}+c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}))- \\ & \quad s \theta_{21}(c \beta_{54 i} c \theta_{76 i} s \theta_{21 i}+(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}))))+ \\ & \quad c \theta_{109 i}\left(-\dot{\theta}_{1110 i} s \theta_{1110 i}(s(\beta_{1413 i}-\theta_{54})(-c \beta_{54 i} c \theta_{21 i} c \theta_{76 i}+(c \theta_{65 i} s \theta_{21 i}+c \theta_{21 i} s \beta_{54 i} s \theta_{65 i}) s \theta_{76 i}))- \right. \\ & \quad c \theta_{43} c(\beta_{1413 i}-\theta_{54})(c \theta_{76 i}(s \beta_{54 i} s \theta_{21}+c \beta_{54 i} c \theta_{21} s \theta_{21 i}))+ \\ & \quad (c \beta_{54 i} s \theta_{21} s \theta_{65 i}+c \theta_{21}(c \theta_{21 i} c \theta_{65 i}-s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}$$

[illegible]

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& c \theta_{21} (-c \theta_{76} i s \theta_{43} (\dot{\theta}_{43} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i)) - \\
& c \theta_{43} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) \\
& s \theta_{76} i + c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) + \\
& c \beta_{54} i (-c \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) + \\
& s \theta_{21} (-\dot{\theta}_{43} c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i + \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s \theta_{43} + \\
& \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s \theta_{43} s \theta_{65} i + \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s \theta_{43} s \theta_{76} i + \\
& c \beta_{54} i (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\
& s \beta_{54} i (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i) + s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \\
& \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))))^2 + \\
& (s \theta_{109} i (c \theta_{21} i c \theta_{65} i c \theta_{76} i s (\theta_{21} + \theta_{43}) - s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} i s \theta_{65} i + c \theta_{21} c \theta_{76} i s \theta_{21} i \\
& s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i \\
& s \theta_{76} i + c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& c \theta_{109} i (c \theta_{1110} i (-s(\beta_{1413} i - \theta_{54})) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{21} i s \theta_{65} i - \\
& c \theta_{21} s \theta_{76} i) + c \beta_{54} i (c \theta_{21} c \theta_{76} i s \theta_{65} i + s \theta_{21} s \theta_{21} i s \theta_{76} i)) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + \\
& c \beta_{54} i s \theta_{76} i) + s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& s \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i))))^2 \\
& ((\ddot{\theta}_{109} i c \theta_{109} i - \dot{\theta}_{109} i^2 s \theta_{109} i) (-s \theta_{43} (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) + \\
& c \theta_{43} (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + \\
& (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + \\
& 2 \dot{\theta}_{109} i c \theta_{109} i (-\dot{\theta}_{43} c \theta_{43} (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + (c \beta_{54} i s \theta_{21} s \theta_{65} i + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& \dot{\theta}_{43} s \theta_{43} (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + \\
& (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + \\
& c \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \\
& \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) - \\
& s \theta_{43} (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i +
\end{aligned}$$

$$\begin{aligned}
& c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + c \theta_{21} \\
& (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))) + (-\dot{\theta}_{109} i^2 c \theta_{109} i - \ddot{\theta}_{109} i s \theta_{109} i) \\
& (c \theta_{1110} i (s(\beta_{1413} i - \theta_{54}) (-c \beta_{54} i c \theta_{21} i c \theta_{76} i + (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \beta_{54} i c \theta_{21} i c \theta_{76} i - (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& s \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + \\
& s \theta_{109} i \left((-\ddot{\theta}_{43} c \theta_{43} + \dot{\theta}_{43}^2 s \theta_{43}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \right. \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& s \theta_{43} (c \theta_{76} i (s \beta_{54} i (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) + c \beta_{54} i (-s \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i) + \\
& c \theta_{21} (\ddot{\theta}_{21} i c \theta_{21} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2) s \theta_{21} i))) + 2 \dot{\theta}_{76} i c \theta_{76} i \\
& (\dot{\theta}_{65} i c \beta_{54} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} c \beta_{54} i c \theta_{21} s \theta_{65} i - c \theta_{21} (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \\
& c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) - \dot{\theta}_{21} s \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) - \\
& 2 \dot{\theta}_{76} i (\dot{\theta}_{21} c \theta_{21} s \beta_{54} i + c \beta_{54} i (\dot{\theta}_{21} i c \theta_{21} c \theta_{21} i - \dot{\theta}_{21} s \theta_{21} s \theta_{21} i)) s \theta_{76} i + \\
& (c \beta_{54} i (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} s \theta_{65} i) + s \theta_{21} (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{65} i^2) s \theta_{65} i)) + \\
& s \theta_{21} (s \theta_{21} i (2 \dot{\theta}_{21} c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{21} s \beta_{54} i s \theta_{65} i) + \\
& c \theta_{21} i (-\ddot{\theta}_{21} c \theta_{65} i + 2 \dot{\theta}_{21} (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i)) - \\
& c \theta_{21} (c \theta_{21} i (c \theta_{65} i (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + \\
& (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) + s \theta_{21} i (c \theta_{65} i (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - \\
& (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \beta_{54} i) s \theta_{65} i))) \\
& s \theta_{76} i - (s \beta_{54} i s \theta_{21} i + c \beta_{54} i c \theta_{21} s \theta_{21} i) (\dot{\theta}_{76} i^2 c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) (\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76} i^2 s \theta_{76} i)) - \\
& (\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) - \\
& 2 \dot{\theta}_{43} s \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \\
& \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) - \\
& 2 \dot{\theta}_{43} c \theta_{43} (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i +
\end{aligned}$$

[illegible]

[illegible]

[illegible]

