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ON THE MODIFICATION OF THE LOCAL GEOMETRIC PROPERTIES OF PLANE CURVES

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ABSTRACT

If the coordinates of the points of a given curve are approximated by spline functions, then the local geometric properties (slope, curvature, etc.) of the curve can be regarded as functions of a finite set of independent variables, the coordinates of the supporting points. Formulae are derived for the computation of the derivatives of these functions with respect to the aforementioned coordinates. An example is included to show how these formulae can be applied to synthesize a plane closed curve with a prescribed curvature distribution.

INTRODUCTION

In the design of structural elements with notches or borigns, henceforth generally referred to as "openings", stress concentrations [1][2] at these openings frequently occur, which could be avoided by a proper determination of the shape of the opening. In this respect, such shapes have been found by application of optimization techniques in connection with the finite-element method (FEM) [2,3]. Schnack [4] has solved similar optimization problems by introducing the monotony relationship between the magnitude of the stress at the opening at a given point and the curvature of the opening at this point; this relation was first established by Neuber [1]. In [4] it was shown how a proper correction of the curvature of the contour, namely diminishing of the curvature at points with high stress and vice versa, can lead to optimal shapes. This way the original mechanical problem can be handled as a purely geometrical one, i.e. given a profile with a known stress distribution, determine a new profile with a "better" stress distribution by correcting the curvature of the profile according to the known stress distribution. What is meant by a "better" stress distribution is one with a lower difference between the highest and the lowest stress

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magnitudes at the profile. In [4] the curvature was approximated by a finite-difference formula at the FEM network nodes lying at the profile. Furthermore, the curvature was assumed to have a linear distribution between the nodes with the highest and the lowest stress magnitude, and a FEM computation was performed at each iteration of the optimization procedure. The curvature corrections were specified as "small" changes, which led to "small" changes in the coordinates of the involved nodes.

In this paper it is shown how, by introducing spline-functions (SF), the curvature and its derivatives, for a given curve, can be computed accurately at arbitrary points of the curve. Furthermore, knowledge of the derivatives of the curvature with respect to the coordinates of the supporting points (SP) allows the use of Newton-Raphson's method to determine the coordinates of the supporting points of a curve to meet specifications on its curvature distribution. This way, "relatively large" curvature corrections can be carried out.

DERIVATION OF THE SLOPE AND CURVATURE DERIVATIVES

Let the (x, y) cartesian coordinates of an arbitrary point P of a curve Γ be approximated by spline functions after the introduction of a parameter t, i.e. [5]

$$x(t)=a_{xk}(t-t_k)^3+b_{xk}(t-t_k)^2+c_{xk}(t-t_k)+d_{xk} \tag{1}$$

$$y(t)=a_{yk}(t-t_k)^3+b_{yk}(t-t_k)^2+c_{yk}(t-t_k)+d_{yk},\ t_k\leq t\leq t_{k+1} \tag{2}$$

where the set t_k, for k=1,...,n, is defined as

$$t_1=0,\ t_{k+1}=t_k+\Delta t_k\ ,\ k=1,\dots,n'(=n-1) \tag{3}$$

$$\Delta t_k=\sqrt{(x_{k+1}-x_k)^2+(y_{k+1}-y_k)^2}\ ,\quad k=1,\dots,n' \tag{4}$$

the set x_k, y_k, for k=1,...,n being the cartesian coordinates of the given supporting points P_k. This paper is concerned only with closed curves, for which reason the SF are periodic. The coefficients $a_{xk}, b_{xk}, \ldots, c_{yk}, d_{yk}, k=1,\ldots,n'$ are then defined as [5]

$$a_{xk}=(\ddot{x}_{k+1}-\ddot{x}_k)/6\Delta t_k \tag{5a}$$

$$b_{xk}=\ddot{x}_k/2 \tag{5b}$$

$$c_{xk}=x_k/\Delta t_k-\Delta t_k(\ddot{x}_{k+1}+2\ddot{x}_k)/6=\dot{x}_k \tag{5c}$$

$$d_{xk}=x_k \tag{5d}$$

with similar expressions for the y-coefficients. In the foregoing formulae $\dot{x}_k$, $\dot{y}_k$, $\ddot{x}_k$ and $\ddot{y}_k$ represent first and second derivatives of the cartesian coordinates with respect to t, computed at t_k. Next, the following n'-dimensional vectors and n'x n' -matrices are defined:

$$\underset{\sim}{x}=[x_1,\ldots,x_{n'}]^T, \underset{\sim}{\dot{x}}=[\dot{x}_1,\ldots,\dot{x}_{n'}]^T, \underset{\sim}{\ddot{x}}=[\ddot{x}_1,\ldots,\ddot{x}_{n'}]^T \tag{6}$$

with similar definitions for the vectors $\underset{\sim}{y}$, $\underset{\sim}{\dot{y}}$ and $\underset{\sim}{\ddot{y}}$.

$$\underset{\sim}{A}=\begin{bmatrix} 2(\Delta t_{n'}+\Delta t_1) & \Delta t_1 & & & & \Delta t_{n'} \\ \Delta t_1 & (2\Delta t_1+\Delta t_2) & \Delta t_2 & & & \\ & \Delta t_2 & 2(\Delta t_2+\Delta t_3) & \Delta t_3 & & \\ & & \ddots & \ddots & \ddots & \\ \Delta t_{n'} & & & & \Delta t_{n''} & 2(\Delta t_{n''}+\Delta t_{n'}) \end{bmatrix} \tag{7}$$

where $n''=n'-1=n-2$

$$\underset{\sim}{H}=\mathrm{diag}(\Delta t_1,\Delta t_2,\ldots,\Delta t_{n'}) \tag{8}$$

$\underset{\sim}{I}=\mathrm{diag}(1,1,\ldots,1)=$ the n'xn' identity matrix (9)

$$\underset{\sim}{J}=\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots \\ 1 & & & & -1 \end{bmatrix}, \underset{\sim}{K}=\underset{\sim}{J}+3\underset{\sim}{I} \tag{10}$$

Thus, the second derivatives are computed from [5] as;

$$\mathbf{A}\ddot{\mathbf{x}}=-6\mathbf{J}^{T}\mathbf{H}^{-1}\mathbf{J}\mathbf{x},\mathbf{A}\ddot{\mathbf{y}}=-6\mathbf{J}^{T}\mathbf{H}^{-1}\mathbf{J}\mathbf{y} \tag{11}$$

Next, "small" changes in $\ddot{\mathbf{x}}$ and $\ddot{\mathbf{y}}$ for correspondingly "small" changes in $\mathbf{x}$ and $\mathbf{y}$ are computed. From the expressions so obtained, formulae for the derivatives $\partial\ddot{\mathbf{x}}/\partial\mathbf{x}$, $\partial\ddot{\mathbf{x}}/\partial\mathbf{y}$, $\partial\ddot{\mathbf{y}}/\partial\mathbf{x}$ and $\partial\mathbf{y}/\partial\mathbf{y}$ are derived. In fact, let δ denote a "small" change of the variable it preceeds. Thus,

$$\delta\ddot{\mathbf{x}}=-6(\delta\mathbf{A}^{-1})\,\mathbf{J}^{T}\mathbf{H}^{-1}\mathbf{J}\mathbf{x}-6\mathbf{A}^{-1}\mathbf{J}^{T}(\delta\mathbf{H}^{-1})\mathbf{J}\mathbf{x}-6\mathbf{A}^{-1}\mathbf{J}^{T}\mathbf{H}^{-1}\mathbf{J}\delta\mathbf{x} \tag{12}$$

