Fundamentos Cinemáticos para el Diseño de las Máquinas

y Mecanismos 1981

Fecha	Tema		Hora	Profesor
Junio 22	1.	Fundamentos de Cinemática Especial de Cuerpos Rígidos	9 a 13:30 h	Dr. Jorge Angeles Alvarez
	1.1	Rotación de cuerpo rígido. Teorema de Euler		Dr. Jorge Angeles Alvarez
	1.2	Movimiento general de cuerpo rígido (movimiento de tornillo). Teorema de Charles.		
		C o m i d a	13:30 a 15 h	
	1.3	Cálculo digital de los parámetros del movimiento de tornillo.	15 a 17 h	Dr. Jorge Angeles Alvarez
Junio 23	1.4	Velocidad de los puntos de un cuerpo rígido que gira alrededor de un punto	9 a 13:30 h	Dr. Jorge Angeles Alvarez
	1.5	Velocidad angular de un cuerpo rígido que se desplaza con movimiento general (movimiento de tornillo instantáneo).		Dr. Jorge Angeles Alvarez
		C o m i d a	13:30 a 15 h	
	1.6	Relación entre los ejes de los tornillos instantáneos relativos de tres cuerpos en movimiento. Teorema de Aronhold Kennedy. Aplicaciones al diseño de las superficies de paso de engranes hipoidales. Otras aplicaciones industriales tendientes a minimizar pérdidas mecánicas por fricción.	15 a 17 h	Dr. Jorge Angeles Alvarez
Junio 24		Aceleración de los puntos de un cuerpo rígido en movimiento espacial. Teorema de Coriolis.	9 a 10 a.m.	Dr. Jorge Angeles Alvarez
	2.	Teoría Cinemática de Posiciones Múltiples de un Cuerpo Rígido y Sus Aplicaciones a la Síntesis de Mecanismos.	10 a 13:30 h	Dr. Bernard Roth
	2.1	Resumen de la teoría plana de posiciones múltiples.		

Fecha	Tema	Horario	Profesor
Junio [illegible]	2.2 Resumen de técnicas de síntesis "exacta" en el plano 2.3 Resumen de técnicas de síntesis "aproximada" en el plano 2.4 Introducción a la teoría de "triángulos de tornillos"	10 a 13:30 h	Dr. Bernard Roth
	C o m i d a	13:30 a 15 h	
	2.5 Empleo de triángulos de tornillos en la síntesis de mecanismos.	15 a 17 h.	Dr. Bernard Roth
Junio 25	2.6 Restricciones debidas a manivelas; ecuaciones y teoría 2.7 Ecuaciones de mallapara mecanismos especiales 2.8 Angulos de transmisión y otras consideraciones prácticas. 2.9 Posiciones con separación infinitesimal	9 a 13:30 h	Dr. Bernard Roth
	C o m i d a	13:30 a 15 h	
	2.10 Introducción a la teoría de curvatura en el espacio	15 a 16 h	Dr. Bernard Roth
	3. Clasificación de los Mecanismos	16 a 17 h	Dr. Jacques M. Herve
	3.1 Introducción		
Junio 26	3.2 La estructura de grupo del conjunto de desplazamientos 3.3 La clasificación sistemática de mecanismos.	9 a 13:30 h	Dr. Jacques M. Herve
	C o m i d a	13:30 a 15 h	
	3.4 Conclusiones	15 a 17	

EVALUACION DEL PERSONAL DOCENTE

CURSO: FUNDMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

FECHA: Del 22 al 26 de junio de 1981.

CONFERENCISTA	DOMINIO DEL TEMA	EFICIENCIA EN EL USO DE AYUDAS AUDIOVISUALES	MANTENIMIENTO DEL INTERES. (COMUNICACION CON LOS ASISTENTES, AMENIDAD, FACILIDAD DE EXPRESION).	PUNTUALIDAD	
1. Dr. Jorge Angeles Alvarez					
2. Bernard Roth					
3. Dr. Jacques M. Herve					
4.					
5.					
6.					
7.					
8.					
9					
ESCALA DE EVALUACION: 1 a 10					

EVALUACION DE LA ENSEÑANZA

SU EVALUACION SINCERA NOS AYUDARA A MEJORAR LOS PROGRAMAS POSTERIORES QUE DISEÑAREMOS PARA USTED.

TEMA	ORGANIZACION Y DESARROLLO DEL TEMA	GRADO DE PROFUNDIDAD LOGRADO EN EL TEMA	GRADO DE ACTUALIZACION LOGRADO EN EL TEMA	UTILIDAD PRACTICA DEL TEMA	
Fundamentos de Cinemática Especial de Cuerpos Rígidos.					
Teoría Cinemática de Posiciones Múltiples de un Cuerpo Rígido y sus Aplicaciones a la Síntesis de Mecanismos.					
lasificación de los Mecanismos.					
ESCALA DE EVALUACION: 1 a 10					

EVALUACION DEL CURSO

CONCEPTO		EVALUACION
1	APLICACION INMEDIATA DE LOS CONCEPTOS EXPUESTOS	
2.	CLARIDAD CON QUE SE EXPUSIERON LOS TEMAS	
3.	GRADO DE ACTUALIZACION LOGRADO CON EL CURSO	
4.	CUMPLIMIENTO DE LOS OBJETIVOS DEL CURSO	
5.	CONTINUIDAD EN LOS TEMAS DEL CURSO	
6.	CALIDAD DE LAS NOTAS DEL CURSO	
7.	GRADO DE MOTIVACION LOGRADO CON EL CURSO	

ESCALA DE EVALUACION DE 1 A 10

1. ¿Qué le pareció el ambiente en la División de Educación Continua?

MUY AGRADABLE	AGRADABLE	DESAGRADABLE

2. Medio de comunicación por el que se enteró del curso:

PERIODICO EXCELSIOR ANUNCIO TITULADO DIVISION DE EDUCACIOÑ CONTINUA	PERIODICO NOVEDADES ANUNCIO TITULADO DIVISION DE EDUCACIOÑ CONTINUA	FOLLETO DEL CURSO

CARTEL MENSUAL	RADIO UNIVERSIDAD	COMUNICACION CARTA, TELEFONO, VERBAL, ETC.

REVISTAS TECNICAS	FOLLETO ANUAL	CARTELERA UNAM "LOS UNIVERSITARIOS HOY"	GACETA UNAM

3. Medio de transporte utilizado para venir al Palacio de Minería:

AUTOMOVIL PARTICULAR	METRO	OTRO MEDIO

4. ¿Qué cambios haría usted en el programa para tratar de perfeccionar el curso?

____

____

____

5. ¿Recomendaría el curso a otras personas?

SI	NO

6. ¿Qué cursos le gustaría que ofreciera la División de Educación Continua?

____

____

7. La coordinación académica fue:

EXCELENTE	BUENA	REGULAR	MALA

8. Si está interesado en tomar algún curso intensivo ¿Cuál es el horario más conveniente para usted?

LUNES A VIERNES DE 9 A 13 H. Y DE 14 A 18 H. (CON COMIDAS)	LUNES A VIERNES DE 17 A 21 H.	LUNES, MIERCOLES Y VIERNES DE 18 A 21 H.	MARTES Y JUEVES DE 18 A 21 H.

VIERNES DE 17 A 21 H. SABADOS DE 9 A 14 H.	VIERNES DE 17 A 21 H. SABADOS DE 9 A 13 Y DE 14 a 18 H.	OTRO

9. ¿Qué servicios adicionales desearía que tuviese la División de Educación Continua, para los asistentes? ______________________

____

10. Otras sugerencias:

____

____

____

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

2. MATHEMATICAL PRELIMINARIES

Dr. Jorge Angeles Alvarez

Junio, 1981

Palacio de Minería Calle de Tacuba 5 primer piso México 1, D. F. Tel: 521-40-20 Apdo. Postal M-2285

b) To each pair $(\alpha, \underline{x})$, where $\alpha \in F$ (usually called "a scalar") and $\underline{x} \in V$, there corresponds one vector $\alpha\underline{x} \in V$, called "the product of the scalar α times $\underline{x}$", such that:

i) This product is <u>associative</u>, i.e. for any $\beta \in F$,

$$\alpha(\beta\underline{x}) = (\alpha\beta)\underline{x}$$

ii) For the identity 1 of F (with respect to multiplication) the following holds:

$$1\underline{x} = \underline{x}$$

c) The product of a scalar times a vector is <u>distributive</u>, i.e.

i) $\alpha(\underline{x} + \underline{y}) = \alpha\underline{x} + \alpha\underline{y}$

ii) $(\alpha + \beta)\underline{x} = \alpha\underline{x} + \beta\underline{x}$

<u>Example 1.1.1.</u> The set of triads of real numbers (x,y,z) constitute a vector space. To prove this, define two such triads, namely (x_1,y_1,z_1) and (x_2,y_2,z_2) and show that their addition is also one such triad and it is commutative as well. To prove associativity, define one third triad, (x_3,y_3,z_3), and so on.

<u>Example 1.1.2</u> The set of all polynomials of a real variable, t, of degree less than or equal to n, for $0 \leq t \leq 1$, constitute a vector space over the field of real numbers.

<u>Example 1.1.3</u> The set of tetrads of the form $(x,y,z,1)$ <u>do not</u> constitute a vector space (Why?)

Given the set of vectors $\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\} \subset V$ and the set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset F$ not necessarily distinct, a <u>linear combination</u> of the n vectors is the vector defined as

$$\underline{c} = \alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2 + \ldots + \alpha_n \underline{x}_n$$

1. MATHEMATICAL PRELIMINARIES

1.0 <u>**INTRODUCTION**</u>. Some relevant mathematical results are collected in this chapter. These results find a wide application within the realm of analysis, synthesis and optimization of mechanisms. Often, rigorous proofs are not provided; however a reference list is given at the end of the chapter, where the interested reader can find the required details.

1.1. <u>**VECTOR SPACE, LINEAR DEPENDENCE AND BASIS OF A VECTOR SPACE.**</u>

<u>A vector space</u>, also called a linear space, over a field F (1.1)* , is a set V of objects, called vectors, having the following properties:

a) To each pair $\{\underline{x}, \underline{y}\}$ of vectors from the set, there corresponds one (and only one) vector, denoted $\underline{x} + \underline{y}$, also from V, called "<u>the addition</u> of $\underline{x}$ and $\underline{y}$" such that

i) This addition is <u>commutative</u>, i.e.

$$\underline{x} + \underline{y} = \underline{y} + \underline{x}$$

ii) It is <u>associative</u>, i.e., for any element $\underline{z}$ of V,

$$\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$$

iii) There exists in V a unique vector $\underline{0}$, called "the zero of V", such that, for any $\underline{x} \in V$,

$$\underline{x} + \underline{0} = \underline{x}$$

iv) To each vector $\underline{x} \in V$, there corresponds a unique vector $-\underline{x}$, also in V, such that

$$\underline{x} + (-\underline{x}) = \underline{0}$$

* Numbers in brackets designate references at the end of each chapter.

The said set of vectors is linearly independent (ℓ. i.) if c equals zero implies that all α's are zero as well. Otherwise, the set is said to be linearly dependent (ℓ. d.)

<u>Example 1.1.4</u> The set containing only one nonzero vector, $\{x\}$, is ℓ.i.

<u>Example 1.1.5</u> The set containing only two vectors, one of which is the origin, $\{x, 0\}$, is ℓ.d.

The set of vectors $\{x_1, x_2, \ldots, x_n\} \subset V$ <u>spans</u> V if and only if every vector $v \in V$ can be expressed as a linear combination of the vectors of the set.

A set of vectors $B = \{x_1, x_2, \ldots, x_n\} \subset V$ is a basis for V if and only if:

i) B is linearly independent, and

ii) B spans V

All bases of a given space V contain the same number of vectors. Thus, if B is a basis for V, the number n of elements of B is the dimension of V (abreviated: $n=\dim V$)

<u>Example 1.1.6</u> In 3-dimensional Euclidean space the unit vectors $\{i, j\}$ lying parallel to the X and Y coordinate axes <u>span</u> the vectors in the X-Y plane, but <u>do not span</u> the vectors in the physical three-dimensional space.

<u>Exercise 1.1.1</u> Prove that the set B given above is a basis for V if and only if each vector in V can be expressed as a unique linear combination of the elements of B.

1.2 LINEAR TRANSFORMATION AND ITS MATRIX REPRESENTATION

Henceforth, only finite-dimensional vector spaces will be dealt with and, when necessary, the dimension of the space will be indicated as an exponent of the space, i.e., V^n means $\dim V=n$.

A <u>transformation</u> T, from an m-dimensional vector space U, to an n-dimensional vector space V is a rule which establishes a correspondence between an element of U and a <u>unique</u> element of V. It is represented as:

$$T: U^m \to V^n \qquad (1.2.1)$$

If $u \in U^m$ and $v \in V^n$ are such that $T: u \to v$, the said correspondence may also be denoted as

$$v = T(u) \qquad (1.2.3a)$$

T is linear if and only if, for any u, u_1 and $u_2 \in U$, and $\alpha \in F$,

i) $T(u_1 + u_2) = T(u_1) + T(u_2)$ and (1.2.3b)

ii) $T(\alpha u) = \alpha T(u)$ (1.2.3c)

Space U^m over which T is defined is called the "<u>domain</u>" of T, whereas the <u>subspace</u> of V^n containing vectors v for which eq. (1.2.3a) holds is called the "<u>range</u>" of T. A subspace of a given vector space V is a subset of V and is in turn a vector space, whose dimension is less than or equal to that of V

<u>Exercise 1.2.1</u> Show that the range of a given linear transformation of a vector space U to a vector space V contitutes a subspace, i.e. it satisfies properties a) and b) of Section 1.1.

For a given $u \in U$, vector v, as defined by (1.2.2) is called the "<u>image of u under T</u>", or, simply, the "<u>image of u</u>" if T is selfunderstood.

An example of a linear transformation is an orthogonal projection onto a plane. Notice that this projection is a transformation of the three-dimensional Euclidean space onto a two-dimensional space (the plane). The domain of T in this case is the physical 3-dimensional space, while its range is the projection plane.

If T, as defined in (1.2.1), is such that all of V contains v's such that (1.2.2) is satisfied (for some u's), T is said to be "onto". If T is such that, for all distinct $\underline{u}_1$ and $\underline{u}_2$, $\underline{T}(\underline{u}_1)$ and $\underline{T}(\underline{u}_2)$ are also distinct, $\underline{T}$ is said to be one-to-one. If $\underline{T}$ is onto and one-to-one, it is said to be invertible.

If $\underline{T}$ is invertible, to each $\underline{v} \in V$ there corresponds a unique $\underline{u} \in U$ such that $\underline{v} = \underline{T}(\underline{u})$, so one can define a mapping $\underline{T}^{-1}: V \to U$ such that

$$\underline{u} = \underline{T}^{-1}(\underline{v}) \qquad (1.2.4)$$

$\underline{T}^{-1}$ is called the "inverse" of $\underline{T}$.

Exercise 1.2.2 Let $\underline{P}$ be the projection of the three-dimensional Euclidean space onto a plane, say, the X-Y plane. Thus, $\underline{v} = \underline{P}(\underline{u})$ is such that the vector with components (x, y, z), is mapped into the vector with components (x, y, 0).

i) Is $\underline{P}$ a linear transformation?

ii) Is $\underline{P}$ onto?, one-to-one?, invertible?

A very important fact concerning linear transformations of finite dimensional vector spaces is contained in the following result:

Let $\underline{L}$ be a linear transformation from U^m to V^n. Let B_u and B_v be bases for U^m and V^n, respectively. Then clearly, for each $\underline{u}_i \in B_u$ its image $\underline{L}(\underline{u}_i) \in V$ can be expressed as a linear combination of the $\underline{v}_k$'s in B_v. Thus

$$\underline{L}(\underline{u}_i) = \alpha_{1i}\underline{v}_1 + \alpha_{2i}\underline{v}_2 + \dots + \alpha_{ni}\underline{v}_n \qquad (1.2.5)$$

Consequently, to represent the images of the m vectors of B_u, mn scalars like those appearing in (1.2.5) are required. These scalars can be arranged in the following manner:

$$[\underline{A}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \dots \alpha_{1m} \\ \alpha_{21} & \alpha_{22} \dots \alpha_{2m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \alpha_{n2} \dots \alpha_{nm} \end{bmatrix} \qquad (1.2.6)$$

where the brackets enclosing $\underline{A}$ are meant to denote a matrix, i.e. an array of numbers, rather than an abstract linear transformation.

$[\underline{A}]$ is called "The matrix of $\underline{L}$ referred to B_u and B_v". This result is summarized in the following:

DEFINITION 1.2.1 *The i'th column of the matrix representation of $\underline{L}$, referred to B_u and B_v, contains the scalar coefficients α_{ji} of the representation (in terms of B_v) of the image of the i th vector of B_u.*

Example 1.2.1 What is the representation of the reflexion $\underline{R}$ of the 3-dimensional Euclidean space E^3 into itself, with respect to one plane, say the X-Y plane, referred to unit vectors parallel to the X,Y,Z axes?

Solution: Let $\underline{i}$, $\underline{j}$, $\underline{k}$, be unit vectors parallel to the X, Y and Z axes, respectively. Clearly,

$$\underline{R}(\underline{i}) = \underline{i}$$
$$\underline{R}(\underline{j}) = \underline{j}$$
$$\underline{R}(\underline{k}) = -\underline{k}$$

Thus, the components of the images of $\underline{i}$, $\underline{j}$ and $\underline{k}$ under $\underline{R}$ are:

$$\{\underline{R}(\underline{i})\} = \begin{Bmatrix}1\\0\\0\end{Bmatrix}, \quad \{\underline{R}(\underline{j})\} = \begin{Bmatrix}0\\1\\0\end{Bmatrix}, \quad \{\underline{R}(\underline{k})\} = \begin{Bmatrix}0\\0\\-1\end{Bmatrix}$$

Hence, the matrix representation of $\underline{R}$, denoted by $[\underline{R}]$, is

$$[\underline{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (1.2.7)$$

Notice that, in this case, U = V and so, it is not necessary to use two different bases for U and V. Thus, $[\underline{R}]$, as given by (1.2.7), is the matrix representation of the reflection $\underline{R}$ under consideration, referred to the basis $\{\underline{i}, \underline{j}, \underline{k}\}$.

1.3 RANGE AND NULL SPACE OF A LINEAR TRANSFORMATION

As stated in Section 1.2, the set of vectors $v \in V$ for which there is at least one $u \in U$ such that $v = L(u)$ is called "the range of L" and is represented as $R(L)$, i.e. $R(L) = \{v=L(u): u \in U\}$.

The set of vectors $u_0 \in U$ for which $L(u_0) = 0 \in V$ is called "the null space of L" and is represented as $N(L)$, i.e. $N(L) = \{u_0 : L(u_0)=0\}$.

It is a simple matter to show that $R(L)$ and $N(L)$ are subspaces of V and U, respectively*.

The dimensions of dom(L), R(L) and N(L) are not independent, but they are related (see [1.2]):

$$\dim \operatorname{dom}(L) = \dim R(L) + \dim N(L) \qquad (1.3.1)$$

Example 1.3.1 In considering the projection of Exercise 1.2.1, U is E^3 and thus R(P) is the X-Y plane, N(P) is the Z axis, hence of dimension 1. The X-Y plane is two-dimensional and dom(L) is three-dimensional, hence (1.3.1) holds.

Exercise 1.3.1 Describe the range and the null space of the reflection of Example 1.2.1 and verify that eq. (1.3.1) holds true.

1.4 EIGENVALUES AND EIGENVECTORS OF A LINEAR TRANSFORMATION

Let L be a linear transformation of V into itself (such an L is called an "endomorphism"). In general, the image $L(v)$ of an element v of V is linearly independent with v, but if it happens that a nonzero vector v and its image under L are linearly dependent, i.e. if

$$L(v) = \lambda v \qquad (1.4.1)$$

* The proof of this statement can be found in any of the books listed in the reference at the end of this chapter.

such a v is said to be an eigenvector of L, corresponding to the eigenvalue λ. If $[A]$ is the matrix representation of L, referred to a particular basis then, dropping the brackets, eq. (1.4.1) can be rewritten as

$$Av = \lambda v \qquad (1.4.2)$$

or else

$$(A - \lambda I)v = 0 \qquad (1.4.3)$$

where I is the identity matrix, i.e. the matrix with the unity on its diagonal and zeros elsewhere. Equation (1.4.3) states that the eigenvectors of L (or of A, clearly) lie in the null space of $A - \lambda I$. One trivial vector v satisfying (1.4.3) is, of course, 0, but since in this context 0 has been discarded, nontrivial solutions have to be sought. The condition for (1.4.3) to have nontrivial solutions is, of course, that the determinant of $A - \lambda I$ vanishes, i.e.

$$\det (A - \lambda I) = 0 \qquad (1.4.4)$$

which is an nth order polynomial in λ, n being the order of the square matrix A [1.3]. The polynomial

$$P(\lambda) \equiv \det (A - \lambda I)$$

is called "the characteristic polynomial" of A. Notice that its roots are the eigenvalues of A. These roots can, of course, be real or complex; in case $P(\lambda)$ has one complex root, say λ_j, then $\overline{\lambda}_j$ is also a root of $P(\lambda)$, $\overline{\lambda}_j$ being the complex conjugate of λ_j. Of course, one or several roots could be repeated. The number of times that a particular eigenvalue λ_i is repeated is called the algebraic multiplicity of λ_i.

In general, corresponding to each λ_i there are several linearly independent eigenvectors of A. It is not difficult to prove (Try it!) that the ℓ_i eigenvectors associated with a particular eigenvalue span a subspace. This subspace is called the "spectral space" of λ_i, and its dimension is called "the geometric multiplicity of λ_i".

Exercise 1.4.1 Show that the geometric multiplicity of a particular eigenvalue cannot be greater than its algebraic multiplicity.

A Hermitian matrix is one which equals its transpose conjugate. If a matrix equals the negative of its transpose conjugate, it is said to be skew Hermitian. For Hermitian matrices we have the very important result:

THEOREM 1.4.1 *The eigenvalues of a Hermitian matrix are real and its eigenvectors are mutually orthogonal (i.e. the inner product, which is discussed in detail in Sec. 1.8, of two distinct eigenvectors, is zero).*

The proof of the foregoing theorem is very widely known and is not presented here. The reader can find a proof in any of the books listed at the end of the chapter.

1.5 CHANGE OF BASIS

Given a vector $\underline{v}$, its representation $(v_1, v_2, \ldots, v_n)^T$ referred to a basis $B = \{\underline{\beta}_1, \underline{\beta}_2, \ldots, \underline{\beta}_n\}$, is defined as the ordered set of scalars that produce $\underline{v}$ as a linear combination of the vectors of B. Thus, $\underline{v}$ can be expressed as

$$\underline{v} = v_1\underline{\beta}_1 + v_2\underline{\beta}_2 + \ldots + v_n\underline{\beta}_n \qquad (1.5.1)$$

A vector $\underline{v}$ and its representation, though isomorphic* to each other, are essentially different entities. In fact, $\underline{v}$ is an abstract algebraic entity satisfying properties a) and b) of Section 1.1, whereas its representation is an array of numbers. Similarly, a linear transformation, $\underline{L}$, and its representation, $(\underline{L})_B$, are essentially different entities. A question that could arise naturally is: Given the representations $(\underline{v})_B$ and $(\underline{L})_B$ of $\underline{v}$ and $\underline{L}$, respectively, referred to the basis B, what are the corresponding

* Two sets are isomorphic to each other if similar operations can be defined on their elements.

representations referred to the basis $C = \{\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_n\}$?

Let $(\underline{A})_B$ be the matrix relating both B and C, referred to B, i.e.

$$(\underline{A})_B = \begin{pmatrix} \alpha_{11} & \alpha_{12} \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{2n} \\ \vdots & \vdots & \\ \alpha_{n1} & \alpha_{n2} & \alpha_{nn} \end{pmatrix} \qquad (1.5.2)$$

and

$$\underline{\gamma}_1 = \alpha_{11}\underline{\beta}_1 + \alpha_{21}\underline{\beta}_2 + \ldots + \alpha_{n1}\underline{\beta}_n$$

$$\underline{\gamma}_2 = \alpha_{12}\underline{\beta}_1 + \alpha_{22}\underline{\beta}_2 + \ldots + \alpha_{n2}\underline{\beta}_n$$

$$\vdots$$

$$\underline{\gamma}_n = \alpha_{1n}\underline{\beta}_1 + \alpha_{2n}\underline{\beta}_2 + \ldots + \alpha_{nn}\underline{\beta}_n$$

Thus, calling v'_i the ith component of $(\underline{v})_C$, then

$$\underline{v} = v'_1\underline{\gamma}_1 + v'_2\underline{\gamma}_2 + \ldots + v'_n\underline{\gamma}_n \qquad (1.5.4)$$

and, from (1.5.3), (1.5.4) leads to

$$\underline{v} = \sum_j v'_j \sum_i \alpha_{ij}\underline{\beta}_i \qquad (1.5.5)$$

or, using index notation* for compactness,

$$\underline{v} = \alpha_{ij} v'_j \underline{\beta}_i \qquad (1.5.6)$$

Comparing (1.5.1) with (1.5.6),

$$v_i = \alpha_{ij} v'_j \qquad (1.5.7)$$

i.e.

$$(\underline{v})_B = (\underline{A})_B (\underline{v})_C$$

* According to this notation, a repeated index implies that a summation over all the possible values of this index is performed.

or, equivalently,

$$(\mathbf{v})_C = (\mathbf{A})_B^{-1} (\mathbf{v})_B \qquad (1.5.8)$$

Now, assuming that $\mathbf{w}$ is the image of $\mathbf{v}$ under $\mathbf{L}$,

$$(\mathbf{w})_B = (\mathbf{L})_B\ (\mathbf{v})_B \qquad (1.5.9)$$

or, referring eq. (1.5.9) to the basis C, instead,

$$(\mathbf{w})_C = (\mathbf{L})_C\ (\mathbf{v})_C \qquad (1.5.10)$$

Applying the relationship (1.5.8) to vector $\mathbf{w}$ and introducing it into eq. (1.5.10),

$$(\mathbf{A}^{-1})_B\ (\mathbf{w})_B = (\mathbf{L})_C\ (\mathbf{A})_B^{-1} (\mathbf{v})_B$$

from which the next relationship readily follows

$$(\mathbf{w})_B = (\mathbf{A})_B\ (\mathbf{L})_C\ (\mathbf{A})_B^{-1}\ (\mathbf{v})_B \qquad (1.5.11)$$

Finally, comparing (1.5.9) with (1.5.11),

$$(\mathbf{L})_B = (\mathbf{A})_B\ (\mathbf{L})_C\ (\mathbf{A})_B$$

or, equivalently,

$$(\mathbf{L})_C = (\mathbf{A})_B^{-1}\ (\mathbf{L})_B\ (\mathbf{A})_B \qquad (1.5.12)$$

Relationships (1.5.8) and (1.5.12) are the answers to the question posed at the beginning of this Section. The right hand side of (1.5.12) is <u>a similarity transformation</u> of $(\mathbf{L})_B$

<u>Exercise 1.5.1</u> Show that, under a similarity transformation, the characteristic polynomial of a matrix remains invariant.

<u>Exercise 1.5.2</u> The trace of a matrix is defined as the sum of the elements on its diagonal. Show that the trace of a matrix remains invariant under a similarity transformation Hint: Show first that, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are nxn matrices,

$$\mathrm{Tr}(\mathbf{ABC}) = \mathrm{Tr}(\mathbf{BCA}).$$

1.6 <u>DIAGONALIZATION OF MATRICES</u>

Let $\mathbf{A}$ be a symmetric nxn matrix and $\{\lambda_i\}$ its set of n eigenvalues, some of which could be repeated. Assume $\mathbf{A}$ has a set of n linearly independent* eigenvectors, $\{\mathbf{e}_i\}$, so that

$$\mathbf{A}\mathbf{e}_i = \lambda_i \mathbf{e}_i \qquad (1.6.1)$$

Arranging the eigenvectors of $\mathbf{A}$ in the matrix

$$\mathbf{Q} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \qquad (1.6.2)$$

and its eigenvalues in the diagonal matrix

$$\mathbf{\Lambda} = \mathrm{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_n) \qquad (1.6.3)$$

eq. (1.6.1) can be rewritten as

$$\mathbf{AQ} = \mathbf{Q\Lambda} \qquad (1.6.4)$$

Since the set $\{\mathbf{e}_i\}$ has been assumed to be l.i., $\mathbf{Q}$ is non-singular; hence from (1.6.4)

$$\mathbf{\Lambda} = \mathbf{Q}^{-1}\mathbf{AQ} \qquad (1.6.5)$$

which states that the diagonal matrix containing the eigenvalues of a matrix $\mathbf{A}$ (which has as many l.i. eigenvectors as its number of columns or rows) is a similarity transformation of $\mathbf{A}$; furthermore, the transformation matrix is the matrix containing the components of the eigenvectors of $\mathbf{A}$ as its columns. On the other hand, if $\mathbf{A}$ is Hermitian, its eigenvalues are real and its eigenvectors are mutually orthogonal. If this is the case and the set $\{\mathbf{e}_i\}$ is normalized, i.e., if $\|\mathbf{e}_i\| = 1$, for all i, then

$$\mathbf{e}_i^T\mathbf{e}_j = 0,\ i \neq j \qquad (1.6.6a)$$

$$\mathbf{e}_i^T\mathbf{e}_i = 1 \qquad (1.6.6b)$$

---

* Some square matrices have less than n l.i eigenvectors, but these are not considered here.

where e_i^T is the transpose of e_i (e_i being a column vector, e_i^T is a row vector). The whole set of equations (1.6.6), for all i and all j can then be written as

$$Q^T Q = I \qquad (1.6.7)$$

where I is the matrix with unity on its diagonal and zeros elsewhere. Eq. (1.6.7) states a very important fact about Q, namely, that it is an orthogonal matrix. Summarizing, a symmetric nxn matrix A can be diagonalized via a similarity transformation, the columns of whose matrix are the eigenvectors of A.

The eigenvalue problem stated in (1.6.1) is solved by first finding the eigenvalues $\{\lambda_i\}_1^n$. These values are found from the following procedure: Write eq. (1.6.1) in the form

$$(A - \lambda_i I)e_i = 0 \qquad (1.6.8)$$

This equation states that the set $\{e_i\}_1^n$ lies in the null space of $A - \lambda_i I$. For this matrix to have nonzero vectors in its null space, its determinant should vanish, i.e.

$$\det(A-\lambda_i I) \equiv P(\lambda_i) = 0 \qquad (1.6.9)$$

whose left hand side is its characteristic polynomial, which was introduced in section 1.4. This equation thus contains n roots, some of which could be repeated.

A very useful result is next summarized, though not proved.

THEOREM (Cayley-Hamilton). A square matrix satisfies its own characteristic equation, i.e. if $P(\lambda_i)$ is its characteristic polynomial, then

$$P(A) = 0 \qquad (1.6.10)$$

A proof of this teorem can be found either in [1.3, pp. 148-150] or in [1.4, pp. 112-115]

Exercise 1.6.1 A square matrix A is said to be strictly lower triangular (SLT) if $a_{ij}=0$, for $j \geq i$. On the other hand, this matrix is said to be nilpotent of index k if k is the lowest integer for which $A^k = 0$.

i) Show that an nxn SLT matrix is nilpotent of index $k \leq n$.

ii) Show that an nxn SLT matrix A statisfies the following indentity:

$$(I+A)^{-1} = \sum_1^n (-1)^{k-1} A^{k-1}$$

The inverse of $I+A$ appears very often in the solution of linear algebraic systems by iterative methods.

1.7. BILINEAR FORMS AND SIGN DEFINITION OF MATRICES.

Given that the space of matrices does not constitute an ordered set (as is the case for the real, rational or integer sets), it is not possible to attribute a sign to a matrix. However, it will be shown that, if a bilinear form (in particular, a quadratic form) is associated with a matrix, then it makes sense to speak of the sign of a matrix. Before proceeding further, some difinitions are needed. Let u and $v \in U$, U being a vector space defined over the complex fied F. A bilinear form of u and v, represented as $\phi(u,v)$ is a mapping from U into F, having the following properties:

i) It is linear in both u and v:

$$\phi(u_1+u_2,v) = \phi(u_1,v) + \phi(u_2,v) \qquad (1.7.1a)$$

$$\phi(u,v_1+v_2) = \phi(u,v_1) + \phi(u,v_2) \qquad (1.7.1b)$$

$$\phi(\alpha u,v) = \alpha\phi(u,v) \qquad (1.7.1c)$$

$$\phi(u,\beta v) = \bar{\beta}\phi(u,v) \qquad (1.7.1d)$$

where α and $\beta \in F$, their conjugates being $\bar{\alpha}$ and $\bar{\beta}$, respectively.

ii) $\phi(\underline{v},\underline{u})$ is the complex conjugate of $\phi(\underline{u},\underline{v})$, i.e.

$$\phi(\underline{v},\underline{u}) = \bar{\phi}(\underline{u},\underline{v}) \qquad (1.7.1e)$$

The foregoing properties of conjugate bilinear forms suggest that one possible way of constructing a bilinear form is as follows:

Let

$$\phi(\underline{u},\underline{v}) = \underline{u}^*\underline{A}\underline{v} \qquad (1.7.2)$$

Exercise 1.7.1 Prove that definition (1.7.2) satisfies properties (1.7.1)

If, in (1.7.1), $\underline{v} = \underline{u}$, the bilinear form becomes the quadratic form

$$\phi(\underline{u}) = \underline{u}^*\underline{A}\,\underline{u} \qquad (1.7.3)$$

It will be shown that the bilinear form (1.7.2) defines a scalar product for a vector space under certain conditions on $\underline{A}$.

Definition: A scalar product, $p(\underline{u},\underline{v})$, of two elements for a vector space U is a complex number with the following properties:

i) It is Hermitian symmetric:

$$p(\underline{u},\underline{v}) = \bar{p}(\underline{v},\underline{u}) \qquad (1.7.4a)$$

ii) It is conjugate linear in both $\underline{u}$ and $\underline{v}$:

$$p(\underline{u}_1+\underline{u}_2,\underline{v}) = p(\underline{u}_1,\underline{v}) + p(\underline{u}_2,\underline{v}) \qquad (1.7.4b)$$

$$p(\underline{u},\underline{v}_1+\underline{v}_2) = p(\underline{u},\underline{v}_1) + p(\underline{u},\underline{v}_2) \qquad (1.7.4c)$$

$$p(\alpha\underline{u},\underline{v}) = \alpha p(\underline{u},\underline{v}) \qquad (1.7.4d)^*$$

$$p(\underline{u},\beta\underline{v}) = \bar{\beta} p(\underline{u},\underline{v}) \qquad (1.7.4e)$$

iii) It is real and positive definite:

$$p(\underline{u},\underline{u})>0, \text{ for } \underline{u} \neq \underline{0} \qquad (1.7.4f)$$

$$p(\underline{u},\underline{u}) = 0, \text{ if and only if } \underline{u} = \underline{0} \qquad (1.7.4g)$$

* Note: conjugate linear in $\underline{v}$

From definition (1.7.2) and properties (1.7.1), it follows that all that is needed for a bilinear form to constitute a scalar product for a vector space is that it is positive definite (and hence, real). Whether a bilinear form is positive definite or not clearly depends entirely on its matrix and not on its vectors. The following definition will be needed:

A square n×n matrix is said to be positive definite if (and only if), the quadratic form for any vector $\underline{u} \neq \underline{0}$ associated to it is real and positive and only vanishes for the zero vector. A positive definite matrix $\underline{A}$ is symbolically designated as $\underline{A} > 0$. If the said quadratic form vanishes for some nonzero vectors, then $\underline{A}$ is said to be positive semidefinite, symbolically designated as $\underline{A} \geq 0$. Negative definite and negative semidefinite matrices are similarly defined. Now:

THEOREM 1.7.1 *Any square matrix is decomposable into the sum of a Hermitian and a skew Hermitian part (this is called the Cartesian decomposition of the matrix)*

Proof. Write the matrix $\underline{A}$ in the form

$$\underline{A} = \frac{1}{2}(\underline{A}+\underline{A}^*) + \frac{1}{2}(\underline{A}-\underline{A}^*) \qquad (1.7.5)$$

Clearly the first term of the right hand side is Hermitian and the second one is skew Hermitian.

THEOREM 1.7.2 *The quadratic form associated with a matrix $\underline{A}$ is real if and only if $\underline{A}$ is Hermitian. It is imaginary if and only if $\underline{A}$ is skew Hermitian.*

Proof.

("if" part) Let $\underline{A}$ be Hermitian, then

$$\phi(\underline{u}) = \underline{u}^*\underline{A}^*\underline{u} = \underline{u}^*\underline{A}\underline{u}$$

and

$$\bar{\phi}(\underline{u}) = \underline{u}^*\underline{A}\underline{u}$$

Since

$$\mathrm{Im}\{\phi(\underline{u})\}=\frac{1}{2}\left(\phi(\underline{u})-\bar{\phi}(\underline{u})\right)$$

then

$$\mathrm{Im}\{\phi(\underline{u})\}=0$$

On the other hand, if $\underline{A}$ is skew-Hermitian, then,

$$\phi(\underline{u})=\underline{u}^*\underline{A}^*\underline{u}=-\underline{u}^*\underline{A}\underline{u}$$

and

$$\bar{\phi}(\underline{u})=\underline{u}^*\underline{A}\underline{u}$$

Since

$$\mathrm{Re}\{\phi(\underline{u})\}=\frac{1}{2}\left(\phi(\underline{u})+\bar{\phi}(\underline{u})\right)$$

then

$$\mathrm{Re}\{\phi(\underline{u})\}=0$$

thus proving the "if" part of the theorem.

Exercise 1.7.2 Prove the "only if" part of Theorem 1.7.2

What Theorem 1.7.2 states is very important, namely that Hermitian matrices are good candidates for defining a scalar product for a vector space, since the associated quadratic form is real. What is now left to investigate is whether this form turns out to be positive definite as well. Though this is not true for any Hermitian matrix, it is (obviously!) so for positive definite Hermitian matrices (by definition!). Futhermore, since the quadratic form of a positive definite matrix must, in the first place, be real, and since, for the quadratic form associated with a matrix to be real, the matrix must be Hermitian (from Theorem 1.7.2), it is not necessary to refer to a positive definite (or semidefinite) matrix as being Hermitian.

Summarizing: In order for the quadratic form (1.7.2) to be a scalar product, $\underline{A}$ must be positive definite. Next, a very important result concerning an easy characterization of positive definite (semidefinite) matrices is given.

THEOREM 1.7.3 A matrix is positive definite (semidefinite) if and only if its eigenvalues are all real and greater than (or equal to) zero.

Proof. ("only if" part).

Indeed, if a matrix $\underline{A}$ is positive definite (semidefinite), it must be Hermitian. Thus, it can be diagonalized (a consequence of Theorem 1.4.1). Furthermore, once the matrix is in diagonal form, the elements on its diagonal are its eigenvalues, which are real and greater than (or equal to) zero. It takes on the form

$$\underline{A}=\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix} \qquad (1.7.10)$$

where

$$\lambda_i > (\geq)\ 0,\ i=1,2,\ldots,n$$

For any vector $\underline{u}\neq\underline{0}$, by definition,

$$\underline{u}^*\underline{A}^*\underline{u}=\underline{u}^*\underline{A}\underline{u}>(\geq)0 \qquad (1.7.11)$$

where the components of $\underline{u}$ (with respect to the basis formed with the complete set of eigenvectors of $\underline{A}$) are

$$\underline{u}=\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_n\end{bmatrix} \qquad (1.7.12)$$

Substitution of (1.7.10) and (1.7.12) into (1.7.11) yields

$$\sum_{1}^{n}\lambda_i|u_i|^2>(\geq)0 \qquad (1.7.13)$$

Now, assume $\underline{u}$ is such that all but its k^{th} component vanish; in this case, (1.7.13) reduces to

$$\lambda_k|u_k|^2>(\geq)0$$

from which

$$\lambda_k>(\geq)0$$

and, since λ_k can be any of the eigenvalues of $\underline{A}$, the proof of this part is done. The proof of the "if" part is obvious and is left as an exercise for the reader.

Exercise 1.7.2 Show that, if the eigenvalues of a square matrix are all real and greater than (or equal to) zero, the matrix is posite definite (semidefinite).

A very special case of a positive definite matrix is the identity matrix, $\underline{I}$, which yields the very well known scalar product

$$p(\underline{u},\underline{v})=\underline{u}^*\underline{I}^*\underline{v} = \underline{u}^*\underline{I}\underline{v}=\underline{u}^*\underline{v} \qquad (1.7.14)$$

In dealing with vector spaces over the real field, the arising inner product is real and hence, from Schwarz's inequality (1.4, p.125),

$$\frac{p(\underline{u},\underline{v})}{\sqrt{p(\underline{u},\underline{u})p(\underline{v},\underline{v})}}\leq 1$$

thus making it possible to define a "geometry" for them, the cosine of the angle between vectors $\underline{u}$ and $\underline{v}$ can be defined as

$$\cos(\underline{u},\underline{v})=\frac{p(\underline{u},\underline{v})}{\sqrt{p(\underline{u},\underline{u})p(\underline{v},\underline{v})}}$$

For vector spaces over the complex field, such an angle cannot be defined, for then the inner product is a complex number.

1.8 NORMS, ISOMETRIES, ORTHOGONAL AND UNITARY MATRICES.

Given a vector space V, a norm for $\underline{v} \in V$ is defined as a real-valued mapping from $\underline{v}$ into a real number, represented by $||\underline{v}||$, such that this norm

i) is positive definite, i.e.

$||\underline{v}||>0$, for any $\underline{v}\neq\underline{0}$

$||\underline{v}||=0$ if and only if $\underline{v}=\underline{0}$

ii) is linear homogeneous, i.e., for some $\alpha \in F$ (the field over which V is defined),

$||\alpha\underline{v}||=|\alpha|\,||\underline{v}||$

$|\alpha|$ being the modulus (or the absolute value, in case α is real) of α.

iii) satisfies the triangle inequality, i.e. for $\underline{u}$ and $\underline{v} \in V$,

$||\underline{u}+\underline{v}||\leq||\underline{u}||+||\underline{v}||$

Example 1.8.1 Let v_i be the ith component of a vector $\underline{v}$ of a space over the complex field. The following are well defined norms for $\underline{v}$:

$$||\underline{v}||=\max_{1\leq i\leq n}|v_i| \qquad (1.8.1)$$

$$||\underline{v}||=\left(\sum_{1}^{n}|v_i|^p\right)^{\frac{1}{p}} \qquad (1.8.2)$$

where p is a positive integer. For p = 2 in (1.8.2) the corresponding norm is the Euclidean norm, or the "magnitude" of $\underline{v}$.

Norm (1.8.1) is easy and fast to compute, and hence it is widely used in numerical computations. However, it is not suitable for physical or geometrical problems since it is not invariant*, i.e. it depends on the coordinate axes being used. The Euclidean norm has the advantage that it is invariant.

---

* Besides, there is no inner product associated with it and hence obviously no "geometry"

However, computing it requires n (the dimension of the space to which the vector under consideration belongs) multiplications (i.e. n square raisings), n-1 additions and one square root computation. In order to proceed further, some more definitions are needed.

An <u>invertible linear transformation</u> is called an "<u>isometry</u>" if it preserves the following scalar product

$$p(x,y) = p(Ax, Ay) = x^*A^*Ay \qquad (1.8.3)$$

It is a very simple matter to show that, in order for a transformation P to be an isometry, it is required that its transpose conjugate, P^*, equals its inverse, i.e.,

$$P^*=P^{-1} \qquad (1.8.4)$$

If P is defined over the complex field and meets condition (1.8.4), then it is said to be <u>unitary</u>. If P is defined over the real field, then $P^*=P^T$, the transpose of P and, if it satisfies (1.8.4), it is said to be <u>orthogonal</u>.

<u>Exercise 1.8.1</u> Show that in order for P to be an isometry, it is necessary that P satisfies (1.8.4), i.e., show that under the similarity transformation

$$\xi=Px,\ \eta = Py,\ B=PAP^{-1},$$

the following scalar product is preserved:

$$p(x,y) = p(\xi,\eta)$$

1.9 PROPERTIES OF UNITARY AND ORTHOGONAL MATRICES.

Some important facts about unitary and orthogonal matrices are discussed in this section. Notice that all results concerning unitary matrices apply to orthogonal matrices, for the latter are a special case of the former.

<u>THEOREM 1.9.1</u> *The set of eigenvalues of a unitary matrix lies on the unit circle $|z|^2 = 1$, centered at the origin of the complex plane.*

Proof: Let U be an nxn unitary matrix. Let λ be one of its eigenvalues and e a corresponding eigenvector, so that

$$Ue = \lambda e \qquad (1.9.1)$$

Taking the transpose conjugate of both sides of (1.9.1),

$$e^*U^* = \bar{\lambda}e^* \qquad (1.9.2)$$

Performing the corresponding products on both sides of eqs. (1.9.1) and (1.9.2),

$$e^*U^*Ue = \lambda\bar{\lambda}e^*e \qquad (1.9.3)$$

But, since U is unitary, (1.9.3) leads to

$$e^*e = |\lambda|^2 e^*e$$

from which

$$|\lambda|^2 = 1, \text{q.e.d.}$$

<u>Corollary 1.9.1</u> *If an nxn unitary matrix is of odd order (i.e. n is odd), then it has at least one real eigenvalue, which is either +1 or -1.*

<u>Exercise 1.9.1</u> Prove Crollary 1.9.1

1.10 <u>STATIONARY POINTS OF SCALAR FUNCTION OF A VECTOR ARGUMENT.</u>

Let $\phi = \phi(x)$ be a (scalar) real function of a vector argument, x, assumed to be continuous and differentiable up to second derivatives within a certain neighborhood around some x_0. The <u>stationary points</u> of this function are defined as those values x_0 of x where the gradient of ϕ, $\phi'(x)$ vanishes. Each stationary point can be an extremum or a <u>saddle point</u>. An extremum, in turn, can be either a local maximum or minimum. The function ϕ attains a local maximum at x_0 if and only if

$$f(x_0) \geq f(x)$$

for any x in the neighborhood of x_0, i.e., for any x such that

$$||x-x_0|| \leq \epsilon$$

ϵ being an arbitrarily small positive number. A local minimum is correspondingly defined. If an extremum is neither a local maximum nor a local minimum, it is said to be a saddle point. Criteria to decide whether an extremum is a maximum, a minimum or a saddle point are next derived.

An expansion of ϕ around x_0 in a Taylor series illustrates the kind of stationary point at hand. In fact, the Taylor expansion of ϕ is

$$\phi(x) = \phi(x_0) + \phi'(x_0)^T(x-x_0) + \frac{1}{2}(x-x_0)^T\phi''(x_0)(x-x_0) + R \quad (1.10.1)$$

where R is the residual, which contains terms of third and higher orders.

Then the increment of ϕ at x_0, for a given increment $\Delta x = x-x_0$, is given by

$$\Delta\phi = \phi'(x_0)^T\Delta x + \frac{1}{2}\Delta x^T\phi''(x_0)\Delta x \quad (1.10.2)$$

if terms of third and higher orders are neglected.

From eq. (1.10.2) it can be concluded that the linear part of $\Delta\phi$ vanishes at a stationary point, which makes clear why such points are called stationary. Whether x_0 constitutes an extremum or not, depends on the sign of $\Delta\phi$. It is a maximum if $\Delta\phi$ is nonpositive for arbitrary Δx. It is a minimum if the said increment is nonnegative for arbitrary Δx. If the sign of the increment depends on Δx, then x_0 is a saddle point for reasons which are brought up in the following. Eq. (1.10.2) shows that the sign of $\Delta\phi$ depends entirely on the quadratic term, at a stationary point. Whether this term is nonpositive or nonnegative, it is sufficient that the Hessian matrix $\phi''(x)$ be sign semidefinite at x_0. Notice, however, that this condition on the Hessian matrix is only sufficient, but not necessary, for it is based on Eq. (1.10.2), which is truncated after third-order terms. In fact, a function whose Hessian at a stationary point is sign-semidefinite can constitute either a maximum, a minimum, or a saddle point as shown next.

From the foregoing discussion, the following theorem is concluded.

THEOREM 1.10.1 *Extrema and saddle points of a differentiable function occur at stationary points. For a stationary point to constitute a local maximum (minimum) it is sufficient, although not necessary, that the corresponding Hessian matrix be negative (positive) semidefinite. For the said point to constitute a saddle point, it is sufficient that the corresponding Hessian matrix sign-indefinite at this stationary point.*

A hypersurface in an n-dimensional space resembles a hyperbolic paraboloid at a saddle point, the resemblance lying in the fact that, at its stationary point, the sign of the curvature of the surface is different for each direction. To illustrate this, consider the hyperbolic paraboloid of Fig 1.10.1 for which, when seen from the X-axis, its stationary point (the origin) appears as a minimum (positive curvature), whereas, if seen from the Y-axis, it appears as a maximum (negative curvature). In fact, it is none of these.

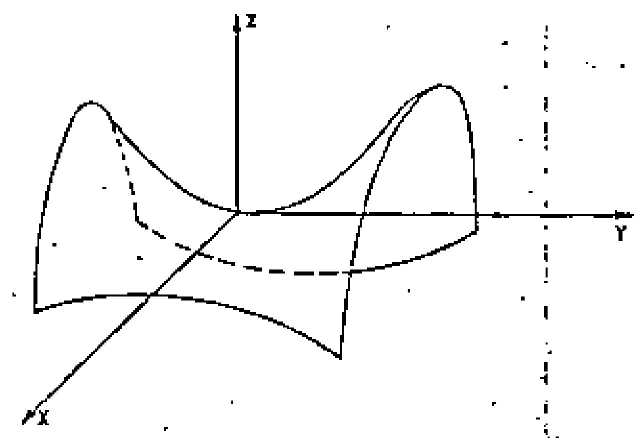


Fig. 1.10.1 Saddle point of a 3-dimensional surface

Corollary 1.10.1 *The quadratic form*

$$\phi(x) = x^TAx + b^Tx + c$$

has a unique extremum at $x_0 = -\frac{1}{2}A^{-1}b$, if A^{-1} exists. This is a maximum (minimum) if A is negative (positive) semidefinite

Exercise 1.10.1 Prove Corollary 1.10.7

Example 1.10.1 The function $\phi = x_1^4 + x_2^4 + \ldots + x_n^4$ has a local minimum at $x_1 = x_2 = \ldots = x_n = 0$. The Hessian matrix of this function, however, vanishes at this minimum.

Example 1.10.2 The function $= x_1^4 - x_2^4$ has a stationary point at the origin, which is a saddle point. Its Hessian matrix, however, vanishes at this point.

Example 1.10.3 The function $x_1^2 + x_2^4$ has a minimum at $(0,0)$. At this point its Hessian matrix is positive semidefinite.

1.11 LINEAR ALGEBRAIC SYSTEMS.

Let A be an mxn matrix and x and b be n-and m-dimensional vectors where, in general, $m \neq n$. Equation

$$Ax=b \tag{1.11.1}$$

is a linear algebraic system. It is linear because, if x_1 and x_2 are its solutions for $b=b_1$ and $b=b_2$, and α and β are scalars, then $\alpha x_1+\beta x_2$ is a solution for $b=\alpha b_1+\beta b_2$. It is algebraic as opposed to differential or dynamic because it does not involve derivatives. There are three different cases regarding the solution of eq. (1.11.1), depending on whether m is greater than, less than or equal to n. These are discussed next:

i) $m>n$. In this case the number of equations is greater than that of unknowns. The system is overdetermined and there is no guarantee of the existence of a certain x_0 such that $Ax_0=b$.

   A very simple example of such a system is the following:

$$x_1=5 \tag{1.11.1a}$$

$$x_1=3 \tag{1.11.1b}$$

   where m=2 and n=1. If $x_1=5$, the first equation is satisfied but the second one is not. If, on the other hand, $x_1=3$, the second equation is satisfied, but the first one is not. However, a system with m>n could have a solution, which could even be unique if, out of the m

equations involved, only n are linearly independent, the remaining m-n linearly dependent on the n l.i. equations. As an example, consider the following system

$$x_1+x_2=5 \tag{1.11.2a}$$

$$x_1-x_2=3 \tag{1.11.2b}$$

$$3x_1+x_2=13 \tag{1.11.2c}$$

whose (unique) solution is

$$x_1=4, x_2=1 \tag{1.11.3}$$

Here equation (1.11.2c) is linearly dependent on (1.11.2a) and (1.11.2b)

In general, however, for m>n it is not possible to satisfy all the equations of a system with more equations than unknowns; but it is possible to "satisfy" them with the minimum possible error. Assume that x_0 does not satisfy all the equations of an mxn system, with m>n, but satisfies the system with the least possible error. Let e be the said error, i.e.

$$e=Ax_0-b \tag{1.11.4}$$

The Euclidean norm of e is

$$||e||^2 = (Ax_0-b)^T (Ax_0-b) \tag{1.11.5}$$

Expanding $||e||^2$, it is noticed that it is a quadratic form of x_0, i.e.

$$\phi(x_0)=||e||^2 = x_0^T A^T A x_0 - 2b^T A x_0 + b^T b \tag{1.11.6}$$

The latter quadratic form has an extremum where $\phi'(x_0)$ vanishes. The corresponding value of x, x_0, is found by setting $\phi'(x_0)$ equal to zero, i.e.

$$\phi'(x_0)=2A^T A x_0 - 2A^T b = 0 \tag{1.11.7}$$

If A is of full rank, i.e., if rank $(A)=n$, then $A^T A$, an nxn matrix, is also of rank n (1.4), i.e. $A^T A$ is invertible and so, from eq. (1.11.5)

$$x_0 = (A^T A)^{-1} A^T b = A^I b \tag{1.11.8}$$

where A^I is a "pseudo-inverse" of A, called the "Moore-Penrose generalized inverse" of $\underline{A}$. A method to determine $\underline{x}_0$ that does not require the computation of $\underline{A}^{-}$ is given in [1.5] and [1.6]. In [1.7], an iterative method to compute $\underline{A}^{I}$ is proposed. The numerical solution of this problem is presented in section 1.12. This problem arises in such fields as control theory, curve-fitting (regressions) and mechanism synthesis.

ii) m<n. In this case the number of equations is less than that of unknowns. Hence, if the system is consistent*, it has an infinity of solutions. For instance, the system

$$x+y=3, \qquad (1.11.9)$$

in which m=1 and n=2, admits infinitely many solution, namely all points lying on the line

$$y=x+3 \qquad (1.11.10)$$

Now consider the system

$$x+y+z=1 \qquad (1.11.11a)$$

$$x+y-z=1 \qquad (1.11.11b)$$

with m=2 and n=3. This system admits an infinity of solutions all with z=0.

In case a system with m<n admits a solution, it in fact admits infinitely many, which is not difficult to prove. Indeed, partition matrix $\underline{A}$ and vector $\underline{x}$ in the form

$$\underline{A}=\underbrace{\left[\underline{A}_1 \,\vdots\, \underline{A}_2\right]}_{m \quad n-m} m, \quad \underline{x}=\begin{bmatrix}\underline{x}_1\\ ---\\ \underline{x}_2\end{bmatrix}\begin{matrix}\}\,m\\ \}\,n-m\end{matrix}$$

Thus, eq. (1.11.1) is equivalent to

$$\underline{A}_1\underline{x}_1 + \underline{A}_2\underline{x}_2=\underline{b} \qquad (1.11.13)$$

---

* i.e. if $\underline{b} \in R(\underline{A})$

In the latter equation, if rank$(\underline{A}_1)$ = m, $\underline{A}_1^{-1}$ exists and a solution to (1.11.13) is

$$\underline{x}_1=\underline{A}_1^{-1}\underline{b}, \qquad \underline{A}_2\underline{x}_2=\underline{0} \qquad (1.11.14)$$

where $\underline{x}_1$ is unique, as will be shown for the case m=n, and $\underline{x}_2$ is a vector lying in the null space of $\underline{A}_2$. Clearly, there are as many linearly independent solutions (1.11.12) as linearly independent vectors in the null space of $\underline{A}_2$.

From the foregoing discussion, if m<n, system (1.11.1) admits an infinity of solutions. However, among those infinitely many solutions, there is exactly one whose Euclidean norm is a minimum. That "optimal" solution is found next, via a quadratic programming problem, namely,

$$\text{Min}\,\phi(\underline{x})=\underline{x}^T\underline{x} \qquad (1.11.15)$$

subject to

$$\underline{A}\underline{x}=\underline{b} \qquad (1.11.16)$$

Applying the Lagrange multiplier technique [1.8], let $\underline{\lambda}$ be an m-dimensional vector whose components are called Lagrange multipliers. Define, then, the new quadratic form

$$\psi(\underline{x})=\underline{x}^T\underline{x}+\underline{\lambda}^T(\underline{A}\underline{x}-\underline{b}) \qquad (1.11.17)$$

which reduces to the original one (1.11.15), when (1.11.16) is satisfied. $\psi(\underline{x})$ has an extremum where its gradient $\psi'(\underline{x})$ vanishes. This condition is

$$\psi'(\underline{x})=2\underline{x}+\underline{A}^T\underline{\lambda}=\underline{0} \qquad (1.11.18)$$

from which

$$\underline{x}=-\frac{1}{2}\underline{A}^T\underline{\lambda} \qquad (1.11.19)$$

However, $\underline{\lambda}$ is yet unknown. Substituting the values of $\underline{x}$ given in (1.11.19), in (1.11.16), one obtains

$$-\frac{1}{2}\underline{A}\underline{A}^T\underline{\lambda}=\underline{b} \qquad (1.11.20)$$

From which, if AA^T is of full rank,

$$\lambda=-2(AA^T)^{-1}b \qquad (1.11.21)$$

Finally, substituting the latter value of λ into eq. (1.11.19),

$$x=A^T(AA^T)^{-1}b=A^+b \qquad (1.11.22)$$

where

$$A^+ \equiv A^T(AA^T)^{-1}$$

is another pseudo-inverse of A.

Exercise 1.11.1 Can both pseudo-inverses of A, the one given in (1.11.8) and that of (1.11.23) exist for a given matrix A? Explain.

The foregoing solution (1.11.22) has many interpretations: in control theory it yields the control taking a system from a known initial state to a desired final one while spending the minimum amount of energy. In Kinematics it finds two interpretations which will be given in Ch. 2, toguhter with applications to hypoid gear design.

Exercise 1.11.2 Show that the image of the error (1.11.4) is perpendicular to x_0 as given by (1.11.8). This result is known as the "Projection Theorem" and finds extensive applications in optimization theory {1.9}.

iii) m=n. This is the best known case and an extensive discussion of it can be found in any elementary linear algebra textbook. The most important result in this case states that if A is of full rank, i.e. if det $A \neq 0$, then the system has a unique solution, which is given by

$$x=A^{-1}b$$

1.12 NUMERICAL SOLUTION OF LINEAR ALGEBRAIC SYSTEMS

Consider the system (1.11.1) for all three cases discussed in section 1.11.

i) m=n. The first case that will be discussed here is that for m=n.

There are many methods to solve such a linear algebraic system, but all

of them fall into one of two categories, namely, a) direct methods and b) iterative methods. Because the first ones are more suitable to be applied in nonlinear algebraic systems, which will be discussed in section 1.13, only direct methods will be treated here. There is an extensive literature dealing with interative methods, of which the treatise by Varga {1.10} discusses the topic very extensively.

As to direct methods, Gauss'algorithm is the one which has received most attention {1.11}, {1.12}. In {1.11} the LU decomposition algorithm is presented and, with further refinements, in {1.12}. The solution is obtained in two steps:

In the first step the matrix of the system, A, is factored into the product of a lower triangular matrix, L, times an upper triangular one, U, in the form

$$A = LU \qquad (1.12.1)$$

where the diagonal of L contains ones in all its entries. Matrix U contains the singular values of A on its diagonal, and all its elements below the main diagonal are zero. The singular values of a matrix A are the nonnegative square roots of the eigenvalues of A^TA. These are real and nonnegative, which is not difficult to prove.

Exercise 1.12.1 Show that if A is a nonsingular nxn matrix, A^TA is positive definite, and if it is singular, then A^TA is positive semi-definite. (Hint: compute the norm of Ax, for arbitraty x).

The LU decomposition of A is performed via the DECOMP subprogram appearing in {1.12}. If A happens to be singular, DECOMP detects this by computing det A, which is done performing the product of the singular values of A, and if this product turns out to be zero, sends a message to the user thereby warning him that he cannot proceed any further.

If $\underline{A}$ is not singular, the user calls the SOLVE subprogram, which computes the solution to the system by back substitution, i.e. from (1.12.1) in the following manner: The equation

$$\underline{L}\underline{U}\underline{x}=\underline{b} \qquad (1.12.2)$$

can be written as

$$\underline{L}\underline{y}=\underline{b}$$

by setting $\underline{U}\underline{x}=\underline{y}$. Thus

$$\underline{y}=\underline{L}^{-1}\underline{b}=\underline{c} \qquad (1.12.3)$$

where $\underline{L}^{-1}$ exists since det $\underline{L}$ (the product of the elements on the diagonal of $\underline{L}$) is equal to one (1.11). Substituting (1.12.3) into $\underline{U}\underline{x}=\underline{y}$, one obtains the final solution:

$$\underline{x}=\underline{U}^{-1}\underline{c}$$

where $\underline{U}^{-1}$ exists because $\underline{A}$ has been detected to be nonsingular*.

The flow diagram of the whole program appears in Fig 1.12.1 and the listings of DECOMP and SOLVE in Figs. 1.12.2 and 1.12.3

ii) m>n. Next, the numerical solution of the overdetermined linear system $\underline{A}\underline{x}=\underline{b}$ is discussed. In this case the number of equations is greater than that of unknowns and hence the sought "solution" is that $\underline{x}_0$ which minimizes the Euclidean norm of the error $\underline{A}\underline{x}_0-\underline{b}$. This is done by application of Householder reflections (1.5) to both $\underline{A}$ and $\underline{b}$. A Householder reflection is an orthogonal transformation $\underline{H}$ which has the property that

$$\underline{H}^{-1}=\underline{H}^{T}=\underline{H} \qquad (1.12.4)$$

Given an m-vector $\underline{a}$ with components $a_1, a_2, \ldots, a_m$, the Householder reflection $\underline{H}$ (a function of $\underline{a}$) defined as

* In fact, there is no need to explicitly compute $\underline{L}^{-1}$ and $\underline{U}^{-1}$, for the triangular structure of $\underline{L}$ and $\underline{U}$ permits a recursive solution.

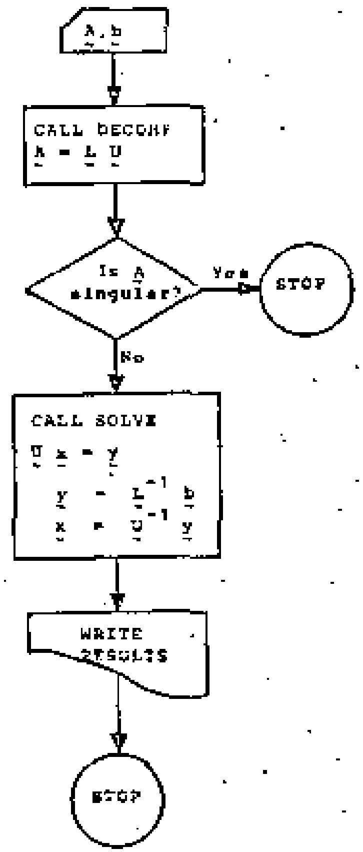


Fig 1.12.1 Flow diagram for the direct solution of a linear algebraic system with equal number of equations as unknowns.

```
      SUBROUTINE DECOMP(N,NDIM,A,IP)
      REAL A(NDIM,NDIM),T
      INTEGER IP(NDIM)
C
C      MATRIX TRIANGULARIZATION BY GAUSSIAN ELIMINATION
C
C      INPUT :
C      N      = ORDER OF MATRIX
C      NDIM   = DECLARED DIMENSION OF ARRAY A, IN THE MAIN PROGRAM
C      A      = MATRIX TO BE TRIANGULARIZED
C
C      OUTPUT :
C      A(I,J), I.LE.J  = UPPER TRIANGULAR FACTOR, U
C      A(I,J), I.GT.J  =MULTIPLIERS = LOWER TRIANGULAR FACTOR, I-L
C      IP(K), K.LT.N   =INDEX OF K-TH PIVOT ROW
C      IP(N)           = (-1)**(NUMBER OF INTERCHANGES) OR 0.
C      USE 'SOLVE' TO OBTAIN SOLUTION OF LINEAR SYSTEM
C      DETERM(A)       = IP(N)*A(1,1)*A(2,2)*...*A(N,N)
C      IF IP(N)=0, A IS SINGULAR, 'SOLVE' WILL DIVIDE BY ZERO
C      INTERCHANGES FINISHED IN U, ONLY PARTLY IN L
C
      IP(N)=1
      DO 60 K=1,N
         IF(K.EQ.N) GO TO 50
         KP1=K+1
         M=K
         DO 10 I=KP1,N
         IF(ABS(A(I,K)).GT.ABS(A(M,K))) M=I
   10    CONTINUE
         IP(K)=M
         IF(M.NE.K) IP(N)=-IP(N)
         T=A(M,K)
         A(M,K)=A(K,K)
         A(K,K)=T
         IF(T.EQ.0) GO TO 50
         DO 20 I=KP1,N
   20       A(I,K)=-A(I,K)/T
         DO 40 J=KP1,N
            T=A(M,J)
            A(M,J)=A(K,J)
            A(K,J)=T
            IF(T.EQ.0.) GO TO 40
            DO 30 I=KP1,N
   30          A(I,J)=A(I,J)+A(I,K)*T
   40    CONTINUE
   50    IF(A(K,K).EQ.0.) IP(N)=0
   60 CONTINUE
      RETURN
      END
```

Fig. 1.12.2 Listing of SUBROUTINE DECOMP

```
      SUBROUTINE SOLVE(N,NDIM,A,B,IP)
      REAL A(NDIM,NDIM),B(NDIM),T
      INTEGER IP(NDIM)
C
C      SOLUTION OF LINEAR SYSTEM, A*X = B
C
C      INPUT :
C      N     = ORDER OF MATRIX.
C      NDIM  = DECLARED DIMENSION OF ARRAY A, IN THE MAIN PROGRAM
C      A     = TRIANGULARIZED MATRIX OBTAINED FROM 'DECOMP'
C      B     = RIGHT HAND SIDE VECTOR
C      IP    = PIVOT VECTOR OBTAINED FROM 'DECOMP'
C      DO NOT USE 'SOLVE' IF 'DECOMP' HAS SET IP(N)=0
C
C      OUTPUT :
C      B     = SOLUTION VECTOR, X
C
      IF(N.EQ.1) GO TO 90
      NM1=N-1
      DO 70 K=1,NM1
         KP1=K+1
         M=IP(K)
         T=B(M)
         B(M)=B(K)
         B(K)=T
         DO 70 I=KP1,N
   70 B(I)=B(I)+A(I,K)*T
      DO 80 KB=1,NM1
         KM1=N-KB
         K=KM1+1
         B(K)=B(K)/A(K,K)
         T=-B(K)
         DO 80 I=1,KM1
   80 B(I)=B(I)+A(I,K)*T
   90 B(1)=B(1)/A(1,1)
      RETURN
      END
```

Fig. 1.12.3 Listing of SUBROUTINE SOLVE

$$\alpha = \text{sgn}\,(a_1)\|\underline{a}\| \qquad (1.12.5a)$$

$$\underline{u} = \underline{a}+\alpha\underline{e}_1 \qquad (1.12.5b)$$

$$\beta = \alpha u_1 \qquad (1.12.5c)$$

$$\underline{H} = \underline{I}-\frac{1}{\beta}\,\underline{u}\underline{u}^T \qquad (1.12.5b)$$

transforms $\underline{a}$ into $-\alpha\underline{e}_1$, and reflects any other vector $\underline{b}$ about a hyperplane perpendicular to $\underline{u}$.

On the other hand, if $\underline{H}_k$ is defined as

$$\alpha_k = \text{sgn}(a_k)(a_k^2 + a_{k+1}^2+\ldots+a_n^2)^{\frac{1}{2}} \qquad (1.12.6a)$$

$$\underline{u}_k = \left(0,\ldots,0,a_k+\alpha_k,\ a_{k+1},\ldots,a_m\right)^T \qquad (1.12.6b)$$

$$\beta_k = \alpha_k(\underline{u}_k)_k \qquad (1.12.6c)$$

$$\underline{H}_k = \underline{I} - \frac{1}{\beta}\,\underline{u}_k\underline{u}_k^T \qquad (1.12.6d)$$

then $\underline{H}_k\underline{a}$ is a vector whose first k-1 components are identical to those of $\underline{a}$, its $k^{\underline{th}}$ component is $-\alpha_k$ and its remaining m-k components are all zero. Furthermore, if v is any other vector, then

$$\underline{H}_k\underline{v} = \underline{v} - \gamma\underline{u}$$

where

$$\gamma = \frac{\underline{v}^T\underline{v}}{\beta}$$

and if, in particular, $v_k = v_{k+1} = \ldots = v_m = 0$, then

$$\underline{H}_k\underline{v} = \underline{v}$$

Let now $\underline{H}_i$ be the Householder reflection wich cancels the last m-i components of the $i^{\underline{th}}$ column of $\underline{H}_{i-1}\underline{A}$, while leaving its i-1 components unchanged and setting its $i^{\underline{th}}$ component equal to $-\alpha_i$, for i=1,...,n. By application of the n Householder reflections thus defined, on $\underline{A}$ and $\underline{b}$ in the form

$$\underline{H}_n\underline{H}_{n-1}\cdots\underline{H}_2\underline{H}_1\underline{A}\underline{x}_0=\underline{H}_n\underline{H}_{n-1}\cdots\underline{H}_2\underline{H}_1\underline{b} \qquad (1.12.7)$$

the original system is transformed into the following two systems

$$\underline{A}_1'\underline{x}_0 = \underline{b}_1'$$

$$\underline{A}_2'\underline{x}_0 = \underline{b}_2'$$

where $\underline{A}_1'$ is nxn and upper triangular, whereas $\underline{A}_2'$ is the (m-n)xn zero matrix and $\underline{b}_2'$ is of dimension m-n and diferent from zero. Once the system is in upper triangular form, it is a simple matter to find the values of the components of $\underline{x}_0$ by back substitution. Let a_{ij}^* and b_k^* be the values of the (i, j) element of $\underline{A}_1'$ and the $k^{\underline{th}}$ component of $\underline{b}_1'$ respectively. Then, starting from the $n^{\underline{th}}$ equation of system (1.12.7),

$$a_{nn}^*x_n=b_n^*$$

x_n is obtained as

$$x_n = \frac{b_n^*}{a_{nn}^*}$$

Substituting this value into the (n-1) st equation,

$$a_{n-1,n-1}^*x_{n-1}+a_{n-1,n}^*\frac{b_n^*}{a_{nn}^*} = b_{n-1}^*$$

from which

$$x_{n-1}=\frac{b_{n-1}^*}{a_{n-1,n-1}^*}\,\frac{b_n^*}{a_{nn}^*}$$

Proceeding similarly with the (n-2)nd,...,2nd and 1st equations, the n components of $\underline{x}_0$ are found. Clearly, then, $\underline{b}_2'$ is the error in the approximation and $\|\underline{b}_2'\| = \|\underline{A}\,\underline{x} - \underline{b}\|$.

The foregoing Householder reflection method can be readily implemented in a digital computer via the HECOMP and HOLVE subroutines appearing in [1.14], whose listings are reproduced in Figs 1.12.4 and 1.12.5.

Exercise 1.12.2 Show that, for any n-vector $\underline{x}$

$$\det(\underline{I}+\underline{x}\underline{x}^T)=1+\underline{x}^T\underline{x}$$

```
      SUBROUTINE HECOMP(MDIM,M,N,A,U)
      INTEGER MDIM,M,N
      REAL A(MDIM,N),U(M)
      REAL ALPHA,BETA,GAMMA,SQRT
C
C   HOUSEHOLDER REDUCTION OF RECTANGULAR MATRIX TO UPPER
C   TRIANGULAR FORM.  USE WITH HOLVE FOR LEAST-SQUARE
C   SOLUTIONS OF OVERDETERMINED SYSTEMS.
C
C   MDIM= DECLARED ROW DIMENSION OF A
C   M    = NUMBER OF ROWS OF A
C   N    = NUMBER OF COLUMNS OF A
C   A    = M-BY-N MATRIX WITH M.>.N
C        INPUT :
C                MATRIX TO BE REDUCED
C        OUTPUT:
C                REDUCED MATRIX AND INFORMATION ABOUT REDUCTION
C   U    = M-VECTOR
C        INPUT :
C                IGNORED
C        OUTPUT:
C                INFORMATION ABOUT REDUCTION
C
C        FIND REFLECTION WHICH ZEROES A(I,K), I= K+1,..........,M
C
      DO 6 K= 1,N
         ALPHA= 0.0
         DO 1 I= K,M
            U(I)= A(I,K)
            ALPHA= ALPHA+U(I)*U(I)
    1    CONTINUE
         ALPHA= SQRT(ALPHA)
         IF(U(K).LT.0.0) ALPHA= -ALPHA
         U(K)= U(K)+ALPHA
         BETA= ALPHA*U(K)
         A(K,K)= -ALPHA
         IF(BETA.EQ.0.0.OR.K.EQ.N) GO TO 6
C
C  APPLY REFLECTION TO REMAINING COLUMNS OF A
         KP1= K+1
         DO 4 J= KP1,N
            GAMMA= 0.0
            DO 2 I= K,M
               GAMMA= GAMMA+U(I)*A(I,J)
    2       CONTINUE
            GAMMA= GAMMA/BETA
            DO 3 I= K,M
               A(I,J)= A(I,J)-GAMMA*U(I)
    3       CONTINUE
    4    CONTINUE
    6 CONTINUE
      RETURN
C
C  TRIANGULAR RESULT STORED IN A(I,J), I.LE.J
C  VECTORS DEFINING REFLECTIONS STORED IN U AND REST OF A
      END
```

Fig 1.12.4 Listing of SUBROUTINE HECOMP


```
      SUBROUTINE HOLVE(MDIM,M,N,A,U,B)
      INTEGER MDIM,M,N
      REAL A(MDIM,N),U(M),B(M)
      REAL BETA,GAMMA,T
C
C
C   LEAST-SQUARE SOLUTION OF OVERDETERMINED SYSTEMS
C   FIND X THAT MINIMIZES NORM(A*X- B)
C
C   MDIM,M,N,A,U,  RESULTS FROM HECOMP
C   B= M-VECTOR
C      INPUT :
C              RIGHT HAND SIDE
C      OUTPUT:
C              FIRST N COMPONENTS = THE SOLUTION, X
C              LAST M-N COMPONENTS= TRANSFORMED RESIDUAL
C   DIVISION BY ZERO IMPLIES A NOT OF FULL RANK
C
C   APPLY REFLECTIONS TO B
C
C
      DO 3 K= 1,N
         T= A(K,K)
         BETA= -U(K)*A(K,K)
         A(K,K)= U(K)
         GAMMA= 0.0
         DO 1 I= K,M
            GAMMA= GAMMA+A(I,K)*B(I)
    1    CONTINUE
         GAMMA= GAMMA/BETA
         DO 2 I= K,M
            B(I)= B(I)-GAMMA*A(I,K)
    2    CONTINUE
         A(K,K)= T
    3 CONTINUE
C
C   BACK SUBSTITUTION
C
      DO 5 KB= 1,N
         K= N+1-KB
         B(K)= B(K)/A(K,K)
         IF(K.EQ.1) GO TO 5
         KM1= K-1
         DO 4 I= 1,KM1
            B(I)= B(I)-A(I,K)*B(K)
    4    CONTINUE
    5 CONTINUE
      RETURN
      END
```

Fig 1.12.5 Listing of SUBROUTINE HOLVE

Exercise 1.12.3* Show that H, as defined in eqs. (1.12.5) is in fact a reflection, i.e. show that H is orthogonal and the value of its determinant is -1. (Hint: Use the result of Exercise 1.12.2).

iii) $m<n$: Now, the linear system of equations $\underline{A}\underline{x}=\underline{b}$ is studied when the number of unknowns is greater that the number of equations.

In this case, the system is underdetermined and has an infinity of solutions. However, as was discussed in Section 1.11, among those solutions, there is one, say $\underline{x}_0$ whose Euclidean norm is a minimum. This is given by eq. (1.11.22), repeated here for ready reference.

$$\underline{x}_0=\underline{A}^T(\underline{A}\underline{A}^T)^{-1}\underline{b} \qquad (1.12.8)$$

One possible way of computing $\underline{x}_0$ is given next:

a) Write eq. (1.11.20) in the form

$$\underline{A}\underline{A}^T\underline{\lambda}=\underline{b} \qquad (1.12.9)$$

b) Using the LU decomposition method, find $\underline{\lambda}$ from (1.12.9)

c) With $\underline{\lambda}$ known from step ii), compute $\underline{x}_0$ by matrix multiplication, as appearing in (1.11.19), i.e.

$$\underline{x}_0=\underline{A}^T\underline{\lambda} \qquad (1.12.10)$$

1.13 NUMERICAL SOLUTION OF NONLINEAR ALGEBRAIC SYSTEMS.

For several reasons, nonlinear systems are more difficult to deal with than are linear systems. Considering the simplest case of equal number of equations and unknowns, there is no guarantee that the nonlinear system has a unique solution; in fact, there is no guarantee that the system has a solution at all.