$$\begin{aligned}
& c \theta_{43} s(\beta_{1413 i} - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21 i} c \theta_{65 i} + s \beta_{54 i} (\dot{\theta}_{76 i} - \dot{\theta}_{21} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i} + \\
& c \theta_{21} (c \theta_{76 i} s \beta_{54 i} (\dot{\theta}_{21} - \dot{\theta}_{76 i} s \theta_{21 i} s \theta_{65 i}) - c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} s \theta_{76 i} + \\
& c \theta_{21 i} (\dot{\theta}_{76 i} c \theta_{65 i} c \theta_{76 i} - (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i} s \theta_{76 i})) + \\
& c \beta_{54 i} (s \theta_{21} (c \theta_{76 i} (-\dot{\theta}_{21} s \theta_{21 i} + \dot{\theta}_{76 i} s \theta_{65 i}) + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}) + \\
& c \theta_{21} (\dot{\theta}_{21 i} c \theta_{21 i} c \theta_{76 i} + (-\dot{\theta}_{76 i} s \theta_{21 i} + \dot{\theta}_{21} s \theta_{65 i}) s \theta_{76 i}))) + \\
& c \theta_{1110 i} \left((\ddot{\theta}_{54} c(\beta_{1413 i} - \theta_{54}) + \dot{\theta}_{54}^2 s(\beta_{1413 i} - \theta_{54})) (c \beta_{54 i} c \theta_{21 i} c \theta_{76 i} - \right. \\
& (c \theta_{65 i} s \theta_{21 i} + c \theta_{21 i} s \beta_{54 i} s \theta_{65 i}) s \theta_{76 i}) - 2 \dot{\theta}_{54} c(\beta_{1413 i} - \theta_{54}) (\dot{\theta}_{21 i} c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + \\
& \dot{\theta}_{76 i} c \theta_{76 i} (c \theta_{65 i} s \theta_{21 i} + c \theta_{21 i} s \beta_{54 i} s \theta_{65 i}) + \dot{\theta}_{76 i} c \beta_{54 i} c \theta_{21 i} s \theta_{76 i} + \\
& (c \theta_{21 i} c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) - (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}) + \\
& c(\beta_{1413 i} - \theta_{54}) (\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43}) (c \theta_{76 i} (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) + \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}) + \\
& s \theta_{43} (\dot{\theta}_{54}^2 c(\beta_{1413 i} - \theta_{54}) - \ddot{\theta}_{54} s(\beta_{1413 i} - \theta_{54})) (c \theta_{21} (c \theta_{76 i} s \beta_{54 i} + c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}) - \\
& s \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i})) + s(\beta_{1413 i} - \theta_{54}) \\
& (c \beta_{54 i} (c \theta_{21 i} (\dot{\theta}_{21 i}^2 + \dot{\theta}_{76 i}^2) c \theta_{76 i} + \ddot{\theta}_{76 i} s \theta_{76 i})) + s \theta_{21 i} (\ddot{\theta}_{21 i} c \theta_{76 i} - 2 \dot{\theta}_{21 i} \dot{\theta}_{76 i} s \theta_{76 i})) + \\
& s \theta_{21 i} (-s \theta_{65 i} (2 \dot{\theta}_{76 i} c \theta_{76 i} (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) + (\ddot{\theta}_{65 i} + \ddot{\theta}_{21 i} s \beta_{54 i}) s \theta_{76 i}) + \\
& c \theta_{65 i} (\dot{\theta}_{76 i} c \theta_{76 i} - (\dot{\theta}_{21 i}^2 + \dot{\theta}_{65 i}^2 + \dot{\theta}_{76 i}^2 + 2 \dot{\theta}_{21 i} \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{76 i})) + \\
& c \theta_{21 i} (c \theta_{65 i} (2 \dot{\theta}_{76 i} c \theta_{76 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) + (\ddot{\theta}_{21 i} + \ddot{\theta}_{65 i} s \beta_{54 i}) s \theta_{76 i}) + \\
& s \theta_{65 i} (\dot{\theta}_{76 i} c \theta_{76 i} s \beta_{54 i} - (2 \dot{\theta}_{21 i} \dot{\theta}_{65 i} + (\dot{\theta}_{21 i}^2 + \dot{\theta}_{65 i}^2 + \dot{\theta}_{76 i}^2) s \beta_{54 i}) s \theta_{76 i}))) - \\
& 2 \dot{\theta}_{54} s(\beta_{1413 i} - \theta_{54}) (\dot{\theta}_{43} c \theta_{43} (c \theta_{21} (c \theta_{76 i} s \beta_{54 i} + c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}) - \\
& s \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i})) + \\
& s \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76 i} s \beta_{54 i} + c \beta_{54 i} s \theta_{65 i} s \theta_{76 i}) - s \theta_{21} (\dot{\theta}_{21 i} c \beta_{54 i} c \theta_{21 i} c \theta_{76 i} + \\
& \dot{\theta}_{76 i} c \theta_{76 i} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) - \dot{\theta}_{76 i} c \beta_{54 i} s \theta_{21 i} s \theta_{76 i} - \\
& (c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} + c \theta_{21 i} (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i}) s \theta_{76 i}) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54 i} c \theta_{76 i} s \theta_{21 i} + (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i}) s \theta_{76 i}) + \\
& c \theta_{21} (-\dot{\theta}_{76 i} s \beta_{54 i} s \theta_{76 i} + c \beta_{54 i} (\dot{\theta}_{76 i} c \theta_{76 i} s \theta_{65 i} + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}))) + \\
& 2 \dot{\theta}_{43} s \theta_{43} (\dot{\theta}_{54} s(\beta_{1413 i} - \theta_{54}) (c \theta_{76 i} (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) + \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}) + \\
& c(\beta_{1413 i} - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21 i} c \theta_{65 i} + s \beta_{54 i} (\dot{\theta}_{76 i} - \dot{\theta}_{21} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i} + \\
& c \theta_{21} (c \theta_{76 i} s \beta_{54 i} (\dot{\theta}_{21} - \dot{\theta}_{76 i} s \theta_{21 i} s \theta_{65 i}) - c \theta_{65 i} (\dot{\theta}_{21 i} + \dot{\theta}_{65 i} s \beta_{54 i}) s \theta_{21 i} s \theta_{76 i} + \\
& c \theta_{21 i} (\dot{\theta}_{76 i} c \theta_{65 i} c \theta_{76 i} - (\dot{\theta}_{65 i} + \dot{\theta}_{21 i} s \beta_{54 i}) s \theta_{65 i} s \theta_{76 i})) + \\
& c \beta_{54 i} (s \theta_{21} (c \theta_{76 i} (-\dot{\theta}_{21} s \theta_{21 i} + \dot{\theta}_{76 i} s \theta_{65 i}) + \dot{\theta}_{65 i} c \theta_{65 i} s \theta_{76 i}) + \\
& c \theta_{21} (\dot{\theta}_{21 i} c \theta_{21 i} c \theta_{76 i} + (-\dot{\theta}_{76 i} s \theta_{21 i} + \dot{\theta}_{21} s \theta_{65 i}) s \theta_{76 i}))) - \\
& c \theta_{43} ((-\dot{\theta}_{54}^2 c(\beta_{1413 i} - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413 i} - \theta_{54})) (c \theta_{76 i} (s \beta_{54 i} s \theta_{21} + c \beta_{54 i} c \theta_{21} s \theta_{21 i}) + \\
& (c \beta_{54 i} s \theta_{21} s \theta_{65 i} + c \theta_{21} (c \theta_{21 i} c \theta_{65 i} - s \beta_{54 i} s \theta_{21 i} s \theta_{65 i})) s \theta_{76 i}) + \\
& c(\beta_{1413 i} - \theta_{54}) (c \theta_{76 i} (s \beta_{54 i} (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) + c \beta_{54 i} (-s \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21 i} c \theta_{21 i} + \\
& \ddot{\theta}_{21} s \theta_{21 i}) + c \theta_{21} (\ddot{\theta}_{21 i} c \theta_{21 i} - (\dot{\theta}_{21}^2 + \dot{\theta}_{21 i}^2) s \theta_{21 i}))) +
\end{aligned}$$