By equating $\delta(\mathbf{A}^{-1}\mathbf{A})$ to zero the following expressions are obtained:

$$\delta\mathbf{A}^{-1}=-\mathbf{A}^{-1}(\delta\mathbf{A})\mathbf{A}^{-1} \tag{13}$$

$$\delta\mathbf{H}^{-1}=-\mathbf{H}^{-1}(\delta\mathbf{H})\mathbf{H}^{-1}=-\mathbf{H}^{-2}\delta\mathbf{H} \tag{14}$$

where the latter equation holds due to the diagonality of matrix $\mathbf{H}$. This way, eq. (11) is transformed into

$$\delta\ddot{\mathbf{x}}=\mathbf{A}^{-1}\left[6\mathbf{J}^{T}\mathbf{H}^{-1}(\mathbf{H}^{-1}\delta\mathbf{H}.\mathbf{J}\mathbf{x}-\mathbf{J}\delta\mathbf{x})-\delta\mathbf{A}.\ddot{\mathbf{x}}\right] \tag{15}$$

where the point between two variables is meant to indicate that the variation δ does not involve the variables after the point. To obtain expressions for the variations of $\mathbf{A}$ and $\mathbf{H}$, which depend only on the set Δt_k, the variation of this set is first derived. From definitions (4),

$$\delta\Delta t_k=\delta\sqrt{\Delta x_k^2+\Delta y_k^2} = (\Delta x_k \;\; \delta\Delta x_k + \Delta y_k \;\; \delta\Delta y_k)/\Delta t_k \tag{16}$$

where Δx_k and Δy_k, for $k=1,...,n'$, are defined similarly to Δt_k. Let

$$c_k=\Delta x_k/\Delta t_k, s_k=\Delta y_k/\Delta t_k \tag{17}$$

Thus:

$$\delta \underset{\sim}{H} . \underset{\sim}{J} \underset{\sim}{x} = \underset{\sim}{A}_{11} \underset{\sim}{J} \delta \underset{\sim}{x} + \underset{\sim}{A}_{12} \underset{\sim}{J} \delta \underset{\sim}{y} \tag{18}$$

where $\underset{\sim}{A}_{11}$ and $\underset{\sim}{A}_{12}$ are diagonal matrices defined as

$$\underset{\sim}{A}_{11} = \mathrm{diag}\ (\Delta x_1 c_1, \ldots, \Delta x_{n'} c_{n'}), \underset{\sim}{A}_{12} = \mathrm{diag}\ (\Delta x_1 s_1, \ldots, \Delta x_{n'} s_{n'}) \tag{19}$$

Similarly,

$$\delta \underset{\sim}{H} . \underset{\sim}{J} \underset{\sim}{y} = \underset{\sim}{A}_{21} \underset{\sim}{J} \delta \underset{\sim}{x} + \underset{\sim}{A}_{22} \underset{\sim}{J} \delta \underset{\sim}{y} \tag{20}$$

where

$$\underset{\sim}{A}_{21} = \mathrm{diag}(\Delta y_1 c_1, \ldots, \Delta y_{n'} c_{n'}),\ \underset{\sim}{A}_{22} = \mathrm{diag}(\Delta y_1 s_1, \ldots, \Delta y_{n'} s_{n'}) \tag{21}$$

To compute $\delta \underset{\sim}{A} . \underset{\sim}{\ddot{x}}$, only the Δt_k terms in $\underset{\sim}{A} \underset{\sim}{\ddot{x}}$ are varied, i.e.

$$\delta \underset{\sim}{A} \underset{\sim}{\ddot{x}} = \begin{bmatrix} \ddot{\underset{\sim}{x}}_{n'} \delta \Delta t_{n'} + 2\ddot{x}_1 \delta (\Delta t_{n'} + \Delta t_1) + \ddot{x}_2 \delta \Delta t_1 \\ \ddot{x}_1 \delta \Delta t_1 + 2\ddot{x}_2 \delta (\Delta t_1 + \Delta t_2) + \ddot{x}_3 \delta \Delta t_2 \\ \vdots \\ \ddot{x}_{n''} \delta \Delta t_{n''} + 2\ddot{x}_{n'} \delta (\Delta t_{n''} + \Delta t_{n'}) + \ddot{x}_1 \delta \Delta t_{n'} \end{bmatrix} \tag{22}$$

Substituting (16) and (17) into the latter expression,

$$\delta \underset{\sim}{A} . \underset{\sim}{\ddot{x}} = \underset{\sim}{B}_{11} \underset{\sim}{J} \delta \underset{\sim}{\ddot{x}} + \underset{\sim}{B}_{12} \underset{\sim}{J} \delta \underset{\sim}{\ddot{y}} \tag{23a}$$

Similarly,

$$\delta \underset{\sim}{A} . \underset{\sim}{\ddot{y}} = \underset{\sim}{B}_{21} \underset{\sim}{J} \delta \underset{\sim}{\ddot{x}} + \underset{\sim}{B}_{22} \underset{\sim}{J} \delta \underset{\sim}{\ddot{y}} \tag{23b}$$

where matrices $\underset{\sim}{B}_{ij}$, for i,j=1, 2 are defined as

$$\underset{\sim}{B}_{ij}=\begin{bmatrix} \beta_{11} & 0 & 0 & & \beta_{1,n'} \\ \beta_{21} & \beta_{22} & 0 & & 0 \\ 0 & \beta_{32} & \beta_{33} & & 0 \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & 0 & 0 \cdots & \cdots & \beta_{n',n''}\;\beta_{n',n'} \end{bmatrix} \tag{24}$$

$$\begin{aligned} i=1,\ j=1,\ \beta_{k,k-1} &= (\ddot{x}_{k-1}+2\ddot{x}_k)\,c_{k-1} \\ \beta_{k,k} &= (2\ddot{x}_k+\ddot{x}_{k+1})\,c_k \end{aligned} \tag{25a}$$

$$\begin{aligned} i=1,\ j=2,\ \beta_{k,k-1} &= (\ddot{x}_{k-1}+2\ddot{x}_k)\,s_{k-1} \\ \beta_{k,k} &= (2\ddot{x}_k+\ddot{x}_{k+1})\,s_k \end{aligned} \tag{25b}$$

$$\begin{aligned} i=2,\ j=1,\ \beta_{k,k-1} &= (\ddot{y}_{k-1}+2\ddot{y}_k)\,c_{k-1} \\ \beta_{k,k} &= (2\ddot{y}_k+\ddot{y}_{k+1})\,c_k \end{aligned} \tag{25c}$$

$$\begin{aligned} i=2,\ j=2,\ \beta_{k,k-1} &= (\ddot{y}_{k-1}+2\ddot{y}_k)\,s_{k-1} \\ \beta_{k,k} &= (2\ddot{y}_k+\ddot{y}_{k+1})\,s_k \end{aligned} \tag{25d}$$

where $k-1=n'$ for $k=1$

Substituting then eqs. (18), (20) and (23) into eq. (12) one obtains

$$\delta\underset{\sim}{\ddot{x}}=\underset{\sim}{A}^{-1}\left[6\underset{\sim}{J}^T\underset{\sim}{H}^{-1}(\underset{\sim}{H}^{-1}\underset{\sim}{A}_{11}-\underset{\sim}{I})-\underset{\sim}{B}_{11}\right]\underset{\sim}{J}\,\delta\underset{\sim}{x}+\underset{\sim}{A}^{-1}(6\underset{\sim}{J}^T\underset{\sim}{H}^{-2}\underset{\sim}{A}_{12}-\underset{\sim}{B}_{12})\underset{\sim}{J}\,\delta\underset{\sim}{y} \tag{26a}$$

$$\delta\underset{\sim}{\ddot{y}}=\underset{\sim}{A}^{-1}(6\underset{\sim}{J}^T\underset{\sim}{H}^{-2}\underset{\sim}{A}_{21}-\underset{\sim}{B}_{21})\underset{\sim}{J}\,\delta\underset{\sim}{x}+\underset{\sim}{A}^{-1}\left[6\underset{\sim}{J}^T\underset{\sim}{H}^{-1}(\underset{\sim}{H}^{-1}\underset{\sim}{A}_{22}-\underset{\sim}{I})-\underset{\sim}{B}_{22}\right]\underset{\sim}{J}\,\delta\underset{\sim}{y} \tag{26b}$$

from which the formulae

$$\partial\underset{\sim}{\ddot{x}}/\partial\underset{\sim}{x}=\underset{\sim}{A}^{-1}\left[6\underset{\sim}{J}^T\underset{\sim}{H}^{-1}(\underset{\sim}{H}^{-1}\underset{\sim}{A}_{11}-\underset{\sim}{I})-\underset{\sim}{B}_{11}\right]\underset{\sim}{J} \tag{27a}$$

$$\partial\ddot{\mathbf{x}}/\partial\mathbf{y}=\mathbf{A}^{-1}(6\mathbf{J}^{T}\mathbf{H}^{-2}\mathbf{A}_{12}-\mathbf{B}_{12})\mathbf{J} \tag{27b}$$

$$\partial\ddot{\mathbf{y}}/\partial\mathbf{x}=\mathbf{A}^{-1}(6\mathbf{J}^{T}\mathbf{H}^{-2}\mathbf{A}_{21}-\mathbf{B}_{21})\mathbf{J} \tag{27c}$$

$$\partial\ddot{\mathbf{y}}/\partial\mathbf{y}=\mathbf{A}^{-1}\left[6\mathbf{J}^{T}\mathbf{H}^{-1}(\mathbf{H}^{-1}\mathbf{A}_{22}-\mathbf{I})-\mathbf{B}_{22}\right]\mathbf{J} \tag{27d}$$

follow immediately.