* See Section 2.3 for more details on reflections.

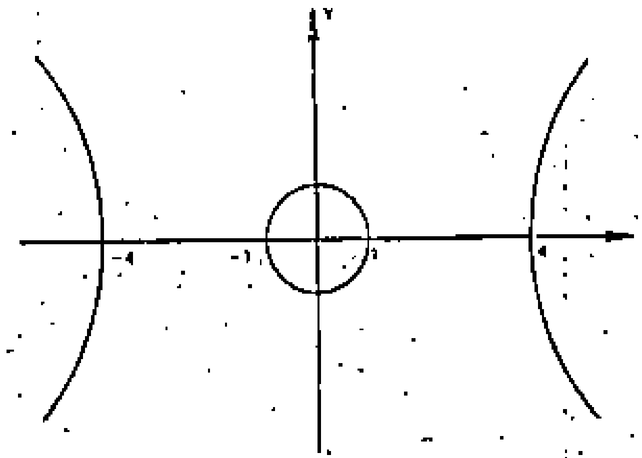


Fig 1.13.1 Non-intersecting hyperbola and circle

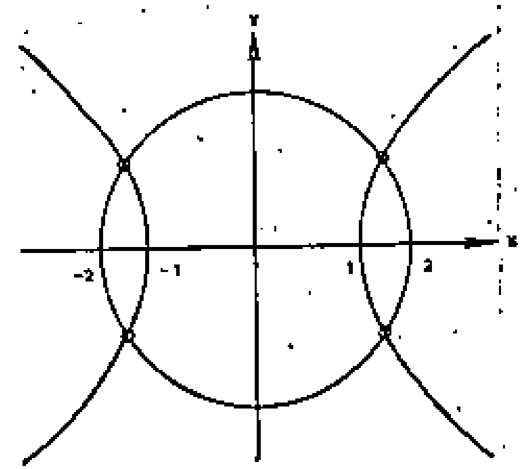


Fig 1.13.2 Intersections of a hyperbola and a circle

Example 1.13.1 The 2nd order nonlinear algebraic system

$$x^2 - y^2 = 16 \qquad \text{(a)}$$

$$x^2 + y^2 = 1 \qquad \text{(b)}$$

has no solution, for the hyperbola (a) does not intersect the circle (b), as is shown in Fig. 1.13.1

Example 1.13.2 The 2nd order linear algebraic system

$$x^2 - y^2 = 1 \qquad \text{(c)}$$

$$x^2 + y^2 = 4 \qquad \text{(d)}$$

has four solutions, namely

$$x_1 = \sqrt{\tfrac{5}{2}},\ y_1 = \sqrt{\tfrac{3}{2}}$$

$$x_2 = \sqrt{\tfrac{5}{2}},\ y_2 = -\sqrt{\tfrac{3}{2}}$$

$$x_3 = -\sqrt{\tfrac{5}{2}},\ y_3 = -\sqrt{\tfrac{3}{2}}$$

$$x_4 = -\sqrt{\tfrac{5}{2}},\ y_4 = \sqrt{\tfrac{3}{2}}$$

which are the four points where the hyperbola (c) intersects the circle (d). These intersections appear in Fig. 1.13.2

The most popular method of solving a nonlinear algebraic system is the so-called Newton-Raphson method. First, the system of equations has to be written in the form

$$\underline{f}(\underline{x}) = \underline{0} \qquad (1.13.1)$$

where $\underline{f}$ and $\underline{x}$ are m- and n- dimensional vectors. For example, system (a), (b) of Example 1.13.1 can be written in the form

$$f_1(x_1,x_2) = x_1^2 - x_2^2 - 16 = 0 \qquad \text{(a')}$$

$$f_2(x_1,x_2) = x_1^2 + x_2^2 - 1 = 0 \qquad \text{(b')}$$

Here f_1 and f_2 are the components of the 2-dimensional vectors $\underline{f}$ and x_1 and

x_2 (clearly, x and y have been replaced by x_1 and x_2, respectively) are the components of the 2-dimensional vector $\underline{x}$. Next, the three cases, m=n, m>n and m<n, are discussed

First case: m=n

Let $\underline{x}_0$ be known to be a "good" approximation to the solutions $\underline{x}_r$ or a "guess". The expansion of $\underline{f}(\underline{x})$ about $\underline{x}_0$ in a Taylor series yields

$$\underline{f}(\underline{x}_0+\Delta\underline{x}) = \underline{f}(\underline{x}_0) + \underline{f}'(\underline{x}_0)\Delta\underline{x} + \underline{R} \qquad (1.13.1)$$

If $\underline{x}_0 + \Delta\underline{x}$ is an even better approximation to $\underline{x}_r$, then $\Delta\underline{x}$ must be small and so, only linear terms could be retained in (1.13.2) and, of course, $\underline{f}(\underline{x}_0+\Delta\underline{x})$ must be closer to $\underline{0}$ than is $\underline{f}(\underline{x}_0)$. Under these assumptions, $\underline{f}(\underline{x}_0+\Delta\underline{x})$ can be assumed to be zero and (1.13.2) leads to

$$\underline{f}(\underline{x}_0) + \underline{f}'(\underline{x}_0)\Delta\underline{x} = \underline{0} \qquad (1.13.3)$$

In the above equation $\underline{f}'(\underline{x}_0)$ is the value of the gradient of $\underline{f}(\underline{x})$, $\underline{f}'(\underline{x})$ at $\underline{x} = \underline{x}_0$. This gradient is an nxn matrix, $\underline{J}$, whose (k,ℓ) element is

$$J_{k\ell} = \frac{\partial f_k}{\partial x_\ell} \qquad (1.13.4)$$

If the Jacobian matrix $\underline{J}$ is nonsingular, it can be inverted to yield

$$\Delta\underline{x} = -\underline{J}^{-1}(\underline{x}_0)\underline{f}(\underline{x}_0) \qquad (1.13.5)$$

Of course, $\underline{J}$ need not actually be inverted, for $\Delta\underline{x}$ can be obtained via the LU decomposition method from eq. (1.13.3) written in the form

$$\underline{J}(\underline{x}_0)\Delta\underline{x} = -\underline{f}(\underline{x}_0) \qquad (1.13.6)$$

With the value of $\Delta\underline{x}$ thus obtained, the improved value of $\underline{x}$, is computed as

$$\underline{x}_1 = \underline{x}_0 + \Delta\underline{x}$$

In general, at the kth iteration, the new value $\underline{x}_{k+1}$ is computed from the formula

$$\underline{x}_{k+1} = \underline{x}_k - \underline{J}(\underline{x}_k)^{-1}\underline{f}(\underline{x}_k) \qquad (1.13.7)$$

[illegible] is the Newton-Raphson iterative scheme. The procedure is stopped when a convergence criterion is met. One possible criterion is that the [illegible] of f [illegible] attain a value below certain prescribed tolerance, i.e.

$$||f(x_k)|| \le \epsilon \qquad (1.13.8)$$

where ϵ is the said tolerance. On the other hand, it can also happen that at iteration k, the norm of the increment becomes smaller than the tolerance. In this case, even if the convergence criterion (1.13.8) is not met, it is useless to perform more interations. Thus, it is more reasonable to verify first that the norm of the correction does not become too small before proceeding further, and stop the procedure if both $||f(x_k)||$ and $||\Delta x_k||$ are small enough, in which case, convergence is reached.

[illegible] by $||\Delta x_k||$ goes below the imposed tolerance, do not accept the corre- [illegible] x_k as the solution. The conditions under which the procedure converges are discussed in (1.15). These conditions, however, cannot be verified easily, in general. What is advised to do is to try different initial guesses x_0 till convergence is reached and to stop the procedure if

[illegible] many iterations have been performed

or

ii) $||\Delta x_k|| \le \epsilon$ but $||f(x_k)|| > \epsilon$

If the method of Newton-Raphson converges for a given problem, it does so quadratically, i.e. two digits are gained per iteration during the aproximation to the solution. It can happen, however, that the procedure does not converge monotonically, in which case,

$$||f(x_{k+1})|| > ||f(x_k)||,$$

thus giving rise to strong oscillations and, possibly, divergence. One way to cope with this situation is to introduce damping, i.e. instead of using

the whole computed increment Δx_k, use a fraction of it, i.e. at the ith iteration, for i=0,1,..., max, instead of using formula (1.13.7) to compute the next value x_{k+1}, use

$$x_{k+1} = x_k - \alpha^i J(x_k)^{-1} f(x_k) \qquad (1.13.9)$$

where α is a real number between 0 and 1. For a given k, eq. (1.13.9) represents the damping part of the procedure, which is stopped when

$$||f(x_{k+1})|| < ||f(x_k)||$$

The algorithm is summarized in the flow chart of Fig 1.13.3 and implemented in the subroutine NRDAMP appearing in Fig 1.13.4

Second case: m>n

In this case the system is overdetermined and it is not possible, in general, to satisfy all the equations. What can be done, however, is to find that x_0 which minimizes $||f(x)||$.

This problem arises, for example, when one tries to design a planar four-bar linkage to guide a rigid body through more than five configurations.

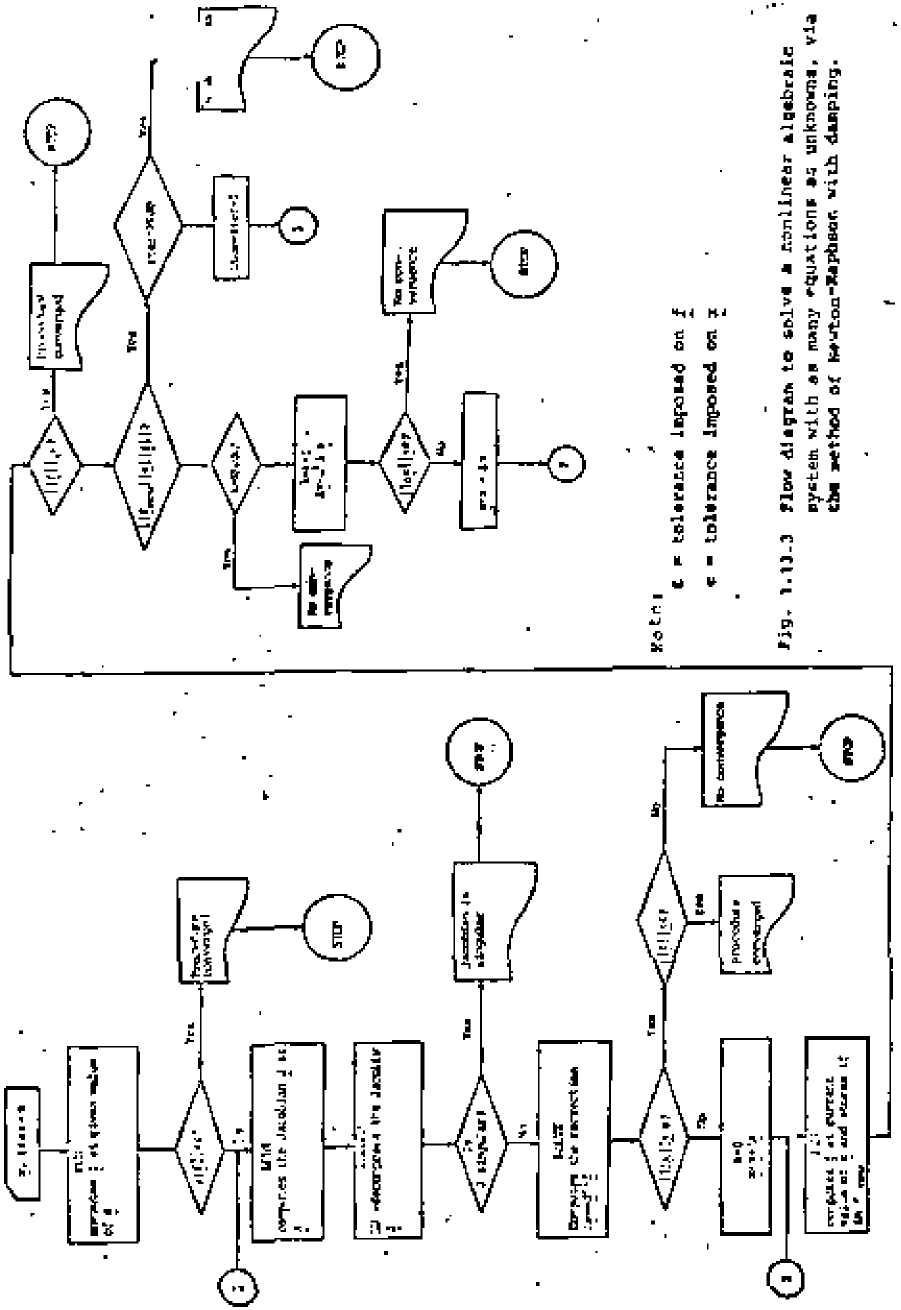
To find the minimizing x_0, define first which norm of $f(x)$ is desired to minimize. One norm which has several advantages is the Euclidean norm, already discussed in case I of Section 1.11, where the linear least-square problem was discussed. In the context of nonlinear systems of equations, minimizing the quadratic norm of $f(x)$ leads to the nonlinear least-square problem. The problem is then to find the minimum of the scalar function

$$\phi(x) = ||f(x)||^2 = f^T(x) f(x) \qquad (1.13.10)$$

As already discussed in Section 1.10, for this function to reach a minimum, it must first reach a stationary point, i.e. its gradient must vanish. Thus,

$$\phi'(x) = 2 J^T(x) f(x) \qquad (1.13.11)$$

where $J(x)$ is the Jacobian matrix of f with respect to x, i.e. an mxn matrix



Note: ε = tolerance imposed on f; δ = tolerance imposed on x

Fig. 1.13.3 Flow diagram to solve a nonlinear algebraic system with as many equations as unknowns, via the method of Newton-Raphson with damping.

```
      SUBROUTINE NRDAMP(X,FUN,DFDX,P,TOLX,TOLF,DAMP,N,ITER,MAX,KMAX)
      REAL X(1),P(1),DF(12,12),DELTA(12),F(12)
      INTEGER IP(12)
C  THIS SUBROUTINE FINDS THE ROOTS OF A NONLINEAR ALGEBRAIC SYSTEM OF
C  ORDER N, VIA NEWTON-RAPHSON METHOD(ISAACSON E. AND KELLER H. B.
C  ANALYSIS OF NUMERICAL METHODS, JOHN WILEY AND SONS, INC., NEW YORK
C  1966,PP. 85-123)WITH DAMPING. SUBROUTINE PARAMETERS
C  X          = N-VECTOR OF UNKNOWS.
C  FUN        = EXTERNAL SUBROUTINE WHICH COMPUTES VECTOR F, CONTAINING
C               THE FUNTIONS WHOSE ROOTS ARE OBTAINED.
C  DFDX       = EXTERNAL SUBROUTINE WHICH COMPUTES THE JACOBIAN MATRIX
C               OF VECTOR F WHIT RESPECT TO X.
C  P          = AN AUXILIAR VECTOR OF SUITABLE DIMENSION. IT CONTAINS
C               THE PARAMETERS THAT EACH PROBLEM MAY REQUIERE.
C  TOLX       = POSITIVE SCALAR, THE TOLERANCE IMPOSED ON THE APPROXIMA-
C               TION TO X.
C  TOLF       = POSITIVE SCALAR, THE TOLERANCE IMPOSED ON THE APPROXIMA-
C               TION TO F.
C  DAMP       =THE DAMPING VALUE, PROVIDED BY THE USER SUCH THAT
C               0.LT.DAMP.LT.1 .
C  ITER       =NUMBER OF ITERATION BEING EXECUTED.
C  MAX        =MAXIMUM NUMBER OF ALLOWED ITERATIONS.
C  KMAX       =MAXIMUM NUMBER OF ALLOWED DAMPINGS PER ITERATION. IT IS
C               PROVIDED BY THE USER.
C  FUN AND DFDX ARE SUPPLIED BY THE USER.
C  SUBROUTINES 'DECOMP' AND 'SOLVE' SOLVE THE NTH. ORDER LINEAR
C  ALGEBRAIC SYSTEM DF(X)*DELTA=F(X), DELTA BEING THE CORRECTION TO
C  THE K-TH ITERATION. THE METHOD USED IS THE LU DECOMPOSITION (MOLER
C  C.B., MATRIX COMPUTATIONS WITH FORTRAN AND PAGING. COMMUNICATIONS OF
C  THE A.C.M., VOLUME 15, NUMBER 4, APRIL 1972).
C
      KONT=1
      ITER=0
      CALL FUN(X,F,P,N)
      FNOR1=FNORM(F,N)
      IF(FNOR1.LE.TOLF) GO TO 4
    1 CALL DFDX(X,DF,P,N)
      CALL DECOMP(N,N,DF,IP)
      K=0
C
C  IF THE JACOBIAN MATRIX IS SINGULAR, THE SUBROUTINE RETURNS TO THE
C  MAIN PROGRAM., OTHERWISE, IT PROCEEDS FURTHER.
C
      IF(IP(N).EQ.0) GO TO 14
      CALL SOLVE (N,N,DF,F,IP)
      DO 2 I=1,N
    2 DELTA(I)=F(I)
      DELNOR=FNORM(DELTA,N)
      IF(DELNOR.LT.TOLX) GO TO 4
      DO 3 I=1,N
    3 X(I)=X(I)-DELTA(I)
      GO TO 5
```

Fig 1.13.4 Listing of SUBROUTINE NRDAMP

```
    4 FNOR2=FNOR1
      GO TO 6
    5 CALL FUN(X,F,P,N)
      KONT=KONT+1
      FNOR2=FNORM(F,N)
    6 IF(FNOR2.LE.TOLF) GO TO 11
C
C   TESTING THE NORM OF THE FUNCTION. IF THIS DOES NOT DECREASE
C   THEN DAMPING IS INTRODUCED.
C
      IF(FNOR2.LT.FNOR1) GO TO 10
      IF(K.EQ.KMAX) GO TO 16
      K=K+1
      DO 8 I=1,N
            IF(K.GE.2) GO TO 7
            DELTA(I)=(DAMP-1.)*DELTA(I)
            GO TO 8
    7       DELTA(I)=DAMP*DELTA(I)
    8 CONTINUE
      DELNOR=FNORM(DELTA,N)
      IF(DELNOR.LE.TOLX) GO TO 14
      DO 9 I=1,N
    9 X(I)=X(I)-DELTA(I)
      GO TO 5
   10 IF(ITER.GT.MAX) GO TO 16
      ITER=ITER+1
      FNOR1=FNOR2
      GO TO 1
   11 WRITE(6,110) ITER,FNOR2,KONT
   12 DO 13 I=1,N
   13 WRITE(6,120) I,X(I)
      RETURN
   14 WRITE(6,130) ITER,KONT
      GO TO 12
   16 WRITE(6,140) ITER,FNOR2,KONT
      GO TO 12
  110 FORMAT(5X,'AT ITERATION NUMBER ',I3,' THE NORM OF THE FUNCTION
     -',E20.6/5X,'THE FUNCTION WAS EVALUATED ',I3,' TIMES'/
     -5X,'PROCEDURE CONVERGED, THE SOLUTION BEING :'/)
  120 FORMAT(5X,'X(',I3,')=',E20.6)
  130 FORMAT(5X,'AT ITERATION NUMBER ',I3,'THE JACOBIAN MATRIX'
     -' IS SINGULAR.'/5X,'THE FUNCTION WAS EVALUATED ',I3,' TIMES'/
     -5X,'THE CURRENT VALUE OF X IS :',/)
  140 FORMAT(10X,'PROCEDURE DIVERGES AT ITERATION NUMBER ',I3/10X,
     -'THE NORM OF THE FUNCTION IS ',E20.6/10X,
     -'THE FUNCTION WAS EVALUATED ',I3,' TIMES'/10X,
     -'THE CURRENT VALUE OF X IS :'/)
      END
```

Fig. 1.13.4 Listing of SUBROUTINE NRDAMP (Continued).

Exercise 1.13.1. Derive the expression (1.13.11)

In order to compute the value of $\mathbf{x}$ that zeroes the gradient (1.13.11) proceed iteratively, as next outlined. Expand $\mathbf{f}(\mathbf{x})$ around $\mathbf{x}_0$:

$$\mathbf{f}(\mathbf{x}_0+\Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}_0) + \mathbf{f}'(\mathbf{x}_0)\Delta\mathbf{x} + \mathbf{R} \qquad (1.13.12)$$

If $\mathbf{x}_0+\Delta\mathbf{x}$ is a better approximation to the value that minimizes the Euclidean norm of $\mathbf{f}(\mathbf{x})$, and if in addition $||\Delta\mathbf{x}||$ is small enough, $\mathbf{R}$ can be neglected in eq. (1.13.12) and as trying to set the whole expression equal to zero, the following equation is obtained

$$\mathbf{f}'(\mathbf{x}_0)\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x}_0)$$

or, denoting by $\mathbf{J}$ the Jacobian matrix $\mathbf{f}'(\mathbf{x})$,

$$\mathbf{J}(\mathbf{x}_0)\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x}_0)$$

which is an overdetermined linear system. As discussed in Section 1.11, such a system has in general no solution, but a value of $\Delta\mathbf{x}$ can be computed which minimizes the quadratic norm of the error $\mathbf{J}(\mathbf{x}_0)\Delta\mathbf{x} + \mathbf{f}(\mathbf{x}_0)$. This value is given by the expression (1.11.8) as

$$\Delta\mathbf{x} = -(\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T\mathbf{f}$$

In general, at the kth iteration, compute $\Delta\mathbf{x}_k$ as

$$\Delta\mathbf{x}_k = -\left(\mathbf{J}^T(\mathbf{x}_k)\mathbf{J}(\mathbf{x}_k)\right)^{-1}\mathbf{J}^T(\mathbf{x}_k)\mathbf{f}(\mathbf{x}_k) \qquad (1.13.13)$$

and stop the procedure when $||\Delta\mathbf{x}_k||$ becomes smaller than a prescribed tolerance, thus indicating that the procedure converged. In fact, if $\Delta\mathbf{x}_k$ vanishes, unless $(\mathbf{J}^T\mathbf{J})^{-1}$ becomes infinity, this means that $\mathbf{J}^T\mathbf{f}$ vanishes. But if this product vanishes, then from eq. (1.13.11), the gradient $\phi'(\mathbf{x})$ also vanishes, thus obtaining a stationary point of the quadratic norm of $\mathbf{f}(\mathbf{x})$.

In order to accelerate the convergence of the procedure, damping can also be introduced. This way, instead of computing $\Delta\mathbf{x}_k$ from eq. (1.13.13), compute it from

$$\Delta \underline{x}_k = -\alpha^i \left(J^T(\underline{x}_k) J(\underline{x}_k)\right)^{-1} J^T(\underline{x}_k) \underline{f}(\underline{x}_k) \qquad (1.13.14)$$

for $i = 0, 1, \ldots, max$ and stop the damping when

$$\|\phi(\underline{x}_{k+1,i})\| < \|\phi(\underline{x}_{k,i})\|$$

The algorithm is illustrated with the flow diagram of Fig 1.13.5 and implemented with the subroutine NRDAMC, appearing in Fig 1.13.6

<u>Third case: m<n</u>

The system, in this case, is underdeterminated and infinitely many solutions can be expected to exist. Out of these solutions, however, one can choose that with a minimum norm, thus converting the problem into a nonlinear quadratic programming problem, stated as

$$\text{minimize } \underline{x}^T \underline{x} \qquad (1.13.15a)$$

$$\text{subject to } \underline{f}(\underline{x}) = \underline{0} \qquad (1.13.15b)$$

One way to find the minimizing $\underline{x}_0$, of problem (1.13.15) is via the method of Lagrange multipliers. Thus, define a new objective function

$$\phi(\underline{x}) = \underline{x}^T \underline{x} + \underline{\lambda}^T \underline{f}(\underline{x}) \qquad (1.13.16)$$

which is stationary at $\underline{x}_0$ where its gradient vanishes. Thus,

$$\phi'(\underline{x}_0) = 2\underline{x}_0 + \underline{f}'^T(\underline{x}_0)\underline{\lambda} = \underline{0} \qquad (1.13.17)$$

The systems of equations (1.13.15b) and (1.13.17) now represent a larger system of m+n equations (m in (1.13.10b) and n in (1.13.17)) in m+n unknowns (m components of $\underline{\lambda}$ and n components of $\underline{x}$). Hence, the problem now reduces to the first case and so can be solved by application of the subroutine NRDAMP.

<u>Exercise 1.13.2</u> Let

$$f(\underline{x}) = \frac{1}{2} x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + 8\left(\frac{7}{2}\cos x_1 + \frac{5}{2}\cos x_2 + \frac{3}{2}\cos x_3 + \frac{1}{2}\cos x_4\right)$$

be a scalar function of a vector argument $\underline{x} = (x_1, x_2, x_3, x_4)^T$. Find its

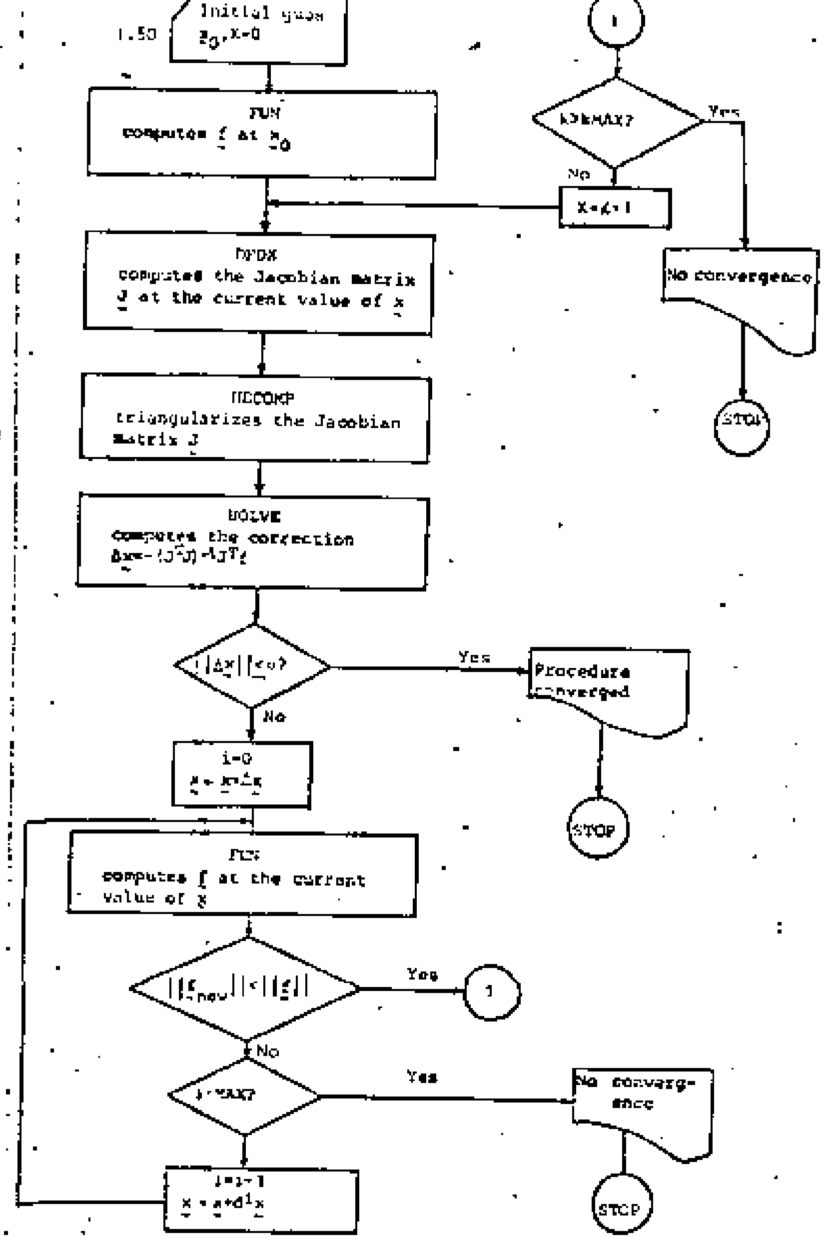


Fig 1.13.5 Flow diagram to compute the least-square solution to an overdetermined nonlinear algebraic system.

```
      SUBROUTINE NRDAMC(X,FUN,DFDX,P,TOL,DAMP,N,M,ITER,MAX,KMAX)
      REAL X(2),F(3),DF(3,2),P,U(3),DELTA(3),FNORM1,FNORM2,
     -     DELNOR
C
C    THIS PROGRAM OBTAINS THE LEAST SQUARE SOLUTION TO THE NONLINEAR
C    SYSTEM F(X)= 0, WHERE F AND X ARE M AND N-DIMENSIONAL VECTORS, M
C    BEING GREATER THAN N. THE PROCEDURE IS ITERATIVE, AND AT EACH
C    ITERATION FINDS THE LEAST SQUARE SOLUTION TO THE LINEAR SYSTEM
C    DF*DELTA= -F, WHERE DF IS THE JACOBIAN M X N MATRIX OF THE ORIGINAL
C    SYSTEM, COMPUTED AT THE CURRENT VALUE OF X. THE LINEAR LEAST SQUARE
C    SOLUTION AT EACH ITERATION IS FOUND VIA HOUSEHOLDER REFLECTIONS (
C    DJOERCK A. AND G. DAHLQUIST, NUMERICAL METHODS, PRENTICE HALL,
C    ENGLEWOOD CLIFFS, N.J., 1974, PP. 201-206, 443-444).
C    PARAMETERS :
C        X     ..= N-DIMENSIONAL VECTOR OF UNKNOWNS.
C        F     ..= M-DIMENSIONAL VECTOR OF FUNCTIONS WHOSE EUCLIDEAN NORM
C                  IS TO BE MINIMIZED.
C        DF    ..= M X N JACOBIAN MATRIX OF F WITH RESPECT TO X.
C        P     ..= VECTOR OF PARAMETERS APPEARING IN FUN AND/OR DFDX. ITS
C                  DIMENSION VARIES FROM PROBLEM TO PROBLEM.
C        TOL   ..= A REAL POSITIVE "SMALL" VARIABLE DENOTING THE IMPOSED
C                  TOLERANCE. IT IS SUPPLIED BY THE USER.
C        DAMP  ..= A REAL POSITIVE VARIABLE, 0.LT. DAMP .LT.1. IT DENOTES
C                  THE DAMPING FACTOR AND IS SUPPLIED BY THE USER.
C        ITER  ..= AN INTEGER VARIABLE DENOTING THE NUMBER OF THE CURRENT
C                  ITERATION.
C        MAX   ..= AN INTEGER VARIABLE DENOTING THE MAXIMUM NUMBER OF
C                  ALLOWED ITERATIONS.
C        KMAX  ..= AN INTEGER VARIABLE DENOTING THE MAXIMUM NUMBER OF
C                  ALLOWED DAMPINGS.
C    SUBSIDIARY SUBROUTINES :
C
C    HECOMP = TRIANGULARIZES A RECTANGULAR MATRIX BY HOUSEHOLDER
C             REFLECTIONS (MOLER C. B., MATRIX EIGENVALUE AND LEAST-
C             SQUARE COMPUTATIONS, COMPUTER SCIENCE DEPARTAMENT,
C             STANFORD UNIVERSITY, MARCH, 1973.)
C    HOLVE  = SOLVES TRIANGULARIZED SYSTEM BY BACK-SUBSTITUTION (MOLER
C             C. B., OP. CIT.)
C    FUN    = COMPUTES F.
C    DFDX   = COMPUTES DF.
C    FNORM  = COMPUTES THE MAXIMUM NORM OF A VECTOR.
C
C
      ITER=0
      CALL FUN(X,F,P,M,N)
    1 ITER=ITER+1
      IF(ITER.GT.MAX) GO TO 10
C
C    FORMS LINEAR LEAST SQUARE PROBLEM
      FNORM1=FNORM(F,M)
      CALL DFDX(X,DF,P,M,N)
      CALL HECOMP(M,M,N,DF,U)
      CALL HOLVE(M,M,N,DF,U,F)
```

Fig 1.13.6 Listing of SUBROUTINE NRDAMC

```
C
C    COMPUTES CORRECTION BETWEEN TWO SUCCESSIVE ITERATIONS
      DO 2 I=1,M
         DELTA(I)=F(I)
    2 CONTINUE
      DELNOR=FNORM(DELTA,N)
      IF(DELNOR.LT.TOL) GO TO 8
      K=1
C
C    IF DELNOR IS STILL LARGE, PERFORMS CORRECTION TO VECTOR X
    3 DO 4 I=1,N
         X(I)=X(I)-DELTA(I)
    4 CONTINUE
      CALL FUN(X,F,P,M,N)
      FNORM2=FNORM(F,M)
C
C    TESTING THE NORM OF THE FUNCTION F AT CURRENT VALUE OF X. IF THIS
C    DOES NOT DECREASE, THEN DAMPING IS INTRODUCED.
      IF(FNORM2.LT.TOL) GO TO 8
      IF(FNORM2.LT.FNORM1) GO TO 1
      IF(K.GT.KMAX) GO TO 7
      DO 6 I=1,N
         IF(K.GE.2) GO TO 5
         DELTA(I)=(DAMP-1.)*DELTA(I)
         GO TO 6
    5.   DELTA(I)=DAMP*DELTA(I)
    6 CONTINUE
      K=K+1
      GO TO 3
    7 WRITE(6,101)DAMP
C
C    AT THIS ITERATION THE NORM OF THE FUNCTION CANNOT BE DECREASED
C    AFTER KMAX DAMPINGS, DAMP IS SET EQUAL TO -1 AND THE SUBROUTINE
C    RETURNS TO THE MAIN PROGRAM.
      DAMP=-1.
      RETURN
    8 WRITE(6,102)FNORM2,ITER,K
      DO 9 I=1,N
         WRITE(6,103) I,X(I)
    9 CONTINUE
      RETURN
   10 WRITE(6,104)ITER
      RETURN
  101 FORMAT(5X,'DAMP =',F10.5,5X,'NO CONVERGENCE WITH THIS DAMPING',
     -       ' VALUE'/)
  102 FORMAT(/5X,'CONVERGENCE REACHED. NORM OF THE FUNCTION :',
     -       F15.6//5X,'NUMBER OF ITERATIONS :',I3,5X,'NUMBER OF ',
     -       'DAMPINGS AT THE LEAST ITERATION :',I3//5X,'THE SOLUTION'
     -       ,' IS :'/)
  103 FORMAT(5X,2HX(I2,3H)= F15.5/)
  104 FORMAT(10X,'NO CONVERGENCE WITH',I3,' ITERATIONS'/)
      END
```

Fig 1.13.6 Listing of SUBROUTINE NRDAMC (Continued) stationary points and decide whether each is either a maximum, a minimum or a saddle point, for β = 1,10,50.

Note: f(x) could represent the potential energy of a mechanical system. In this case the stationary points correspond to the following equilibrium states: minima yield a stable equilibrium state, whereas maxima and saddle points yield unstable states.

Example 1.13.3 Find the point closest to all three curves of Fig 1.13.7. These curves are the parabola (P), the circle (C) and the hyperbola (H) with the following equations:

$$y = \frac{1}{2.4} x^2 \qquad (P)$$

$$x^2 + y^2 = 4 \qquad (C)$$

$$x^2 - y^2 = 1 \qquad (H)$$

From Fig 1.13.7 it is clear that no single pair (x,y) satisfies all three equations simultaneously. There exist points of coordinates x_0, y_0, however, that minimize the quadratic norm of the error of the said equations. These can be found with the aid of SUBROUTINE NRDAMC. A program was written that calls NRDAMC, RECOMP and SOLVE to find the least-square solution to eqs. (P), (C) and (H). The found solutions were:

First solution: x=-1.61537, y=1.17844

Second solution: x= 1.61537, y=1.17844

which are shown in Fig 1.13.7. These points have symmetrical locations, as expected, and lie almost on the circle at abount equal distances from A_i and C_i and B_i and D_i (i=1,2)

The maximum error of the foregoing approximation was computed as 0.22070

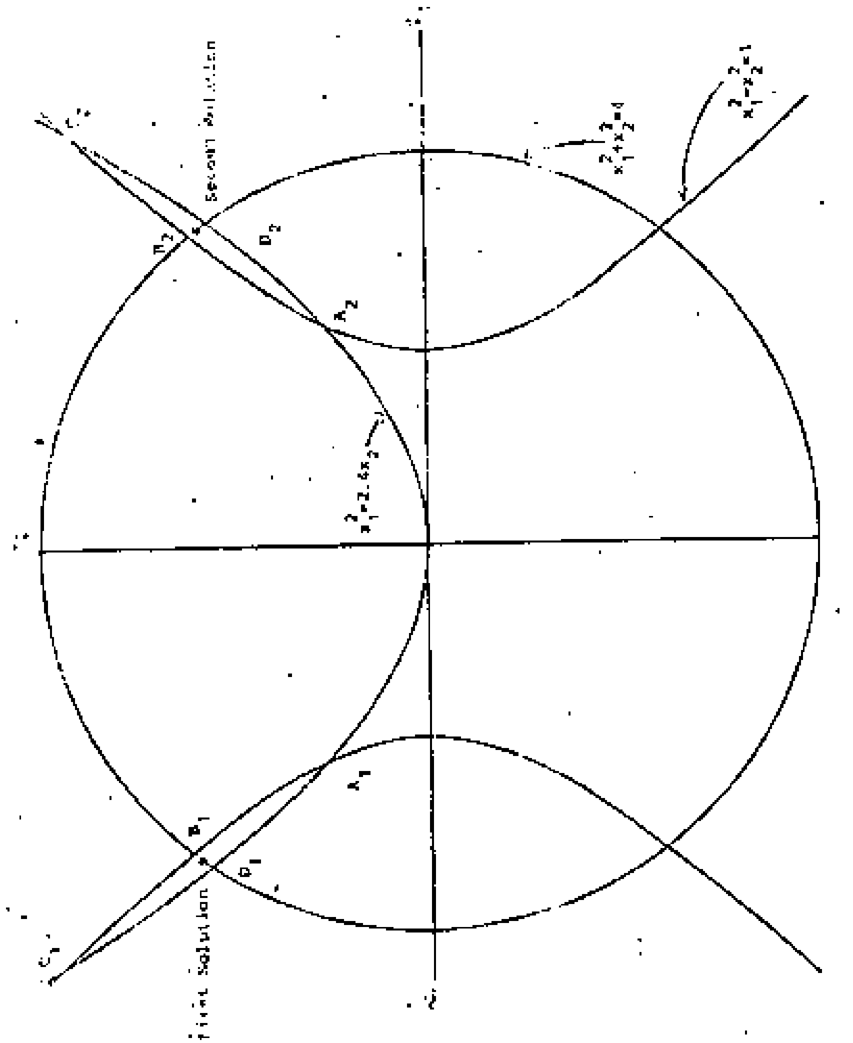


Fig 1.13.7 Location of the point closest to a parabola, a circle, and a hyperbola.

REFERENCES

1.1 Lang S., Linear Algebra, Addison-Wesley Publishing Co., Menlo Park, 1970, pp. 39 and 40.

1.2 Lang S., op. cit., pp. 99 and 100

1.3 Finkbeiner, D.F., Matrices and Linear Transformations, W.H. Freeman and Company, San Francisco, 1960, pp. 139-142

1.4 Halmos, P.R., Finite-Dimensional Vector Spaces, Springer-Verlag, N. York, 1974.

1.5 Businger P. and G.H. Golub, "Linear Least Squares Solutions by Householder Transformations", in Wilkinson J.H. and C. Reinsch, eds., Handbook for Automatic Computation, Vol. II, Springer-Verlag, N. York, 1971, pp. 111-118

1.6 Stewart, G.W., Introduction to Matrix Computations, Academic Press, N.York, 1973, pp. 208-249.

1.7 Soderstrom T. and G.W. Stewart, "On the numerical properties of an iteretive method for computing the Moore-Penrose generalized inverse", SIAM J. on Numerical Analysis, Vol. II, No. 1, March 1974.

1.8 Brand L., Advanced Calculus, John Wiley and Sons, Inc., N. York, 1955, pp. 147-197.

1.9 Luenberger, D.G., Optimization by Vector Space Methods, John Wiley and Sons, Inc., N. York, 1969, pp. 8, 49-52

1.10 Varga, R.S., Matrix Iterative Analysis, Prentice Hall, Inc., Englewood Cliffs, 1962, pp. 56-160

1.11 Forsythe, G.E. and C.B. Moler, Computer Solution of Linear Algebraic Systems, Prentice Hall, Inc., Englewood Cliffs, 1967, pp. 27-33

1.12 Moler C.B., "Algorithm 423. Linear Equation Solver (F 4)" Communications of the ACM, Vol. 15, Number 4, April 1973, p. 274.

1.13 Björck A. and G. Dahlquist, Numerical Methods, Prentice-Hall, Inc., Englewood Cliffs, 1974, pp. 201-206.

1.14 Moler C.B., Matrix Eigenvalue and Least Square Computations, Computer Science Departament, Stanford University, Stanford, California, 1973 pp. 4.1-4.15

1.15 Isaacson, E. and H. B. Keller, Analysis of Numerical Methods, John Wiley and Sons, Inc., N. York, 1966, pp. 85-123

1.16 Angeles, J., "Optimal synthesis of linkages using Householder reflections", Proceedings of the Fifth World Congress on the Theory of Machines and Mechanisms, Vol. I, Montreal, Canada, July 8-13, 1979, pp. 111-114.

DIVISION. DE. EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

2. FUNDAMENTALS OF RIGID-BODY THREE-DIMENSIONAL KINEMATICS

Dr. Jorge Angeles Alvarez

Junio, 1981

2. FUNDAMENTALS OF RIGID-BODY THREE-DIMENSIONAL KINEMATICS.

2.1 INTRODUCTION.

The rigid body is defined as a continuum for which, under any physically possible motion, the distance between any pair of its points remains unchanged. The rigid body is a mathematical abstraction which models very accurately the behaviour of a wide variety of natural and man-made mechanical systems under certain conditions. However, as such it does not exist in nature, as neither do the elastic body nor the perfect fluid. The theorems related to rigid body motions are rigorously proved and the foundations for the analysis of the motion of systems of coupled rigid bodies (linkages) are laid down. The main results in this chapter are the theorems of Euler, Chasles, the one on the existence of an instant screw, the Theorem of Aronhold-Kennedy and that of Coriolis.

2.2 MOTION OF A RIGID BODY.

Consider a subset D of the Euclidean three-dimensional physical space occupied by a rigid body, and let x be the position vector of a point of that body. A rigid body motion is a mapping M which maps every point x of D into a unique point y of a set D', called "the image" of D under M,

$$M : x \rightarrow y \qquad (2.2.1)$$

such that, for any pair x_1 and x_2, mapped by M into y_1 and y_2, respectively, one has

$$||x_2 - x_1|| = ||y_2 - y_1|| \qquad (2.2.2)$$

The symbol $||.||$ denotes the Euclidean norm* of the space under consideration.

It is next shown that, under the above definition, a rigid-body motion preserves the angle between any two lines of a body. Indeed, let x_1, x_2 and x_3 be three noncollinear points of a rigid body. Let M map these points into y_1, y_2 and y_3, respectively. Clearly,

$$||x_3-x_2||^2 = (x_3-x_2, x_3-x_2) = ((x_3-x_1) - (x_2-x_1), (x_3-x_1)-(x_2-x_1)) =$$
$$= ||x_3-x_1||^2 - 2(x_3-x_1, x_2-x_1) + ||x_2-x_1||^2$$

Similarly,

$$||y_3-y_2||^2 = (y_3-y_2, y_3-y_2) = ||y_3-y_1||^2 - 2(y_3-y_1, y_2-y_1) + ||y_2-y_1||^2$$

From the definition of a rigid-body motion, however,

$$||y_3-y_2||^2 = ||x_3-x_2||^2$$

Thus,

$$||x_3-x_1||^2 - 2(x_3-x_1, x_2-x_1) + ||x_2-x_1||^2 = ||y_3-y_1||^2$$
$$-2(y_3-y_1, y_2-y_1) + ||y_2-y_1||^2 \qquad (2.2.3)$$

However, again from the rigid-body motion definition,

$$||x_3-x_1||^2 = ||y_3-y_1||^2 \qquad (2.2.4)$$

and

$$||x_2-x_1||^2 = ||y_3-y_1||^2 \qquad (2.2.5)$$

Thus clearly, from (2.2.3), (2.2.4) and (2.2.5),

$$(x_3-x_1, x_2-x_1) = (y_3-y_1, y_2-y_1) \qquad (2.2.6)$$

which states that the angle (See Section 1.7) between vectors x_3-x_1 and x_2-x_1 remains unchanged.

The foregoing mapping M is, in general, nonlinear, but there exists a class Q of mappings M, leaving one point in a body fixed, that are linear. In fact, let O be a point of a rigid body which remains fixed under M, its position vector being the zero vector 0 of the space under study (this can always be rearranged since one has the freedom to place the origin of coordinates in any suitable position). Let x_1 and x_2 be any two points of

* See Section 1.8

this rigid body.

From the previous results,

$$||\underline{x}_i|| = ||\underline{Q}(\underline{x}_i)||, \; i = 1, 2 \qquad (2.2.7)$$

$$(\underline{x}_i, \underline{x}_j) = (\underline{Q}(\underline{x}_i), \underline{Q}(\underline{x}_j)), \; i,j=1,2 \qquad (2.2.8)$$

Assume for a moment that $\underline{Q}$ is not linear.

Let

$$\underline{e} = \underline{Q}(\underline{x}_1+\underline{x}_2) - (\underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2))$$

Then

$$||\underline{e}||^2 = ||\underline{Q}(\underline{x}_1+\underline{x}_2)||^2 + ||\underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2)||^2 - 2(\underline{Q}(\underline{x}_1+\underline{x}_2), \underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2)) =$$

$$= ||\underline{x}_1+\underline{x}_2||^2 + ||\underline{Q}(\underline{x}_1)||^2 + ||\underline{Q}(\underline{x}_2)||^2 + 2(\underline{Q}(\underline{x}_1), \underline{Q}(\underline{x}_2))$$

$$- 2(\underline{Q}(\underline{x}_1+\underline{x}_2), \underline{Q}(\underline{x}_1)) - 2(\underline{Q}(\underline{x}_1+\underline{x}_2), \underline{Q}(\underline{x}_2))$$

where the rigidity condition has been applied, i.e. the condition that states that, under a rigid body motion, any two points of the body remain equidistant. Applying this condition again, together with the condition of constancy of the angle between any two lines of the rigid body (eq. (2.2.6)),

$$||\underline{e}||^2 = ||\underline{x}_1||^2 + ||\underline{x}_2||^2 + 2(\underline{x}_1, \underline{x}_2) + ||\underline{x}_1||^2 + ||\underline{x}_2||^2 + 2(\underline{x}_1, \underline{x}_2)$$

$$-2(\underline{x}_1+\underline{x}_2, \underline{x}_1) - 2(\underline{x}_1+\underline{x}_2, \underline{x}_2) =$$

$$= 2||\underline{x}_1||^2 + 2||\underline{x}_2||^2 + 4(\underline{x}_1, \underline{x}_2) - (2||\underline{x}_1||^2 + 2||\underline{x}_2||^2 + 4(\underline{x}_1, \underline{x}_2))$$

$$= 0$$

From the positive-definiteness of the norm, then

$$\underline{e} = \underline{0}$$

thereby showing that

$$\underline{Q}(\underline{x}_1+\underline{x}_2) = \underline{Q}(\underline{x}_1) + \underline{Q}(\underline{x}_2)$$

i.e. $\underline{Q}$ is an <u>additive</u> operator*

On the other hand, since $\underline{Q}$ preserves the angle between any pair of lines of a rigid body, for any given real number $\alpha>0$, $\underline{Q}(\underline{x})$ and $\underline{Q}(\alpha\underline{x})$ are parallel, i.e. linearly dependent (for $\underline{x}$ and $\alpha\underline{x}$ are parallel as well). Hence,

$$\underline{Q}(\alpha\underline{x}) = \beta\underline{Q}(\underline{x}), \; \beta>0 \qquad (2.2.9)$$

Since $\underline{Q}$ preserves the Euclidean norm,

$$||\underline{Q}(\alpha\underline{x})|| = ||\alpha\underline{x}|| = |\alpha| \cdot ||\underline{x}|| \qquad (2.2.10)$$

On the other hand, from eq. (2.2.9),

$$||\underline{Q}(\alpha\underline{x})|| = ||\beta\underline{Q}(\underline{x})|| = |\beta| \cdot ||\underline{Q}(\underline{x})|| = |\beta| \cdot ||\underline{x}|| \qquad (2.2.11)$$

Hence, equating (2.2.10) and (2.2.11), and dropping the absolute-value brackets, for $\alpha,\beta>0$,

$$\alpha = \beta$$

and

$$\underline{Q}(\alpha\underline{x}) = \alpha\underline{Q}(\underline{x}) \qquad (2.2.12)$$

and hence, $\underline{Q}$ is a <u>homogeneous</u> operator. Being homogeneous and additive, $\underline{Q}$ is <u>linear</u>. The following has thus been proved.

<u>THEOREM 2.2.1</u> *If $\underline{Q}$ is a rigid body motion that leaves a point fixed, then $\underline{Q}$ is a linear transformation.*

From the foregoing discussion, $\underline{Q}$ is representable by means of a 3x3 matrix referred to a certain basis (theorem 1.2.1)

If $B=\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is an orthonormal** basis for the 3-dimensional Euclidean

---

* This proof is due to Prof. G.S. Sidhu, Institute for Applied Mathematics and Systems Research, U. of Mexico

** $(\underline{e}_i, \underline{e}_j) = \delta_{ij}$ (The Kronecker delta) space, the ith column of the matrix $\underset{\sim}{Q}$ is formed from the coefficients of $Q(e_i)$ expressed in terms of B according to Definition 1.2.1. In fact, the resulting matrix is orthogonal. Since Q is linear, $Q(x)$ can be expressed simply as Qx. Now if

$$y = Qx$$

then

$$||y|| = ||x||$$

Hence

$$y^T y = x^T Q^T Q x = x^T x, \text{ for any } x$$

Hence, clearly

$$Q^T Q = I$$

the identity matrix. This result can then be stated as

THEOREM 2.2.1 A rigid body motion leaving one point fixed is represented with respect to an orthonormal basis by an orthogonal matrix.

2.3 THE THEOREM OF EULER AND THE REVOLUTE MATRIX.

In the previous sections it was shown that the motion of a rigid body which keeps one of its points fixed can be represented by an orthogonal 3 x 3 matrix. In view of Sect. 1.9 there are two classes of orthogonal matrices, depending on whether their determinant is plus or minus unity. Orthogonal matrices whose determinant is +1 are called proper orthogonal and those whose determinant is -1 are called improper orthogonal.

Proper orthogonal matrices represent rigid body rotations, whereas improper orthogonal matrices represent reflections. Indeed, consider the rotation of axes X_1, Y_1, Z_1 into X_2, Y_2, Z_2 as shown in Fig 2.3.1

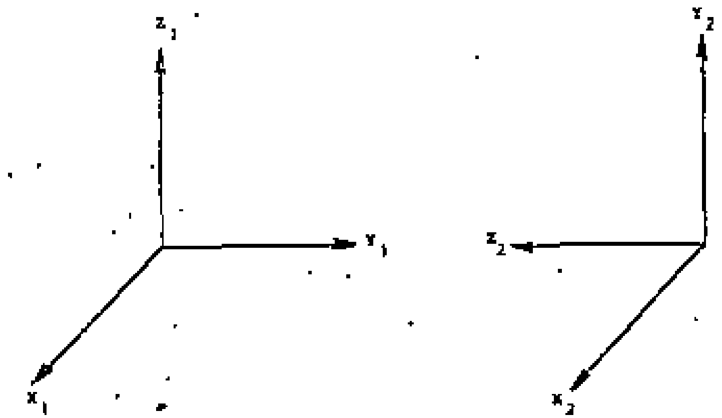


Fig 2.3.1 Rotation of Axes

The matrix representation of the above rotation is obtained from the relationship

$$x_2 = x_1 \quad (2.3.1)$$
$$y_2 = z_1$$
$$z_2 = -y_1$$

where x_1, x_2 represent unit vectors along the X_1 and X_2 axes, respectively, etc. From eqs. (2.3.1),

$$(Q)_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (2.3.2)$$

$(Q)_1$ means the rotation expressed in terms of the basis $\{x_1, y_1, z_1\}$.

Clearly,

$$\det Q = +1$$

and thus it is a proper orthogonal matrix.

On the other hand, consider the reflection of axes X_1, Y_1, Z_1 into

(X_2, Y_2, Z_2), as shown in Fig 2.3.2

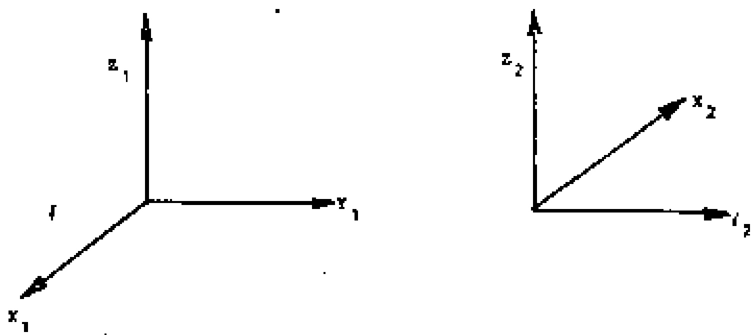


Fig. 2.3.2 Reflection of axes

Now,

$$x_2 = -x_1 \qquad (2.3.3)$$
$$y_2 = y_1$$
$$z_2 = z_1$$

Hence,

$$(\underset{\sim}{Q})_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so,

$$\det \underset{\sim}{Q} = -1$$

i.e., $\underset{\sim}{Q}$ as obtained from (2.3.3) is a reflection. Applications of reflections were studied in Sect. 1.12.

From Corollary 1.9.1 it can be seen that a 3 x 3 proper orthogonal matrix has exactly one eigenvalue equal to +1. Now if $\underset{\sim}{e}$ is the eigenvector of $\underset{\sim}{Q}$ corresponding to the eigenvalue +1, it follows that

$$\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e}$$

and, furthermore, for any scalar α,

$$\underset{\sim}{Q}\alpha\underset{\sim}{e} = \alpha\underset{\sim}{e}$$

Hence all points of the rigid body located along a line parallel to $\underset{\sim}{e}$ passing through the fixed point O, remain fixed under the rotation $\underset{\sim}{Q}$. Hence, the following result, due to Euler (2.1) :

<u>THEOREM 2.3.1 (Euler)</u>. *If a rigid body undergoes a displacement leaving one of its points fixed, then there exists a line passing through the fixed point, such that all of the points on that line remain fixed during the displacement. This line is called "<u>the axis of rotation</u>" and the angle of rotation is measured on a plane perpendicular to the axis.*

The matrix representing a rotation is sometimes referred to as "<u>the revolute</u>". Clearly, the revolute is completely determined by a scalar parameter, the angle of rotation and a vector, the direction of the axis of rotation*. From the foregoing discussion it is clear that the direction vector of the revolute is obtained as the (unique linearly independent) eigenvector of the revolute associated with its + 1 eigenvalue. The angle of rotation is obtained as follows:

From Euler's Theorem, it is always possible to obtain an (orthonormal) basis $B = \{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ such that, say $\underset{\sim}{b}_3$, is parallel to the axis of rotation, $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ thus lie in a plane perpendicular to this axis. The rotation would then rotate the vectors through an angle θ. Let $\underset{\sim}{b}_1'$ and $\underset{\sim}{b}_2'$ be the corresponding images of $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ after the rotation under consideration, represented graphically in Fig 2.3.3

---

* These parameters are also called "the invariants" of the revolute, for they remain unchanged under different choices of coordinate axes.

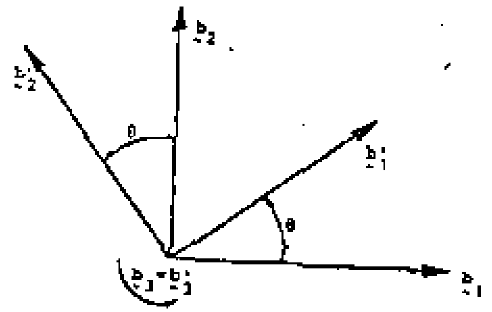


Fig 2.3.3 Rotation through an angle θ about axis $\underline{b}_3$.

Then

$$\underline{b}'_1=\cos\theta\,\underline{b}_1+\sin\theta\,\underline{b}_2$$
$$\underline{b}'_2=-\sin\theta\,\underline{b}_1+\cos\theta\,\underline{b}_2 \qquad (2.3.4)$$
$$\underline{b}'_3=\underline{b}_3$$

and it follows that

$$(\underline{Q})_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2.3.5)$$

Due to its simple and illuminating form, it seems justified to call matrix (2.3.5) a "<u>canonical form</u>" of the rotation matrix.

<u>Exercise 2.3.1</u> Devise an algorithm to carry any orthogonal matrix into its canonical form (2.3.5).

Let a revolute matrix $\underline{Q}$ be given refered to an arbitrary orthonormal basis $A=\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}$, different from B as defined above. Furthermore, let

$$(\underline{P})_A = (\underline{b}_1 \;\vdots\; \underline{b}_2 \;\vdots\; \underline{b}_3) \qquad (2.3.6)$$

where

$$\underline{b}_j = (b_{1j}, b_{2j}, b_{3j})^T,\; j=1,2,3$$

b_{ij} being the <u>ith</u> component of $\underline{b}_j$ referred to the basis A, i.e.

$$\underline{b}_j = b_{1j}\underline{a}_1+b_{2j}\underline{a}_2+b_{3j}\underline{a}_3$$

Since both A and B are orthonormal, $(\underline{P})_A$ is an orthogonal matrix. Thus, the canonical form can be obtained from the following similarity transformation

$$(\underline{Q})_B = (\underline{P}^T)_A(\underline{Q})_A(\underline{P})_A \qquad (2.3.7)$$

From the canonical form given above, it is apperent that

$$\mathrm{Tr}(\underline{Q})_B=1+2\cos\theta$$

from which

$$\theta = \cos^{-1}\{\tfrac{1}{2}(\mathrm{Tr}(\underline{Q})_B-1)\} \qquad (2.3.8)$$

is readily obtained. It should be pointed out that, since the trace is invariant under similarity transformations, i.e. since

$$\mathrm{Tr}(\underline{Q})_B = \mathrm{Tr}\,(\underline{Q})_A$$

one can compute the rotation angle without transforming the revolute matrix into its canonical form.

Eq. (2.3.8), however, yields the angle of rotation through the $\cos^{-1}$ function, which is even, i.e. $\cos^{-1}(-x)=\cos^{-1}(x)$; hence, the said formula does not provide the sign of the angle. This is next determined by application of Theorem 2.3.2. The proof of this theorem needs some background, which is now laid down.

In what follows, dyadic notation will be used*. Let L be the axis of a

---

* For readers unfamiliar with this notation, a short account of algebra of dyadics is provided in Appendix I.

rotation about point O, whose existence is guarenteed by Euler's Theorem. Moreover, let θ be the corresponding angle of rotation, as indicated in Fig 2.3.4, and $\mathbf{e}$ a unit vector parallel to L.

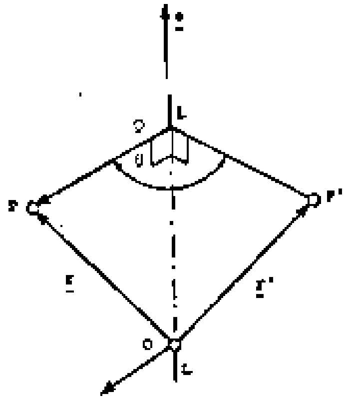


Fig 2.3.4 Rotation about a point.

In Fig 2.3.4 P' is the rotated position of point P. If PQ is perpendicular to L, so is P'Q, because rotations preserve angles of rigid bodies. Thus points P, P' and Q determine a plane perpendicular to L, on which the angle of rotation, θ, is measured. From that figure,

$$\mathbf{r}' = \overrightarrow{OQ} + \overrightarrow{QP'}$$

and

$$\overrightarrow{OQ} = \mathbf{r} - \overrightarrow{QP}$$

Hence

$$\mathbf{r}' = \mathbf{r} - \overrightarrow{QP} + \overrightarrow{QP'} \quad (2.3.9)$$

Let QP" be a line contained in plane PP'Q, at right angles with line PQ and of length equal to that of QP. Thus, vector $\overrightarrow{QP'}$ can be expressed as a linear combination of vectors $\overrightarrow{QP}$ and $\overrightarrow{QP''}$. But

$$\overrightarrow{QP''} = \mathbf{e} \times \mathbf{r} \quad (2.3.10)$$

whereas

$$\overrightarrow{QP} = -\mathbf{e} \times \overrightarrow{QP''} = -\mathbf{e} \times (\mathbf{e} \times \mathbf{r}) \quad (2.3.11)$$

which can readily be proved. Besides, $\overrightarrow{QP'}$ can be expressed as

$$\overrightarrow{QP'} = \overrightarrow{QP}\cos\theta + \overrightarrow{QP''}\sin\theta$$

which, in view of eqs. (2.3.10) and (2.3.11), yields

$$\overrightarrow{QP'} = -\cos\theta\,\mathbf{e} \times (\mathbf{e} \times \mathbf{r}) + \sin\theta\,\mathbf{e} \times \mathbf{r} \quad (2.3.12)$$

Substituting eqs. (2.3.11) and (2.3.12) into eq. (2.3.9) leads to

$$\mathbf{r}' = \mathbf{r} + \mathbf{e} \times (\mathbf{e} \times \mathbf{r}) - \cos\theta\,\mathbf{e} \times (\mathbf{e} \times \mathbf{r}) + \sin\theta\,\mathbf{e} \times \mathbf{r} \quad (2.3.13)$$

But

$$\mathbf{e} \times (\mathbf{e} \times \mathbf{r}) = (\mathbf{e} \cdot \mathbf{r})\mathbf{e} - (\mathbf{e} \cdot \mathbf{e})\mathbf{r} = (\mathbf{e}\mathbf{e} - \mathbf{1})\cdot\mathbf{r} \quad (2.3.14)$$

where $\mathbf{1}$ is the identity dyadic, i.e. a dyadic that is isomorphic to the identity matrix. Furthermore

$$\mathbf{e} \times \mathbf{r} = \mathbf{1} \cdot \mathbf{e} \times \mathbf{r} = \mathbf{1} \times \mathbf{e} \cdot \mathbf{r} \quad (2.3.15)$$

where the dot and the point have been exchanged, what is possible to do by virtue of the algebra of cartesian vectors. Substituting eqs. (2.3.14) and (2.3.15) into eq. (2.3.13) one obtains

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + (1-\cos\theta)(\mathbf{e}\mathbf{e}-\mathbf{1})\cdot\mathbf{r} + \sin\theta\,\mathbf{1} \times \mathbf{e}\cdot\mathbf{r} = \\ &= \left((1-\cos\theta)\mathbf{e}\mathbf{e} + \cos\theta\,\mathbf{1} + \sin\theta\,\mathbf{1} \times \mathbf{e}\right)\cdot\mathbf{r} = \\ &= \mathbf{Q}\cdot\mathbf{r}\end{aligned} \quad (2.3.16)$$

i.e. $\mathbf{r}'$ has been expressed as a linear transformation of vector $\mathbf{r}$. The dyadic $\mathbf{Q}$ is then, isomorphic to the rotation matrix defined in Section 2.2. That is

$$\mathbf{Q} = \mathbf{e}\mathbf{e} + (\mathbf{1}-\mathbf{e}\mathbf{e})\cos\theta + \sin\theta\,\mathbf{1} \times \mathbf{e} \quad (2.3.17)$$

One can now prove the following

<u>THEOREM 2.3.2</u> *Let a rigid body undergo a pure rotation about a fixed point O and let* $\mathbf{r}$ *and* $\mathbf{r}'$ *be the initial and the final position vectors of a point of the body (measured from O) not lying on the axis of rotation*

Furthermore let θ and $\mathbf{e}$ be the angle of rotation and the unit vector pointing in the direction of the rotation. Then

$$\mathrm{sgn}(\mathbf{r}\times\mathbf{r}'\cdot\mathbf{e})=\mathrm{sgn}(\theta)$$

Proof.

Application of eq. (2.3.16) leads to

$$\mathbf{r}\times\mathbf{r}'=(1-\cos\theta)(\mathbf{e}\cdot\mathbf{r})\mathbf{r}\times\mathbf{e}+\sin\theta\,\mathbf{r}\times(\mathbf{e}\times\mathbf{r})=$$

$$=(1-\cos\theta)(\mathbf{e}\cdot\mathbf{r})\mathbf{r}\times\mathbf{e}+\sin\theta\left(r^2\mathbf{e}-(\mathbf{r}\cdot\mathbf{e})\mathbf{r}\right)$$

where

$$r^2=||\mathbf{r}||^2$$

Thus,

$$\mathbf{r}\times\mathbf{r}'\cdot\mathbf{e}=\sin\theta\left(r^2-(\mathbf{r}\cdot\mathbf{e})^2\right)$$

which can be reduced to

$$\mathbf{r}\times\mathbf{r}'\cdot\mathbf{e}=r^2\sin\theta\sin^2(\mathbf{r},\mathbf{e})$$

where $(\mathbf{r},\mathbf{e})$ is the angle between vectors $\mathbf{r}$ and $\mathbf{e}$. Hence,

$$\mathrm{sgn}(\mathbf{r}\times\mathbf{r}'\cdot\mathbf{e})=\mathrm{sgn}(\sin\theta)$$

But

$$\mathrm{sgn}(\sin\theta)=\mathrm{sgn}(\theta)$$

for sin( ) is an odd function, i.e. $\sin(-x)=-\sin(x)$.

Finally, then

$$\mathrm{sgn}(\mathbf{r}\times\mathbf{r}'\cdot\mathbf{e})=\mathrm{sgn}(\theta),\ \text{q.e.d} \tag{2.3.18}$$

In conclusion, Theorem 2.3.2 allows to distinguish whether a rotation in the specified direction $\mathbf{e}$ is either through an angle θ or through an angle $-\theta$.

Exercise 2.3.2 Let $\mathbf{p}$ and $\mathbf{p}'$ be the initial and the final position vectors of a point P of a rigid body undergoing a screw motion whose rotation matrix is $\mathbf{Q}$. Show that the displacement $\mathbf{p}'-\mathbf{Q}\mathbf{p}$ lies in the null space of $\mathbf{Q}-\mathbf{I}$.

Exercise 2.3.3 Show that the trace of a matrix is invariant under similarity transformations.

Exercise 2.3.4 Show that a revolute matrix $\mathbf{Q}$ has two complex conjugate eigenvalues, λ and $\bar{\lambda}$ ($\bar{\lambda}$ = complex conjugate of λ).

Furthermore, show that

$$\mathrm{Re}\{\lambda\}=\frac{1}{2}\,(\mathrm{Tr}\mathbf{Q}-1)$$

What is the relationship between the complex eigenvalues of the revolute matrix and its angle of rotation?.

In the foregoing paragraphs the revolute matrix was analyzed, i.e. it was shown how to obtain its invariants when the matrix is known. The inverse problem is discussed next: Given the axis and the angle of rotation, obtain the revolute matrix referred to a specified set of coordinate axes.

It is apparent that the most convenient basis (or coordinate axes) for representing the revolute matrix is the one for which this takes on its canonical form. Let $B=\{\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}$ be this basis, where $\mathbf{b}_3$ coincides with the given revolute axis, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are any pair of orthonormal vectors lying in the plane perpendicular to $\mathbf{b}_3$.

Hence, $(\mathbf{Q})_B$ appears as in eq. (2.3.5), with given θ. Let $A=\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ be an othonormal basis with respect to which $\mathbf{Q}$ is to be represented, and let

$$(\underset{\sim}{P})_A = (\underset{\sim}{b}_1 \;\vdots\; \underset{\sim}{b}_2 \;\vdots\; \underset{\sim}{b}_3)_A$$

be a matrix formed with the vectors of B. Then, it is clear that

$$(\underset{\sim}{Q})_A = (\underset{\sim}{P}^T)_A (\underset{\sim}{Q})_B (\underset{\sim}{P}^T)_A$$

<u>Example 2.3.1</u> Let

$$\underset{\sim}{Q} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

Verify whether it is orthogonal. If it is, does it represent a rotation? If so, describe the rotation

<u>Solution:</u>

$$\underset{\sim}{Q}\underset{\sim}{Q}^T = \frac{1}{9}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \underset{\sim}{I}$$

Hence $\underset{\sim}{Q}$ is in fact orthogonal. Next,

$$\det \underset{\sim}{Q} = \frac{2}{3}\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{vmatrix} + \frac{2}{3}\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{vmatrix} - \frac{1}{3}\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \frac{2}{3}\left(\frac{4}{9} + \frac{2}{9}\right) + \frac{2}{3}\left(\frac{2}{9} + \frac{4}{9}\right) - \frac{1}{3}\left(\frac{1}{9} - \frac{4}{9}\right) = +1$$

Thus $\underset{\sim}{Q}$ is a proper orthogonal matrix and consequently represents a rotation.

To find the axis of the rotation it is necessary to find a unit vector $\underset{\sim}{e} = (e_1, e_2, e_3)^T$ such that

$$\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e},$$

i.e.,

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

Hence

$$\begin{aligned} -e_1 + e_2 + 2e_3 &= 0 \\ -2e_1 - e_2 + e_3 &= 0 \\ -e_1 - 2e_2 - e_3 &= 0 \end{aligned}$$

from which

$$e_1 = e_3$$

$$e_2 = -e_3$$

and so

$$\underset{\sim}{e} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e_3.$$

Setting $||\underset{\sim}{e}|| = 1$, it follows that $e_3 = \frac{\sqrt{3}}{3}$, and

$$\underset{\sim}{e} = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, the axis of rotation is parallel to the vector $\underset{\sim}{e}$ given above.

To find the angle of rotation is an even simpler matter:

$$\operatorname{Tr} \underset{\sim}{Q} = \frac{1}{3}(2+2+2) = 1 + 2\cos\theta$$

Thus $\theta = \cos^{-1}\left(\frac{1}{2}\right) = -60°$

where use was made of Theorem 2.3.2 to find the sign of θ.

<u>Example 2.3.2.</u> Determine the revolute matrix representing a rotation of 90° about an axis having three equal direction cosines with respect to the X,Y,Z axes. The matrix should be expressed with respect to these axes.

<u>Solution:</u>

Let $\underline{B} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3\}$ be an orthonormal basis with respect to which the revolute is represented in its canonical form. Let $\underline{b}_3$ be coincident with the axis of rotation. Clearly

$$\underline{b}_3 = \frac{\sqrt{3}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

It remains only to determine $\underline{b}_1$ and $\underline{b}_2$. Clearly, these must satisfy

$$\underline{b}_1 \cdot \underline{b}_2 = \underline{b}_1 \cdot \underline{b}_3 = \underline{b}_2 \cdot \underline{b}_3 = 0.$$

Let

$$(\underline{b}_1) = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}, \quad (\underline{b}_2) = \begin{pmatrix}\lambda\\\mu\\\nu\end{pmatrix}.$$

Thus, the components of $\underline{b}_1$ must satisfy

$$\alpha + \beta + \gamma = 0,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

It is apparent that one component can be freely chosen. Let, for example,

$$\alpha = 0$$

Hence,

$$\beta + \gamma = 0$$
$$\beta^2 + \gamma^2 = 1$$

from which

$$2\beta^2 = 1. \quad \text{Thus } \beta = \frac{\pm\sqrt{2}}{2}, \gamma = \mp\frac{\sqrt{2}}{2}$$

Thus, choosing the + sign for β,

$$\underline{b}_1 = \frac{1}{2}\begin{pmatrix}0\\\sqrt{2}\\\sqrt{2}\end{pmatrix}$$

$\underline{b}_2$ can be obtained now very easily from the fact that $\underline{b}_1, \underline{b}_2$ and $\underline{b}_3$ constitute an orthonormal right-hand triad, i.e.

$$\underline{b}_2 = \underline{b}_3 \times \underline{b}_1 = \frac{\sqrt{6}}{6}(-2,1,1)^T$$

With respect to this basis, then, from eq. (2.3.5) the rotation matrix has the form

$$(\underline{Q})_B = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

Thus, letting A be the basis defined by the given X, Y and Z axes,

$$(P)_A = \begin{bmatrix}0 & -\sqrt{6}/3 & \sqrt{3}/3\\\sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3\\-\sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3\end{bmatrix}$$

and, from eq. (1.5.12), defining the following similarity transformation,

$$(\underline{Q})_A = (\underline{P})_A (\underline{Q})_B (\underline{P}^{-1})_A$$

With $(\underline{Q})_B$ in its canonical form, the revolute matrix $\underline{Q}$, expressed with respect to the X,Y,Z axes, is found to be

$$(\underline{Q})_A = \frac{1}{2}\begin{bmatrix}1 & 1-\sqrt{3} & 1+\sqrt{3}\\1+\sqrt{3} & 1 & 1-\sqrt{3}\\1-\sqrt{3} & 1+\sqrt{3} & 1\end{bmatrix}$$

<u>Exercise 2.3.6</u> If the plane

$$x + y + z + 1 = 0$$

is rotated through 60° about an axis passing through the point (-1, -1, -1) and with direction cosines $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, what is the equation of the plane in its new position?.

<u>Exercise 2.3.7</u>. The four vertices of an equilateral tetrahedron are labelled A, B, C, and D. If the tetrahedron is rotated in such a way that A, B, C, and D are mapped into C, B, D, and A, respectively, find the axis and the angle of the rotation.

What are the other rotations similar to the previous one, i.e., which map every vertex of the tetrahedron into another vertex?.

All these rotations, together with the identity rotation (the one leaving the vertices of the tetrahedron unchanged), constitute the <u>symmetry group</u>* of the tetrahedron.

<u>Exercise 2.3.8</u> Given an axis A whose direction cosines are $(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})$, with respect to a set of coordinate axes XYZ, what is the matrix representation, with respect to these coordinate axes, of a rotation about A through an angle $2\pi/n$?

<u>Exercise 2.3.9</u> A square matrix $\underline{A}$ is said to be idempotent of index k whenever k is the smallest integer for which the $k^{\underline{th}}$ power of $\underline{A}$ becomes the identity matrix. Explain why the matrix obtained in Exercise 2.3.8 should be idempotent of index n.

<u>Exercise 2.3.10</u> Show that any rotation matrix $\underline{Q}$ can be expressed as

$$\underline{Q} = e^{\underline{A}\theta}$$

where $\underline{A}$ is a nilpotent matrix and θ is the rotation angle. What is the relationship between matrix $\underline{A}$ and the axis of rotation of $\underline{Q}$?

---

*See Sect. 2.4 for the definition of this term.

<u>Exercise 2.3.11</u> The equation of a three-axes ellipsoid is given as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

what is its equation after rotating it through an angle ϕ about an axis of direction numbers (a,b,c)?

2.4 <u>GROUPS OF ROTATIONS</u>.

A group is a set $\mathcal{G}$ with a binary operation θ such that

i) if a and b $\in \mathcal{G}$, then $a\theta b \in \mathcal{G}$

ii) if a,b,c then $a\theta(b\theta c) = (a\theta b)\theta c$

iii) $\mathcal{G}$ contains an element i, called the <u>identity</u> of $\mathcal{G}$ under θ, such that, for every $a \in \mathcal{G}$

$$a\theta i = i\theta a = a$$

iv) for every $a \in \mathcal{G}$, there exists an element denoted $a^{-1} \in \mathcal{G}$, called the <u>inverse</u> of a under θ such that

$$a\theta a^{-1} = a^{-1}\theta a = i$$

Notice that in the above definition it is not required that the group be commutative, i.e. that $a\theta b = b\theta a$ for all $a,b \in \mathcal{G}$. Commutative groups are a special class of groups, called abelian groups.

Some examples of groups are:

a) The natural numbers 1,2,..., 12 on the face of a (mechanical, not quartz or similar) clock and the operation $k\theta m$ corresponding to "shift the clock hand from location k to location k + m", where k and m are natural numbers between 1 and 12. Of course, if k + m>12, the resulting operation is meant to be (k + m)(mod 12).

b) The set of rational numbers with the usual multiplication operation.

c) The set of integers with the usual algebraic addition operation.

The set of integers with the multiplication operation do not constitute a group (Why?)

Exercise 2.4.1 Show that the set of all the rotations referred to in Exercise 2.3.5 actually constitute a group.

Exercise 2.4.2 What is the symmetry group* of

i) an icosahedron?

ii) a regular pentagonal prism?

iii) a circular cylinder?

iv) a sphere?