$$\begin{aligned}
& 2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{65} i c \beta_{54} i c \theta_{65} i s \theta_{21} + \dot{\theta}_{21} c \beta_{54} i c \theta_{21} s \theta_{65} i - \\
& c \theta_{21} (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) - \\
& \dot{\theta}_{21} s \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{21} c \theta_{21} s \beta_{54} i + \\
& c \beta_{54} i (\dot{\theta}_{21} i c \theta_{21} c \theta_{21} i - \dot{\theta}_{21} s \theta_{21} s \theta_{21} i)) s \theta_{76} i + (c \beta_{54} i (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{65} i \\
& c \theta_{65} i + \ddot{\theta}_{21} s \theta_{65} i) + s \theta_{21} (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2) s \theta_{65} i)) + \\
& s \theta_{21} (s \theta_{21} i (2 \dot{\theta}_{21} c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{21} s \beta_{54} i s \theta_{65} i) + \\
& c \theta_{21} i (-\ddot{\theta}_{21} c \theta_{65} i + 2 \dot{\theta}_{21} (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i)) - \\
& c \theta_{21} (c \theta_{21} i (c \theta_{65} i (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + \\
& (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) + s \theta_{21} i (c \theta_{65} i (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - \\
& (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21}^2 + \dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \beta_{54} i) s \theta_{65} i))) s \theta_{76} i - \\
& (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) (\dot{\theta}_{76} i^2 c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + (c \beta_{54} i s \theta_{21} s \theta_{65} i + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) (\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76} i^2 s \theta_{76} i)) + \\
& 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (-s \theta_{21} (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i)) s \theta_{76} i + \\
& c \theta_{21} (c \theta_{76} i s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) - c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} i (\dot{\theta}_{76} i c \theta_{65} i c \theta_{76} i - (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i s \theta_{76} i)) + \\
& c \beta_{54} i (s \theta_{21} (c \theta_{76} i (-\dot{\theta}_{21} s \theta_{21} i + \dot{\theta}_{76} i s \theta_{65} i) + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i) + \\
& c \theta_{21} (\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i + (-\dot{\theta}_{76} i s \theta_{21} i + \dot{\theta}_{21} s \theta_{65} i) s \theta_{76} i))) - \\
& c(\beta_{1413} i - \theta_{54}) ((\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) + \\
& 2 \dot{\theta}_{43} c \theta_{43} (-\dot{\theta}_{21} s \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \\
& \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - \\
& (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) - \\
& \dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) + \\
& c \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i))) + \\
& s \theta_{43} ((-\dot{\theta}_{21}^2 c \theta_{21} - \ddot{\theta}_{21} s \theta_{21}) (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& 2 \dot{\theta}_{21} c \theta_{21} (\dot{\theta}_{21} i c \beta_{54} i c \theta_{21} i c \theta_{76} i + \dot{\theta}_{76} i c \theta_{76} i (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) - \\
& \dot{\theta}_{76} i c \beta_{54} i s \theta_{21} i s \theta_{76} i - (c \theta_{65} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \\
& c \theta_{21} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i) s \theta_{76} i) + (-\ddot{\theta}_{21} c \theta_{21} + \dot{\theta}_{21}^2 s \theta_{21}) \\
& (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i) - \\
& 2 \dot{\theta}_{21} s \theta_{21} (-\dot{\theta}_{76} i s \beta_{54} i s \theta_{76} i + c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{65} i c \theta_{65} i s \theta_{76} i)) + \\
& c \theta_{21} (-s \beta_{54} i (\dot{\theta}_{76} i^2 c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \beta_{54} i (c \theta_{65} i (2 \dot{\theta}_{65} i \dot{\theta}_{76} i c \theta_{76} i + \\
& \ddot{\theta}_{65} i s \theta_{76} i) + s \theta_{65} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i))) - \\
& s \theta_{21} (c \beta_{54} i (-s \theta_{21} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} i \\
& (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i))) + c \theta_{21} i
\end{aligned}$$

$$\begin{aligned}
& \ddot{\theta}_{65} i s \theta_{76} i) + s \theta_{65} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i)) - \\
& s \theta_{21} (c \beta_{54} i (-s \theta_{21} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} i \\
& (\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i)) + c \theta_{21} i \\
& (-s \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) + (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i) + \\
& c \theta_{65} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) \\
& s \theta_{76} i)) + s \theta_{21} i (-c \theta_{65} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) + \\
& (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i) + s \theta_{65} i (-\ddot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + \\
& (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \beta_{54} i) s \theta_{76} i)))))) - \\
& (s \theta_{109} i (-s \theta_{43} (c \theta_{76} i (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) + (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - \\
& s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) + \\
& c \theta_{43} (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + \\
& (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + \\
& c \theta_{109} i (c \theta_{1110} i (s(\beta_{1413} i - \theta_{54}) (-c \beta_{54} i c \theta_{21} i c \theta_{76} i + (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \beta_{54} i c \theta_{21} i c \theta_{76} i - (c \theta_{65} i s \theta_{21} i + c \theta_{21} i s \beta_{54} i s \theta_{65} i) s \theta_{76} i) - \\
& c \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{76} i (s \beta_{54} i s \theta_{21} + c \beta_{54} i c \theta_{21} s \theta_{21} i) + \\
& (c \beta_{54} i s \theta_{21} s \theta_{65} i + c \theta_{21} (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i)) s \theta_{76} i) - \\
& s \theta_{43} s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{76} i s \beta_{54} i + c \beta_{54} i s \theta_{65} i s \theta_{76} i) - \\
& s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{21} i + (c \theta_{21} i c \theta_{65} i - s \beta_{54} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)))))) \\
& (s \theta_{109} i (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i + \\
& c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& c \theta_{109} i (c \theta_{1110} i (-s(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& c(\beta_{1413} i - \theta_{54}) s \theta_{43} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} + s \beta_{54} i (c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i - c \theta_{21} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{21} c \theta_{76} i s \theta_{65} i + s \theta_{21} s \theta_{21} i s \theta_{76} i)) + \\
& c \theta_{43} c(\beta_{1413} i - \theta_{54}) (s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) + \\
& c \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)))) - \\
& s \theta_{1110} i (c(\beta_{1413} i - \theta_{54}) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& s(\beta_{1413} i - \theta_{54}) (c \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + \\
& s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& s \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i)))))) \\
& ((\ddot{\theta}_{109} i c \theta_{109} i - \dot{\theta}_{109} i^2 s \theta_{109} i) (c \theta_{21} i c \theta_{65} i c \theta_{76} i s(\theta_{21} + \theta_{43}) - s \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{21} s \theta_{21} i s \theta_{65} i + \\
& c \theta_{21} c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{21} c \theta_{43} s \theta_{76} i + s \theta_{21} s \theta_{43} s \theta_{76} i) - \\
& c \beta_{54} i (c \theta_{21} c \theta_{43} c \theta_{76} i s \theta_{65} i - c \theta_{76} i s \theta_{21} s \theta_{43} s \theta_{65} i + c \theta_{43} s \theta_{21} s \theta_{21} i s \theta_{76} i + \\
& c \theta_{21} s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& 2 \dot{\theta}_{109} i c \theta_{109} i ((\dot{\theta}_{21} + \dot{\theta}_{43}) c \theta_{21} i (c(\theta_{21} + \theta_{43}) c \theta_{65} i c \theta_{76} i - \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i
\end{aligned}$$