Next, formulae for the $\partial\dot{\mathbf{x}}/\partial\mathbf{x}$, $\partial\dot{\mathbf{x}}/\partial\mathbf{y}$, $\partial\dot{\mathbf{y}}/\partial\mathbf{x}$ and $\partial\dot{\mathbf{y}}/\partial\mathbf{y}$ derivatives are derived. From eq. (5c), vector $\dot{\mathbf{x}}$ can be written as

$$\dot{\mathbf{x}}=\mathbf{H}^{-1}\mathbf{J}\mathbf{x}-\mathbf{H}\mathbf{K}\ddot{\mathbf{x}}/6 \tag{28}$$

A "small" variation $\delta\dot{\mathbf{x}}$, for given "small" variations of $\mathbf{x}$ and $\mathbf{y}$ is obtained from the latter equation as

$$\delta\dot{\mathbf{x}}=\mathbf{H}^{-1}\mathbf{J}\delta\mathbf{x}-\mathbf{H}^{-1}\delta\mathbf{H}.\mathbf{H}^{-1}\mathbf{J}\mathbf{x}-\mathbf{H}\mathbf{K}\delta\ddot{\mathbf{x}}/6-\delta\mathbf{H}.\mathbf{K}\ddot{\mathbf{x}}/6 \tag{29}$$

Making use of the diagonality of $\mathbf{H}$ and of eqs. (18) and (26a), eq. (29) is transformed into

$$\begin{aligned}\delta\dot{\mathbf{x}}=\;&\mathbf{H}^{-1}\mathbf{J}\delta\mathbf{x}-\mathbf{H}^{-2}\mathbf{A}_{11}\mathbf{J}\delta\mathbf{x}-\mathbf{H}^{-2}\mathbf{A}_{12}\mathbf{J}\delta\mathbf{y}-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{x})\delta\mathbf{x}/6\\&-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{y})\delta\mathbf{y}/6-\delta\mathbf{H}.\mathbf{K}\ddot{\mathbf{x}}/6\end{aligned} \tag{30}$$

An expression for $\delta\mathbf{H}.\mathbf{K}\ddot{\mathbf{x}}$ can be obtained by varying only the Δt_k terms in $\mathbf{H}\mathbf{K}\ddot{\mathbf{x}}$. Thus,

$$\delta\mathbf{H}.\mathbf{K}\ddot{\mathbf{x}}=\begin{bmatrix}(2\ddot{x}_1+\ddot{x}_2)\delta\Delta t_1\\(2\ddot{x}_2+\ddot{x}_3)\delta\Delta t_2\\ \cdot\\ \cdot\\ \cdot\\(2x_{n'}+x_1)\delta\Delta t_{n'}\end{bmatrix}=\begin{bmatrix}(2\ddot{x}_1+\ddot{x}_2)(c_1\delta\Delta x_1+s_1\delta\Delta y_1)\\(2\ddot{x}_2+\ddot{x}_3)(c_2\delta\Delta x_2+s_2\delta\Delta y_2)\\ \cdot\\ \cdot\\ \cdot\\(2\ddot{x}_{n'}+\ddot{x}_1)(c_{n'}\delta\Delta x_{n'}+s_{n'}\delta\Delta y_{n'}\end{bmatrix}=$$

$$=\mathbf{C}_{11}\mathbf{J}\delta\mathbf{x}+\mathbf{C}_{12}\mathbf{J}\delta\mathbf{y} \tag{31a}$$

Similarly

$$\delta \mathbf{H}.\mathbf{K}\ddot{\mathbf{y}}=\mathbf{C}_{21}\mathbf{J}\delta\mathbf{x}+\mathbf{C}_{22}\mathbf{J}\delta\mathbf{y} \tag{31b}$$

where

$$\mathbf{C}_{ij}=\mathrm{diag}(d_1,\ldots,d_{n'}) \tag{32}$$

with the d_k elements defined as

$$i=1 \text{ and } j=1, d_k=(2\ddot{x}_k+\ddot{x}_{k+1})c_k \tag{33a}$$

$$i=1 \text{ and } j=2, d_k=(2\ddot{x}_k+\ddot{x}_{k+1})s_k \tag{33b}$$

$$i=2 \text{ and } j=1, d_k=(2\ddot{y}_k+\ddot{y}_{k+1})c_k \tag{33c}$$

$$i=2 \text{ and } j=2, d_k=(2\ddot{y}_k+\ddot{y}_{k+1})s_k \tag{33d}$$

where $k+1=1$ for $k=n'$

Substituting eqs. (27a,b), and (31a) into eq. (30) one obtains

$$\begin{aligned}\delta\dot{\mathbf{x}}=&-\{[\mathbf{H}^{-1}(\mathbf{H}^{-1}\mathbf{A}_{11}-\mathbf{I})+\mathbf{C}_{11}/6]\mathbf{J}+\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{x})/6\}\delta\mathbf{x}\\ &-[(\mathbf{H}^{-2}\mathbf{A}_{12}+\mathbf{C}_{12}/6)\mathbf{J}+\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{y})]\delta\mathbf{y}\end{aligned} \tag{34a}$$

Similarly

$$\begin{aligned}\delta\dot{\mathbf{y}}=&-[(\mathbf{H}^{-2}\mathbf{A}_{21}+\mathbf{C}_{21}/6)\mathbf{J}+\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{y}}/\partial\mathbf{x})/6]\delta\mathbf{x}\\ &-\{[\mathbf{H}^{-1}(\mathbf{H}^{-1}\mathbf{A}_{22}-\mathbf{I})+\mathbf{C}_{22}]\mathbf{J}+\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{y}}/\partial\mathbf{x})/6\}\delta\mathbf{y}\end{aligned} \tag{34b}$$

from which the sought formulae

$$\partial\dot{\mathbf{x}}/\partial\mathbf{x}=-[\mathbf{H}^{-1}(\mathbf{H}^{-1}\mathbf{A}_{11}-\mathbf{I})+\mathbf{C}_{11}/6]\mathbf{J}-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{x})/6 \tag{35a}$$

$$\partial\dot{\mathbf{x}}/\partial\mathbf{y}=-(\mathbf{H}^{-2}\mathbf{A}_{12}+\mathbf{C}_{12}/6)\mathbf{J}-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{x}}/\partial\mathbf{y})/6 \tag{35b}$$

$$\partial\dot{\mathbf{y}}/\partial\mathbf{x}=-(\mathbf{H}^{-2}\mathbf{A}_{21}+\mathbf{C}_{21}/6)\mathbf{J}-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{y}}/\partial\mathbf{x})/6 \tag{35c}$$

$$\partial\dot{\mathbf{y}}/\partial\mathbf{y}=-[\mathbf{H}^{-1}(\mathbf{H}^{-1}\mathbf{A}_{22}-\mathbf{I})+\mathbf{C}_{22}/6]\mathbf{J}-\mathbf{H}\mathbf{K}(\partial\ddot{\mathbf{y}}/\partial\mathbf{y})/6 \tag{35d}$$

follow directly.

Formulae (35) express the sensitivity of the slope or, correspondingly, of the unit tangential and normal vectors of the curve to changes in the coordinates of the supporting points, thus allowing the synthesis of curves having a prescribed slope. These formulae, however, require the evaluation of the derivatives of formulae (27), for which reason these were derived first. The said eight formulae can now be applied to compute the derivatives of the curvature at the SP. Let κ_k be the curvature at point P_k, i.e.

$$\kappa_k=\frac{\dot{x}_k\ddot{y}_k-\ddot{x}_k\dot{y}_k}{(\dot{x}_k^2+\dot{y}_k^2)^{3/2}} \tag{36}$$

and then define

$$\underset{\sim}{\kappa}=[\kappa_1,\kappa_2,\ldots,\kappa_{n'}]^T \tag{37}$$

Applying the" chain rule" the following formulae are obtained:

$$\frac{\partial\underset{\sim}{\kappa}}{\partial\underset{\sim}{x}}=\frac{\partial\underset{\sim}{\kappa}}{\partial\dot{\underset{\sim}{x}}}\frac{\partial\dot{\underset{\sim}{x}}}{\partial\underset{\sim}{x}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\dot{\underset{\sim}{y}}}\frac{\partial\dot{\underset{\sim}{y}}}{\partial\underset{\sim}{x}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\ddot{\underset{\sim}{x}}}\frac{\partial\ddot{\underset{\sim}{x}}}{\partial\underset{\sim}{x}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\ddot{\underset{\sim}{y}}}\frac{\partial\ddot{\underset{\sim}{y}}}{\partial\underset{\sim}{x}} \tag{38a}$$

$$\frac{\partial\underset{\sim}{\kappa}}{\partial\underset{\sim}{y}}=\frac{\partial\underset{\sim}{\kappa}}{\partial\dot{\underset{\sim}{x}}}\frac{\partial\dot{\underset{\sim}{x}}}{\partial\underset{\sim}{y}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\dot{\underset{\sim}{y}}}\frac{\partial\dot{\underset{\sim}{y}}}{\partial\underset{\sim}{y}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\ddot{\underset{\sim}{x}}}\frac{\partial\ddot{\underset{\sim}{x}}}{\partial\underset{\sim}{y}}+\frac{\partial\underset{\sim}{\kappa}}{\partial\ddot{\underset{\sim}{y}}}\frac{\partial\ddot{\underset{\sim}{y}}}{\partial\underset{\sim}{y}} \tag{38b}$$