It is clear, from the above discussion, that the set of all orthogonal matrices constitute a group under matrix multiplication. In particular, the set of proper orthogonal matrices constitutes a group under matrix multiplication, but the improper set does not (Why?).

As an application of the group property of rotations or, equivalently, of proper orthogonal matrices, arbitrary rotations can be formed by the composition of successive simple rotations (See Example 2.4.1).

Another application is found in the compositon of rotations using Euler's angles (2.2)

Example 2.4.1 Referring to Fig 2.4.1, find the matrix representation, with respect to the X_1, Y_1, Z_1 axes, of the rotation that carries vertices A and B of the cube into A' and B', respectively, while leaving vertex O fixed. A' and B' lie in the Y_1Z_1 plane and points A', O and D, are collinear, as are B', F and E.

---

* See Exercise 2.3.5 for a definition of a symmetry group.

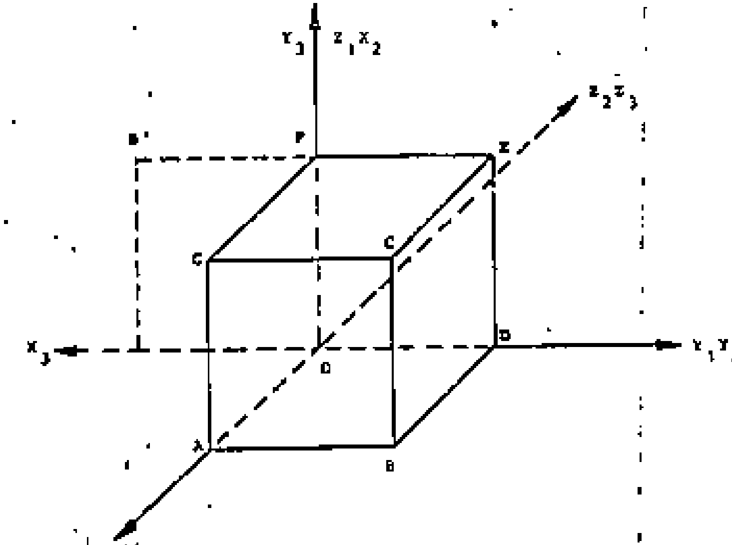


Fig 2.4.1 Rotation of a cube

Solution:

Let $(Q_{12})_1$ be the matrix representing the rotation of axes labelled 1 into those labelled 2 (referred to axes 1). Then, letting $\mathbf{x}_1$, $\mathbf{y}_1$ and $\mathbf{z}_1$ be unit vectors directed along the X_1-, Y_1- and Z_1 axes, respectively,

$$Q_{12}\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{z}_1$$

$$Q_{12}\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{y}_1$$

$$Q_{12}\mathbf{z}_1 = \mathbf{z}_2 = -\mathbf{x}_1$$

from which

$$(Q_{12})_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Next, rotate axes labelled 2 into axes labelled 3. Call this rotation Q_{23}. This rotation would leave axis X_1 fixed whereas it would carry axis Y_1 into Z_1 and axis Z_1 into $-Y_1$. Hence,

$$\underset{\sim}{Q}_{23}\underset{\sim}{x}_1 = \underset{\sim}{x}_1$$

$$\underset{\sim}{Q}_{23}\underset{\sim}{y}_1 = \underset{\sim}{z}_1$$

$$\underset{\sim}{Q}_{23}\underset{\sim}{z}_1 = \underset{\sim}{y}_1$$

and so,

$$(Q_{23})_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $\underset{\sim}{Q}_{13}$ be the rotation meant to be obtained. Its matrix can be computed then as

$$(Q_{13})_1 = (Q_{23})_1(Q_{12})_1 = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

which could also have been obtained by noticing that

$$\underset{\sim}{Q}_{13}\underset{\sim}{x}_1 = \underset{\sim}{x}_3 = -\underset{\sim}{y}_1$$

$$\underset{\sim}{Q}_{13}\underset{\sim}{y}_1 = \underset{\sim}{y}_3 = \underset{\sim}{z}_1$$

$$\underset{\sim}{Q}_{13}\underset{\sim}{z}_1 = \underset{\sim}{z}_3 = -\underset{\sim}{x}_1$$

Matrix $(Q_{13})_1$ represents a rotation through an angle $\theta = 120°$ about an axis with direction cosines $-a,a,a$. Although in this example the rotation could be obtained by an alternate method, in many cases, such as the one in Exercise 2.4.3, the use of rotation composition seems to be the simplest method.

<u>Exercise 2.4.3</u> Determine the axis and the angle of the rotation carrying axes X,Y,Z into axes ξ,η,ζ, as shown in Fig. 2.4.2

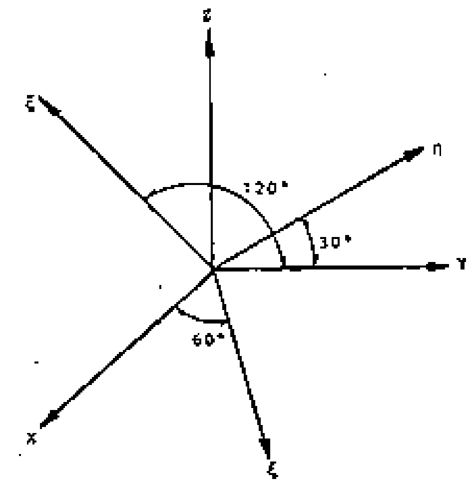


Fig 2.4.2 Rotation of axes

<u>Exercise 2.4.4</u> The cube appearing in Fig 2.4.1 is rotated 45° about diagonal OC. Find the matrix representation, with respect to X_1,Y_1,Z_1, of this rotation and the distance that vertex B is displaced through.

2.5 <u>RODRIGUES' FORMULA AND CARTESIAN DECOMPOSITION OF THE ROTATION MATRIX.</u>

The image $\underset{\sim}{r}_2$ of a Cartesian vector $\underset{\sim}{r}_1$ under a rotation through an angle θ about an axis parallel to the unit vector $\underset{\sim}{e}$ passing through the origin of coordinates was shown to be (See Section 2.3)

$$\underset{\sim}{r}_2 = \left[(1-\cos\theta)\underset{\sim}{e}\underset{\sim}{e}+\cos\theta\underset{\sim}{1}+\sin\theta\underset{\sim}{1}\times\underset{\sim}{e}\right]\cdot\underset{\sim}{r}_1 \qquad (2.5.1)$$

multiplying both sides of eq. (2.5.1) times $\underset{\sim}{e}\times$ yields

$$\underset{\sim}{r}_2-\underset{\sim}{r}_1 = \tan\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{r}_1+\underset{\sim}{r}_2) \qquad (2.5.2)$$

which is called <u>Rodrigues' formula</u> (2.3,2.4)

Form (2.5.1) of the rotation dyadic is advantageous since it shows explicitly the invariants $\underset{\sim}{e}$ and θ of the rotation.

Other useful expression of the rotation matrix is now derived. Letting

$$\mathbf{e} = (u, v, w)^T \qquad (2.5.3)$$

the rotation matrix can be written as (2.5)

$$\mathbf{Q} = \mathbf{R} + \mathbf{T}\cos\theta + \mathbf{P}\sin\theta \qquad (2.5.4)$$

where

$$\mathbf{R} = \begin{bmatrix} u^2 & uv & uw \\ uv & v^2 & uw \\ uw & vw & w^2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} v^2+w^2 & -uv & -uw \\ -uv & u^2+w^2 & -vw \\ -uw & -vw & u^2+v^2 \end{bmatrix} \qquad (2.5.5a)$$

and

$$\mathbf{P} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \qquad (2.5.5b)$$

In fact, computing the dyadics involved in expression (2.3.17),

$$\begin{aligned} \mathbf{ee} &= (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = \\ &= u^2\mathbf{ii} + uv\mathbf{ij} + uw\mathbf{ik} + \\ &+ uv\mathbf{ji} + v^2\mathbf{jj} + vw\mathbf{jk} + \\ &+ uw\mathbf{ki} + vw\mathbf{kj} + w^2\mathbf{kk} \end{aligned} \qquad (2.5.6)$$

$$\mathbf{1} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} \qquad (2.5.7)$$

Hence

$$\begin{aligned} \mathbf{1} \times \mathbf{e} &= (\mathbf{ii} + \mathbf{jj} + \mathbf{kk}) \times (u\mathbf{i}+v\mathbf{j}+w\mathbf{k}) = \\ &= u\mathbf{i}\mathbf{i}\times\mathbf{i} + v\mathbf{i}\mathbf{i}\times\mathbf{j} + w\mathbf{i}\mathbf{i}\times\mathbf{k} + \\ &+ u\mathbf{j}\mathbf{j}\times\mathbf{i} + v\mathbf{j}\mathbf{j}\times\mathbf{j} + w\mathbf{j}\mathbf{j}\times\mathbf{k} + \\ &+ u\mathbf{k}\mathbf{k}\times\mathbf{i} + v\mathbf{k}\mathbf{k}\times\mathbf{j} + w\mathbf{k}\mathbf{k}\times\mathbf{k} \end{aligned} \qquad (2.5.8)$$

But

$$\mathbf{i} \times \mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0} \qquad (2.5.9)$$

and

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j} \qquad (2.5.10)$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \qquad (2.5.11)$$

Thus

$$\begin{aligned} \mathbf{1} \times \mathbf{e} &= -w\mathbf{ij} + v\mathbf{ik} \\ &+ w\mathbf{ji} - u\mathbf{jk} \\ &- v\mathbf{ki} + u\mathbf{kj} \end{aligned}$$

Dyadics (2.5.6) and (2.5.7) can be written in matrix form as

$$(\mathbf{ee}) = \begin{bmatrix} u^2 & uv & uw \\ uv & v^2 & uv \\ uw & vw & w^2 \end{bmatrix}, \quad (\mathbf{1}\times\mathbf{e}) = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \qquad (2.5.12)$$

and

$$(\mathbf{1} - \mathbf{ee}) = \begin{bmatrix} v^2+w^2 & -uv & -uv \\ -uv & u^2+w^2 & -vw \\ -uw & -uw & u^2+v^2 \end{bmatrix} \qquad (2.5.13)$$

Substitution of matrices (2.5.12) and (2.5.13) into eq. (2.3.17) leads directly to eq. (2.5.4). This expression of matrix $\mathbf{Q}$ is very useful because it allows one to determine the sign of θ without requiring to compute the image $\mathbf{r}'$ of a vector $\mathbf{r}$ under $\mathbf{Q}$.

Indeed, from eqs. (2.5.5a) and (2.5.5b), it is clear that matrices $\mathbf{R}$ and $\mathbf{T}$ are symmetric, whereas $\mathbf{P}$ is skew symmetric. Hence, and from Theorem 1.7.1, $\mathbf{P}\sin\theta$ can be obtained as

$$\mathbf{P}\sin\theta = \frac{1}{2}(\mathbf{Q} - \mathbf{Q}^T) \qquad (2.5.14)$$

i.e. eq. (2.5.4) can be regarded as the cartesian decomposition (see Sect. 1.7) of matrix $\mathbf{Q}$. Now, calling e_i the ith component of vector $\mathbf{e}$, as given by eq. (2.5.3) and taking definition (2.5.5b) and eq. (2.5.14) into account, one obtains

$$-e_1 \sin\theta = \frac{1}{2}(q_{23}-q_{32}) \tag{2.5.15a}$$

$$-e_2 \sin\theta = \frac{1}{2}(q_{13}-q_{31}) \tag{2.5.15b}$$

$$-e_3 \sin\theta = \frac{1}{2}(q_{12}-q_{21}) \tag{2.5.15c}$$

Introducing the alternating tensor ϵ_{ijk} defined as

$$\epsilon_{ijk} = \begin{cases} +1, \text{if } i,j \text{ and } k \text{ are in cyclic order} \\ -1, \text{if } i,j \text{ and } k \text{ are in acyclic order} \\ 0, \text{if at least one index is repeated} \end{cases}$$

eqs. (2.5.15) can be written as

$$e_i \sin\theta = -\frac{1}{2}\epsilon_{ijk}(q_{jk}-q_{kj})$$

from which, if e_i does not vanish,

$$\text{sgn}\theta = \epsilon_{ijk}\, \text{sgn}\left(\frac{q_{kj}-q_{jk}}{e_i}\right)$$

follows directly.

<u>Exercise 2.5.1</u> Given matrices $\tilde{T}$ and $\tilde{P}$, as defined in eqs. (2.5.5a) and (2.5.5b), prove that $\tilde{T} = -\tilde{P}^2$ and device an algorithm to compute $\tilde{P}$ given $\tilde{T}$.

<u>Exercise 2.5.2</u> Use eq. (2.5.16) to determine the sign of θ for the rotation matrix of Example 2.3.1 and verify the result thus obtained with the one obtained previously.

2.6 GENERAL MOTION OF A RIGID BODY AND CHASLES' THEOREM

In the previous sections only the motion of a rigid body about a fixed point was discussed. There are rigid body motions, however, with no fixed point. Such motions are studied in this section.

Consider a motion under which one point is displaced from A to A' and another one is displaced from B to B', as shown in Fig 2.6.1

This motion can take place in any of three different ways, namely i) any pair of points A,B of the body undergo a displacement to A', B', respectively in such a way that line A' B' is parallel to line AB; this motion is referred to as <u>pure translation</u>; ii) a line of the body remains fixed, in which case, according to Euler's Theorem (Theorem 2.3.1), the motion is referred to as <u>pure rotation</u>;

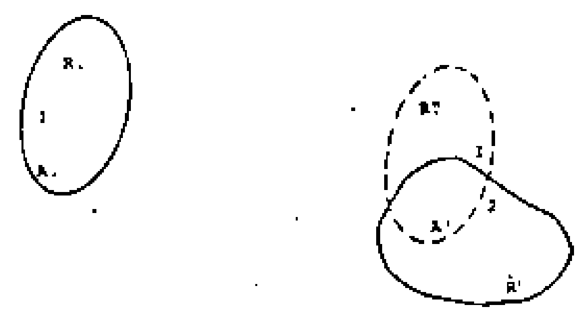


Fig 2.6.1 General motion of a rigid body

iii) no point of the body remains fixed under the motion, in which case it is referred to as general motion.

The motion from configuration 1 to configuration 2 can be regarded as the composition of two motions: first the rigid body is displaced from 1 to I without any rotation. Hence, the lines connecting any pair of points in 1 are parallel to those connecting them in the intermediate configuration I. Since this is a rigid body motion, the length of each segment remains unchanged. Thus, letting $\mathbf{a}, \mathbf{a}'$, $\mathbf{r}, \mathbf{r}'$ and $\mathbf{r}''$ be the position vectors of points A, A', R, R' and R", respectively,

$$\mathbf{r}''-\mathbf{a}'=\mathbf{r}-\mathbf{a} \tag{2.6.1}$$

Next, to take the body into its final configuration, 2, a rigid body rotation $\mathbf{Q}$, about point A', must be performed.

Thus,

$$\mathbf{r}'-\mathbf{a}'=\mathbf{Q}(\mathbf{r}''-\mathbf{a}') \tag{2.6.2}$$

Substitution of (2.6.1) into (2.6.2) and rearrangement of the terms yield

$$\mathbf{r}'=\mathbf{a}'+\mathbf{Q}(\mathbf{r}-\mathbf{a}) \tag{2.6.3}$$

which is an expression for the final position R' of any point R of the rigid body in terms of: i) its initial position, $\mathbf{r}$, ii) the initial and the final position of any other point A, and iii) the rotation $\mathbf{Q}$ accompanying the motion. The above expression could have also been obtained considering first a rigid body rotation about point A from 1 to an intermediate configuration I' in which all lines connecting any pair of points are parallel to the corresponding lines in 2 and, since the motion is rigid, the segments thus defined are of equal lengths; then, perform a pure translation from I' to 2. Summarizing: The general motion of a rigid body is completely defined by the initial and final positions of any one of its points and the rotation involved.

Exercise 2.6.1 Obtain eq. (2.6.3) by performing first a rotation and then a translation.

The main result in this section is Chasles' Theorem, which states that, given any rigid body displacement, it can always be obtained as the rotation about a line of the body, known as "the screw axis", followed by a translation parallel* to the rotation axis. Moreover, the displacements of all points of the body along the screw axis are of minimum magnitude. The displacement vector of a point is defined as the vector between the final and the initial positions of the point, e.g. the displacement of point R in the previous discussion is

$$\mathbf{u}=\mathbf{r}'-\mathbf{r}=\mathbf{a}'+\mathbf{Q}(\mathbf{r}-\mathbf{a})-\mathbf{r}=$$
$$=\mathbf{a}'-\mathbf{Q}\mathbf{a}+(\mathbf{Q}-\mathbf{I})\mathbf{r} \tag{2.6.3a}$$

From eq. (2.6.3) notice that $\mathbf{u}$ is a linear function of one single variable, $\mathbf{r}$. Hence, the norm of $\mathbf{u}$ is a linear function of $\mathbf{r}$ only. The square of this norm is quadratic in $\mathbf{r}$ and is given as

$$\phi(\mathbf{r})=\mathbf{u}^T\mathbf{u}=\mathbf{r}^T(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{r}+2(\mathbf{a}'-\mathbf{Q}\mathbf{a})^T(\mathbf{Q}-\mathbf{I})\mathbf{r}+(\mathbf{a}'-\mathbf{Q}\mathbf{a})^T(\mathbf{a}'-\mathbf{Q}\mathbf{a}) \tag{2.6.4}$$

The theorem is now proved via the minimization of $\phi(\mathbf{r})$. This function has one extremum at the point $\mathbf{r}_0$ where $\phi'(\mathbf{r}_0)=\mathbf{0}$. The derivative $\phi'(\mathbf{r})$ is next computed, and zeroed at $\mathbf{r}_0$.

Applying the "chain rule" to ϕ,

$$\phi'(\mathbf{r}) = \left(\frac{\partial \mathbf{u}}{\partial \mathbf{r}}\right)^T\frac{\partial \phi}{\partial \mathbf{u}},$$

* The direction of a pure translation of a rigid body is understood here as the direction of the displacement vectors of the points of the body.

where, from eq. (2.6.3a),

$$\frac{\partial \mathbf{u}}{\partial \mathbf{r}} = \mathbf{Q}-\mathbf{I} \qquad (2.6.5a)$$

and

$$\frac{\partial \phi}{\partial \mathbf{u}} = 2\mathbf{u} \qquad (2.6.5b)$$

Thus, letting $\mathbf{u}_0=\mathbf{u}(\mathbf{r}_0)$, the zeroing of the gradient of ϕ at $\mathbf{r}=\mathbf{r}_0$ leads to

$$(\mathbf{Q}-\mathbf{I})^T\mathbf{u}_0=\mathbf{0} \qquad (2.6.6)$$

or

$$\mathbf{Q}^T\mathbf{u}_0=\mathbf{u}_0 \qquad (2.6.6a)$$

Now, if both sides of eq. (2.6.6) are multiplied by $\mathbf{Q}$, one obtains

$$\mathbf{Q}\mathbf{u}_0=\mathbf{u}_0 \qquad (2.6.6b)$$

thereby concluding that the minimum-norm displacement $\mathbf{u}_0$ lies in the real spectral space of $\mathbf{Q}$, i.e., it is parallel to the axis of rotation of $\mathbf{Q}$. What is now left to complete the proof of Chasles' Theorem is to determine the set of points of the rigid body having a displacement vector parallel to the rotation axis. This is done next.

Substituting $\mathbf{u}$, evaluated at $\mathbf{r}_0$, as given by eq. (2.6.3a) into eq. (2.6.6), and rearranging terms leads to

$$(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{r}_0=(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}\mathbf{a}-\mathbf{a}') \qquad (2.6.6c)$$

from which $\mathbf{r}_0$ cannot be solved for, since $(\mathbf{Q}-\mathbf{I})$, and hence $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$ is singular. In fact, it can be readily proved that this matrix is of rank 2.

<u>Exercise 2.6.1</u> Prove that $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$ is of rank 2, except for $\mathbf{Q}=\mathbf{I}$.

Although $\mathbf{r}_0$ cannot be solved for from the latter equation, interesting results can be derived from it. Indeed, given a point R_0, with position vector $\mathbf{r}_0$, of minimum-magnitude displacement $\mathbf{u}_0$, define a new point S_0, with position vector $\mathbf{s}_0$ given as

$$\mathbf{s}_0 = \mathbf{r}_0 + \alpha\mathbf{e}$$

where $\mathbf{e}$ is the unit vector parallel to the axis of rotation of $\mathbf{Q}$. Multiplying $\mathbf{s}_0$, as given before, times $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$ gives

$$(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{s}_0=(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})(\mathbf{r}_0+\alpha\mathbf{e})=(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{r}_0+ (\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{e}$$

but $\mathbf{e}$, being parallel to the rotation axis of $\mathbf{Q}$, is in the null space of $\mathbf{Q}-\mathbf{I}$, hence, in the null space of $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$. Therefore, the second term in the right-hand side of the latter equation vanishes, the latter equation thus reducing to

$$(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\,\mathbf{s}_0 = (\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}\mathbf{a}-\mathbf{a}')$$

i.e, $\mathbf{s}_0$ also satisfies eq. (2.6.6c). In conclusion, <u>all points S_0 of minimum-magnitude displacement, $\mathbf{u}_0$, lie on a line parallel to the axis of rotation of $\mathbf{Q}$.</u>

<u>Exercise 2.6.2</u>. Show that the $\mathbf{r}_0$ satisfying eq. (2.6.6) actually yields a minimum.

From Exercise 2.6.1, if $\mathbf{Q}\neq\mathbf{I}$, the rank of $\mathbf{Q}-\mathbf{I}$ is exactly 2. Therefore, two of the three scalar equations of (2.6.5) are linearly independent. These two equations can be expressed in matrix form as

$$\mathbf{A}\mathbf{r}_0 = \mathbf{c} \qquad (2.6.7)$$

where $\mathbf{A}$ is a 2 x 3-rank-two matrix and vectors $\mathbf{r}_0$ and $\mathbf{c}$ are 3-and 2-dimensional, respectively. Now, since the rank of $\mathbf{A}$ is 2, $\mathbf{A}\mathbf{A}^T$, being 2x2, is nonsingular and hence, the minimum norm solution to eq. (2.6.7) is (See Section 1.11)

$$\mathbf{r}_0=\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{c} \qquad (2.6.8)$$

The geometric interpretation of the previous result is that $\mathbf{r}_0$, as given by (2.6.8), is perpendicular to the sought axis. This axis is "<u>the screw</u>

axis" and is totally determined by the rotation axis, which gives its direction, and the point r_0 whose position vector, r_0, is given by eq. (2.6.8). The name "screw" comes from the fact that the body moves as if it were fastened to the bolt of a screw whose axis were the screw axis. Other facts motivating the name of the screw axis will be shown later.

Another method of finiding a point on the screw axis is via Rodrigues' formula as it appears in (2.6). This procedure can be developed as follows: As was pointed out from eq. (2.6.6), the minimum-norm displacement is parallel to the axis of rotation. Hence, the displacement of R_0 must satisy

$$u_0 = r_0' - r_0 = \alpha e \tag{2.6.9}$$

where α is a scalar. Substituting the initial and the final position vectors of R in Rodrigues' formula, eq. (2.5.3),

$$r_0' - r_0 = \tan\frac{\theta}{2} e \times (r_0' + r_0) \tag{2.6.10}$$

which, together with eq. (2.5.3) for vectors a and a', denoting the initial and the final positions of point A, yields

$$a' - r_0' - (a - r_0) = \tan\frac{\theta}{2} e \times \left((a' - r_0') + (a - r_0)\right) \tag{2.6.11}$$

From eq. (2.6.9),

$$e \times (r_0' - r_0) = 0 \tag{2.6.12}$$

Hence, eq. (2.6.11) becomes

$$a' - a - \alpha e = \tan\frac{\theta}{2} e \times (a' + a) - 2\tan\frac{\theta}{2} e \times r_0 \tag{2.6.13}$$

Multiplying both sides of eq. (2.6.13) times $\cot\frac{\theta}{2} e \times$,

$$\cot\frac{\theta}{2} e \times (a' - a) = e \times \left(e \times (a' + a)\right) - 2 e \times (e \times r_0) =$$

$$= e \times \left(e \times (a' + a)\right) - 2(e \cdot r_0)e + 2(e \cdot e) r_0 \tag{2.6.14}$$

To determine r_0 from eq. (2.6.14), it is necessary to impose one extra

condition on it, which is done next. Let it be the particular point on the screw axis which is closet to the origin; hence,

$$r_0 \cdot e = 0$$

and so, substituting this vector into eq. (2.6.14) and solving for r_0 in the same equation, leads to

$$r_0 = \frac{1}{2}\cot\frac{\theta}{2} e \times (a' - a) - \frac{1}{2} e \times \left(e \times (a' + a)\right) \tag{2.6.15}$$

which is an alternate expression for r_0. The foregoing result is summarized next.

<u>THEOREM 2.6.1 (CHASLES)</u>. *The most general displacement of a rigid body is equivalent to a translation together with a rotation about an axis parallel to the translation.*

Alternatively, Chasles' Theorem can be stated as follows:

"Given an arbitrary displacement of a rigid body, there exists a set of points of the body, constituting a line, such that all points on that line undergo a displacement parallel to the line, which is of minimum Euclidean norm"

A property of the screw axis is established in the next theorem.

<u>THEOREM 2.6.2</u> *The displacement vectors of all the points of a rigid body undergoing an arbitrary motion have the same projection along the screw axis.*

Proof:

Let P be an arbitrary point of a rigid body and S a point on the screw axis; let P' and S' represent the corresponding points after the displacement. From eq. (2.6.3), the displacement of P, u_p, is given in terms of the position vectors of P, S and S', by

$$u_p = s' - Qs + (Q - I)p \tag{2.6.16}$$

The projection of $\underline{u}_p$ onto the screw axis is computed now by obtaining the scalar product of $\underline{u}_p$ times $\underline{u}_s$. From eq. (2.6.16) this becomes

$$\underline{u}_p^T\underline{u}_s=(\underline{s}'-\underline{Q}\underline{s})^T\underline{u}_s+\underline{p}^T(\underline{Q}-\underline{1})^T\underline{u}_s \qquad (2.6.17)$$

where the second term on the right hand side vanishes because, as already shown, $\underline{u}_s$ is an eigenvector of $\underline{Q}$ and $\underline{Q}^T$. Thus, eq. (2.6.17) becomes

$$\underline{u}_p^T\underline{u}_s=(\underline{s}')^T\underline{u}_s-\underline{s}^T\underline{u}_s=(\underline{s}'-\underline{s})^T\underline{u}_s=\underline{u}_s^T\underline{u}_s$$

From the above expression it follows that the projection of $\underline{u}_p$ onto the screw axis has lenght $||\underline{u}_s||$, q.e.d

Using the same notation as above, the final position vector of a point of a rigid body undergoing an arbitrary motion and its displacement can be expressed as

$$\underline{p}'=\underline{p}+\underline{u}_p \qquad (2.6.18)$$

$$\underline{u}_p=\underline{u}_s+(\underline{Q}-\underline{1})(\underline{p}-\underline{s}) \qquad (2.6.19)$$

Exercise 2.6.2 Derive eq. (2.6.19)

Hence it is clear that the displacement of any point of the rigid body is known if the following quantities are given:

i) The magnitude of the screw displacement, $||\underline{u}_s||$

ii) One point of the screw axis, R_0, whose position vector is $\underline{r}_0$

iii) The axis of rotation, $\underline{e}$

iv) The angle of rotation, ϕ

Given the above data, vector $\underline{u}_s$ is obtained as

$$\underline{u}_s=||\underline{u}_s||\underline{e} \qquad (2.6.20)$$

and matrix $\underline{Q}$ is given by eqs. (2.5.1) or (2.5.4). Point R_0 and vector $\underline{e}$ completely determine the screw axis, henceforth called L. From Theorem 2.6.2 it is clear that a rigid body undergoing an arbitrary motion, moves as if it were welded to the bolt of a screw whose axis where L and whose pitch were given by

$$\lambda=\frac{2\pi||\underline{u}_s||}{\phi} \qquad (2.6.21)$$

For this reason, the pair (L,$\underline{Q}$), which completely determines a rigid body motion, is called a "screw", and rigid-body motions are thus referred to as "screw motions". It was shown in section 2.3 how to obtain the matrix $\underline{Q}$, given a rigid body motion with a fixed point. Vectors $\underline{r}_0$ and $\underline{u}_s$, which define L, are obtained from eqs. (2.6.15) or, alternatively, from eq. (2.6.7) and eq. (2.6.20).

The following interesting result is derived immediately from Theorem 2.6.2

Corollary 2.6.1 *A rigid body motion is a rotation about a fixed point if and only if the displacement of one point of the body is perpendicular to the screw axis of the motion.*

Another useful result is the following

Corollary 2.6.2 *The difference vector of the displacements of any two of the points of a rigid body undergoing an arbitrary motion is perpendicular to the screw axis.*

Exercise 2.6.3 Prove corollaries 2.6.1 and 2.6.2

Clearly, the motion of any one plane of a rigid body completely determines the motion of the body. Futhermore, three noncollinear points determine a plane; thus it follows that the motion of any three noncollinear points of a rigid body determine the motion of the body. In other words, knowing the initial and the final positions of three noncollinear points of a rigid body, one can determine the axis, the displacement and the rotation of the corresponding screw. In the following, formulae are derived to compute the screw parameters of a rigid body motion in terms of the motion of three noncollinear points of the body. It will be shown that these formulae require that the displacements of the involved points be noncoplanar. Now, if three points of a rigid body are collinear, their displacements under any motion are coplanar. The converse, however, is not true, for three noncollinear points of a rigid body could have, under special circumstances, coplanar displacements, as is proved in Theorem 2.6.4. To prove this, a previous result is derived in the following

THEOREM 2.6.3 *If the displacements of three noncollinear points of a rigid body are identical, the body undergoes a pure translation.*

Proof:

Let A, B, C be three noncollinear points of a rigid body, and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ their respective position vectors. Using eq. (2.6.19), the displacements of these points can be written as

$$\mathbf{u}_A = \mathbf{u}_S + (\mathbf{Q}-\mathbf{I})(\mathbf{a}-\mathbf{s})$$

$$\mathbf{u}_B = \mathbf{u}_S + (\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{s})$$

$$\mathbf{u}_C = \mathbf{u}_S + (\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{s})$$

where $\mathbf{s}$ is the position vector of a point S on the screw axis.

Subtracting the third equation from the first and the second, and recalling that the three displacements are identical, one obtains

$$(\mathbf{Q}-\mathbf{I})(\mathbf{a}-\mathbf{c})=\mathbf{0}$$

and

$$(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{c})=\mathbf{0}$$

Hence both $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{c} \in N(\mathbf{Q}-\mathbf{I})$*, i.e. $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{c}$ lie in the same one-dimensional space spanned by the real eigenvector of $\mathbf{Q}$. This cannot be so because A, B, C were assumed to be noncollinear. Thus, the only possibility for the two latter equations to hold is that $\mathbf{Q}=\mathbf{I}$, i.e. the motion contains no rotation and hence is a pure translation. q.e.d.

* $N(\mathbf{T})$ and $R(\mathbf{T})$ denote the null space and the range of $\mathbf{T}$, as defined in Section 1.3

THEOREM 2.6.4 *The non-identical* displacements of three points of a rigid body are coplanar if and only if one of the following three conditions is met:*

i) The motion is a pure rotation

ii) The motion is general, but the points are collinear

iii) The motion is general and the points are not collinear, but lie in a plane parallel to the screw axis.

Proof:

("if" part)

i) If the motion is a pure rotation and the origin of coordinates is located along the axis of rotation, the displacement $\mathbf{u}$ of any point of position vector $\mathbf{r}$ is then

$$\mathbf{u}=(\mathbf{Q}-\mathbf{I})\mathbf{r}$$

i.e. $\mathbf{u} \in R(\mathbf{Q}-\mathbf{I})$. Since $N(\mathbf{Q}-\mathbf{I})$ is of dimension 1, namely the axis of rotation, then from eq. (1.3.1), $R(\mathbf{Q}-\mathbf{I})$ is of dimension 2, namely a plane passing through the origin, normal to the axis of rotation. Thus, all displacements are coplanar, thereby proving this part.

ii) Let A, B and C be the given three collinear points of the rigid body undergoing a general motion. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be their respective position vectors. Hence, vectors $\mathbf{c}-\mathbf{a}$ and $\mathbf{b}-\mathbf{a}$ are linearly dependent and they are related by

*If the displacements of the three noncollinear points were identical, the motion would be a pure translation, according to Theorem 2.6.3.

$$\mathbf{c}-\mathbf{a}=\alpha(\mathbf{b}-\mathbf{a}) \qquad (ii.1)$$

From eq. (2.6.3),

$$\mathbf{u}_C=\mathbf{a}'+\mathbf{Q}(\mathbf{c}-\mathbf{a})-\mathbf{c}$$

$$=\mathbf{a}'-\mathbf{a}+\mathbf{c}+(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{a})-\mathbf{c}$$

$$=\mathbf{u}_A+\alpha(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a}) \qquad (ii.2)$$

But, also from eq. (2.6.3),

$$(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a})=\mathbf{u}_B-\mathbf{u}_A \qquad (ii.3)$$

Hence, eq. (ii.2) can be written as

$$\mathbf{u}_C=(1-\alpha)\mathbf{u}_A+\alpha\mathbf{u}_B$$

thus making evident that the three involved displacements are coplanar.

iii) Using eq. (2.6.19), the displacements of points A, B and C are

$$\mathbf{u}_A=\mathbf{u}_S+(\mathbf{Q}-\mathbf{I})(\mathbf{a}-\mathbf{s})$$

$$\mathbf{u}_B=\mathbf{u}_S+(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{s})$$

$$\mathbf{u}_C=\mathbf{u}_S+(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{s})$$

Since A, B and C lie in a plane parallel to the screw axis, vectors $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{a}$ and $\mathbf{u}_S$ are coplanar and hence they can be related as

$$\mathbf{c}-\mathbf{a}=\alpha(\mathbf{b}-\mathbf{a})+\beta\mathbf{u}_S$$

or

$$\mathbf{c}=(1-\alpha)\mathbf{a}+\alpha\mathbf{b}+\beta\mathbf{u}_S$$

Substituting the latter expression into $\mathbf{u}_C$, after cancellations and rearrangement of terms,

$$\mathbf{u}_C=\mathbf{u}_A-\alpha(\mathbf{Q}-\mathbf{I})(\mathbf{a}-\mathbf{b})$$

But, from the above expressions for $\mathbf{u}_A$ and $\mathbf{u}_B$,

$$\mathbf{u}_A-\mathbf{u}_B=(\mathbf{Q}-\mathbf{I})(\mathbf{a}-\mathbf{b})$$

and so, from the latter expressions for $\mathbf{u}_C$,

$$\mathbf{u}_C=(1-\alpha)\mathbf{u}_A+\alpha\mathbf{u}_B$$

thereby showing the linear dependence, i.e. the coplanarity of the three displacements involved.

("only if" part)

If $\mathbf{u}_A$, $\mathbf{u}_B$ and $\mathbf{u}_C$ are coplanar, then

$$\det(\mathbf{u}_A,\mathbf{u}_B,\mathbf{u}_C)=0$$

Introducing eq. (2.6.3), $\mathbf{u}_B$ and $\mathbf{u}_C$ can be written as

$$\mathbf{u}_B=\mathbf{u}_A+(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a})$$

$$\mathbf{u}_C=\mathbf{u}_A+(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{a})$$

Hence, the coplanarity condition can be written, after proper simplifications, as

$$\det\left(\mathbf{u}_A,(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a}),(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{a})\right)=0$$

or, in Gibbs notation,

$$(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a})\times(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{a})\cdot\mathbf{u}_A=0$$

From (2.6.p.5), the first product can be expressed as

$$(\mathbf{Q}-\mathbf{I})(\mathbf{b}-\mathbf{a})\times(\mathbf{Q}-\mathbf{I})(\mathbf{c}-\mathbf{a})=\alpha\mathbf{e}$$

where

$$\alpha=2(1-\cos\theta)\mathbf{e}\times(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})$$

θ and $\mathbf{e}$ being the angle of rotation and the unit vector parallel to the axis of this rotation.

Exercise 2.6.4 Derive the above expression for [illegible]

The double product thus can vanish if any one of the following conditions is met:

i) $\mathbf{e}\cdot\mathbf{u}_A=0$

which, from Corollary 2.6.1, states that the body undergoes a pure rotation

ii) $\quad \mathbf{a}=\mathbf{0}$

which is satisfied under one of the two following conditions:

ii.1) $\quad 1-\cos\theta=0$

which implies $\theta=0$, i.e. the motion reduces to a pure translation. This case, however, has been discarded in the present analysis, for the displacements have been assumed to be non-identical (See Theorem 2.6.3)

ii.2) $\quad \mathbf{e}\times(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})=0$

which in turn is satisfied under one of the following two conditions:

ii.2.a) $\mathbf{e}$, $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are coplanar, i.e. points A, and C lie on a plane parallel to the rotation axis (Picture it!)

ii.2.b) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$

which implies that A, B and C are collinear, thus completing the proof.

<u>COROLLARY 2.6.3</u> *Assume a rigid body undergoes an arbitrary motion and choose any three noncollinear points of the body, A, B and C. Letting $\mathbf{u}_A$, $\mathbf{u}_B$ and $\mathbf{u}_C$ be the three resulting displacements, then the two difference vectors $\mathbf{u}_A-\mathbf{u}_C$ and $\mathbf{u}_B-\mathbf{u}_C$ are parallel if and only if the points lie in a plane parallel to the screw axis if the motion is general. If the motion is a pure rotation, these points lie in a plane parallel to the axis of rotation.*

<u>Exercise 2.6.5</u> Prove Corollary 2.6.3

Further consequences of Theorem 2.6.4 are next stated.

<u>Corollary 2.6.4</u> *The displacements of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation*

<u>Exercise 2.6.6</u> Prove Corollary 2.6.4. Hint: Use eq. (2.6.19) and the fact that the two terms of its right-hand side are linearly independent; in fact, orthogonal.

<u>Corollary 2.6.5</u> *If two, and only two, displacements of three noncollinear points of a rigid body are parallel, then either i) the parallel displacements are identical and belong to points lying on a line parallel to the screw axis, or ii) the parallel displacements are different from each other, in which case the motion is a pure rotation whose axis is parallel to the line connecting the two points of parallel displacements.*

<u>Corollary 2.6.6</u> *If one, and only one of three points of a rigid body has a zero displacement and other two points, noncollinear with the former, have parallel but different displacements, then the body undergoes a pure rotation, whose axis is determined by the intersection of the plane containing the three given points with a second plane defined by the displaced positions of the points.*

<u>Exercise 2.6.7</u> Prove Corollaries 2.6.5 and 2.6.6

The formulas that allow the computation of the screw parameters are next derived. It will be assumed that the displacements of <u>three</u> <u>noncollinear</u> points are known and the two cases that could arise are dealt with. These cases are: i) the resulting displacements are noncoplanar, ii) these displacements are coplanar and the motion is either pure rotation or general but the three points lie in a plane parallel to the screw axis.

<u>First Case. The displacements are noncoplanar</u>

Let A, B, C and A', B', C' be the initial and the displaced positions of three noncollinear points. Denoting by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ the corresponding position vectors, the displacement vectors are, then

$$\underline{u}_A = \underline{a}' - \underline{a} \tag{2.6.22a}$$

$$\underline{u}_B = \underline{b}' - \underline{b} \tag{2.6.22b}$$

$$\underline{u}_C = \underline{c}' - \underline{c} \tag{2.6.22c}$$

Now, the direction vector, $\underline{e}$, of the screw axis can be obtained as follows: Theorem 2.6.2 suggests one way to obtain $\underline{e}$, namely, determine the unique vector along which $\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$ all have the same projections. If the tails of all these vectors are attached to one point, say O, then it becomes evident that the vector $\underline{e}$ is perpendicular to the plane determined by the tips of the three vectors $\underline{u}_A, \underline{u}_B, \underline{u}_C$. Thus, $\underline{e}$ is parallel to the cross product of vectors $\underline{u}_A - \underline{u}_C$ and $\underline{u}_B - \underline{u}_C$. Normalizing $\underline{e}$ to have unit lenght, one immediately has

$$\underline{e} = \frac{(\underline{u}_A - \underline{u}_C) \times (\underline{u}_B - \underline{u}_C)}{||(\underline{u}_A - \underline{u}_C) \times (\underline{u}_B - \underline{u}_C)||} \tag{2.6.23}$$

To determine the magnitudes of the screw displacement, $||\underline{u}_S||$, all that is needed is to project any one of $\underline{u}_A$, $\underline{u}_B$ or $\underline{u}_C$ onto $\underline{e}$.

Hence,

$$||\underline{u}_S|| = |\underline{u}_A \cdot \underline{e}| \tag{2.6.24}$$

where the absolute value has been taken because eq. (2.6.23) determines the vector $\underline{e}$ up to a change of sign (the order of the vectors in the cross product could have been changed). Vector $\underline{u}_S$ can now be computed as

$$\underline{u}_S = ||\underline{u}_S||\,\underline{e}\,\operatorname{sgn}(\underline{u}_A \cdot \underline{e}) \tag{2.6.25}$$

where sgn is the signum function, i.e., sgn(x) is -1 if x<0; it is +1 if x>0 and it is irrelevant if x=0. The latter indeterminacy of sgn(0) causes no difficulty for, if $\underline{u}_A \cdot \underline{e} = 0$, $||\underline{u}_S|| = 0$, and, from corollary 2.6.1, the motion is one of rotation about one fixed point.

Now, notice that, if vector $\underline{u}_S$ is subtracted from the vectors defined in (2.6.23), the new vectors $\underline{u}'_A$, $\underline{u}'_B$ and $\underline{u}'_C$ lie in the same plane (Why?). To completely determine the screw, the rotation angle θ and the location of the screw axis are next determined. Since $\underline{u}'_A$, $\underline{u}'_B$ and $\underline{u}'_C$ are coplanar, they can be regarded as the displacements of three points of a rigid body undergoing pure rotation. Let A", B" and C" be the final positions of points A, B and C undergoing displacements $\underline{u}'_A$, $\underline{u}'_B$ and $\underline{u}'_C$. Since these vectors are coplanar, the mediator planes π_1 and π_2 of segments AA" and BB" can be projected as the dotted lines appearing in Fig 2.6.2

Then the screw axis L is defined by the intersection I of π_1 and π_2. This intersection is next determined.

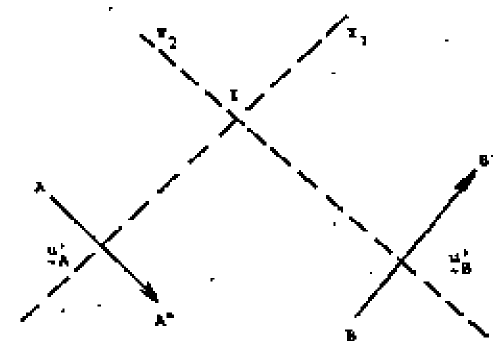


Fig 2.6.2. Determination of the screw axis.

Let M and N be the middle points of segments AA" and BB", their positon vectors being denoted by $\underline{m}$ and $\underline{n}$, respectively.

Clearly,

$$\underline{m}=\underline{a}+\frac{1}{2}\underline{u}'_A \qquad (2.6.26a)$$

$$\underline{n}=\underline{b}+\frac{1}{2}\underline{u}'_B \qquad (2.6.26b)$$

The equations of planes π_1 and π_2 are thus

$$(\underline{r}-\underline{m})\cdot\underline{u}'_A=0 \qquad (2.6.27a)$$

$$(\underline{r}-\underline{n})\cdot\underline{u}'_B=0 \qquad (2.6.27b)$$

respectively.

The set of points $\underline{r}$, satisfying both eqs. (2.6.27), yield line L, the screw-axis. The angle of rotation, ϕ, is then simply, angle AIA" (or equivalently, angle BIB" or angle CIC"). This angle is readily obtained once the line L is known, for it can then be computed from the dot product of vectors $\overrightarrow{IA}$ and $\overrightarrow{IA''}$, both lying in a plane normal to L.

Example 2.6.1 Determine the screw of the displacement of the cube of Fig 2.6.3 as it is moved from configuration 1 to configuration 2. Assume the sides of the cube have lenght h.

Solution I:

The problem is first solved via a minimisation procedure.

Step i): Determination of the revolute. For this purpose, assume a rigid body rotation about point B, as shown in Fig 2.6.4

From Fig 2.6.4 it is clear that the cube undergoes a rotation about line EB, thereby saving the analysis performed in example 2.6.1 to determine the axis of rotation. Thus,

$$\underline{u}=\frac{\overrightarrow{EB}}{||\overrightarrow{EB}||}=\frac{\sqrt{3}}{3}(-\underline{e}_x+\underline{e}_y+\underline{e}_z) \qquad (2.6.28)$$

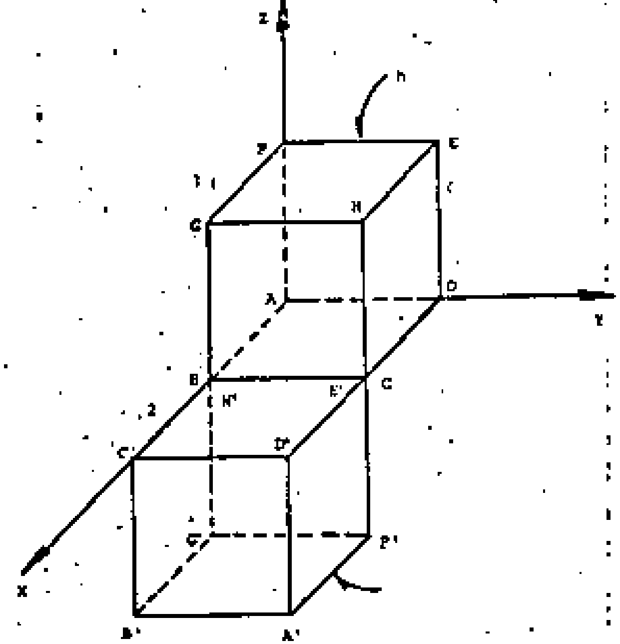


Fig 2.6.3 Motion of a cube

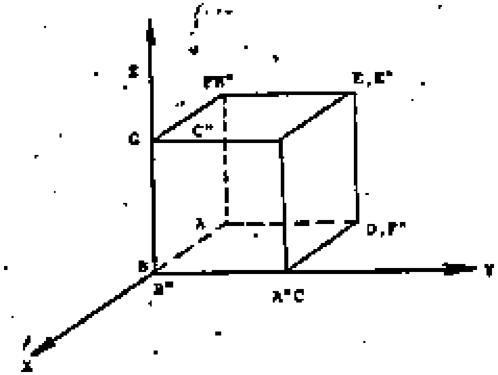


Fig 2.6.4 Rotation of a cube about a fixed point

or, in matrix form,

$$(e) = \frac{\sqrt{3}}{3}\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}$$

Let M be the intersection of the line BE and a plane perpendicular to it but containing point A (the plane also contains point A", since AA" is perpendicular to BE, as can readily be checked).

Let $\underline{m}$ be the position vector of M. Now $\underline{m}$-$\underline{a}$ is perpendicular to BE, and M is contained in line BE. BE is specified as the intersection of the planes

$$x+z=0 \qquad (2.6.29a)$$

$$y-z=0 \qquad (2.6.29b)$$

Since AM is perpendicular to BE, $\underline{m}$-$\underline{a}$ must be perpendicular to vector $\underline{e}$ of (2.6.28). Hence, the coordinates of M(x,y,z) must satisfy the relation

$$x+h-y-z=0 \qquad (2.6.29c)$$

which, together with eqs. (2.6.29), determines M, namely

$$x=-\frac{h}{3},\quad y=\frac{h}{3},\quad z=\frac{h}{3} \qquad (2.6.30)$$

Hence,

$$\underline{m}=\frac{h}{3}(-\underline{e}_x+\underline{e}_y+\underline{e}_z) \qquad (2.6.31)$$

It can also be readily checked that $\underline{m}$-$\underline{a}$" is perpendicular to $\underline{e}$, as expected, if point A" were to lie in the plane perpendicular to EB. The angle of rotation, θ, can now be computed from the relationship

$$\cos\theta=\frac{(\underline{a}-\underline{m})\cdot(\underline{a}''-\underline{m})}{||\underline{a}-\underline{m}||^2}$$

i.e.,

$$\cos\theta=-\frac{1}{2},\ \sin\theta=\frac{\sqrt{3}}{2} \qquad (2.6.32)$$

Knowing the axis of rotation, EB, and the angle fo rotation, θ, the revolute matrix is now readily constructed form eqs. (2.5.4), where

$$u=-\frac{\sqrt{3}}{3},\ v=\frac{\sqrt{3}}{3},\ w=\frac{\sqrt{3}}{3} \qquad (2.6.33)$$

and so,

$$(\underline{P})=\frac{\sqrt{3}}{3}\begin{pmatrix}0 & 1 & +1\\ +1 & 0 & +1\\ 1 & 1 & 0\end{pmatrix},\quad (\underline{R})=\frac{1}{3}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{pmatrix}\quad (\underline{T})=\frac{1}{3}\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}$$

Therefore,

$$(\underline{Q})=\begin{pmatrix}0 & 0 & -1\\ -1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix},\quad (\underline{Q}-\underline{I})=\begin{pmatrix}-1 & 0 & -1\\ -1 & -1 & 0\\ 0 & 1 & -1\end{pmatrix}$$

<u>Step ii)</u>: <u>Determination of the screw axis</u>. The minimum-magnitude displacement $\underline{u}_R$ of point R is obtained from eq. (2.6.3a), expressed in terms of the coordinate axes of Fig 2.6.3. Thus,

$$(\underline{u}_R)=\begin{pmatrix}2h-x-z\\ h-x-y\\ -h+y-z\end{pmatrix} \qquad (2.6.36)$$

and

$$\phi(\underline{r})\equiv||\underline{u}_R||^2=2x^2+2y^2+2z^2+2xz+2xy-2yz-6hx-4hy-2hz+6h^2 \qquad (2.6.37)$$

Hence

$$(\phi'(\underline{r}))=2\begin{pmatrix}2x+y+z-3h\\ x+2y-z-2h\\ x-y+2z-h\end{pmatrix} \qquad (2.6.38)$$

and, equating $\phi'(\mathbf{r})$ to zero, a set of three linearly dependent equations is obtained, from which the following linearly independent set is sorted out.

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} h \qquad (2.6.39)$$

This has a minimum-norm solution (according to eq. (2.6.8) given by

$$(\mathbf{r}_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{h}{3} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad (2.6.40)$$

thereby determining the screw axis, which passes through point R_0 (whose position vector is $\mathbf{r}_0$), as given by eq. (2.6.40) and is parallel to vector $\mathbf{e}$, as given by eq. (2.6.28). In order to compute the pitch of the screw, λ, it is necessary to compute $||\mathbf{u}_S||$ which, from Theorem 2.6.2, is given as

$$||\mathbf{u}_S|| = |\mathbf{u}_D \cdot \mathbf{e}| = \frac{2\sqrt{3}}{3} h \qquad (2.6.41a)$$

and

$$\mathbf{u}_S = \frac{2\sqrt{3}}{3} h \mathbf{e} \qquad (2.6.41b)$$

The pitch is, then, from eq. (2.6.21a),

$$\lambda = 2\sqrt{3}\, h \qquad (2.6.42)$$

Solution II

An alternative solution is now given, using eqs. (2.6.23), (2.6.24) and (2.6.27). In order to simplify the computations, choose the displacements of points C, D and G to determine the screw. Thus,

$$\begin{aligned} \mathbf{u}_C &= \mathbf{c}' - \mathbf{c} = h(\mathbf{e}_x - \mathbf{e}_y) \\ \mathbf{u}_D &= \mathbf{d}' - \mathbf{d} = 2h\mathbf{e}_x \\ \mathbf{u}_G &= \mathbf{g}' - \mathbf{g} = -2h\mathbf{e}_z \\ (\mathbf{u}_C - \mathbf{u}_G) \times (\mathbf{u}_D - \mathbf{u}_G) &= 2h^2(-\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) \end{aligned} \qquad (2.6.43)$$

from which

$$||(\mathbf{u}_C - \mathbf{u}_G) \times (\mathbf{u}_D - \mathbf{u}_G)|| = 2\sqrt{3}\, h^2$$

and so,

$$\mathbf{e} = \frac{\sqrt{3}}{3}(-\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z), \text{ or } \mathbf{e} = \frac{\sqrt{3}}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad (2.6.44)$$

which is identical with the value previously obtained in (2.6.28).

$||\mathbf{u}_S||$ is obtained from

$$||\mathbf{u}_S|| = |\mathbf{u}_D \cdot \mathbf{e}| = \frac{2\sqrt{3}}{3} h$$

and so

$$\mathbf{u}_S = \frac{2}{3} h(\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z) \qquad (2.6.45)$$

where the sign of $\mathbf{u}_S$ has been reversed, as compared with that of $\mathbf{e}$, because $\mathbf{u}_D \cdot \mathbf{e} < 0$. Next form the vectors,

$$\mathbf{u}'_C = \mathbf{u}_C - \mathbf{u}_S = h\left(\frac{1}{3}\mathbf{e}_x - \frac{1}{3}\mathbf{e}_y + \frac{2}{3}\mathbf{e}_z\right) \qquad (2.6.46a)$$

$$\mathbf{u}'_D = \mathbf{u}_D - \mathbf{u}_S = h\left(\frac{4}{3}\mathbf{e}_x + \frac{2}{3}\mathbf{e}_y + \frac{2}{3}\mathbf{e}_z\right) \qquad (2.6.46b)$$

$$\mathbf{u}'_G = \mathbf{u}_G - \mathbf{u}_S = h\left(-\frac{2}{3}\mathbf{e}_x + \frac{2}{3}\mathbf{e}_y - \frac{4}{3}\mathbf{e}_z\right) \qquad (2.6.46c)$$

which can be readily verified to be coplanar, as expected. Next, the equations of planes π_1 and π_2 are obtained. Let

$$\mathbf{p} = \mathbf{c} + \frac{1}{2}\mathbf{u}'_C = \frac{h}{6}(7\mathbf{e}_x + 5\mathbf{e}_y + 2\mathbf{e}_z) \qquad (2.6.47a)$$

$$\underline{p}=\underline{d}-\frac{1}{2}\underline{u}'_p=\frac{h}{3}(2\underline{e}_x+4\underline{e}_y+\underline{e}_z) \qquad (2.6.47b)$$

The equation of plane π_1 is, then,

$$x-y+2z-h=0 \qquad (2.6.48a)$$

and that of plane π_2 is

$$2x+y+z-3h=0 \qquad (2.6.48b)$$

Next, a point I on the axis of rotation, contained in a plane perpendicular to this axis and passing through points C and C", is located. Let $\underline{r}_I$ be the position vector of this point. Then, $\underline{r}_I$ clearly must satisfy eqs. (2.6.48a and b), for it is a point of the intersection of π_1 and π_2. In addition, $\underline{r}_I-\underline{c}$ must be perpendicular to that intersection, whose direction cosines are already known from eq. (2.6.44). This latter condition is expressed then as

$$-x+y+z=0 \qquad (2.6.48c)$$

which, together with eqs. (2.6.48 a and b) constitutes a linear algebraic system of 3 equations and 3 unknowns. Its solution is

$$\{\underline{r}_I\}=\frac{h}{3}\begin{bmatrix}3\\2\\1\end{bmatrix} \qquad (2.6.49)$$

which is a solution identical to that obtained in eq. (2.6.40).

The angle of rotation is now obtained from the relationship

$$\cos\theta=\frac{(\underline{c}-\underline{r}_I)\cdot(\underline{c}+\underline{u}'_C-\underline{r}_I)}{||\underline{c}-\underline{r}_I||} \qquad (2.6.50)$$

where

$$\{\underline{c}-\underline{r}_I\}=\frac{h}{3}\begin{bmatrix}0\\1\\-1\end{bmatrix},\quad \underline{c}+\underline{u}'_C-\underline{r}_I=\frac{h}{3}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Thus,

$$(\underline{c}-\underline{r}_I)\cdot(\underline{c}+\underline{u}'_C-\underline{r}_I)=-\frac{h^2}{9} \qquad (2.6.51)$$

and

$$||\underline{c}-\underline{r}_I||^2=\frac{2}{9}h^2 \qquad (2.6.52)$$

Substitution of eqs. (2.6.51) and (2.6.52) into eq. (2.6.50) yields, then,

$$\cos\theta=-\frac{1}{2} \text{ or } \theta=-120° \qquad (2.6.53)$$

where the minus sign was found by application of the result of eq. (2.5.16)

The screw displacement, $||\underline{u}_S||$ is obtained from eq. (2.6.45) as

$$||\underline{u}_S||=\frac{2\sqrt{3}}{3} \qquad (2.6.54)$$

and the pitch, λ, is obtained from eq. (2.6.21a) as,

$$\lambda=2\sqrt{3}h$$

which results are identical to those of eqs. (2.6.41a) and (2.6.42)

One third method to obtain the point $\underline{r}_0$ on the screw axis closest to the origin is now presented as it appears in (2.2,p.11)

Let $\underline{a}$ and $\underline{a}'$ be the initial and the final position vectors, respectively, of a given point A of a rigid body, which are known. Also let $\underline{r}$ and $\underline{r}'$ be the initial and the final position vectors of another point R, both yet unknown. If point R is to lie on the screw axis, then $\underline{u}_R=\underline{r}'-\underline{r}$ is parallel to the axis of rotation, $\underline{e}$, as was found previously. From Rodriques' Formula, eq. (2.6.3),

$$\underline{a}'-\underline{a}=\tan\frac{\theta}{2}\underline{e}\times(\underline{a}'+\underline{a}) \qquad (2.6.56)$$

and

$$\underline{r}'-\underline{r}=\tan\frac{\theta}{2}\underline{e}\times(\underline{r}'+\underline{r})$$

Subtracting eq. (2.6.56b) from eq. (2.6.56a) and taking into account that $\underline{r}'-\underline{r}$ is parallel to $\underline{e}$, i.e., writing $\underline{r}'-\underline{r}=\alpha\underline{e}$, α being a scalar.

$$\underline{a}'-\underline{a}=\sigma\underline{e}+\tan\frac{\theta}{2}\underline{e}\times(\underline{a}'+\underline{a})-\tan\frac{\theta}{2}\underline{e}\times(\underline{r}'+\underline{r}) \qquad (2.6.57)$$

Since $\underline{r}'=\underline{r}+\sigma\underline{e}$, it follows that

$$\underline{e}\times\underline{r}'=\underline{e}\times\underline{r}$$

Hence, eq. (2.6.57) can be written as

$$\underline{a}'-\underline{a}=\sigma\underline{e}+\tan\frac{\theta}{2}\underline{e}\times(\underline{a}'+\underline{a})-2\tan\frac{\theta}{2}\underline{e}\times\underline{r} \qquad (2.6.57a)$$

Multiplying both sides of eq. (2.6.57a) times $\cot\frac{\theta}{2}\underline{e}\times$, one obtains

$$\cot\frac{\theta}{2}\underline{e}\times(\underline{a}'-\underline{a})=\underline{e}\times\left(\underline{e}\times(\underline{a}'+\underline{a})\right)-2\underline{e}\times(\underline{e}\times\underline{r})=\underline{e}\times\left(\underline{e}\times(\underline{a}'+\underline{a})\right)-2(\underline{e}\cdot\underline{r})\underline{e}+2\underline{r} \qquad (2.6.58)$$

If $\underline{r}$ is chosen to be the position vector of the point on the screw axis closest to the origin, then

$$\underline{e}\cdot\underline{r}=0$$

and vector $\underline{r}_0$ thus can be obtained from eq. (2.6.58) as

$$\underline{r}_0=\frac{1}{2}\cot\frac{\theta}{2}\underline{e}\times(\underline{a}'-\underline{a})-\frac{1}{2}\underline{e}\times\left(\underline{e}\times(\underline{a}'+\underline{a})\right) \qquad (2.6.59)$$

<u>Second Case. The displacements are coplanar</u>

This could be due to two possibilities: either the points lie in a plane parallel to the screw axis (or to the axis of rotation if the motion is a pure rotation) or they do not, but the motion is then necessarily a pure rotation.

<u>First possibility. The three points lie in a plane parallel either to the screw axis or to the axis of rotation.</u> From Corollary 2.6.3 the differences of displacement vectors are parallel and hence the cross product appearing in eq. (2.6.23) vanishes thus rendering the computation of $\underline{e}$ indeterminate. This vector can be computed, nevertheless, attending the aforementioned Corollary and the fact that it is perpendicular to vector $\underline{u}_A-\underline{u}_C$, according to Corollary 2.6.2. The condition that $\underline{e}$ is contained in the plane of the given points A, B and C is expressed as

$$\underline{e}=\alpha(\underline{a}-\underline{c})+\beta(\underline{b}-\underline{c}) \qquad (2.6.60)$$

The perpendicularity condition between $\underline{e}$ and $\underline{u}_A-\underline{u}_C$ is expressed in turn as

$$(\underline{u}_A-\underline{u}_C)^T\underline{e}=0 \qquad (2.6.61)$$

Substitution of eq. (2.6.60) into eq. (2.6.61) yields

$$\alpha(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})+\beta(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})=0 \qquad (2.6.62)$$

Hence

$$\alpha=-\beta\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})} \qquad (2.6.63)$$

provided $\underline{u}_A-\underline{u}_C$ is not orthogonal to $\underline{a}-\underline{c}$. If this is so, then from eq. (2.6.62), $\beta=0$ and, since $\underline{e}$ has been defined as of magnitude equal to unity, then

$$\underline{e}=\frac{\underline{a}-\underline{c}}{||\underline{a}-\underline{c}||} \qquad (2.6.64)$$

Now, since points A, B and C are not collinear, then it cannot happen that $\underline{u}_A$ and $\underline{u}_C$ be orthogonal to both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$. If, however, $\underline{u}_A-\underline{u}_C$ is orthogonal to $\underline{b}-\underline{c}$, then $\alpha=0$, from eq. (2.6.62), and, in this instance,

$$\underline{e}=\frac{\underline{b}-\underline{c}}{||\underline{b}-\underline{c}||} \qquad (2.6.65)$$

If neither α nor β vanish, then from eq. (2.6.63) and the condition imposed on $\underline{e}$ as being of magnitude unity,

$$\frac{1}{\beta^2}=||\underline{a}-\underline{c}||^2\left(\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})}\right)^2-2(\underline{a}-\underline{c})^T(\underline{b}-\underline{c})\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})}+||\underline{b}-\underline{c}||^2 \qquad (2.6.66)$$

Eq. (2.6.66) yields β. With the value of β known, α is then computed from eq. (2.6.63). Thus, $\underline{s}$ is finally computed from eq. (2.6.60).

<u>Second possibility. The motion is pure rotation.</u> If the three points are noncollinear and the displacements are nonidentical and parallel but vectors u_A-u_C and u_B-u_C are nonparallel, then, from Theorem 2.6.4 and Corollary 2.6.3, the motion is one of pure rotation. In this case the axis of rotation can be obtained simply from the intersection of the mediator planes of segments AA' and BB'. The perpendiculars to the axis of rotation, traced from A and A' intersect that axis at a common point, I. The angle of rotation is then, simply AIA', thereby completing the motion parameters. The computation of the screw parameters is realized by SUBROUTINE SCREW, which considers all cases that could arise regarding the relationships amongst all three displacement vectors. These possible cases are shown in the "tree" diagram appearing in Fig 2.6.5. SCREW uses the following auxiliary subroutines:

1. SUBROUTINE COPL 1 computes the screw parameters when the motion is pure rotation. It distinguishes amongst the different particular cases with the aid of the integer variable INDEX
2. SUBROUTINE COPL 2 computes the screw parameters when the points lie in a plane parallel either to the screw axis or to the axis of rotation. Two different cases could arise, which are distinguished with the aid of the integer variable INDE.
3. SUBROUTINE GENMOT computes the screw parameters when the motion is general and the three given displacements are noncoplanar.

The computation procedure for each case is next described. All over, the vectors referred to are the given displacement vectors, $\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$, of the three given points, A, B and C, whose position vectors in the reference configuration are $\underline{a}_1$, $\underline{b}_1$ and $\underline{c}_1$, whereas those in their displaced configura

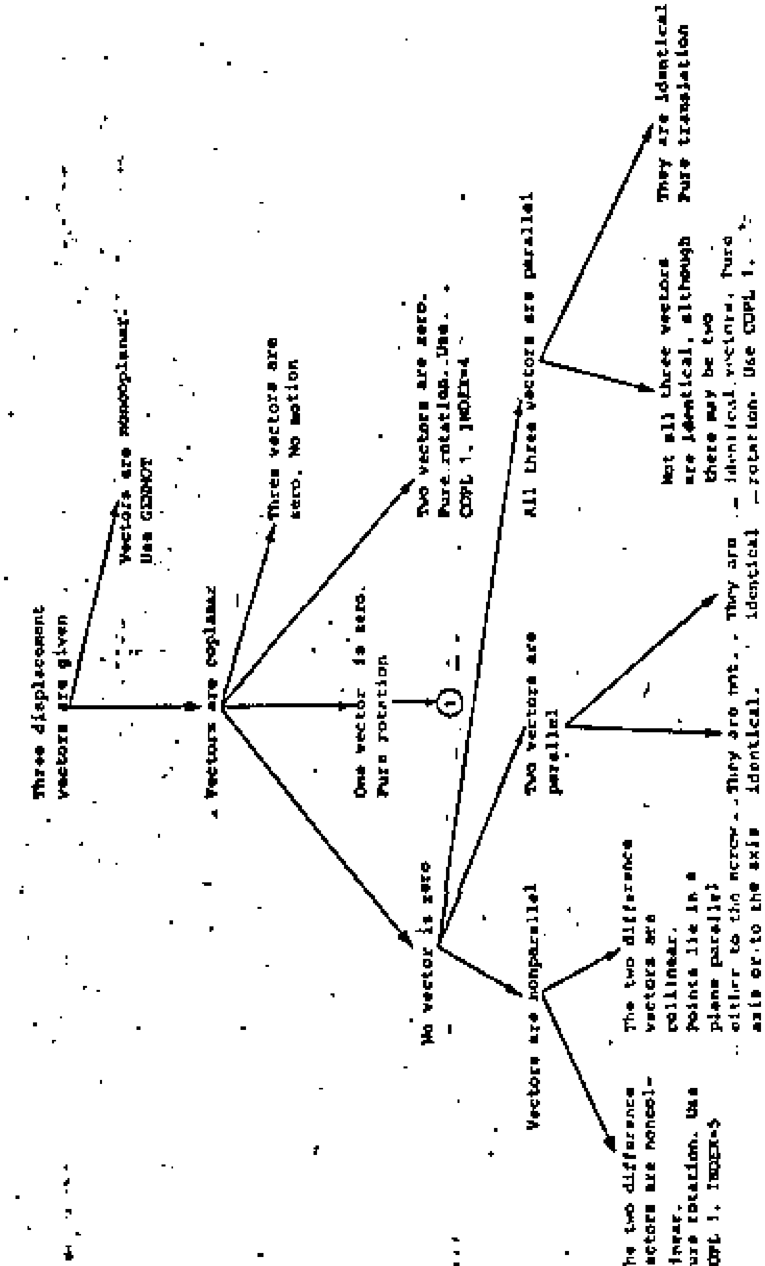


Fig 2.6.5 Tree diagram showing the different possible relationships amongst the displacements of three noncollinear points defining a rigid-body motion.

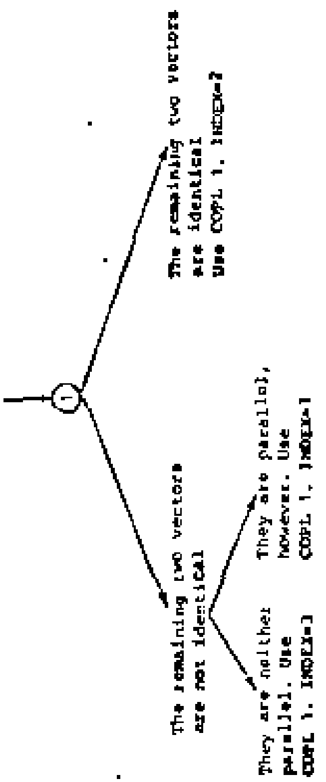


Fig 2.6.5 (continued)

tions are a_2, b_2 and c_2.

INDEX = 1. One vector is zero and the remaining two vectors are not identical; they are parallel, however. It follows from Corollary 2.6.1 that the motion is pure rotation. The location of the axis of rotation follows from the fact that the axis of rotation is contained in the intersection of two planes, Π_1 and Π_2, where Π_1 is the plane of the three given points and Π_2 is defined by the displaced positions of these points. Notice that the point of zero displacement is contained in both Π_1 and Π_2.

Proof

Let C be the point of zero displacement and the origin of coordinates. Furthermore, the displacements u_A and u_B are given as

$$u_A=(Q-I)a_1;\ u_B=(Q-I)b_1 \qquad (2.6.67)$$

Since u_A and u_B are parallel, there exists a scalar α such that

$$u_A+\alpha u_B=0 \qquad (2.6.68)$$

Substituting eqs. (2.6.67) into eq. (2.6.68) yields

$$(Q-I)(a_1+\alpha b_1)=0 \qquad (2.6.69)$$

which means vector $a_1+\alpha b_1$ is parallel to the axis of rotation, i.e. the axis of rotation is contained in the plane determined by A,B and C. Moreover, since

$$a_2=a_1+u_A,\ b_2=b_1+u_B \qquad (2.6.70)$$

The displacements of those points are given by

$$u'_A=(Q-I)a_2,\ u'_B=(Q-I)b_2 \qquad (2.6.71)$$

Introducing eqs. (2.6.70) into eqs. (2.6.71) and then simplifying the resulting expression with the aid of eqs. (2.6.67), one obtains

$$u'_A = Qu_A, \quad u'_B = Qu_B \qquad (2.6.72)$$

eqs. (2.6.68) and (2.6.72) lead to

$$u'_A + \alpha u'_B = Q(u_A + \alpha u_B) = 0 \qquad (2.6.73)$$

But, introducing eqs. (2.6.71) into eq. (2.6.73),

$$(Q-I)(a_2 + \alpha b_2) = 0 \qquad (2.6.74)$$

which implies that vector $a_2 + \alpha b_2$ is parallel to the axis of rotation, i.e. this axis is contained in the plane defined by the points A_2, B_2 and C_2, thereby completing the proof. Other parameters are computed using the general procedure previously outlined.

INDEX = 2 One vector is zero and the remaining two are identical. The motion is pure rotation, due to Corollary 2.6.1, and the axis of rotation is defined by a line passing through the point of zero displacement in the direction of the line connecting the other two points.

Proof

Let C be the point of zero displacement. The displacements of the other two points are

$$u_A = (Q-I)a_1, \quad u_B = (Q-I)b_1 \qquad (2.6.75)$$

Since $u_A = u_B$, it follows that

$$u_A - u_B = (Q-I)(a_1 - b_1) = 0 \qquad (2.6.76)$$

which implies that vector $a_1 - b_1$ is parallel to the axis of rotation, i.e. the line connecting points A and B is parallel to the axis of rotation, q.e.d.

INDEX = 3. One vector is zero and the remaining two vectors are not parallel. The motion is pure rotation, due to Corollary 2.6.1, and the axis of rotation passes through the point of zero displacement, in the direction of the cross product of the two nonzero displacement vectors, which is a consequence of Theorem 2.6.4 and Corollary 2.6.1.

INDEX = 4. Two vectors are zero. The motion is pure rotation and the axis of rotation is defined by the two points of zero displacement.

INDEX = 5. No vector is zero and all three vectors are nonparallel amongst them but coplanar. Furthermore, the two arising difference vectors are noncollinear. According to Theorem 2.6.4 and Corollary 2.6.3, then, the motion is pure rotation and the screw parameters can be computed using the general procedure.

INDEX = 6. No vector is zero but two vectors are parallel and different. Moreover, the vectors are coplanar. The motion is pure rotation due to Corollary 2.6.5 and the axis of rotation is perpendicular to the plane of the given vectors. Its location can be determined using the general procedure, already outlined for pure rotation.

Proof

Let u_A and u_B be parallel but different. Then the following relationship holds

$$u_B = \alpha u_A \qquad (2.6.77)$$

Let e be the unit vector along the screw axis. Then, from Theorem 2.6.2,

$$\mathbf{u}_B \cdot \mathbf{e} = \alpha \mathbf{u}_A \cdot \mathbf{e} \qquad (2.6.78)$$

Substituting eq. (2.6.77) into eq. (2.6.78), one obtains

$$(1-\alpha)\mathbf{u}_A \cdot \mathbf{e} = 0 \qquad (2.6.79)$$

which vanishes if either $\alpha=1$ or if $\mathbf{u}_A \cdot \mathbf{e}=0$. The first condition is impossible to meet because $\mathbf{u}_A$ and $\mathbf{u}_B$ have been assumed to be different. Hence the only possibility for eq. (2.6.79) to hold is

$$\mathbf{u}_A \cdot \mathbf{e} = 0$$

which indicates that the motion is one of pure rotation, according to Corollary 2.6.1, q.e.d.

INDEX = 7. No vector is zero and all three vectors are parallel to each other. Furthermore, not all three vectors are identical to each other, although there may be a pair of identical vectors. The motion is one of pure rotation and the axis of rotation is determined by the intersection of the plane defined by the given points in their reference configuration with that defined by the points in their final configuration.

Proof

It was shown in the case for which INDEX=6 that the existence of at least two parallel nonidentical vectors guarantees that the motion is one of pure rotation.

It will be shown first that the plane of the three given points contains the axis of rotation. In fact, the corresponding displacements are given by

$$\mathbf{u}_A = (\mathbf{Q}-\mathbf{I})\mathbf{a}_1, \quad \mathbf{u}_B = (\mathbf{Q}-\mathbf{I})\mathbf{b}_1, \quad \mathbf{u}_C = (\mathbf{Q}-\mathbf{I})\mathbf{c}_1 \qquad (2.6.80)$$

which are all parallel to each other. Thus, the differences

$$\mathbf{u}_A - \mathbf{u}_C = (\mathbf{Q}-\mathbf{I})(\mathbf{a}_1-\mathbf{c}_1), \quad \mathbf{u}_B - \mathbf{u}_C = (\mathbf{Q}-\mathbf{I})(\mathbf{b}_1-\mathbf{c}_1) \qquad (2.6.81)$$

are also parallel to each other. Thus, there exists a scalar α such that

$$\mathbf{u}_A - \mathbf{u}_C + \alpha(\mathbf{u}_B - \mathbf{u}_C) = \mathbf{0} \qquad (2.6.82)$$

But, substituting eqs. (2.6.81) into eq. (2.6.82),

$$(\mathbf{Q}-\mathbf{I})\left[\mathbf{a}_1-\mathbf{c}_1 + \alpha(\mathbf{b}_1-\mathbf{c}_1)\right] = \mathbf{0} \qquad (2.6.83)$$

which implies that the vector $\mathbf{a}_1-\mathbf{c}_1+\alpha(\mathbf{b}_1-\mathbf{c}_1)$, contained in the plane ABC, is parallel to the axis of rotation. Next, consider the position vectors of the points in their displaced positions

$$\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{u}_A, \quad \mathbf{b}_2 = \mathbf{b}_1 + \mathbf{u}_B, \quad \mathbf{c}_2 = \mathbf{c}_1 + \mathbf{u}_C \qquad (2.6.84)$$

The displacements of these points are, after substitutions and cancellations,

$$\mathbf{u}'_A = \mathbf{Q}\mathbf{u}_A, \quad \mathbf{u}'_B = \mathbf{Q}\mathbf{u}_B, \quad \mathbf{u}'_C = \mathbf{Q}\mathbf{u}_C \qquad (2.6.85)$$

i.e. $\mathbf{u}'_A$, $\mathbf{u}'_B$ and $\mathbf{u}'_C$ are all parallel to each other. Hence, the differences

$$\mathbf{u}'_A - \mathbf{u}'_C = (\mathbf{Q}-\mathbf{I})(\mathbf{a}_2-\mathbf{c}_2), \quad \mathbf{u}'_B - \mathbf{u}'_C = (\mathbf{Q}-\mathbf{I})(\mathbf{b}_2-\mathbf{c}_2) \qquad (2.6.86)$$

are also parallel to each other. Hence, there exists a scalar β such that

$$\mathbf{u}'_A - \mathbf{u}'_C + \beta(\mathbf{u}'_B - \mathbf{u}'_C) = \mathbf{0} \qquad (2.6.87)$$

Substitution of eqs. (2.6.86) into eq. (2.6.87) yields then

$$(\mathbf{Q}-\mathbf{I})\left[\mathbf{a}_2-\mathbf{c}_2+\beta(\mathbf{b}_2-\mathbf{c}_2)\right] = \mathbf{0} \qquad (2.6.88)$$

which means that the vector $\mathbf{a}_2-\mathbf{c}_2+\beta(\mathbf{b}_2-\mathbf{c}_2)$, contained in the plane $A_2B_2C_2$, is parallel to the axis of rotation. Moreover, both planes, ABC and $A_2B_2C_2$, are nonparallel, for the vectors

$\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$ have been assumed to be not all three identical to each other. Hence both planes intersect along a line which is the axis of rotation, q.e.d.