$$\begin{aligned}
& s(\theta_{21} + \theta_{43}) - \dot{\theta}_{65} i \\
& c \theta_{21} i c \theta_{76} i s(\theta_{21} + \theta_{43}) \\
& s \theta_{65} i - \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i \\
& s(\theta_{21} + \theta_{43}) s \theta_{76} i - \\
& c \beta_{54} i \left(c \theta_{21} \left(s \theta_{43} \left(c \theta_{76} i \left(\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i \right) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i \right) + \right. \right. \\
& \quad \left. c \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i \right) \right) + \\
& \quad s \theta_{21} \left(c \theta_{43} \left(c \theta_{76} i \left(\dot{\theta}_{76} i s \theta_{21} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{65} i \right) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i \right) - \right. \\
& \quad \left. s \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + ((\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i \right) \right) \Big) - \\
& s \beta_{54} i \left(s \theta_{21} \left(c \theta_{76} i s \theta_{43} \left(\dot{\theta}_{76} i - (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i \right) + c \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \right. \right. \right. \\
& \quad \left. \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i \right) \Big) + \\
& \quad c \theta_{21} \left(c \theta_{43} c \theta_{76} i \left(-\dot{\theta}_{76} i + (\dot{\theta}_{21} + \dot{\theta}_{43}) s \theta_{21} i s \theta_{65} i \right) + s \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \right. \right. \\
& \quad \left. \left. \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{21} + \dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i \right) \right) \Big) \Big) + \\
& s \theta_{109} i \left(-c \theta_{76} i s(\theta_{21} + \theta_{43}) \left(c \theta_{21} i \left((\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + s \theta_{21} i \right. \right. \\
& \quad \left. \left(\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i \right) \right) \Big) + \\
& 2 \left(-\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i - \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i \right) \left((\dot{\theta}_{21} + \dot{\theta}_{43}) c(\theta_{21} + \theta_{43}) c \theta_{76} i - \dot{\theta}_{76} i s(\theta_{21} + \theta_{43}) s \theta_{76} i \right) + \\
& c \theta_{21} i c \theta_{65} i \\
& \left(-s(\theta_{21} + \theta_{43}) \left((\dot{\theta}_{21} i^2 + 2 \dot{\theta}_{21} \dot{\theta}_{43} + \dot{\theta}_{43} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + \right. \\
& \quad \left. c(\theta_{21} + \theta_{43}) \left((\ddot{\theta}_{21} + \ddot{\theta}_{43}) c \theta_{76} i - 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{76} i s \theta_{76} i \right) \right) - \\
& c \beta_{54} i \left(-c \theta_{76} i \left(c \theta_{21} \left((\dot{\theta}_{21} i^2 + \dot{\theta}_{43} i^2) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43} \right) + s \theta_{21} \left(\ddot{\theta}_{21} c \theta_{43} - 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{43} \right) \right) s \theta_{65} i + \right. \\
& \quad c \theta_{76} i s \theta_{21} \left(-c \theta_{43} \left(2 \dot{\theta}_{43} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{43} s \theta_{65} i \right) + s \theta_{43} \left(-\ddot{\theta}_{65} i c \theta_{65} i + (\dot{\theta}_{43} i^2 + \dot{\theta}_{65} i^2) s \theta_{65} i \right) \right) - \\
& \quad \left(s \theta_{21} \left(2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i \right) + c \theta_{21} \left(-\ddot{\theta}_{21} i c \theta_{21} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{21} i^2) s \theta_{21} i \right) \right) s \theta_{43} s \theta_{76} i + \\
& \quad s \theta_{21} i \left(-s \theta_{21} \left((\dot{\theta}_{21} i^2 + \dot{\theta}_{43} i^2) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43} \right) + c \theta_{21} \left(\ddot{\theta}_{21} c \theta_{43} - 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{43} \right) \right) s \theta_{76} i + \\
& \quad 2 (\dot{\theta}_{21} c \theta_{21} c \theta_{43} - \dot{\theta}_{43} s \theta_{21} s \theta_{43}) (\dot{\theta}_{76} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& \quad 2 (\dot{\theta}_{21} i c \theta_{21} c \theta_{21} i - \dot{\theta}_{21} s \theta_{21} s \theta_{21} i) (\dot{\theta}_{76} i c \theta_{76} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{76} i) + \\
& \quad 2 (\dot{\theta}_{65} i c \theta_{65} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{65} i) (-\dot{\theta}_{21} c \theta_{21} c \theta_{76} i + \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) - \\
& \quad 2 (\dot{\theta}_{21} c \theta_{43} s \theta_{21} + \dot{\theta}_{43} c \theta_{21} s \theta_{43}) (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i - \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i) + \\
& \quad s \theta_{43} s \theta_{65} i \left(s \theta_{21} \left((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + c \theta_{21} \left(-\ddot{\theta}_{21} c \theta_{76} i + 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i \right) \right) + \\
& \quad c \theta_{21} c \theta_{43} \left(-s \theta_{65} i \left((\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + c \theta_{65} i \left(\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i \right) \right) + \\
& \quad c \theta_{43} s \theta_{21} \left(c \theta_{21} i \left(2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i \right) + s \theta_{21} i \left(\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i \right) \right) + \\
& \quad c \theta_{21} s \theta_{21} i \left(c \theta_{43} \left(2 \dot{\theta}_{43} \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{43} s \theta_{76} i \right) + s \theta_{43} \left(\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{43} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i \right) \right) \Big) - \\
& s \beta_{54} i \left(c \theta_{21} \left(-s \theta_{43} \left(c \theta_{76} i \left(-2 \dot{\theta}_{21} \dot{\theta}_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i - \ddot{\theta}_{65} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} i^2 s \theta_{21} i s \theta_{65} i + \dot{\theta}_{21} i^2 s \theta_{21} i \right. \right. \right. \right. \\
& \quad \left. \left. s \theta_{65} i + 2 \dot{\theta}_{21} \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i + \dot{\theta}_{43} i^2 s \theta_{21} i s \theta_{65} i + \dot{\theta}_{65} i^2 s \theta_{21} i s \theta_{65} i + \right. \right. \\
& \quad \left. \left. \dot{\theta}_{76} i^2 s \theta_{21} i s \theta_{65} i - c \theta_{21} i \left(2 \dot{\theta}_{21} i \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} i s \theta_{65} i \right) \right) - \left(\ddot{\theta}_{21} + \ddot{\theta}_{43} - \right. \right. \\
& \quad \left. \left. 2 \dot{\theta}_{65} i \dot{\theta}_{76} i c \theta_{65} i s \theta_{21} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{21} i s \theta_{65} i - \ddot{\theta}_{76} i s \theta_{21} i s \theta_{65} i \right) s \theta_{76} i \right) + \\
& \quad c \theta_{43} \left(c \theta_{76} i \left(-\ddot{\theta}_{76} i + 2 (\dot{\theta}_{21} + \dot{\theta}_{43}) \dot{\theta}_{65} i c \theta_{65} i s \theta_{21} i + 2 \dot{\theta}_{21} i (\dot{\theta}_{21} + \dot{\theta}_{43}) \right. \right.
\end{aligned}$$

[illegible]

[illegible]