where $\partial\underset{\sim}{\kappa}/\dot{\underset{\sim}{x}},\ldots,\partial\underset{\sim}{\kappa}/\partial\ddot{\underset{\sim}{y}}$ are diagonal matrices defined as

$$\partial\underset{\sim}{\kappa}/\partial\dot{\underset{\sim}{x}}=\mathrm{diag}(D_1,\ldots,D_{n'}) \tag{39a}$$

$$\partial\underset{\sim}{\kappa}/\partial\dot{\underset{\sim}{y}}=\mathrm{fiag}(E_1,\ldots,E_{n'}) \tag{39b}$$

$$\partial\underset{\sim}{\kappa}/\partial\ddot{\underset{\sim}{x}}=\mathrm{diag}(F_1,\ldots,F_{n'}) \tag{40a}$$

$$\partial\underset{\sim}{\kappa}/\partial\ddot{\underset{\sim}{y}}=\mathrm{diag}(G_1,\ldots,G_{n'}) \tag{40b}$$

with

$$D_k=[(\dot{y}_k^2-2\dot{x}_k^2)\ddot{y}_k+3\dot{x}_k\dot{y}_k\ddot{x}_k]/(\dot{x}_k^2+\dot{y}_k^2)^{5/2} \tag{41a}$$

$$E_k=-[(\dot{x}_k^2-2\dot{y}_k^2)\ \ddot{x}_k+3\dot{x}_k\dot{y}_k\ddot{y}_k]/(\dot{x}_k^2+\dot{y}_k^2)^{5/2} \tag{41b}$$

$$F_k=-\dot{y}_k/(\dot{x}_k^2+\dot{y}_k^2)^{3/2} \tag{42a}$$

$$G_k=\dot{x}_k/(\dot{x}_k^2+\dot{y}_k^2)^{3/2} \tag{42b}$$

So far, the cartesian coordinates of the SP were regarded as independent variables. In many applications, however, the sought curves contain symmetries, thus turning a set of such coordinates to depend upon the remaining ones. Moreover, by selecting a point enclosed by the curve as the origin of polar coordinates, ρ,ϕ, and by fixing the angles ϕ_k, the only independent variables are ρ_k, for $k=1,\ldots,m$, where, due to possible symmetries, $m\leq n'$, and $m=n'$ only if the curve possesses no symmetries. Storing the independent variables in the m-dimensional vector $\underset{\sim}{z}$, then, the dependence of the cartesian coordinates of the SP upon the independent variables $\underset{\sim}{z}$, can be written as

$$\underset{\sim}{x} = \underset{\sim}{V}\underset{\sim}{z} + \underset{\sim}{a}, \quad \underset{\sim}{y} = \underset{\sim}{W}\underset{\sim}{z} + \underset{\sim}{b} \tag{43}$$

where $\underset{\sim}{V}$ and $\underset{\sim}{W}$ are constant $n'\times m$ matrices, whereas $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are constant n-dimensional vectors, accounting for those coordinates which remain fixed throughout a particular problem. The formulae for the total derivatives with respect to $\underset{\sim}{z}$

$$\frac{\partial \dot{\underset{\sim}{x}}}{\partial \underset{\sim}{z}} = \frac{\partial \dot{\underset{\sim}{x}}}{\partial \underset{\sim}{x}}\underset{\sim}{V} + \frac{\partial \dot{\underset{\sim}{x}}}{\partial \underset{\sim}{y}}\underset{\sim}{W}, \quad \frac{\partial \dot{\underset{\sim}{y}}}{\partial \underset{\sim}{z}} = \frac{\partial \dot{\underset{\sim}{y}}}{\partial \underset{\sim}{x}}\underset{\sim}{V} + \frac{\partial \dot{\underset{\sim}{y}}}{\partial \underset{\sim}{y}}\underset{\sim}{W} \tag{44}$$

$$\frac{\partial \ddot{\underset{\sim}{x}}}{\partial \underset{\sim}{z}} = \frac{\partial \ddot{\underset{\sim}{x}}}{\partial \underset{\sim}{x}}\underset{\sim}{V} + \frac{\partial \ddot{\underset{\sim}{x}}}{\partial \underset{\sim}{y}}\underset{\sim}{W}, \quad \frac{\partial \ddot{\underset{\sim}{y}}}{\partial \underset{\sim}{z}} = \frac{\partial \ddot{\underset{\sim}{y}}}{\partial \underset{\sim}{x}}\underset{\sim}{V} + \frac{\partial \ddot{\underset{\sim}{y}}}{\partial \underset{\sim}{y}}\underset{\sim}{W} \tag{45}$$

$$\frac{\partial \underset{\sim}{\kappa}}{\partial \underset{\sim}{z}} = \frac{\partial \underset{\sim}{\kappa}}{\partial \underset{\sim}{x}}\underset{\sim}{V} + \frac{\partial \underset{\sim}{\kappa}}{\partial \underset{\sim}{y}}\underset{\sim}{W} \tag{46}$$

follow directly.

ERROR IN THE APPROXIMATION OF THE CURVATURE AND ITS DERIVATIVES

A series of subroutines were written, which compute the foregoing derivatives and the curvature [6]. These subroutines were used to establish the dependence of the error in the approximation upon the number of SP. Tests were carried on a circle of radius =1.

The double symmetry of the curve was exploited and so, the results comprise only the first quadrant. The error in the approximation was recorded for 2,3,5,7 and 10 SP defined on the first quadrant. The corresponding approximating curves and their error distribution in the approximation of the curvature are shown in Figs 1-5. Defining as the error in the approximation the greatest absolute value of the error over the whole quadrant, this is recorded vs. the number of SP in Table 1.

The foregoing computations contain not only an error of approximation, but also a round-off error; this was, however, disregarded because the only critical computation in the formulae derived above is the inversion of matrix $\underset{\sim}{A}$, as defined in eq. (7). This matrix, nevertheless, is very well conditioned, for it is positive definite, tridiagonal and diagonally dominant [7]. Furthermore, the matrix was not explicitly inverted, but LU-decomposed [8] once and for all and later on its factors, $\underset{\sim}{L}$ and $\underset{\sim}{U}$, were used repeatedly in the back-substitution stage of Gauss' algorithm [8] to compute successively the different columns of the matrices appearing in eqs. (27). For this purpose, the subroutines DECOMP and SOLVE, due to Moler [9], were applied.

An example is next included, which can be computed with zero round-off error, to illustrate the procedure.

EXAMPLE 1. COMPUTATION OF THE CURVATURE OF A CIRCLE AND OF ITS DERIVATIVE WITH RESPECT TO THE RADIUS, USING TWO INDEPENDENT VARIABLES

A circle is approximated with spline functions using the following 5SP: $P_1(1,0)$, $P_2(0,1)$, $P_3(-1,0)$, $P_4(0,-1)$, $P_5(1,0)=P_1$. The involved vectors and matrices are

$$\underset{\sim}{x}=[1,0,-1,0]^T \quad , \quad \underset{\sim}{y}=[0,1,0,-1]^T$$

$$\underset{\sim}{A}=\sqrt{2}\begin{bmatrix}4&1&0&1\\1&4&1&0\\0&1&4&1\\1&0&1&4\end{bmatrix},\qquad \underset{\sim}{A}^{-1}=\frac{\sqrt{2}}{48}\begin{bmatrix}7&-2&1&-2\\-2&7&-2&1\\1&-2&7&-2\\-2&1&-2&7\end{bmatrix}$$

$$\underset{\sim}{H}=\sqrt{2}\ \mathrm{diag}\ (1,1,1,1),\ \underset{\sim}{I}=\mathrm{diag}(1,1,1,1)$$

$$\underset{\sim}{J}=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\\1&0&0&-1\end{bmatrix},\qquad \underset{\sim}{K}=\begin{bmatrix}2&1&0&0\\0&2&1&0\\0&0&2&1\\1&0&0&2\end{bmatrix}$$

Solving for $\ddot{\underset{\sim}{x}}$ and $\ddot{\underset{\sim}{y}}$ in eqs. (11).

$$\ddot{\underset{\sim}{x}}=\frac{3}{2}[-1,0,1,0]^T,\ddot{\underset{\sim}{y}}=\frac{3}{2}\ [0,-1,0,1]^T$$