So far all cases leading necessarily to a pure rotation motion have been discussed. Next the case in which the given displacement vectors are coplanar but the motion is either a pure rotation or general, is discussed. In this case the arising difference vectors are parallel and hence the given points lie in a plane parallel either to the axis of rotation or to the screw axis. This case is handled by subroutine COPL 2, which identifies each possible different subcase with the aid of the integer variable INDE.

INDE = 1. No vector is zero and two vectors are identical. The motion is either general or a pure rotation, but the screw axis or, correspondingly, the axis of rotation, is parallel to the line defined by the points with identical displacements.

<u>Proof</u>

Let B and C be the two points with identical displacements. These displacements can be expressed using eqs. (2.6.3a) as

$$\underline{u}_B=\underline{u}_A+(\underline{Q}-\underline{I})(\underline{b}_1-\underline{a}_1) \quad (2.6.95a)$$

$$\underline{u}_C=\underline{u}_A+(\underline{Q}-\underline{I})(\underline{c}_1-\underline{a}_1) \quad (2.6.95b)$$

Subtracting eq. (2.6.95b) from eq. (2.6.95a) one obtains

$$\underline{u}_B-\underline{u}_C=(\underline{Q}-\underline{I})(\underline{b}_1-\underline{c}_1)=\underline{0} \quad (2.6.96)$$

which states that the vector connecting points B and C is parallel to the axis of rotation of matrix $\underline{Q}$. Hence, line BC is parallel to the screw axis. This axis is located following the general procedure previously outlined.

INDE = 2. No vector is zero and no two vectors are parallel, but they are coplanar. The motion is either general or a pure rotation and the given points lie in a plane parallel either to the screw axis or to the axis of rotation, according to Theorem 2.6.4 and Corollary 2.6.4. The direction of the screw axis or, correspondingly, of the axis of rotation, is found using eqs. (2.6.60)-(2.6.66). Summarizing, one has the following

<u>THEOREM 2.6.5</u> *The motion of a rigid body is determined, i.e. its screw parameters can be computed, if, and only if, the positions of three noncollinear points of the body are known in both its reference and its final configurations.*

Subroutines SCREW, COPL 1, COPL 2, and GENMOT, implementing the foregoing computations, use LOCAT 1, LOCAT 2, ANGLE, CYCLIC, EXCHGE, CROSS and SCAL as subsidiary subroutines. Listings of all these subroutines appear in Figs 2.6.6 - 2.6.16

<u>Exercise 2.6.8</u> In a manufacturing process it is required to position the workpiece of Fig 2.6.17 in configuration 2 starting from configuration 1, by means of an arm fastened to the bolt of a screw. Determine the location of the axis of this screw as well as its pitch. If the operation is to take place in *n* screw revolutions plus a fraction, what is the value of this fraction?

```
      SUBROUTINE SCREW(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
C
C
C   THIS SUBROUTINE COMPUTES THE SCREW-PARAMETERS OF A RIGID BODY MOTION
C
C
C   INPUT:
C   THE X,Y AND Z-COORDINATES OF THREE NONCOLLINEAR POINTS OF THE
C    RIGID BODY IN BOTH ITS INITIAL (AIN,BIN,CIN 3-DIMENSIONAL VECTORS)
C    AND IN ITS FINAL (AFIN,BFIN,CFIN 3-DIMENSIONAL VECTORS)
C    CONFIGURATIONS.
C
C   OUTPUT:
C    1.) THE DIRECTION E, OF THE SCREW AXIS
C    2.) THE LOCATION RHO OF THE POINT ON THE SCREW AXIS LYING CLOSEST
C        TO THE ORIGIN.
C    3.) THE ANGLE OF ROTATION (SIGN WITH RESPECT TO THE DIRECTION OF E
C        INCLUDED), THETA.
C    4.) THE SCALAR DISPLACEMENT DISPL, ALONG E (SIGN WITH RESPECT TO
C        THE DIRECTION OF E INCLUDED).
C
C   SUBSIDIARY SUBROUTINES:
C
C  COPL1(*****).- CONTAINS THE SAME PARAMETERS AS SCREW PLUS  INDEX AND
C                 IN, WHICH DEFINE EACH PARTICULAR POSSIBLE CASE.
C                 COMPUTES THE SCREW PARAMETERS WHEN THE MOTION IS PURE
C                 ROTATION.
C  COPL2(*****).- COMPUTES THE SCREW PARAMETERS WHEN THE GIVEN POINTS
C                 LIE IN A PLANE PARALLEL TO THE SCREW AXIS. THE MOTION
C                 IS EITHER GENERAL OR PURE ROTATION.
C  GENMOT(****).- COMPUTES THE SCREW PARAMETERS WHEN THE MOTION IS
C                 GENERAL AND THE GIVEN DISPLACEMENTS ARE NONCOPLANAR.
C  CROSS(A,B,C).- COMPUTES THE CROSS PRODUCT OF VECTORS A AND B, IN THIS
C                 ORDER, AND STORES THE PRODUCT IN VECTOR C.
C  SCAL(A,B,C) .- COMPUTES THE SCALAR PRODUCT OF VECTORS A AND B AND
C                 STORES THE PRODUCT IN THE SCALAR S.
C
C
C  3-DIMENSIONAL VECTORS A,B,C ARE AUXILIARY FIELDS.
C
      REAL  AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),UA(3),UB(3)
     -        ,UC(3),A(3),B(3),C(3),E(3),RHO(3)
      LOGICAL LO(3)
      COMMON ZERO
C
C   COLLINEARITY OF GIVEN POINTS IS VERIFIED. WHEN POINTS ARE COLLINEAR,
C   ZERO IS SET EQUAL TO -1. AND SUBROUTINE RETURNS TO MAIN PROGRAM.
      DO 10 I=1,3
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (first part)


```
            LO(I)=.FALSE.
            A(I)=AIN(I)-CIN(I)
   10    B(I)=BIN(I)-CIN(I)
         CALL CROSS(A,B,C)
         CALL SCAL(C,C,S)
         S=SQRT(S)
         IF(S-ZERO) 20,20,30
   20    ZERO=-1.
         WRITE(6,1000)
         RETURN
C
C  DONE
C  COMPABILITY IS VERIFIED. IF THIS IS NOT MET, THEN ZERO IS SET EQUAL
C  TO -2.,-3.,OR -4., DEPENDING UPON WETHER DISTANCE AC,BC, OR AB DOES
C  NOT REMAIN CONSTANT THROUGHOUT THE MOTION.
   30   DO 40 I=1,3
             C(I)=AFIN(I)-CFIN(I)
   40   CONTINUE
        CALL SCAL(A,A,S1)
        S1=SQRT(S1)
        CALL SCAL(C,C,S2)
        S2=SQRT(S2)
        IF(ABS(S1-S2).LE.ZERO) GO TO 50
        ZERO=-2.
        WRITE(6,1010)
        RETURN
   50   DO 60 I=1,3
             C(I)=BFIN(I)-CFIN(I)
   60   CONTINUE
        CALL SCAL(B,B,S1)
        CALL SCAL(C,C,S2)
        S1=SQRT(S1)
        S2=SQRT(S2)
        IF(ABS(S1-S2).LE.ZERO) GO TO 70
        ZERO=-3
        WRITE(6,1020)
        RETURN
   70   DO 80 I=1,3
             A(I)=AIN(I)-BIN(I)
   80   B(I)=AFIN(I)-BFIN(I)
        CALL SCAL(A,A,S1)
        CALL SCAL(B,B,S2)
        S1=SQRT(S1)
        S2=SQRT(S2)
        IF(ABS(S1-S2).LE.ZERO) GO TO 90
        ZERO=-4
        WRITE(6,1030)
        RETURN
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (second part)

```
C
C  DONE
C  DISPLACEMENT VECTORS ARE COMPUTED
 90   DO 100 I=1,3
            UA(I)=AFIN(I)-AIN(I)
            UB(I)=BFIN(I)-BIN(I)
 100  UC(I)=CFIN(I)-CIN(I)
C
C  DONE
C  NUMBER OF ZERO-DISPLACEMENTS IS DETERMINED AND STORED IN NUZE.
C  DISPLACEMENT MAGNITUDES ARE TEMPORARILY STORED IN E. IF NUZE.EQ.0
C  THEN NUZE IS SET EQUAL TO 4.
      CALL SCAL(UA,UA,E(1))
      CALL SCAL(UB,UB,E(2))
      CALL SCAL(UC,UC,E(3))
      NUZE=0
      DO 110 I=1,3
            E(I)=SQRT(E(I))
            IF(E(I).GT.ZERO) GO TO 110
            NUZE=NUZE+1
            LO(I)=.TRUE.
 110  CONTINUE
      IF(NUZE.EQ.0) NUZE=4
C
C  DONE
C  EACH CASE(NUZE=0,1,2,3) IS NOW INVESTIGATED
      GO TO(111,211,311,411),NUZE
 111  DO 120 I=1,3
            IF(LO(I)) IN=I
 120  CONTINUE
      GO TO(121,131,141),IN
 121  CALL CROSS(UB,UC,C)
      DO 130 I=1,3
            A(I)=UB(I)-UC(I)
 130  CONTINUE
      GO TO 160
 131  CALL CROSS(UA,UC,C)
      DO 140 I=1,3
            A(I)=UA(I)-UC(I)
 140  CONTINUE
      GO TO 160
 141  CALL CROSS(UA,UB,C)
      DO 150 I=1,3
            A(I)=UA(I)-UB(I)
 150  CONTINUE
 160  CALL SCAL(C,C,S1)
      CALL SCAL(A,A,S2)
      S1=SQRT(S1)
*
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (third part)


```
      S2=SQRT(S2)
      INDEX=3
      IF(S1.LE.ZERO) INDEX=1
      IF(S2.LE.ZERO) INDEX=2
      GO TO 230
 211  INDEX=4
      DO 221 I=1,3
            IF(LO(I)) GO TO 221
            IN=I
 221  CONTINUE
 230  CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -          IN)
      RETURN
 311  ZERO=-5
      WRITE(6,1040)
      RETURN
C
C  DONE
C  ONE, TWO AND THREE-ZERO-DISPLACEMENT CASES WHERE ALREADY DEALT WITH
C  NO-ZERO DISPLACEMENT CASE IS NEXT INVESTIGATED.
C  PARALLELISM OF DISPLACEMENTS IS FIRST DETERMINED. CROSS PRODUCT
C  MAGNITUDES ARE TEMPORARILY STORED IN E.
 411  CALL CROSS(UA,UB,A)
      CALL CROSS(UA,UC,B)
      CALL CROSS(UB,UC,C)
      CALL SCAL(A,A,E(1))
      CALL SCAL(B,B,E(2))
      CALL SCAL(C,C,E(3))
      DO 510 I=1,3
 510        LO(I)=.FALSE.
      DO 520 I=1,3
            E(I)=SQRT(E(I))
            IF(E(I).GT.ZERO) GO TO 520
            LO(I)=.TRUE.
 520  CONTINUE
      IF(LO(1).OR.LO(2).OR.LO(3)) GO TO 525
C
C  DONE
C  NO TWO DISPLACEMENT VECTORS WERE FOUND TO BE PARALLEL.  COPLANARITY
C  IS NEXT VERIFIED.
      CALL SCAL(UC,A,S)
      IF(ABS(S).LE.ZERO) GO TO 523
      CALL GENMOT(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
      RETURN
C
C  COLLINEARITY OF DIFFERENCE VECTORS IS VERIFIED.
C  DIFFERENCES OF DISPLACEMENT VECTORS ARE TEMPORARILY STORED IN A
C  AND B.  THE CROSS PRODUCT OF THE LATTER IS STORED IN C.
*
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (fourth part)

```
 523    DO 524 I=1,3
              A(I)=UA(I)-UC(I)
 524    B(I)=UB(I)-UC(I)
        CALL CROSS(A,B,C)
        CALL SCAL(C,C,S)
        S=SQRT(S)
        IF(S.LE.ZERO) GO TO 700
        INDEX=5
        CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -               IN)
        RETURN
C
C  DONE
C  DETERMINES WHICH VECTORS ARE PARALLEL BY SETTING LO(I) EQUAL TO .TRUE
 525    DO 530 I=1,3
              IF(LO(I)) GO TO 528
              GO TO 530
 528          IN=I
              INDEX=6
 530    CONTINUE
C
C  DONE
C  INVESTIGATES IF ALL THREE VECTORS ARE PARALLEL
        DO 540 I=1,2
              IP1=I+1
              DO 540 J=IP1,3
                     IF(LO(I).AND.LO(J)) INDEX=7
 540    CONTINUE
        JNDEX=INDEX-5
        GO TO(550,600),JNDEX
C
C  DONE
C  DETERMINES IF, FOR TWO PARALLEL VECTORS, THESE ARE IDENTICAL
 550    GO TO(551,561,571),IN
 551    DO 552 I=1,3
              A(I)=UA(I)
              C(I)=UC(I)
              UA(I)=C(I)
 552    UC(I)=A(I)
        GO TO 571
 561    DO 562 I=1,3
              A(I)=UA(I)
              B(I)=UB(I)
             UA(I)=B(I)
 562    UB(I)=A(I)
 571    DO 572 I=1,3
              A(I)=UB(I)-UC(I)
 572    CONTINUE
*
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (fifth part)


```
        CALL SCAL(A,A,S)
        S=SQRT(S)
        IF(S.LE.ZERO) GO TO 750
        CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -               IN)
        RETURN
C
C  DONE
C  DETERMINES IF ALL THREE PARALLEL VECTORS ARE IDENTICAL
 600    DO 610 I=1,3
              A(I)=UA(I)-UC(I)
 610    B(I)=UB(I)-UC(I)
        CALL SCAL(A,A,S1)
        CALL SCAL(B,B,S2)
        S1=SQRT(S1)
        S2=SQRT(S2)
        IF(S1.LE.ZERO.AND.S2.LE.ZERO) GO TO 910
        CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -               IN)
        RETURN
C
C  DONE
 700    INDE=2
        GO TO 900
 750    INDE=1
 900    CALL COPL2(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDE,
     -               IN)
        RETURN
C
C  IF MOTION IS PURE TRASLATION, ZERO IS SET EQUAL TO -6
 910    ZERO=-6
        WRITE(6,1050)(UA(I),I=1,3)
        RETURN
 1000   FORMAT(5X,'POINTS ARE COLLINEAR. MOTION IS UNDEFINED'/)
 1010   FORMAT(5X,'MOTION IS NOT RIGID. LENGTH AC DOES NOT REMAIN',
     -         ' CONSTANT.'/)
 1020   FORMAT(5X,'MOTION IS NOT RIGID. LENGTH BC DOES NOT REMAIN',
     -         ' CONSTANT.'/)
 1030   FORMAT(5X,'MOTION IS NOT RIGID. LENGTH AB DOES NOT REMAIN',
     -         ' CONSTANT.'/)
 1040   FORMAT(5X,'NO MOTION. ALL THREE DISPLACEMENTS VECTORS ARE',
     -         ' ZERO.'/)
 1050   FORMAT(5X,'THE MOTION IS PURE TRANSLATION. '//15X,'THE ',
     -         'DISPLACEMENT HAS THE FOLLOWING X-,Y-AND Z COMPONENTS :'
     -         /15X,F12.5,5X,F12.5,5X,F12.5/)
        END
*
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (sixth and last part)

```
      SUBROUTINE COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL
     -                 ,INDEX,IN)
C
C  THIS SUBROUTINE COMPUTES THE SCREW PARAMETERS E,RHO,THETA AND DISPL
C  WHEN THE RIGID BODY UNDER STUDY UNDERGOES A PURE ROTATION.
C  THE SUBROUTINE PARAMETERS WERE DEFINED IN SUBROUTINE SCREW, EXCEPT
C  FOR INDEX AND IN. THESE ARE DEFINED NEXT.
C  INDEX = 1, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE PARALLEL, BUT NOT IDENTICAL.
C  INDEX = 2, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE IDENTICAL
C  INDEX = 3, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE NOT IDENTICAL.
C  INDEX = 4, IF EXACTLY TWO DISPLACEMENTS ARE ZERO.
C  INDEX = 5, IF NO DISPLACEMENTS IS ZERO AND ALL DISPLACEMENTS ARE
C             NONPARALLEL, PROVIDED THE TWO DISTINCT DISPLACEMENT
C             DIFFERENCES ARE NONCOLLINEAR.
C  INDEX = 6, IF NO DISPLACEMENT IS ZERO AND EXACTLY TWO VECTORS ARE
C             PARALLEL BUT DIFFERENT.
C  INDEX = 7, IF ALL THREE DISPLACEMENTS ARE PARALLEL BUT NOT ALL THREE
C             ARE IDENTICAL.
C
C  IN, DETECTS WHICH VECTORS ARE PARALLEL OR IDENTICAL, IF AT ALL.
C  SUBSIDARY SUBROUTINES :
C            LOCAT1(*****).- COMPUTES VECTOR RHO, WHEN NO TWO
C                            DISPLACEMENT VECTORS ARE PARALLEL.
C            LOCAT2(*****).- COMPUTES VECTOR E AND RHO WHEN AT LEAST
C                            TWO DISPLACEMENT VECTORS ARE PARALLEL.
C            ANGLE (*****).- COMPUTES THE ANGLE OF ROTATION.
C            CYCLIC(A,B,C).- PERFORMS A CYCLIC CHANGE OF VECTORS A,B
C                            & C, I.E., A IS SET EQUAL TO B, B IS SET
C                            EQUAL TO C,...
C            EXCHGE (A,B) .- EXCHANGES THE LOCATIONS OF FIELDS A AND B
C
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3),
     -     UA(3),UB(3),UC(3),PERP(3),A(3),B(3)
C
C  COMPUTES THE DISPLACEMENTS
C
      DO 10 I=1,3
         UA(I)=AFIN(I)-AIN(I)
         UB(I)=BFIN(I)-BIN(I)
   10 UC(I)=CFIN(I)-CIN(I)
      GO TO(100,100,100,400,500,600,700),INDEX
C
C  DONE
C  IN WAS SET IN SUBROUTINE SCREW EQUAL TO 1,2 OR 3, DEPENDING ON WHICH
#
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (first part)


```
C  VECTOR IS ZERO, UA, UB, OR UC, RESPECTIVELY
  100 GO TO(120,110,110),IN
  110 CALL CYCLIC(UA,UB,UC)
      CALL CYCLIC(AIN,BIN,CIN)
      IF(IN.EQ.2) GO TO 120
      CALL CYCLIC(UA,UB,UC)
      CALL CYCLIC(AIN,BIN,CIN)
  120 GO TO(130,200,300),INDEX
C
C  COMPUTATION OF RHO AND E WHEN INDEX=1
  130 WRITE(6,1000)
      CALL LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
      GO TO 320
C
C  DONE
C  COMPUTATION OF THE DIRECTION OF THE AXIS OF ROTATION WHEN INDEX=2
  200 WRITE(6,1010)
      DO 210 I=1,3
  210       E(I)=BIN(I)-CIN(I)
      CALL SCAL(E,E,X)
      X=SQRT(X)
      GO TO 305
C
C  DONE
C  COMPUTATION OF THE DIRECTION OF THE AXIS OF ROTATION WHEN INDEX=3
  300 WRITE(6,1020)
      CALL CROSS(UB,UC,E)
      CALL SCAL(E,E,X)
      X=SQRT(X)
  305 DO 307 I=1,3
  307       E(I)=E(I)/X
C
C  DONE
C  COMPUTATION OF THE POINT OF THE AXIS OF ROTATION LYING CLOSEST TO
C  THE ORIGIN.
      CALL SCAL(AIN,E,S)
      DO 310 I=1,3
  310       RHO(I)=AIN(I)-S*E(I)
C
C  DONE
C  COMPUTATION OF THE ANGLE OF ROTATION
  320 CALL ANGLE(BIN,UB,E,RHO,THETA)
      DISPL=0.
      RETURN
C
C  DONE
C  INVESTIGATES THE CASE WHEN TWO DISPLACEMENTS ARE ZERO. IN WAS SET
C  IN SUBROUTINE SCREW EQUAL TO 1,2 OR 3, DEPENDING ON WETHER UA,UB, OR
#
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (second part)

```
C     UC RESPECTIVELY, IS DIFFERENT FROM ZERO. THEN THE SCREW PARAMETERS
C     ARE COMPUTED.
  400 WRITE(6,1030)
      GO TO(420,410,410),IN
  410 CALL CYCLIC(AIN,BIN,CIN)
      CALL CYCLIC(UA,UB,UC)
      IF(IN.EQ.2) GO TO 420
      CALL CYCLIC(AIN,BIN,CIN)
      CALL CYCLIC(UA,UB,UC)
  420 DO 430 I=1,3
  430        E(I)=CIN(I)-BIN(I)
      CALL SCAL(E,E,X)
      X=SQRT(X)
      DO 440 I=1,3
  440        E(I)=E(I)/X
      CALL SCAL(BIN,E,S)
      DO 450 I=1,3
  450        RHO(I)=BIN(I)-S*E(I)
      CALL ANGLE(AIN,UA,E,RHO,THETA)
      DISPL=0.
      RETURN
C
C     DONE
C     COMPUTES THE SCREW PARAMETERS WHEN NO DISPLACEMENT IS ZERO, AND
C     ALL THREE VECTORS ARE NONPARALLEL. FURTHERMORE, THE TWO DIFFERENCE
C     VECTORS ARE NONCOLLINEAR. HENCE THE MOTION IS A PURE ROTATION.
  500 WRITE(6,1040)
      GO TO 610
C
C     COMPUTES THE SCREW PARAMETERS WHEN NO DISPLACEMENT IS ZERO BUT
C     EXACTLY TWO VECTORS ARE PARALLEL AND DIFFERENT. IN IS SET IN SUB-
C     ROUTINE SCREW EQUAL TO 1,2 OR 3 DEPENDING UPON WETHER UC,UB OR UA IS
C     THE NONPARALLEL VECTOR.
  600 WRITE(6,1050)
      IF(IN.NE.2) GO TO 610
      CALL CYCLIC(AIN,BIN,CIN)
      CALL CYCLIC(UA,UB,UC)
  610 CALL CROSS(UA,UC,E)
      CALL SCAL(E,E,S)
      S=SQRT(S)
      DO 620 I=1,3
             E(I)=E(I)/S
  620 CONTINUE
      CALL LOCAT1(AIN,CIN,UA,UC,RHO)
      CALL ANGLE(AIN,UA,E,RHO,THETA)
      DISPL=0.
      RETURN
C
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (third part)

```
C     DONE
C     COMPUTES PARAMETERS FOR NO-ZERO-DISPLACEMENT-CASE WITH ALL THREE
C     VECTORS PARALLEL BUT NO TWO VECTORS IDENTICAL TO EACH OTHER.
  700 WRITE(6,1060)
      CALL LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
      CALL ANGLE(AIN,UA,E,RHO,THETA)
      DISPL=0.
      RETURN
 1000 FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     - 'PARALLEL'/5X,'BUT DISTINTCT. THE MOTION IS PURE ROTATION.',
     - ' INDEX=1'/)
 1010 FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     - 'IDENTICAL.'/5X,'THE MOTION IS PURE ROTATION. INDEX=2'/)
 1020 FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     - 'NEITHER IDENTICAL'/5X,'NOR PARALLEL. THE MOTION IS PURE ',
     - 'ROTATION. INDEX=3'/)
 1030 FORMAT(5X,'TWO DISPLACEMENTS ARE ZERO. THE MOTION IS PURE',
     - ' ROTATION.'/5X,'INDEX=4'/)
 1040 FORMAT(5X,'THE DISPLACEMENTS ARE COPLANAR, THE TWO DISPLACEMENT',
     - ' DIFFERENCES'/5X,'ARE NONCOLLINEAR AND NO DISPLACEMENT IS ',
     - 'ZERO. THE MOTION IS PURE'/5X,'ROTATION. INDEX=5'/)
 1050 FORMAT(5X,'TWO DISPLACEMENTS ARE PARALLEL BUT DIFFERENT AND',
     - ' NO DISPLACEMENT'/5X,'IS ZERO. THE MOTION IS PURE ROTATION.',
     - ' INDEX=6'/)
 1060 FORMAT(5X,'THREE DISPLACEMENTS ARE PARALLEL, BUT NOT ALL THREE',
     - ' ARE IDENTICAL.'/5X,'THE MOTION IS PURE ROTATION. INDEX=7'/)
      END
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (fourth and last part)

```
      SUBROUTINE COPL2(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL
     -                 ,INDE,IN)
C
C   THIS SUBROUTINE COMPUTES THE SCREW PARAMETERS E,RHO,THETA AND DISPL
C   WHEN THE DISPLACEMENTS OF THE THREE GIVEN POINTS ARE COPLANAR. IN
C   WHICH CASE THE POINTS LIE IN A PLANE PARALLEL EITHER TO THE SCREW
C   AXIS OR TO THE AXIS OF ROTATION.
C   THE SUBROUTINE PARAMETERS WERE DEFINED IN SUBROUTINE SCREW EXCEPT
C   FOR INDE AND IN. THESE ARE DEFINED NEXT.
C   INDE = 1, WHEN TWO OF THE SAID DISPLACEMENTS ARE IDENTICAL. IN,
C             DETECT WHICH VECTORS ARE PARALLEL, IF AT ALL.
C   INDE = 2, WHEN ALL THREE DISPLACEMENTS ARE NONPARALLEL BUT THE
C             CORRESPONDING TWO DIFFERENCE DISPLACEMENT VECTORS ARE
C             COLLINEAR.
C   SUBSIDIARY SUBROUTINES WERE ALREADY DESCRIBED IN SUBROUTINE SCREW.
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3),
     -     UA(3),UB(3),UC(3)
      COMMON ZERO
C
C   COMPUTES THE DISPLACEMENTS
C
      DO 10 I=1,3
            UA(I)=AFIN(I)-AIN(I)
            UB(I)=BFIN(I)-BIN(I)
   10 UC(I)=CFIN(I)-CIN(I)
      GO TO(100,200),INDE
C
C   DONE
C   COMPUTES THE SCREW PARAMETERS WHEN INDE =1
C   RELABELS THE POINTS AND THEIR DISPLACEMENTS
  100 WRITE(6,1000)
      GO TO(110,120,130),IN
  110 CALL EXCHGE(AIN,CIN)
      CALL EXCHGE(UA,UC)
      GO TO 130
  120 CALL EXCHGE(AIN,BIN)
      CALL EXCHGE(UA,UB)
  130 DO 140 I=1,3
            E(I)=CIN(I)-BIN(I)
  140 CONTINUE
      CALL SCAL(E,E,S)
      S=SQRT(S)
      DO 150 I=1,3
            E(I)=E(I)/S
  150 CONTINUE
      CALL SCAL(UA,E,DISPL)
C
#
```

Fig 2.6.8 Listing of SUBROUTINE COPL2 (first part)


```
C   DONE
C   ELIMINATES THE TRANSLATION PART OF THE MOTION
C
      DO 160 I=1,3
            S=DISPL*E(I)
            UA(I)=UA(I)-S
            UB(I)=UB(I)-S
  160 UC(I)=UC(I)-S
      GO TO 310
C
C   DONE
C   COMPUTES THE SCREW PARAMETERS WHEN INDE =2
C   DIFFERENCES ARE TEMPORARILY STORED IN AFIN, BFIN AND CFIN
C
  200 WRITE(6,1010)
      DO 210 I=1,3
            AFIN(I)=UA(I)-UC(I)
            BFIN(I)=BIN(I)-CIN(I)
  210 CFIN(I)=AIN(I)-CIN(I)
      CALL SCAL(AFIN,BFIN,PRO1)
      CALL SCAL(AFIN,CFIN,PRO2)
      CALL SCAL(BFIN,BFIN,BC)
      CALL SCAL(CFIN,CFIN,AC)
      BC=SQRT(BC)
      AC=SQRT(AC)
      IF(ABS(PRO2).GT.ZERO) GO TO 240
      DO 230 I=1,3
            E(I)=CFIN(I)/AC
  230 CONTINUE
      GO TO 290
  240 IF(ABS(PRO1).GT.ZERO) GO TO 260
      DO 250 I=1,3
            E(I)=BFIN(I)/BC
  250 CONTINUE
      GO TO 290
  260 QUOT=PRO1/PRO2
      CALL SCAL(BFIN,CFIN,PRO3)
      BETA=(AC*AC*QUOT-PRO3-PRO3)*QUOT+BC*BC
      BETA=1./BETA
      BETA=SQRT(BETA)
      ALPHA=-BETA*QUOT
      DO 280 I=1,3
            E(I)=ALPHA*CFIN(I)+BETA*BFIN(I)
  280 CONTINUE
C
C   COMPUTES THE SCREW DISPLACEMENT
  290 CALL SCAL(UA,E,DISPL)
C
#
```

Fig 2.6.8 Listing of SUBROUTINE COPL2 (second part)

```
C   ELIMINATES THE TRANSLATION PART OF THE MOTION
        DO 300 I=1,3
               S=DISPL*E(I)
               UA(I)=UA(I)-S
  300   UB(I)=UB(I)-S
  310   CALL LOCAT1(AIN,BIN,UA,UB,RHO)
        CALL ANGLE(AIN,UA,E,RHO,THETA)
        RETURN
 1000   FORMAT(5X,'TWO DISPLACEMENTS ARE IDENTICAL. THE POINTS ',
     -          'CORRESPONDING'/5X,'TO THESE DISPLACEMENTS LIE IN A LINE ',
     -          'PARALLEL TO THE SCREW'/5X,'AXIS (OR TO THE AXIS OF '
     -          ,'ROTATION, IF THE MOTION IS PURE'/5X,'ROTATION), INDE=1'/)
 1010   FORMAT(5X,'THE TWO DISPLACEMENT DIFFERENCES ARE COLLINEAR. THE ',
     -          'GIVEN POINTS'/5X,'LIE IN A PLANE PARALLEL TO THE SCREW ',
     -          'AXIS (OR TO THE AXIS OF'/5X,'ROTATION, IF THE MOTION IS ',
     -          'PURE ROTATION), INDE=2'/)
C
C   DONE
        END
```

Fig 2.6.8 Listing of SUBROUTINE COPL2 (third and last part)

```
        SUBROUTINE GENMOT( AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
C
C    THIS PROGRAM COMPUTES THE SCREW PARAMETERS WHEN THE MOTION
C    IS GENERAL AND THE RESULTING DISPLACEMENTS ARE NONCOMPLANAR.
C    SUBSIDIARY SUBROUTINES WERE  ALREADY DESCRIBED IN SUBROUTINE
C    SCREW.
C
        REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3)
        REAL  UA(3), UB(3),UC(3)
        COMMON  ZERO
C
C   COMPUTES THE DISPLACEMENTS
        DO 10 I=1,3
              UA(I)=AFIN(I)-AIN(I)
              UB(I)=BFIN(I)-BIN(I)
   10   UC(I)=CFIN(I)-CIN(I)
C
C   DONE
C   COMPUTES VECTOR E
C   STORES DIFFERENCE VECTORS TEMPORARILY IN AFIN AND BFIN
        DO 20 I=1,3
              AFIN(I)=UA(I)-UC(I)
              BFIN(I)=UB(I)-UC(I)
   20   CONTINUE
        CALL CROSS(AFIN,BFIN,CFIN)
        CALL SCAL(CFIN,CFIN,S)
        S=SQRT(S)
        DO 30 I=1,3
              E(I)=CFIN(I)/S
   30   CONTINUE
C
C   DONE
C   COMPUTES DISPL
        CALL SCAL(UA,E,DISPL)
C
C   DONE
C   STORES DISPLACEMENT VECTOR (UA*E)*E TEMPORARILY IN RHO AND COMPUTES
C   VECTORS UA', UB' AND UC', AND STORES THEM IN UA, UB AND UC,
C   RESPECTIVELY.
        DO 40 I=1,3
              RHO(I)=DISPL*E(I)
               UA(I)=UA(I)-RHO(I)
               UB(I)=UB(I)-RHO(I)
               UC(I)=UC(I)-RHO(I)
   40   CONTINUE
C
C   DONE
C   DETECTS PARALLELISM AMONGST THE MODIFIED DISPLACEMENT VECTORS AND
```

Fig 2.6.9 Listing of SUBROUTINE GENMOT (first part)

```
C   COMPUTES THE SCREW PARAMETERS.
      CALL CROSS(UA,UB,AFIN)
      CALL CROSS(UB,UC,BFIN)
      CALL CROSS(UC,UA,CFIN)
      CALL SCAL(AFIN,AFIN,RHO(1))
      CALL SCAL(BFIN,BFIN,RHO(2))
      CALL SCAL(CFIN,CFIN,RHO(3))
      DO 60 I=1,3
            RHO(I)=SQRT(RHO(I))
            IF(RHO(I).GT.ZERO) GO TO 50
            CALL CYCLIC(AIN,BIN,CIN)
            CALL CYCLIC(UA,UB,UC)
   50       I=3
   60 CONTINUE
      CALL LOCAT1(AIN,BIN,UA,UB,RHO)
      CALL ANGLE(AIN,UA,E,RHO,THETA)
C
C   DONE
      WRITE(6,100)
  100 FORMAT(5X,'THE MOTION IS GENERAL AND THE GIVEN DISPLACEMENTS ARE'
     -       /5X,'NONCOPLANAR'/)
      RETURN
      END
```

Fig 2.6.9 Listing of SUBROUTINE GENMOT (second and last part)

```
      SUBROUTINE LOCAT1(AIN,BIN,UA,UB,RHO)
C
C   THIS SUBROUTINE COMPUTES VECTOR RHO, I.E., THE POINT ON THE AXIS OF
C   A PURE ROTATION LYING CLOSEST TO THE ORIGIN.
C   PROCEDURE :
C   THE PSEUDO-INVERSE FORMULA (BEN-ISRAEL A. AND GREVILLE T.N.E.,
C   GENERALIZED INVERSES THEORY AND APPLICATIONS, WILEY N. YORK, 1974)
C   IS APPLIED TO FIND THE MINIMUM-NORM SOLUTION TO THE OVERDETERMINED
C   LINEAR 2X3 SYSTEM A*X=B, THESE EQUATIONS BEING THOSE OF TWO NON-
C   PARALLEL PLANES. THIS FORMULA THUS FINDS THE POINT OF THE LINE
C   DEFINED BY THE INTERSECTION OF TWO NON-PARALLEL PLANES LYING CLOSEST
C   TO THE ORIGIN.
C   THESE PLANES ARE THE MEDIATOR PLANES OF SEGMENTS AFIN-AIN AND BFIN-
C   BIN.
C
C
C   COMPUTES THE POSITION VECTORS OF THE MID-POINTS OF THE GIVEN
C   SEGMENTS. EACH IS THEN TEMPORARILY STORED IN RHO AND TEMP. THEN
C   CONSTRUCTS VECTOR B.
C
C
      REAL AIN(3),BIN(3),UA(3),UB(3),RHO(3),TEMP(3)
      DO 10 I=1,3
            RHO(I)=AIN(I)+UA(I)*0.5
            TEMP(I)=BIN(I)+UB(I)*0.5
   10 CONTINUE
C
C   BUILDS MATRIX A*(ATRANSP)
      CALL SCAL(UA,UA,A11)
      CALL SCAL(UB,UB,A22)
      CALL SCAL(UA,UB,A12)
      CALL SCAL(UA,RHO,B1)
      CALL SCAL(UB,TEMP,B2)
      DEN=A11*A22-A12*A12
      IF(ABS(DEN).LE.ZERO) GO TO 30
      X1=(B1*A22-B2*A12)/DEN
      X2=(B2*A11-B1*A12)/DEN
      DO 20 I=1,3
            RHO(I)=UA(I)*X1+UB(I)*X2
   20 CONTINUE
      RETURN
   30 WRITE(6,50)
      DO 40 I=1,3
            RHO(I)=UA(I)+UB(I)
            RHO(I)=RHO(I)*0.5
   40 CONTINUE
   50 FORMAT(//5X,'MATRIX A*(AT) IS SINGULAR'/)
      RETURN
      END
```

Fig 2.6.10 Listing of SUBROUTINE LOCAT1

```
      SUBROUTINE LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
C
C     THIS SUBROUTINE COMPUTES VECTORS RHO AND E WHEN ALL THREE RESULTING
C     DISPLACEMENTS ARE PARALLEL BUT NOT TWO VECTORS ARE IDENTICAL TO EACH
C     OTHER.
C     PROCEDURE :
C     EACH PLANE IS DETERMINED BY A TRIAD OF NONCOLLINEAR POINTS (AIN,BIN,
C     CIN, -AFIN,BFIN,CFIN). VECTOR E IS DETERMINED BY THE CROSS PRODUCT
C     OF THE NORMALS TO THE PLANES. RHO IS COMPUTED EXACTLY AS IN LOCAT1.
C
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),DIF1(3),
     -     DIF2(3),PROD(3),RHO(3),E(3)
      DO 10 I=1,3
            DIF1(I)=BIN(I)-AIN(I)
            DIF2(I)=CIN(I)-AIN(I)
   10 CONTINUE
      CALL CROSS(DIF1,DIF2,PROD)
      DO 20 I=1,3
            DIF1(I)=BFIN(I)-AFIN(I)
            DIF2(I)=CFIN(I)-AFIN(I)
   20 CONTINUE
      CALL CROSS(DIF1,DIF2,RHO)
      CALL CROSS(PROD,RHO,E)
      CALL SCAL(E,E,S)
      S=SQRT(S)
      IF(ABS(S).LE.ZERO) GO TO 40
C
C     BUILDS MATRIX A*(ATRANSP) AND VECTOR B.
      CALL SCAL(PROD,PROD,A11)
      CALL SCAL(RHO,RHO,A22)
      CALL SCAL(PROD,RHO,A12)
      CALL SCAL(PROD,CIN,B1)
      CALL SCAL(RHO,CFIN,B2)
      DEN=A11*A22-A12*A12
      IF(ABS(DEN).LE.ZERO) GO TO 40
      T1=(B1*A22-B2*A12)/DEN
      T2=(B2*A11-B1*A12)/DEN
      DO 30 I=1,3
            E(I)=E(I)/S
            RHO(I)=PROD(I)*T1+RHO(I)*T2
   30 CONTINUE
      RETURN
   40 DO 50 I=1,3
            DIF1(I)=CFIN(I)-CIN(I)
   50 CONTINUE
      CALL CROSS(DIF1,PROD,E)
      CALL SCAL(E,E,S)
      S=SQRT(S)
      DO 60 I=1,3
            E(I)=E(I)/S
   60 CONTINUE
      CALL SCAL(AIN,E,T)
      DO 70 I=1,3
            RHO(I)=AIN(I)-T*E(I)
   70 CONTINUE
      RETURN
      END
```

Fig 2.6.11 Listing of SUBROUTINE LOCAT2

```
      SUBROUTINE ANGLE(AIN,UA,E,RHO,THETA)
C
C     THIS SUBROUTINE COMPUTES THE ANGLE OF ROTATION OF A PURE-ROTATION
C     MOTION.
C     PROCEDURE:
C     USE IS MADE OF RODRIGUES' FORMULA (BISHOP K.E.,'RODRIGUES' FORMULA
C     AND THE SCREW MATRIX', JOURNAL OF ENGINEERING FOR INDUSTRY, TRANS.
C     ASME, SERIES B, VOL. 91, FEB. 1969) :
C                                          R2-R1=TAN(THETA/2)EX(R1+R2)
C     WHERE R1 AND R2 ARE THE INITIAL AND THE FINAL POSITION VECTORS OF
C     ONE POINT OF THE BODY NOT LYING ON THE AXIS OF ROTATION. THE ORIGIN
C     IS ASSUMED TO BE LOCATED AT ONE POINT OF THE AXIS OF ROTATION. THETA
C     AND E ARE THE ANGLE OF ROTATION AND THE UNIT VECTOR PARALLEL TO THE
C     AXIS OF ROTATION, RESPECTIVELY.
C
C
      REAL AIN(3),UA(3),E(3),RHO(3),TEMP(3)
      DO 10 I=1,3
            AIN(I)=AIN(I)-RHO(I)
            UA(I)=AIN(I)+UA(I)
            TEMP(I)=AIN(I)+UA(I)
            UA(I)=UA(I)-AIN(I)
   10 CONTINUE
      CALL CROSS(E,TEMP,AIN)
      QUOT=0
      DO 20 I=1,3
            IF(ABS(AIN(I)).LE.ZERO) GO TO 20
            QUOT=UA(I)/AIN(I)
            THETA=ATAN(QUOT)*2.0
            GO TO 30
   20 CONTINUE
   30 IF(ABS(QUOT).GT.ZERO) RETURN
      THETA=ATAN(1.0)*4.0
      RETURN
      END
```

Fig 2.6.12 Listing of SUBROUTINE ANGLE

```
      SUBROUTINE  CYCLIC (A,B,C)
C
C  THIS SUBROUTINE PERFORMS A CYCLIC RELABELLING OF VECTORS A, B, C,
C  I.E. VECTORS A,B AND C ARE RELABELLED B, C AND A RESPECTIVELY.
C
C
      REAL A(3),B(3),C(3),AUX(3)
      DO 10 I=1,3
            AUX(I)=A(I)
            A(I)=B(I)
            B(I)=C(I)
            C(I)=AUX(I)
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.13 Listing of SUBROUTINE CYCLIC

```
      SUBROUTINE  EXCHGE (A,B)
C
C  THIS SUBROUTINE EXCHANGES THE FIELDS OF A AND B, I.E., IT
C  RETURNS B AS A AND A AS B .
C
C
      REAL A(3),B(3),AUX(3)
      DO 10 I=1,3
            AUX(I)=A(I)
            A(I)=B(I)
            B(I)=AUX(I)
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.14 Listing of SUBROUTINE EXCHGE


```
      SUBROUTINE CROSS  (A,B,C)
C
C  THIS SUBROUTINE PERFORMS THE CROSS PRODUCT A AND B, IN THIS ORDER,
C  AND STORES THIS PRODUCT IN C.
C
C
      REAL A(3),B(3),C(3)
      DO 10 K=1,3
            C(K)=0.
            DO 10 L=1,3
                  DO 10 M=1,3
                        N=(L-K)*(M-L)*(K-M)
                        C(K)=C(K)-N*A(L)*B(M)/2.
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.15 Listing of SUBROUTINE CROSS

```
      SUBROUTINE SCAL (A,B,S)
C
C  THIS SUBROUTINE PERFORMS THE SCALAR PRODUCT OF VECTORS A AND B
C  AND STORES THIS PRODUCT IN S.
C
C
      REAL A(3),B(3)
      S=0.
      DO 10 I=1,3
            S=S+A(I)*B(I)
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.16 Listing of SUBROUTINE SCAL

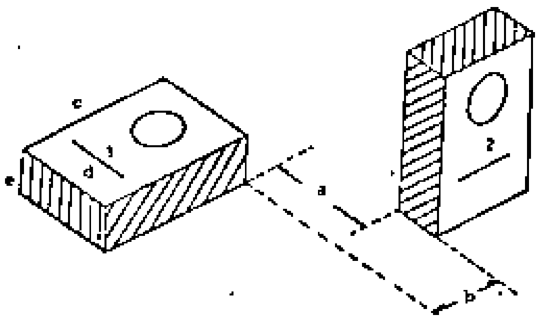


Fig. 2.6.17 Motion of a workpiece

Exercise 2.6.9 (Taken from [2.7]) Let $\underline{a}, \underline{r}, \underline{a}'$ and $\underline{r}'$ be the position vectors of the initial and the displaced positions of points A and B of a rigid body under the screw motion

$$\underline{r}' = \underline{a}' + \underline{Q}(\underline{r} - \underline{a})$$

$\underline{Q}$ being the rotation of the screw. Show that the set of points of the body that, under the given motion remain equidistant from a fixed point P, lie in a plane.

2.7 VELOCITY OF A POINT OF A RIGID BODY ROTATING ABOUT A FIXED POINT.

In the previous sections the motion of a rigid body when moving between two finitely separated configurations was analyzed. In this section and the following ones, the motion of a rigid body between two infinitesimally separated configurations is analyzed. The variables involved in the body motion are considered to be functions of time and results concerning their time derivatives are obtained.

Let $\underline{y}(t)$ be the image of vector $\underline{x}$ under a pure rotation $\underline{Q}(t)$. Clearly,

$\underline{x}$ is an independent variable; however, its image, $\underline{y}(t)$, is a function of time. If the origin of coordinates is placed at the fixed point, then

$$\underline{y}(t) = \underline{Q}(t)\underline{x} \qquad (2.7.1)$$

Differentiating the above equation with respect to time, one obtains

$$\dot{\underline{y}}(t) = \dot{\underline{Q}}(t)\underline{x} \qquad (2.7.2)$$

which is an expression for the velocity of the point located by vector $\underline{x}$ in its initial configuration, at time t. Expression (2.7.2), however, is not practical to compute the velocity of the said point, for it requires knowledge of the point position in its initial configuration. Solving for $\underline{x}$ in eq. (2.7.1) and introducing the corresponding value in eq. (2.7.2) yields

$$\underline{v}(t) = \dot{\underline{y}}(t) = \dot{\underline{Q}}(t)\underline{Q}^T(t)\underline{y}(t) \qquad (2.7.3)$$

which is an expression for the velocity of a point of a rigid body moving about a fixed point, in terms of the current position vector of the moving point. The matrix poduct, $\dot{\underline{Q}}(t)\underline{Q}^T(t)$, called the angular velocity* of the rigid body, represented by $\underline{\Omega}(t)$, is a skew symmetric matrix. Then, the velocity $\underline{v}(t)$ can be expressed as

$$\underline{v}(t) = \underline{\Omega}(t)\underline{y}(t) \qquad (2.7.4a)$$

where

$$\underline{\Omega}(t) = \dot{\underline{Q}}(t)\underline{Q}^T(t)$$

Exercise 2.7.1 Show that, if $\underline{Q}(t)$ is orthogonal, then $\dot{\underline{Q}}(t)\underline{Q}^T(t)$ is skew symmetric.

* Truesdell [2.8] prefers to call it "the spin" and so it is found also under this name in the literature

Exercise 2.7.2 Show that the velocity of a point of a rigid body moving about a fixed point is perpendicular to its position vector (directed from the fixed point). Since $\Omega(t)$ is skew symmetric and 3x3 it is totally determined by three independent scalars, thus being isomorphic to a cartesian vector, $\omega(t)$, called also the angular velocity of the rigid body. Using cartesian vector notation, the velocity $v(t)$ then can be expressed as

$$v(t)=\omega(t)\times r(t) \tag{2.7.5}$$

Exercise 2.7.3 Obtain the components ω_i of vector ω in terms of the components Ω_{ij} of matrix Ω.

Equation 2.7.5 makes the result of Exercise 2.7.2 apparent.

Since $\Omega(t)$ is skew symmetric and 3x3, it has one zero eigenvalue, as is shown below. Furthermore, its other two eigenvalues are complex (and conjugate, of course). Indeed, assume $Q(t)$ is in its canonical form, i.e.

$$\left(Q(t)\right)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ \sin\theta & 0 & 1\end{pmatrix} \tag{2.7.6}$$

$$\left(\dot{Q}(t)\right)=\begin{pmatrix}-\sin\theta & -\cos\theta & 0\\ \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta} \tag{2.7.7}$$

From the above expressions,

$$\dot{Q}(t)Q^T(t) = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta} \tag{2.7.8}$$

which makes evident that all vectors of the form $(a) = (0,0,a)^T$, a being any scalar, correspond to a zero eigenvalue. The other two eigenvalues

are readily found to be

$$\lambda_1=\dot{\theta}i,\ \lambda_2=-\dot{\theta}i \tag{2.7.9}$$

where i is the imaginary unity, $\sqrt{-1}$.

The null space of $\Omega(t)$ (see Sec. 1.3) is, from the foregoing discussion, of dimension 1, i.e., a line. All the points lying on that line have zero velocity, the line thus being called "the instant axis of rotation" of the rigid body.

A cone rolling without slipping on a plane is a simple example of a rigid body rotating about a fixed point, its apex; its instant axis of rotation is clearly, the element of the cone touching instantaneously the plane. Another example would be a sphere rotating on a plane in such a way that the contact point remains fixed; the instant axis of rotation of the sphere is thus the diameter passing through the point of contact.

Exercise 2.7.4 A cone of revolution rolls on a conic surface, also of revolution, without slipping, in such a way that both appices are coincident. What is the instant axis of rotation of the cones in motion?

Exercise 2.7.5 Show that the spin matrix Ω can be written as

$$\Omega=A\dot{\theta}$$

where A is a constant matrix and $\dot{\theta}$ is the time derivative of θ, the rotation angle.

2.8 VELOCITY OF A MOVING POINT REFERRED TO A MOVING OBSERVER.

In what follows, an observer will be understood to be a set of coordinate axes provided with a clock (2.8, p.26). Assume a point P_o, located by vector x_o, is the origin of a coordinate system in motion with respect to another coordinate system, which will be arbitrarily referred to as "fixed".

The latter system constitutes a fixed observer, whereas the first, a moving one.

Let $\underline{x}$ be the position vector of a point P, in motion with respect to both observers. Vectors and matrices expressed with respect to the fixed observer will be indexed with letter F, whereas those expressed with respect to the moving one will be indexed with letter M. Let $\underline{\xi}$ be the position vector of P in the moving observer and $\underline{Q}$ the rotation dyadic from the fixed observer to the moving one.

Hence,

$$(\underline{x})_F = (\underline{x}_o)_F + (\underline{\xi})_F \qquad (2.8.1)$$

where it is understood that all three vectors are functions of time.

The velocity of P is obtained differentiating both sides of eq. (2.8.1) with respect to time, i.e.

$$(\underline{v})_F = (\underline{v}_o)_F + (\dot{\underline{\xi}})_F \qquad (2.8.2)$$

where v_o is the velocity of point P_o and, since

$$(\underline{\xi})_F = (\underline{Q})_F(\underline{\xi})_M \qquad (2.8.3)$$

then

$$(\dot{\underline{\xi}})_F = (\dot{\underline{Q}})_F(\underline{\xi})_M + (\underline{Q})_F(\dot{\underline{\xi}})_M =$$

$$= (\underline{\Omega})_F(\underline{\xi})_F + (\underline{Q})_F(\dot{\underline{\xi}})_M \qquad (2.8.4)$$

Thus, eq. (2.8.2) becomes

$$(\underline{v})_F = (\underline{v}_o)_F + (\underline{\Omega})_F(\underline{\xi})_F + (\underline{Q})_F(\dot{\underline{\xi}})_M =$$

$$= (\underline{v}_o)_F + (\underline{\Omega})_F(\underline{Q})_F(\underline{\xi})_M + (\underline{Q})_F(\dot{\underline{\xi}})_M \qquad (2.8.5)$$

where the first two terms of the right hand side represent the velocity of P as if it were one point of the rigid body defined by the moving observer*

* See Section 2.9

and the last term is the velocity of P as measured by the moving observer. Matrix $(\underline{Q})_F$ transfers the description of velocity $(\dot{\underline{\xi}})_M$ to the fixed observer. Thus, eq. (2.8.5) states that the velocity of point P equals that of point P as if it were fixed to the moving coordinate axes, plus the velocity of point P as if the moving observer were fixed.

Given any two points, P_1 and P_2, moving with velocities $\underline{v}_1$ and $\underline{v}_2$, respectively, "<u>the relative velocity of P_2 with respect to P_1</u>" is defined as

$$\underline{v}_{2/1} = \underline{v}_2 - \underline{v}_1 \qquad (2.8.6)$$

Similarly, the relative angular velocity of body 2, moving with angular velocity $\underline{\Omega}_2$, with respect to body 1, moving with angular velocity $\underline{\Omega}_1$, is defined as

$$\underline{\Omega}_{2/1} = \underline{\Omega}_2 - \underline{\Omega}_1 \qquad (2.8.7)$$

or, alternatively,

$$\underline{\omega}_{2/1} = \underline{\omega}_2 - \underline{\omega}_1 \qquad (2.8.7)$$

2.9 <u>GENERAL MOTION OF A RIGID BODY</u>

Let a rigid body B undergo the most general motion, i.e., in general, no point of B remains fixed. Let $\underline{v}_p$ be the velocity of one of its points, P, with position vector $\underline{x}_p$ and angular velocity $\underline{\Omega}$.

Thus, the relative velocity, $\underline{v}-\underline{v}_p$, of any other point R (located by $\underline{x}$) with respect to P, is given by

$$\underline{v} - \underline{v}_p = \underline{\Omega}(\underline{x} - \underline{x}_p) \qquad (2.9.1)$$

for $\underline{v}-\underline{v}_p$ is the velocity that R would have if P were fixed. From eq. (2.9.1),

$$\underline{v} = \underline{v}_p + \underline{\Omega}(\underline{x} - \underline{x}_p) \qquad (2.9.2)$$

is the velocity of point R, and is given in terms of the velocity of another point, P, the angular velocity Ω and the position vector of R with respect to P.

Given two rigid bodies in motion, body 1 rolls without slipping with respect to body 2 if, and only if, there exists a set of points on both 1 and 2 such that the relative velocity of points on that set is zero. Regarding the velocity $\mathbf{v}$, as given by eq. (2.9.2) as the relative velocity of one point R of body B with respect to body C, the fixed observer, the condition for B to roll without slipping on C is that there exists a set of points, whose position vector is given by $\mathbf{x}$, for which $\mathbf{v}=\mathbf{0}$. But for this to happen, the condition is

$$\mathbf{v}_P = -\Omega(\mathbf{x}-\mathbf{x}_P) \qquad (2.9.3)$$

which states that $\mathbf{v}_P$ is in the range (See section 1.3) of Ω. However, it was shown in Section 2.8 that the null space of Ω if of dimension 1; hence -eq. (1.3.1)- the range of Ω is of dimension 2, thereby existing vectors in E^3 not belonging to the range of Ω. If $\mathbf{v}_P$ happens to lie outside the range of Ω, it is impossible to find a vector $\mathbf{x}$ for which eq. (2.9.3) is satisfied. Those vectors lying outside the range of Ω lie necessarily on its null space, i.e., on a line parallel to the eigenvector of Ω corresponding to its zero eigenvalue or, equivalently, are parallel to the vector ω associated with Ω. In case $\mathbf{v}_P$ has a nonzero component along the null space of Ω, body B is said to slide on body C, which is the case of the worm-gear or of the hypoid gear couplings. In those couplings there are power losses due to the involved sliding and, since the dissipated power is proportional to the sliding velocity, the contact between the two mechanical elements under consideration should take place along points of minimum magnitude of sliding velocity. For hypoid gears this set of points lie on a line which, paralleling Chasles' Theorem, is called "the instant axis of the screw motion of body B with respect to C" or, for short, "the instant screw axis". Indeed, let the sliding velocity be given by eq. (2.9.2). Finding the points of minimum magnitude of sliding velocity corresponds to finding the vectors $\mathbf{x}$ of expression (2.8.2) which minimize the quadratic form $\phi(\mathbf{x})=\mathbf{v}^T\mathbf{v}$, which has a stationary value (Section 1.7C) when $\phi'(\mathbf{x})$ vanishes.

Applying the "chain rule" to $\phi(\mathbf{x})$,

$$\phi'(\mathbf{x})=2\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\mathbf{v} \qquad (2.9.4a)$$

where, from eq. (2.9.2),

$$\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \Omega \qquad (2.9.4b)$$

Thus, points $\mathbf{x}_0$ yielding an extremum of $\phi(\mathbf{x})$ satisfy the equation

$$\Omega^T\mathbf{v} = -\Omega\mathbf{v}=\mathbf{0} \qquad (2.9.5)$$

Exercise 2.9.1 Show that the gradient of $\phi(\mathbf{x})$ is twice the left hand side of eq. (2.9.5).

Since Ω has one zero eigenvalue (and only one), eq. (2.9.5) states that the minimum-magnitude velocity, given by

$$\mathbf{v}_0 = \mathbf{v}_P + \Omega(\mathbf{x}_0-\mathbf{x}_P) \qquad (2.9.6)$$

is in the null space of Ω, i.e. is a vector parallel to ω. Notice that eq. (2.9.6) does not yield a unique vector $\mathbf{x}_0$ for, if any vector $\alpha\omega$ (in the null space of Ω) is added to $\mathbf{x}_0$, the velocity $\mathbf{v}_N$ of the new point, is given by

$$\mathbf{v}_N=\mathbf{v}_P+\Omega(\mathbf{x}_0+\alpha\omega-\mathbf{x}_P)=\mathbf{v}_P+\Omega(\mathbf{x}_0-\mathbf{x}_P)$$

Hence, the points of minimum-magnitude velocity lie in a line passing through one point $\mathbf{x}_0$, in the direction of ω, this line being the instant screw axis of the motion under study. The particular point P_0 on the screw axis, located by x_0, is chosen such that x_0 be normal to the screw axis; thus, x_0 happens to be the minimum-norm vector satisfying (2.9.6). This vector can be found in a similar way as vector r_0 of eq. (2.6.8) was found, namely, choose two linearly independent equations out of eq (2.9.5) and form the system

$$Ax_0=b \tag{2.9.7}$$

where A is a 2x3 matrix and b is a two-dimensional vector. Hence, the minimum-norm solution x_0 is given as

$$x_0=A^T(AA^T)^{-1}b \tag{2.9.8}$$

An alternative way of finding x_0 is now presented, expressing eq (2.9.5) in cartesian vector form, namely,

$$\omega\times\left(v_P+\omega\times(r_0-r_P)\right)=0 \tag{2.9.9}$$

which can be expanded as

$$\omega\times v_P+\omega\times\left(\omega\times(r_0-r_P)\right)=0$$

or, expanding the double cross product*,

$$\omega\times v_P+\left(\omega\cdot(r_0-r_P)\right)\omega-\omega^2(r_0-r_P)=0$$

If now r_0 is specified to be normal to ω, i.e., to be the minimum-norm vector of all those satifying eq. (2.9.9), then it can be obtained from the above equation as

$$r_0=r_P+\frac{1}{\omega^2}\left(\omega\times v_P-(\omega\cdot r_P)\omega\right) \tag{2.9.10}$$

which is an expression similar to that appearing in eq. (2.6.15).

---

* $\omega^2 = \omega\cdot\omega$

The foregoing results are summarized in a theorem similar to that of Chasles'.

<u>THEOREM 2.9.1</u> *Any rigid body motion is equivalent to a screw motion, composed of a velocity v_0 and a spin about an axis parallel to v_0. The points whose velocity is v_0 are located on a line parallel to v_0, called the instant screw axis and v_0 is of* ***minimum magnitude****. The screw axis passes through point P_0 whose position vector is given either by eq. (2.9.8) or by eq. (2.9.10)*

The counterpart of Theorem 2.6.2 now follows:

<u>THEOREM 2.9.2</u> *The velocities of all the points of a rigid body undergoing an arbitrary motion have identical projections along the instant screw axis.*

<u>Proof</u>:

The velocity of any point of the rigid body can be written as

$$v=v_P+\omega\times(r-r_P)$$

Dot multiplying both sides of the above equation times ω(a vector parallel to the screw axis) yields

$$v\cdot\omega=v_P\cdot\omega+\omega\times(r-r_P)\cdot\omega$$

But the second term of the right hand side clearly vanishes. Hence

$$v\cdot\omega=v_P\cdot\omega$$ q.e.d.

By virtue of the latter result, the projection of the velocities of all the points of a rigid body in motion along the screw axis is given by $||v_0||$, which is called "the sliding". The pitch of the instant screw is given by

$$\lambda=\frac{2\pi||v_0||}{||\omega||} \tag{2.9.11}$$

which is the counterpart of eq. (2.6.21a)

After Theorem 2.9.2, there follows one method of determining the orientation $\underline{e}$, of the screw axis of a rigid body motion when the non-coplanar velocities of three points A, B and C of a rigid body are known. Indeed, paralleling the derivation of eq. (2.7.24) one obtains

$$\underline{e}=\frac{(\underline{v}_A-\underline{v}_C)\times(\underline{v}_B-\underline{v}_C)}{||(\underline{v}_A-\underline{v}_C)\times(\underline{v}_B-\underline{v}_C)||} \quad (2.9.12)$$

Exercise 2.9.1 Show that the velocity of all the points of a rigid body, three of whose points, A, B and C, have velocities $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$, respectively, have identical projections along vector $\underline{e}$, as given by eq. (2.9.12).

The sliding velocity, $\underline{v}_0$, can then be obtained as

$$\underline{v}_0=(\underline{v}_A\cdot\underline{e})\underline{e}\ \mathrm{sgn}\ (\underline{v}_A\cdot\underline{e}) \quad (2.9.13)$$

where the signum function has been introduced in order to make the directions of both $\underline{v}_0$ and $\underline{e}$ coincident.

To completely determine the instant screw, only the angular velocity $\underline{\omega}$ needs be computed. This is done in what follows, after deriving some results similar to those of Section 2.6

Corollary 2.9.1 If at least one point of a rigid body has a velocity which is normal to its angular velocity or zero, the body undergoes a pure rotation.

The foregoing Corollary is a direct consequence of Theorem 2.9.2 and so its proof is left as an exercise for the reader.

Exercise 2.9.2 Prove Corollary 2.9.1

THEOREM 2.9.3 The difference vector of the velocities of any two points of a rigid body undergoing an arbitrary motion is perpendicular to the instant screw axis.

Proof

Let $\underline{v}_A$ and $\underline{v}_B$ be the velocities of two points, A and B, of a rigid body. From eq. (2.9.2) these are related by

$$\underline{v}_B=\underline{v}_A+\underline{\Omega}(\underline{b}-\underline{a})$$

Hence, the difference, $\underline{d}$, is given by

$$\underline{d}=\underline{v}_B-\underline{v}_A=\underline{\Omega}(\underline{b}-\underline{a})$$

which makes it clear that $\underline{d}$ lies in the range (See Section 1.3) of $\underline{\Omega}$, thereby being normal to the null space of $\underline{\Omega}$, i.e., normal to the screw axis. Alternately this result can be proved resorting to Gibbs' notation. This way, $\underline{d}$ can be written as

$$\underline{d}=\underline{\omega}\times(\underline{b}-\underline{a})$$

and hence

$$\underline{d}\cdot\underline{\omega}=\underline{\omega}\times(\underline{b}-\underline{a})\cdot\underline{\omega}$$

which vanishes because the double product at the right hand side contains two identical vectors, q.e.d.

THEOREM 2.9.4 If the velocities of three noncollinear points of a rigid body are identical, the body undergoes a pure translation.

Proof

Let $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$ be the respective velocities of points A, B and C. Referring these to the velocity of an arbitrary point P, one obtains

$$\underline{v}_A=\underline{v}_P+\underline{\Omega}(\underline{a}-\underline{p})$$

$$\underline{v}_B=\underline{v}_P+\underline{\Omega}(\underline{b}-\underline{p})$$

$$\underline{v}_C=\underline{v}_P+\underline{\Omega}(\underline{c}-\underline{p})$$

Subtracting the third equation from the first two yields

$$\underline{v}_A-\underline{v}_C=\underline{\Omega}(\underline{a}-\underline{c})=\underline{0}$$

$$\underline{v}_B-\underline{v}_C=\underline{\Omega}(\underline{b}-\underline{c})=\underline{0}$$

which implies that both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ lie the null space of $\underline{\Omega}$. This space, however, is of dimension 1, as discussed in Section 2.7. Since points A, B and C are noncollinear, vectors $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ are linearly independent and hence cannot be simultaneously in the null space of $\underline{\Omega}$, unless $\underline{\Omega}=\underline{0}$, the motion thus reducing to a pure translation, q.e.d.

THEOREM 2.9.5 *The nonidentical velocities of three points of a rigid body are coplanar if and only if one of the following conditions is met:*

i] The motion is a pure rotation

ii] The motion is general, but the points are collinear

iii] The motion is general and the points are not collinear, but lie in a plane parallel to the screw axis.

Proof

"If" part:

i) If the motion is a pure rotation, the velocity of any point with position vector $\underline{r}$ is given by

$$\underline{v}=\underline{\Omega}\underline{r}$$

which states that $\underline{v}$ lies in the range (See Section 1. [illegible]) of $\underline{\Omega}$, which is of dimension 2, as was discussed in Section 2.7. This means that all velocity vectors lie in a plane perpendicular to the null space of $\underline{\Omega}$, i.e. perpendicular to the axis of rotation, thereby showing that these velocities are coplanar.

ii) Let A, B and C be three collinear points of the rigid body and $\underline{a}$, $\underline{b}$ and $\underline{c}$ be their respective position vectors. The velocities of these points, referred to an arbitrary point with position vector $\underline{p}$ are

$$\underline{v}_A=\underline{v}_P+\underline{\Omega}(\underline{a}-\underline{p})$$

$$\underline{v}_B=\underline{v}_P+\underline{\Omega}(\underline{b}-\underline{p})$$

$$\underline{v}_C=\underline{v}_P+\underline{\Omega}(\underline{c}-\underline{p})$$

Since the points are collinear, their position vectors are related by

$$\underline{c}-\underline{a}=\alpha(\underline{b}-\underline{a})$$

Now, adding $\underline{\Omega}\underline{a}$ to $\underline{v}_C$ and subtracting it simultaneously from the same expression, one obtains

$$\underline{v}_C=\underline{v}_P+\underline{\Omega}(\underline{a}-\underline{p})+\underline{\Omega}(\underline{c}-\underline{a})$$

whose first two terms can be readily identified with $\underline{v}_A$. Moreover, substituting $\underline{c}-\underline{a}$ in the third term of the latter equation by $\alpha(\underline{b}-\underline{a})$, as given above, leads to

$$\underline{v}_C=\underline{v}_A+\alpha\underline{\Omega}(\underline{b}-\underline{a})$$

But

$$\underline{\Omega}(\underline{b}-\underline{a})=\underline{v}_B-\underline{v}_A$$

Hence, the expression for $\underline{v}_C$ is transformed into

$$\underline{v}_C=\underline{v}_A+\alpha(\underline{v}_B-\underline{v}_A)$$

or, equivalently,

$$\underline{v}_C=(1-\alpha)\underline{v}_A+\alpha\underline{v}_B$$

thereby proving the linear dependence, i.e. the coplanarity of $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$.

iii) The velocities of the three given points, A, B and C, are referred to that of a point P on the screw axis. These velocities take on the form appearing in ii. Thus, the velocity of P, $\underline{v}_P$, is parallel to the screw axis. On the other hand, the fact that A, B and C lie in a plane parallel to the screw axis allows one to establish the following relationship

$$\underline{c}-\underline{a}=\alpha(\underline{b}-\underline{a})+\beta\underline{v}_P$$

or, equivalently,

$$\underline{c}=(1-\alpha)\underline{a}+\alpha\underline{b}+\beta\underline{v}_P$$

Substituting the latter expression in v_C as given in ii leads to

$$v_C = v_P + \Omega(c-p) =$$

$$= v_P + \Omega(a-p) - \alpha\Omega(b-a) + \beta\Omega v_P$$

whose two first terms can be readily identified as v_A, its fourth term vanishing because it lies in the null space of Ω. Hence

$$v_C = v_A - \alpha\Omega(b-a)$$

But

$$\Omega(b-a) = v_B - v_A$$

Thus, the latter expression for v_C is transformed into

$$v_C = v_A - \alpha\Omega(v_B - v_A)$$

which shows the linear dependence of the three given velocity vectors, i.e. its coplanarity.

"only if" part:

Assuming that the velocities v_A, v_B and v_C of three given points A, B and C are coplanar, the following relationship holds

$$\det(v_A, v_B, v_C) = 0$$

Referring v_B and v_C to v_A one has

$$v_B = v_A + \Omega(b-a)$$

$$v_C = v_A + \Omega(c-a)$$

Thus, the above expression for the determinant becomes

$$\det\{v_A, v_A + \Omega(b-a), v_A + \Omega(c-a)\} = 0$$

Subtracting the first column of this determinant from the remaining ones does not change the value of the determinant. Hence

$$\det(v_A, \Omega(b-a), \Omega(c-a)) = 0$$

which is equivalent to

$$\Omega(b-a) \times \Omega(c-a) \cdot v_A = 0$$

Introducing Gibbs' notation, and expanding the resulting expression,

$$\Omega(b-a) \times \Omega(c-a) = \left(\omega \times (b-a)\right) \times \left(\omega \times (c-a)\right) =$$

$$= \left(\omega \times (b-a) \cdot (c-a)\right)\omega - \left(\omega \times (b-a) \cdot \omega\right)(c-a)$$

where the expression in brackets in the second term of the rightmost hand side clearly vanishes. Hence

$$\Omega(b-a) \times \Omega(c-a) \cdot v_A = \left(\omega \times (b-a) \cdot (c-a)\right)\omega \cdot v_A$$

which vanishes under one of the following conditions:

i) $\omega \cdot v_A = 0$

which implies, under Corollary 2.9.1, that the motion is a pure rotation

ii) $(b-a) \times (c-a) = 0$

which means that points A, B and C are collinear

iii) $\omega \times (b-a) \cdot (c-a) = 0$

which indicates that vectors ω, $b-a$ and $c-a$ are coplanar, q.e.d.

A direct consequence of the foregoing result is the following

<u>Corollary 2.9.2</u> *Assume a rigid body under motion and choose any three noncollinear points A, B and C of the body. Letting v_A, v_B and v_C be the three involved velocities, then the difference vectors $v_A - v_C$ and $v_B - v_C$ (and, consequently, $v_A - v_B$) are parallel if and only if the points lie in a plane parallel to the screw axis when the motion is general. If the motion reduces to a pure rotation, then the said plane is parallel to the axis of rotation.*

<u>Exercise 2.9.1</u> Prove Corollary 2.9.2

More results connected with the present discussion are the following

<u>Corollary 2.9.3</u> The velocities of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation

<u>Corollary 2.9.4</u> If two, and only two, velocities of three noncollinear points of a rigid body are parallel, then either i) the parallel velocities are identical and belong to points lying on a line parallel to the screw axis, or ii) the parallel velocities are different from each other, in which case the motion is a pure rotation whose axis is parallel to the line connecting the two points of parallel velocities.

<u>Corollary 2.9.5</u> Given three noncollinear points, A, B and C, of a rigid body in motion, such that $\underline{v}_C=\underline{0}$ and $\underline{v}_A$ and $\underline{v}_B$ are parallel but distinct, i.e. $\underline{v}_B=\alpha\underline{v}_A$, then the body undergoes a pure rotation and its axis passes through C and is parallel to vector $\underline{b}-\alpha\underline{a}$, $\underline{a}$ and $\underline{b}$ being the position vectors of A and B, respectively. If $\underline{v}_A=\underline{v}_B$, then the axis of rotation is parallel to line AB.

The computation of $\underline{\omega}$ given the velocities of three noncollinear points is next discussed. Two cases are considered: i) the arising difference vectors are noncollinear, and ii) these vectors are collinear.

In what follows let A, B and C be the three involved points, $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$ being their corresponding velocities. Then,

i) The difference vectors are not collinear

$$\underline{v}_{C/A}=\underline{\omega}\times(\underline{c}-\underline{a}) \qquad (2.9.14)$$

Hence

$$\underline{v}_{B/A}\times\underline{v}_{C/A}=\underline{v}_{B/A}\times\left(\underline{\omega}\times(\underline{c}-\underline{a})\right)=$$
$$=\left(\underline{v}_{B/A}\cdot(\underline{c}-\underline{a})\right)\underline{\omega}-(\underline{v}_{B/A}\cdot\underline{\omega})(\underline{c}-\underline{a}) \qquad (2.9.15)$$

But

$$\underline{v}_{B/A}\cdot\underline{\omega}=\left(\underline{\omega}\times(\underline{b}-\underline{a})\right)\cdot\underline{\omega}=0 \qquad (2.9.16)$$

Thus, $\underline{\omega}$ can be solved for from eq. (2.9.15) as

$$\underline{\omega}=\frac{\underline{v}_{B/A}\times\underline{v}_{C/A}}{\underline{v}_{B/A}\cdot(\underline{c}-\underline{a})} \qquad (2.9.17)$$

which is the desired expression for $\underline{\omega}$, irrespective of whether the motion is general or a pure rotation.

ii) The difference vectors are collinear. In this case the points lie in a plane parallel to the instant screw axis. Due to the collinearity of the difference vectors, the cross product of the left-hand side of eq. (2.9.15) vanishes, thus making it impossible to compute $\underline{\omega}$ using the procedure of case i). Thus, a different approach is introduced. According to Corollary 2.9.2, the given points lie in a plane parallel to the instant screw axis, i.e. to $\underline{\omega}$. Hence, the following holds.

$$\underline{\omega}=\alpha(\underline{a}-\underline{c})+\beta(\underline{b}-\underline{c}) \qquad (2.9.18)$$

According to Theorem 2.9.3,

$$(\underline{v}_A-\underline{v}_C)\cdot\underline{\omega}=0 \qquad (2.9.19)$$

or, substituting eq. (2.9.18) into eq. (2.9.19),

$$\alpha(\underline{v}_A-\underline{v}_C)\cdot(\underline{a}-\underline{c})+\beta(\underline{v}_A-\underline{v}_C)\cdot(\underline{b}-\underline{c})=0 \qquad (2.9.20)$$

Now, several possibilities can arise, namely

.1) The relative velocity $\underline{v}_A-\underline{v}_C$ is perpendicular to $\underline{a}-\underline{c}$, in which case (2.9.20) holds only if β vanishes. Indeed, $\underline{v}_A-\underline{v}_C$ cannot be simultaneously perpendicular to both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ for these vectors are nonparallel, given the assumed noncollinearity of points A, B and C. Hence

$$\underline{\omega}=\alpha(\underline{a}-\underline{c})$$

where α is computed from

103

$$\mathbf{v}_{B/C} = \boldsymbol{\omega}\times(\mathbf{b}-\mathbf{c}) \qquad (2.9.21)$$

i.e.,

$$\mathbf{v}_B - \mathbf{v}_C = \alpha(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c}) \qquad (2.9.21a)$$

Dot-multiplying the latter equation times $\mathbf{v}_C$,

$$(\mathbf{v}_B-\mathbf{v}_C)\cdot\mathbf{v}_C = \alpha(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})\cdot\mathbf{v}_C$$

from which

$$\alpha = \frac{(\mathbf{v}_B-\mathbf{v}_C)\cdot\mathbf{v}_C}{(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})\cdot\mathbf{v}_C}$$

readily follows. In the latter equation it might have happened that the dot product vanishes. In this case, α cannot be solved for due to the arising indeterminacy. This indeterminacy, however, can be resolved by dot multiplying times $\mathbf{v}_A$ or $\mathbf{v}_B$, eq. (2.9.21a), instead.

ii.2) The relative velocity $\mathbf{v}_A-\mathbf{v}_C$ is perpendicular to $\mathbf{b}-\mathbf{c}$, in which case eq. (2.9.20) holds only if α vanishes, resorting to the same argument as in ii.1. Hence

$$\boldsymbol{\omega} = \beta(\mathbf{b}-\mathbf{c})$$

the constant β being determined as in ii.1.

ii.3) No inner product in (2.9.20) vanishes. Hence β can be solved for as

$$\beta = -\frac{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{a}-\mathbf{c})}{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{b}-\mathbf{c})}\,\alpha$$

Substituting the latter expression into eq. (2.9.18),

$$\boldsymbol{\omega} = \left((\mathbf{a}-\mathbf{c}) - \frac{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{a}-\mathbf{c})}{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{b}-\mathbf{c})}(\mathbf{b}-\mathbf{c})\right)\alpha \qquad (2.9.22)$$

where α can be computed as before. Indeed, substituting eq. (2.9.22) into eq. (2.9.21), one obtains

104

$$\mathbf{v}_{B/C} = \alpha\left((\mathbf{a}-\mathbf{c}) - \frac{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{a}-\mathbf{c})}{(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{b}-\mathbf{c})}(\mathbf{b}-\mathbf{c})\right)\times(\mathbf{b}-\mathbf{c}) \qquad (2.9.23)$$

Hence

$$\mathbf{v}_{B/C} = \alpha(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c}) \qquad (2.9.24)$$

Dot-multiplying both sides of eq. (2.9.24) times $\mathbf{v}_C$, α can be solved for as

$$\alpha = \frac{\mathbf{v}_{B/C}\cdot\mathbf{v}_C}{(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})\cdot\mathbf{v}_C} \qquad (2.9.25)$$

Again, if $\mathbf{v}_{B/C}\cdot\mathbf{v}_C$ happens to vanish then eq. (2.9.25) should be dot-multiplied times either $\mathbf{v}_A$ or $\mathbf{v}_B$ instead

The computation of the instant-screw parameters (the instant-screw axis, the sliding velocity and the spin) is carried on by SUBROUTINE INSCRU, which parallels SUBROUTINE SCREW and thus considers all cases that could arise regarding the relationships amongst all three velocity vectors. These possible cases are shown in the "tree" diagram appearing in Fig 2.9.1. INSCRU uses the following auxiliary subroutines:

1. SUBROUTINE COP1 computes the instant-screw parameters when the motion is pure rotation. It distinguishes amongst the different cases with the aid of the integer variable INDEX
2. SUBROUTINE COP2 computes the instant-screw parameters when the points lie in a plane parallel either to the instant-screw axis or to the instant axis of rotation. Two different cases could arise, which are distinguished with the aid of the integer variable INDE.
3. SUBROUTINE GENO computes the instant-screw parameters when the motion is general and the three given velocities are noncoplanar.