$$\begin{aligned}
& \left(-c \theta_{43} \left(\left(\dot{\theta}_{43}^2 + \dot{\theta}_{54}^2 \right) c(\beta_{1413} i - \theta_{54}) - \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) \right) - s \theta_{43} \left(\ddot{\theta}_{43} c(\beta_{1413} i - \theta_{54}) + \right. \right. \\
& \quad \left. \left. 2 \dot{\theta}_{43} \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) \right) \right) (s \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) + \\
& \quad c \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i))) - s(\beta_{1413} i - \theta_{54}) \\
& \quad (c \beta_{54} i (-s \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i) + c \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i))) - \\
& \quad s \theta_{21} i (c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& \quad s \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i))) + \\
& \quad c \theta_{21} i (-s \theta_{65} i (c \theta_{76} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \beta_{54} i) + \ddot{\theta}_{76} i s \beta_{54} i s \theta_{76} i) + \\
& \quad c \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i))) + \\
& \quad 2 (-\dot{\theta}_{43} c(\beta_{1413} i - \theta_{54}) s \theta_{43} + \dot{\theta}_{54} c \theta_{43} s(\beta_{1413} i - \theta_{54})) \\
& \quad (\dot{\theta}_{21} c \theta_{21} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i) - \\
& \quad \dot{\theta}_{21} s \theta_{21} (c \theta_{21} i c \theta_{65} i c \theta_{76} i - s \theta_{21} i (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i))) - \\
& \quad c \theta_{21} (c \beta_{54} i (\dot{\theta}_{76} i c \theta_{76} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& \quad c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i) + \\
& \quad s \theta_{65} i (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) - \dot{\theta}_{76} i s \beta_{54} i s \theta_{21} i s \theta_{76} i)) - \\
& \quad s \theta_{21} (\dot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + c \beta_{54} i (-\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i))) + \\
& \quad 2 (\dot{\theta}_{43} c \theta_{43} c(\beta_{1413} i - \theta_{54}) + \dot{\theta}_{54} s \theta_{43} s(\beta_{1413} i - \theta_{54})) \\
& \quad (-c \theta_{21} c \theta_{76} i (\dot{\theta}_{21} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{21} s \theta_{21} i s \theta_{65} i))) + \\
& \quad s \theta_{21} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{21} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i + \\
& \quad c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) + \\
& \quad c \beta_{54} i (s \theta_{21} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{21} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& \quad c \theta_{21} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{21} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) + \\
& \quad c(\beta_{1413} i - \theta_{54}) s \theta_{43} (c \theta_{76} i s \theta_{21} (c \theta_{21} i ((\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i) + \\
& \quad s \theta_{21} i (\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i))) + \\
& \quad 2 (\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i) (\dot{\theta}_{21} c \theta_{21} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) - c \theta_{21} i \\
& \quad c \theta_{65} i (-s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} (\ddot{\theta}_{21} c \theta_{76} i - 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i))) + \\
& \quad s \beta_{54} i (2 \dot{\theta}_{21} \dot{\theta}_{76} i c \theta_{76} i s \theta_{21} + c \theta_{76} i s \theta_{21} (c \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{21} i s \theta_{65} i) + \\
& \quad s \theta_{21} i (\ddot{\theta}_{65} i c \theta_{65} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2) s \theta_{65} i)) + (\dot{\theta}_{21} i^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21}) s \theta_{76} i + \\
& \quad c \theta_{21} (-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i) + 2 (\dot{\theta}_{65} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{65} i) \\
& \quad (\dot{\theta}_{21} c \theta_{21} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} s \theta_{76} i) + s \theta_{21} i s \theta_{65} i \\
& \quad (-s \theta_{21} ((\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i) + c \theta_{21} (\ddot{\theta}_{21} c \theta_{76} i - 2 \dot{\theta}_{21} \dot{\theta}_{76} i s \theta_{76} i))) + \\
& \quad c \beta_{54} i (2 \dot{\theta}_{76} i c \theta_{76} i (\dot{\theta}_{21} i c \theta_{21} i s \theta_{21} + \dot{\theta}_{21} c \theta_{21} s \theta_{21} i) + c \theta_{21} c \theta_{76} i \\
& \quad (\ddot{\theta}_{65} i c \theta_{65} i - \dot{\theta}_{65} i^2 s \theta_{65} i) + (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{21} i c \theta_{21} i + \ddot{\theta}_{21} s \theta_{21} i) + \\
& \quad s \theta_{21} (\ddot{\theta}_{21} i c \theta_{21} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{21} i^2) s \theta_{21} i)) s \theta_{76} i + s \theta_{21} s \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - \\
& \quad \dot{\theta}_{76} i^2 s \theta_{76} i) - 2 \dot{\theta}_{65} i c \theta_{65} i (\dot{\theta}_{21} c \theta_{76} i s \theta_{21} + \dot{\theta}_{76} i c \theta_{21} s \theta_{76} i) - s \theta_{65} i
\end{aligned}$$

$$\begin{aligned}
& \left(c \theta_{21} \left(\left(\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2 \right) c \theta_{76} + \ddot{\theta}_{76} s \theta_{76} \right) + s \theta_{21} \left(\ddot{\theta}_{21} c \theta_{76} - 2 \dot{\theta}_{21} \dot{\theta}_{76} s \theta_{76} \right) \right) + \\
& c \theta_{43} c(\beta_{1413} - \theta_{54}) \left(\left(\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21} \right) (c \beta_{54} c \theta_{76} s \theta_{65} - s \beta_{54} s \theta_{76}) - \right. \\
& \left(\dot{\theta}_{21}^2 c \theta_{21} + \ddot{\theta}_{21} s \theta_{21} \right) (c \theta_{21} c \theta_{65} c \theta_{76} - s \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76})) + \\
& 2 \dot{\theta}_{21} s \theta_{21} (c \beta_{54} (\dot{\theta}_{76} c \theta_{76} s \theta_{21} + \dot{\theta}_{21} c \theta_{21} s \theta_{76})) + \\
& c \theta_{65} (c \theta_{76} (\dot{\theta}_{21} + \dot{\theta}_{65} s \beta_{54}) s \theta_{21} + \dot{\theta}_{76} c \theta_{21} s \theta_{76}) + \\
& s \theta_{65} (c \theta_{21} c \theta_{76} (\dot{\theta}_{65} + \dot{\theta}_{21} s \beta_{54}) - \dot{\theta}_{76} s \beta_{54} s \theta_{21} s \theta_{76}) - \\
& 2 \dot{\theta}_{21} c \theta_{21} (\dot{\theta}_{76} c \theta_{76} s \beta_{54} + c \beta_{54} (-\dot{\theta}_{65} c \theta_{65} c \theta_{76} + \dot{\theta}_{76} s \theta_{65} s \theta_{76})) + \\
& s \theta_{21} (s \beta_{54} (-\ddot{\theta}_{76} c \theta_{76} + \dot{\theta}_{76}^2 s \theta_{76}) + c \beta_{54} (-s \theta_{65} (\dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) c \theta_{76} + \\
& \ddot{\theta}_{76} s \theta_{76})) + c \theta_{65} (\ddot{\theta}_{65} c \theta_{76} - 2 \dot{\theta}_{65} \dot{\theta}_{76} s \theta_{76})) + \\
& c \theta_{21} (c \beta_{54} (-c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{76} c \theta_{76} + \ddot{\theta}_{21} s \theta_{76}) + s \theta_{21} \\
& (-\ddot{\theta}_{76} c \theta_{76} + (\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2) s \theta_{76})) - \\
& c \theta_{21} (c \theta_{65} (c \theta_{76} (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2 + 2 \dot{\theta}_{21} \dot{\theta}_{65} s \beta_{54}) + \ddot{\theta}_{76} s \theta_{76}) + \\
& s \theta_{65} (c \theta_{76} (\ddot{\theta}_{65} + \ddot{\theta}_{21} s \beta_{54}) - 2 \dot{\theta}_{76} (\dot{\theta}_{65} + \dot{\theta}_{21} s \beta_{54}) s \theta_{76})) + s \theta_{21} \\
& (s \theta_{65} (c \theta_{76} (2 \dot{\theta}_{21} \dot{\theta}_{65} + (\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) s \beta_{54}) + \ddot{\theta}_{76} s \beta_{54} s \theta_{76}) - \\
& c \theta_{65} (c \theta_{76} (\ddot{\theta}_{21} + \ddot{\theta}_{65} s \beta_{54}) - 2 \dot{\theta}_{76} (\dot{\theta}_{21} + \dot{\theta}_{65} s \beta_{54}) s \theta_{76}))) - \\
& 2 \dot{\theta}_{1110} c \theta_{1110} (\dot{\theta}_{54} s(\beta_{1413} - \theta_{54}) (c \theta_{65} c \theta_{76} s \theta_{21} + c \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76})) + \\
& c(\beta_{1413} - \theta_{54}) (-s \theta_{21} (c \theta_{76} (\dot{\theta}_{65} + \dot{\theta}_{21} s \beta_{54}) s \theta_{65} + \dot{\theta}_{76} c \theta_{65} s \theta_{76}) + \\
& c \beta_{54} (\dot{\theta}_{76} c \theta_{21} c \theta_{76} - \dot{\theta}_{21} s \theta_{21} s \theta_{76}) + \\
& c \theta_{21} (c \theta_{65} c \theta_{76} (\dot{\theta}_{21} + \dot{\theta}_{65} s \beta_{54}) - \dot{\theta}_{76} s \beta_{54} s \theta_{65} s \theta_{76})) - \\
& \dot{\theta}_{54} c(\beta_{1413} - \theta_{54}) (c \theta_{21} (c \theta_{21} c \theta_{43} c \theta_{65} c \theta_{76} - c \theta_{43} s \theta_{21} \\
& (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76}) + s \theta_{43} (c \beta_{54} c \theta_{76} s \theta_{65} - s \beta_{54} s \theta_{76})) + \\
& s \theta_{21} (-c \theta_{21} c \theta_{65} c \theta_{76} s \theta_{43} + s \beta_{54} (c \theta_{76} s \theta_{21} s \theta_{43} s \theta_{65} - c \theta_{43} s \theta_{76}) + \\
& c \beta_{54} (c \theta_{43} c \theta_{76} s \theta_{65} + s \theta_{21} s \theta_{43} s \theta_{76})) + \\
& s(\beta_{1413} - \theta_{54}) (-\dot{\theta}_{21} s \theta_{21} (c \theta_{21} c \theta_{43} c \theta_{65} c \theta_{76} - c \theta_{43} s \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + \\
& c \beta_{54} s \theta_{76}) + s \theta_{43} (c \beta_{54} c \theta_{76} s \theta_{65} - s \beta_{54} s \theta_{76})) + \\
& \dot{\theta}_{21} c \theta_{21} (-c \theta_{21} c \theta_{65} c \theta_{76} s \theta_{43} + s \beta_{54} (c \theta_{76} s \theta_{21} s \theta_{43} s \theta_{65} - c \theta_{43} s \theta_{76}) + \\
& c \beta_{54} (c \theta_{43} c \theta_{76} s \theta_{65} + s \theta_{21} s \theta_{43} s \theta_{76})) + \\
& c \theta_{21} (-c \theta_{76} s \theta_{43} (\dot{\theta}_{43} c \theta_{21} c \theta_{65} + s \beta_{54} (\dot{\theta}_{76} - \dot{\theta}_{43} s \theta_{21} s \theta_{65})) - \\
& c \theta_{43} (c \theta_{21} c \theta_{76} (\dot{\theta}_{65} + \dot{\theta}_{21} s \beta_{54}) s \theta_{65} + s \beta_{54} (\dot{\theta}_{43} - \dot{\theta}_{76} s \theta_{21} s \theta_{65}) s \theta_{76} + \\
& c \theta_{65} (c \theta_{76} (\dot{\theta}_{21} + \dot{\theta}_{65} s \beta_{54}) s \theta_{21} + \dot{\theta}_{76} c \theta_{21} s \theta_{76})) + \\
& c \beta_{54} (-c \theta_{43} (c \theta_{76} (\dot{\theta}_{76} s \theta_{21} - \dot{\theta}_{43} s \theta_{65}) + \dot{\theta}_{21} c \theta_{21} s \theta_{76}) + \\
& s \theta_{43} (\dot{\theta}_{65} c \theta_{65} c \theta_{76} + (\dot{\theta}_{43} s \theta_{21} - \dot{\theta}_{76} s \theta_{65}) s \theta_{76})) + \\
& s \theta_{21} (-\dot{\theta}_{43} c \theta_{21} c \theta_{43} c \theta_{65} c \theta_{76} + \dot{\theta}_{21} c \theta_{65} c \theta_{76} s \theta_{21} s \theta_{43} + \\
& \dot{\theta}_{65} c \theta_{21} c \theta_{76} s \theta_{43} s \theta_{65} + \dot{\theta}_{76} c \theta_{21} c \theta_{65} s \theta_{43} s \theta_{76} + \\
& c \beta_{54} (s \theta_{43} (c \theta_{76} (\dot{\theta}_{76} s \theta_{21} - \dot{\theta}_{43} s \theta_{65}) + \dot{\theta}_{21} c \theta_{21} s \theta_{76}) + \\
& c \theta_{43} (\dot{\theta}_{65} c \theta_{65} c \theta_{76} + (\dot{\theta}_{43} s \theta_{21} - \dot{\theta}_{76} s \theta_{65}) s \theta_{76})) + \\
& s \beta_{54} (c \theta_{43} c \theta_{76} (-\dot{\theta}_{76} + \dot{\theta}_{43} s \theta_{21} s \theta_{65}) + s \theta_{43} (\dot{\theta}_{65} c \theta_{65} c \theta_{76} s \theta_{21} +
\end{aligned}$$