Similarly, from eq. (28) and an analogous one for $\dot{\underset{\sim}{y}}$, which is not shown, one obtains

$$\dot{\underset{\sim}{x}}=\frac{3\sqrt{2}}{2^2}\ [0,-1,0,1]^T,\dot{\underset{\sim}{y}}=\frac{3\sqrt{2}}{2^2}\ [1,0,-1,0]^T$$

Moreover,

$$\underset{\sim}{A}_{11}=\sqrt{2}\underset{\sim}{I}/2,\underset{\sim}{A}_{12}=\sqrt{2}\ \mathrm{diag}(-1,1,-1,1)/2,\underset{\sim}{A}_{21}=\sqrt{2}\ \mathrm{diag}(-1,1,-1,1)/2\ \underset{\sim}{A}_{22}=\sqrt{2}\ \underset{\sim}{I}/2$$

$$\underset{\sim}{B}_{11}=\frac{3\sqrt{2}}{2^2}\begin{bmatrix}2&0&0&-2\\1&-1&0&0\\0&-2&2&0\\0&0&1&-1\end{bmatrix},\qquad \underset{\sim}{B}_{12}=\frac{3\sqrt{2}}{2^2}\begin{bmatrix}-2&0&0&-2\\-1&-1&0&0\\0&-2&-2&0\\0&0&-1&-1\end{bmatrix}$$

$$\underset{\sim}{B}_{21}=\frac{3\sqrt{2}}{2^2}\begin{bmatrix}1&0&0&1\\2&2&0&0\\0&1&1&0\\0&0&2&2\end{bmatrix},\qquad \underset{\sim}{B}_{22}=\frac{3\sqrt{2}}{2^2}\begin{bmatrix}-1&0&0&1\\-2&2&0&0\\0&1&-1&0\\0&0&-2&2\end{bmatrix}$$

Substitution of the above matrices in eqs. (27) yields

$$\partial\ddot{\underset{\sim}{x}}/\partial\underset{\sim}{x}=\frac{3}{2^4}\begin{bmatrix}-2 & 2 & -2 & 2\\ 4 & -7 & 4 & -1\\ -2 & 2 & -2 & 2\\ 4 & -1 & 4 & -7\end{bmatrix}, \quad \partial\ddot{\underset{\sim}{x}}/\partial\underset{\sim}{y}=\frac{3}{2^4}\begin{bmatrix}0 & 4 & 0 & -4\\ 1 & 0 & -1 & 0\\ 0 & -4 & 0 & 4\\ -1 & 0 & 1 & 0\end{bmatrix}$$

$$\partial\ddot{\underset{\sim}{y}}/\partial\underset{\sim}{x}=\frac{3}{2^4}\begin{bmatrix}0 & 1 & 0 & -1\\ 4 & 0 & -4 & 0\\ 0 & -1 & 0 & 1\\ -4 & 0 & 4 & 0\end{bmatrix}, \quad \partial\ddot{\underset{\sim}{y}}/\partial\underset{\sim}{y}=\frac{3}{2^4}\begin{bmatrix}-7 & 4 & -1 & 4\\ 2 & -2 & 2 & -2\\ -1 & 4 & -7 & 4\\ 2 & -2 & 2 & -2\end{bmatrix}$$

In order to compute the derivatives of formulae (35), the $\underset{\sim}{C}_{ij}$ matrices of formulae (32) and (33) should be first computed. These are:

$$\underset{\sim}{C}_{11}=3\sqrt{2}\ \mathrm{diag}(2,-1,2,-1)/2^2, \underset{\sim}{C}_{12}=3\sqrt{2}\mathrm{diag}(-2,-1,-2,-1)/2^2$$

$$\underset{\sim}{C}_{21}=3\sqrt{2}\ \mathrm{diag}(1,2,1,2)/2^2, \ \underset{\sim}{C}_{22}=3\sqrt{2}\ \mathrm{diag}(-1,2,-1,2)/2^2$$

Thus,

$$\partial\dot{\underset{\sim}{x}}/\partial\underset{\sim}{x}=\frac{\sqrt{2}}{2^5}\begin{bmatrix}0 & 3 & 0 & -3\\ -6 & 0 & 6 & 0\\ 0 & -3 & 0 & 3\\ 6 & 0 & -6 & 0\end{bmatrix}, \quad \partial\dot{\underset{\sim}{x}}/\partial y=\frac{\sqrt{2}}{2^5}\begin{bmatrix}-17 & 8 & 1 & 8\\ -2 & 8 & -2 & -4\\ 1 & 8 & -17 & 8\\ -2 & -4 & -2 & 8\end{bmatrix}$$

$$\partial\dot{\underset{\sim}{y}}/\partial x=\frac{\sqrt{2}}{2^5}\begin{bmatrix}-8 & 2 & 4 & 2\\ -8 & 17 & -8 & -1\\ 4 & 2 & -8 & 2\\ -8 & -1 & -8 & 17\end{bmatrix} \quad \partial\dot{\underset{\sim}{y}}/\partial y=\frac{\sqrt{2}}{2^5}\begin{bmatrix}0 & 6 & 0 & -6\\ -3 & 0 & 3 & 0\\ 0 & -6 & 0 & 6\\ 3 & 0 & -3 & 0\end{bmatrix}$$

The partial derivatives of the curvature are:

$$\partial\underset{\sim}{\kappa}/\partial\dot{\underset{\sim}{x}}=\sqrt{2}\times 2^4\mathrm{diag}(0,1,0,-1)/3^2, \partial\underset{\sim}{\kappa}/\partial\dot{\underset{\sim}{y}}=\sqrt{2}\times 2^4\ \mathrm{diag}(-1,0,1,0)/3^2$$

$$\partial\underset{\sim}{\kappa}/\partial\ddot{\underset{\sim}{x}}=2^3\mathrm{diag}(-1,0,1,0)/3^2, \partial\underset{\sim}{\kappa}/\partial\ddot{\underset{\sim}{y}}=2^3\mathrm{diag}(0,-1,0,1)/3^2$$

The partial derivatives of formulae (38) are then

$$\partial \underset{\sim}{\kappa}/\partial \underset{\sim}{x}=\frac{1}{3^2}\begin{bmatrix} 11 & -5 & -1 & -5 \\ -12 & 0 & 12 & 0 \\ 1 & 5 & -11 & 5 \\ -12 & 0 & 12 & 0 \end{bmatrix}, \quad \partial \underset{\sim}{\kappa}/\partial \underset{\sim}{y}=\frac{1}{3^2}\begin{bmatrix} 0 & -12 & 0 & 12 \\ -5 & 11 & -5 & -1 \\ 0 & -12 & 0 & 12 \\ 5 & 1 & 5 & -11 \end{bmatrix}$$

Considering the abscissa of P_1 and the ordinate of P_2 as the independent variables and assuming that the supporting points are variable, but restricted to remain in the coordinate axes, matrices $\underset{\sim}{V}$ and $\underset{\sim}{W}$ appearing in formulae (43) are then obtained as

$$\underset{\sim}{V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underset{\sim}{W}=\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus, the total derivative of $\underset{\sim}{\kappa}=[\kappa_1,\kappa_2]^T$ with respect to $\underset{\sim}{z}$ is

$$\partial \underset{\sim}{\kappa}/\partial \underset{\sim}{z}=\frac{4}{3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

If z_1 and z_2 are set equal to the radius of the circle, the number of independent variables reduces to 1 and so the derivative of the curvature with respect to the radius can be computed. The values found for the curvature and its derivative were

$$\kappa=4/3, \partial \kappa/\partial r=-4/3$$

which approximate the true values, 1 and -1 with an error of 33%. The elements of the matrix $\partial \kappa/\partial \underset{\sim}{z}$ also contain an error of 33%. To compute the true values of these elements, the circle is regarded as a particular case of an ellipse of semiaxes a and b. The curvature at the point where the ellipse intersects the x-axis, κ_1, is

$$\kappa_1=a/b^2$$

Regarding a and b as the independent variables z_1 and z_2, and setting $z_1=z_2=1$, one obtains

$$\partial\kappa_1/\partial z_1=1/b^2=1, \partial\kappa_1/\partial z_2=-2a/b^2=-2$$

which are the true values of these components.

The next example is included to illustrate how arbitrary values of the curvature of a curve can be synthesized by properly assigning the coordinates of the SP, which is done with the aid of Newton-Raphson's method.