The computation procedure for each case is next described. All over, the three given points are A, B and C, their respective position vectors being

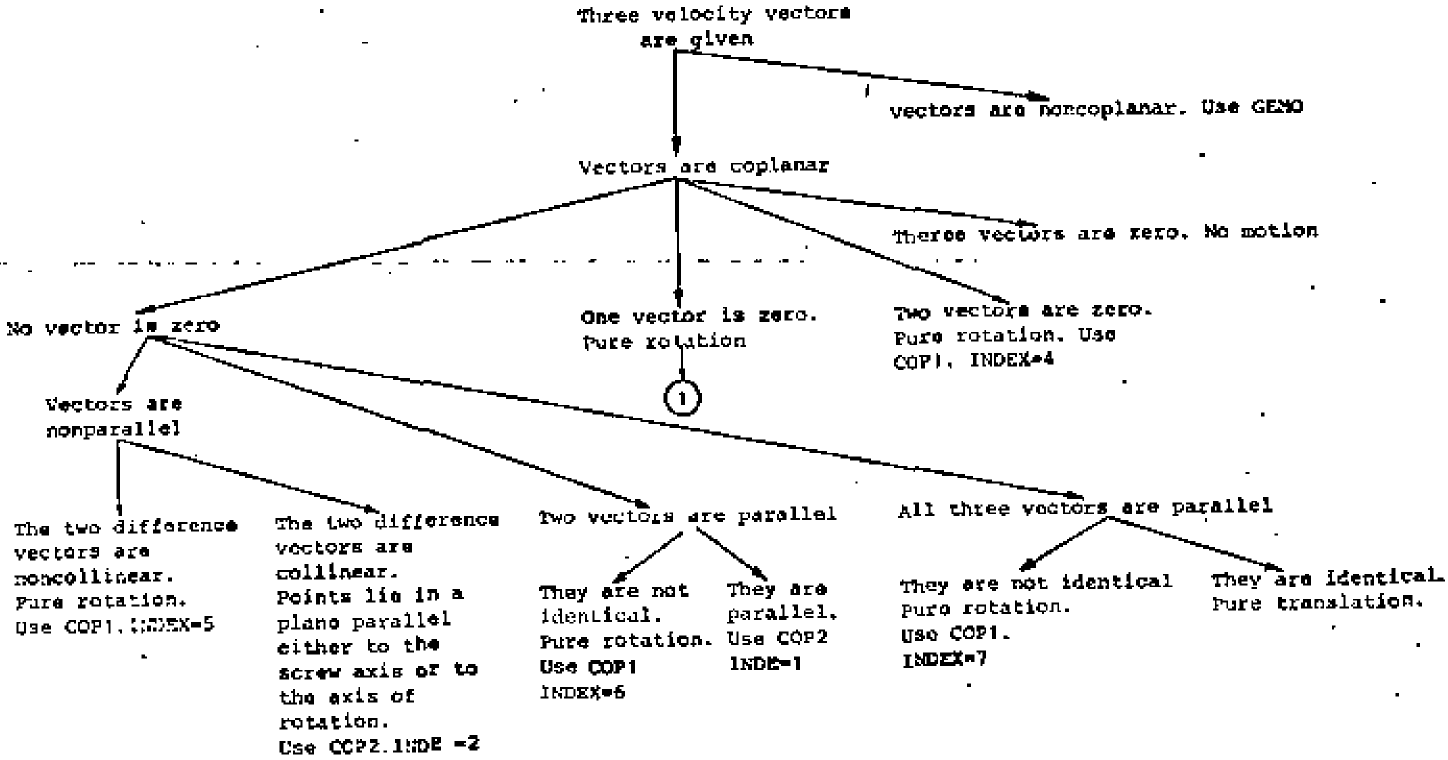


Fig 2.9.1 Tree diagram showing all possible relationships amongst the velocities of three noncollinear points of a rigid body.

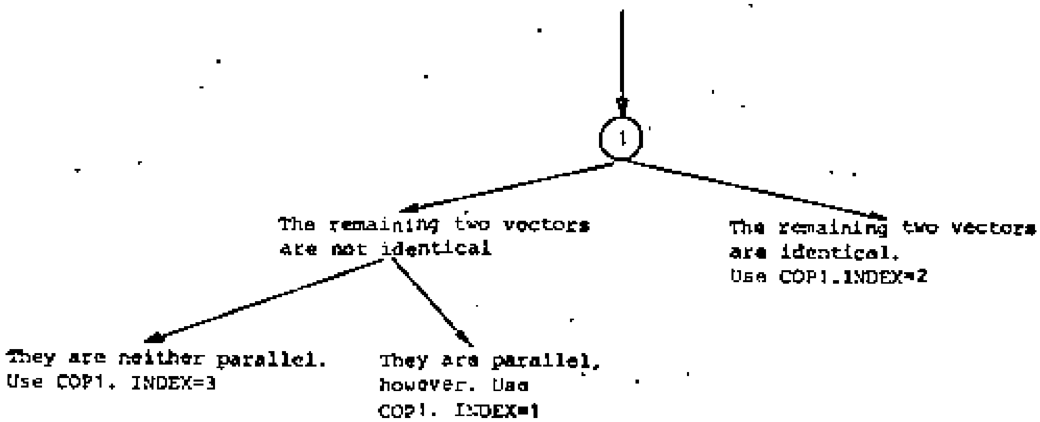


Fig 2.9.1 (continues)

a, **b** and **c**. Their velocities are v_A, v_B and v_C

INDEX = 1. One vector is zero and the remaining two vectors are not identical; they are parallel, however. The motion is pure rotation according to Corollary 2.9.1 and the axis of rotation is located following Corollary 2.9.5

INDEX = 2. One vector is zero and the remaining two vectors are identical. The motion is pure rotation again and the axis of rotation passes through the point of zero velocity in the direction of the line joining the other two points, according to Corollary 2.9.5

INDEX = 3. One vector is zero and the remaining two vectors are not parallel. The motion is pure rotation as before, Attending Corollary 2.9.1, and the axis of rotation passes through the point of zero velocity in the direction of the normal to the plane defined by the other two velocities.

INDEX = 4. Two vectors are zero. The motion is pure rotation and the axis of rotation is defined by the two points of zero velocity.

INDEX = 5. No vector is zero and all three vectors are nonparallel amongst them but coplanar. Furthermore, the two arising difference vectors are noncollinear. According to Theorem 2.9.5 and Corollary 2.9.2 then, the motion is pure rotation and the instant-screw parameters can be computed using the general procedure.

INDEX = 6. The vectors are coplanar and no vector is zero but two vectors are parallel and different. According to Corollary 2.9.4 the motion is pure rotation and the axis of rotation is perpendicular to the plane of the velocity vectors. This axis is located using the general procedure.

INDEX = 7. No vector is zero and all three vectors are parallel to each other. Furthermore, not all three vectors are identical to each other. There may be, nevertheless, a couple of identical vectors. The motion is one of pure rotation and the instant axis of rotation is determined by the intersection of the plane of the given points with a second plane defined next. Let A', B' and C' be points whose position vectors are

$\mathbf{a}' = \mathbf{a}+\mathbf{v}_A,\ \mathbf{b}' = \mathbf{b}+\mathbf{v}_B,\ \mathbf{c}' = \mathbf{c}+\mathbf{v}_C$

The second plane is that defined by the noncollinear points A', B' and C'

<u>Proof</u>

According to Corollary 2.9.3 the existence of at least two parallel and distinct velocities guarantees that the motion is pure rotation. Thus, there exists a point O in the body whose velocity is zero. Placing the origin of coordinates at O, then,

$\mathbf{v}_A = \boldsymbol{\omega}\times\mathbf{a},\ \mathbf{v}_B = \boldsymbol{\omega}\times\mathbf{b},\ \mathbf{v}_C = \boldsymbol{\omega}\times\mathbf{c}$

From the parallelism condition, one has

$\mathbf{v}_B = \beta\mathbf{v}_A,\ \mathbf{v}_C = \gamma\mathbf{v}_A$

Hence

$\boldsymbol{\omega}\times(\mathbf{b}-\beta\mathbf{a})=\mathbf{0}$ and $\boldsymbol{\omega}\times(\mathbf{c}-\gamma\mathbf{a})=\mathbf{0}$

which implies that vectors $\mathbf{b}-\beta\mathbf{a}$ and $\mathbf{c}-\gamma\mathbf{a}$ are parallel to the axis of rotation. In other words, the planes defined by points A, B, O and A,C,O contain the axis of rotation. Since $\mathbf{v}_A \neq \mathbf{0}$, AO cannot be the axis of rotation; hence, points A,B, C and O are coplanar and their plane contains the axis of rotation.

On the other hand, recalling the definition of points A', B' and C', whose position vectors are

$\mathbf{a}' = \mathbf{a}+\mathbf{v}_A, \ \mathbf{b}' = \mathbf{b}+\mathbf{v}_B, \ \mathbf{c}' = \mathbf{c}+\mathbf{v}_C$

The velocities of these points are then

$\mathbf{v}_A' = \boldsymbol{\omega}\times(\mathbf{a}+\mathbf{v}_A) = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{v}_A$

$\mathbf{v}_B' = \boldsymbol{\omega}\times(\mathbf{b}+\mathbf{v}_B) = \mathbf{v}_B + \boldsymbol{\omega}\times\mathbf{v}_B$

$\mathbf{v}_C' = \boldsymbol{\omega}\times(\mathbf{c}+\mathbf{v}_C) = \mathbf{v}_C + \boldsymbol{\omega}\times\mathbf{v}_C$

i.e. these velocities are parallel and related by

$\mathbf{v}_B' = \beta\mathbf{v}_A', \ \mathbf{v}_C' = \gamma\mathbf{v}_A'$

Hence

$\boldsymbol{\omega}\times(\mathbf{b}' - \beta\mathbf{a}') = \mathbf{0}, \ \boldsymbol{\omega}\times(\mathbf{c}' - \gamma\mathbf{a}') = \mathbf{0}$

which, by arguments similar to those resorted to previously, imply that points A', B', C' and O are coplanar, their plane containing the axis of rotation. Furthermore, since not all three vectors $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_C$ are identical to each other, planes ABC and A' B' C' are not parallel. Their intersection, clearly, is the axis of rotation, q.e.d.

So far all cases leading necessarily to a pure rotation motion have been discussed. Next the case in which the given velocity vectors are coplanar but the motion is either a pure rotation or general, is discussed. In this case the arising, difference vectors are parallel and hence the given points lie in a plane parallel either to the instant axis of rotation or to the instant screw axis, depending upon whether the motion is a pure rotation or general. This case is handled by Subroutine COP2, which identifies each possible different subcase with the aid of the integer variable INDE.

INDE = 1. No vector is zero and two vectors are identical. The motion is either general or a pure rotation, but the screw axis or, correspondingly, the axis of rotation, is parallel to the line defined by the points with identical velocities.

<u>Proof</u>

Let B and C be the two points with identical velocities. These can be expressed as

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a}), \ \mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega}\times(\mathbf{c}-\mathbf{a})$

Subtracting the latter from the former equation,

$\mathbf{v}_B-\mathbf{v}_A = \boldsymbol{\omega}\times(\mathbf{b}-\mathbf{c}) = \mathbf{0}$

which implies that line BC is parallel to $\boldsymbol{\omega}$, i.e. to either the instant axis of rotation or to the instant screw axis.

INDE = 2. No vector is zero and no two vectors are parallel, but they are coplanar. The motion is either general or a pure rotation and the given points lie in a plane parallel either to the instant screw axis or to the instant axis of rotation, according to Theorem 2.9.5 and Corollary 2.9.2. The angular velocity is found by application of eqs. (2.9.18) -(2.9.25)

Thus, one has the following

<u>THEOREM 2.9.6</u> *The instant motion of a rigid body is determined, i.e. its instant-screw parameters can be computed, if, and only if, the velocities of three noncollinear points of the body are known*

Subroutines INSCRU, COP1, COP2, and GEMO implement the foregoing computations. They use LOCAT1, LOCAT2, SPIN, CYCLIC, EXCHGE, CROSS and SCAL as subsidiary subroutines. Listings of INSCRU, COP1, COP2, GEMO, LOCAT1, LOCAT2 and SPIN appear in Fig 2.9.2-2.9.8.

2.10 THEOREMS RELATED TO THE VELOCITY DISTRIBUTION IN A MOVING RIGID BODY.

Some results concerning the velocity field in a rigid body in motion are now obtained, the main result of this section being the Aronhold-Kennedy Theorem. First a very useful result is proved.

THEOREM 2.10.1 *The velocities of two points of a rigid body have identical components along the line connecting them.*

Proof:

Let $\underline{a}$ and $\underline{b}$ be the position vectors of two points, A and B, of a rigid body in motion. Thus, for any configuration,

$$(\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a})=\text{const} \qquad (2.10.1)$$

from the rigidity condition. Differentiating both sides of eq. (2.10.1),

$$(\dot{\underline{b}}-\dot{\underline{a}})\cdot(\underline{b}-\underline{a})=0$$

or, alternatively,

$$\underline{v}_B\cdot(\underline{b}-\underline{a})=\underline{v}_A\cdot(\underline{b}-\underline{a}),\text{q.e.d.} \qquad (2.10.2)$$

This theorem is used to check the compatibility of the given velocities of a rigid body in subroutine INSCRU of Sect. 2.9.

Exercise 2.10.1 The triangular plate of Fig 2.10.1 is constrained to move in such a way that vertex C remains on the Z-axis, while vertex A remains on the X-axis and side AB remains on the X-Y plane. Vertex C has a velocity $\underline{v}_C=-3\underline{e}_z$ m/sec.

i) Determine the velocity of vertices A and B

ii) Determine the angular velocity of the plate

iii) Locate the instant screw axis of the motion of the plate, and compute the pitch of its screw.

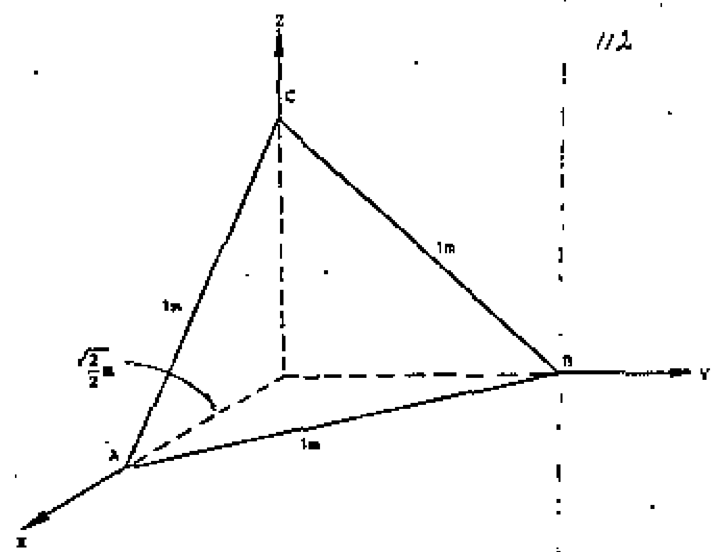


Fig 2.10.1 Triangular plate in constrained motion

THEOREM 2.10.2 (Aronhold-Kennedy). *Given three rigid bodies in motion, the resulting three instant screw axes have one common normal intersecting all three axes.*

Proof:

Referring to Fig 2.10.2 let S_B and S_C be the instant screw axes of bodies B and C, respectively, with respect to body A; let $\underline{v}_B$ and $\underline{v}_C$ be the relative sliding velocities of the instant screws S_B and S_C, with respect to A. Finally, let $\underline{\Omega}_B$ and $\underline{\Omega}_C$ be the relative angular velocities of bodies B and C, respectively, with respect to body A and $\underline{c}$, the common normal to S_B and S_C, joining both axes. It will be shown that the third instant screw axis, $S_{B/C}$, passes through the common normal B*C*.

Let P be any point of the three-dimensional Euclidean space, with position vector $\underline{r}$. Points P_A, P_B and P_C of bodies A, B and C, coincide at P. Let

v_{PB}, v_{PB} and v_{PC} be the velocities of each of these points. Furthermore, let v be the relative velocities of P_B with respect to P_C and let B^* and C^* be the points in which the common normal intersects S_B and S_C.

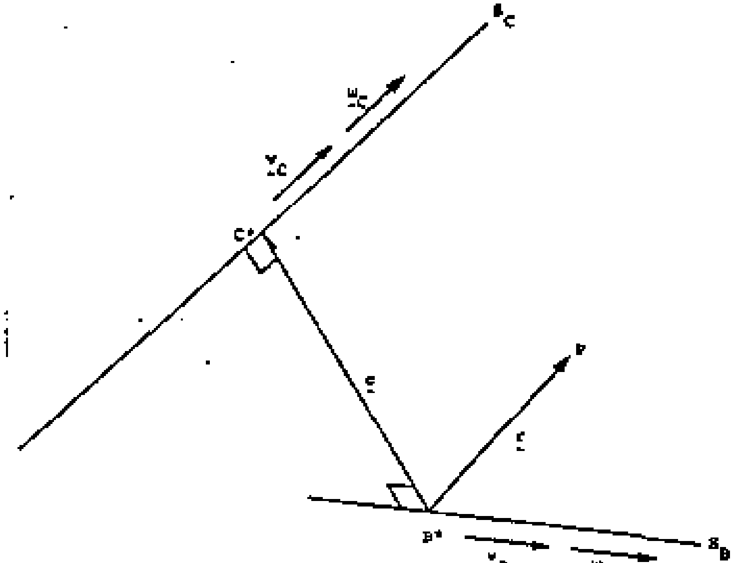


Fig 2.10.2 Instant screws of two bodies in motion with respect to a third one.

Thus,

$$
\begin{aligned}
v &= v_{PB} - v_{PC} = \\
&= (v_B + \Omega_B r) - \{v_C + \Omega_C (r - c)\} = \\
&= v_{B/C} + \Omega_{B/C} r + \Omega_C c
\end{aligned}
\qquad (2.10.3)^*
$$

It is next shown that, if P is a point of the relative instant screw axis $S_{B/C}$, then it lies on the line defined by points B^* and C^*. This is done

* $v_{B/C}$ is to be interpreted as the relative velocity of B^* with respect to C^*.

via the minimization of the quadratic form

$$\phi(r) = v^T v \qquad (2.10.4)$$

$\phi(r)$ has an extremum at a point r_0 where its gradient vanishes. The said gradient is, applying the "chain rule" again,

$$\phi'(r) = 2\Omega^T_{B/C} v \qquad (2.10.5a)$$

Thus, at point r_0,

$$\Omega_{B/C}(v_{B/C} + \Omega_C c + \Omega_{B/C} r_0) = 0 \qquad (2.10.5b)$$

from which r_0 cannot be solved for, since $\Omega_{B/C}$ is singular, of rank two. One possible way to find r_0 is imposing on it the minimum-norm condition, as done previously in similar instances. Another possible way is to write eq. (2.10.5) in Gibbs' notation as

$$w_{B/C} \times (w_{B/C} \times r_0) + w_{B/C} \times (v_{B/C} + w_C \times c) = 0$$

Expanding the first term and imposing the condition that $w_{B/C} \cdot r_0$ be zero, one obtains

$$-w^2_{B/C} r_0 + w_{B/C} \times (v_{B/C} + w_C \times c) = 0$$

from which,

$$r_0 = \frac{1}{w^2_{B/C}} (v_{B/C} + w_C \times c) \times w_{B/C} \qquad (2.10.6)$$

which is parallel to vector c. Since r_0 is parallel to vector c, the common normal to axes S_B and S_C, then $S_{B/C}$ passes through line B^*C^*, q.e.d.

Exercise 2.10.1 Show that r_0, as given by eq. (2.10.6), is parallel to c.

One application of the Aronhold-Kennedy Theorem arises in the pitch surface synthesis of the coupling of two bodies whose relative motion is the composition of sliding and rotation, as is the case in hypoid gears. This is shown in the following example.

Example 2.10.1 Let L_1 and L_2 be the non-intersecting non-parallel axes of two shafts required to be coupled. These axes are shown in Fig 2.10.3.

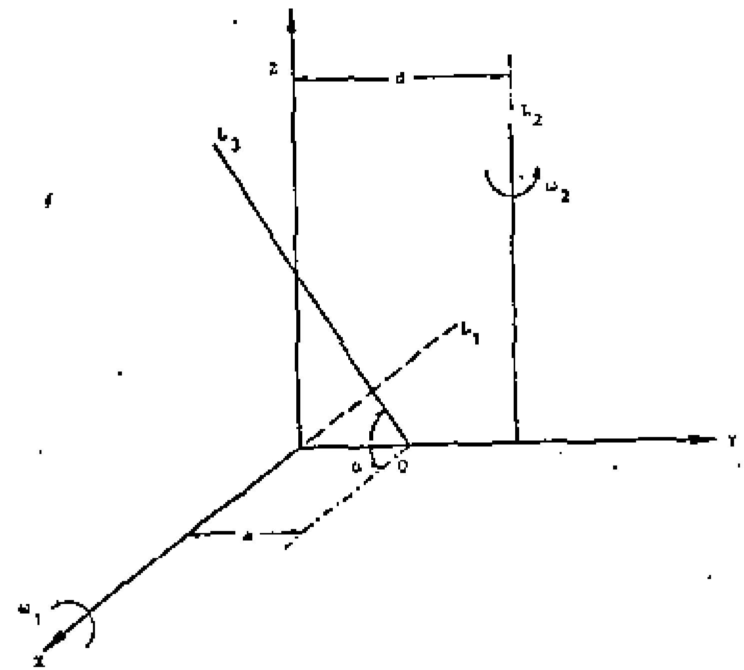


Fig 2.10.3 An application of the Aronhold-Kennedy Theorem

In order to have the most efficient coupling, it is required that this takes place along points of minimum-magnitude relative velocity, i.e. on the instant screw axis of shaft 2 with respect to shaft 1.

From the A-K Theorem, that set of points constitutes line L_3, normal to the Y-axis, a distance a from L_1. Hence, line L_3 is determined by distance a and angle α. Point Q, the intersection of line L_3 and the Y-axis is found from the minimality condition on the relative velocity magnitude. Let $\underline{v}_{Q1}$

and $\underline{v}_{Q2}$ be the velocities of points Q1 and Q2, with respect to the fixed axes X, Y, Z.

Thus,

$$\underline{v}_{Q1}=a\omega_1\underline{e}_x \quad (2.10.7a)$$

$$\underline{v}_{Q2}=(d-a)\omega_2\underline{e}_x \quad (2.10.7b)$$

Assuming a required reduction m, i.e.,

$$\omega_2=m\omega_1 \quad (2.10.8)$$

eq. (2.10.7b) can be rewritten as

$$\underline{v}_{Q2}=(d-a)m\omega_1\underline{e}_x \quad (2.10.7c)$$

Next the quadratic form $\phi(a)$, obtained squaring the relative velocity magnitude, is minimized.

$$\phi(a)=(\underline{v}_{Q2}-\underline{v}_{Q1})\cdot(\underline{v}_{Q2}-\underline{v}_{Q1})=$$
$$=\omega_1^2\left(m^2(d-a)^2+a^2\right) \quad (2.10.9)$$

$\phi(a)$ has an extremum when $\phi'(a)$ becomes zero, i.e.

$$\phi'(a)=\omega_1^2\left[-2m^2(d-a)+2a\right]=0 \quad (2.10.10)$$

from which the minimizing value of a is obtained as

$$a=\frac{m^2 d}{1+m^2} \quad (2.10.11)$$

Angle α is now obtained from the relationship

$$\cos\alpha=\frac{|\underline{\omega}_{2/1}\cdot\underline{e}_x|}{||\underline{\omega}_{2/1}||}$$

Thus

$$\cos\alpha=\frac{1}{\sqrt{1+m^2}} \quad (2.10.12)$$

Summarizing, the pitch surface (on 1) is a ruled surface whose elements are lines a distance a form the X axis, making an angle α with this axis.

this is a one-fold hyperboloid of revolution. Hence the name "hypoid" given to such gears.

One very important consequence of the A-K Theorem now follows.

Corollary 2.10.2 *Given three rigid bodies in motion, A, B and C, there exists an instant axis of pure rotation (i.a.p.r.) of B with respect to C if, and only if, there exist i.a's.p.r. of both B and C with respect to A and these intersect, the i.a.p.r. of B with respect to C passing through the said intersection. Furthermore, all three axes are coplanar.*

Exercise 2.10.4 Prove Corollary 2.10.2.

As an application of Corollary 2.10.2, solve the following problem.

Example 2.10.2 (Kane [2.9]). A shaft, terminating in a truncated cone C of semivertical angle θ, see Fig 2.10.4, is supported by a thrust bearing consisting of a fixed race R and four identical spheres S of radius r. When the shaft rotates about its axis, S rolls on R at both of its points of contact with R, and C rolls on S.

Proper choice of the dimension b allows to obtain pure rolling of C on S. Determine b

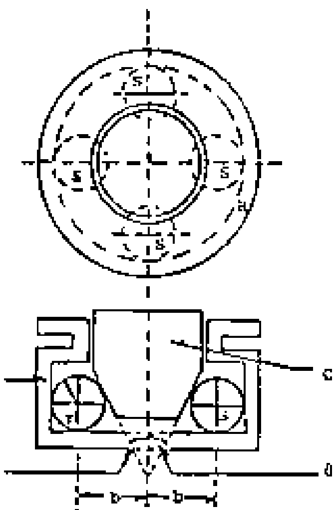


Fig 2.10.4 Shaft rotating on thrust bearings.

Solution:

From Corollary 2.10.2, if all C, S and R move with pure rolling relative motion, then the i.a's.p.r. all coincide at one common point. Clearly, the i.a.p.r. of C with respect to S is the cone element passing through the contact point (between C and S), whereas the i.a.p.r. of C with respect to R is the symmetry axis of C. The intersection of those two axes is the cone apex, which henceforth is referred to as point O. Length b is now determined by the condition that the i.a.p.r. of C with respect to R passes through O.

But two points of this axis are already known, namely, the two points of contact of S on R, henceforth referred to as points S_1 and S_2. Then, the geometry of Fig 2.10.5 follows

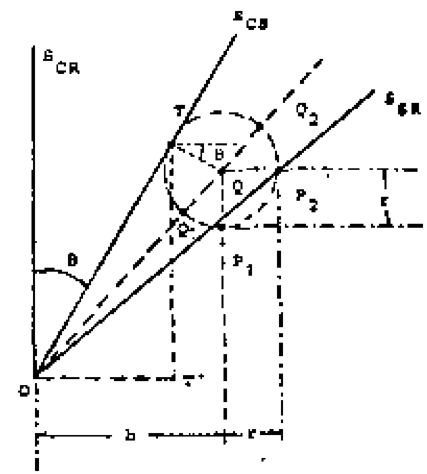


Fig 2.10.5 Instant axes of pure rotation of bodies C, S and R of Fig 2.10.4.

From Fig 2.10.5 it is clear that axis S_{SR} makes a 45° angle with axis S_{CR}. Let T be the contact point between C and S. From a well known theorem of plane geometry,

$$\overline{OT}^2 = \overline{OP}_1\,\overline{OP}_2 \tag{2.10.13}$$

Applying the Pythagorean Theorem to triangle OT'T,

$$\overline{OT}^2 = \overline{OT'}^2 + \overline{T'T}^2 \tag{2.10.14}$$

But

$$\overline{OT'} = b - r\cos\theta, \tag{2.10.15a}$$

and

$$\overline{T'T} = b + r + r\sin\theta \tag{2.10.15b}$$

Hence,

$$\overline{OT}^2 = 2r^2(1+\sin\theta)+2br(1+\sin\theta-\cos\theta)+2b^2 \tag{2.10.16}$$

Also,

$$\overline{OP}_1 = \sqrt{2}b \tag{2.10.17a}$$

$$\overline{OP}_2 = \sqrt{2}(b+r) \tag{2.10.17b}$$

Substitution of eqs. (2.10.16) and (2.10.17 a and b) into eq. (2.10.13) yields

$$r(1+\sin\theta)+b(\sin\theta-\cos\theta)=0$$

from which,

$$b = r\frac{1+\sin\theta}{\cos\theta-\sin\theta} \tag{2.10.18}$$

One more consequence of the A-K Theorem is summarized in the following

Corollary 2.10.3 (Three center Theorem). *In plane motion the three instant axes (centers in this context) of three rigid bodies in motion lie on a line* (1.10)

2.11 ACCELERATION DISTRIBUTION IN A RIGID BODY MOVING ABOUT A FIXED POINT

It was shown in Section 2.7 that the velocity of a point of a rigid body moving about a fixed point is given by

$$\underline{v}(t)=\underline{\Omega}(t)\underline{r}(t) \tag{2.11.1}$$

where $\underline{\Omega}(t)$ is the rigid body angular velocity and $\underline{r}(t)$ is the current position vector of the point under consideration.

The acceleration $\underline{a}(t)$ of the said point is now obtained differentiating both sides of eq. (2.11.1) with respect to time; thus

$$\underline{a}(t)=\dot{\underline{\Omega}}(t)\underline{r}(t)+\underline{\Omega}(t)\dot{\underline{r}}(t)$$

But $\dot{\underline{r}}(t)$ is $\underline{v}(t)$, the above equation taking on the form

$$\underline{a}(t)=\left(\dot{\underline{\Omega}}(t)+\underline{\Omega}^2(t)\right)\underline{r}(t) \tag{2.11.2}$$

The matrix in brackets appearing in eq. (2.11.2) is referred to, by analogy with eq. (2.11.1), as "the angular acceleration matrix". The acceleration of the point under study is formed by two components, as appearing in eq. (2.11.2), namely, the "tangential acceleration", $\dot{\underline{\Omega}}(t)\underline{r}(t)$, and the "normal acceleration", $\underline{\Omega}^2(t)\underline{r}(t)$, the former being tangential and the latter being normal to the velocity.

Exercise 2.11.1 Show that $\dot{\underline{\Omega}}(t)\underline{r}(t)$ and $\underline{\Omega}^2(t)\underline{r}(t)$ are, respectively, parallel and normal to the velocity.

There is one implicit fact in the above result, namely, in the square matrix vector space, one scalar product (See Section 1.7) can be defined as $Tr(\underline{A}\underline{B}^T)$, $\underline{A}$ and $\underline{B}$ being matrices of the same space.

In this context, matrices $\dot{\underline{\Omega}}(t)$ and $\underline{\Omega}^2(t)$ are orthogonal, i.e., its scalar product vanishes.

Exercise 2.11.2 Show that $Tr(\dot{\underline{\Omega}}\underline{\Omega}^2)=0$

Result (2.11.2) can be expressed, in Gibbs' notation as

$$\underline{a}(t)=\dot{\underline{\omega}}(t)\times\underline{r}(t)+\underline{\omega}(t)\times\left(\underline{\omega}(t)\times\underline{r}(t)\right) \tag{2.11.3}$$

thereby making the result of Exercise 2.11.1 apparent

2.12 ACCELERATION DISTRIBUTION IN A RIGID BODY UNDER GENERAL MOTION.

Consider now the most general case of rigid body motion, in which none of the points of the body remains fixed.

From eq. (2.9.2), the velocity of a point whose position vector is $\underline{y}(t)$ of a rigid body under general motion is

$$\underline{v}(t)=\underline{v}_P(t)+\underline{\Omega}(t)\left(\underline{y}(t)-\underline{y}_P(t)\right) \tag{2.12.1}$$

where $\underline{y}_P(t)$ and $\underline{v}_P(t)$ are the position vector and the velocity, both known, of a given point P of the rigid body. The acceleration of a general point, $\underline{a}(t)$, of the body under consideration is next obtained differentiating both sides of eq. (2.12.1) with respect to time, i.e.

$$\underline{a}(t)=\underline{a}_P(t)+\dot{\underline{\Omega}}(t)\left(\underline{y}(t)-\underline{y}_P(t)\right)+\underline{\Omega}(t)\left(\dot{\underline{y}}(t)-\dot{\underline{y}}_P(t)\right) \tag{2.12.2}$$

and, from eq. (2.12.1),

$$\dot{\underline{y}}(t)-\dot{\underline{y}}_P(t)=\underline{v}(t)-\underline{v}_P(t)=\underline{\Omega}(t)\left(\underline{y}(t)-\underline{y}_P(t)\right) \tag{2.12.3}$$

which, when substituted in eq. (2.12.2), leads to

$$\underline{a}(t)=\underline{a}_P(t)+\left(\dot{\underline{\Omega}}(t)+\underline{\Omega}^2(t)\right)\left(\underline{y}(t)-\underline{y}_P(t)\right) \tag{2.12.4}$$

which, except for the term $\underline{a}_P(t)$, is identical to eq. (2.11.2), with $\underline{y}(t)-\underline{y}_P(t)$ instead of $\underline{y}(t)$ of that equation.

The relative acceleration, $\underline{a}(t)-\underline{a}_P(t)$, of the general point with respect to P is clearly given as

$$\underline{a}(t)-\underline{a}_P(t)=\left(\dot{\underline{\Omega}}(t)+\underline{\Omega}^2(t)\right)\left(\underline{y}(t)-\underline{y}_P(t)\right) \tag{2.12.5}$$

which again, is seen to be composed of both a tangential and a normal component.

Parallelling previous sections, the set of points of minimum-magnitude acceleration is now determined. Thus, the function ϕ defined as

$$\phi(\underline{y})=\underline{a}\cdot\underline{a} \tag{2.12.6}$$

is now minimized over $\underline{y}$. Applying the "chain rule" to it,

$$\phi'(\underline{y})=2\left(\frac{\partial \underline{a}}{\partial \underline{y}}\right)^T\underline{a} \tag{2.12.7}$$

where, from eq. (2.12.4),

$$\frac{\partial \underline{a}}{\partial \underline{y}}=\dot{\underline{\Omega}}+\underline{\Omega}^2 \tag{2.12.8}$$

Hence, the minimum-magnitude acceleration satisfies

$$(\dot{\underline{\Omega}}+\underline{\Omega}^2)\underline{a}=\underline{0} \tag{2.12.9}$$

i.e. the minimum-magnitude acceleration is in the null space of $\dot{\underline{\Omega}}+\underline{\Omega}^2$.

If both $\dot{\underline{\Omega}}$ and $\underline{\Omega}^2$ have the same null space, then that minimum-magnitude acceleration lies in that space. Since both $\dot{\underline{\Omega}}$ and $\underline{\Omega}$ are skew symmetric, vectors $\dot{\underline{\omega}}$ and $\underline{\omega}$ lying in their null space, can be defined in such a way that, for any vector $\underline{r}$,

$$\underline{\Omega}\underline{r}=\underline{\omega}\times\underline{r},\quad \dot{\underline{\Omega}}\underline{r}=\dot{\underline{\omega}}\times\underline{r} \tag{2.12.10}$$

Hence it becomes clear that for $\dot{\underline{\Omega}}$ and $\underline{\Omega}$ to have the same null space, $\underline{\omega}$ and $\dot{\underline{\omega}}$ should be parallel. Furthermore, $\underline{\Omega}^2$ and $\underline{\Omega}$ have the same null space (Prove it) and hence, for $\dot{\underline{\Omega}}$ and $\underline{\Omega}^2$ to have the same null space, $\underline{\omega}$ and $\dot{\underline{\omega}}$ should be parallel. A simple case for which $\dot{\underline{\Omega}}$ and $\underline{\Omega}^2$ have the same null space is that for which the rotation axis, $\underline{e}$, has a constant direction. In fact, if this is so, then,

$$\underline{\Omega}\underline{e}=\underline{0} \tag{2.12.11}$$

for all time t. Differentiating the latter expression with respect to time yields

$$\dot{\underline{\Omega}}\underline{e}+\underline{\Omega}\dot{\underline{e}}=\underline{0}$$

But, since the magnitude of $\underline{e}$ is unity and its direction is constant, $\dot{\underline{e}}=\underline{0}$ and hence, the latter equation leads to

$$\dot{\underline{\Omega}}\underline{e}=\underline{0} \tag{2.12.12}$$

thereby showing that, under the conditions stated, if e is in the null space of $\dot{\Omega}$, then it is also in the null space of Ω. Hence, when the instant screw axis has a contant direction, the minimum-magnitude acceleration is parallel to that direction. If the involved matrices do not have a common (non-empty) null space, then the only possibility of eq. (2.12.9) to hold is if $a=0$. The latter condition is equivalent to

$$a_P+(\dot{\Omega}+\Omega^2)(r_O-r_P)=0$$

or

$$(\dot{\Omega}+\Omega^2)r_O=(\dot{\Omega}+\Omega^2)r_P-a_P \qquad (2.12.13)$$

which has a unique solution if $\dot{\Omega}$ and Ω^2 do not have a common null space, for then, the sum of them becomes of full rank, i.e. in that case rank rank$(\dot{\Omega}+\Omega^2)=3$, and hence this sum is nonsigular. In this case, then, one single point of the body, located by the position vector r_O, has a zero acceleration. This point is called an *acceleration pole* and its position vector is given as

$$r_O=r_P-(\dot{\Omega}+\Omega^2)^{-1}a_P \qquad (2.12.14)$$

Example 2.12.1 For a rigid circular cone rolling without slipping on a plane, its acceleration pole is its apex (prove it)

Exercise 2.12.2 The system shown in Fig 2.12.1 is an inversion of the worm-gear mechanism and is composed of a rigid arm OA of lenght b that can rotate freely about the axis EE', this axis being normal to the plane of motion of OA. A rigid wheel is coupled to OA at A in such a way that the wheel can rotate freely about axis FF' passing though A; this axis is perpendicular to both OA and EE'. If OA rotates at a constant rate p and the wheel rotates about FF' at a constant rate q, show that the point of the disk on OA, a distance d from O, has zero acceleration, the distance d being given by

$$d=\frac{q^2}{p^2+q^2}b$$

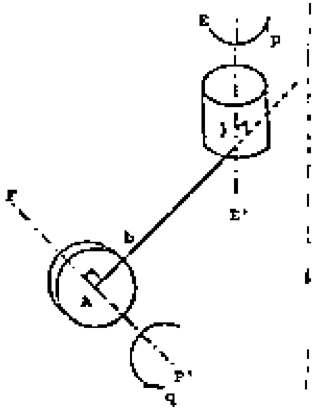


Fig 2.12.1 Inversion of the worm- gear mechanism

An extensive account of this topic is presented in (2.11)

2.13. ACCELERATION OF A MOVING POINT REFERRED TO A MOVING OBSERVER. CORIOLIS' THEOREM.

In Section 2.10 it was shown that the velocity $(\underline{v})_F$* of a moving point, referred to a fixed observer, given in terms of its position vector, $(\underline{\xi})_M$, referred to a moving observer is given by

$$(\underline{v})_F = (\underline{v}_o)_F + (\underline{\Omega})_F(\underline{\xi})_F + (\underline{Q})_F(\dot{\underline{\xi}})_M \qquad (2.13.1)$$

where $\underline{Q}$ and $\underline{\Omega}$ are the rotation and the angular velocity matrices, respectively, of the moving axes with respect to the fixed ones.

The acceleration, $(\underline{a})_F$, of the moving point, in the fixed observer, is now obtained differentiating eq. (2.13.1) with respect to time, namely

$$(\underline{a})_F = (\underline{a}_o)_F + (\dot{\underline{\Omega}})_F(\underline{\xi})_F + (\underline{\Omega})_F(\dot{\underline{\xi}})_F + (\dot{\underline{Q}})_F(\dot{\underline{\xi}})_M + (\underline{Q})_F(\ddot{\underline{\xi}})_M \qquad (2.13.2)$$

where

$$(\underline{a}_o)_F = (\dot{\underline{v}}_o)_F \qquad (2.13.3)$$

But

$$(\underline{\xi})_F = (\underline{Q})_F\,(\underline{\xi})_M \qquad (2.13.4)$$

Hence

$$(\dot{\underline{\xi}})_F = (\dot{\underline{Q}})_F\,(\underline{\xi})_M + (\underline{Q})_F(\dot{\underline{\xi}})_M \qquad (2.13.5)$$

Substitution of eqs. (2.13.4) and (2.13.5) into eq. (2.13.2) yields

$$(\underline{a})_F = (\underline{a}_o)_F + (\dot{\underline{\Omega}})_F(\underline{\xi})_F + (\underline{\Omega})_F[(\dot{\underline{Q}})_F(\underline{\xi})_M + (\underline{Q})_F(\dot{\underline{\xi}})_M] + (\dot{\underline{Q}})_F(\dot{\underline{\xi}})_M + (\underline{Q})_F(\ddot{\underline{\xi}})_M \qquad (2.13.6)$$

But

$$(\dot{\underline{Q}})_F(\dot{\underline{\xi}})_M = (\dot{\underline{Q}})_F(\underline{Q}^T)_F(\underline{Q})_F(\dot{\underline{\xi}})_M = (\underline{\Omega})_F(\underline{Q})_F(\dot{\underline{\xi}})_M$$

---

* All vectors and matrices appearing in this section are functions of time, but for simplicity the argument (t) has been dropped.

and

$$(\underline{\Omega})_F(\dot{\underline{Q}})_F(\underline{\xi})_M = (\underline{\Omega})_F(\dot{\underline{Q}})_F(\underline{Q}^T)_F(\underline{Q})_F(\underline{\xi})_M = (\underline{\Omega}^2)_F(\underline{Q})_F(\underline{\xi})_M = (\underline{\Omega}^2)_F(\underline{\xi})_F$$

Substituting the two latter expressions into eq. (2.13.6) one obtains

$$(\underline{a})_F = (\underline{a}_o)_F + (\dot{\underline{\Omega}} + \underline{\Omega}^2)_F(\underline{Q})_F(\underline{\xi})_M + (\underline{Q})_F(\ddot{\underline{\xi}})_M + 2(\underline{\Omega})_F(\underline{Q})_F(\dot{\underline{\xi}})_M \qquad (2.13.7)$$

which is an expression for the acceleration of a point in terms of measurements of its position, velocity and acceleration, taken by a moving observer. The first two terms of eq. (2.13.7) are identical to the right hand side of eq. (2.12.4) with $\underline{r}-\underline{r}_p$ substituted for $\underline{\xi}$; hence, the two said terms constitute the acceleration of a point fixed in the moving observer, coincident with the moving point under study, at a particular time. The third term stands for the acceleration of the moving point, as measured by the moving observer, and the fourth term is an acceleration term arising from the rotation of the moving observer, as is apparent from eq. (2.13.7); this term is known as "Coriolis' acceleration". Equation (2.13.7) constitutes, then, the Theorem of Coriolis. (2.11)

Exercise 2.13.1 The mechanism shown in Fig 2.13.1 is a component of a quick-return mechanism used in a crank shaper. Assuming that disk 2 rotates at a constant angular velocity ω_2=1800 rpm, determine graphically the angular acceleration of link 3, for the given configuration.

Hint: Two points, B2 and B3, coincide at B. Find the acceleration of B3 via eq. (2.13.7), referred to an observer fixed in 2.

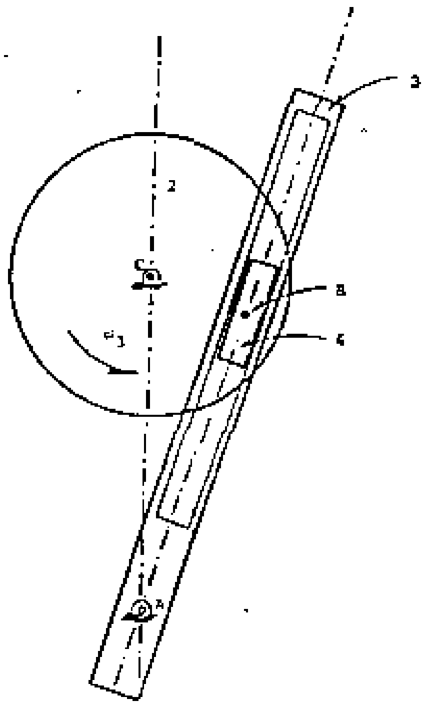


Fig 2.13.1 Driving system of a quick-return mechanism

Exercise 2.13.2 The rectangular plate shown in Fig 2.13.2 is displaced from configuration 1 to configuration 2. Determine the locus of the points of the plate that undergo a displacement of minimum magnitude from 1 to 2. What is the value of this displacement?

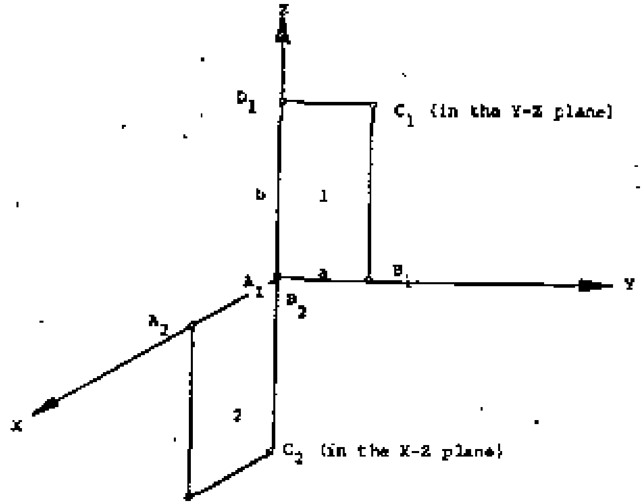


Fig. 2.13.2 Rigid plate undergoing a general displacement.

REFERENCES

2.1 Euler L., "Du movement de rotation des corps solides autour d'un axe variable", Mémoire de l'Académie de Sciences de Berlin [15], 1759, pp. 210-240

2.2 Corben H.C. and P. Stehle., Classical Mechanics, 2nd Edition, John Wiley & Sons, Inc., New York, 1960, pp. 140

2.3 Rodrigues O., "Des lois géométriques qui réglement les déplacements d'un systéme solide dans l'espace, et de la variation des coordonnés provenant de ces déplacements considerés indépendemment des causes qui peuvent les produire", Journal de Mathématiques Pures et Appliquées, vol. 5, 1st Series, 1840, pp. 380-440

2.4 Bisshop K.E., "Rodrigues' formula and the screw matrix" Journal of Engineering for Industry, Trans. ASME, Series B, vol. 91, February 1969.

2.5 Gupta V.K., "Kinematic analysis of plane and spatial mechanisms", Journal of Engineering for Industry, Trans. ASME, Series B, vol. 95, May 1973.

2.6 Fox E.A., Mechanics, Harper & Row, N. York, 1967, p. 11

2.7 Roth B., "The Kinematics of Motion Through Finitely Separated Positions", Journal of Applied Mechanics, Trans. ASME, Series E, September, 1967.

2.8 Truesdell C., A First Course in Rational Continuum Mechanics, vol. I, Academic Press, N. York, 1977. p. 38.

2.9 Kane T.R., Dynamics, Holt, Rinehart and Winston, Inc., N. York, 1968, p. 66.

2.10 Cowie A., Kinematics And Design of Mechanisms, International Textbook Company, Scranton, Penn, 1961, p.144.

2.11 Veldkamp G.R., "Acceleration axes and distribution in spatial motion" Journal of Engineering for Industry, Trans. ASME, Series B, vol. 91, Feb. 1969.

2.12 Coriolis G., "Mémoire sur les équations du mouvement relatif des systèmes des corps", J. Ecole Polytechnique, 15, cahier 24, 1835, pp. 142-154.

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

1. The Kinematics of Motion Through Finitely Separated Positions
2. Finite-Position Theory Applied to Mechanism Synthesis
3. On the Screw Axes and Other Special Lines Associated with Spatial Displacements of a Rigid Body
4. A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis
5. Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains

Dr Bernard Roth

Junio, 1981

Palacio de Minería Calle de Tacuba 5 primer piso México 1, D. F. Tel: 521-40-20 Apdo. Postal M-2285

Paper No.
67—APM-2

The Kinematics of Motion Through Finitely Separated Positions

BERNARD ROTH
Associate Professor,
Mechanical Engineering,
Stanford University,
Stanford, Calif.
Mem. ASME

A rigid body is studied in a series of finitely separated positions, in order to determine those points which lie on a special locus (a sphere, circle, plane, line, or cylinder). Equations governing these special points are derived and their numerical evaluation is discussed. Several numerical examples are presented. In a companion paper [21],[1] these results are applied to the synthesis of spatial linkages, and special motions (e.g., planar and spherical) are incorporated into the general theory presented herein.

Introduction

There are two basic methods which have been successfully applied to the analytical synthesis of planar linkages. One involves writing the synthesis equations so that they explicitly include the unknown linkage dimensions [1, 2, 3, and so on], while the other method, which stems from the earlier graphical theory, requires determining those points in the moving plane which lie on curves which can be readily mechanized [4, 5, and so on]. To date, most spatial-linkage synthesis work has followed the first method [6, 7, 8, 9, 10-11], while the study of points with easily mechanized motions—which is the subject of this paper—seems to have been largely ignored.

Several recent works [13, 14, 15] have treated isolated aspects of this second method, but the only previous work in the same vein as this present study is Schoenflies' [12] classical text. Unfortunately, Schoenflies' development is entirely descriptive (i.e., synthetic as opposed to analytic) and does not lend itself to the solution of practical problems.

In this paper we study points which lie on spheres, circles, planes, lines, or cylinders. Here we consider only general spatial displacements, while a companion paper [21] deals with important special displacements (e.g., planar). Applications to linkage synthesis are discussed in [21]. The main contribution of this present work is that it combines algebraic geometry and computer techniques to yield quantitative results, thereby, for the first time, making it possible to apply the theory to engineering practice. All the derivations are new. In addition, the discussion about points on a cylinder is entirely new, as is the unified treatment of the loci of points with two, three, four, and five positions on a circle.

Kinematic Preliminaries

We refer to two systems of points which for convenience we call the moving system, Σ, and the fixed system, Σ'. However, since our concern is only with their relative positions, both Σ and Σ' may actually be moving. The distance between points in any one system does not vary and hence we may consider Σ and Σ' as rigid bodies, each of which contain all the points in three-dimensional space. The position of Σ is uniquely defined, relative to a coordinate system fixed in Σ', by the coordinates of any three noncollinear points.

[1] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division for presentation at the Applied Mechanics Conference, Pasadena, Calif., June 26-28, 1967, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until one month after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by ASME Applied Mechanics Division, November 12, 1965; final draft, September 26, 1966. Paper No. 67—APM-2.

Linear Transformations and Screw Motion

In considering two finitely separated positions of Σ it is convenient to call one the reference position, Σ_1, and describe the other, Σ_j, as the jth. It is possible to move from position one to position "j" in an infinite number of ways, but for our purposes it will be convenient to regard this motion as a screw (i.e., a rotation about and a translation along the same axis). The screw is denoted by $\$_{1j}$.

If we are given two positions (i.e., two sets of three noncollinear points) then the screw may be determined from a generalized form of Rodrigues's equation. Alternatively, if we are given the screw we obtain the new position, (x_j, y_j, z_j), of some point (x_1, y_1, z_1) from the following well-known linear transformation.

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} (a_{xj}+1) & b_{xj} & c_{xj} \\ a_{yj} & (b_{yj}+1) & c_{yj} \\ a_{zj} & b_{zj} & (c_{zj}+1) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} d_{xj} \\ d_{yj} \\ d_{zj} \end{bmatrix} \quad (1)$$

where

$$\begin{aligned}
a_{xj} &= (u_{xj}^2 - 1)(1 - \cos\theta_j) \\
b_{xj} &= u_{xj}u_{yj}(1 - \cos\theta_j) - u_{zj}\sin\theta_j \\
c_{xj} &= u_{xj}u_{zj}(1 - \cos\theta_j) + u_{yj}\sin\theta_j \\
d_{xj} &= d_j u_{xj} - a_j a_{xj} - b_j b_{xj} - c_j c_{xj} \\
a_{yj} &= u_{yj}u_{xj}(1 - \cos\theta_j) + u_{zj}\sin\theta_j \\
b_{yj} &= (u_{yj}^2 - 1)(1 - \cos\theta_j) \\
c_{yj} &= u_{yj}u_{zj}(1 - \cos\theta_j) - u_{xj}\sin\theta_j \\
d_{yj} &= d_j u_{yj} - a_j a_{yj} - b_j b_{yj} - c_j c_{yj} \\
a_{zj} &= u_{zj}u_{xj}(1 - \cos\theta_j) - u_{yj}\sin\theta_j \\
b_{zj} &= u_{zj}u_{yj}(1 - \cos\theta_j) + u_{xj}\sin\theta_j \\
c_{zj} &= (u_{zj}^2 - 1)(1 - \cos\theta_j) \\
d_{zj} &= d_j u_{zj} - a_j a_{zj} - b_j b_{zj} - c_j c_{zj}
\end{aligned}$$

In the foregoing we have taken the displacement from position 1 to j as equivalent to a translation d_j along, and a rotation θ_j about an axis parallel to the unit vector (u_{xj}, u_{yj}, u_{zj}) which passes through the point (a_j, b_j, c_j). The terms θ_j and d_j are referred to as the screw parameters, their positive senses being defined by the right-handed screw rule. Equation (1) emphasizes that the rotational and translational aspects of the motion may be regarded as occurring separately and their effects superimposed.

We now present a theorem which will be very useful in studying points with special motions:

The locus of all points in Σ_1 which under a general displacement remain a fixed distance from a given point (in Σ') is a plane. The

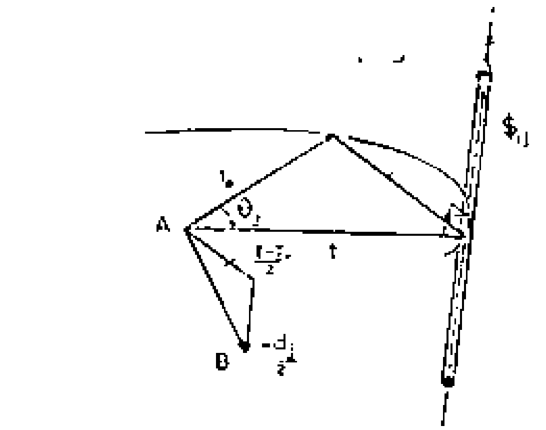


Fig. 1 Normal to plane given by equation (2) is determined by fixed point A and screw $\$_{1j}$. Vector AB, the required normal, is sum of $\frac{t - t_\theta}{2}$ and $\frac{-d_j}{2}$.

proof is as follows. Let $\mathbf{A}$ be the vector from the origin to the point A in Σ', and $\mathbf{r}_1$ the vector to some point in Σ_1. For motion from Σ_1 to Σ_j the constant distance condition requires:

$$|\mathbf{r}_1 - \mathbf{A}| = |\mathbf{r}_j - \mathbf{A}|$$

which is equivalent to

$$\frac{r_j^2 - r_1^2}{2} + \mathbf{A}\cdot(\mathbf{r}_j - \mathbf{r}_1) = 0 \qquad (2)$$

Now from (1) we know that $\mathbf{r}_j$ is a linear function of $\mathbf{r}_1$, and it remains only to show that $r_1^2 - r_j^2$ is linear.

Substituting $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $\mathbf{r}_j = x_j\mathbf{i} + y_j\mathbf{j} + z_j\mathbf{k}$, and making use of the usual orthogonality conditions for the rotational part of the rigid-body transformation, we obtain:

$$\begin{aligned}\frac{r_j^2 - r_1^2}{2} = {} & (d_j u_{1j} + a_j a_{1j} + b_j a_{2j} + c_j a_{3j})x_1 \\ & + (d_j u_{2j} + a_j b_{1j} + b_j b_{2j} + c_j b_{3j})y_1 \\ & + (d_j u_{3j} + a_j c_{1j} + b_j c_{2j} + c_j c_{3j})z_1 \\ & + \frac{d_j^2}{2} + (a_j^2 + b_j^2 + c_j^2)(1 - \cos\theta_j) \qquad (3)\end{aligned}$$

Hence (2) is linear in (x_1, y_1, z_1) and the theorem is proved.

We note that, if the screw passes through the origin, (3) simplifies to

$$\frac{r_j^2 - r_1^2}{2} = \frac{d_j^2}{2} + d_j(u_{1j}x_1 + u_{2j}y_1 + u_{3j}z_1)$$

and if in addition the motion is a pure rotation: $r_j^2 - r_1^2 = 0$.

The plane given by equation (2) is a distance of $\left[\left(\frac{d_j}{2}\right)^2 + t^2 \sin\left(\frac{\theta_j}{2}\right)\right]^{1/2}$ from A and is normal to the vector $\frac{1}{2}(\mathbf{t} - \mathbf{t}_\theta - \mathbf{d}_j)$. Here $\mathbf{t}$ is the normal vector from the fixed point A to the screw axis $\$_{1j}$, and $\mathbf{t}_\theta$ is the vector obtained by rotating $\mathbf{t}$ by $-\theta_j$ about an axis through A parallel to $\$_{1j}$, Fig. 1. The term $\mathbf{d}_j$ is the translation vector $d_j(u_{1j}\mathbf{i} + u_{2j}\mathbf{j} + u_{3j}\mathbf{k})$. If the screw axis passes through the fixed point, the plane given by (2) becomes normal to the screw axis and intersects it a point $-\frac{\mathbf{d}_j}{2}$ from the fixed point.

Equation (2) is also useful if we regard $\mathbf{r}_1$ and $\mathbf{r}_j$ as known and seek the locus of points in Σ' which are at an equal distance from the known points in Σ_1 and Σ_j. Under this interpretation (2) becomes the locus of A and is, of course, the perpendicular bisector plane of the cord connecting (x_1, y_1, z_1) and (x_j, y_j, z_j).

Correspondence—The Cubic Transformation

Consider four finitely separated positions of Σ. Associated with each point, p_1, in Σ_1 is the set of four homologous points p_1, p_2, p_3, p_4 given by the four positions of p. Since it is possible to pass a sphere through any four points, the four positions of each point define a sphere. Associated with each such set of homologous points in Σ is the point in the fixed system which is at the center of the sphere. Since the relationship between points in Σ and Σ' is important to what follows, we undertake to describe this correspondence.

We require that (in four positions) a point in the moving system remain a fixed distance from a point (A_x, A_y, A_z) in the fixed system. This is given by Equation (2) taken three times:

$$(x_2 - x_1)A_x + (y_2 - y_1)A_y + (z_2 - z_1)A_z + \left(\frac{r_1^2 - r_2^2}{2}\right) = 0$$

$$(x_3 - x_1)A_x + (y_3 - y_1)A_y + (z_3 - z_1)A_z + \left(\frac{r_1^2 - r_3^2}{2}\right) = 0 \qquad (4)$$

$$(x_4 - x_1)A_x + (y_4 - y_1)A_y + (z_4 - z_1)A_z + \left(\frac{r_1^2 - r_4^2}{2}\right) = 0$$

Now, given screws $\$_{12}$, $\$_{13}$, $\$_{14}$ (or any equivalent set, e.g., $\$_{12}$, $\$_{23}$, $\$_{34}$) and any point (x_1, y_1, z_1) we are able to generally solve the three linear nonhomogeneous equations (4) for a unique center point (A_x, A_y, A_z). Similarly, if the screws and the center of the sphere are given, we know from (3) and (4) that the equations (4) become a linear set of three nonhomogeneous equations in (x_1, y_1, z_1). Hence, a unique (x_1, y_1, z_1) may generally be determined. Thus we generally have a (1, 1) correspondence between points in Σ and points in Σ'. This correspondence is invariant to kinematic inversion, since under an inversion the center becomes the moving point and the moving point becomes the center. This transformation is called *the cubic transformation* [10].

Singularities in this correspondence occur when the rank of the augmented matrix of system (4) is less than three. The necessary and sufficient conditions for the rank to be two are that the coefficient matrix

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1)\\(x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1)\\(x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1)\end{vmatrix} = 0, \qquad (5)$$

and one other 3×3 be singular, say,

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (r_1^2 - r_2^2)\\(x_3 - x_1) & (y_3 - y_1) & (r_1^2 - r_3^2)\\(x_4 - x_1) & (y_4 - y_1) & (r_1^2 - r_4^2)\end{vmatrix} = 0, \qquad (6)$$

provided the common 2×2's of (5) and (6) do not all vanish. If the rank of the system is two, a point in the moving system corresponds to a line in the fixed system, and under kinematic inversion a point in Σ' corresponds to a line in Σ. As we shall see later this leads us to points whose four positions fall on circles.

If the rank of the system is one, a point in the moving body Σ corresponds to a plane in Σ'. This requires that all of the 2×2's of the augmented matrix vanish which is impossible under general motions.

Having laid the groundwork, we now proceed to determine those points in Σ whose several positions lie on special loci.

Points on Special Loci

Points Which Lie on a Sphere

A general point will not have more than four positions on a sphere. Those points with five positions on one sphere will satisfy equation (2) written four times (i.e., $j = 2, 3, 4, 5$). The condition for the four nonhomogeneous linear equations in the three unknowns A_x, A_y, A_z to be compatible is:

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) & (r_1^2 - r_2^2)\\(x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) & (r_1^2 - r_3^2)\\(x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) & (r_1^2 - r_4^2)\\(x_5 - x_1) & (y_5 - y_1) & (z_5 - z_1) & (r_1^2 - r_5^2)\end{vmatrix} = 0 \qquad (7)$$

Table 1 Input defining seven positions of a body in terms of six screws

POSITION	AXIS ORIENTATION			AXIS LOCATION			PARAMETERS	
1 to j	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	Rotation [illegible]	Translation [illegible]
1 to 2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.6811	[illegible]
1 to 3	0.1281	[illegible]	[illegible]	[illegible]	0.4914	-0.1115	[illegible]	0.1757
1 to 4	0.6626	0.4428	[illegible]	[illegible]	1.0765	[illegible]	1.7059	0.4551
1 to 5	0.1744	-0.400	[illegible]	[illegible]	3.4535	-0.0549	2.1144	0.7853
1 to 6	0.3733	0.6901	[illegible]	0.3242	0.9499	[illegible]	[illegible]	0.1052
1 to 7	[illegible]	[illegible]	[illegible]	[illegible]	0.9714	[illegible]	1.0351	[illegible]

Table 2 Equation of the locus of points in a body that lie on a sphere when the body is in positions 1, 2, 3, 4, and 5

$E^4_{2345} =$ [illegible] $= 0$

As shown previously, substituting (1) and (3) explicitly makes all of these elements linear in (x_1, y_1, z_1). Hence (7), which is the locus of all points in Σ_1 with five positions on a sphere, is of fourth degree. It will be convenient to refer to (7) as $E_{2345}{}^4 = 0$ and sometimes simply as E^4. Geometrically, the locus E^4 is a fourth-order algebraic surface embedded in Σ_1.

In order to expand (7) and obtain explicit expressions for the 35 coefficients of E^4 in terms of the motion parameters, we have to expand 4!(4 × 4) determinants and sort out 6144 terms which each consist of a product of four parameters. This prodigious amount of algebra is simplified by the symmetry in x_1, y_1, z_1, but still the task is formidable. We have elected to always determine these coefficients of E^4 numerically, and have found that a very small amount of programming and a trivial amount of computational time are required for this development.[2] Table 1 gives the screws which define a set of finitely separated positions. Table 2 lists a set of coefficients of E^4 corresponding to the screws of Table 1.

By, for example, arbitrarily choosing x_1 and y_1 and then solving the resulting quartic for (at most) four possible z_1's we may compute the coordinates of points on E^4. The center point corresponding to any point on E^4 is obtained from the cubic transformation, equation (4). Alternatively, we could have started by regarding x_1, y_1, z_1 as the unknowns and instead of (7) obtained a (4 × 4) determinant in A_1, A_2, A_3. This would yield a fourth-order algebraic surface embedded in Σ' which is the locus of all sphere centers corresponding to a given set of five positions of Σ. For any point on this surface we could compute the corresponding moving point from (4).

Now considering a sixth position of Σ, we use any four of the original five positions (for example, the first four) and the sixth position in equation (7), and obtain a second equation, say, (7[1]). (7[1]) represents a second fourth-order algebraic surface $E_{2346}{}^4 = 0$ which is embedded in Σ_1. Four positions will generally uniquely determine a sphere, and since the two fourth-order surfaces share four positions, we conclude that the locus of all points with six positions on a sphere is included in the intersection of $E_{2345}{}^4 = 0$ with $E_{2346}{}^4 = 0$. Algebraically, these two fourth-degree equations (7) and (7[1]) are the compatibility conditions for the five nonhomogeneous linear equations obtained by writing (2) five times.

The intersection may be written as $E_{2345}{}^4 \times E_{2346}{}^4$. It consists of two components: One is a tenth-order space curve $k_{23456}{}^{10}$ or simply k^{10}, of genus eleven, and the other is a sixth-order curve $k_{234}{}^6$, or simply k^6, of genus three. k^{10} contains all the points with six positions on a sphere. As will be shown in the next section, k^6 is the locus of all points which lie on a circle for the four positions 1, 2, 3, 4. Physically, the reason k^6 does not contain points with six positions on a sphere is that the four common positions of the two surfaces do not define a unique sphere. Hence the fifth and sixth positions of any point on k^6 will, when taken in combination with the circle, define two different spheres.

Analytically, k^6 corresponds to the singular case of the cubic transformation given by equations (5) and (6). Under these circumstances equations (7) and (7[1]) are no longer sufficient to guarantee the compatibility of equation (2) written five times (with j = 2, 3, 4, 5, 6) and, therefore, points on k^6 will not satisfy all five equations.

Table 3 lists points on k^{10} corresponding to the motion given in Table 1. The corresponding center points are computed from (4). Alternatively, we could, as described previously, obtain two fourth-order surfaces imbedded in Σ', find their intersections and compute the corresponding points in Σ_1 from (4).

For seven positions on a sphere the reasoning is analogous to the foregoing. We substitute the subscript 7 for 5 in equation (7) and obtain a new equation (7[2]) which yields a third surface $E_{2347}{}^4 = 0$. Intersecting these three surfaces one finds that there are at most 20 points with seven positions on a sphere.[3] Appendix 1.

Analytically, equations (7), (7[1]), (7[2]) represent the compatibility conditions for (2) written six times, and the 44 spurious solutions correspond to the singular case given by (5) and (6).

As in the case of points with five and six positions on a sphere, the corresponding center points are obtained from (4). Alternatively, by suitable relabeling or by kinematic inversion we could obtain three surfaces in Σ' whose intersection would contain the "seven-position" centers.

If we introduce an eighth position, we require the common intersections of four surfaces. Since these will not generally

[2] This and all other programs referred to in this paper may be obtained by writing to the author. The program language is FORTRAN II.

[3] The original proof due to Schoenflies [12] depends upon surfaces which are computationally not as convenient as the foregoing. In Appendix 1 we give a new proof which is due to E. J. F. Primrose.

Table 3 Points on a sphere

Points with j positions on a sphere	Initial position x_1	y_1	z_1	Sphere center A_x	A_y	A_z	Radius of sphere
	[illegible]	1.0000	0.0000	[illegible]	[illegible]	[illegible]	[illegible]
j = 5	1.[illegible]	1.5000	1.0000	[illegible]	[illegible]	1.[illegible]	1.04[illegible]
	-1.[illegible]	2.5000	0.5000	2.1[illegible]	1.7[illegible]	3.[illegible]	[illegible]
	1.0000	[illegible]	-1.[illegible]	[illegible]	1.0[illegible]	[illegible]	[illegible]
j = 6	[illegible]	-4.[illegible]	-1.7[illegible]	[illegible]	[illegible]	-1.[illegible]	[illegible]
	3.0000	[illegible]	-1.3[illegible]	0.3[illegible]	[illegible]	-1.1[illegible]	[illegible]
	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	1.0000
	[illegible]	[illegible]	[illegible]	[illegible]	1.[illegible]	[illegible]	[illegible]
j = 7	[illegible]	1.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	-5.[illegible]	[illegible]	[illegible]	-5.[illegible]	[illegible]
	[illegible]	[illegible]	-4.[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

exist, it follows that under general motion there are no points with more than seven positions on one sphere.

Table 3 contains several points which under the motion given in Table 1 have seven positions on a sphere. As a result of the prohibitive size of the eliminants required to develop explicit expressions for the intersections, iterative techniques are employed to determine points common to the three surfaces.

Points Which Lie on a Circle

When dealing with points on a circle it is advantageous to formulate equations which explicitly give the correspondence between the points (fixed in Σ) and their axes (fixed in Σ'). A circle axis is specified by its direction cosines (l, m, n) and the coordinates of one of its points (A_x, A_y, A_z).

All points whose several positions fall on a circle must satisfy two conditions: (*a*) Their distance from any (and all) points on the axis is the same in each position; (*b*) in the several positions, they lie in a plane which is normal to the axis. These conditions may be expressed analytically as follows:

$$(x_j - x_1)A_x + (y_j - y_1)A_y + (z_j - z_1)A_z + \left(\frac{r_1^2 - r_j^2}{2}\right) = 0 \quad (8)$$

$$(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n = 0 \quad (9)$$

We have previously given a geometrical interpretation of (8), which is in fact equation (2), and we now give a similar interpretation to (9). From equation (9) it follows that *the locus of all points which have two positions on a line (or in a plane) normal to a given direction is a plane*. By substituting (1), we write (9) explicitly in terms of the screw $\$_{1j}$:

$$\begin{aligned}
&[(1 - \cos\theta_j)(u_{xj}\cos\alpha_j - l) + \sin\theta_j(mu_{zj} - nu_{yj})](x_1 - a_j) \\
&+ [(1 - \cos\theta_j)(u_{yj}\cos\alpha_j - m) + \sin\theta_j(nu_{xj} - lu_{zj})](y_1 - b_j) \\
&+ [(1 - \cos\theta_j)(u_{zj}\cos\alpha_j - n) + \sin\theta_j(lu_{yj} - mu_{xj})](z_1 - c_j) \\
&+ d_j\cos\alpha_j = 0 \quad (10)
\end{aligned}$$

Here α_j is the angle between the screw axis (u_{xj}, u_{yj}, u_{zj}) and circle axis (l, m, n). Dividing the aforementioned by $2\sin\left(\frac{\theta_j}{2}\right)\sin\alpha_j$ puts the equation into normal form. For convenience, we call this plane P_j.

Now from (10) it may easily be shown that the distance from P_j to the screw axis is $\frac{d_j}{2}\cot(\alpha_j)\csc\left(\frac{\theta_j}{2}\right)$ and that the dihedral angle between P_j and a plane parallel to the screw axis and the circle axis is $\left(\frac{\theta_j}{2}\right)$. Alternatively, this last result asserts that $\left(\frac{\theta_j}{2}\right)$ is the angle between the common normal to the two axes and the normal to P_j. It should be noted that the inclination of P_j depends only on the direction of the screw axis and the magnitude of the rotation; it is independent of the location of the screw and the translation.

If the screw axis and the circle axis are parallel, P_j becomes the plane at infinity. However, if in addition to the axes being parallel, the motion is a pure rotation, every point in Σ identically satisfies equation (9). If the motion is a pure translation, regardless of axis orientation, P_j again becomes the plane at infinity.

Finally, we note that since P_j is parallel to the screw axis, P_j cannot meet the (screw) axis in a finite point unless $\alpha_j = 90$ deg or the motion is a pure rotation in which case P_j contains the screw.

For any specified axis, equations (8) and (9) yield two planes which intersect in a line which is embedded in Σ. Hence, *the locus of all points which have two positions on a circle with a specified axis is a line*. If we take a third position we require the intersection of two skew lines and hence for an arbitrary choice of axis there are generally no points which lie on a circle for three positions.

Now we take as specified a point through which the circle axis passes, and leave the direction of the axis unspecified. For three positions, we write (8) twice ($j = 2, 3$). These equations with (A_x, A_y, A_z) known, represent two planes whose line of intersection satisfies (8). Equation (9) written twice will generally yield unique values for $\frac{l}{n}, \frac{m}{n}$ for any points in Σ and in particular for the line determined from (8) with $j = 2, 3$. Hence *the locus of all points having three positions on a circle whose axis passes through a specified point is a line*.

Considering a fourth position, equation (8) with $j = 2, 3, 4$ yields a unique point, but equation (9) written with $j = 2, 3, 4$ gives three homogeneous equations in (l, m, n). Hence, we do not have the possibility of finding a solution unless (x_1, y_1, z_1) are such that:

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix} = 0$$

This is the same cubic surface as given by equation (5). We have a solution only if (A_x, A_y, A_z) are chosen so that the corresponding point (x_1, y_1, z_1) lies on the previous cubic. If (A_x, A_y, A_z) are finite, this second requirement can only be met if all the (3×3) determinants formed from the coefficients of (8) (written three times) are zero. This requires that, in addition to (5), equation (6) also be satisfied. [As we have shown previously, these are the necessary conditions for the system matrix (4) to be of rank two. Physically, we require that the three perpendicular bisectors given by equation (2), or (8), with $j = 2, 3, 4$, intersect in a common line.] Thus, for four positions, we are not at liberty to arbitrarily specify (A_x, A_y, A_z).

Equations (5) and (6) are third-degree algebraic polynomials

Table 4 Points on circles, lines, planes, and cylinders

	Initial position			Center			Radius	Direction cosines of axis		
	x_1	y_1	z_1	A_x	A_y	A_z		l	m	n
Points with Four Positions on a Circle	1.2000	2.2[illegible]	0.[illegible]	0.8144	1.6518	0.[illegible]	1.3632	0.4402	0.4744	0.5948
	-0.7506	2.0[illegible]	-1.8139	0.4165	0.7331	-1.0951	1.[illegible]	-0.5204	0.8634	-0.[illegible]
	1.4598	3.1[illegible]	7.2[illegible]	1.2992	1.2604	2.[illegible]	1.1705	0.6411	0.2845	0.6743
Points with Three Positions on a Straight Line	-1.4[illegible]	0.4[illegible]	-5.0000							
	-0.2[illegible]	0.2[illegible]	-5.0000							
	2.0[illegible]	-1.1[illegible]	-5.0000							
Points with Six Positions on a Plane	0.6966	1.7500	3.1232							
	-0.[illegible]	1.7[illegible]	-2.50[illegible]							
	-4.[illegible]	1.7[illegible]	-3.5[illegible]							
Points with Six Positions on a Cylinder	0.[illegible]	1.[illegible]	-0.3[illegible]	0.2612	0.0025	0.[illegible]	0.[illegible]	-0.4[illegible]	-0.[illegible]	0.5[illegible]
	2.4[illegible]	3.4[illegible]	7.5[illegible]	0.0[illegible]	-0.7[illegible]	7.2[illegible]	3.3[illegible]	"	"	"
	0.[illegible]	0.17[illegible]	0.00[illegible]	0.3[illegible]	0.3[illegible]	0.7[illegible]	0.2[illegible]	"	"	"

which represent the two cubic surfaces F^3 and G^3, respectively. The intersection of $F^3 \times G^3$ (or more analytically, the common roots of (5) and (6)) contains all the points in Σ_1 which have four positions on a circle. The intersection consists of k^6 which is a sixth-order space curve of genus three, and a cubic j^3. The "residual" part of the intersection, j^3, is the space cubic which is the intersection of the two hyperboloids

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1)\\(x_3 - x_1) & (y_3 - y_1)\end{vmatrix} = 0$$

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1)\\(x_4 - x_1) & (y_4 - y_1)\end{vmatrix} = 0 \qquad (11)$$

The line $(x_2 - x_1) = 0$, $(y_2 - y_1) = 0$ common to these hyperboloids is not included in j^3. Analytically, equations (11) give the conditions under which (5) and (6) are no longer sufficient to guarantee a rank of two for the system matrix (4). For all points on k^6 the rank of (4) is two and, therefore, *the locus of all points which have four positions on a circle is a space curve of order six.*[4]

We return to the study of three positions, but now we specify the direction of the axis and leave its location arbitrary. Equation (8) with $j = 2, 3$ gives two nonhomogeneous equations in the unknowns (A_x, A_y, A_z). Therefore, for any point (x_1, y_1, z_1) there is a singly infinite set of solutions corresponding to the points on an axis. Equation (9), with $j = 2, 3$ and (l, m, n) known, yields two planes whose line of intersection gives us permissible values for (x_1, y_1, z_1) which when substituted into (8) (with $j = 2, 3$) yield the location of the axis. Hence, *the locus of all points which have three positions on a circle of a specified inclination is a line.* This same line is also the locus of all points which have three positions in a plane whose normal is (l, m, n).

For four positions with the axis direction specified, equation (9) yields three planes and therefore generally a unique point (x_1, y_1, z_1). At first, it might seem that (8) taken three times $(j = 2, 3, 4)$ would always yield a unique (A_x, A_y, A_z). This is not the case. For any (l, m, n) the values of (x_1, y_1, z_1) computed from (9) must be such that they automatically satisfy (5), and if (A_x, A_y, A_z) are finite the rank of (8), with $j = 2, 3, 4$, is at most two. In this case (8) yields a line which, of course, defines the axis location. However, the center will generally be at infinity since (A_x, A_y, A_z) are only finite when (x_1, y_1, z_1) also satisfy (6). Hence, *for any given axis inclination there is a unique point whose four positions lie in a plane.* Since this point is determined by the intersection of three planes, this last result affords a rather simple means of obtaining points on the surface F^3 (which is discussed in a later section). For certain axis inclinations the corresponding point also falls on G^3 and, therefore, on k^6.