$$\begin{aligned}
& \ddot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i)) - \\
& s \theta_{1110} i \left((-\dot{\theta}_{54}^2 c(\beta_{1413} i - \theta_{54}) + \ddot{\theta}_{54} s(\beta_{1413} i - \theta_{54})) (c \theta_{65} i c \theta_{76} i s \theta_{21} i + c \theta_{21} i \right. \\
& \quad (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i)) + \\
& \quad 2 \dot{\theta}_{54} s(\beta_{1413} i - \theta_{54}) (-s \theta_{21} i (c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + \dot{\theta}_{76} i c \theta_{65} i s \theta_{76} i) + \\
& \quad c \beta_{54} i (\dot{\theta}_{76} i c \theta_{21} i c \theta_{76} i - \dot{\theta}_{21} i s \theta_{21} i s \theta_{76} i) + \\
& \quad c \theta_{21} i (c \theta_{65} i c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) - \dot{\theta}_{76} i s \beta_{54} i s \theta_{65} i s \theta_{76} i)) + c(\beta_{1413} i - \theta_{54}) \\
& \quad (c \beta_{54} i (-s \theta_{21} i (2 \dot{\theta}_{21} i \dot{\theta}_{76} i c \theta_{76} i + \ddot{\theta}_{21} i s \theta_{76} i) + c \theta_{21} i (\ddot{\theta}_{76} i c \theta_{76} i - (\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2) s \theta_{76} i))) - \\
& \quad s \theta_{21} i (c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 + 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \beta_{54} i) + \ddot{\theta}_{76} i s \theta_{76} i) + \\
& \quad s \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{65} i + \ddot{\theta}_{21} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{76} i)) + \\
& \quad c \theta_{21} i (-s \theta_{65} i (c \theta_{76} i (2 \dot{\theta}_{21} i \dot{\theta}_{65} i + (\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2) s \beta_{54} i) + \ddot{\theta}_{76} i s \beta_{54} i s \theta_{76} i) + \\
& \quad c \theta_{65} i (c \theta_{76} i (\ddot{\theta}_{21} i + \ddot{\theta}_{65} i s \beta_{54} i) - 2 \dot{\theta}_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{76} i))) + \\
& \quad (-\ddot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) - \dot{\theta}_{54}^2 s(\beta_{1413} i - \theta_{54})) (c \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i \\
& \quad (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& \quad s \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& \quad c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i))) - \\
& \quad 2 \dot{\theta}_{54} c(\beta_{1413} i - \theta_{54}) (-\dot{\theta}_{21} s \theta_{21} (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i \\
& \quad (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& \quad \dot{\theta}_{21} c \theta_{21} (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - c \theta_{43} s \theta_{76} i) + \\
& \quad c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i)) + \\
& \quad c \theta_{21} (-c \theta_{76} i s \theta_{43} (\dot{\theta}_{43} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i)) - \\
& \quad c \theta_{43} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i + \\
& \quad c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) + \\
& \quad c \beta_{54} i (-c \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& \quad s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i))) + \\
& \quad s \theta_{21} (-\dot{\theta}_{43} c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i + \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s \theta_{43} + \\
& \quad \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s \theta_{43} s \theta_{65} i + \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s \theta_{43} s \theta_{76} i + \\
& \quad c \beta_{54} i (s \theta_{43} (c \theta_{76} i (\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i) + \\
& \quad c \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + (\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i) s \theta_{76} i)) + \\
& \quad s \beta_{54} i (c \theta_{43} c \theta_{76} i (-\dot{\theta}_{76} i + \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i) + s \theta_{43} (\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \\
& \quad \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i))) + \\
& \quad s(\beta_{1413} i - \theta_{54}) ((-\dot{\theta}_{21}^2 c \theta_{21} - \ddot{\theta}_{21} s \theta_{21}) (c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i - c \theta_{43} s \theta_{21} i \\
& \quad (c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i) + s \theta_{43} (c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i)) + \\
& \quad (\ddot{\theta}_{21} c \theta_{21} - \dot{\theta}_{21}^2 s \theta_{21}) (-c \theta_{21} i c \theta_{65} i c \theta_{76} i s \theta_{43} + s \beta_{54} i (c \theta_{76} i s \theta_{21} i s \theta_{43} s \theta_{65} i - \\
& \quad c \theta_{43} s \theta_{76} i) + c \beta_{54} i (c \theta_{43} c \theta_{76} i s \theta_{65} i + s \theta_{21} i s \theta_{43} s \theta_{76} i)) - \\
& \quad 2 \dot{\theta}_{21} s \theta_{21} (-c \theta_{76} i s \theta_{43} (\dot{\theta}_{43} c \theta_{21} i c \theta_{65} i + s \beta_{54} i (\dot{\theta}_{76} i - \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i)) - \\
& \quad c \theta_{43} (c \theta_{21} i c \theta_{76} i (\dot{\theta}_{65} i + \dot{\theta}_{21} i s \beta_{54} i) s \theta_{65} i + s \beta_{54} i (\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i) s \theta_{76} i + \\
& \quad c \theta_{65} i (c \theta_{76} i (\dot{\theta}_{21} i + \dot{\theta}_{65} i s \beta_{54} i) s \theta_{21} i + \dot{\theta}_{76} i c \theta_{21} i s \theta_{76} i)) +
\end{aligned}$$