EXAMPLE 2. SYNTHESIS OF A PLANE CLOSED CURVE TO MEET PRESCRIBED CONDITIONS IMPOSED ON ITS CURVATURE

From the foregoing analysis it follows that the curvature at every point of a curve approximated with spline functions is a function of the coordinates of the involved supporting points. If these coordinates are dependent upon a certain set of free parameters $\underset{\sim}{z}$, like the ones introduced in eq. (43), then

$$\underset{\sim}{\kappa}=\underset{\sim}{\kappa}(\underset{\sim}{z})$$

where $\underset{\sim}{z}$ is an m-dimensional vector, with $m\leq n'$, n' having been previously defined as the number of SP minus 1. Numbering the SP in such a way that the first m are independent, then κ is an m-dimensional vector containing the curvature at the first m SP. Assuming that it is intended to synthesize a double symmetrical curve, the m free SP are located in the first quadrant of the x-y plane. The curvature distribution is then assigned through vector $\underset{\sim}{c}$, where

$$\underset{\sim}{c}=\left[c_1, c_2, \ldots, c_m\right]^T$$

and c_i is the prescribed curvature at the $i^{\underline{th}}$ SP, i.e. at the point with polar coordinates $\rho=z_i, \phi=\phi_i$.

The unknown variables z_i can now be computed as the solution of the m-dimensional system of nonlinear equations

$$\underset{\sim}{f}(\underset{\sim}{z})=\underset{\sim}{\kappa}(\underset{\sim}{z})-\underset{\sim}{c}=0$$

The best known method to solve this system is that of Newton-Raphson [10], which only requires the computation of the Jacobian matrix of $\underset{\sim}{f}$ with respect to $\underset{\sim}{z}$, which is

$$\underset{\sim}{f}'(\underset{\sim}{z})=\partial\underset{\sim}{\kappa}/\partial\underset{\sim}{z}$$

as given by formula (46). In the above Jacobian, the derivative of $\underset{\sim}{c}$ vanishes since this is a constant prescribed value.

Fig 6 shows the successive curves which were obtained during the Newton-Raphson's iterations to synthesize a curve with the following curvature distribution

$$\underset{\sim}{c}=[1,1,1,1,1]^T$$

i.e. a circle. The 5 SP were equally distributed over the first quadrant, the initial "guess" of $\underset{\sim}{z}$ was assigned so as to represent an ellipse with semiaxes a =2, b=1. The number of iterations needed until convergence was reached was 7, and the procedure was stopped when the correction to the unknown vector attained a -maximal-norm smaller than 10^{-4}

CONCLUSIONS

The advantages of introducing splines in synthesis or optimization problems involving the determination of curves to meet prescribed geometrical conditions are many-fold, some of which are:

- The number of independent variables which have to be handled is relatively low.
- Derivatives with respect to the free parameters can be efficiently computed, which helps in iterative methodseither to solve equations or to optimize objective functions- since the introduction

of gradients, normally accelerates the convergence of the method.
- The equations appearing in spline computations are linear and well conditioned, which allows computations with small round-off errors.
- The analysis of errors -which was not the subject of this paper- can be performed systematically, for spline analyses have been extensively carried out [11,12].

Further research in this direction should involve the introduction of spline functions in the solution of classical problems of the calculus of variations [13] , thus reducing such problems to the search of the solution over a finite-dimensional space, instead of over a Hilbert space.

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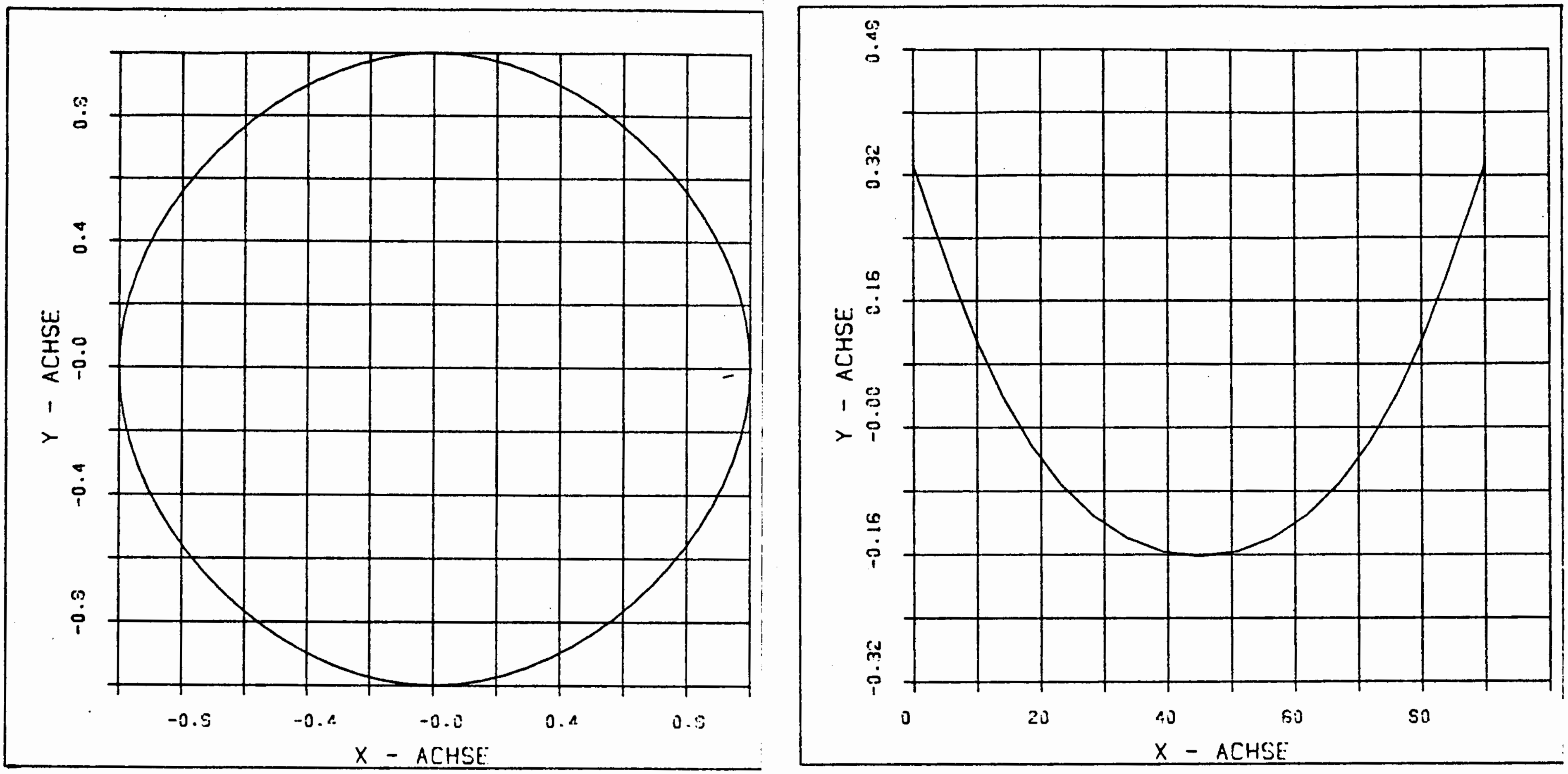
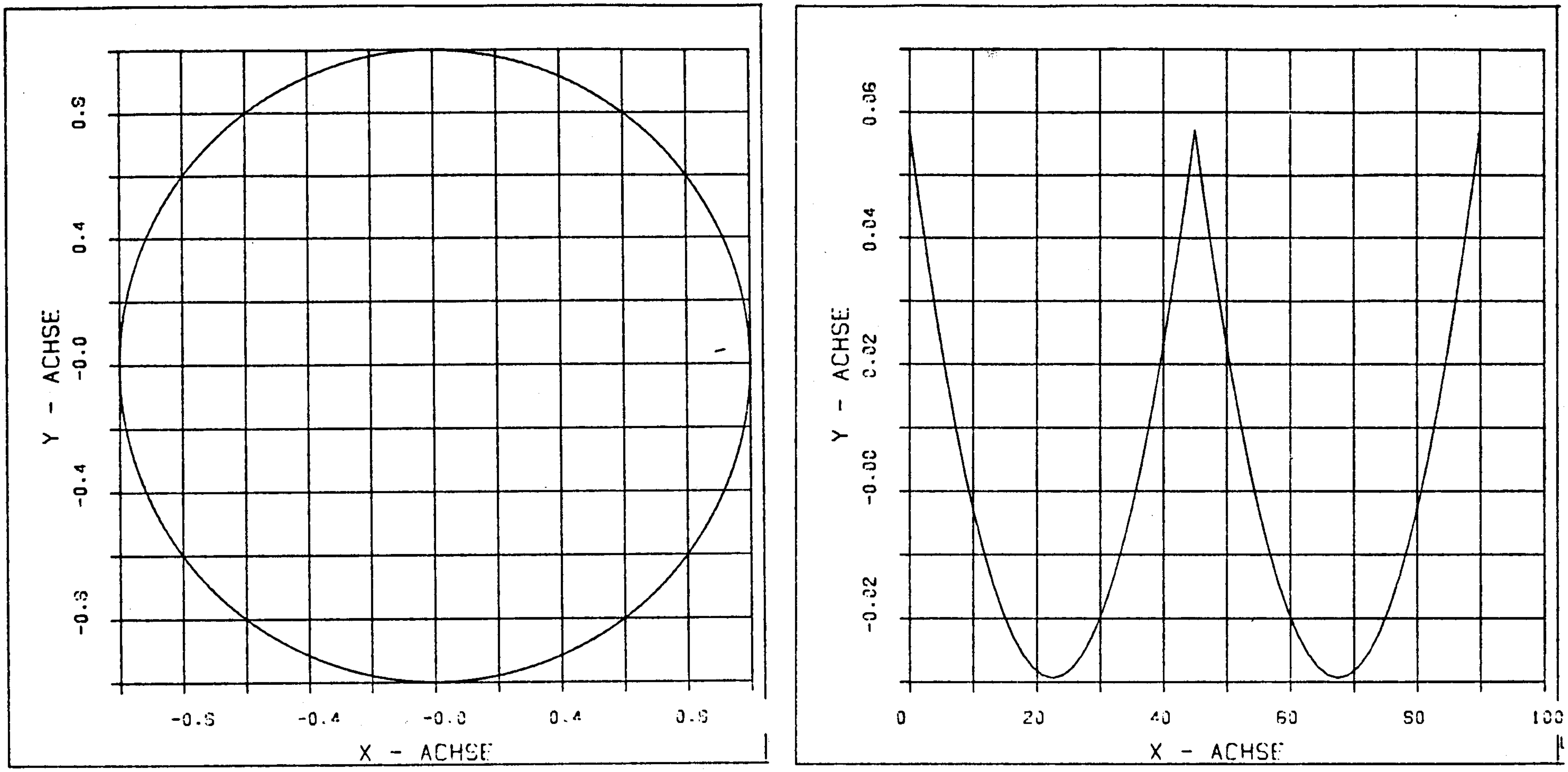