[4] A curve such as k^6, whose points correspond to curves (lines in this case), is singular in regard to the given (1, 1) correspondence and is called the *fundamental curve* of the transformation.

There generally are no points having five positions on a circle. Analytically, we would require two additional equations (5') and (6') which are like (5) and (6) except that they have the subscript 5 instead of 4. The four equations (5), (6), (5'), (6') generally would not be compatible. Geometrically, two space curves such as k_{34}^6 and k_{35}^6 will generally not intersect[5] even when they are contained on the same surface E^3. There are, however, some important special motions (e.g., planar and spherical) for which these four equations are compatible. We shall discuss these cases elsewhere [21].

Computationally, points on k^6 are obtained by numerically determining the coefficients of the cubics (5) and (6). These equations are then "solved" simultaneously for common roots. The roots are then sorted according to whether they belong to k^6 or j^3, and the corresponding axes are determined from (8) (with $j = 2, 3$). In Table 4 we list several points on the curve k^6 which correspond to the motion given in Table 1.

Points Which Lie on a Straight Line

Points with three positions on a line are given by the condition that their cords have parallel perpendicular-bisector planes, or by the condition that they lie on a three-point circle with axis at infinity. In either case we require:

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1)\\(x_3 - x_1) & (y_3 - y_1)\end{vmatrix} = 0$$

$$\begin{vmatrix}(x_2 - x_1) & (z_2 - z_1)\\(x_3 - x_1) & (z_3 - z_1)\end{vmatrix} = 0 \qquad (12)$$

These two second-degree polynomials are equations of hyperboloids (of one sheet) which intersect in a cubic space curve i^3 and the common generator $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. The curve i^3, which is embedded in Σ_1, is the locus of all points with three positions on a line. Since i^3 is a cubical hyperbola[6] it may also be obtained as the intersection of two hyperbolic cylinders.

Generally there are no points with four positions on a straight

[5] The real points at infinity along the screw axes have two positions which coincide and, hence, having only three distinct positions, such points always lie on a circle for four positions. The curve k_{34}^6 contains the points where $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{23}^1$, $\$_{24}^1$, $\$_{34}^1$ pierce the plane at infinity and k_{35}^6 contains the points at infinity along $\$_{12}$, $\$_{13}$, $\$_{15}$, $\$_{23}^1$, $\$_{25}^1$, $\$_{35}^1$. Since k^6 can be intersected by a plane at most six times, we conclude from the foregoing that the six screw axes give the directions of the only asymptotes. The intersection $k_{34}^6 \times k_{35}^6$ contains the point at infinity along $\$_{12}$, $\$_{13}$, $\$_{23}^1$, but these three points do not fall on five-point circles. Schoenflies [12] has shown that these curves have four imaginary intersections. (The screw axis $\$_{jk}^1$ is taken as that line in Σ_1 which coincides with $\$_{jk}$ when Σ is in position j or k.)

[6] The points at infinity along the screw axes are stationary points for two positions and are, therefore, always on i^3. Hence $\$_{12}$, $\$_{13}$ and $\$_{23}^1$ are the directions of the asymptotes and, since they generally cut the plane at infinity in distinct points, i^3 is a cubical hyperbola.

line. Four points on a straight line would require that the rank of the system matrix of equation (4) be two and that its coefficient matrix be of rank one. In other words, we require all (2 × 2)'s of the determinant (5) to vanish. If, say, $(x_2 - x_1) \neq 0$, this requirement is met by equations (12) and a similar set, say, (12') with the subscript 3 replaced by 4. These four equations will generally not be compatible. Geometrically, four quadratics do not generally have a common intersection, nor do the two space cubics $i_3{}^3$ and $i_4{}^3$.[7] This leads us to conclude that, unlike the well-known planar case, *there are generally no "four-point" circles with infinite radii.*

As we shall see later [21], there are, however, important special cases (e.g., planar and spherical motion) when (12) and (12') are compatible.

Computationally, equations (12) are simple enough to allow (after substituting from (1)) one of the unknowns, say, x_1, to be explicitly eliminated. Considering one of the remaining unknowns, say, y_1, as a parameter, the eliminant may be regarded as a quartic in z_1. For any given value of y_1 one obtains at most four values of z_1, three of which belong to i^3. The extraneous root is a point on the line $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. Alternatively, we could, after substituting from (1), eliminate x_1 between $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. Then for any value of y_1 compute the spurious z_1 and eliminate it from the quartic by synthetic division. Table 4 lists several points on i^3 corresponding to the motion given in Table 1.

Points Which Lie on a Plane

For three positions all points lie on planes. The condition that a point lie on a plane for four positions is obtained from the cubic transformation by requiring that the center of the sphere be at infinity:

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix} = 0$$

This is of course the surface we have called F^3 which is embedded in Σ_1 and is identical to equation (5). (Assuming no other (3 × 3) determinants of (4) vanish and that all the (2 × 2)'s of (5) are not zero, this condition leaves the rank of the system matrix of (4) unaltered while reducing the rank of the coefficient matrix to two.) Hence, *the locus of all points having four positions on a plane is a third-order algebraic surface.* From physical reasoning (as well as from the foregoing) it is obvious that this surface contains k^4 and i^3.

Introducing a fifth position, we follow the procedure described in the discussion on points on a sphere and require $F_{234}{}^3 \times F_{235}{}^3$. This intersection contains c^6 a space curve of order six and genus three which is the locus of all points with five positions on a plane; curve i^3 is also common to both surfaces but it does not contain the required points. (For points on i^3, the rank of the coefficient matrix of (4) is one.) An alternative approach, which is computationally inferior, is to require $F_{234}{}^3 \times E_{345}{}^3$ in which case the intersection is composed of c^6 and k^4 with k^4 being spurious.

Introducing a sixth position we require $F_{234}{}^3 \times F_{235}{}^3 \times F_{236}{}^3$. Since these three surfaces have i^3 as a common component, there are at most ten points in Σ_1 with six positions on a plane.[8]

This result also follows if we consider $k^{10} \times F_{236}{}^3$.[9] In either case, this means that there is no upper limit to the radii of the "six-position" spheres and that ten points on k^{10} fall on spheres of infinite radii. In addition since four surfaces such as F^3 will generally not have a common intersection point, *there are (generally) no points which have seven positions on a plane* and, hence, all 20 "seven-position" spheres will (generally) be finite.

Computationally, curve c^6 may be developed as a function of a single parameter: Eliminating, say, x_1 between $F_{234}{}^3$ and $F_{235}{}^3$ yields a ninth-degree polynomial in which we regard, say, y_1 as a parameter and z_1 as an unknown. Then using i^3 (after eliminating x_1 as described previously) corresponding to a given value of the "parameter" y_1, we synthetically divide the ninth-degree polynomial and obtain a sextic in z_1 whose roots lie on c^6. This procedure has the advantage of being systematic and is useful for plotting the entire curve. However, for obtaining arbitrary points on c^6 and points with six positions on a plane, simple iteration techniques generally suffice. For the motion defined in Table 1, points with six positions on a plane are listed in Table 4.

From the discussion of four positions on a circle, it follows that to each set of directions, (l, m, n), of the plane there corresponds a point on F^3. Hence, (8) taken three times may be used to parametrically generate F^3. Similarly, c^6 may be generated parametrically by writing (8) four times ($j = 2, 3, 4, 5$) and eliminating, say, m. The resulting three quadratics, containing $\left(\frac{n}{l}\right)$ as a single parameter, may be solved for (x_1, y_1, z_1).

Points Which Lie on a Cylinder

We specify the axis of a right-circular cylinder by its direction cosines (l, m, n) and the coordinates of one of its points (a, b, c), and let the normal vector $\mathbf{r}_A$ from the origin terminate at (a, b, c). We take $\mathbf{r}_j$ as the vector from the origin to a point in Σ_j, and define the vector $\mathbf{D}_j = \mathbf{r}_j - \mathbf{r}_A$. In Fig. 2, the directed distance between planes normal to the axis and through the tips of $\mathbf{r}_j$ (i.e., $\mathbf{r}_j$ with $j = 1$) and $\mathbf{r}_j$ is taken as $\mathbf{c}$. The vector $\mathbf{d}_j$ is taken from (a, b, c) to the point obtained by projecting the tip of $\mathbf{r}_j$ (parallel to the cylinder axis) into the plane containing the tip of $\mathbf{r}_1$. From the figure it follows that $\mathbf{D}_j = \mathbf{d}_j + \mathbf{c}$ and, hence, $\mathbf{d}_j = \mathbf{r}_j - \mathbf{r}_A - \mathbf{c}$.

If the point (x_j, y_j, z_j), i.e., the tip of $\mathbf{r}_j$, is to lie on a cylinder about the given axis, we require $|\mathbf{D}_1| = |\mathbf{d}_j|$. Therefore, $d_j{}^2 - D_1{}^2 = 0$ or using the foregoing

$$(\mathbf{r}_j - \mathbf{r}_A - \mathbf{c})\cdot(\mathbf{r}_j - \mathbf{r}_A - \mathbf{c}) - (\mathbf{r}_1 - \mathbf{r}_A)\cdot(\mathbf{r}_1 - \mathbf{r}_A) = 0$$

which when expanded and simplified yields

$$(x_j - x_1)a + (y_j - y_1)b + (z_j - z_1)c + \frac{r_1{}^2 - r_j{}^2}{2}$$

$$+ \frac{1}{2}\,[(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n][(x_j + x_1)l$$

$$+ (y_j + y_1)m + (z_j + z_1)n] = 0 \quad (13)$$

If the axis is specified, then equation (13) (with (l, m, n) and

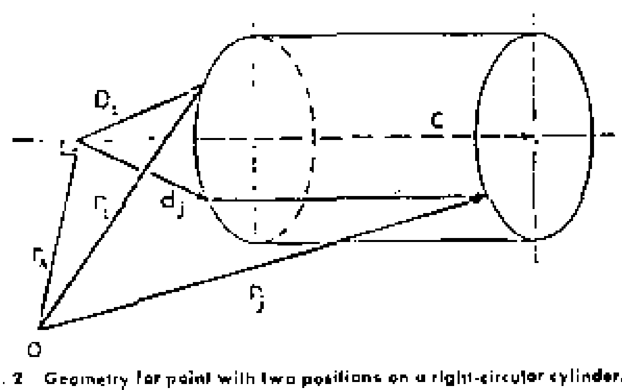


Fig. 2 Geometry for point with two positions on a right-circular cylinder. C is cylinder axis; r_j and r_1 are position vectors for required point in positions j and 1, respectively.

[7] The point at infinity along $\mathbf{S}_A$ will be common to both cubics, but it does not lie on a four-point circle.

[8] Schoenflies' original proof [12] deals with somewhat different surfaces, so we require a new proof: For $F_{234}{}^3 \times F_{235}{}^3$ we have $N_1 = N_2 = 3$. Using $r_{12} = 3$, $p = 0$, equation (15) yields 8 as the number of intersections of curves c^6 and i^3. Now c^6 intersects $F_{236}{}^3$ in 18 points, but since $F_{236}{}^3$ contains i^3, 8 of these fall on i^3. Hence, 18 − 8 = 10 points have six positions on a plane.

[9] The curve k^{10} intersects $F_{236}{}^3$ in 30 points and, as shown in Appendix 1, k^{10} intersects k^4 in 20 points. Now since $F_{236}{}^3$ contains k^4, there are only 30 − 20 = 10 points of $k^{10} \times F_{236}{}^3$ which do not lie on k^4.

(a, b, c) fixed) represents a quadratic surface embedded in Σ_1 which is the locus of all points with two positions on a cylinder about the given axis. If the cylinder axis coincides with screw axis $\$_{1j}$, equation (13) is identically satisfied by every point in Σ_1.[10]

For three positions we take (13) with $j = 2, 3$, and find that *the locus of all points with three positions on a cylinder is a quartic space curve.*

For four positions we have equation (13) taken three times, $j = 2, 3, 4$, with the result that *there are at most eight points with four positions on a cylinder with a specified axis.*

For five positions we are not at liberty to choose an arbitrary axis. If only the direction (l, m, n) is specified, then the orthogonality condition $al + bm + cn = 0$ must be explicitly satisfied. With $j = 2, 3, 4$ in equation (13) and the orthogonality condition one obtains a set of four nonhomogeneous linear equations in (a, b, c). The compatibility condition is that the determinant of the system matrix equal zero. Hence we require:

$$\begin{vmatrix} l & m & n & 0 \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) & f_2^2 \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) & f_3^2 \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) & f_4^2 \end{vmatrix} = 0 \qquad (14)$$

where

$$f_j^2 = \frac{r_1^2 - r_j^2}{2} + \frac{1}{2}\,[(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n] \times [(x_j + x_1)l + (y_j + y_1)m + (z_j + z_1)n]$$

Equation (14) represents a quartic surface embedded in Σ_1. This surface which we call H_{234}^4, or simply H^4, is the locus of all points with four positions on right-circular cylinders with axes parallel to (l, m, n). For a specified location (a, b, c), the intersection of H^4 with the aforementioned quartic ((13) taken twice) yields the eight points previously described.[11]

Resuming consideration of five positions, we conclude that the intersection $H_{234}^4 \times H_{235}^4$ will contain the required locus. This intersection is composed of a thirteenth-order space curve, k^{13}, which is the locus of all points with five positions on a cylinder with specified axis orientation, and a residual of three lines which are the loci of all points with two of their first three positions on one generator; that is, a line parallel to (l, m, n).[12]

For six positions we have the intersections of $H_{234}^4 \times H_{235}^4 \times H_{236}^4$ which yield 34 points with six positions on a cylinder for each specified axis orientation.[13]

If we do not specify the inclination, we may instead specify the normal to the axis (a, b, c). However, in this case and in the case of both (l, m, n) and (a, b, c) left unspecified, the nonlinearity of (l, m, n) in the f_j^2 terms makes the analysis more complicated.

Reference Positions and Kinematic Inversion

In the foregoing we have dealt with Σ_1 as the reference position and determined those loci which are embedded in it. It is felt that this choice affords the simplest means of explanation, derivation, and computation. However, it should be noted that analogous loci exist in each position Σ_j. These loci can be computed either directly, by a suitable change in the subscripts which denote the position, or indirectly, by remembering that all loci embedded in Σ_1 move with the rigid-body motion of the system and can, therefore, be transformed from Σ_1 to Σ_j by screw $\$_{1j}$.

We also point out that for some of the previous cases, one could work instead with loci which are embedded in Σ'. This is especially true in the case of the sphere in which points in Σ_1 and Σ' are in (1, 1) correspondence. Here we note the analogy with the well-known planar center and circle-point curves which also contain points in (1, 1) correspondence. The loci in Σ' are obtained either directly, by interchanging "known" and "unknown" variables in the derivations (as described in the section on points on a sphere), or indirectly, by inversion. In an inversion we fix Σ in position Σ_1 and move Σ' under that set of screws (embedded in Σ_1) which have the same axes as the original screws but opposite senses.[14] If we invert the motion, then the loci in Σ_1 and Σ' are interchanged. Actually, both Σ and Σ' may be in motion, since the results of the preceding sections depend only on the relative positions of the two systems.

Conclusions

We have considered points whose several positions fall on special surfaces or curves. Equations for the loci of points which lie on spheres, planes, circles, cylinders, or lines were derived. In addition, it has been shown how to compute points on these loci. Computer programs have been written to perform these computations and several numerical examples have been presented.

In addition to their intrinsic interest, these results have direct application to problems of mechanism design. The application of these theories is discussed in a companion paper [21]. Special motions (e.g., planar and spherical) are also studied, and it is shown, in [21], that some of these motions yield the conditions under which equations (5), (6), (5') and (6'), as well as (12) and (12'), become compatible.

Acknowledgments

The author wishes to express his appreciation to Mr. Pietlaw Chen for doing the bulk of the computer programming, and Mr. Kenneth Waldron for programming assistance. The author is also indebted to Prof. F. Freudenstein for having introduced him to the problems discussed herein and for his continued interest in this work; to Dr. E. J. F. Primrose for his interest and willing assistance; to the National Science Foundation whose grant NSF-GP 2005 made this work possible.

APPENDIX 1

Seven Positions on a Sphere

In general, if F_1 and F_2 are surfaces of order N_1 and N_2 which pass through a nonsingular curve of order n and genus p, they will contain another curve of order n', and genus p'. The number of intersections of these curves is I, which is given by [20]

$$I = n(N_1 + N_2 - 4) - (2p - 2). \qquad (15)$$

The following condition connects the genera

$$2(p' - p) = (n' - n)(N_1 + N_2 - 4) \qquad (16)$$

Consider first the four-point circle condition $F^3 \times G^3$ which we know yields k^6 and l^3. Now for l^3 $n = 3$ and $p = 0$, hence with $n' = 6$, and $N_1 = N_2 = 3$, (16) yields $p' = 3$ as the genus of k^6.

For $E_{123}^4 \times E_{124}^4$ we use (15) to determine the number of points of intersection of k^6 with k^{10}. Substituting $N_1 = N_2 = 4$, $n = 6$, $p = 3$, one obtains $I = 20$. Now k^{10} intersects E_{125}^4 in 40 points, but since E_{125}^4 contains k^6, 20 of these fall on k^6. Hence, there are $40 - 20 = 20$ points with seven positions on a sphere.

---

[10] For any general orientation the screw axis $\$_{1j}$, considered as a line in Σ_1, meets the quadratic at two points, one of which is at infinity.

[11] There are actually 16 points, but half of them fall on the lines described in the next paragraph and therefore do not have the required axis location. These 8 redundant points lie 4 each on the lines associated with positions 1, 2 and 1, 3.

[12] These three lines are easily obtained from the intersection of planes $(x_j - x_1)m - (y_j - y_1)l = 0$ and $(z_j - z_1)m - (y_j - y_1)n = 0$, $j = 2, 3$, and from plane $[(x_2 - x_1) - (x_3 - x_1)]m - [(y_2 - y_1) - (y_3 - y_1)]l = 0$ with $[(z_2 - z_1) - (z_3 - z_1)]m - [(y_2 - y_1) - (y_3 - y_1)]n = 0$.

[13] There are 52 points common to k^{13} and H_{236}^4; however, 18 of them fall on the three lines previously described. Of the remaining 34 three are at infinity along the axes $\$_{12}$, $\$_{13}$, $\$_{23}$.

[14] That is, instead of $\$_{1j}$ we screw about $\$_{1j}^{-1}$.

References

1 F. Freudenstein, "Approximate Synthesis of Four-Bar Linkages," Trans. ASME, vol. 77, 1955, pp. 853–859.

2 F. Freudenstein and G. N. Sandor, "Synthesis of Path-Generating Mechanisms by Means of a Programmed Digital Computer," *Journal of Engineering for Industry*, Trans. ASME, vol. 81, Series B, 1959, pp. 159–168.

3 B. Roth and F. Freudenstein, "Synthesis of Path-Generating Mechanisms by Numerical Methods," *Journal of Engineering for Industry*, Trans. ASME, vol. 85, Series B, 1963, pp. 298–306.

4 F. Freudenstein and G. N. Sandor, "On the Burmester Points of a Plane," Journal of Applied Mechanics, vol. 28, Trans. ASME, vol. 83, Series E, 1961, pp. 41–49.

5 G. N. Sandor, "On the Existence of a Cycloidal Burmester Theory in Planar Kinematics," Journal of Applied Mechanics, vol. 31, Trans. ASME, vol. 86, Series E, 1964, pp. 694–699.

6 J. Denavit and R. S. Hartenberg, "Approximate Synthesis of Spatial Linkages," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, Series E, 1960, pp. 201–206.

7 R. S. Hartenberg and J. Denavit, *Kinematic Synthesis of Linkages*, McGraw-Hill Book Company, Inc., New York, N. Y. 1964, 435 pages.

8 N. I. Levitskii and K. K. Shakvazian, "Synthesis of Four-Element Spatial Mechanisms With Lower Pairs," *International Journal of Mechanical Sciences*, vol. 2, pp. 76–92. Translation by F. Freudenstein of a paper which was originally published in 1954.

9 E. P. Novodvorskii, "A Method of Synthesis of Mechanisms," in Russian, *Trudi. Seminar. po Theor. Mash. Mekh.*, Akademii Nauk, SSSR, vol. 42, 1951.

10 F. M. Dimentberg, "A General Method for the Investigation of Finite Displacements of Spatial Mechanisms and Certain Cases of Passive Constraints," in Russian, *Trudi. Seminar. po Teor. Mash. Mekh.*, Akademii Nauk, SSSR, vol. 5, no. 17, 1948, pp. 5–39.

11 K. K. Shakvazian, "Analytical Method for the Design of Spatial Link Mechanisms With Lower Pairs," in Russian, *Trudi Inst. Mashin.*, vol. 20, 1960, pp. 5–9.

12 A. Schoenflies, *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, Germany, 1886, 194 pages.

13 M. L. Keler, "Analyse und Synthese der Raumkurbelgetriebe mittels Raumliniengeometrie und dualer Grössen," *Forschung Ing. Wes.*, vol. 25, 1959, pp. 27–32 and 55–63.

14 G. Hein, "Analytische und Zeichnerische Unterlagen für zwei und drei Lagen eines sphärisch bewegten Getriebegliedes als Hilfsmittel zur Synthese sphärischer Kurbelgetriebe," *Forsch Ing. Wes.*, vol. 25, 1959, pp. 92–96 and 115–126.

15 J. T. Wilson, III, "Analytical Kinematic Synthesis by Finite Displacements," *Journal of Engineering for Industry*, Trans. ASME, vol. 87, Series B, 1965, pp. 161–169.

16 R. S. Ball, *Theory of Screws*, Cambridge University Press, Cambridge, England, 1900.

17 A. T. Yang, "Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms," doctoral dissertation, Columbia University, New York, N. Y., 1963, 241 pages.

18 E. Study, *Geometrie der Dynamen*, B. G. Teubner, Leipzig, Germany, 1903, 603 pages.

19 M. Nöther, "Ueber die eindeutigen Raumtransformationen insbesondere in ihrer Anwendung auf die Abbildung algebraischer Flachen," *Mathematische Annalen*, vol. 3, 1871, pp. 547–580.

20 J. G. Semple and L. Roth, *Introduction to Algebraic Geometry*, Oxford University Press, England, 1949, 446 pages.

21 B. Roth, "Finite Position Theory Applied to Mechanism Synthesis," ASME Paper No. 67—APM-39.

Printed in U. S. A.

Paper No.
67—APM-39

Finite-Position Theory Applied to Mechanism Synthesis

B. ROTH
Associate Professor,
Mechanical Engineering Department,
Stanford University, Stanford, Calif.
Mem. ASME

The well-known finite-position planar theory of kinematic synthesis (the so called Burmester theory) and the corresponding spherical theory are derived from the results of the general spatial theory which has been given in a companion paper [1].[1] Other special displacements studied are those for which the author has coined the names "similarity transformation," "pseudoplanar," and "pseudospherical." These results, as well as those obtained in [1], are shown to be applicable to the synthesis of spatial, spherical, and planar linkages. Several numerical examples are presented.

Introduction

In a previous paper [1], we developed equations governing the loci of points which, under a series of rigid-body displacements, have several positions on spheres, planes, circles, cylinders, or lines. In this paper, we particularize these general results to three types of special displacements having important practical applications. Since the author is unaware of any previous discussion of the first special case, he has (with misgiving) coined the name similarity transformation. The other two special cases—intersecting screws and parallel screws—lead to the well-known spherical and planar motion finite-position theories which may now be viewed in a more general context. In addition, these latter two special cases lead, respectively, to the intriguing, and seemingly new, concept of pseudospherical and pseudoplanar mechanisms.

Applications of the general theory, as well as the foregoing special cases, are discussed. It is shown that these results are applicable to various types of design problems, and several mechanism syntheses are presented. The function-generation problem has been treated previously elsewhere ([3, 5] and especially [6]), but most of the other syntheses seem to be new.

Nomenclature

We describe a finite displacement of a rigid body, from position i to position j, as a screw displacement $\$_{ij}$. The system of points in the moving system is called Σ, in general, and Σ_i when we wish to emphasize that the system is in the ith position. The system of points in the reference system (also referred to as the fixed system) is called Σ'. The screw axis is considered as a line in Σ' unless it contains a superscript, as in $\$_{ij}{}^{k}$, in which case it is taken as that line in Σ_k which coincides with $\$_{ij}$ when Σ is in position i.

Following the notation developed in [1]: E^4 is the fourth-order surface which contains all the points in Σ_1 which have five positions on a sphere; k^6 is a sixth-order space curve which contains all the points in Σ_1 which lie on a circle for four positions, and l^3 is a third-order space curve which contains all the points in Σ_1 which have three positions on a line. For the sake of clarity, it is sometimes necessary to use subscripts (e.g., $k_{234}{}^6$) to denote which positions are being considered. However, in what follows, position 1 is generally omitted; since it is taken as the reference position, it is always understood to be the first position.

Finally, all numbered equations refer to equations derived in [1]. In this context, it is useful to know that:

(*a*) Equation (1) defines the linear transformation under which point (x_1, y_1, z_1) is "screwed" to (x_j, y_j, z_j) by $\$_{1j}$:

$$x_j = (a_{x_j} + 1)x_1 + b_{x_j}y_1 + c_{x_j}z_1 + d_{x_j}$$

$$y_j = a_{y_j}x_1 + (b_{y_j} + 1)y_1 + c_{y_j}z_1 + d_{y_j}$$

$$z_j = a_{z_j}x_1 + b_{z_j}y_1 + (c_{z_j} + 1)z_1 + d_{z_j}$$

(*b*) The square of the distance from the origin is given by:

$$r_j{}^2 = x_j{}^2 + y_j{}^2 + z_j{}^2$$

(*c*) Equations (5) and (6) are the equations which define $k_{234}{}^6$, while (5′) and (6′) define $k_{235}{}^6$. These two curves generally do not have any common finite points.

(*d*) Equation (9) gives the locus of all points with positions 1 and j in a plane normal to some given direction (l, m, n).

(*e*) Equation (8) gives the locus of all points (x_1, y_1, z_1) which in positions 1 and j have the same distance from a point (A_x, A_y, A_z) fixed in Σ'.

(*f*) Equation (12) is a set of two equations which defines $l_{23}{}^3$, while (12′) defines $l_{24}{}^3$. These two curves generally do not have a common finite point.

Special Motions

We now undertake the study of three types of screw motions which are commonly used in mechanical linkwork. These motions yield special cases of the general theory which are of theoretical and practical interest.

Similarity Transformations

We consider the case when the screw is determined by pure rotations about two axes, $\$_A$ and $\$_B$, which are embedded, respectively, in Σ' and Σ (Fig. 1). Further, the distance and inclination between $\$_A$ and $\$_B$ are fixed.

Let A and B be the transformation matrices, for motion of points in Σ about $\$_A$ and $\$_B$, respectively, such that the total transformation due to a rotation θ about $\$_B$ followed by a rotation Φ about $\$_A$ is given by the matrix product AB. Now, if we reverse the order of the two rotations so that we first rotate Σ by Φ about $\$_A$, we note that axis $\$_B$ has a new position (in Σ'), say, $\$_{B_1}$, and we must determine the transformation matrix, say, B_1, which corresponds to a rotation θ about $\$_{B_1}$. Since the only change between $\$_B$ and $\$_{B_1}$ is the rotation defined by A, the new transformation B_1 is given by the so-called similarity transformation ABA^{-1}. Hence the total transformation is $(ABA^{-1})A =$

[1] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division for presentation at the Applied Mechanics Conference, Pasadena, Calif., June 26–28, 1967, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until one month after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by ASME Applied Mechanics Division, November 12, 1965; final draft, September 26, 1966. Paper No. 67—APM-39.

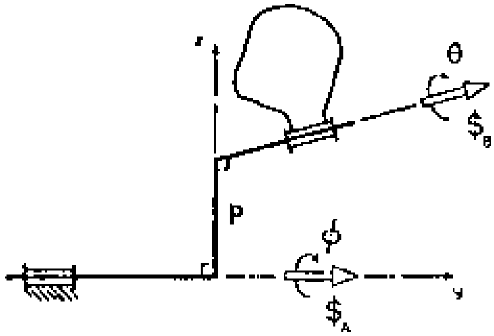


Fig. 1 "Similarity-transformation" motion is defined by rotations about a fixed axis ($\$_A$) and an axis moving with the body ($\$_B$). The distance, p, and angle between the axes are constant.

AB. We conclude that the order of the transformations about $\$_A$ and $\$_B$ is immaterial, and that, in the special case of one axis fixed in the moving body, finite rotations (as well as general screw displacements) do commute.

We may simplify the derivations without losing any generality by choosing, say, the y-axis along $\$_A$ and the z-axis along the common normal from $\$_A$ to $\$_B$. The length of the common normal is p, and the direction cosines of $\$_B$ (in position Σ_1) are $(u_1, u_2, 0)$. There is a unique screw which is equivalent to the two rotation screws $\$_A$, $\$_B$; we compute its parameters (d_{eq}, θ_{eq}) and its direction cosines $(u_{1eq}, u_{2eq}, u_{3eq})$:

$$d_{eq} = 2pu_1{}^2 \frac{\sin\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$\cos\left(\frac{\theta_{eq}}{2}\right) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) - u_2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)$$

$$u_{1eq} = u_1\frac{\cos\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$u_{2eq} = \frac{u_2\cos\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$u_{3eq} = -u_1\frac{\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

With these results, the transformation coefficients of equation (1) become:

$$a_{x_j} = -u_1[\sin\theta_j\sin\Phi_j + u_2\cos\Phi_j(1-\cos\theta_j)] - (1-\cos\Phi_j)$$

$$b_{x_j} = u_1[\sin\theta_j\sin\Phi_j + u_2\cos\Phi_j(1-\cos\theta_j)]$$

$$c_{x_j} = u_2\sin\theta_j\cos\Phi_j + \cos\theta_j\sin\Phi_j$$

$$d_{x_j} = -p[u_2\sin\theta_j\cos\Phi_j + (\cos\theta_j - 1)\sin\Phi_j]$$

$$a_{y_j} = u_1u_2(1-\cos\theta_j)$$

$$b_{y_j} = -u_1{}^2(1-\cos\theta_j)$$

$$c_{y_j} = -u_1\sin\theta_j$$

$$d_{y_j} = pu_1\sin\theta_j$$

$$a_{z_j}' = -u_1[\sin\theta_j\cos\Phi_j - u_2\sin\Phi_j(1-\cos\theta_j)] - \sin\Phi_j$$

$$b_{z_j} = u_1[\sin\theta_j\cos\Phi_j - u_2\sin\Phi_j(1-\cos\theta_j)]$$

$$c_{z_j} = -u_2\sin\theta_j\sin\Phi_j + \cos\theta_j\cos\Phi_j - 1$$

$$d_{z_j} = p[u_2\sin\Phi_j\sin\theta_j + \cos\Phi_j(1-\cos\theta_j)]$$

Further, it may be shown that

$$\frac{r_j{}^2 - r_1{}^2}{2} = u_2\sin\theta_jx_1 - u_1\sin\theta_jy_1 + (1-\cos\theta_j)(z_1 - p).$$

Under these conditions, E^4 degenerates into two quadratic surfaces, and, as shown in Appendix 1, we may write $E^4 = u_1F^2G^2$. Since $F^2 = [u_2x_1 - u_1y_1]^2 + [z_1 - p]^2$, it is independent of the rotations. The only real points on F^2 are given by the intersection of planes $u_2x_1 - u_1y_1 = 0$ and $z_1 - p = 0$, which is in fact the moving axis $\$_B$ (in Σ_1). Physically, line $\$_B$ satisfies E^4 identically because all of its points always lie on circles.

If $u_1 = 0$, E^4 is identically zero. This corresponds to the case of $\$_A$ and $\$_B$ parallel which restricts all points to planar motion and, hence, to spheres of infinite radii.

The surface G^2 is generally a hyperboloid. Its equation is derived in Appendix 1.

Obviously, the same degeneracies must occur for arbitrary orientation of the coordinate axes: One of the quadratics is imaginary except for an isolated line which is always $\$_B$. The equation of this quartic is, in fact, $(f^2 + g^2) = 0$. Where $f = 0$ is the plane through $\$_B$ perpendicular to the common normal between $\$_A$ and $\$_B$, and $g = 0$ is the plane through $\$_B$ and the common normal (i.e., it is normal to $\$_A$), the equations of both planes are taken in normal form. The other quadratic is generally a real hyperboloid and, of course, $E^4 = 0$ is identically satisfied if the axes are parallel.

Assuming the rotation axes are not parallel, the points of interest lie on G^2. For six positions on a sphere, we consider $G_{234}{}^2 \times G_{235}{}^2$, which yields a space cubic, and a line l_1 which is spurious. For seven positions, $G_{234}{}^2 \times G_{235}{}^2 \times G_{236}{}^2$ yields (at most) four points.[2] Since the surfaces are only of order two, these computations may be carried out without recourse to iterative techniques.

For this motion, equation (6) factors into a plane P^* and the foregoing "line quadratic" explicitly (6) yields:

$$u_1F^2P^* = 0$$

where

$$P^* = \sum_{\substack{i=2\\j=2\\k=2}}^{4} (x_1 - x_1)\sin\theta_j(1-\cos\theta_k)\epsilon_{ijk}$$

Here ϵ_{ijk} is $+1$ or -1 if (i, j, k) is, respectively, an even or odd permutation of (2, 3, 4), and is zero if two of the subscripts are equal.

Equation (5) remains a cubic, but it may be put into the form:

$$u_1[(x_1u_2 - y_1u_1)F_A{}^{**} - (z_1 - p)F_B{}^{**}] = 0$$

where $F_A{}^{**}$ and $F_B{}^{**}$ are quadratic functions of x_1, y_1, z_1.

From the aforementioned, we may easily verify that all points on the moving rotation axis satisfy (5) and (6). Further, the intersection of P^* with (5) is a quadratic curve, which is the residual part of the intersection of (5) and (6), and a line l_1. Hence *the loci of points in* Σ *with four (or more) positions on a circle are the lines* $\$_B$ and l_1.

Parallel Screws

In the case of three parallel, pure-rotation ($\$_{12}$, $\$_{13}$, $\$_{14}$) screws and a fourth arbitrary screw ($\$_{15}$), the quartic surface E^4 degenerates to a plane and a cubic cylinder. Taking (l, m, n) as the direction cosines of the three parallel screw axes, we may write:

$$E^4 = PK^3$$

where

[2] The space cubic cuts $G_{236}{}^2$ in six points; however, two of these are on l_1. The three surfaces contain the line l_1 as a common component.

Table 1 Five-position circles for three-parallel pure-rotation screws and one-skew screw

Screw $\$_{1j}$ j	Direction cosines of screw axis a_{1x}	a_{1y}	a_{1z}	Point on screw x	y	z	Rotation ϕ_{1j}(rad)	Translation d_{1j}
2	0.0000	0.0000	1.0000	5.4594	-1.2182	0.0000	0.3554	0.0000
3	"	"	"	1.1880	-2.4352	0.0000	0.2451	0.0000
4	"	"	"	2.7803	-3.4158	0.0000	0.3150	0.0000
5	-0.8749	-0.1760	-0.4241	0.2411	-0.4746	-0.3179	0.6443	-1.4472

Root number	Initial position x_1	y_1	z_1	Center c_x	c_y	c_z	Radius	Direction cosines of axis l	m	n
1	-0.3881	0.1353	-0.0027	0.2565	0.0658	-0.4675	0.9410	0.0000	0.0000	1.0000
2	7.9426	1.4769	0.0197	5.1442	-0.2409	-0.8139	1.5925	"	"	"
3	49.4831	27.5186	1.7893	-74.6193	-12.4654	3.9393	104.3664	"	"	"
4	-0.3346	1.0577	-2.0329	0.3983	0.6646	-3.0519	[illegible]	"	"	"
5	1.4471	1.6014	-3.2125	3.5412	0.6827	-2.3935	1.6652	"	"	"
6	22.4914	0.4352	20.3590	46.3409	1.3918	30.1550	75.4301	"	"	"

$$P = (x_4 - x_1)l + (y_4 - y_1)m + (z_4 - z_1)n$$

and

$$k^2 = \frac{1}{2n}\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (r_2^2 - r_1^2) \\ (x_3 - x_1) & (y_3 - y_1) & (r_3^2 - r_1^2) \\ (x_4 - x_1) & (y_4 - y_1) & (r_4^2 - r_1^2) \end{vmatrix}$$

The cubic cylinder k^2 has generators which are all parallel to (l, m, n) (six of which are the axes $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{23}{}^1$, $\$_{24}{}^1$, $\$_{34}{}^1$). Any plane section normal to the generators yields the circle-point curve, associated with the motion in positions 1, 2, 3, 4, which is, in fact, planar.

The plane P is the locus of all points whose positions 1 and 5 fall on a plane normal to the three parallel screws and hence have five positions on an infinitely large sphere. Since P is basically the same plane we studied in connection with circular motion, equation (9), the previous geometrical interpretation is immediately applicable (with the understanding that (l, m, n) now represents the direction of the three parallel screws and not the direction of a circle axis).

For this motion, (5) is identically zero and (6) is essentially nk^2. Therefore, the cubic cylinder k^2 is the locus of all points with four positions on a circle.

Points with five positions on a circle will generally exist because (5) is identically zero. Such points are determined by the intersection of (6), (5¹) and (6¹). In this case (5¹) (which is the same as (5) except that subscript 4 is changed to 5), may be shown to degenerate to $P \times F_2{}^{*1}$, but, since all points with five positions on a circle obviously fall in one plane, equation (5¹) will always be satisfied by points on P and we may ignore the quadratic $F_2{}^{*1}$. Denoting the cubic surface (6¹) as $G_{35}{}^3$, the required points are given by $P \times G_{35}{}^3 \times E^3$. This yields (at most) six real points with five positions on a circle.[3] In Table 1, we list the results of one such computation.[4]

If $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{15}$ are all parallel, pure-rotation screws, the term E^4 is identically zero since P is indeterminate and is satisfied by every point in Σ_1. Both (5) and (5¹) are identically zero, and (6) and (6¹) become (circular) cubic cylinders; $k_{234}{}^3$ and $k_{235}{}^3$, with parallel generators. The normal sections of these cubics yield the circle-point curves of the planar theory. Of the nine lines of the intersection, at most seven are real and three are the screws $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$. This leaves four lines which correspond to the well-known Burmester points. If, in addition, the screws are defined by a similarity transformation, one of these four lines is always the (moving) axis $\$_M$.

If the four screws ($\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{15}$) are parallel, but one, say, $\$_{15}$, has pure translation while the others have pure rotation, P becomes the plane at infinity and there are no finite points with five positions on a circle.

[3] Since the points lie on P they are all coplanar. Three of the nine intersections have been discounted since they fall on $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$ and will not generally yield a five-point circle.

[4] The computations described in this paper have been programmed in FORTRAN II. Programs may be obtained by writing to the author.

For parallel screws with rotation and translation, E^4 will generally not degenerate. However, points with four and five positions on right, circular cylinders with axes parallel to the screws may be obtained by merely neglecting the translation and determining points which lie on circles.

Two parallel, pure-rotation screws generally do not degenerate E^4. In addition, since (6) does not vanish, there are, generally, no points with five positions on a circle. However, since (5) is of the form $P \times F_2{}^{*1}$, all points with four positions on a circle are given by $P \times G_{34}{}^3$ and are therefore coplanar.

Schoenflies has shown that any two, parallel screws will cause P to decompose and that, conversely, if P decomposes, the screws must be parallel. For this case, the locus, of all points with three positions on a line, is a line. This same result follows directly from equations (12): For screws parallel to, say, the z-axis, the hyperboloids become, respectively, a right, circular cylinder (with generators parallel to the screws) and a plane parallel to the screws. The intersection, apart from the residual line $(x_2 - x_1) = 0$, $(y_2 - y_1) = 0$, is a line common to the cylinder and plane (see Appendix 2).

We now restrict the parallel screws to pure rotation and, for convenience, again take the screws along the z-direction. For pure rotation, the first equation in set (12) represents the same right, circular cylinder as previously mentioned but the second equation of (12) vanishes identically. Hence the locus of all points with three positions on a line becomes a right, circular cylinder embedded in Σ_1. This cylinder obviously contains the screws $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$. Any normal section yields a planar, image-pole triangle and its circumscribing circle. For three parallel, pure-rotation screws, equations (12¹) also have one member identically equal to zero. Hence points with four positions on a line are given by the intersection of the aforementioned cylinder and

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_4 - x_1) & (y_4 - y_1) \end{vmatrix} = 0$$

These two parallel cylinders intersect in two lines. This corresponds with the well-known planar theory since the line $(x_2 - x_1) = 0$, $(y_2 - y_1) = 0$, which is in fact the axis for $\$_{12}$, is the residual part of the intersection.

Intersecting Screws

For screws that intersect in a common point to lead to degenerate cases, they must generally be restricted to pure rotations.

For four positions corresponding to intersecting, pure-rotation screws, E^4 is identically satisfied since all points are undergoing spherical motion about the point of intersection of the screws. If, instead, one of the motions, for example, $\$_{14}$, is arbitrary, $E^4 = P_{A_1}C^3$. P_{A_1} is the plane given by equation (2) with $\$_f$ equal to $\$_{14}$ and (A_x, A_y, A_z) taken as the coordinates of the point of intersection of the screws. C^3 is formally the cubic surface given by equation (5) (i.e., F^3); however, due to the pure-rotation intersecting screws ($\$_{12}$, $\$_{13}$, $\$_{14}$), the surface becomes a cubic cone with apex at the intersection point of the screws. It can be easily shown that for this motion equation (6) yields either C^3 or is identically zero.[5] In either case, C^3 is the locus of all points with four positions on a circle. Hence, in the case of $\$_{15}$ skew, points with five positions on a sphere are either points which lie on a circle for the four positions associated with the intersecting pure rotation screws, or points whose distances from the intersection point (of $\$_{12}$, $\$_{13}$, $\$_{14}$) are not altered by the skew screw $\$_{15}$.

The intersection of cone C^3 with any sphere whose center is at the apex of C^3 is a spherical curve of order six. This curve is composed of two symmetrical and equal portions, either of which may be regarded as the spherical analog of a planar cubic. These spherical curves are of course analogous to the circle-point curves of the planar theory.

If $\$_{15}$ is also taken as a pure-rotation, intersecting screw, P_{A_5} is identically satisfied by every point in the body, and hence E^3 is identically zero. For five-position spherical motion, equations (5) and (5'), that is, $C_{234}^3 \times C_{235}^3$, give the locus of all points with five positions on a circle.[6] Now since the surfaces C^3 are cones with common vertexes, they will intersect in (at most) nine lines which are elements of the cones. $\$_{12}$, $\$_{13}$, $\$_{14}$ are three of these lines, the remaining six lines are the loci of points which have five positions on a circle. If we define a sphere on which the spherical motion is taking place, for example, the unit sphere, then there are at most 12 real points in which these lines pierce the sphere. Only half of these are independent since each pair is symmetric about the sphere center; hence we conclude *there are (at most) six independent points which, under spherical motion, have five positions on a circle*. These points are analogous to those studies by Burmester for the case of planar motion.[7] If, in addition, the screws are defined by a similarity transformation, one of these points is always on the (moving) axis $\$_{P}$.

Computationally the intersections $C_{234}^3 \times C_{235}^3$ can be obtained by "intersecting" each of these surfaces with an arbitrary plane. Eliminating one variable, say, x_1, between the plane and each C^3 yields two planar cubics from which a second variable, say, y_1, can then be eliminated. The roots of the resulting ninth-degree polynomial in z_1 give one of the required coordinates; the corresponding y_1 and x_1 are then obtained by back-substituting z_1 into the aforementioned. Since we know the location of the apex of the cone, we now have two points on each line, which means we have determined the intersection of C_{234}^3 and C_{235}^3. Table 2 lists the results of one such computation.[8]

Alternatively, since the cubics are homogeneous, we could divide through by, say, $(x_1)^3$ and reduce the cones to planar cubics with variables (y_1/x_1) and (z_1/x_1). (If we take $x_1 = 1$, the two methods become identical.)

Returning to the case where screw $\$_{15}$ is skew, we note that the locus of all points with four positions on a great circle of their five-position sphere is the planar cubic $P_A \times C^3$. All other points on C^3 lie on small circles of their sphere.

Introducing a sixth position defined by an arbitrary screw $\$_{16}$ we conclude that, since E_{235}^3 and E_{236}^3 have a common component in C^3, the locus of all points with six positions on a sphere is given by the line $P_{A_5} \times P_{A_6}$. This line cuts C^3 in (at most) three points; these points have four positions on a great circle. Similarly, introducing a seventh position defined by skew screw $\$_{17}$, we find that only the point given by $P_{17} \times P_{A_5} \times P_{A_6}$ has seven positions on a sphere.

[5] If we choose the origin at the point of intersection, the terms $t_j^2 - r_j^2 = 0$ for $j = 2, 3, 4$, and hence (6) is identically zero. The fact that a plane intersects a sphere in a circle is a physical reason for the conditions for points on a plane (5) and points on a circle (5 and 6) to be identical under spherical motion.

[6] As shown previously, (6) and therefore also (6') are either identically zero or equivalent to (5) and (5').

[7] For the case of infinitesimal, spherical motion, Dobrovolskii [2] has pointed out the existence of six points which have five-point contact with a circle in the tangent plane.

[8] In determining centers for points on the moving lines, we merely determine the axis for any point on the line. For three positions of a body in planar or spherical motion, there is obviously a (1, 1) correspondence between lines in Σ^2 and Σ. This correspondence is called the quadratic transformation and replaces the cubic transformation which has no significance in spherical or planar motion.

Table 2 Spherical motion; points with five positions on a circle

SCREW $\$_{1j}$

j	DIRECTION COSINES			ROTATION
	u_{1j}	v_{1j}	w_{1j}	ϕ_{1j} (deg)
2	-0.4063	-0.6131	+0.7041	-31.0500
3	0.0648	-0.7615	0.6118	-1.4800
4	0.5124	0.6520	0.4527	15.7000
5	0.2835	0.7745	0.5607	21.3400

All the screws have zero translation and intersect at the origin.

Line Number	MOVING POINTS $y_1 = mx_1$, $z_1 = nx_1$		FIXED POINTS $y = m_F x$, $z = n_F x$	
	m	n	m_F	n_F
1	0.4764	-0.1670	0.4815	+0.7499
2	+0.4520	-4.3610	-10.7756	+5.4119
3	0.5673	0.9821	0.5145	0.7918
4	1.5676	6.8006	1.3530	1.5607
5, 6	IMAGINARY			

Table 3 Synthesis of two-revolute, two-spheric-pair spatial four bar

PROBLEM STATEMENT

Function: $y = \cos x$ $\quad 0 \le x \le 180°$

Range: Input crank $(\phi) = 150°$; Output crank $(\psi) = 100°$

INPUT TO COMPUTER

Precision positions (using Chebyshev spacing)

j	1	2	3	4	5	6	7
$\Delta\phi_{1j}$	0.0	1.6[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\Delta\psi_{1j}$	0.0	12.4167	32.4167	50.4167	102.4167	127.4167	140.4[illegible]

Angle between axes = 90°

RESULTS OF COMPUTATION

	1	2	3
Point on a sphere	0.9019	0.1057	+0.4561
Center of sphere	0.2591	0.7569	0.8711
Radius of sphere	1.5195		

MECHANISM PARAMETERS

Link lengths: Input crank = 0.5606; Coupler = 1.5195; Output crank = 0.8177

Angle between axes = 90°

Distance between axes = 1

Distance to common normal between fixed axes from plane of:

a) input-crank circle = 0.9915

b) output-crank circle = 0.3968

ANALYSIS OF SOLUTIONS

Maximum error = .0032 which corresponds to 0.16% of the range of y.

For $0 \le x \le 177°$, maximum error is 0.0003 which corresponds to 0.015% of the range of y.

In the case of intersecting, pure-rotation screws, equation (12) yields two quadratic cones with a common apex. These cones intersect in $\$_{12}$, $\$_{13}$, $\$_{14}$ and the spurious line $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. Hence I^3 degenerates into three lines. There are no nontrivial points with three positions on a line, since a line will not pierce a sphere in more than two points.

Application to Mechanism Synthesis

Examples. In the following paragraphs, we illustrate various ways in which the foregoing results can be applied to the synthesis of linkwork. Since it is the versatility of the method which we seek to illustrate, we limit ourselves to one or two examples of a given type and do not consider all the structural variants which may be synthesized by essentially the same procedure.

Motion of a coupler link relative to a fixed link is the most straightforward application of the theory. We illustrate the procedure with a type of seven-bar linkage which has been used in automotive suspensions and elsewhere. The linkage (Fig. 5) has a coupler, a fixed link, and five binary cranks in parallel between the coupler and fixed link. All 10 joints are spherical. For any specified seven positions of the coupler, we may find, at most, 20 points with spherical motion. These points and their corresponding centers define the cranks. Taken five at a time, these 20 points yield 15,504 seven bars whose couplers pass through the seven specified positions. On the other hand, there will generally be fewer, and maybe no, solutions since, for a given set of positions, there may be less than 20, or even five, real points with seven positions on a sphere. For less than seven prescribed positions, we generally have an infinite number of solutions. The points given in Table 3 of [1] are, in effect, the results of such a synthesis.

The relative motion of two moving links **can be treated by inversion.** This is illustrated by a procedure described by Wilson [3] for the function-generation synthesis of a 2R2S four bar. In this linkage, the input and output crank are joined to the frame by turning pairs (revolutes) which have skew axes. Each crank connects to the coupler with a spherical pair. In synthesizing a function, say, $\Phi = f(\Psi)$, we choose sets of angle changes $\Delta\Phi_{1j}$ and $\Delta\Psi_{1j}$ which satisfy this functional relationship, and require that the two cranks rotate through these angles from some unspecified initial position Φ_1, Ψ_1.

The procedure is as follows: We arbitrarily choose the skew, rotation axes[3] and then invert the mechanism by holding either the input or output crank fixed. If we fix the input crank, the two sets of rotation angles are $-\Delta\Psi_{1j}$ (about the fixed axis) and $\Delta\Phi_{1j}$ (about the moving axis). Alternatively, fixing the output crank we have, respectively, $-\Delta\Phi_{1j}$ and $\Delta\Psi_{1j}$. In either case, the motion of what is now the coupler link is defined under the special case of a similarity transformation and the order in which the rotations are taken is immaterial. The synthesis is completed by determining one point, in what is now the coupler, which lies on a sphere in the several design positions. This point and its corresponding center are taken as the spherical joints in the original coupler. For five angle changes, we use any point on G^2. For six specified angle changes, we use any point given by $G_{sss}^1 \times G_{sss}^2$; and for seven sets of $\Delta\Phi_{1j}$, $\Delta\Psi_{1j}$, we choose one of the points given by $G_{sss}^2 \times G_{sss}^3 \times G_{sss}^4$. One such synthesis is described in Table 3.

The same procedure may be applied, for example, to the four bar, shown in Fig. 2, except that here we hold the revolute-cylinder link fixed and seek a point which lies on a cylinder.

In order to move a body according to a given timing cycle, it is desirable to be able to specify the corresponding input motion. Consider, for example, the four bar with two turning, one cylindrical, and one spherical pair, shown in Fig. 2. Here we arbitrarily choose dimensions of the crank with the cylindrical pair, and specify up to four positions. For these design positions, we define the motion of the coupler by specifying a rotation about, and a translation along, the cylindrical joint axis. The synthesis is completed by choosing any point on k^4 as the location of the spherical joint and the corresponding axis as the fixed revolute.

Similar procedures may be applied, for example, to a three bar with a revolute-cylinder input crank whose third joint is a sphere-in-cylinder pair, Fig. 3. Here we are limited to three positions and seek a point on t^2.

Inversions of the same linkage are treated differently. For example, if, instead of the aforementioned, the revolute-cylinder crank of the four bar in Fig. 2 is fixed, one could either specify three arbitrary positions of the coupler and synthesize the linkage by determining a revolute-revolute crank (this is discussed later) and one point with three positions on a cylinder (any point will do, since all points have three positions on a cylinder), or choose revolute-revolute crank and consider only similarity-transformation coupler motions. In this latter case, the synthesis is solved by any point which falls on a cylinder in the several design positions.

In certain linkages, it is required that a point simultaneously lie on two special loci. Graphical constructions for such syntheses have been given by Altmann [4]. The methods of this paper may be used to study such problems. The most direct approach is to choose the appropriate equations from the ones developed in [1] and then numerically determine their simultaneous solutions.

Dwell mechanisms and other linkwork **can be designed by using points in the body with special motions.** For example, to design a dwell mechanism we take any convenient spatial linkage, and from its known "coupler" motion determine which points on the coupler have, say, seven positions on a sphere. To one such point, we attach a dyad of the type shown in Fig. 4. The dimensions of the floating link of the dyad are completely determined by a point which lies on a sphere and its corresponding center. The dimensions of the rotating dyad crank (which is the link with the approximate dwell) are arbitrary.

Spherical and planar linkages are synthesized exactly as above. Therefore, we may solve function generation problems and the like; the only difference is that, instead of a general screw, the motion must be defined by, respectively, intersecting and parallel pure-rotation screws. The results in Table 2 may, of course, be interpreted as a five-position synthesis of a spherical four bar.

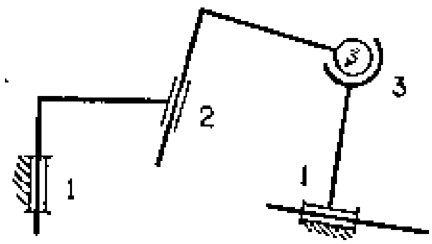


Fig. 2 Four bar with one cylinder, one spheric, and two revolute joints. The numbers denote the freedom in the joints.

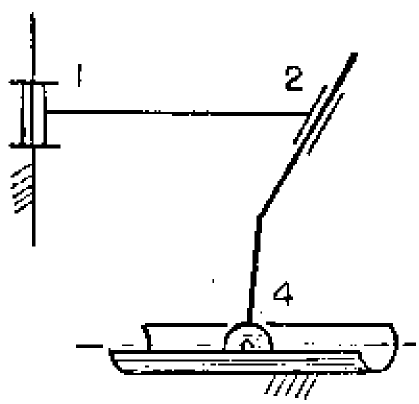


Fig. 3 Three bar with revolute, cylinder, and ball-in-cylinder joint. The numbers denote the freedom in the joints.

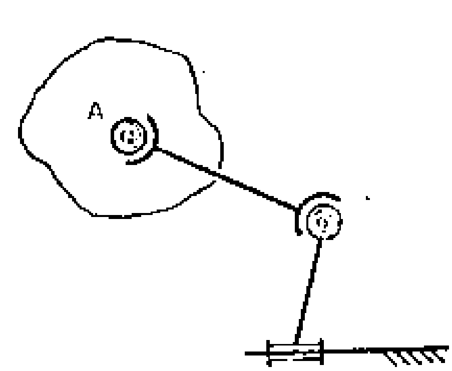


Fig. 4 Spatial dwell linkage. The rotating-crank dwells if the center of spheric joint *A* moves on a sphere whose center is at the other spheric joint.

[3] The distance between skew axes and their absolute orientation does not affect the relative rotation between input and output. However, the angle between the skew axes does affect the final solution. This procedure is extremely useful in designing linkages which act like noncircular gears connecting skew shafts at a specified angle.

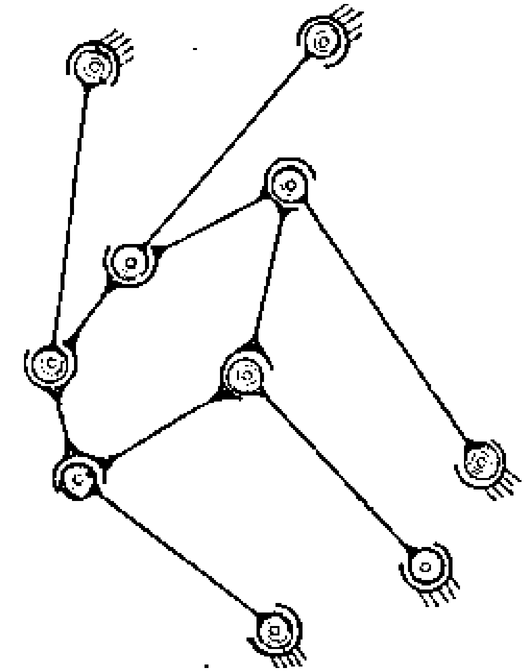


Fig. 5 Seven bar with 10 spheric joints. This linkage has five binary cranks, each of which is free to rotate about its own center line.

When *kinematic pairs other than those included* in our treatment are present, the synthesis will require some arbitrary choices (or an extension of the foregoing development to these pairs). These choices reduce the number of precision conditions. However, they do have the advantage of allowing an element of control over the final solution.

Pseudoplanar and pseudospherical linkages are names we coin for linkages which have several of their design positions given by, respectively, parallel or intersecting pure-rotation screws. Such linkages are spatial but may be designed to have several coplanar or cospherical precision positions. Their synthesis is accomplished in the usual way, except that we now make use of the degenerate cases discussed in the sections on special motions.

New Kinematic Pairs—The Revolute-Revolute Crank. In this section, we illustrate how the results of [1] may be adapted to the study of new pairs. The design of a link with two turning pairs (a so-called revolute-revolute crank) requires the determination of a moving rotation axis (in Σ) and a fixed rotation axis (in Σ') for which the relative distance and angle (between these axes) remains constant. Such a link may be synthesized if we can find two points in Σ which lie on circles about the same axis, since all the points on the line connecting these two points will also lie on circles about this same axis.

From (9), we have the condition that a point has two positions on a plane normal to a given direction. This may be written:

$$e_j x_1 + f_j y_1 + g_j z_1 + h_j = 0$$

where

$$e_j = a_{x_j}(l/n) + a_{y_j}(m/n) + a_{z_j}$$
$$f_j = b_{x_j}(l/n) + b_{y_j}(m/n) + b_{z_j}$$
$$g_j = c_{x_j}(l/n) + c_{y_j}(m/n) + c_{z_j}$$
$$h_j = d_{x_j}(l/n) + d_{y_j}(m/n) + d_{z_j}$$

Here (l, m, n) are the direction cosines of the fixed axis, and (x_1, y_1, z_1) is a point on the moving axis. The $a_{x_j}, b_{x_j}, c_{x_j}, d_{x_j}, \ldots, d_{z_j}$ are the coefficients of the matrix which defines the motion (see equation (1)).

Further, it has been shown in [1] that the locus of all points with three positions on a circle, of a specified inclination, is a line. Hence, if we regard (l, m, n) as known, the previous equation written twice ($j = 2, 3$) yields the equation of the moving axis:

$$x_1 = (s/v)y_1 + r/v$$
$$z_1 = (t/v)y_1 + u/v \qquad (a)$$

where

$$s = f_3 g_2 - f_2 g_3$$
$$r = g_2 h_3 - g_3 h_2$$
$$v = e_2 g_3 - e_3 g_2$$
$$t = e_3 f_2 - e_2 f_3$$
$$u = e_3 h_2 - e_2 h_3$$

Substituting these results for x_1 and z_1 into (8) [which is the condition that point (x_1, y_1, z_1) remain at a constant distance from a point (A_x, A_y, A_z) in Σ'] yields:

$$A_j A_x + B_j A_y + C_j A_z + D_j = 0 \qquad (b)$$

where (A_x, A_y, A_z) is any point on the fixed axis, and

$$A_j = (a_{x_j} s + b_{x_j} v + c_{x_j} t)y_1 + (a_{x_j} r + c_{x_j} u + d_{x_j} v),$$
$$B_j = (a_{y_j} s + b_{y_j} v + c_{y_j} t)y_1 + (a_{y_j} r + c_{y_j} u + d_{y_j} v),$$
$$C_j = (a_{z_j} s + b_{z_j} v + c_{z_j} t)y_1 + (a_{z_j} r + c_{z_j} u + d_{z_j} v),$$
$$D_j = ((r_j^2)_x s + (r_j^2)_y v + (r_j^2)_z t)y_1 + ((r_j^2)_x r + (r_j^2)_z u + (r_j^2)_d v)$$

In D_j, the coefficients of the x, y, z and constant terms of $(r_1^2 - r_j^2)/2$ have been written as $(r_j^2)_x, (r_j^2)_y, (r_j^2)_z, (r_j^2)_d$, respectively.

If (l, m, n) and y_1 were known, the foregoing equation taken twice ($j = 2, 3$) would yield the fixed axis. Now, clearly, the choice of y_1 should in no way affect the location of the fixed axis. Similarly, we are at liberty to arbitrarily choose one of the coordinates (A_x, A_y, A_z) of the fixed axis. For simplicity, we take $A_z = 0$ and $y_1 = 0$ in (b):

$$A_2^0 A_x + B_2^0 A_y + D_2^0 = 0$$
$$A_3^0 A_x + B_3^0 A_y + D_3^0 = 0$$

(The superscript denotes the value of y_1.)

Similarly, if we take $A_z = 0$ and $y_1 = 1$:

$$A_2^1 A_x + B_2^1 A_y + D_2^1 = 0$$
$$A_3^1 A_x + B_3^1 A_y + D_3^1 = 0$$

Now, these two sets of equations will yield the same A_x and A_y if the following two compatibility equations are satisfied:

$$\begin{vmatrix} A_2^0 & B_2^0 & D_2^0 \\ A_3^0 & B_3^0 & D_3^0 \\ A_3^1 & B_3^1 & D_3^1 \end{vmatrix} = 0$$
$$\begin{vmatrix} A_2^0 & B_2^0 & D_2^0 \\ A_3^0 & B_3^0 & D_3^0 \\ A_2^1 & B_2^1 & D_2^1 \end{vmatrix} = 0 \qquad (c)$$

When expanded, these determinants give two sixth-degree polynomials in (l/n) and (m/n). For each valid set of roots determined from (c), we have a unique revolute-revolute crank. The moving axis is computed from (a), and the fixed axis from (b) with $j = 2, 3$ and y_1 arbitrary. Table 4 lists the results of one such computation for the motion defined in Table 1 of [1].

Table 4 Axes of a revolute-revolute crank corresponding to positions 1, 2, 3 of Table 1 in [1]

```
MOVING AXIS

x1 = -0.5843y1 - 3.3546
z1 = -0.2892y1 - 2.9623


FIXED AXIS

x = 0.3921y - 1.0092
z = 4.5116y - 7.6542
```

Conclusions

The similarity transformation leads to a simplification of the screw-transformation coefficients. These, in turn, considerably simplify E^4, which, in effect, degenerates into a second-order surface. It is important to note that the similarity transformation provides a unified approach to function generation problems for spatial, spherical, and planar mechanisms.

The intersecting pure-rotation screws bring spherical motion into the general theory. Obviously, the loci of interest become conics, or curves and points on conics. The study of parallel pure-rotation screws shows how the well-known planar theory is derived from more general considerations. In this case, the surfaces of interest becomes cylinders—i.e., cones with apexes at infinity. The inclusion of "skew-screws" with planar and spherical motion allows the design of mechanisms with pseudoplanar and pseudospherical motion. Such linkages are spatial, but most of their design positions are, respectively, planar and spherical.

This paper and [1] deal almost exclusively with special points. The study of special lines may be pursued by imposing simultaneous conditions on two points, as has been done in the revolute-revolute crank synthesis described in the preceding section. Alternatively, it is possible to study special lines directly. This latter approach is utilized in a subsequent study [7] to effect the synthesis of cylindric-cylindric cranks.

These results and those presented in [1] and [7] are suited to various types of syntheses, and may be regarded as generalizations of the classical finite-position Burmester theory. The results given in these papers are applicable to function-generation, path-generation, and coupler-plane-motion syntheses of spatial, spherical, and planar linkages.

Acknowledgments

The author wishes to express his appreciation to Pictaw Chen for doing the bulk of the computer programming, and to the National Science Foundation whose Grant NSF-GP 2005 made this work possible.

References

1 Roth, B., "The Kinematics of Motion Through Finitely Separated Positions," ASME Paper No. 67—APM-2.

2 Dobrovolskii, V. V., "Burmester's Points in Spherical Motion" (in Russian), *Prikladnaya Matematika i Mekhanika*, Vol. 9, 1945, pp. 189–191.

3 Wilson, J. T., III, "Analytical Kinematic Synthesis by Finite Displacements," *Journal of Engineering for Industry*, Trans. ASME, Series B, Vol. 87, No. 2, May 1965, pp. 161–169.

4 Altmann, F. G., "Zur Massynthese der Raumgetriebe," *Maschinenbautechnik*, Vol. 6, 1957, pp. 93–100.

5 Denavit, J., and Hartenberg, R. S., "Approximate Synthesis of Spatial Linkages," Journal of Applied Mechanics, Vol. 27, No. 1, Trans. ASME, Vol. 82, Series E, Mar. 1960, pp. 201–206.

6 Levitskii, N. I., and Shakvazian, K. K., "Synthesis of Four-Element Spatial Mechanisms With Lower Pairs," *International Journal of Mechanical Sciences*, Vol. 2, pp. 76–92. (This is a translation by F. Freudenstein of a paper which was originally published in 1954.)

7 Roth, B., "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body," *Journal of Engineering for Industry*, Trans. ASME, Series B, Vol. 89, No. 1, Feb. 1967, pp. 102–110.

APPENDIX 1

Proof That Under a Similarity Transformation $E^4 = u_1 \times F^2 \times G^2$. We write (7):

$$E^4 = \frac{1}{2}\begin{vmatrix} (r_2^2 - r_1^2) & (y_2 - y_1) & (x_2 - x_1) & (z_2 - z_1) \\ (r_3^2 - r_1^2) & (y_3 - y_1) & (x_3 - x_1) & (z_3 - z_1) \\ (r_4^2 - r_1^2) & (y_4 - y_1) & (x_4 - x_1) & (z_4 - z_1) \\ (r_5^2 - r_1^2) & (y_5 - y_1) & (x_5 - x_1) & (z_5 - z_1) \end{vmatrix} = 0$$

Expanding this determinant according to the Laplace development yields:

$$E^4 = \sum_{j=2}^{4}\sum_{k=3}^{4} (-1)^{(j+k-1)}\left[\left(\frac{r_j^2 - r_1^2}{2}\right)(y_k - y_1) - \left(\frac{r_k^2 - r_1^2}{2}\right)(y_j - y_1)\right]\cdot[(x_l - x_1)(z_m - z_1) - (x_m - x_1)(z_l - z_1)] = 0$$

Here l is 2, 3, or 4 and m is 2, 3, or 4 as determined from the restrictions that $j \neq k \neq l \neq m$, $j < k$ and $l < m$.

Substituting motion parameters, one finds that the first square bracket may be written:

$$u_1\{(u_2x_1 - u_1y_1)^2 + (z_1 - p)^2\}[\sin\theta_j(1 - \cos\theta_k) - \sin\theta_k(1 - \cos\theta_j)]$$

and hence $u_1[(u_2x_1 - u_1y_1)^2 + (z_1 - p)^2]$, which we write as u_1F^2, may be brought outside of the summation signs. Denoting the terms in the summation as G^2 (since they are quadratic), we have $E^4 = u_1F^2G^2$.

APPENDIX 2

The Curve I^2 Under Parallel Screws. We write equation (12):

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix} = 0 \qquad (1)$$

$$\begin{vmatrix} (x_2 - x_1) & (z_2 - z_1) \\ (x_3 - x_1) & (z_3 - z_1) \end{vmatrix} = 0 \qquad (2)$$

For simplicity, we take the screws, $\$_{12}$ and $\$_{13}$, parallel to the z-axis. Expanding (1) and substituting the motion parameters yields:

$$[(x_1 - a_2)(x_1 - a_3) + (y_1 - b_2)(y_1 - b_3)]\left[\sin\left(\frac{\theta_3 - \theta_2}{2}\right)\right] + [x_1(b_2 - b_3) + y_1(a_3 - a_2) + (a_2b_3 - a_3b_2)] \times \left[\cos\left(\frac{\theta_3 - \theta_2}{2}\right)\right] = 0 \qquad (3)$$

This is obviously the equation of right, circular cylinder. We note that the translations (d_2 and d_3) do not enter into this result. Further, since (a_2, b_2, 0) and (a_3, b_3, 0) are, respectively, the coordinates of the ft of the perpendiculars from the origin to $\$_{12}$ and $\$_{13}$, this cylinder contains $\$_{12}$ and $\$_{13}$ (and also $\$_{23}$). θ_2 and θ_3 are the rotations corresponding to $\$_{12}$ and $\$_{13}$, respectively.

Expanding (2) yields:

$$(x_1 - a_3)(1 - \cos\theta_3)d_2 - (x_1 - a_2)(1 - \cos\theta_2)d_3 + (y_1 - b_3)(\sin\theta_3)d_2 - (y_1 - b_2)(\sin\theta_2)d_3 = 0 \qquad (4)$$

which is a plane parallel to the z-axis.

The plane (4) cuts the cylinder (3) in two lines—one of which ($x_2 - x_1 = 0$, $x_3 - x_1 = 0$) is spurious. *Hence, for parallel screws, I^2 becomes a line.* If $d_2 = d_3 = 0$, the motion is pure rotation and (4) vanishes. This leaves (3); any normal section yields the circle of the classical planar theory. If the screws are collinear, (3) becomes a "line" cylinder, and I^2 coincides with the screw axis.

Printed in U. S. A.

Paper No.
66—Mech-7

B. ROTH
Assistant Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif.
Mem. ASME

On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body

For a rigid body in two, three, four, and five finitely separated positions, the loci of the screw axes and other special lines are derived. It is shown that the classical planar theory is a special case of a more general theory which includes planar, spherical, and spatial displacements.

Introduction

In this paper, we study the kinematics of a rigid body in a series of finitely separated positions, and seek to determine spatial equivalents for the basic quantities of the classical planar theory. Of special interest are those spatial geometries which are analogous to the pole triangles, pole curves, center and circle-point curves, and Burmester-point pairs.

These studies are in the spirit of two of this author's previous works [1, 2].[1] However, here we are primarily concerned with lines, in the body, while [1] and [2] deal with points.

Several other works deal with extensions of the planar theory but they are concerned with generalized planar concepts [3, 4] or with spherical motion [5, 6, 7]. Although Keler [8] has used line geometry and dual numbers to consider general spatial motion for two and three positions, and Schoenflies [9] and more recently others [10, 1] have considered points which lie on special loci, this author is unaware of any previous work in which line congruences are taken as the spatial analogs of the pole, center-point, and circle-point curves.

Screw Axis Geometry

Nomenclature

In this paper we are concerned with the relative position of two rigid bodies. It is convenient to refer to one as a moving body and to the other as a fixed body. The moving system is denoted by Σ and the fixed system by Σ'. We number the various positions of the moving system and use subscripts to indicate which position of Σ we are referring to.

It is well known that, regardless of how a motion actually occurs, the displacement may always be regarded as a rotation about a given axis and a translation along the same axis. Such a motion is called a screw displacement. As shown in Fig. 1, we denote the screw which takes Σ from the ith to the jth position as $\$_{ij}$. The corresponding rotation is taken as θ_{ij} and the translation is d_{ij}. Further, the unit vector $\mathbf{U}_{ij}$ parallel to this screw axis has components $(u_{xij}, u_{yij}, u_{zij})$, and the screw passes through the point (a_{ij}, b_{ij}, c_{ij}); these six components are measured along coordinate axes fixed in Σ'.

θ_{ij} and d_{ij} are called the screw parameters, and their ratio d_{ij}/θ_{ij} is the pitch of the screw. The notion of pitch implies that the rotation and translation occur simultaneously. However, it is equally important to remember that these motions may be regarded as occurring separately, and that their effects may be superimposed.

For plane motion $d_{ij} = 0$, and it is customary to take $u_{zij} = 1$, $c_{ij} = 0$.

Two Positions

We now describe an alternative characterization of a finite displacement, which, although less well known than the screw, will prove to be equally important in what follows.

It is known [11] that any finite displacement may be considered as the result of successive reflections about two fixed lines.[4] The two lines must be normal to and intersect the screw axis. In addition, the distance and angle between them must be, respectively, $d_{ij}/2$ and $\theta_{ij}/2$ (measured from the first line to the second in the positive sense of $\$_{ij}$). Otherwise, the two lines are arbitrary.

The foregoing is illustrated and proved, with the aid of Fig. 2, as follows. L_i is any line normal to $\$_{ij}$, and L_j is the line obtained by operating on L_i with a screw coincident to $\$_{ij}$ but with

[1] Numbers in brackets designate References at end of paper.

Contributed by the Machine Design Division for presentation at the Mechanisms Conference, Lafayette, Ind., October 10-12, 1966, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received at ASME Headquarters, June 2, 1966. Paper No. 66—Mech-7.

[4] A reflection about a line is equivalent to a 180 deg rotation about the line.

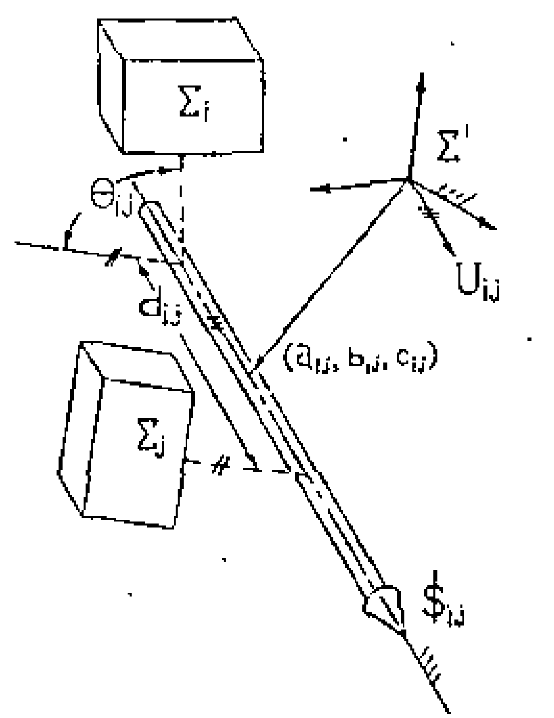


Fig. 1 Screw displacement $\$_{ij}$ and the associated nomenclature

Discussion on this paper will be accepted at ASME Headquarters until November 14, 1966

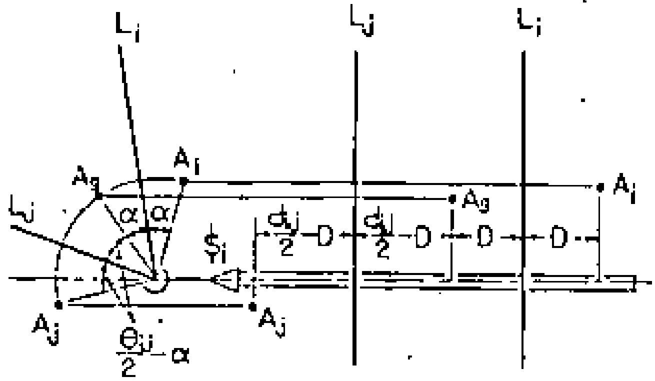


Fig. 2 Two reflections are equivalent to a screw displacement. Point A_i is reflected about L_i to position A_g, and A_g is reflected about L_j to position A_j. The figure shows side and end views of the screw $\$_{ij}$.

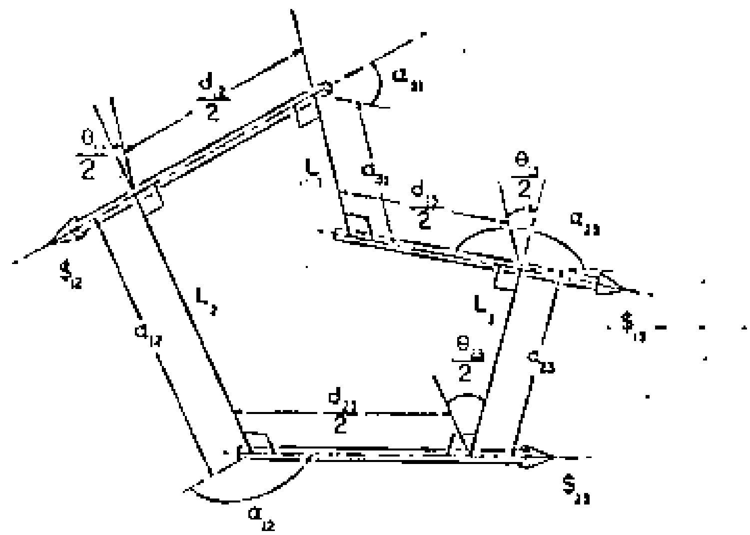


Fig. 3 Screw triangle geometry showing lengths of sides $d_{12}/2$, $d_{23}/2$, $d_{31}/2$; angles between sides α_{12}, α_{23}, α_{31}; lengths of normals between sides a_{12}, a_{23}, a_{31}; angles between normals $\theta_{12}/2$, $\theta_{23}/2$, $\theta_{31}/2$

parameters $\theta_{ij}/2$, $d_{ij}/2$. Take A_i as a generic point in Σ_i. Then, reflecting A_i about L_i brings it to A_g and reflecting A_g about L_j brings the point to position A_j. From the figure, which shows an end and side view of the screw, it follows that the relative position of A_i and L_i, which is given by D and α, does not affect the displacement. Explicitly, the displacement along the screw axis is given by

$$D + D + 2\left(\frac{d_{ij}}{2} - D\right) = d_{ij},$$

and the angular displacement, of the radial line to A_i, about the screw axis is

$$\alpha + \alpha + 2\left(\frac{\theta_{ij}}{2} - \alpha\right) = \theta_{ij}.$$

Hence, the two reflections effect a displacement identical to the one defined by $\$_{ij}$, and therefore, we conclude that the position of L_i (along $\$_{ij}$) is indeed arbitrary.