$$\begin{aligned}
& c \beta_{54} i \left(-c \theta_{43} \left(c \theta_{76} i \left(\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i \right) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i \right) + \right. \\
& \quad \left. s \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \left(\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i \right) s \theta_{76} i \right) \right) + \\
& 2 \dot{\theta}_{21} c \theta_{21} \left(-\dot{\theta}_{43} c \theta_{21} i c \theta_{43} c \theta_{65} i c \theta_{76} i + \dot{\theta}_{21} i c \theta_{65} i c \theta_{76} i s \theta_{21} i s \theta_{43} + \right. \\
& \quad \dot{\theta}_{65} i c \theta_{21} i c \theta_{76} i s \theta_{43} s \theta_{65} i + \dot{\theta}_{76} i c \theta_{21} i c \theta_{65} i s \theta_{43} s \theta_{76} i + \\
& \quad c \beta_{54} i \left(s \theta_{43} \left(c \theta_{76} i \left(\dot{\theta}_{76} i s \theta_{21} i - \dot{\theta}_{43} s \theta_{65} i \right) + \dot{\theta}_{21} i c \theta_{21} i s \theta_{76} i \right) + \right. \\
& \quad \left. c \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \left(\dot{\theta}_{43} s \theta_{21} i - \dot{\theta}_{76} i s \theta_{65} i \right) s \theta_{76} i \right) \right) + \\
& \quad s \beta_{54} i \left(c \theta_{43} c \theta_{76} i \left(-\dot{\theta}_{76} i + \dot{\theta}_{43} s \theta_{21} i s \theta_{65} i \right) + s \theta_{43} \left(\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i s \theta_{21} i + \right. \right. \\
& \quad \left. \left. \dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i s \theta_{65} i + \left(\dot{\theta}_{43} - \dot{\theta}_{76} i s \theta_{21} i s \theta_{65} i \right) s \theta_{76} i \right) \right) + \\
& s \theta_{21} \left(c \theta_{76} i s \theta_{43} \left(c \theta_{21} i \left(\left(\dot{\theta}_{21} i^2 + \dot{\theta}_{65} i^2 \right) c \theta_{65} i + \ddot{\theta}_{65} i s \theta_{65} i \right) + s \theta_{21} i \right. \right. \\
& \quad \left. \left(\ddot{\theta}_{21} i c \theta_{65} i - 2 \dot{\theta}_{21} i \dot{\theta}_{65} i s \theta_{65} i \right) \right) + 2 \left(\dot{\theta}_{21} i c \theta_{65} i s \theta_{21} i + \dot{\theta}_{65} i c \theta_{21} i s \theta_{65} i \right) \\
& \quad \left(\dot{\theta}_{43} c \theta_{43} c \theta_{76} i - \dot{\theta}_{76} i s \theta_{43} s \theta_{76} i \right) - c \theta_{21} i c \theta_{65} i \\
& \quad \left(-s \theta_{43} \left(\left(\dot{\theta}_{43}^2 + \dot{\theta}_{76} i^2 \right) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + c \theta_{43} \left(\ddot{\theta}_{43} c \theta_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i s \theta_{76} i \right) \right) + \\
& s \beta_{54} i \left(2 \dot{\theta}_{43} \dot{\theta}_{76} i c \theta_{76} i s \theta_{43} + c \theta_{76} i s \theta_{21} i \left(c \theta_{43} \left(2 \dot{\theta}_{43} \dot{\theta}_{65} i c \theta_{65} i + \ddot{\theta}_{43} s \theta_{65} i \right) + \right. \right. \\
& \quad \left. \left. s \theta_{43} \left(\ddot{\theta}_{65} i c \theta_{65} i - \left(\dot{\theta}_{43}^2 + \dot{\theta}_{65} i^2 \right) s \theta_{65} i \right) \right) + \right. \\
& \quad \left(\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43} \right) s \theta_{76} i + c \theta_{43} \left(-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i \right) + \\
& \quad 2 \left(\dot{\theta}_{65} i c \theta_{65} i s \theta_{43} + \dot{\theta}_{43} c \theta_{43} s \theta_{65} i \right) \left(\dot{\theta}_{21} i c \theta_{21} i c \theta_{76} i - \dot{\theta}_{76} i s \theta_{21} i s \theta_{76} i \right) + \\
& \quad s \theta_{43} s \theta_{65} i \left(-s \theta_{21} i \left(\left(\dot{\theta}_{21} i^2 + \dot{\theta}_{76} i^2 \right) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + \right. \\
& \quad \left. c \theta_{21} i \left(\ddot{\theta}_{21} i c \theta_{76} i - 2 \dot{\theta}_{21} i \dot{\theta}_{76} i s \theta_{76} i \right) \right) + c \beta_{54} i \left(2 \dot{\theta}_{76} i c \theta_{76} i \right. \\
& \quad \left(\dot{\theta}_{43} c \theta_{43} s \theta_{21} i + \dot{\theta}_{21} i c \theta_{21} i s \theta_{43} \right) + c \theta_{43} c \theta_{76} i \left(\ddot{\theta}_{65} i c \theta_{65} i - \dot{\theta}_{65} i^2 s \theta_{65} i \right) + \\
& \quad \left(c \theta_{21} i \left(2 \dot{\theta}_{21} i \dot{\theta}_{43} c \theta_{43} + \ddot{\theta}_{21} i s \theta_{43} \right) + s \theta_{21} i \left(\ddot{\theta}_{43} c \theta_{43} - \left(\dot{\theta}_{21} i^2 + \dot{\theta}_{43}^2 \right) s \theta_{43} \right) \right) \\
& \quad s \theta_{76} i + s \theta_{21} i s \theta_{43} \left(\ddot{\theta}_{76} i c \theta_{76} i - \dot{\theta}_{76} i^2 s \theta_{76} i \right) - 2 \dot{\theta}_{65} i c \theta_{65} i \\
& \quad \left(\dot{\theta}_{43} c \theta_{76} i s \theta_{43} + \dot{\theta}_{76} i c \theta_{43} s \theta_{76} i \right) - s \theta_{65} i \left(c \theta_{43} \left(\left(\dot{\theta}_{43}^2 + \dot{\theta}_{76} i^2 \right) c \theta_{76} i + \right. \right. \\
& \quad \left. \left. \ddot{\theta}_{76} i s \theta_{76} i \right) + s \theta_{43} \left(\ddot{\theta}_{43} c \theta_{76} i - 2 \dot{\theta}_{43} \dot{\theta}_{76} i s \theta_{76} i \right) \right) + \\
& c \theta_{21} \left(-c \theta_{65} i c \theta_{76} i \left(c \theta_{21} i \left(\left(\dot{\theta}_{21} i^2 + \dot{\theta}_{43}^2 \right) c \theta_{43} + \ddot{\theta}_{43} s \theta_{43} \right) + s \theta_{21} i \right. \right. \\
& \quad \left. \left(\ddot{\theta}_{21} i c \theta_{43} - 2 \dot{\theta}_{21} i \dot{\theta}_{43} s \theta_{43} \right) \right) + \\
& s \theta_{21} i \left(\dot{\theta}_{43}^2 c \theta_{43} + \ddot{\theta}_{43} s \theta_{43} \right) \left(c \theta_{76} i s \beta_{54} i s \theta_{65} i + c \beta_{54} i s \theta_{76} i \right) + \\
& 2 \left(\dot{\theta}_{21} i c \theta_{43} s \theta_{21} i + \dot{\theta}_{43} c \theta_{21} i s \theta_{43} \right) \left(\dot{\theta}_{65} i c \theta_{76} i s \theta_{65} i + \dot{\theta}_{76} i c \theta_{65} i s \theta_{76} i \right) + \\
& \left(\ddot{\theta}_{43} c \theta_{43} - \dot{\theta}_{43}^2 s \theta_{43} \right) \left(c \beta_{54} i c \theta_{76} i s \theta_{65} i - s \beta_{54} i s \theta_{76} i \right) - \\
& c \theta_{21} i c \theta_{43} \left(c \theta_{65} i \left(\left(\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 \right) c \theta_{76} i + \ddot{\theta}_{76} i s \theta_{76} i \right) + \right. \\
& \quad \left. s \theta_{65} i \left(\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i \right) \right) - 2 \dot{\theta}_{43} c \theta_{43} \\
& \quad \left(\dot{\theta}_{76} i c \theta_{76} i s \beta_{54} i + c \beta_{54} i \left(-\dot{\theta}_{65} i c \theta_{65} i c \theta_{76} i + \dot{\theta}_{76} i s \theta_{65} i s \theta_{76} i \right) \right) + \\
& s \theta_{43} \left(s \beta_{54} i \left(-\ddot{\theta}_{76} i c \theta_{76} i + \dot{\theta}_{76} i^2 s \theta_{76} i \right) + c \beta_{54} i \left(-s \theta_{65} i \left(\left(\dot{\theta}_{65} i^2 + \dot{\theta}_{76} i^2 \right) c \theta_{76} i + \right. \right. \right. \\
& \quad \left. \left. \ddot{\theta}_{76} i s \theta_{76} i \right) + c \theta_{65} i \left(\ddot{\theta}_{65} i c \theta_{76} i - 2 \dot{\theta}_{65} i \dot{\theta}_{76} i s \theta_{76} i \right) \right) +
\end{aligned}$$