(a) (b)

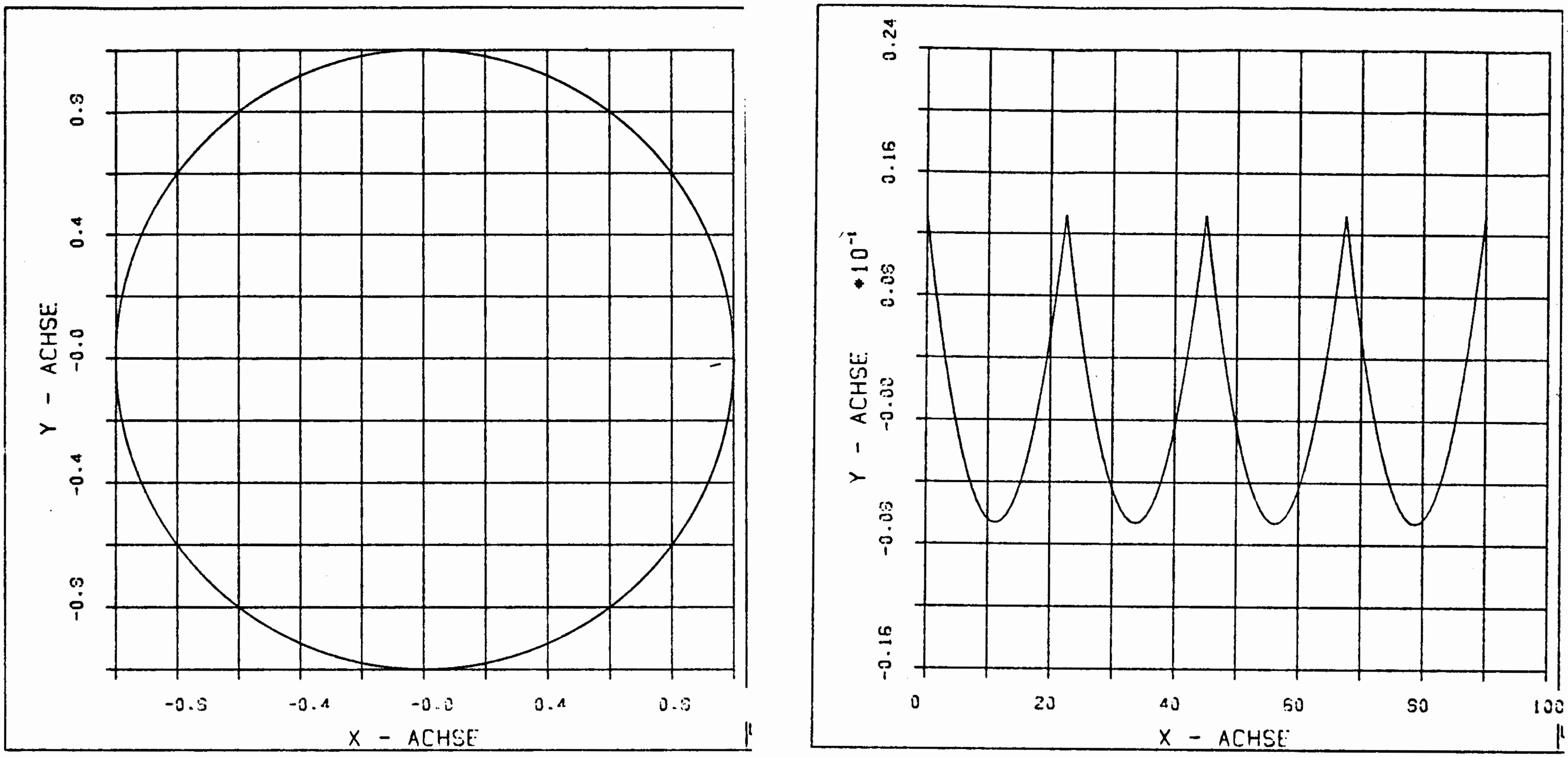
Fig. 1 Spline-approximation to a circle of radius 1. a) Approximating curve with 2 equally distributed SP in the first quadrant. b) Error distribution of the curvature approximation.



(a)

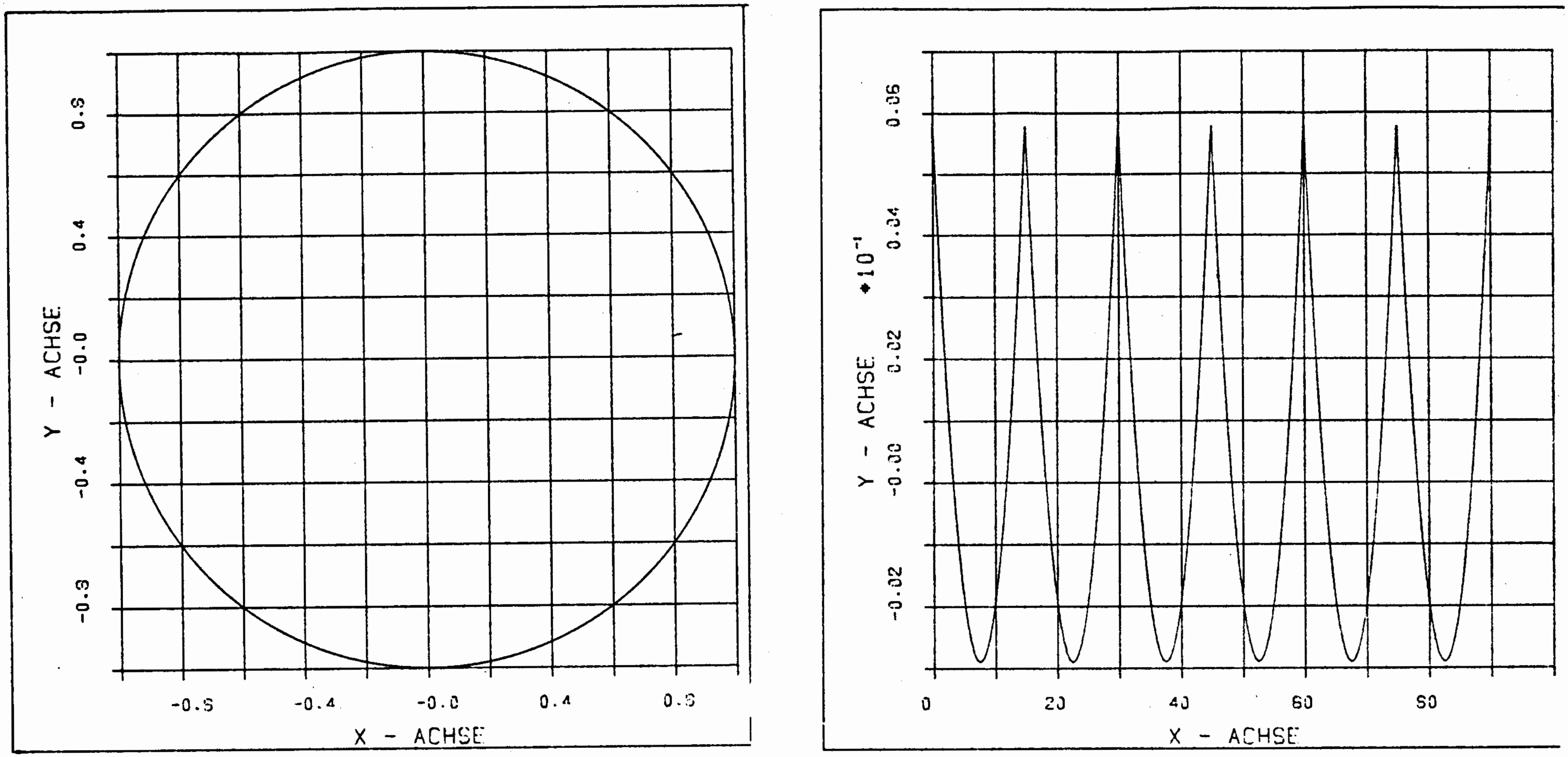
(b)

Fig. 2 Spline-approximation to a circle of radius 1.a) Approximating curve with 3 equally distributed SP in the first quadrant. b) Error distribution of the curvature approximation.