Three Positions

For three finitely separated positions, Σ_1, Σ_2, Σ_3, there are three screw axes, $\$_{12}$, $\$_{23}$, $\$_{31}$, but only two are independent. The geometry relating these screws, as shown in Fig. 3, may be easily derived by replacing each screw by two reflections. Given $\$_{12}$ and $\$_{23}$ we call their common perpendicular L_2, and define L_1 as the line L_2 screwed about $\$_{12}$ by an amount $-d_{12}/2$, $-\theta_{12}/2$. L_3 is the line normal to $\$_{23}$ which is $d_{23}/2$, $\theta_{23}/2$ from L_2. The third screw axis $\$_{31}$ is then given as the line normal to L_3 and L_1. The validity of this can be seen by considering an arbitrary point A_1. We reflect about L_1 and then L_2, the resulting position A_2, as we have seen above, is the same as that after screw $\$_{12}$. Reflecting A_2 about L_2 and then L_3 results in the point being in position A_3. Thus A_1 has moved to A_3 by undergoing four reflections, but two successive reflections are about the same line, L_2, and hence may be ignored. Therefore, the same results may be obtained from a reflection about L_1 followed by a reflection about L_3. Hence the normal to L_1 and L_3 defines the screw axis for $\$_{13}$ and the parameters $\theta_{13}/2$, $d_{13}/2$ are given, respectively, by the angle and the distance between L_1 and L_3. These quantities are always measured from L_1 to L_3.

Since any three lines taken two at a time generally have three unique normals, it is apparent that any set of three axes defines a unique set of lines L_1, L_2, L_3 and therefore a set of parameters d_{12}, d_{23}, d_{31}, θ_{12}, θ_{23}, θ_{31}. Hence *there is a set of screw motions determined by any configuration of three axes.*

The geometrical configuration formed by three lines and their common normals has been called by Yang [12] a "spatial triangle." The configuration of three screw axes and their normals will be called a "screw triangle" in analogy to the pole triangle of planar motion. The pole triangle is a special case of the "screw triangle." By investigating the geometry of the screw triangle shown in Fig. 3, we obtain analytical expressions for the screw $\$_{31}$[3] in terms of $\$_{12}$ and $\$_{23}$.

Taking projections along the screw axes yields:

$$\sin\left(\frac{\theta_{31}}{2}\right) = \frac{\sin\alpha_{12}}{\sin\alpha_{31}}\sin\left(\frac{\theta_{12}}{2}\right)$$

$$\cos\left(\frac{\theta_{31}}{2}\right) = \cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right) - \cos\alpha_{12}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)$$

$$\sin\alpha_{31} = -\frac{1}{\cos\left(\frac{\theta_{31}}{2}\right)}\left[\sin\alpha_{12}\cos\alpha_{23} + \cos\alpha_{12}\sin\alpha_{23}\cos\left(\frac{\theta_{23}}{2}\right)\right]$$

$$\cos\alpha_{31} = -\frac{1}{\sin\left(\frac{\theta_{31}}{2}\right)}\left[\sin\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right) + \cos\alpha_{12}\cos\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right]$$

$$d_{31} = \frac{2}{\sin\left(\frac{\theta_{31}}{2}\right)}\left[\frac{d_{12}}{2}\left(\sin\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right) + \cos\alpha_{12}\cos\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right) + \frac{d_{23}}{2}\left(\cos\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right) + \cos\alpha_{12}\sin\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right)\right) - a_{12}\sin\alpha_{12}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right]$$

$$a_{31} = \frac{1}{\sin^2\left(\frac{\theta_{31}}{2}\right)}\left[a_{12}\left(\cos\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{31}}{2}\right) - \cos\left(\frac{\theta_{12}}{2}\right)\right) + \sin\left(\frac{\alpha_{12}}{2}\right)\left(d_{12}\sin\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{31}}{2}\right) - d_{23}\sin\left(\frac{\theta_{12}}{2}\right)\right)\right]$$

α_{12} and a_{12} are, respectively, the angle and distance between $\$_{12}$ and $\$_{23}$ taken in the sense of screwing $\$_{12}$ about L_2 into $\$_{23}$. Similarly, for (α_{23}, a_{23}) and (α_{31}, a_{31}) we measure from $\$_{23}$ to $\$_{31}$ and from $\$_{31}$ to $\$_{12}$, respectively.

[3] $\$_{13}$ is a screw along the same axis and with the same magnitude as $\$_{31}$; however, the sense (of rotation and translation) is opposite in both screws.

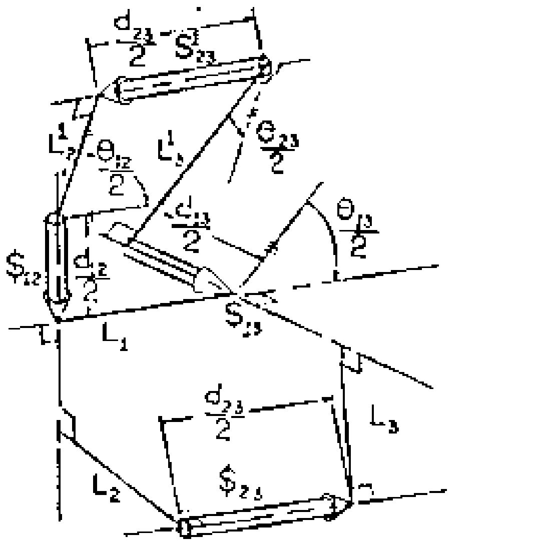


Fig. 4 Image screw triangle $\$_{12}$, $\$_{23}{}^1$, $\$_{13}$ and screw triangle $\$_{12}$, $\$_{23}$, $\$_{13}$

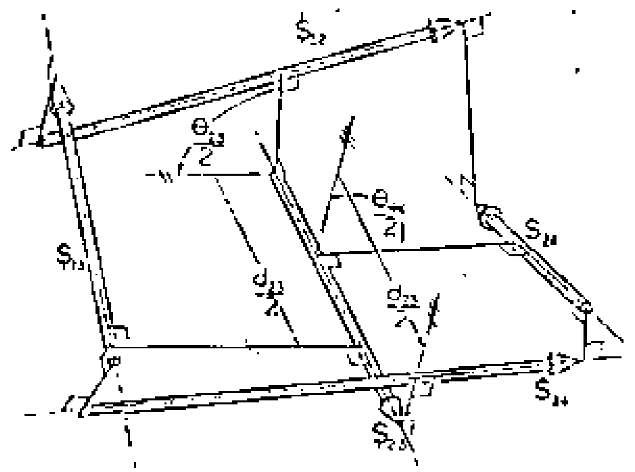


Fig. 5 Opposite screw quadrilateral $\$_{12}$, $\$_{23}$, $\$_{34}$, $\$_{14}$ and a fifth screw, $\$_{24}$ which subtends equal dual angles at opposite sides of the quadrilateral

We have chosen to work with $\$_{31}$ instead of $\$_{13}$ to preserve the symmetry, hence all the above expressions are valid under cyclic permutation of indices. (These equations depend exclusively on the geometry of the screw triangle and may be viewed as special cases of the equations obtained from the so-called spatial-triangle, sine and cosine laws as given by Yang [12].) A host of similar relationships may be obtained.

For planar motion $\alpha_{ij} = 0$ or 180 deg, and the second equation yields the well-known summation rule $\theta_{13} = \theta_{12} + \theta_{23}$.

The Ground (or Cardinal[3]) Point

Just as in planar work, there is a unique point A_p which can be reflected about the "sides" of the screw triangle to give the three positions of any point A. The "sides" of the screw triangle are the lines L_1, L_2, L_3 (i.e., the common normals to the screw axes). If we call the three positions of point A: A_1, A_2, A_3, respectively, then A_p may be found by reflecting A_1 about L_1 or A_2 about L_2 or A_3 about L_3. Conversely, by reflecting A_p about L_1 we obtain A_1, etc., for A_2 and A_3. The proof of this follows directly from the equivalence of two reflections to a screw as outlined above.

Similarly, it follows that to every geometric entity there corresponds an equivalent cardinal member. For example, given a line l_1, a reflection about L_1 yields l_p which may be reflected about L_2 and L_3 to yield, respectively, l_2 and l_3. The same is of course true for planes.

Image Screw Triangles

We now consider the screw axes as fixed in the moving system Σ. In position one, Σ_1, the three axes are $\$_{12}{}^1$, $\$_{23}{}^1$, $\$_{13}{}^1$. $\$_{12}{}^1$ and $\$_{13}{}^1$ are of course identical to $\$_{12}$ and $\$_{13}$ in the fixed system Σ'. The axis $\$_{23}{}^1$ is obtained by screwing that line in Σ_2 which is coincident with $\$_{23}$ (in Σ') about $\$_{12}$ or screwing that line in Σ_3 which is coincident with $\$_{23}$ about $\$_{13}$.[4] Alternatively, $\$_{23}{}^1$ may be obtained by reflecting axis $\$_{23}$ about L_3 or L_2 (depending upon if we wish to regard the line in Σ coincident with the line $\$_{23}$ in Σ' as being in Σ_2 or Σ_3),[4] and then reflecting it about L_1. Since L_2 and L_3 both intersect (and are normal to) $\$_{23}$, it follows that the first reflection only reverses the "head and tail" of the screw and that the line of action of $\$_{23}{}^1$ depends exclusively on the reflection about L_1. See Fig. 4.

[3] Like other terms in kinematics, the various names are due to different translations of the original German words. Here the problem is whether "grund" in "grundpunkt" means ground or basic (i.e. cardinal).

[4] If we regard the screw axis as a sliding line vector, then there is no need to distinguish between the line being in Σ_2 or Σ_3. However, if we consider the axis as a line fixed in the body, the distinction between Σ_2 and Σ_3 is important since corresponding points on the axis are d_{23} apart.

Similarly, $\$_{13}{}^2$ is obtained by reversing the "head and tail" of $\$_{13}$ and then reflecting it about L_2. $\$_{12}{}^3$ is obtained by reflecting $\$_{12}$ about L_3 after having reversed its sense. Fig. 4 shows the "image screw triangle" $\$_{12}$, $\$_{23}{}^1$, $\$_{13}$ with sides L_1, $L_2{}^1$, $L_3{}^1$ and some pertinent dimensions. The well-known planar case, which is a special case of the above, follows if we take all the axes as parallel, pure rotation screws.

Four Positions

There are now six screws associated with the motion: $\$_{12}$, $\$_{23}$, $\$_{34}$, $\$_{14}$, $\$_{24}$, $\$_{13}$. We assume knowledge of any four which, in analogy to the planar opposite-pole quadrilateral, form an "opposite screw quadrilateral"[5] and inquire as to the locus of the other two screws.

Referring to Fig. 5 and recalling the relationship between the normals and the screw parameters of a screw triangle, we conclude that given, for example, $\$_{12}$, $\$_{14}$, $\$_{34}$, $\$_{23}$, the locus of $\$_{24}$ must be such that:

(*a*) The angle which the perpendicular between $\$_{24}$ and $\$_{12}$ makes with the perpendicular between $\$_{24}$ and $\$_{23}$ must equal the angle which the perpendicular between $\$_{24}$ and $\$_{34}$ makes with the perpendicular between $\$_{24}$ and $\$_{14}$ (since in each case the angle is $\theta_{24}/2$).

(*b*) The distance between the points on $\$_{24}$ where it is met by its common perpendiculars with $\$_{12}$ and $\$_{23}$ is equal to the distance between the points on $\$_{24}$ where the normals from $\$_{34}$ and $\$_{14}$ fall (since in each case the distance is $d_{24}/2$).

The distance and angle between two lines may be combined to form the so-called dual angle. (For example, in the foregoing, dual angle $\hat{\theta}_{24}$ would be given by $\hat{\theta}_{24} = \theta_{24} + \epsilon d_{24}$, where $\epsilon^2 = 0$.) We define the dual angle which any two lines subtend at a third line as the dual angle between the normals from the two lines to the third. With this definition (*a*) and (*b*) may be combined into the following: The dual angle subtended by $\$_{12}$ and $\$_{23}$ at $\$_{24}$ equals the dual angle subtended by $\$_{34}$ and $\$_{14}$ at $\$_{24}$.

If we call two adjacent screws a "side" of the quadrilateral, and call two sides which do not have a common screw "opposite sides," then (*a*) and (*b*) require that one set of opposite sides subtend equal dual angles at $\$_{24}$. From the figure it follows that if one pair of opposite sides subtends equal dual angles, the other pair must do likewise. Generalizing the foregoing, and taking into account the possible ± 180 deg variation in the angle, we have: *The position of all screw axes must be such that they form*

[5] An "opposite screw quadrilateral" is a spatial polygon bounded by four screw axes and the four normals between adjacent axes. (Axes are adjacent if they have one subscript in common.)

equal (or supplementary[7]) dual angles with opposite sides of an opposite screw quadrilateral.[8]

In the foregoing we have been working with the screw axes in Σ'. If we invert the motion so that, say, Σ_1 is fixed, it becomes obvious the positions of the screw axes in the moving plane are also governed by the above. For example, the locus of $\$_{34}'$ is such that it must "see" opposite sides of the opposite screw quadrilateral $\$_{12}$, $\$_{23}'$, $\$_{14}'$, $\$_{13}$ at equal (or supplementary) dual angles.

Screw Congruences

Since, in the aforementioned example, $\$_{34}$ must satisfy two conditions, the freedom in its location is reduced from four to two, and the locus of $\$_{34}$ is a two-dimensional assemblage of lines called a congruence. Similarly, $\$_{12}$ will have to fulfill conditions analogous to (*a*) and (*b*), and, by symmetry, so will all the other screws.

Equations governing these geometric loci are derived in Appendix 1 and also in the section on the cylindric-cylindric crank. These results may be summarized as follows:

(i) The six screw axes associated with four positions of a body are parallel to six generators of a cubic cone. The cone is completely defined by the directions of any four "opposite" screws, and the remaining two screws may be parallel to any two elements of this cone. We refer to this cone as a "screw cone."

(ii) Each generator of the screw cone defines a unique direction, and *to each such direction there corresponds a singly infinite set of possible screw axis positions.* (Such a one-dimensional assemblage of lines is known as a line series.) The members of a given line series are all subtended by the same (or supplementary) angles at the opposite sides of an opposite screw quadrilateral. However, the distance subtended (and hence the dual angle) is different for each member of the series.

All the members of a single line series are coplanar and parallel.

The first result, (i), is completely independent of the translation along the screws and the location of the screws. Therefore, in questions dealing exclusively with inclination—as opposed to location—one may consider all the screws as intersecting, pure rotation screws. This leads immediately to the conclusion that the screw cone for spherical motion is identical to the cone for corresponding general spatial motion. Further, if the apex is taken at infinity the cone degenerates into a (circular) cubic cylinder, and any normal section yields a planar pole curve.

The screw cone associated with the motion defined in Table 1 is shown in Fig. 6. We note that the cone is completely defined by the planar cubic which is its curve of intersection with any plane not passing through the origin. The curve cut by the plane $Z = 1$ is shown in the figure. Alternatively, one could—as is customary in spherical-motion studies—intersect the cone with a unit-sphere; however, this seems much less convenient.

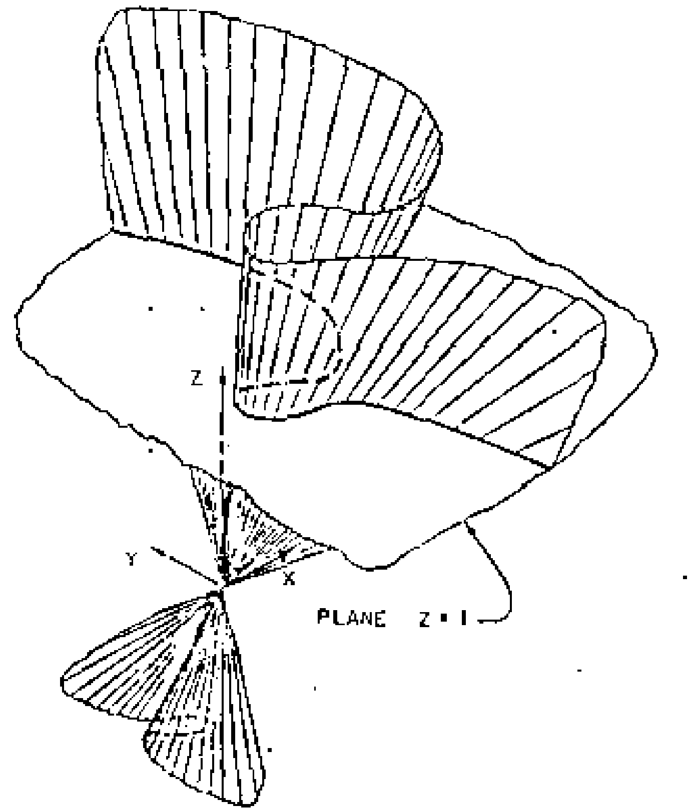


Fig. 6 Cubic cone, and planar cubic falling in plane $Z = 1$. This cone contains the directions of the "moving" screw axes and the moving cylindric-cylindric crank axes associated with the first four positions of the motion given in Table 1. The X, Y, Z-coordinates correspond, respectively, to the cosines L_1, M_1, N_1.

Using the notion that the screw axes may be considered as intersecting, pure rotation screws, one can rephrase condition (*a*):

(*a'*) The dihedral angle between the plane $\$_{12} \times \$_{23}$ and the plane $\$_{23} \times \$_{34}$ is equal (or supplementary) to the dihedral angle between planes $\$_{14} \times \$_{13}$ and $\$_{14} \times \$_{34}$.

Conditions (*a*) and (*a'*) are, of course, equally valid for spatial, spherical, or planar motion, while (*b*) is trivial if there is no translation.

By inversion, one immediately obtains analogous results for the loci of the screw axes fixed in the moving body.

[7] Two dual angles are supplementary if their primary parts differ by 180 deg.

[8] The rules governing the sense of these angles are exactly the same as in the planar case [13]. The sense of a dual angle is defined as the sense of the right-handed screw required to bring the line *from which* the angle is being measured into coincidence with the line *to which* the angle is being measured.

Table 1 The screws for five arbitrarily defined positions are listed. Corresponding to the first four, we give the equation of the screw cone taken in the moving body in Position 1, and the plane containing all the moving axes parallel to one of the generators of this screw cone. Fig. 6 shows this same screw cone.

Position	Axis cosines			Axis location			Screw parameters	
i to j	l_{ij}	m_{ij}	n_{ij}	a_{ij}	b_{ij}	c_{ij}	θ_{ij} deg	d_{ij}
1 to 2	0.3510	0.0865	0.9280	0.9414	0.5214	−0.4194	133.2	1.384
1 to 3	0.1457	−0.0990	0.9886	−0.6103	0.8471	0.1230	70.6	1.809
1 to 4	0.4267	−0.2164	0.8702	0.7327	0.6889	−0.1613	87.9	−1.337
1 to 5	0.4027	−0.0251	−0.9150	5.963	0.1010	2.505	25.2	2.559

Equation of *Screw Cone* (corresponding to Positions 1, 2, 3, and 4)

$$-0.3461L_1^3 - 0.0519M_1^3 + 0.0213N_1^3 + 0.0518L_1^2M_1 + 0.5107L_1^2N_1 - 0.3805L_1M_1^2 + 0.3340M_1^2N_1 - 0.2168L_1N_1^2 - 0.0175M_1N_1^2 + 0.0071L_1M_1N_1 = 0$$

All axes parallel to the generator $L_1 = -0.1489$, $M_1 = -0.0502$, $N_1 = 0.9876$ lie in the plane:

$$0.0287A_1 - 0.0880B_1 - 0.0080C_1 + 0.0848 = 0$$

The Cylindric-Cylindric Crank

Introduction

It is well known that, in the planar case, the pole curve is also the locus of the fixed centers of all "four-point" circles. In this context, the pole curve is called the center-point curve. Similarly, the pole curve fixed in the moving body is also the locus of points whose four positions fall on a circle, i.e., the so-called circle-point curve. *These dualisms are special cases of more general ones associated with spatial motion.*

In space, the entity analogous to a circle point is a line in the moving body whose four positions lie at the same distance and same angle from a line in the fixed body. This fixed line is analogous to the corresponding center point. Each such set of fixed and moving lines define the axes of a crank with two cylindrical joints. Fig. 7 shows a cylindric-cylindric crank in positions associated with four finitely separated positions of Σ.

In the context of a general theory, the revolute-revolute crank defined by a center and circle point should, of course, be viewed as a special case of the cylindric-cylindric crank. By adopting this viewpoint we may state the following general result:

The loci of screw axes defined by the congruence of (i) *and* (ii) *are identical to the loci of the fixed axes of all four-position cylindric-cylindric cranks.* Correspondingly, the loci of the moving axes for all four-position cylindric-cylindric cranks coincide with the loci of screw axes in the moving system.

If, and only if, the screws are intersecting pure rotation screws,[9] there will be no translation in the cylindrical joints, and the cylindric-cylindric cranks become, in effect, revolute-revolute cranks.

Nomenclature

The following quantities are introduced to describe a cylindric-cylindric crank (see Fig. 8).

$\mathbf{L}_j(L_j, M_j, N_j)$, a unit vector parallel to the jth position of the moving axis; $\mathbf{A}_j(A_j, B_j, C_j)$, the normal vector to the moving axis; and a vector (R_j, S_j, T_j) defined as $\mathbf{A}_j \times \mathbf{L}_j$. Similarly, for the fixed axis we have the unit vector along the axis (λ, μ, ν); the normal vector from the origin (α, β, γ); and their cross-product (ρ, σ, τ). All of these quantities are measured in the fixed coordinate system.

Using Φ_j and D_j to denote, respectively, the angle and distance between the moving and fixed axes, we write expressions for the cosine of the angle

$$\cos \Phi_j = L_j\lambda + M_j\mu + N_j\nu \tag{1}$$

and the moment

$$D_j \sin \Phi_j = L_j\rho + M_j\sigma + N_j\tau + \lambda R_j + \mu S_j + \nu T_j \tag{2}$$

between the axes.

Correspondence (Three Positions)

We now show that for three positions there is a (1,1) correspondence between the moving and fixed axes of a cylindric-cylindric crank.

The "twist" in the crank does not vary with the motion, and hence in positions l, m, and n[10] it is necessary that

$$\cos \Phi_l = \cos \Phi_m = \cos \Phi_n$$

Substituting from (1) we find that

$$\begin{aligned} \lambda(L_m - L_l) + \mu(M_m - M_l) + \nu(N_m - N_l) &= 0 \\ \lambda(L_n - L_l) + \mu(M_n - M_l) + \nu(N_n - N_l) &= 0 \end{aligned} \tag{3}$$

Actually, (3) requires $\mathbf{L}_l$, $\mathbf{L}_m$, $\mathbf{L}_n$ to be the generators and (λ, μ, ν) the axis of a right circular cone.

If the moving axis is arbitrarily chosen, the L's, M's, and N's are all known. Equation (3) may be solved for, say, λ/ν and μ/ν, which together with $\lambda^2 + \mu^2 + \nu^2 = 1$ yield a unique result for the fixed-axis cosines.

If (λ, μ, ν) are specified, and we substitute the linear transformation

$$L_m - L_l = a_{l1m}L_l + b_{l1m}M_l + c_{l1m}N_l,$$

$$M_m - M_l = a_{l2m}L_l + b_{l2m}M_l + c_{l2m}N_l,$$

and so on,[11]

Equation (3) becomes a linear homogeneous set in L_l, M_l, N_l which yields, say, the ratios L_l/N_l, M_l/N_l. Finally, calling upon $L_l^2 + M_l^2 + N_l^2 = 1$ one determines a unique set of moving-axis direction cosines corresponding to the given fixed axis.

It may easily be shown that the screw axes are singular lines in

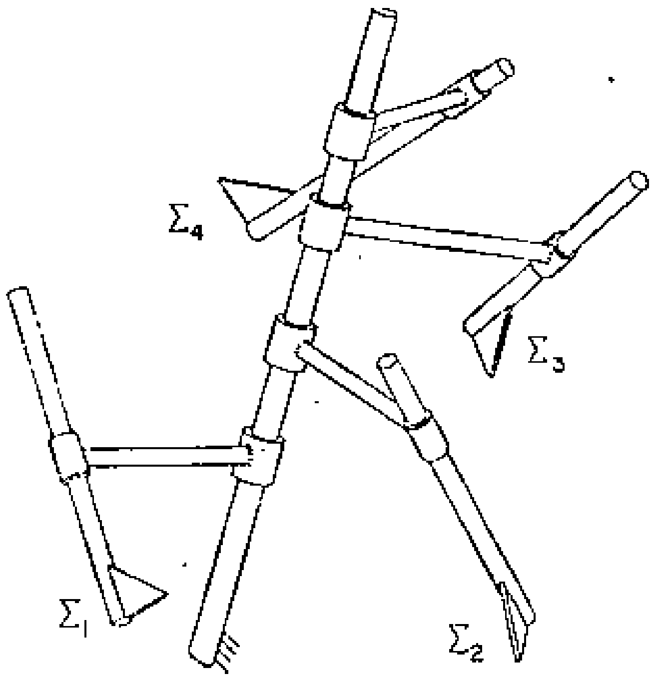


Fig. 7 Cylindric-cylindric crank for the four positions Σ1, Σ2, Σ3, Σ4

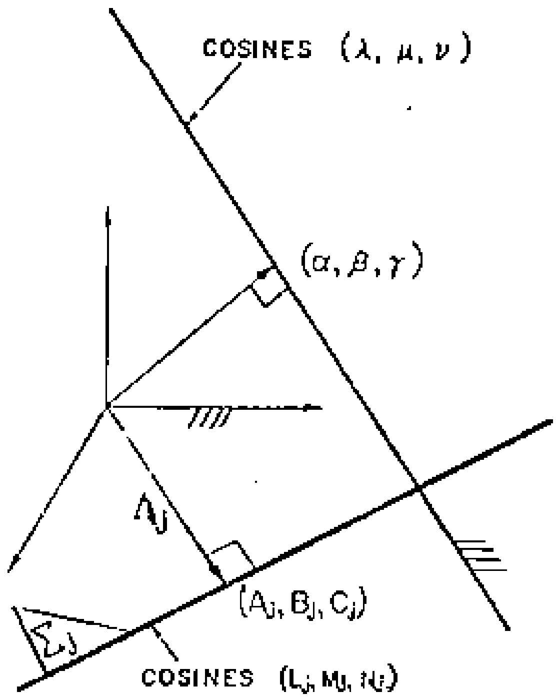


Fig. 8 Nomenclature used in the cylindric-cylindric crank derivations

[9] This motion is either planar or spherical depending upon whether the screw axes intersect in an infinite or finite point.

[10] Usually, $l = 1$, $m = 2$, and $n = 3$.

[11] The a's, b's, and c's are function of the known screw-axis cosines and the angle of rotation. See Appendix 2 for explicit expressions.

regard to the quadratic correspondence defined by equation (3).

For any given crank the distance and "twist" between the axes are fixed; hence we require that $D_l \sin \hat{\theta}_l = D_m \sin \hat{\theta}_m = D_n \sin \hat{\theta}_n$ which, when we substitute (2) yields

$$\rho(L_m - L_l) + \sigma(M_m - M_l) + \tau(N_m - N_l)$$
$$= \lambda(R_l - R_m) + \mu(S_l - S_m) + \nu(T_l - T_m) \qquad (4)$$
$$\rho(L_n - L_l) + \sigma(M_n - M_l) + \tau(N_n - N_l)$$
$$= \lambda(R_l - R_n) + \mu(S_l - S_n) + \nu(T_l - T_n)$$

Since, from the above, the direction cosines of both axes are known, (4) yields two linear, nonhomogeneous equations in either (ρ, σ, τ) or (R_l, S_l, T_l). When the moving axis is specified, the R's, S's and T's are known and (ρ, σ, τ) is restricted to the line of intersection of the two planes (4). Finally, using

$$\rho = \beta\nu - \gamma\mu, \; \sigma = \gamma\lambda - \alpha\nu, \; \tau = \alpha\mu - \beta\lambda,$$

and adding the restriction $\alpha\lambda + \beta\mu + \gamma\nu = 0$, one obtains a unique normal vector (α, β, γ).

If, instead, we specify the fixed axis, it is required that we first substitute

$$R_j = B_j N_j - C_j M_j$$
$$S_j = C_j L_j - A_j N_j$$
$$T_j = A_j M_j - B_j L_j \; (j = l, m, n) \text{ into (4),}$$

and then apply the linear transformation of Appendix 2.[12]

Substituting (18) explicitly converts the right-hand sides of (4) into linear functions of (A_l, B_l, C_l), and these two equations determine a line. Finally, by requiring that $A_l L_l + B_l M_l + C_l N_l = 0$ a unique normal vector is obtained, and the moving axis corresponding to the given fixed axis is uniquely determined.

In the above we used the specified motion throughout. It is of course, also possible to determine this correspondence from (3) and (4) by use of inversion.

We have shown that for three positions of the moving body we are at liberty to choose any line in Σ' as the fixed axis, or any line in Σ as the moving axis. However, *once either axis is specified the other is uniquely defined by the above (1,1) correspondence.*

It can be shown that this same correspondence is given by two applications of the following theorem:

Each pair of fixed and moving axes subtends a screw axis at a dual angle equal to one-half, or the supplement of one-half the dual angle associated with that screw. For example, with the moving axis in position 1, the dual angle measured along $\$_{12}$ is $\hat{\theta}_{12}/2$ ($\pm$ 180 deg). The one along $\$_{13}$ is $\hat{\theta}_{13}/2$ ($\pm$ 180 deg), and the one along $\$_{23}{}'$ equals $\hat{\theta}_{23}/2$($\pm$180 deg). The sense of the dual angle is from the moving to the fixed axis.

Finally, once one set of fixed and moving axes are known, the following theorem could be used to establish the same correspondence: ***Corresponding sets of two moving and two fixed axes appear from either $\$_{lm}$ or $\$_{ln}$, as well as from $\$'_{mn}$, to subtend two equal or two supplementary dual angles.***[13] (In this we have taken the moving axis in position l.)

Four Positions

Axis Direction

For four finitely separated positions, l, m, n, p[14], we add a third equation to (3) and obtain:

$$\lambda(L_m - L_l) + \mu(M_m - M_l) + \nu(N_m - N_l) = 0$$
$$\lambda(L_n - L_l) + \mu(M_n - M_l) + \nu(N_n - N_l) = 0 \qquad (5)$$
$$\lambda(L_p - L_l) + \mu(M_p - M_l) + \nu(N_p - N_l) = 0$$

Regarding (λ, μ, ν) as the unknowns, the compatibility condition for (5) becomes:

$$\begin{vmatrix} (L_m - L_l) & (M_m - M_l) & (N_m - N_l) \\ (L_n - L_l) & (M_n - M_l) & (N_n - N_l) \\ (L_p - L_l) & (M_p - M_l) & (N_p - N_l) \end{vmatrix} = 0 \qquad (6)$$

By remembering that $L_j - L_l = a_{lj}L_l + b_{lj}M_l + c_{lj}N_l$ and so on, it is possible to write (6) as a homogeneous polynomial in (L_l, M_l, N_l), hence (6) represents a cone of third order. In fact, ***this cone is the screw cone* studied earlier.**

The generators of this cone define the directions in which there exist lines in Σ_l which maintain the same angle with a fixed line for all four positions. The directions of the corresponding fixed **lines may be obtained from the (1,1) correspondence described above.**

If we change from general to spherical motion, equation (6) remains unaltered. In fact, (6) is identical to the condition obtained, by the author [2], for the locus of moving lines whose **points all fall on circles for four-position spherical motion.**

It is computationally convenient to reduce the cone to a planar cubic. Dividing (6) by, say, $(N_l)^3$ yields a third-order planar curve in the variables (L_l/N_l), (M_l/N_l). Fig. 6 shows the cone and planar curve associated with the four positions given in Table 1.

Axis Location

If, in addition to the constant angle, a moving line remains at a constant distance from some fixed line, the moving and fixed lines **may be used as four-position cylindric-cylindric crank axes.**

For a direction corresponding to any generator (L_l, M_l, N_l) the axes locations are obtained as follows:

For four positions, the equation

$$\rho(L_p - L_l) + \sigma(M_p - M_l) + \tau(N_p - N_l)$$
$$= \lambda(R_l - R_p) + \mu(S_l - S_p) + \nu(T_l - T_p)$$

added to (4) yields a set of three equations. In addition, the **condition that (ρ, σ, τ) is normal to (λ, μ, ν)** yields

$$\rho\lambda + \sigma\mu + \tau\nu = 0.$$

Regarding (ρ, σ, τ) as the unknowns in these four equations, the compatibility conditions require that[15]:

$$\begin{vmatrix} \lambda & \mu & \nu & 0 \\ (L_m - L_l) & (M_m - M_l) & (N_m - N_l) & \lambda(R_l - R_m) + \mu(S_l - S_m) + \nu(T_l - T_m) \\ (L_n - L_l) & (M_n - M_l) & (N_n - N_l) & \lambda(R_l - R_n) + \mu(S_l - S_n) + \nu(T_l - T_n) \\ (L_p - L_l) & (M_p - M_l) & (N_p - N_l) & \lambda(R_l - R_p) + \mu(S_l - S_p) + \nu(T_l - T_p) \end{vmatrix} = 0 \qquad (7)$$

After substituting (18), (7) is converted to a linear equation in (A_l, B_l, C_l). This, coupled with the second linear condition,

$$A_l L_l + B_l M_l + C_l N_l = 0 \qquad (8)$$

yields a line as the locus of the tip of the normal vector (A_l, B_l, C_l) to the moving axis. The moving axis may be chosen through

---

[12] Similarly, instead of (2) and (4) we could have used $D_l = D_m = D_n$ where D_j = [illegible] Here [illegible] are all known. **This could then be carried forward to the four and more-position derivations to yield equations analogous to (7) and (7').** Such a formulation has the advantage of using the position vectors explicitly.

[13] These angles are measured in the same sense (just as in the equivalent planar theorem). **This theorem is also valid if one takes the moving [illegible] lines instead of the moving axes. However, in** this latter case the sense is reversed.

[14] **Usually one takes $l = 1$, $m = 2$, $n = 3$, $p = 4$.**

[15] **Since our choice of (L_l, M_l, N_l), in effect, determines (λ, μ, ν), we regard all the cosines as known.**

any point on this line. Hence, *corresponding to each direction defined by an element of the cubic cone, there is a singly infinite array of moving axes.* All the axes of a given array are (*a*) parallel to a single element of the cone, and (*b*) coplanar.

Once the moving axis is selected, the fixed axis is uniquely determined (by using any three positions) from the (1,1) correspondence.

Five Positions

We extend the four-position analysis to include a fifth position.

Axis Direction

We now write (5) as a set of four equations by adding

$$\lambda(L_r - L_j) + \mu(M_r - M_j) + \nu(N_r - N_j) = 0.$$

$$(r \neq p \neq m \neq n \neq l)^{[8]}$$

Using the same argument as before, the compatibility condition now requires that any two (3 × 3) determinants vanish. It is convenient to use (6) and the cubic cone obtained by changing one of the subscripts in (6), say p, to r. Since the two cubic cones have the same apex, there are at most nine real directions (L_1, M_1, N_1) for which the compatibility conditions may be satisfied. However, three of these directions are spurious since they correspond to the screw axes $\$_{jm}$, $\$_{jn}$, $\$^1_{mn}$,[9] and *there are generally at most six moving-axis directions for any set of five spatial displacements.*

Again, these nine directions are identical to the ones previously determined [2] for the case of spherical motion.

By using any three positions in the (1,1) correspondence a corresponding fixed-axis direction may be determined for each of the six directions.

Axis Location

The lines corresponding to these six directions are located as follows:

We add a fourth equation to (4),

$$\rho(L_r - L_j) + \sigma(M_r - M_j) + \tau(N_r - N_j)$$
$$= \lambda(R_j - R_r) + \mu(S_j - S_r) + \nu(T_j - T_r)$$

and proceed as before. The compatibility condition now requires that two (4 × 4) determinants vanish. It is convenient to use (7) and a similar determinant (7') which is the same as (7) except for the subscript in the last row, which is changed from p to r. Substituting (18) into (7') yields a linear equation in (A_1, B_1, C_1). The simultaneous solution of (7), (7'), and (8) yields a unique point (A_1, B_1, C_1) for each axis direction.

Therefore, *to each of the at most six nontrivial directions common to the two cubic cones, there corresponds a unique moving axis.* Again, by using any three positions the corresponding fixed axis may be determined.

It follows that, associated with any five positions, there are either six, four, two, or zero cylindric-cylindric cranks. If the motion is planar, two of these cranks become imaginary, while the remaining four (at most) become the revolute-revolute axes corresponding to the well-known Burmester points.

The solutions to a five-position problem are listed in Table 2.

[8] Usually $r = 5$.

[9] For the directions of these three screw axes the compatibility conditions are no longer sufficient, since the rank of the system becomes one instead of two.

Inversion

In the foregoing we first computed the moving axis and then determined the fixed one from the (three-position) correspondence. In an analogous manner we could have treated the fixed axis first, and then determined the moving axis from the correspondence. Alternatively, if we inverted the motion, the fixed axes could have been computed from exactly those equations developed for the moving axis.

Parametric Developments

The two basic sets of parameters associated with the cylindric-cylindric cranks are:

1 ***The twist* (Φ) *and the length* (*D*) *of the crank.*** In order to determine which, and how many, lines of the congruence correspond to cranks with a given twist and/or length, it is necessary to derive loci analogous to the planar R_m and R^1 curves. (It is anticipated that these questions will be discussed in a subsequent publication.)

2 *The displacements of the crank along the fixed or moving axis.* The fixed-axis congruence may be developed as a function of the displacement along the fixed axis by a procedure based on the following theorem: ***The dual angle subtended at the fixed cylindric-cylindric crank axis by screws $\$_{ij}$ and $\$_{ik}$ is equal to one-half, or one-half the supplement, of the crank displacement from position j to k.***

Accordingly, any desired crank rotation is substituted for θ_{jk} and θ_{1k} in (14) and (14'), respectively. The unit vectors, $\hat{\$}_1$, common to these two equations determine the required fixed-axis directions. The loci given by (14) and (14') are completely analogous to the families of intersecting circles used to parametrically construct the planar pole curves.

By using any one of these directions and any desired crank translation for d_{jk} and d_{1m} in (10), one obtains two linear equations which determine a fixed axis for which the corresponding crank displacement from position j to k is as specified.

By inversion, the above may be used to parametrically develop the moving-axis congruence. The parameters now correspond to displacements of the crank relative to the moving body, Σ.

Table 2 For the five positions specified in Table 1, the cubic cones have seven (real) common generators, therefore, there are four cylindric-cylindric cranks. For each of these cranks we list the axes for the fixed and moving cylindrical joints and also the corresponding crank dimensions. The sign on the twist, Φ, is defined by the right-handed screw rule, with the screw pointing from the moving toward the fixed axis.

Solution Number	Fixed Axis cosines λ	μ	ν	Fixed Axis location α	β	γ
1	0.4058	−0.2511	0.8788	0.5034	2.547	0.4538
2	0.1451	−0.0401	0.9886	2.181	−0.4675	−0.3392
3	0.0894	0.3493	0.9357	3.150	−1.210	0.1393
4	−0.4672	0.0977	0.8787	1.176	−5.920	1.283

Moving Axis cosines L_1	M_1	N_1	Moving Axis locations A_1	B_1	C_1	Crank twist Φ (deg)	Crank length D
0.1489	−0.0502	0.9876	2.016	1.022	−0.2214	−49.8	0.1525
0.3658	0.0022	0.9261	−0.4225	−1.610	0.3270	15.2	0.3548
0.4718	0.7008	0.5230	−0.0259	−2.380	3.768	39.3	1.0265
0.5040	−0.1612	0.7302	−6.500	−2.478	2.065	−68.8	0.7600

Conclusion

The strong analogy between these results and the classical planar theory makes it seem appropriate to regard the planar theory as but one special case of a more general spatial-motion theory.

The basic geometry associated with spatial motion is spatial. However, since the inclinations of lines and axes are independent of the screw locations and translations, questions of inclination may be resolved without considering the location. Hence, *as far as inclination is concerned, spherical motion and spatial motion are identical*, and the fundamental geometric entities are made up of intersecting lines. For planar motion, the point of intersection is at infinity, and the lines become parallel. Thus, *questions of inclination in spatial and spherical motion are analogous to questions of location for the planar case.*

For spatial motion, in order to determine location, it is necessary to "dualize" the inclination by taking into account the translation and screw location.

The equations derived for the spatial equivalents of the Burmester points, pole curves, center-point and circle-point curves have been programmed in FORTRAN IV. Copies of this program may be obtained by writing to the author.

In addition to being of theoretical interest, these results should be useful in the actual design of spatial linkages. The synthesis techniques described elsewhere [2] are directly applicable to mechanisms with cylindric-cylindric links.

Acknowledgments

The author wishes to express his appreciation to Mr. Daniel Silva for assisting with the derivations and figures, and to the National Science Foundation whose grants NSF-GP 2005 and NSF-GK 650 made this work possible.

References

1 B. Roth, "The Kinematics of Motion Through Finitely Separated Positions," submitted to the *Journal of Applied Mechanics*, Trans. ASME, Series E.

2 B. Roth, "Finite Position Theory Applied to Mechanism Synthesis," submitted to the *Journal of Applied Mechanics*, Trans. ASME, Series E.

3 G. N. Sandor, "On the Existence of a Cycloidal Burmester Theory in Planar Kinematics," *Journal of Applied Mechanics*, vol. 31, Trans. ASME, Series E, vol. 86, 1964, pp. 694–699.

4 B. Roth and F. Freudenstein, "Synthesis of Path-Generating Mechanisms by Numerical Methods," Journal of Engineering for Industry, Trans. ASME, Series B, vol. 85, 1963, pp. 298–306.

5 G. Hein, "Analytische und Zeichnerische Unterlagen für zwei und drei Lagen eines Sphärisch bewegten Getriebegliedes als Hilfsmittel zur Synthese sphärischer Kurbelgetriebe," *Forsch. Ing. Wes.*, vol. 25, 1959, pp. 92–96, 115–126.

6 G. Dittrich, "Über die momentane Bewegungsgeometrie eines sphärisch bewegten starren Systems," doctoral dissertation, Technical University, Aachen, Germany, 1964.

7 V. V. Dobrovolskii, "Burmester's Points in Spherical Motion" (in Russian), *Prikl. Mat. Mekh.*, vol. 9, 1945, pp. 489–491.

8 M. L. Keler, "Analyse und Synthese der Raumkurbelgetriebe mittels Raumliniengeometrie und dualer Grössen," *Forsch. Ing. Wes.*, vol. 25, 1959, pp. 27–32, 55–63.

9 A. Schoenflies, *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, 1886, 194 pp.

10 J. T. Wilson, III, "Analytical Kinematic Synthesis by Finite Displacements," Journal of Engineering for Industry, Trans. ASME, Series B, vol. 87, 1965, pp. 161–169.

11 J. S. Ames and F. D. Murnaghan, *Theoretical Mechanics*, Ginn and Co., 1929. (Dover Publications, 1958.)

12 A. T. Yang, "Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms," doctoral dissertation, Columbia University, New York, N. Y., 1963, 241 pp.

13 R. S. Hartenberg and J. Denavit, *Kinematic Synthesis of Linkages*, McGraw-Hill Book Company, New York, N. Y., 1964, 435 pp.

14 E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, 1937. (Dover Publications, 1944.)

15 E. Study, *Geometrie der Dynamen*, Leipzig, 1903, 603 pp.

APPENDIX 1

The Screw Congruence

In this section we derive analytical expressions for the line complexes which are spatial analogs of the planar pole curves. As we showed earlier, the basic requirements are that all six screw axes, associated with four finitely separated positions, are subtended at equal, or supplementary, dual angles by opposite sides of opposite-screw quadrilaterals.

Let $\hat{S}_j$, $\hat{S}_k$, $\hat{S}_l$, $\hat{S}_m$ represent unit line vectors along any four screws which form an opposite-screw quadrilateral, and let $\hat{S}_1$ represent a unit line vector along one of the remaining screws.

Choosing any position vector (A_j, B_j, C_j) from the origin of coordinates to line $\hat{S}_j$ and the unit vector (l_j, m_j, n_j) parallel to $\hat{S}_j$, we define (r_j, s_j, t_j) as the cross product $(A_j, B_j, C_j) \times (l_j, m_j, n_j)$. The unit line vector $\hat{S}_j$ can now be expressed as the dual vector $\hat{S}_j = l_j\mathbf{i} + m_j\mathbf{j} + n_j\mathbf{k} + \epsilon(r_j\mathbf{i} + s_j\mathbf{j} + t_j\mathbf{k})$, where $\epsilon^2 \equiv 0$. We express $\hat{S}_1$ similarly.

The normal from line $\hat{S}_1$ to $\hat{S}_j$ is one of the quantities given by the product $\hat{S}_1\hat{S}_j$. Denoting the unit line vector along this normal by $\hat{I}_j$, we obtain:

$$\hat{S}_1\hat{S}_j = -\cos\hat{\alpha}_{1j} + \hat{I}_j \sin\hat{\alpha}_{1j}.$$

Here $\hat{\alpha}_{1j}$ is the dual angle between $\hat{S}_1$ and $\hat{S}_j$, and as has been shown by Yang [12],

$$\hat{I}_j = \frac{a_j\mathbf{i} + b_j\mathbf{j} + c_j\mathbf{k}}{(a_j^2 + b_j^2 + c_j^2)^{1/2}} + \epsilon\left[\frac{a_{0j}\mathbf{i} + b_{0j}\mathbf{j} + c_{0j}\mathbf{k}}{(a_j^2 + b_j^2 + c_j^2)^{1/2}} + d_j d_{0j}\frac{(a_j\mathbf{i} + b_j\mathbf{j} + c_j\mathbf{k})}{(a_j^2 + b_j^2 + c_j^2)^{3/2}}\right]$$

where

$$\begin{aligned}
d_j &= -(l_1l_j + m_1m_j + n_1n_j)\\
d_{0j} &= -(l_1r_j + l_jr_1 + m_1s_j + m_js_1 + n_1t_j + n_jt_1)\\
a_j &= m_1n_j - n_1m_j\\
a_{0j} &= n_js_1 + m_1t_j - n_1s_j - m_jt_1\\
b_j &= n_1l_j - l_1n_j\\
b_{0j} &= l_jt_1 + n_1r_j - l_1t_j - n_jr_1\\
c_j &= l_1m_j - m_1l_j\\
c_{0j} &= l_1s_j + m_jr_1 - m_1r_j - l_js_1
\end{aligned}$$

By altering the subscripts one obtains similar expressions for $\hat{I}_k$, $\hat{I}_l$, and $\hat{I}_m$, the unit line vectors along the normals from $\hat{S}_1$ to, respectively, $\hat{S}_k$, $\hat{S}_l$, and $\hat{S}_m$.

Now if we take $\hat{S}_j$ and $\hat{S}_k$ as one "side" of the opposite screw quadrilateral, and $\hat{S}_l$ and $\hat{S}_m$ as the other, the required condition is that the dual angle, $\hat{\theta}_{jk}/2$, between $\hat{I}_j$ and $\hat{I}_k$ be equal, or supplementary, to the dual angle, $\hat{\theta}_{lm}/2$, between $\hat{I}_l$ and $\hat{I}_m$. The product $\hat{I}_j\hat{I}_k$ yields

$$\hat{I}_j\hat{I}_k = -\cos\left(\frac{\hat{\theta}_{jk}}{2}\right) + \hat{S}_1 \sin\left(\frac{\hat{\theta}_{jk}}{2}\right)$$

from which it follows [12] that

$$\cos\left(\frac{\theta_{jk}}{2}\right) = \frac{a_ja_k + b_jb_k + c_jc_k}{(a_j^2 + b_j^2 + c_j^2)^{1/2}(a_k^2 + b_k^2 + c_k^2)^{1/2}} \qquad (9)$$

and

$$\frac{-d_{0jk}}{2}\sin\left(\frac{\theta_{jk}}{2}\right) = \Big\{(a_ja_{0k} + b_jb_{0k} + c_jc_{0k}) + (a_ka_{0j} + b_kb_{0j} + c_kc_{0j}) + \left[\frac{d_kd_{0k}}{(a_k^2 + b_k^2 + c_k^2)} + \frac{d_jd_{0j}}{(a_j^2 + b_j^2 + c_j^2)}\right]$$

$$\times (a_j a_k + b_j b_k + c_j c_k)\Big\}\Big/ [(a_j^2 + b_j^2 + c_j^2)^{1/2} \times (a_k^2 + b_k^2 + c_k^2)^{1/2}] \quad (10)$$

$\theta_{jk}/2$ and $d_{jk}/2$, the angle and distance between l_j and l_k, are the principal and dual parts of $\hat{\theta}_{jk}/2$, i.e.,

$$\frac{\hat{\theta}_{jk}}{2} = \frac{\theta_{jk}}{2} + \frac{\epsilon d_{jk}}{2}$$

The equal angle condition requires that

$$\cos\left(\frac{\theta_{jk}}{2}\right) = \cos\left(\frac{\theta_{lm}}{2}\right).$$

Substituting from (9), and squaring in order to remove the radicals, one obtains:

$$(a_j a_k + b_j b_k + c_j c_k)^2 (a_l^2 + b_l^2 + c_l^2)(a_m^2 + b_m^2 + c_m^2) -$$
$$(a_l a_m + b_l b_m + c_l c_m)^2 (a_j^2 + b_j^2 + c_j^2)(a_k^2 + b_k^2 + c_k^2) = 0 \quad (11)$$

This is a homogeneous algebraic polynomial of degree eight in the unknown directions (l_1, m_1, n_1). When expanded, the common factor $(l_1^2 + m_1^2 + n_1^2)$ may be removed, and (11) reduced to a sextic. This sextic is made up of two cubic cones: One contains the directions (l_1, m_1, n_1) for which $\theta_{jk}/2 = \theta_{lm}/2$ ($\pm$180 deg), and the other contains the directions for which $\theta_{jk}/2 = -\theta_{lm}/2$ ($\pm$180 deg). We are interested in only the one for which $\theta_{jk}/2 = \theta_{lm}/2$ ($\pm$180 deg); we refer to this cone as the screw cone.

We also require that

$$\frac{d_{jk}}{2}\sin\left(\frac{\theta_{jk}}{2}\right) = \pm\frac{d_{lm}}{2}\sin\left(\frac{\theta_{lm}}{2}\right), \quad (12)$$

where the sign is chosen plus or minus according to whether the angles are equal or supplementary.

Substituting from (10) and considering (l_1, m_1, n_1) as known, one obtains a linear equation in the a's, b's, c's, and d's, which are linear functions of (r_1, s_1, t_1). Finally, since the position vector (A_j, B_j, C_j) is a linear function of (r_1, s_1, t_1), we have the result that *parallel to each generator of the screw cone there is an infinite set of coplanar lines. This doubly infinite array of lines is the locus of the six screws.*

The foregoing derivation shows analytically that the directions of the screws are independent of their positions. Using this fact we now obtain the screw-cone equation free of extraneous factors:

Taking the unit vectors $\hat{S}_j, \hat{S}_k, \hat{S}_l, \hat{S}_m, \hat{S}_1$ as intersecting lines, the dihedral angle, $\theta_{jk}/2$, between the planes containing $(\hat{S}_j, \hat{S}_1)$ and $(\hat{S}_k, \hat{S}_1)$ is given by:

$$\frac{(\hat{S}_j \times \hat{S}_1) \times (\hat{S}_k \times \hat{S}_1)}{(\hat{S}_j \times \hat{S}_1)\cdot(\hat{S}_k \times \hat{S}_1)} = \hat{S}_1 \tan\left(\frac{\theta_{jk}}{2}\right) \quad (13)$$

Expanding and simplifying yields

$$\frac{(\hat{S}_j \times \hat{S}_k)\cdot\hat{S}_1}{(\hat{S}_j\cdot\hat{S}_k) - (\hat{S}_j\cdot\hat{S}_1)(\hat{S}_k\cdot\hat{S}_1)} = \tan\left(\frac{\theta_{jk}}{2}\right) \quad (14)$$

The dihedral angle, $\theta_{lm}/2$, between planes $(\hat{S}_l, \hat{S}_1)$ and $(\hat{S}_m, \hat{S}_1)$ is given by equation (14'), which is obtained from (14) by replacing j and k with l and m.

From the section on screw congruences, we apply condition a', and require that $\tan(\theta_{jk}/2) = \tan(\theta_{lm}/2)$. This yields

$$(\hat{S}_j \times \hat{S}_k)\cdot\hat{S}_1[(\hat{S}_l\cdot\hat{S}_m) - (\hat{S}_l\cdot\hat{S}_1)(\hat{S}_m\cdot\hat{S}_1)] - (\hat{S}_l \times \hat{S}_m)\cdot\hat{S}_1[(\hat{S}_j\cdot\hat{S}_k)$$
$$- (\hat{S}_j\cdot\hat{S}_1)(\hat{S}_k\cdot\hat{S}_1)] = 0 \quad (15)$$

which is the equation of the screw cone.

It is possible, but not necessary, to also phrase the moment condition in terms of the tangent of the half angle [by dividing (10) by (9)].

This same derivation yields the planar pole curve, if in (13) one substitutes the vector differences $(\hat{S}_j - \hat{S}_1)$ and $(\hat{S}_k - \hat{S}_1)$ for the corresponding cross products, and replaces the unit vectors by the vectors from the origin to the poles.

APPENDIX 2

Linear Transformations

The following well-known [14] linear transformation gives the coordinates of a point (A_m, B_m, C_m) in terms of its coordinates in the lth position and the screw $\hat{S}_{lm}$. The point is fixed in the moving system but all coordinates are measured along axes in the fixed system.

$$\begin{bmatrix} A_m \\ B_m \\ C_m \end{bmatrix} = \begin{bmatrix} (a_{1lm}+1) & b_{1lm} & c_{1lm} \\ a_{2lm} & (b_{2lm}+1) & c_{2lm} \\ a_{3lm} & b_{3lm} & (c_{3lm}+1) \end{bmatrix} \begin{bmatrix} A_l \\ B_l \\ C_l \end{bmatrix} + \begin{bmatrix} d_{1lm} \\ d_{2lm} \\ d_{3lm} \end{bmatrix} \quad (16)$$

where $a_{1lm} = (u_{1lm}^2 - 1)(1 - \cos\theta_{lm})$

$$b_{1lm} = u_{1lm}u_{2lm}(1 - \cos\theta_{lm}) - u_{3lm}\sin\theta_{lm}$$
$$c_{1lm} = u_{1lm}u_{3lm}(1 - \cos\theta_{lm}) + u_{2lm}\sin\theta_{lm}$$
$$d_{1lm} = d_{lm}u_{1lm} - a_{1lm}A_{lm} - b_{1lm}B_{lm} - c_{1lm}C_{lm}$$
$$a_{2lm} = u_{1lm}u_{2lm}(1 - \cos\theta_{lm}) + u_{3lm}\sin\theta_{lm}$$

and so on.[8]

Similarly, the transformation of a set of direction cosines (L, M, N) is given by

$$\begin{bmatrix} L_m \\ M_m \\ N_m \end{bmatrix} = \begin{bmatrix} (a_{1lm}+1) & b_{1lm} & c_{1lm} \\ a_{2lm} & (b_{2lm}+1) & c_{2lm} \\ a_{3lm} & b_{3lm} & (c_{3lm}+1) \end{bmatrix} \begin{bmatrix} L_l \\ M_l \\ N_l \end{bmatrix} \quad (17)$$

Now for the quantities (R, S, T) defined by the cross product of a position vector (A, B, C) and the unit vector (L, M, N) we use

$$R_m = B_m N_m - C_m M_m$$
$$S_m = C_m L_m - A_m N_m$$
$$T_m = A_m M_m - B_m L_m,$$

similarly for the subscript l. Then substituting (16) we determine that

$$\begin{aligned} \lambda(R_m - R_l) + \mu(S_m - S_l) + \nu(T_m - T_l) \\ = A_l[\lambda(N_m a_{2lm} - M_m a_{3lm}) \\ + \mu(L_m a_{3lm} - N_m a_{1lm} + N_l - N_m) \\ + \nu(M_m a_{1lm} - L_m a_{2lm} + M_m - M_l)] \\ + B_l[\mu(L_m b_{3lm} - N_m b_{1lm}) \\ + \lambda(N_m b_{2lm} - M_m b_{3lm} + N_m - N_l) \\ + \nu(M_m b_{1lm} - L_m b_{2lm} + L_l - L_m)] \\ + C_l[\lambda(N_m c_{2lm} - M_m c_{3lm} + M_l - M_m) \\ + \mu(L_m c_{3lm} - N_m c_{1lm} + L_m - L_l) \\ + \nu(M_m c_{1lm} - L_m c_{2lm})] \\ + \lambda(N_m d_{2lm} - M_m d_{3lm}) + \mu(L_m d_{3lm} - N_m d_{1lm}) \\ + \nu(M_m d_{1lm} - L_m d_{2lm}) \end{aligned} \quad (18)$$

In our application of (18) (L_l, M_l, N_l) are known and (L_m, M_m, N_m) are determined from (17).

[8] The a's, d's, θ's, etc., are defined according to Fig. 1.

Printed in U. S. A.

Paper No.
68—i'' '1-7

A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis

P. CHEN
Assistant Agricultural Engineer,
University of California,
Davis, Calif.

B. ROTH
Associate Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif. Mem. ASME

A rigid body is studied in a series of different positions. These positions can be finitely separated, infinitesimally separated, or a combination of the two. A general method for determining the locations of points or lines (in the rigid body) which have their different multiple positions satisfying the constraints of binary links or combined link chains is developed. In a companion paper [10][1] equations governing the locations of these special points and lines are derived.

Introduction

In dimensional synthesis of spatial mechanisms the central problem is to determine the dimensions of a selected type of linkage which constrains the relative motion of a moving body σ and a fixed body Σ in a specified manner.[2] The "motion" of σ is specified by a given series of positions of σ relative to Σ. These several positions are either finitely separated, infinitesimally separated, or mixed finitely and infinitesimally separated.

Finitely separated position problems have previously been extensively studied. (See, for example, Schoenflies [1], Roth [2–4], Suh [5], Sandor [14], and Wilson [6].) However, comparatively little research has been done on infinitesimally separated position problems [1, 7, 8, 14]. The primary objective of this paper is to recast the existing finite position work and combine it with new results to form a general theory which unifies the analytical study of finite, infinitesimal, and mixed displacement problems.

In this paper a general theory is developed for determining the number and locus of the points or lines in σ which have their several positions satisfying the constraints of binary links or combined link chains. In a companion paper [10], we use these general results to obtain explicit formulations for chains of practical interest. It has been previously shown [3] that such results may be applied to a variety of different synthesis problems.

Linear Transformations

Central to what follows are the relationships between the coordinates of points or lines before and after a displacement. Such relationships are known as transformations. For rigid-body displacements these transformations are linear. Hence, the coordinates, as measured in the fixed system, of the jth position of a point may be expressed as a linear function of the coordinates of the initial (or any reference) position. In this section, we first introduce nomenclature which analytically describes displacements and then obtain explicit expressions for the linear transformations.

Finite Displacements. We select (x, y, z) as the Cartesian coordinates of a point in a moving system σ and (X, Y, Z) as its coordinates in a fixed system Σ. Let σ_j denote the jth position of σ, and (X_j, Y_j, Z_j) the coordinates in Σ of the jth position of the point (x, y, z) in σ. Knowing the position of σ_j relative to Σ, we can express (X_j, Y_j, Z_j) in terms of (x, y, z):

$$\begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix} = \begin{bmatrix} a_{xj} & b_{xj} & c_{xj} \\ a_{yj} & b_{yj} & c_{yj} \\ a_{zj} & b_{zj} & c_{zj} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_{xj} \\ d_{yj} \\ d_{zj} \end{bmatrix} \qquad (1)$$

where the a's, b's, c's, and d's are functions of the parameters governing the relative position of σ_j and Σ. If we let the original position of σ coincide with Σ, the displacement to σ_j may be described as a screw displacement which is equivalent to a translation d_j along, and a rotation θ_j about an axis which is parallel to the unit vector $v_j(u_j, v_j, w_j)$ and passes through the point (a_j, b_j, c_j) in Σ. In this case we have

$$\begin{aligned} a_{xj} &= (u_j^2 - 1)(1 - \cos\theta_j) + 1 \\ b_{xj} &= u_j v_j(1 - \cos\theta_j) - w_j \sin\theta_j \\ c_{xj} &= u_j w_j(1 - \cos\theta_j) + v_j \sin\theta_j \\ d_{xj} &= d_j u_j - a_j(a_{xj} - 1) - b_j b_{xj} - c_j c_{xj} \\ a_{yj} &= v_j u_j(1 - \cos\theta_j) + w_j \sin\theta_j, \end{aligned} \qquad (2)$$

etc. (See, for example, [2]).

If (l, m, n) are the components of a vector fixed to σ_j and (L_j, M_j, N_j) are the components of the same vector when measured in Σ, (L_j, M_j, N_j) can be expressed as linear functions of (l, m, n):

$$\begin{bmatrix} L_j \\ M_j \\ N_j \end{bmatrix} = \begin{bmatrix} a_{xj} & b_{xj} & c_{xj} \\ a_{yj} & b_{yj} & c_{yj} \\ a_{zj} & b_{zj} & c_{zj} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \qquad (3)$$

The screw displacement leads to one of the most convenient forms of the linear transformation. However it is not the only possibility. For example, one could describe the displacement in regard to two axes as has been done by Hartenberg and Denavit [13], or by the displacement of some arbitrary reference point and the rotation about it, or in any number of other ways.

Infinitesimal Displacements. As in the case of finite displacement, infinitesimal motion of a rigid body can be described in many different ways. For example, it can be described by a series of successive infinitesimal screw displacements or by the displacement of a point in the body and the rotation of the body. In any case we can take one parameter as the reference parameter of the motion, designated by ϕ, and express all other parameters as functions of ϕ. Let σ_1 be the position of σ at $\phi = \phi_1$, and $d\phi$ an infinitesimal change in ϕ. The second infinitesimally separated position of σ (infinitesimally separated from σ_1) can be defined as the position of σ at $\phi = \phi_2 = (\phi_1 + d\phi)$, and the third infinitesimally separated position as the position of σ at $\phi = \phi_3 = (\phi_2 + d\phi)$, and so on. In general, we can specify n infinitesimally separated positions of σ in terms of the first $(n - 1)$ derivatives (with respect to ϕ) of the displacement parameters of σ.

We consider the motion of the moving body as a series of consecutive infinitesimal screw displacements. If we take the rotation θ about the screw as the independent parameter of the motion and express the other screw parameters (i.e., the direc-

[1] Numbers in brackets designate References at end of paper.

[2] The moving and fixed bodies are represented by two Cartesian-coordinate systems σ and Σ, respectively.

Contributed by the Design Engineering Division for presentation at the Mechanisms Conference, Atlanta, Ga., October 6–9, 1968, of The American Society of Mechanical Engineers. Manuscript received by the ASME Mechanisms Division, June 18, 1968. Paper No. 68—Mech-7.

Discussion on this paper will be accepted at ASME Headquarters until November 12, 1968

tion of the screw axis u, the translation d along the axis, and the position vector s of the axis) as function of θ, the position vector $\mathbf{X}(X, Y, Z)$ (relative to Σ) of a point (x, y, z) in σ would depend on θ. Let $\mathbf{X}_0$ be the position vector of the point (x, y, z) at $\theta = 0$, $\mathbf{X}_n$ be the position vector of the same point at $\theta = n\Delta\theta$, where $\Delta\theta$ represents a small change in θ. Denoting as $\mathbf{X}_0^{(n)}$ the nth derivative of $\mathbf{X}_0$ with respect to θ, we can express the derivatives of $\mathbf{X}_0$ with respect to θ by taking the limit as $\Delta\theta \rightarrow 0$ of the successive forward differences as follows:

$$
\begin{aligned}
\mathbf{X}_0^{(1)} &= \lim_{\Delta\theta\rightarrow 0} \frac{\mathbf{X}_1 - \mathbf{X}_0}{\Delta\theta}\\
\mathbf{X}_0^{(2)} &= \lim_{\Delta\theta\rightarrow 0} \frac{\mathbf{X}_2 - 2\mathbf{X}_1 + \mathbf{X}_0}{(\Delta\theta)^2}\\
\mathbf{X}_0^{(3)} &= \lim_{\Delta\theta\rightarrow 0} \frac{\mathbf{X}_3 - 3\mathbf{X}_2 + 3\mathbf{X}_1 - \mathbf{X}_0}{(\Delta\theta)^3}\\
\mathbf{X}_0^{(4)} &= \lim_{\Delta\theta\rightarrow 0} \frac{\mathbf{X}_4 - 4\mathbf{X}_3 + 6\mathbf{X}_2 - 4\mathbf{X}_1 + \mathbf{X}_0}{(\Delta\theta)^4}\\
\mathbf{X}_0^{(5)} &= \lim_{\Delta\theta\rightarrow 0} \frac{\mathbf{X}_5 - 5\mathbf{X}_4 + 10\mathbf{X}_3 - 10\mathbf{X}_2 + 5\mathbf{X}_1 - \mathbf{X}_0}{(\Delta\theta)^5}
\end{aligned}
\tag{4}
$$

and so forth.

If we know $\mathbf{X}_i$ $(i = 0, 1, 2, \ldots, p)$ we can calculate $\mathbf{X}_0^{(i)}$ from the foregoing equations. The problem now is to find the $\mathbf{X}_i$. We first introduce two column matrices

$$
\underline{X}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

where (X_i, Y_i, Z_i) and (x, y, z) are, respectively, the components of $\mathbf{X}_i$ and $\mathbf{x}$, and the screw transformation matrix

$$
\underline{A}_j = \begin{bmatrix} a_{xj} & b_{xj} & c_{xj} & d_{xj} \\ a_{yj} & b_{yj} & c_{yj} & d_{yj} \\ a_{zj} & b_{zj} & c_{zj} & d_{zj} \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

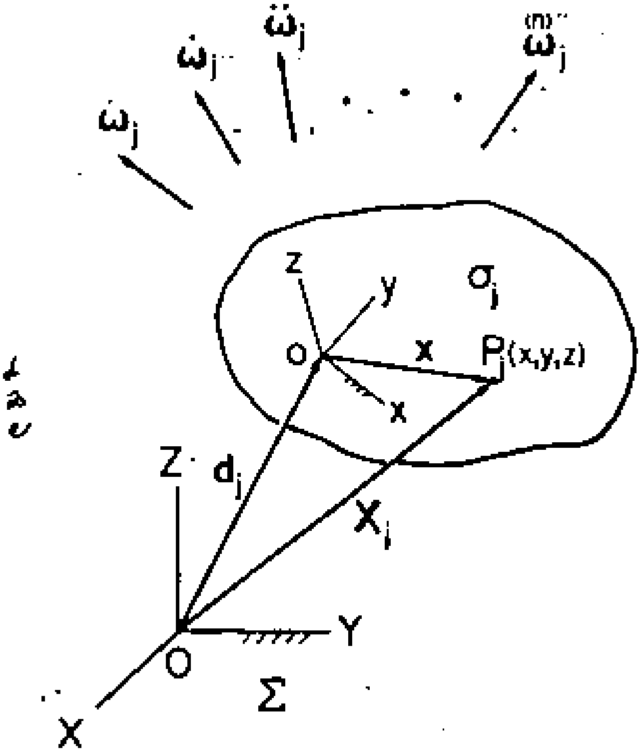


Fig. 1

where the elements in $\underline{A}_j$ are the same as those in (2), but here

$$
\begin{aligned}
u_1 &= u\\
u_2 &= u + u^{(1)}d\theta\\
u_3 &= u + 2u^{(1)}d\theta + u^{(2)}d\theta^2\\
u_4 &= u + 3u^{(1)}d\theta + 3u^{(2)}d\theta^2 + u^{(3)}d\theta^3\\
u_5 &= u + 4u^{(1)}d\theta + 6u^{(2)}d\theta^2 + 4u^{(3)}d\theta^3 + u^{(4)}d\theta^4 \qquad \text{etc.}
\end{aligned}
$$

and similarly for v_j, w_j, a_j, b_j, and c_j. Also

$$
\begin{aligned}
d_1 &= d^{(1)}d\theta\\
d_2 &= d^{(1)}d\theta + d^{(2)}d\theta^2\\
d_3 &= d^{(1)}d\theta + 2d^{(2)}d\theta^2 + d^{(3)}d\theta^3\\
d_4 &= d^{(1)}d\theta + 3d^{(2)}d\theta^2 + 3d^{(3)}d\theta^3 + d^{(4)}d\theta^4\\
d_5 &= d^{(1)}d\theta + 4d^{(2)}d\theta^2 + 6d^{(3)}d\theta^3 + 4d^{(4)}d\theta^4 + d^{(5)}d\theta^5
\end{aligned}
$$

and so forth, where $[\ \ ]^{(n)} \equiv \dfrac{d^n[\ \ ]}{d\theta^n}$.

Therefore, from the screw displacement (1) and (2), we have

$$
\begin{aligned}
\underline{X}_0 &= \underline{x}\\
\underline{X}_1 &= \underline{A}_1\underline{x}\\
\underline{X}_2 &= \underline{A}_2\underline{X}_1 = \underline{A}_2\underline{A}_1\underline{x}\\
&\ \,\vdots\\
\underline{X}_n &= \underline{A}_n\underline{X}_{n-1} = \underline{A}_n\underline{A}_{n-1}\ldots\underline{A}_2\underline{A}_1\underline{x} = \underline{A}^{(n)}\underline{x}
\end{aligned}
\tag{5}
$$

where $\underline{A}^{(n)} = \underline{A}_n\underline{A}_{n-1}\ldots\underline{A}_1$.

Hence, from the equations (4) and (5), we can express the $(X_0^{(i)}, Y_0^{(i)}, Z_0^{(i)})$ as linear functions of (x, y, z). These expressions are shown in the Appendix.

Instead of a series of screw motions, the motion of a body can be described in terms of the motion of any one point in the body and the rotation of the body. For example, with time as the independent parameter, if we know the time-derivatives of the displacement vector of a point in the body and also of the rotating vector of the body, we can find the time-derivatives of the position vector of any point in the body.

Considering the motion of σ_j, we let $\mathbf{d}_j$ be the position vector of the origin of σ_j, $\boldsymbol{\omega}_j$ be the angular velocity of σ_j, x be the vector from the origin of σ_j to a point P_j in σ_j, $\mathbf{x}_j$ be the same vector measured in Σ, and $\mathbf{X}_j$ be the position vector of P_j relative to the origin of Σ, as shown in Fig. 1. Using dots to denote differentiation with respect to time, we can express the time-derivatives of $\mathbf{X}_j$ as follows:[1]

$$
\begin{aligned}
\dot{\mathbf{X}}_j &= \dot{\mathbf{d}}_j + \dot{\mathbf{x}}_j = \dot{\mathbf{d}}_j + \boldsymbol{\omega}_j \times \mathbf{x}_j\\
\ddot{\mathbf{X}}_j &= \ddot{\mathbf{d}}_j + \ddot{\mathbf{x}}_j = \ddot{\mathbf{d}}_j + \dot{\boldsymbol{\omega}}_j \times \mathbf{x}_j + \boldsymbol{\omega}_j \times \dot{\mathbf{x}}_j\\
\dddot{\mathbf{X}}_j &= \dddot{\mathbf{d}}_j + \dddot{\mathbf{x}}_j = \dddot{\mathbf{d}}_j + \ddot{\boldsymbol{\omega}}_j \times \mathbf{x}_j + 2\dot{\boldsymbol{\omega}}_j \times \dot{\mathbf{x}}_j + \boldsymbol{\omega}_j \times \ddot{\mathbf{x}}_j \qquad \text{(6a)}\\
\overset{(4)}{\mathbf{X}}_j &= \overset{(4)}{\mathbf{d}}_j + \overset{(4)}{\mathbf{x}}_j = \overset{(4)}{\mathbf{d}}_j + \dddot{\boldsymbol{\omega}}_j \times \mathbf{x}_j + 3\ddot{\boldsymbol{\omega}}_j \times \dot{\mathbf{x}}_j + 3\dot{\boldsymbol{\omega}}_j \times \ddot{\mathbf{x}}_j + \boldsymbol{\omega}_j \times \dddot{\mathbf{x}}_j\\
\overset{(5)}{\mathbf{X}}_j &= \overset{(5)}{\mathbf{d}}_j + \overset{(5)}{\mathbf{x}}_j = \overset{(5)}{\mathbf{d}}_j + \overset{(4)}{\boldsymbol{\omega}}_j \times \mathbf{x}_j + 4\dddot{\boldsymbol{\omega}}_j \times \dot{\mathbf{x}}_j + 6\ddot{\boldsymbol{\omega}}_j \times \ddot{\mathbf{x}}_j + 4\dot{\boldsymbol{\omega}}_j \times \dddot{\mathbf{x}}_j + \boldsymbol{\omega}_j \times \overset{(4)}{\mathbf{x}}_j \qquad \text{etc.}
\end{aligned}
$$

[1] $\overset{(i)}{\mathbf{X}}_j$ denotes the ith time-derivative of $\mathbf{X}_j$, similarly for any quantity $\overset{(i)}{[\ \ ]}$ we use $\overset{(i)}{[\ \ ]} \equiv \dfrac{d^i[\ \ ]}{dt^i}$.

Since $\overset{(i)}{x}_j$ is linear in x_j, $\overset{(i)}{\mathbf{X}}_j$ is also linear in x_j. Therefore we can express $\overset{(i)}{\mathbf{X}}_j$ as given in (6a) as follows:

$$\begin{bmatrix} \overset{(i)}{X}_j \\ \overset{(i)}{Y}_j \\ \overset{(i)}{Z}_j \end{bmatrix} = \overset{(i)}{A}_j \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} + \begin{bmatrix} \overset{(i)}{d}_{x_j} \\ \overset{(i)}{d}_{y_j} \\ \overset{(i)}{d}_{z_j} \end{bmatrix}$$

where $\overset{(i)}{A}_j$ is a 3 × 3 matrix whose elements are functions of the components of the time-derivatives of the rotation of σ_j.

But

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A_j is the 3 × 3 matrix in (3). Therefore

$$\begin{bmatrix} \overset{(i)}{X}_j \\ \overset{(i)}{Y}_j \\ \overset{(i)}{Z}_j \end{bmatrix} = \overset{(i)}{A}_j A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \overset{(i)}{d}_{x_j} \\ \overset{(i)}{d}_{y_j} \\ \overset{(i)}{d}_{z_j} \end{bmatrix} \qquad (6b)$$

Equation (6b) explicitly shows $\overset{(i)}{\mathbf{X}}_j$ is a linear function of $\mathbf{x}$.

Conditions for Several Positions of a Point in σ to Satisfy the Constraint of a Given Link

Finitely Separated Positions. When the motion of a point P in σ is constrained by a link, the constraint which the link imposes on P can be represented by one or two constraint equations of the form

$$F(X, Y, Z; \quad a_1, a_2, \ldots, a_n) = 0 \qquad (7)$$

where (X, Y, Z) are the coordinates in Σ of P, and $a_1, a_2, \ldots, a_n$ are the independent parameters of the constraint. These constraint parameters define the dimensions of the link.

When σ assumes m finitely separated positions $\sigma_1, \sigma_2, \ldots, \sigma_m$, a point P in σ which has its m positions satisfying the constraint described by (7) must satisfy

$$F(X_j, Y_j, Z_j; \quad a_1, a_2, \ldots, a_n) = 0 \qquad j = 1, 2, \ldots, m \quad (8)$$

where (X_j, Y_j, Z_j) are the coordinates in Σ of the jth position of P.

Substituting the linear expressions for X_j, Y_j, and Z_j from (1) into (8), we can write (8) in terms of x, y, and z.