$$\begin{aligned}
& 2 \dot{\theta}_{43} s \theta_{43} (\dot{\theta}_{21} c \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76}) + s \theta_{21} (\dot{\theta}_{76} c \beta_{54} c \theta_{76} + s \beta_{54} (\dot{\theta}_{65} c \theta_{65} c \theta_{76} - \dot{\theta}_{76} s \theta_{65} s \theta_{76}))) - \\
& c \theta_{43} (c \beta_{54} (c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{76} c \theta_{76} + \ddot{\theta}_{21} s \theta_{76}) + s \theta_{21} (\ddot{\theta}_{76} c \theta_{76} - (\dot{\theta}_{21}^2 + \dot{\theta}_{76}^2) s \theta_{76})) - s \beta_{54} \\
& (-c \theta_{21} (2 \dot{\theta}_{21} \dot{\theta}_{65} c \theta_{65} c \theta_{76} + s \theta_{65} (\ddot{\theta}_{21} c \theta_{76} - 2 \dot{\theta}_{21} \dot{\theta}_{76} s \theta_{76}))) + \\
& s \theta_{21} (s \theta_{65} ((\dot{\theta}_{21}^2 + \dot{\theta}_{65}^2 + \dot{\theta}_{76}^2) c \theta_{76} + \ddot{\theta}_{76} s \theta_{76}) + \\
& c \theta_{65} (-\ddot{\theta}_{65} c \theta_{76} + 2 \dot{\theta}_{65} \dot{\theta}_{76} s \theta_{76})))))) - \\
& 2 \dot{\theta}_{98}^2 \sec(\theta_{98})^2 (s \theta_{109} (c \theta_{21} c \theta_{65} c \theta_{76} s(\theta_{21} + \theta_{43}) - s \beta_{54} (c \theta_{43} c \theta_{76} s \theta_{21} s \theta_{65} + \\
& c \theta_{21} c \theta_{76} s \theta_{21} s \theta_{43} s \theta_{65} - c \theta_{21} c \theta_{43} s \theta_{76} + s \theta_{21} s \theta_{43} s \theta_{76}) - \\
& c \beta_{54} (c \theta_{21} c \theta_{43} c \theta_{76} s \theta_{65} - c \theta_{76} s \theta_{21} s \theta_{43} s \theta_{65} + c \theta_{43} s \theta_{21} s \theta_{21} \\
& s \theta_{76} + c \theta_{21} s \theta_{21} s \theta_{43} s \theta_{76})) + \\
& c \theta_{109} (c \theta_{1110} (-s(\beta_{1413} - \theta_{54}) (c \theta_{65} c \theta_{76} s \theta_{21} + c \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76})) + \\
& c(\beta_{1413} - \theta_{54}) s \theta_{43} (-c \theta_{21} c \theta_{65} c \theta_{76} s \theta_{21} + s \beta_{54} (c \theta_{76} s \theta_{21} s \theta_{21} s \theta_{65} - \\
& c \theta_{21} s \theta_{76}) + c \beta_{54} (c \theta_{21} c \theta_{76} s \theta_{65} + s \theta_{21} s \theta_{21} s \theta_{76})) + \\
& c \theta_{43} c(\beta_{1413} - \theta_{54}) (s \theta_{21} (c \beta_{54} c \theta_{76} s \theta_{65} - s \beta_{54} s \theta_{76}) + \\
& c \theta_{21} (c \theta_{21} c \theta_{65} c \theta_{76} - s \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76})))) - \\
& s \theta_{1110} (c(\beta_{1413} - \theta_{54}) (c \theta_{65} c \theta_{76} s \theta_{21} + c \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + c \beta_{54} s \theta_{76})) + \\
& s(\beta_{1413} - \theta_{54}) (c \theta_{21} (c \theta_{21} c \theta_{43} c \theta_{65} c \theta_{76} - c \theta_{43} s \theta_{21} (c \theta_{76} s \beta_{54} s \theta_{65} + \\
& c \beta_{54} s \theta_{76}) + s \theta_{43} (c \beta_{54} c \theta_{76} s \theta_{65} - s \beta_{54} s \theta_{76})) + \\
& s \theta_{21} (-c \theta_{21} c \theta_{65} c \theta_{76} s \theta_{43} + s \beta_{54} (c \theta_{76} s \theta_{21} s \theta_{43} s \theta_{65} - c \theta_{43} s \theta_{76}) + \\
& c \beta_{54} (c \theta_{43} c \theta_{76} s \theta_{65} + s \theta_{21} s \theta_{43} s \theta_{76})))))) \tan(\theta_{98})
\end{aligned}$$