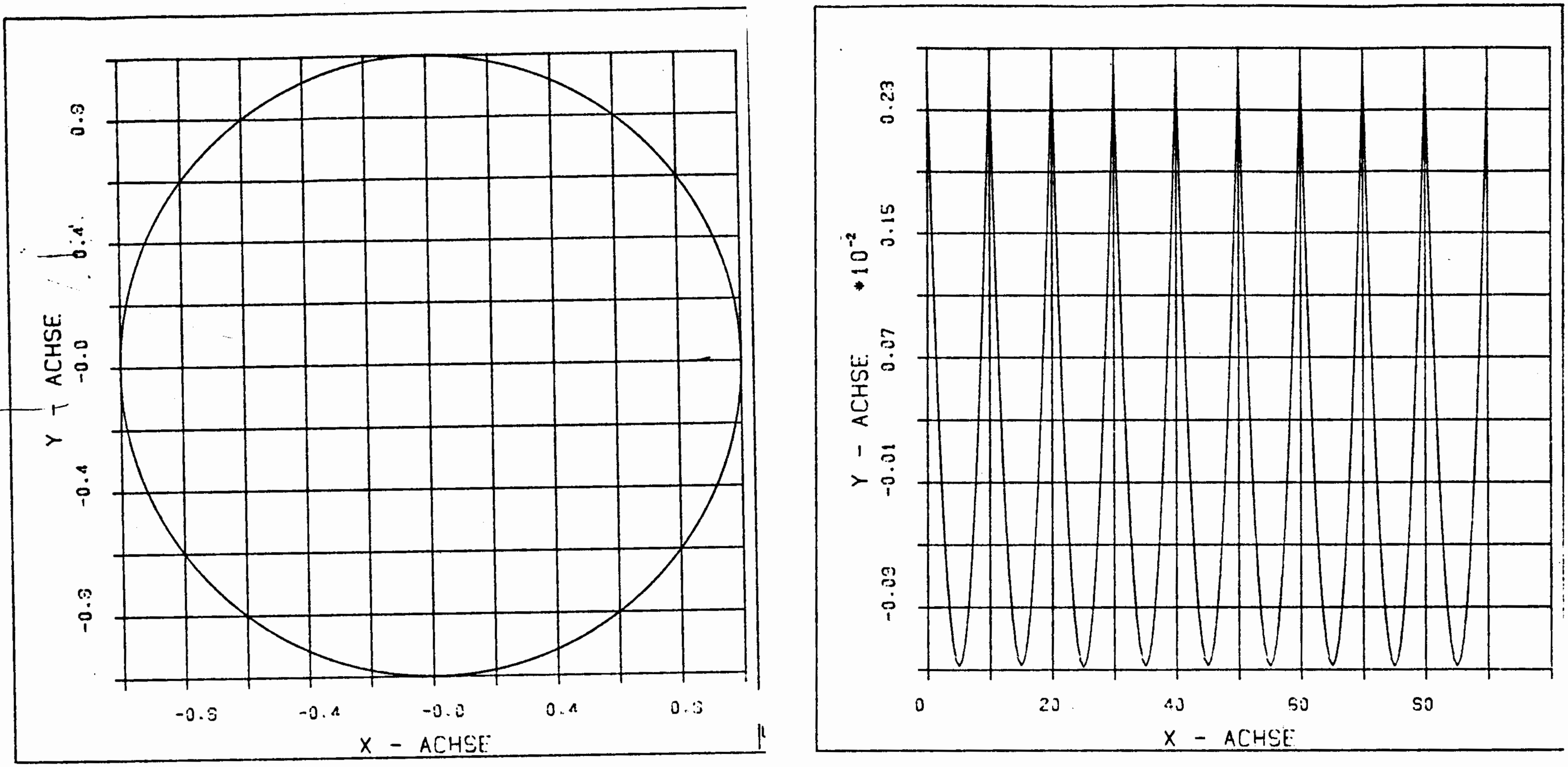
(a) (b)

Fig. 3 Spline-approximation to a circle of radius 1. a) Approximating curve with 5 equally distributed SP in the first quadrant. b) Error distribution of the curvature approximation.



(a) (b)

Fig. 4 Spline-approximation to circle of radius 1. a) Approximating curve with 7 equally distributed SP in the first quadrant. b) Error distribution of the curvature approximation.



(a) (b)

Fig. 5 Spline-approximation to a circle of radius 1 . a) Approximating curve with 10 equally distributed SP in the first quadrant. b) Error distribution of the curvature approximation.

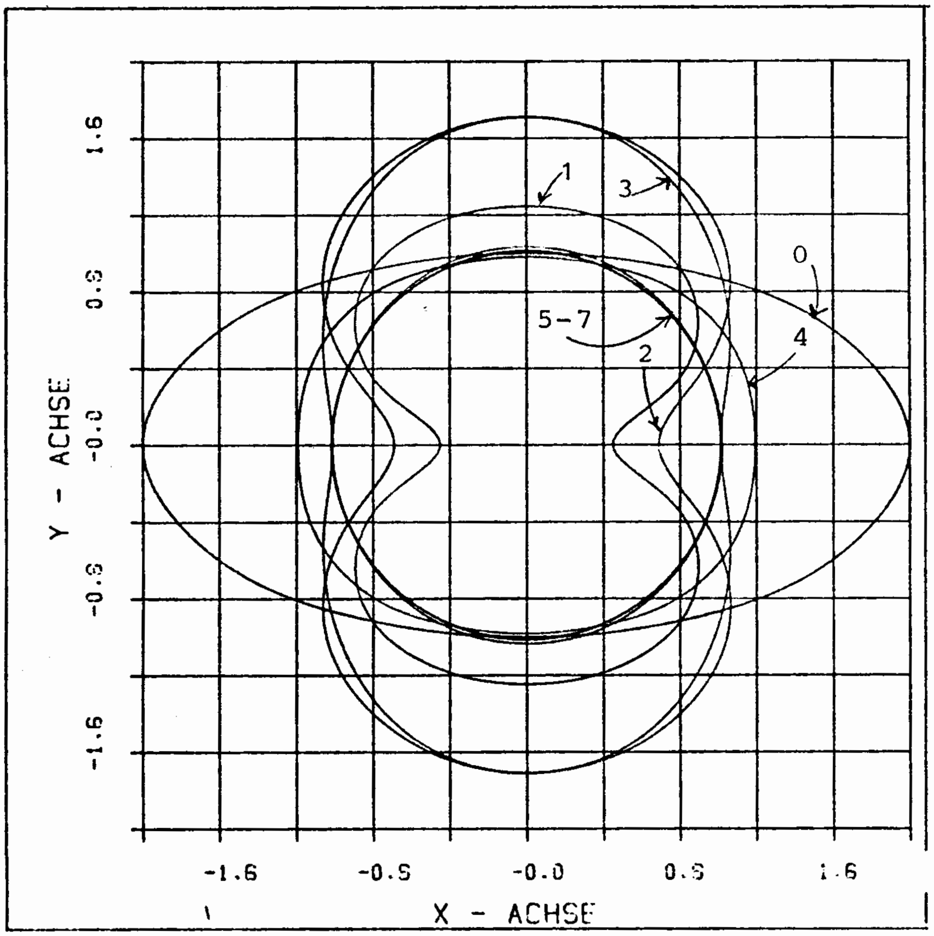


Fig. 6 Successive Newton-Raphson iterations until convergence was reached in synthesizing a circle with radius 1 starting with an ellipse with semiaxes a=2, b=1.

TABLE 1 Curvature error depending upon the number of supporting points in the first quadrant

Number of SP	Error %
2	33
3	3.6
5	1.4
7	0.58
10	0.25