Infinitesimally Separated Positions. At the initial position ($\phi = 0$) we let σ coincide with Σ. A point P in σ whose initial position satisfies the constraint of a given link, equation (7), must satisfy

$$F(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) = 0 \qquad (9)$$

where (X_0, Y_0, Z_0) are the coordinates (in Σ) of the initial position of P. After an infinitesimal displacement of σ, the coordinates of P become $(X_0 + dX_0, Y_0 + dY_0, Z_0 + dZ_0)$. The new position of P will satisfy the constraint $F = 0$ only if

$$F(X_0 + dX_0, \quad Y_0 + dY_0, \quad Z_0 + dZ_0; \quad a_1, a_2, \ldots, a_n) = 0 \qquad (10)$$

which is

$$F(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) + \frac{d}{d\phi}\left[F(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n)\right] d\phi = 0 \qquad (11)$$

If $d\phi \neq 0$, the substitution of (9) yields

$$F'(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) = 0 \qquad (12)$$

where the prime denotes differentiation with respect to ϕ. Similarly, after a second infinitesimal displacement of σ, the new position of P which again satisfies the constraint $F = 0$ must satisfy

$$F(X_0 + dX_0, \quad Y_0 + dY_0, \quad Z_0 + dZ_0; \quad a_1, a_2, \ldots, a_n) + F'(X_0 + dX_0, \quad Y_0 + dY_0, \quad Z_0 + dZ_0; \quad a_1, a_2, \ldots, a_n)d\phi = 0 \qquad (13)$$

which is

$$F(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) + F'(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n)d\phi + F'(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n)d\phi + F''(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n)(d\phi)^2 = 0 \qquad (14)$$

If $d\phi \neq 0$, the substitution of (9) and (12) yields

$$F''(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) = 0 \qquad (15)$$

By induction, we can show that the necessary conditions for all k infinitesimally separated positions of P to satisfy the constraint $F = 0$ are given by

$$F^{(i)}(X_0, Y_0, Z_0; \quad a_1, a_2, \ldots, a_n) = 0 \qquad i = 0, 1, 2, \ldots, k-1 \quad (16)$$

where $F^{(i)}$ denotes the ith derivative of F with respect to the independent motion parameter ϕ.

After the differentiation is carried out, each $F^{(i)}$ becomes a new function of $X_0^{(j)}$, $Y_0^{(j)}$, and $Z_0^{(j)}$, where j, which denotes the order of the derivative, can be any integer less than or equal to i. Hence (16) can be written in the form

$$F_i(X_0^{(j)}, Y_0^{(j)}, Z_0^{(j)}; \quad a_1, a_2, \ldots, a_n) = 0 \qquad i = 0, 1, 2, \ldots, k-1 \quad (17)$$

Mixed Finite and Infinitesimal Displacements. In the foregoing we have discussed finite and infinitesimal displacement problems separately. We shall now investigate the possibility of combining the two.

We have shown earlier that if (X_j, Y_j, Z_j) are the coordinates in Σ of P_j (the jth position of a point P in σ), the condition for the point P to have m finitely separated positions satisfying the link constraint (7) is that it must satisfy (8). Furthermore, we have shown that if (X_0, Y_0, Z_0) are the coordinates in Σ of the initial position of P, the condition for P to have its initial and next $k - 1$ infinitesimally separated positions satisfying the same constraint is that it must satisfy (16). Now if we consider P_i as the initial position, the condition for the next k infinitesimally separated positions of P to satisfy (7) is

$$F^{(l)}(X_i, Y_i, Z_i; \quad a_1, a_2, \ldots, a_n) = 0 \qquad l = 1, 2, \ldots, k \quad (18)$$

If the ith position is one of the m finitely separated positions considered in (8), the combination of (8) and (18) gives the necessary and sufficient condition for a point P in σ to have $m + k$ positions (m finitely separated and k infinitesimally separated from the ith finitely separated position) satisfying the constraint (7).

From this we can draw a general rule for combined finite and infinitesimal displacements as follows:

If σ assumes $m + k_1 + k_2 + \ldots + k_m$ positions such that m positions are finitely separated and each group of k_j positions are infinitesimally separated from the jth finitely separated position, then the condition for a point P in σ to have all these $m + \sum_{j=1}^{m} k$ positions satisfying (7) is that it must satisfy

$$F^{(l_j)}(X_j, Y_j, Z_j; \quad a_1, a_2, \ldots, a_n) = 0; \qquad l_j = 0, 1, 2, \ldots, k_j; \quad j = 1, 2, \ldots, m \quad (19)$$

Maximum Number of Design Positions. The number of design positions, designated by p, is the number of positions of σ which one specifies for the design of a given link. The maximum number of design positions depends on the number of equations N_e and the number of the total unknowns N_u. In order to be compatible, N_e must be less than or equal to N_u. For a given link

$$N_e = p \times N_a \tag{20}$$

where N_a is the number of constraint equations associated with the given link. The number of unknowns is

$$N_u = n + 3, \tag{21}$$

where n is the total number of parameters in the constraint equation(s), and the number 3 is equal to the number of coordinates (x, y, z) of the point being constrainted. From the compatibility condition we require that

$$N_e \leq N_u$$

or

$$p \times N_a \leq n + 3 \tag{22}$$

$$p \leq (n + 3)/N_a$$

Therefore, the maximum p is equal to the largest integer contained in $(n + 3)/N_a$.

General Procedure for Determining Locations of Special Points. We have shown that all special points whose different positions satisfy a given link constraint (7) must satisfy (19). In what follows we will discuss the procedure for determining the coordinates (x, y, z) of such special points. After carrying out the differentiation, we can rewrite (19) in the following form:

$$F_{l_j}(X_j^{(l)}, Y_j^{(l)}, Z_j^{(l)};\ a_1, a_2, \ldots, a_n) = 0;$$

$$l_j = 0, 1, 2, \ldots, k_j;\quad j = 1, 2, \ldots, m;\quad i \leq l_j \tag{23}$$

Using (1) and (6), we can express $(X_j^{(l)}, Y_j^{(l)}, Z_j^{(l)})$ linearly in terms of (x, y, z). Hence (23) can be written as

$$F_r(x, y, z;\ a_1, a_2, \ldots, a_n) = 0 \qquad r = 1, 2, \ldots, p \tag{24}$$

where $p = m + \sum_{j=1}^{m} k_j$.

Theoretically, when $N_e \leq N_u$ (here $N_e = p$, $N_u = n + 3$), we can solve (24) for all the unknowns. In the case where $N_e < N_u$ we can arbitrarily specify any $(N_u - N_e)$ unknowns and solve for the rest.

Fortunately, in many cases, the constraint equations are linear in the a's. In such cases we can write (24) in the following form:

$$f_{r1}a_1 + f_{r2}a_2 + \ldots + f_{rn}a_n + f_{r(n+1)} = 0 \qquad r = 1, 2, \ldots, p \tag{25}$$

where f_{rj} are functions of (x, y, z) and the motion parameters. When $p = n$, given any point (x, y, z) in σ, we can solve (25) for at least one set of a's. This indicates that a generic point in σ has n positions satisfying (7). For $p > n$, the necessary and sufficient condition for a point (x, y, z) in σ to satisfy all p-equations in (25) is that the rank of the augmented matrix of the system (regarding the a's as unknowns) must be equal to or less than n. In other words, the p-equations must be compatible. The compatibility condition requires that the determinants of all the $(n + 1)$ by $(n + 1)$ matrices of the system must vanish. This requirement can be met if all

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & f_{1(n+1)} \\ f_{21} & f_{22} & \cdots & f_{2n} & f_{2(n+1)} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & f_{n(n+1)} \\ f_{(n+k)1} & f_{(n+k)2} & \cdots & f_{(n+k)n} & f_{(n+k)(n+1)} \end{vmatrix} = 0 \tag{26}$$

$$k = 1, 2, \ldots, p - n$$

and the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} & f_{1(n+1)} \\ f_{21} & f_{22} & \cdots & f_{2n} & f_{2(n+1)} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & f_{n(n+1)} \end{bmatrix} \tag{27}$$

is n. Points at which the rank of (27) is less than n will be called "residual points."

Since all f_{rj} are functions of (x, y, z), each of the equations in (26) can be expanded into a polynomial in (x, y, z) which describes a surface embedded in σ. This surface which we name the "compatibility surface" will be designated by S_k. It should be noted that all the residual points also satisfy (26). Therefore every S_k contains all the residual points. The surface S_k is the locus of all points whose $(n + k)$th position satisfies the constraint (7) defined by the points' first n positions. However, the residual points distinguish themselves in that their first n positions do not define a unique member of the family $F = 0$. Hence, all points in σ, other than the residual points, that satisfy (26) will have their p positions on the locus given by equation (7). The residual points satisfy

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} & f_{2l} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{n(n-1)} & f_{nl} \end{vmatrix} = 0 \tag{28}$$

$$l = n, n + 1$$

provided that the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} \\ \vdots & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{n(n-1)} \end{bmatrix} \tag{29}$$

is $(n - 1)$.

The equations in (28) represent two surfaces whose intersection contains the locus of all residual points and a "subresidual curve" which is the locus of points which do not satisfy the rank condition (29). If we call the intersection of the two surfaces in (28) C_n, the residual curve R_n, and the subresidual curve R_{n-1}, then symbolically

$$C_n = R_n + R_{n-1}$$

The next step is to determine R_{n-1}. Proceeding in the same way, we find that

$$R_{n-1} = C_{n-1} - R_{n-2}$$

or

$$C_{n-1} = R_{n-1} + R_{n-2}$$

where C_{n-1} is the intersection of the two surfaces

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} \\ \vdots & & & \vdots \\ f_{(n-2)1} & f_{(n-2)2} & \cdots & f_{(n-2)(n-1)} \\ f_{l1} & f_{l2} & \cdots & f_{l(n-1)} \end{vmatrix} = 0 \tag{30}$$

$$l = (n - 1), n$$

and R_{n-2} is the curve at which the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} \\ \vdots & & & \vdots \\ f_{(n-2)1} & f_{(n-2)2} & \cdots & f_{(n-2)(n-1)} \end{bmatrix} \tag{31}$$

is less than $(n - 2)$.

The continuation of this procedure leads to the following result:

$$C_i = R_i + R_{i-1} \qquad i = 2, 3, \ldots, n$$
$$C_1 = R_1$$

where C_i is the intersection of

$$\begin{vmatrix} f_{11} & f_{12} & \cdot & \cdot & \cdot & f_{1l} \\ f_{21} & f_{22} & \cdot & \cdot & \cdot & f_{2l} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{(i-1)1} & f_{(i-1)2} & \cdot & \cdot & \cdot & f_{(i-1)l} \\ f_{l1} & f_{l2} & \cdot & \cdot & \cdot & f_{ll} \end{vmatrix} = 0 \qquad (32)$$
$$l = i, \quad i + 1$$

if $(n - i)$ is odd, or the intersection of

$$\begin{vmatrix} f_{11} & f_{12} & \cdot & \cdot & \cdot & f_{1(i-1)} & f_{1l} \\ f_{21} & f_{22} & \cdot & \cdot & \cdot & f_{2(i-1)} & f_{2l} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f_{l1} & f_{l2} & \cdot & \cdot & \cdot & f_{l(i-1)} & f_{ll} \end{vmatrix} = 0 \qquad (33)$$
$$l = i, \quad i + 1$$

if $(n - i)$ is even. Therefore

$$R_n = \sum_{i=1}^{n} (-1)^{(i+\delta)} C_i$$
$$\delta = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

Knowing the degree of each f_{ij}, we can calculate the order of C_i and thereby determine the order of R_n. In this context the C's and R's may be given numerical values equal to the orders of the curves.

When $p = n + 1$, the compatibility surface S_1 represents the locus of all points in σ which have $n + 1$ positions satisfying $F = 0$. For $p = n + 2$, the intersection of S_1 and S_2 yields a space curve K_{12} whose order equals the product of the orders of S_1 and S_2. It can easily be shown that the order of K_{12} is greater than that of R_n. Since both S_1 and S_2 have R_n in common, K_{12} must be degenerate and contain R_n and another curve k_{12} which is the locus of all points in σ which have their $n + 2$ positions satisfying $F = 0$. For $p = n + 3$, we have three compatibility surfaces S_1, S_2, and S_3. These surfaces intersect one another in the curve R_n and a finite set of points. Since points on R_n do not satisfy the first $n + 3$ equations of (25), only those points belonging to the finite set have $n + 3$ positions satisfying $F = 0$. The number of such points can be calculated by a method given by Semple and Roth [9]. The same method is shown in reference [11, Appendix 3].

In some cases the constraint equations are nonlinear in the unknown constraint parameters (i.e., the a's); no general method for solving such equations exists. Although elimination techniques and numerical methods may be successfully employed, it does not seem feasible to attempt a general discussion.

Special Lines. Although the foregoing is concerned with the determination of the special points in σ, the same analysis can be applied to problems relating to those special lines in σ which have their multiple positions satisfying the constraint of a given link. In the case of lines the constraint equations can be written in the following form:

$$F(X, Y, Z, L, M, N; \; a_1, a_2, \ldots, a_n) = 0, \qquad (34)$$

where (X, Y, Z) are the coordinates in Σ of a point on the line and (L, M, N) are the components of a vector (usually of unit length) parallel to the line. The six quantities X, Y, Z, L, M, and N will be called the "coordinates of the line." Since a line has only four degrees of freedom, clearly only four of the six coordinates are independent. For example, we can arbitrarily specify one of the point coordinates (X, Y, Z) and one of the vector components (L, M, N).

Replacing equation (7) by (34), and following the procedure used in the previous sections, we find that equation (19) becomes

$$F^{(l_j)}(X_j, Y_j, Z_j, L_j, M_j, N_j; \; a_1, a_2, \ldots, a_n) = 0;$$
$$l_j = 0, 1, 2, \ldots, k_j; \quad j = 1, 2, \ldots, m \qquad (35)$$

which, after substituting (1), (3), and (6), can be transformed to

$$F_r(x, y, z, l, m, n; \; a_1, a_2, \ldots, a_n) = 0 \qquad r = 1, 2, \ldots, p \qquad (36)$$

Here (x, y, z, l, m, n) are the coordinates of the line in σ. These coordinates are measured in the σ system which is taken as initially coincident with Σ.

Since the number of independent coordinates of a line is four

$$N_n = n + 4,$$

the maximum number of design positions p is equal to the largest integer contained in $(n + 4)/N_e$.

The procedure for determining lines which satisfy (36) is essentially the same as that for determining points which satisfy (24). However, in the case of lines, the results will be in the form of line loci (i.e., complexes, congruences, reguli, or finite sets of lines which are represented, respectively, by one, two, three, or four equations in x, y, z, l, m, and n).

Conclusion

A general method for solving multiple-position problems in spatial kinematic synthesis has been presented. This method unifies the treatment of finitely and infinitesimally separated positions, and is applicable to the dimensional design of essentially all types of binary links and combined link chains. For each link one must first derive the constraint equation(s). Once the constraint equations are obtained, we can predict the maximum design positions and determine the locations of those special points or lines which define the dimensions of the desired link. The procedure is straightforward, and all computations can be performed on a digital computer. The application of the method to various types of links is discussed in a companion paper [10].

Acknowledgment

The financial support of the National Science Foundation (NSF GK-660) is gratefully acknowledged.

APPENDIX

Expressions for $(X^{(i)}, Y^{(i)}, Z^{(i)})$ in Terms of (x, y, z) and Screw Parameters Governing Motion of σ (see equations (4) and (5))

After performing the algebraic computations in equations (4) and (5), we obtain the following expressions for $(X^{(i)}, Y^{(i)}, Z^{(i)})$:

$$\begin{bmatrix} X^{(1)} \\ Y^{(1)} \\ Z^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} bw - cv + ud^{(1)} \\ cu - aw + vd^{(1)} \\ av - bu + wd^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} X^{(2)} \\ Y^{(2)} \\ Z^{(2)} \end{bmatrix} = \begin{bmatrix} u^2 - 1 & uv - w^{(1)} & uw + v^{(1)} \\ uv + w^{(1)} & v^2 - 1 & vw - u^{(1)} \\ uw - v^{(1)} & vw + u^{(1)} & w^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} D_{x2} \\ D_{y2} \\ D_{z2} \end{bmatrix}$$

where

$$D_{x2} = -a(u^2 - 1) - buv - cuw + [bw - cv + ud^{(1)}]^{(1)}$$
$$D_{y2} = -auv - b(v^2 - 1) - cvw + [cu - aw + vd^{(1)}]^{(1)}$$
$$D_{z2} = -auw - bvw - c(w^2 - 1) + [av - bu + wd^{(1)}]^{(1)}$$

$$\begin{bmatrix} X^{(1)} \\ Y^{(1)} \\ Z^{(1)} \end{bmatrix} = \begin{bmatrix} -vv^{(1)} - ww^{(1)} + [u^2 - 1]^{(1)} & w + vu^{(1)} + [uv - w^{(1)}]^{(1)} & -v + uw^{(1)} + [uw + v^{(1)}]^{(1)} \\ -w + vu^{(1)} + [uv + w^{(1)}]^{(1)} & -uu^{(1)} - ww^{(1)} + [v^2 - 1]^{(1)} & u + vw^{(1)} + [(vw - u^{(1)}]^{(1)} \\ v + wu^{(1)} + [uw - v^{(1)}]^{(1)} & -u + wv^{(1)} + [vw - u^{(1)}]^{(1)} & -uu^{(1)} - vv^{(1)} + [w^2 - 1]^{(1)} \end{bmatrix} + \begin{bmatrix} D_{x1} \\ D_{y1} \\ D_{z1} \end{bmatrix}$$

where

$$D_{x1} = a(vv^{(1)} + ww^{(1)}) - b(w + uv^{(1)}) - c(-v + uw^{(1)}) + (wv^{(1)} - vw^{(1)})d^{(1)} + D_{x1}^{(1)}$$

$$D_{y1} = -a(-w + vu^{(1)}) + b(uu^{(1)} + ww^{(1)}) - c(u + vw^{(1)}) + (uw^{(1)} - wu^{(1)})d^{(1)} + D_{y1}^{(1)}$$

$$D_{z1} = -a(v + wu^{(1)}) - b(-u + wv^{(1)}) + c(uu^{(1)} + vv^{(1)}) + (vu^{(1)} - uv^{(1)})d^{(1)} + D_{z1}^{(1)}$$

The algebraic manipulations were performed on a digital computer using a program called "REDUCE" [12]. Higher derivatives can be obtained in a similar manner.

References

1 Schoenflies, A., *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, 1886.

2 Roth, B., "The Kinematics of Motion Through Finitely Separated Positions," *Journal of Applied Mechanics*, Vol. 34, No. 3, TRANS. ASME, Series E, Vol. 89, Sept. 1967, pp. 591–598.

3 Roth, B., "Finite Position Theory Applied to Mechanism Synthesis," *Journal of Applied Mechanics*, Vol. 34, No. 3, TRANS. ASME, Vol. 89, Sept. 1967, pp. 599–605.

4 Roth, B., "On the Screw Axis and Other Special Lines Associated With Spatial Displacements of a Rigid Body," JOURNAL OF ENGINEERING FOR INDUSTRY, TRANS. ASME, Series B, Vol. 89, No. 1, 1967, pp. 102–110.

5 Suh, C. H., "Synthesis and Analysis of Space Mechanisms With Use of the Displacement Matrix," dissertation, University of California, Berkeley, 1966.

6 Wilson, J. T., III, "Analytical Kinematic Synthesis by Finite Displacements," JOURNAL OF ENGINEERING FOR INDUSTRY, TRANS. ASME, Series B, Vol. 87, No. 2, 1965, pp. 161–169.

7 Disteli, M., "Über das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes," *Zeitschrift für Mathematik und Physik*, Vol. 62, 1914, pp. 261–309.

8 Skreiner, M., "A Study of the Geometry and the Kinematics of Instantaneous Spatial Motion," *Journal of Mechanisms*, Vol. 1, No. 2, 1966, pp. 115–143.

9 Semple, J. G., and Roth, L., *Introduction to Algebraic Geometry*, Oxford University Press, England, 1949.

10 Chen, P., and Roth, B., "Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains." Paper no. 68-mech-8.

11 Chen, P., "An Analytical Investigation of Infinitesimal Spatial Motion Theory and Its Application to Three-Dimensional Linkages," dissertation, Stanford University, 1968.

12 Hearn, A. C., "Reduce User's Manual," Memo No. 50, Stanford Artificial Intelligence Project, Feb. 1967.

13 Hartenberg, R. S., and Denavit, J., *Kinematic Synthesis of Linkages*, McGraw-Hill, New York, 1964.

14 Sandor, G. N., "Principles of a General Quaternion-Operator Method of Spatial Kinematic Synthesis," *Journal of Applied Mechanics*, Vol. 35, No. 1, TRANS. ASME, Vol. 90, Series E, Mar. 1968, pp. 40–46.

Printed in U. S. A.

Paper No.
68—Mech-8

Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains

P. CHEN
Assistant Agricultural Engineer,
University of California,
Davis, Calif.

B. ROTH
Associate Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif. Mem. ASME

A general method for synthesizing spatial linkages (derived in a previous paper [1][1]) is applied to the design of various types of binary links and combined link chains. Equations governing those special points and lines which define the link dimensions are derived. Types and order of loci of such special points and lines and the maximum number of design positions for each link are given. Numerical method for computing these special points and lines are discussed, and a numerical example is presented.

Introduction

In spatial kinematic synthesis problems it is often necessary to determine the loci of those special points and lines which have multiple positions satisfying the constraints of certain links and link chains. In this paper we use a method described elsewhere [1] to determine such loci for various types of links and link chains. (Since the Nomenclature and methods used in this paper are as described in [1], the reader is advised to first familiarize himself with that work.)

The moving and fixed bodies are represented by a moving Cartesian coordinate system σ and a fixed system Σ, respectively. The coordinates of a point in σ are given by (x, y, z) and by (X, Y, Z) when measured in Σ. In what follows we consider the moving body, σ, in p general multiple positions. These p positions consist of $m + k_1 + k_2 + \ldots + k_m$ positions $\left(p = m + \sum_{j=1}^{m} k_j\right)$ such that m positions are finitely separated and each group of k_j positions are infinitesimally separated from the jth finitely separated position. If all the positions of σ are finitely separated, $k_j = 0$ for all j. On the other hand, when all the positions are infinitesimally separated, $m = 1$.

If $P(x, y, z)$ is a point in σ, the coordinates in Σ of P_j (the jth finitely separated position of P) will be denoted by (X_j, Y_j, Z_j), and the ith derivative of the position vector of P_j will be denoted by $(X_j^{(i)}, Y_j^{(i)}, Z_j^{(i)})$.

Application to Link Design

Sphere-Sphere Binary Link. A sphere-sphere binary link, shown in Fig. 1, constrains the moving pivot P in σ to remain on a sphere whose center is at the fixed pivot C. The constraint equation is

$$(X - X_c)^2 + (Y - Y_c)^2 + (Z - Z_c)^2 - R^2 = 0 \qquad (1)$$

where (X, Y, Z) and (X_c, Y_c, Z_c) are, respectively, the coordinates of P and C in Σ; R is the distance from P to C. Equation (1) can be written in the form

$$X^2 + Y^2 + Z^2 + a_1X + a_2Y + a_3Z + a_4 = 0 \qquad (2)$$

where

$$a_1 = -2X_c \qquad a_2 = -2Y_c \qquad a_3 = -2Z_c$$
$$\text{and} \quad a_4 = X_c^2 + Y_c^2 + Z_c^2 - R^2$$

According to [1], the condition for a point $P(x, y, z)$ in σ to have p multiple positions on the sphere (2) is that it must satisfy

$$(X_j^2 + Y_j^2 + Z_j^2)^{(l_j)} + a_1X_j^{(l_j)} + a_2Y_j^{(l_j)} + a_3Z_j^{(l_j)} + a_4^{(l_j)} = 0$$
$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$
$$p = m + \sum_{j=1}^{m} k_j \qquad (3)$$

where (X_j, Y_j, Z_j) are the coordinates in Σ of the jth finitely separated position of P and the superscript (l_j) denotes the l_jth derivative with respect to the reference parameter of the motion.

For $l_j = 0$ we have

$$X_j^2 + Y_j^2 + Z_j^2 + a_1X_j + a_2Y_j + a_3Z_j + a_4 = 0$$
$$j = 1, 2, \ldots, m \qquad (4)$$

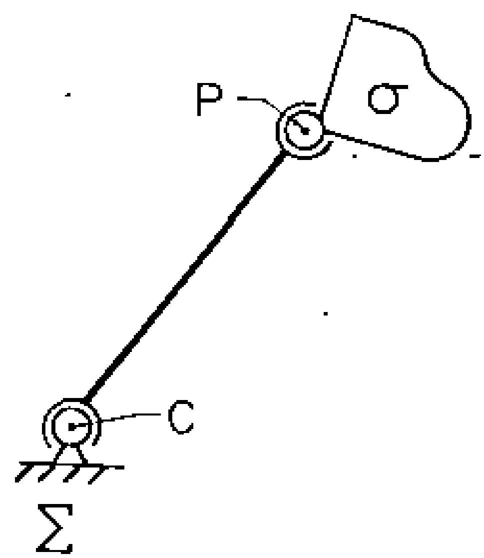


Fig. 1 Sphere-sphere binary link

[1] Numbers in brackets designate References at end of paper.

Contributed by the Design Engineering Division for presentation at the Mechanisms Conference, Atlanta, Ga., October 6-9, 1968, of The American Society of Mechanical Engineers. Manuscript received by the ASME Mechanisms Division, June 18, 1968. Paper No. 68—Mech-8.

Subtracting the first equation ($j = 1$) from the others, we get

$$r_j^2 - r_1^2 + a_1(X_j - X_1) + a_2(Y_j - Y_1) + a_3(Z_j - Z_1) = 0 \qquad j = 2, 3, \ldots, m \qquad (5)$$

where $r_j^2 = X_j^2 + Y_j^2 + Z_j^2$.

For $l_j > 0$ we have

$$(r_j^2)^{(l_j)} + a_1 X_j^{(l_j)} + a_2 Y_j^{(l_j)} + a_3 Z_j^{(l_j)} = 0 \qquad (6)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

It has been shown [1] that X_j, Y_j, Z_j, $X_j^{(l_j)}$, $Y_j^{(l_j)}$, and $Z_j^{(l_j)}$ are linear in x, y, and z (the coordinates of P in σ), and it can be shown (Appendix 1 and 2 of [2]) that $(r_j^2 - r_1^2)$ and $(r_j^2)^{(l_j)}$ are also linear in x, y, and z.

Therefore both (5) and (6) can be written as

$$f_{i1}a_1 + f_{i2}a_2 + f_{i3}a_3 + f_{i4} = 0 \qquad i = 1, 2, \ldots, p - 1 \qquad (7)$$

where f_{ij} are linear functions of x, y, and z. Equations (7) will be referred to as design equations.

For four positions ($p = 4$) we have three equations with the three unknowns a_1, a_2, and a_3. Therefore, given any point in σ, we can solve the three equations for the three unknowns and thereby determine the coordinates (X_c, Y_c, Z_c) of the center of the sphere. The radius of the sphere can be calculated from the first equation of (4).

For p greater than four, the number of equations is greater than the number of unknowns. The compatibility condition requires that the rank of the system be equal to the number of the unknowns, which is three. This requirement can be met if

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{k1} & f_{k2} & f_{k3} & f_{k4} \end{vmatrix}_{k=4,5,\ldots,p-1} = 0, \qquad (8)$$

$$k = 4, 5, \ldots, p - 1$$

and the rank of

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} \qquad (9)$$

is three.

Since f_{ij} are linear in x, y, and z, each equation of (8) represents a fourth-order surface embedded in σ. This surface will be referred to as $E_{123(k+1)}^4$.

The points at which the rank of (9) is less than three have been called [1] the residual points. Following the procedure given in [1], we find that these points lie on a residual curve k^6 which is a sixth-order space curve. As will be shown later, k^6 is the locus of all points which have the first four positions on a circle.

Therefore, according to [1], E_{12345}^4 is the locus of all points which have five multiple positions satisfying the constraint of a sphere-sphere link. The locus of all points which have six positions on a sphere is the tenth-order space curve k^{10} which is the intersection of E_{12345}^4 and E_{12346}^4 excluding k^6. The intersection of E_{12345}^4, E_{12346}^4, and E_{12347}^4 yields k^6 and 20 discrete points which have their seven positions on a sphere. (The number of the discrete points can be calculated from equations given in Appendix 3 of [2].)

The foregoing is an example showing the detailed procedure for analyzing a multiple-position design problem of a sphere-sphere link. For other types of links the procedures are essentially the same as the foregoing. Therefore, for the remaining links and link chains, we will only derive the design equations and present the results in a tabulated form.

Slider-Slider-Sphere Dyad. A slider-slider-sphere dyad, as shown in Fig. 2, constrains the moving pivot P to move on a plane which is parallel to the sliding directions of the two sliders. The constraint equation is given by

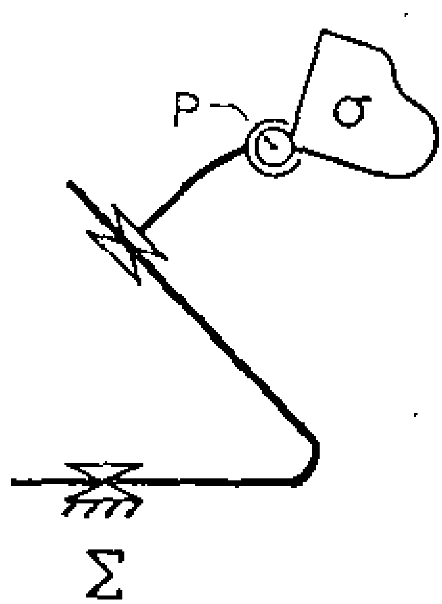


Fig. 2 Slider-slider-sphere dyad

$$(X - X_0)L + (Y - Y_0)M + (Z - Z_0)N = 0 \qquad (10)$$

where (L, M, N) are the direction cosines of the normal to the plane, and (X_0, Y_0, Z_0) the coordinates of a point on the plane. Equation (10) can be written as

$$X + b_1 Y + b_2 Z + b_3 = 0 \qquad (11)$$

where

$$b_1 = \frac{M}{L}, \qquad b_2 = \frac{N}{L}, \quad \text{and} \quad b_3 = -(X_0 + Y_0 b_1 + Z_0 b_2).$$

The design equations for p multiple positions are

$$X_j^{(l_j)} + Y_j^{(l_j)} b_1 + Z_j^{(l_j)} b_2 + b_3^{(l_j)} = 0 \qquad (12)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

which can be reduced to

$$f_{i1} + f_{i2}b_1 + f_{i3}b_2 = 0 \qquad i = 1, 2, \ldots, p - 1 \qquad (13)$$

where f_{ij} are as defined in (7).

Slider-Sphere Binary Link. A slider-sphere binary link constrains point P (Fig. 3) to move on a straight line parallel to the sliding axis. The constraint equations are

$$X - \frac{L}{N} Z - X_0 = 0$$
$$Y - \frac{M}{N} Z - Y_0 = 0 \qquad (14)$$

where (L, M, N) are the direction cosines of the lines and (X_0, Y_0) are the coordinates of the point where the line intersects the XY-plane. The design equations can be written

$$X_j^{(l_j)} + a_1 Z_j^{(l_j)} + a_2^{(l_j)} = 0$$
$$Y_j^{(l_j)} + b_1 Z_j^{(l_j)} + b_2^{(l_j)} = 0 \qquad (15)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

where

$$a_1 = -\frac{L}{N}, \qquad a_2 = -X_0, \qquad b_1 = -\frac{M}{N}, \quad \text{and} \quad b_2 = -Y_0.$$

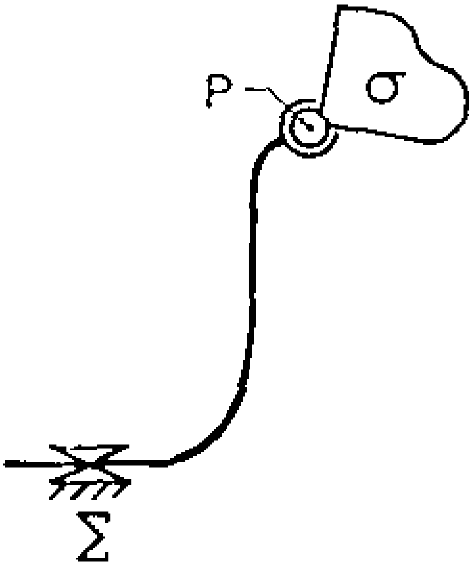


Fig. 3 Slider-sphere binary link

Equations (15) can be reduced to

$$f_{r1} + f_{r2} a_1 = 0$$
$$f_{r1} + f_{r2} b_1 = 0 \qquad (16)$$
$$r = 1, 2, \ldots, p - 1$$

Revolute-Sphere Binary Link. A revolute-sphere binary link has the same constraint on the moving pivot P in σ as does a sphere-sphere-slider-slider four bar (Fig. 4). The constraint locus is a circle which may be determined as the intersection of a sphere

$$X^2 + Y^2 + Z^2 + a_1 X + a_2 Y + a_3 Z + a_4 = 0 \qquad (17)$$

a plane

$$X + b_1 Y + b_2 Z + b_3 = 0 \qquad (18)$$

where the a's and b's are as defined in (2) and (11).

Equations (17) and (18) are identical to (2) and (11), respectively; consequently, the design equations for p multiple positions are

$$f_{r1} a_1 + f_{r2} a_2 + f_{r3} a_3 + f_{r4} = 0$$
$$f_{r1} + f_{r2} b_1 + f_{r3} b_2 = 0 \qquad (19)$$
$$r = 1, 2, \ldots, p - 1$$

Revolute-Slider-Sphere Dyad. A revolute-slider-sphere dyad, shown in Fig. 5, constrains a point P in σ to move on a one-sheet hyperboloid of revolution whose axis is coincident with that of the revolute joint, and whose generator through P is parallel to the sliding direction. The constraint equation for point P is

$$(X - a)^2 + (Y - b)^2 + (Z - c)^2$$
$$- [l(X - a) + m(Y - b) + n(Z - c)]^2 \left[1 + \frac{\alpha^2}{\beta^2}\right]$$
$$- \{k^2 - 2k[l(X - a) + m(Y - b) + n(Z - c)]\} \frac{\alpha^2}{\beta^2} = \alpha^2 \qquad (20)$$

where (X, Y, Z) are the coordinates of P in Σ; (l, m, n) are the direction cosines of the hyperboloid axis; (a, b, c) are the coordinates of a point A on this axis; k is the distance from A to the center of the hyperboloid; α and $\frac{\alpha}{\beta}$ are, respectively, the distance and the tangent of the angle between the axis and a generator.

If we select A such that

$$al + bm + cn = 0, \qquad (21)$$

then (20) can be reduced to

$$aX + bY + cZ - k(lX + mY + nZ) \frac{\alpha^2}{\beta^2}$$
$$- \frac{1}{2}(X^2 + Y^2 + Z^2) + \frac{1}{2}(lX + mY + nZ)^2 \left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- \frac{1}{2}\left[a^2 + b^2 + c^2 - \left(k\frac{\alpha}{\beta}\right)^2\right] = -\frac{\alpha^2}{2} \qquad (22)$$

Hence, the design equations for p multiple positions are

$$aX_j^{(l_j)} + bY_j^{(l_j)} + cZ_j^{(l_j)}$$
$$- k(lX_j + mY_j + nZ_j)^{(l_j)} \frac{\alpha^2}{\beta^2} - \frac{1}{2}(X_j^2 + Y_j^2 + Z_j^2)^{(l_j)}$$
$$+ \frac{1}{2}[(lX_j + mY_j + nZ_j)^2]^{(l_j)} \left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- \frac{1}{2}\left[a^2 + b^2 + c^2 - \left(k\frac{\alpha}{\beta}\right)^2\right]^{(l_j)} = -\left(\frac{\alpha^2}{2}\right)^{(l_j)}$$
$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j \qquad (23)$$

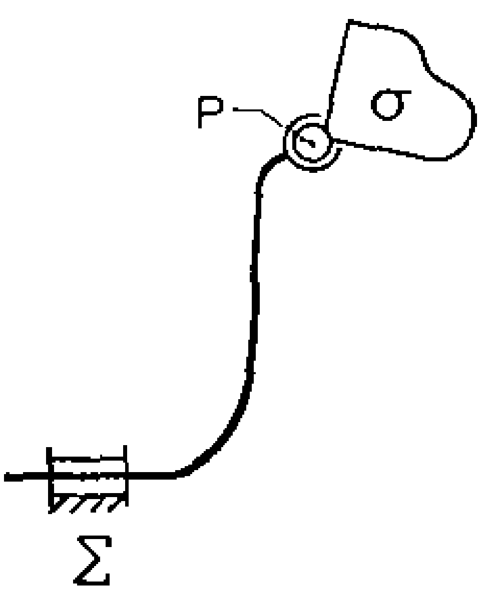


Fig. 4(a)

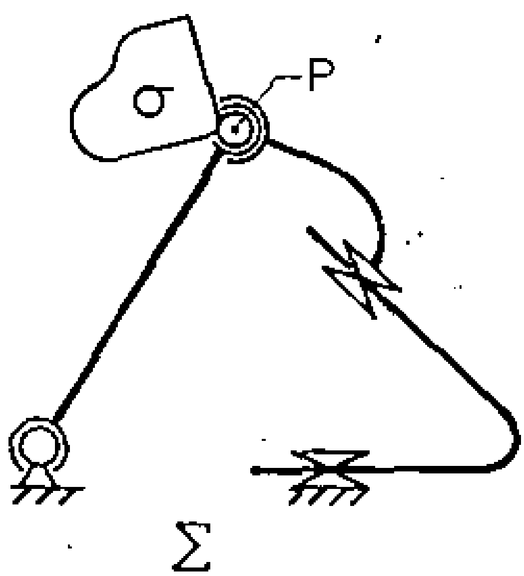


Fig. 4(b)

Fig. 4 Revolute-sphere binary link and sphere-sphere-slider-slider four bar

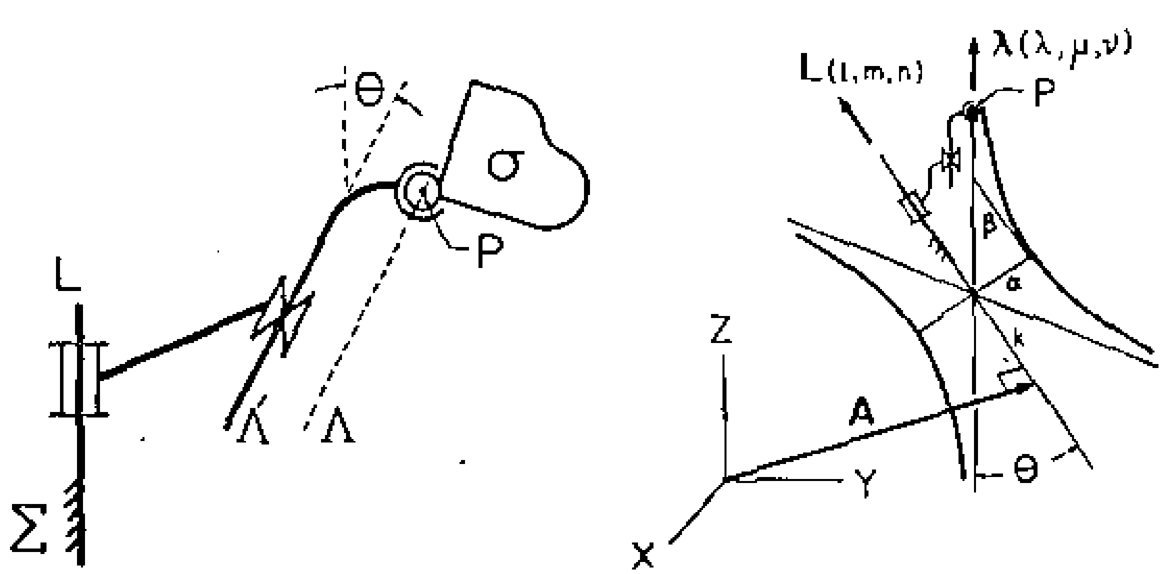


Fig. 5(a) Fig. 5(b)

Fig. 5 Revolute-slider-sphere dyad

Equations (23) can be transformed into

$$f_{r1}a + f_{r2}b + f_{r3}c + \epsilon_r k\frac{\alpha^2}{\beta^2} + t_r^2\left(1 + \frac{\alpha^2}{\beta^2}\right) - \frac{1}{2}f_{r4} = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (24)$$

where

$$\epsilon_r = \begin{cases} -(lX_r + mY_r + nZ_r) \times (lX_1 + mY_1 + nZ_1) & \text{if } l_j = 0 \\ -(lX_r^{(l_j)} + mY_r^{(l_j)} + nZ_r^{(l_j)}) & \text{if } l_j > 0 \end{cases}$$

$$t_r^2 = \begin{cases} \frac{1}{2}\left[(lX_r + mY_r + nZ_r)^2 - (lX_1 + mY_1 + nZ_1)^2\right] & \text{if } l_j = 0 \\ \frac{1}{2}\left[(lX_j + mY_j + nZ_j)^2\right]^{(l_j)} & \text{if } l_j > 0 \end{cases}$$

It can be shown that ϵ_r and t_r^2 are, respectively, linear and quadratic in x, y, and z.

Since equations (24) are nonlinear in the parameters, the methods of [1] are not directly applicable. However, if we specify (l, m, n) and k $\left(\text{or } \frac{\alpha^2}{\beta^2}\right)$, equations (24) are linear in the remaining parameters. Thus we can use the method shown in [1] to determine the loci of points in σ which have several positions on a hyperboloid with a specified axis directon and k $\left(\text{or } \frac{\alpha^2}{\beta^2}\right)$. Once such points are known, the direction cosines of the sliding direction (λ, μ, ν), which are not contained in (24), can be determined from the following:

(*a*) The cosine of the angle θ is

$$l\lambda + m\mu + n\nu = \frac{1}{\left[1 + \left(\frac{\alpha}{\beta}\right)^2\right]^{1/2}} \qquad (25)$$

(*b*) The distance α is

$$\alpha = \frac{1}{\sin\theta}\left[l(Y\nu - Z\mu) + m(Z\lambda - X\nu) + n(X\mu - Y\lambda) + \lambda(bn - cm) + \mu(cl - an) + \nu(am - bl)\right]$$

and hence

$$\lambda[m(Z - c) - n(Y - b)] + \mu[n(X - a) - l(Z - c)] + \nu[l(Y - b) - m(X - a)] = \frac{\alpha}{\left[\left(\frac{\beta}{\alpha}\right)^2 + 1\right]^{1/2}} \qquad (26)$$

(*c*)

$$\lambda^2 + \mu^2 + \nu^2 = 1 \qquad (27)$$

Equations (25), (26), and (27) yield two sets of (λ, μ, ν). This is due to the fact that there are always two generators passing through a point on a hyperboloid.

Cylinder-Sphere Binary Link. A cylinder-sphere binary link (Fig. 6) is equivalent to the special case of a revolute-slider-sphere dyad where the angle between L and Λ' is zero $\left(\frac{\alpha}{\beta} = 0\right)$; see Fig. 5. In this case the constraint equation (22) becomes

$$aX + bY + cZ - \frac{1}{2}\left[(X^2 + Y^2 + Z^2) - (lX + mY + nZ)^2 + (a^2 + b^2 + c^2)\right] = -\frac{\alpha^2}{2} \qquad (28)$$

and the design equations (24) reduce to

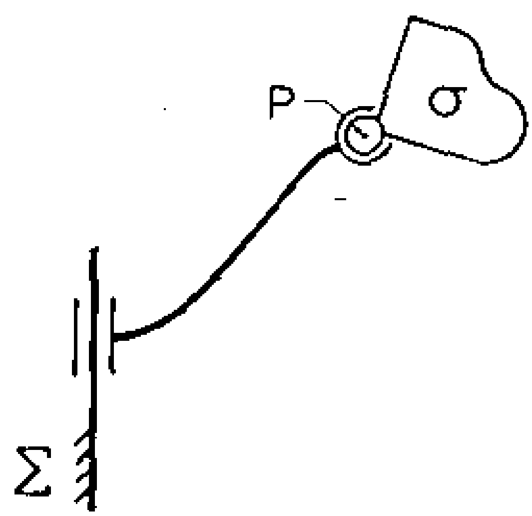


Fig. 6 Cylinder-sphere binary link

$$f_{r1}a + f_{r2}b + f_{r3}c + t_r^{\dagger} - \frac{1}{2} f_{rr} = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (29)$$

Cylinder-Cylinder Binary Link. A cylinder-cylinder binary link (Fig. 7) constrains a line L in σ such that the angle θ and the distance D between L and the fixed axis Λ are constant. If we let $\mathbf{L}(l, m, n)$ and $\boldsymbol{\lambda}(\lambda, \mu, \nu)$ be two unit vectors parallel to L and Λ, respectively, $\mathbf{A}(a, b, c)$ and $\boldsymbol{\alpha}(\alpha, \beta, \gamma)$ be, respectively, the position vectors of a point on L and a point on Λ, the constraint equations, in vector notation, for a cylinder-cylinder link are

$$\boldsymbol{\lambda}\cdot\mathbf{L} = \cos\theta = \text{const} \quad (30)$$

$$(\boldsymbol{\lambda} \times \mathbf{L})\cdot(\mathbf{A} - \boldsymbol{\alpha}) = D \sin\theta = \text{const} \quad (31)$$

A line in σ which has p multiple positions satisfying the constraint of a cylinder-cylinder link must satisfy

$$(\boldsymbol{\lambda}\cdot\mathbf{L}_j)^{(l_j)} = \begin{cases} \text{const} & \text{if } l_j = 0 \\ 0 & \text{if } l_j > 0 \end{cases} \quad (32)$$

$$[(\boldsymbol{\lambda} \times \mathbf{L}_j)\cdot(\mathbf{A}_j - \boldsymbol{\alpha})]^{(l_j)} = \begin{cases} \text{const} & \text{if } l_j = 0 \\ 0 & \text{if } l_j > 0 \end{cases} \quad (33)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

Thus the design equations for p multiple positions can be written as

$$\boldsymbol{\lambda}\cdot(\mathbf{L}_j - \mathbf{L}_1) = 0 \qquad j = 2, 3, \ldots, m \quad (34a)$$

$$(\boldsymbol{\lambda}\cdot\mathbf{L}_j)^{(l_j)} = 0 \quad (34b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

$$(\boldsymbol{\lambda} \times \mathbf{L}_j)\cdot(\mathbf{A}_j - \boldsymbol{\alpha}) - (\boldsymbol{\lambda} \times \mathbf{L}_1)\cdot(\mathbf{A}_1 - \boldsymbol{\alpha}) = 0$$
$$j = 2, 3, \ldots, m \quad (35a)$$

$$[(\boldsymbol{\lambda} \times \mathbf{L}_j)\cdot(\mathbf{A}_j - \boldsymbol{\alpha})]^{(l_j)} = 0 \quad (35b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

Equations (35) are linear in $\mathbf{A}_j^{(l_j)}$ and $\boldsymbol{\alpha}$. Since $\mathbf{A}_j^{(l_j)}$ is linear in a_1, b_1, and c_1, equations (35) are linear in a_1, b_1, c_1, α, β, and γ.

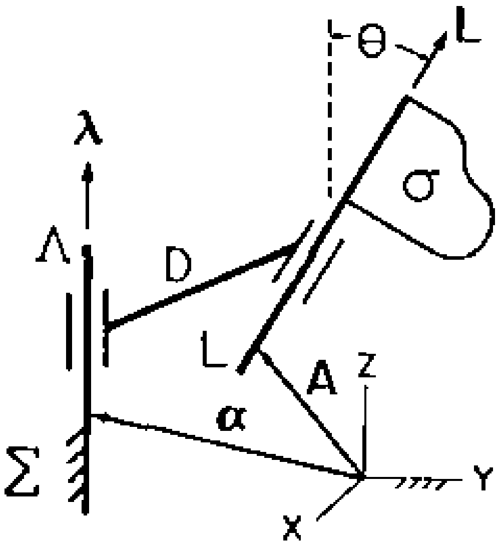


Fig. 7 Cylinder-cylinder binary link

Cylinder-Revolute Binary Link. A cylinder-revolute link (Fig. 8) is simply a cylinder-cylinder link with an additional constraint of no sliding along the moving axis. This additional constraint is governed by the equation

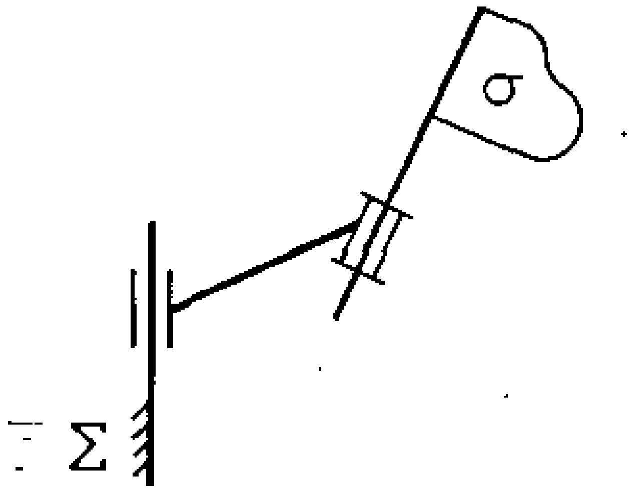


Fig. 8 Cylinder-revolute binary link

$$[(\mathbf{A} - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}]\cdot(\mathbf{L} \times \boldsymbol{\lambda}) = \text{const}$$

Therefore the design equations for p multiple positions are (34a), (34b), (35a), (35b),

$$[(\mathbf{A}_j - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}]\cdot(\mathbf{L}_j \times \boldsymbol{\lambda})$$
$$- [(\mathbf{A}_1 - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}]\cdot(\mathbf{L}_1 \times \boldsymbol{\lambda}) = 0 \qquad j = 2, 3, \ldots, m \quad (36a)$$

and

$$\{[(\mathbf{A}_j - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}]\cdot(\mathbf{L}_j \times \boldsymbol{\lambda})\}^{(l_j)} = 0 \quad (36b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

Revolute-Revolute Binary Link. A revolute-revolute binary link (Fig. 9) is another special case of a cylinder-cylinder link. Such a link does not allow sliding on both the moving and fixed axes. The constraint equations for this condition are given by

$$(\mathbf{A} - \boldsymbol{\alpha})\cdot\boldsymbol{\lambda} = \text{const} \quad (37)$$

and

$$(\mathbf{A} - \boldsymbol{\alpha})\cdot\mathbf{L} = \text{const} \quad (38)$$

Hence the design equations are (34a), (34b), (35a), (35b),

$$(\mathbf{A}_j - \mathbf{A}_1)\cdot\boldsymbol{\lambda} = 0, \qquad j = 2, 3, \ldots, m \quad (39a)$$

$$\mathbf{A}_j^{(l_j)}\cdot\boldsymbol{\lambda} = 0, \quad (39b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

$$(\mathbf{A}_j\cdot\mathbf{L}_j - \mathbf{A}_1\cdot\mathbf{L}_1) - \boldsymbol{\alpha}\cdot(\mathbf{L}_j - \mathbf{L}_1) = 0, \qquad j = 2, 3, \ldots, m \quad (40a)$$

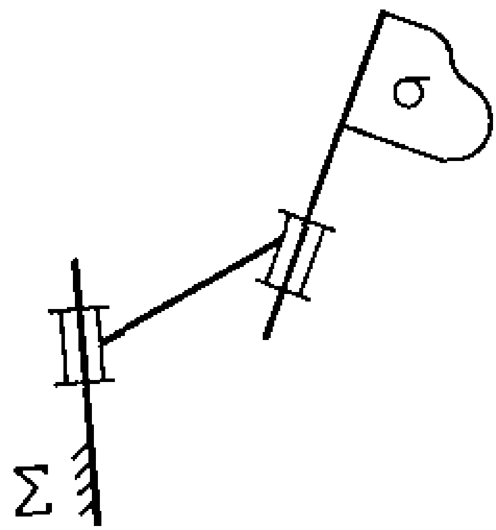


Fig. 9 Revolute-revolute binary link

Table 1

Link or dyad	No. of design positions	Locus (or number) of points that satisfy link constraint	Symbol for locus
Slider-slider-sphere	3	All points	
	4	3rd-order surface	(F^3)
	5	6th-order curve	(f^6)
	6	10 points	
Slider-sphere	2	All points	
	3	3rd-order curve	(i^3)
Sphere-sphere	4	All points	
	5	4th-order surface	(E^4)
	6	10th-order curve	(e^{10})
	7	20 points	
Revolute-sphere	3	All points	
	4	6th-order curve	(k^6)
Revolute-slider-sphere (with l, m, n, and k specified)	4	All points	
	5	5th-order surface	(H^5)
	6	16th-order curve	(h^{16})
	7	42 points	
Revolute-slider-sphere (with l, m, n, and α/β specified)	4	All points	
	5	5th-order surface	(HH^5)
	6	16th-order curve	(hh^{16})
	7	42 points	
Cylinder-sphere (with l, m, and n specified)	3	All points	
	4	4th-order surface	(H^4)
	5	11th-order curve	(h^{11})
	6	26 points	
Cylinder-cylinder	3	All lines	
	4	Line congruence (which contains all lines which are parallel to the generators of a cubic cone and pass through an infinite number of straight lines each of which corresponds to a generator of the cone).	
	5	6 lines	
Cylinder-revolute (with l, m, and n specified)	3	One unique line	
Revolute-revolute	3	24 lines	

and
$$[(\mathbf{A}_j - a)\cdot\mathbf{L}_j]^{(t_j)} = 0 \qquad (40b)$$
$$j = 1, 2, \ldots, m$$
$$t_j = 1, 2, \ldots, k_j$$

Results. Starting with the design equations derived for each link, we follow the procedure illustrated in the foregoing discussion of the sphere-sphere link. The results obtained are found in Table 1. (Detailed discussions of these derivations are given in [2].)

Inversion. In Fig. 1–Fig. 9 we have chosen the inversion which leads to the simplest design equations. The design of a chain which is the kinematic inversion of any of these can be accomplished by either kinematic inversion, which requires inverting the motion and interchanging the fixed and moving bodies, or by the direct application of the general method, which requires the derivation of new design equations. The following is an example illustrating the derivation of design equations for an inverted chain.

We consider the inversion of a revolute-slider-sphere dyad; namely, a sphere-slider-revolute dyad as shown in Fig. 10. This dyad constrains a line L in σ such that the line always satisfies the equation

$$(X_c - a)^2 + (Y_c - b)^2 + (Z_c - c)^2$$
$$- [l(X_c - a) + m(Y_c - b) + n(Z_c - c)]^2\left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- [k^2 - 2k[l(X_c - a) + m(Y_c - b) + n(Z_c - c)]]\frac{\alpha^2}{\beta^2} = \alpha^2 \qquad (41)$$

where (X_c, Y_c, Z_c) are the coordinates of the center of the spherical joint P_c; (l, m, n) are the direction cosines of line L; (a, b, c) are the coordinates of point A on L; Λ is a line passing through P_c parallel to the sliding axis Λ'; α is the shortest distance from Λ to L; $\frac{\alpha}{\beta}$ is the tangent of θ, the angle between L and Λ; k is the distance from A to A' (A' is the point on L at which the common normal from Λ terminates).

The design equations for p multiple positions are

$$[(X_c - a_j)^2 + (Y_c - b_j)^2 + (Z_c - c_j)^2]^{(t_j)}$$
$$- \{[l_j(X_c - a_j) + m_j(Y_c - b_j) + n_j(Z_c - c_j)]^2\}^{(t_j)}\left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- [k^2 - 2k[l_j(X_c - a_j) + m_j(Y_c - b_j)$$
$$+ n_j(Z_c - c_j)]^{(t_j)}]\frac{\alpha^2}{\beta^2} = (\alpha^2)^{(t_j)} \qquad (42)$$
$$j = 1, 2, \ldots, m$$
$$t_j = 0, 1, \ldots, k_j$$

For $t_j = 0$, the subtraction of the first equation from all the others of (42) yields

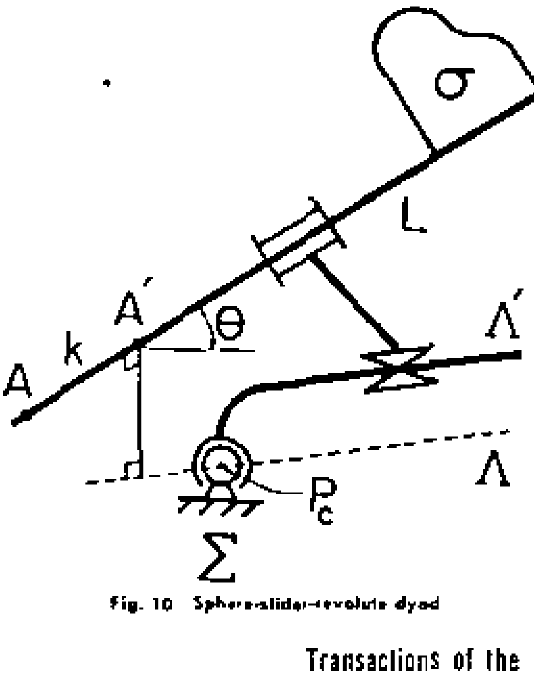


Fig. 10 Sphere-slider-revolute dyad

$$-2[X_e(a_j - a_1) + Y_e(b_j - b_1) + Z_e(c_j - c_1)] + (a_j^2 + b_j^2 + c_j^2) - (a_1^2 + b_1^2 + c_1^2) - [X_e(l_j - l_1) + Y_e(m_j - m_1) + Z_e(n_j - n_1) - (l_ja_j + m_jb_j + n_jc_j) + (l_1a_1 + m_1b_1 + n_1c_1)][X_e(l_j + l_1) + Y_e(m_j + m_1) + Z_e(n_j + n_1) - (l_ja_j + m_jb_j + n_jc_j) - (l_1a_1 + m_1b_1 + n_1c_1)]\left(1 + \frac{\alpha^2}{\beta^2}\right) + 2k[l_j(X_e - a_j) + m_j(Y_e - b_j) + n_j(Z_e - c_j) - l_1(X_e - a_1) - m_1(Y_e - b_1) - n_1(Z_e - c_1)]\frac{\alpha^2}{\beta^2} = 0 \qquad j = 2, 3, \ldots, m \quad (43)$$

But, as shown in Appendix 1 of [2], $(l_ja_j + m_jb_j + n_jc_j)$ is equal to $(C_{Lj} + l_1a_1 + m_1b_1 + n_1c_1)$, where C_{Lj} is a linear function of (l, m, n). If we choose point A such that

$$l_1a_1 + m_1b_1 + n_1c_1 = 0$$

then (43) becomes

$$X_e(a_j - a_1) + Y_e(b_j - b_1) + Z_e(c_j - c_1) - \frac{1}{2}(r_{Aj}^2 - r_{A1}^2) + \frac{1}{2}[X_e(l_j - l_1) + Y_e(m_j - m_1) + Z_e(n_j - n_1) - C_{Lj}][X_e(l_j + l_1) + Y_e(m_j + m_1) + Z_e(n_j + n_1) - C_{Lj}]\left(1 + \frac{\alpha^2}{\beta^2}\right) - k[X_e(l_j - l_1) + Y_e(m_j - m_1) + Z_e(n_j - n_1) - C_{Lj}]\frac{\alpha^2}{\beta^2} = 0 \quad (44)$$

$$j = 2, 3, \ldots, m$$

where $\qquad r_{Aj}^2 = a_j^2 + b_j^2 + c_j^2.$

For $l_j > 0$, equations (42) can be written as

$$(r_{Aj}^2)^{(l_j)} - 2(a_jX_e + b_jY_e + c_jZ_e)^{(l_j)} - \{[(l_jX_e + m_jY_e + n_jZ_e) - (l_ja_j + m_jb_j + n_jc_j)]^2\}^{(l_j)}\left(1 + \frac{\alpha^2}{\beta^2}\right) + 2k[l_j(X_e - a_j) + m_j(Y_e - b_j) + n_j(Z_e - c_j)]^{(l_j)}\frac{\alpha^2}{\beta^2} = 0 \quad (45)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

Since $a_j, b_j, c_j, r_{Aj}^2, a_j^{(l_j)}, b_j^{(l_j)}, c_j^{(l_j)}$, and $(r_{Aj}^2)^{(l_j)}$ are linear in a_1, b_1, and c_1, we can write (44) and (45) as:

$$\underline{f}_{r1}a_1 + \underline{f}_{r2}b_1 + \underline{f}_{r3}c_1 + \underline{g}_rk\frac{\alpha^2}{\beta^2} + \underline{t}_r^2\left(1 + \frac{\alpha^2}{\beta^2}\right) - \frac{1}{2}\underline{f}_{r4} = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (46)$$

where $\underline{f}_{r1}, \underline{f}_{r2}, \underline{f}_{r3}$, and $\underline{f}_{r4}$ are linear functions of X_e, Y_e, and Z_e

$$\underline{g}_r = \begin{cases} -[X_e(l_j - l_1) + Y_e(m_j - m_1) + Z_e(n_j - n_1) - C_{Lj}] & \text{if } l_j = 0 \\ -[l_jX_e + m_jY_e + n_jZ_e - (l_ja_j + m_jb_j + n_jc_j)]^{(l_j)} & \text{if } l_j > 0 \end{cases}$$

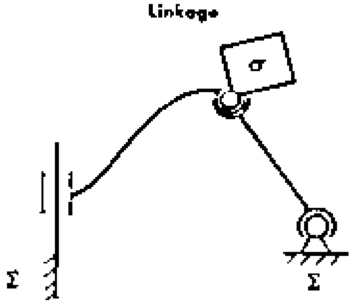
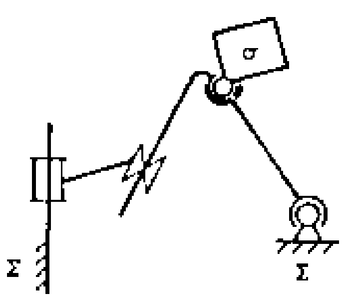
$$\underline{t}_r^2 = \begin{cases} \frac{1}{2}[X_e(l_j - l_1) + Y_e(m_j - m_1) + Z_e(n_j - n_1) - C_{Lj}] \times [X_e(l_j + l_1) + Y_e(m_j + m_1) + Z_e(n_j + n_1) - C_{Lj}] & \text{if } l_j = 0 \\ \frac{1}{2}\{[(l_jX_e + m_jY_e + n_jZ_e) - (l_ja_j + m_jb_j + n_jc_j)]^2\}^{(l_j)} & \text{if } l_j > 0 \end{cases}$$

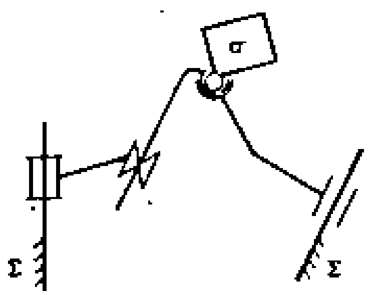
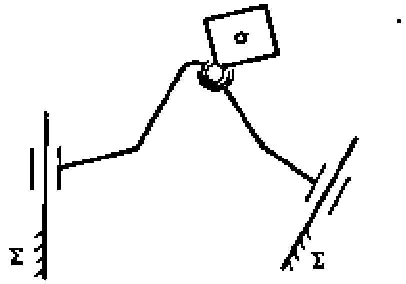
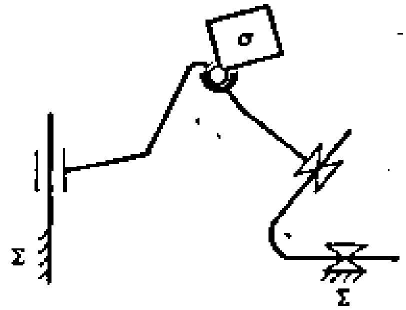
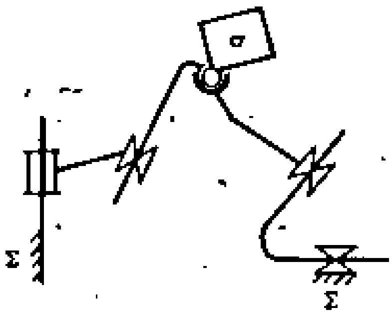
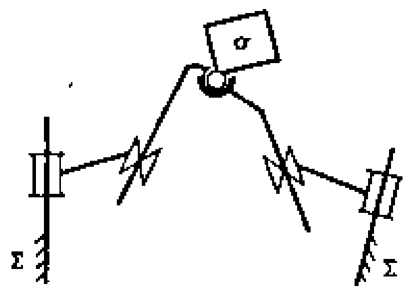
It can be shown (Appendix 1 and 2 of [2]) that $(l_ja_j + m_jb_j + n_jc_j)$ and $(l_ja_j + m_jb_j + n_jc_j)^{(l_j)}$ are independent of a_1, b_1, and c_1; therefore $\underline{g}_r$ and $\underline{t}_r^2$ are independent of a_1, b_1, and c_1, and are, respectively, linear and quadratic in X_e, Y_e, and Z_e.

Equation (46) is exactly of the same form as (24). The functions $\underline{f}_{r i}$, $\underline{g}_r$, and $\underline{t}_r^2$ are in terms of the coordinates of the pivot which in this case is fixed to Σ. Therefore, if we specify (l_1, m_1, n_1) and $k\left(\text{or } \frac{\alpha}{\beta}\right)$ and follow the procedure given in [1], we will obtain results similar to those for a revolute-slider-sphere dyad problem. However, in this case the final equations represent loci which are embedded in Σ instead of in σ.

Design equations for other inverted chains are given in [2].

Combined Link Chains. Many links discussed in the foregoing are links which constrain a point P in σ to move on a special surface. By combining any two such links so that both links share a common pivot at P, we can construct a link chain which constrains P so that it can only move along the intersection of the two surfaces associated with the two links. The following is a list of such combined link chains:

Linkage	Link combination	Geometrical locus of point P
	Cylinder-sphere link and sphere-sphere link (C-S-S three-bar)	Intersection of a sphere and a cylinder
	Revolute-slider-sphere dyad and sphere-sphere link (R-P-S-S- four-bar)	Intersection of a sphere and a hyperboloid

	Revolute-slider-sphere dyad and cylinder sphere link (R-P-S-C four-bar)	Intersection of a cylinder and a hyperboloid
	Two cylinder-sphere links (C-S-C three-bar)	Intersection of two cylinders with different axis inclinations
	Cylinder-sphere link and slider-slider-sphere dyad (C-S-P-P four-bar)	Intersection of a cylinder and a plane (an ellipse)
	Revolute-slider-sphere dyad and slider-slider-sphere dyad (R-P-S-P-P five bar)	Intersection of a plane and a hyperboloid
	Two revolute-slider-sphere dyads (R-P-S-P-R five bar)	Intersection of two hyperboloids

Points in σ that have n positions satisfying the constraint of such a linkage must satisfy the constraints of both chains which form the linkage. These points can be determined by "intersecting" the locus of points which have n positions satisfying the constraint of the first chain with that of the second chain. For example, the locus of all points in σ that have four multiple position on an ellipse, defined by the intersection of a plane and a cylinder with specified axis inclination, may be found by intersecting F^8 with H^4 which will give a twelfth-order space curve. Since there are six unknown parameters in the equations of a plane and a cylinder with specified axis inclination, according to equation (22) in [1], there are no points in σ which have five positions on such an ellipse.

In the case of a C-S-S three-bar, the procedure is slightly different. For four positions only points on H^4 have four positions on a cylinder with specified axis inclination, but every point in σ has four positions on a sphere. The locus of all points that have four positions on the intersection of a sphere and a cylinder is, therefore, identical to H^4.

For five positions the locus of five-position cylinder points is an eleventh-order space curve k^{11}, and the locus of five-position sphere points is a fourth-order surface E^4. The intersection of k^{11}

and E' yields 44 points which have five positions satisfying the constraint of a *C-S-S* three-bar.

Following the foregoing procedure, one can easily determine points in σ which have multiple positions satisfying the constraints of the remaining combined linkages. All such results can be found in Table 2.

Table 2

Linkage	No. of positions	Locus (or number) of points that satisfy linkage constraint
C-S-S three-bar	4	Fourth-order surface
	5	44 points
R-P-S-S four-bar	4	All points in σ
	5	Twentieth-order space curve
R-P-S-C four-bar	4	Fourth-order surface
	5	55 points
C-S-C three-bar	3	All points in σ
	4	Sixteenth-order space curve
C-S-P-P four-bar	3	All points in σ
	4	Twelfth-order space curve
R-P-S-P-P five-bar	4	Third-order surface
	5	30 points
R-P-S-P-R five bar	4	All points in σ
	5	Twenty-fifth-order space curve

Numerical Procedure for Computing Special Points and Lines. The equations which govern the loci of the special points or lines are obtained from the compatibility conditions which generally are in the form of determinants. Theoretically, it is possible to expand these determinants and obtain explicit expressions for all the coefficients in terms of the specified motion parameters. The expansion of these determinants, however, requires prodigious amounts of algebra which would result in impossibly lengthy expressions. Therefore, no attempt has been made to expand these equations algebraically. Instead, with the aid of a digital computer, both the coefficients and the "solutions" of these equations are determined numerically. The following is a general description of the computational procedure:

Starting with a set of specified motion parameters, we compute the elements of the determinant which represents the desired equation. In the case of finitely separated positions these elements can easily be obtained from the linear relation shown in equation (1) of [1]. In the case of infinitesimal displacements, if the motion is described by series of consecutive infinitesimal screw displacements, the elements can be computed from the expressions shown in equation (4) of [1]. On the other hand, when the motion is described by kinematic parameters, the elements are computed from the equations (6a) of [1].

Knowing all the elements, we can proceed to expand the determinant. First, we arrange the elements such that all like variables lie in the same subcolumns. For example, the elements in the equation of the cubic cone (51) can be arranged into the following form:

$$\begin{vmatrix} a_{1x}l + b_{1x}m + c_{1x}n & a_{1y}l + b_{1y}m + c_{1y}n & a_{1z}l + b_{1z}m + c_{1z}n \\ a_{2x}l + b_{2x}m + c_{2x}n & a_{2y}l + b_{2y}m + c_{2y}n & a_{2z}l + b_{2z}m + c_{2z}n \\ a_{3x}l + b_{3x}m + c_{3x}n & a_{3y}l + b_{3y}m + c_{3y}n & a_{3z}l + b_{3z}m + c_{3z}n \end{vmatrix}$$

where (l, m, n) are the variables and the a's, b's, and c's are known coefficients. The foregoing determinant can be expanded into a series of three-by-three subdeterminants with the same variable in each column. An example of such a subdeterminant is

$$\begin{vmatrix} a_{1x}l & b_{1y}m & b_{1z}m \\ a_{2x}l & b_{2y}m & b_{2z}m \\ a_{3x}l & b_{3y}m & b_{3z}m \end{vmatrix}$$

from which lm^2 may be factored:

$$\begin{vmatrix} a_{1x} & b_{1y} & b_{1z} \\ a_{2x} & b_{2y} & b_{2z} \\ a_{3x} & b_{3y} & b_{3z} \end{vmatrix} lm^2$$

After evaluating all such subdeterminants the final result is obtained simply by summing all coefficients of like terms. Having the equations, the next step is to determine their solutions. The most general case involves three polynomials in three unknowns. Theoretically, we can always eliminate one unknown from two independent equations and obtain a resulting equation known as the eliminant [3]. Therefore, with three equations, we can first eliminate one unknown and obtain two eliminants in the two remaining unknowns. Then we can eliminate a second unknown between the two eliminants to obtain a single polynomial, the roots of which can be determined numerically. By back substituting each of these roots into the previous equations the corresponding two other unknowns can be determined. This process of elimination, however, is not practical for equations of high degree because each step of the elimination introduces extraneous roots and increases the degree of the equations. If the equations are all of the same degree in the three unknowns then each elimination increases the degree by the square. For instance, the elimination of two unknowns from three third-degree equations would result in an 81st degree single-unknown polynomial. For this reason the roots of high-degree equations are determined by (Newton's method of) iteration. It is desirable to use the method of elimination whenever possible since it gives all solutions whereas iteration does not.

Numerical Example. The following is an illustrative example in which, for a given motion, we determine the lines which have five infinitesimally separated positions satisfying the constraint of a cylinder-cylinder crank. From (34b) and (35b), the design equations for five infinitesimally separated positions ($m = 1$, $k_1 = 4$) are

$$(\lambda \cdot \mathbf{L}_1)^{(k)} = 0 \qquad (47)$$

and

$$[(\lambda \times \mathbf{L}_1)\cdot(\mathbf{A}_1 - \alpha)]^{(k)} = 0 \qquad k = 1, 2, 3, 4 \qquad (48)$$

In scalar form (47) and (48) can be written as[7]

$$l^{(i)}\lambda + m^{(i)}\mu + n^{(i)}\nu = 0 \qquad (49)$$

and

$$l^{(i)}\rho + m^{(i)}\sigma + n^{(i)}\tau + \lambda R^{(i)} + \mu S^{(i)} + \nu T^{(i)} = 0 \qquad (50)$$

$$i = 1, 2, 3, 4$$

where (ρ, σ, τ) and $(R^{(i)}, S^{(i)}, T^{(i)})$ are the components of $\alpha \times \lambda$ and $(\mathbf{A} \times \mathbf{L})^{(i)}$, respectively.

The compatibility condition for (49) yields

$$\begin{vmatrix} l^{(1)} & m^{(1)} & n^{(1)} \\ l^{(2)} & m^{(2)} & n^{(2)} \\ l^{(k)} & m^{(k)} & n^{(k)} \end{vmatrix} = 0 \qquad (51)$$

$$k = 3, 4$$

providing that the rank of

$$\begin{bmatrix} l^{(1)} & m^{(1)} & n^{(1)} \\ l^{(2)} & m^{(2)} & n^{(2)} \end{bmatrix} \qquad (52)$$

is two.

The four equations (50) are linear in a, b, c, α, β, and γ, because (ρ, σ, τ) are, by definition, linear in (α, β, γ), and $(R^{(i)}, S^{(i)}, T^{(i)})$ are linear in (a, b, c). By choosing $\mathbf{A}$ and α such that

$$al + bm + cn = 0 \qquad (53)$$

and

$$\alpha\lambda + \beta\mu + \gamma\nu = 0 \qquad (54)$$

we obtain a total of six linear equations in a, b, c, α, β, and γ.

Since $(l^{(i)}, m^{(i)}, n^{(i)})$ are linear and homogeneous in (l, m, n),

[7] For convenience the subscript "1" is omitted.

the two equations (51) represent two cubic cones having a common apex (the origin). These two cubic cones intersect each other in (a maximum of) nine lines, six of which represent the directions of the, at most, six possible moving axes. The other three are spurious since they correspond to the lines at which the rank of (52) is less than two. Further investigation shows that these three lines are coincident (a triple intersection, see Fig. 12) and are parallel to the instantaneous screw axis. Corresponding to each of the moving axis directions, we can determine a unique cylinder-cylinder crank by first solving any two equations of (49) for the fixed axis direction (λ, μ, ν) and then solving (50), (53), and (54) for the moving and fixed axis positions (a, b, c) and (α, β, γ), respectively.

Table 3 Specified motion of the moving body σ (five infinitesimally separated positions)

	X-component	Y-component	Z-component
$\dot{d}$	−4.01920	−7.52880	−12.21129
$\ddot{d}$	71.45568	−47.92845	1.58135
$\dddot{d}$	79.23863	94.10794	476.1080
$\ddddot{d}$	−2202.585	1527.748	192.3612
ω	0.00000	0.00000	8.67932
$\dot{\omega}$	−31.65956	20.51287	−0.88769
$\ddot{\omega}$	−16.00582	−39.98313	−164.1540
$\dddot{\omega}$	641.3854	−409.7535	57.80889

where $\dot{d}$ is the velocity of the origin of σ and ω the angular velocity of σ (see Fig. 1 in [1]).

For the motion given in Table 3, the equations of the two cubic cones (51) are for $k = 3$:

$$(l^2 + m^2)(l - 0.63971m) - 0.43510n(l^2 - m^2) - 0.30053lmn = 0 \quad (55)$$

for $k = 4$:

$$0.34675l^3 + 0.56442m^3 + 0.62204l^2m - 0.88635l^2n + 0.72127m^2l + m^2n + 0.20591n^2l + 0.31810n^2m - 0.35901lmn = 0 \quad (56)$$

The real intersections of the foregoing two cones are:

Intersection	l	m	n	
1	0.00000	0.00000	1.00000	trivial
2	0.00000	0.00000	1.00000	trivial
3	0.00000	0.00000	1.00000	trivial
4	0.27990	0.57608	0.76798	
5	0.27652	0.41478	0.86689	

The moving and fixed axes corresponding to the two nontrivial (the 4th and 5th) intersections are:

Direction

Moving axis			Fixed axis		
l	m	n	λ	μ	ν
0.27991	0.57608	0.76798	−0.33935	−0.69843	0.63011
0.27652	0.41478	0.86689	−0.53727	−0.80591	0.24868

Location

Moving axis			Fixed axis		
a	b	c	α	β	γ
−0.32512	−0.05752	0.16187	1.9963	−0.79623	0.19255
0.00000	0.00000	0.00000	1.6410	−1.0385	0.18008

After replacing l, m, and n by x, y, and z, respectively, equations (55) and (56) represent two cubic cones in the moving system σ.

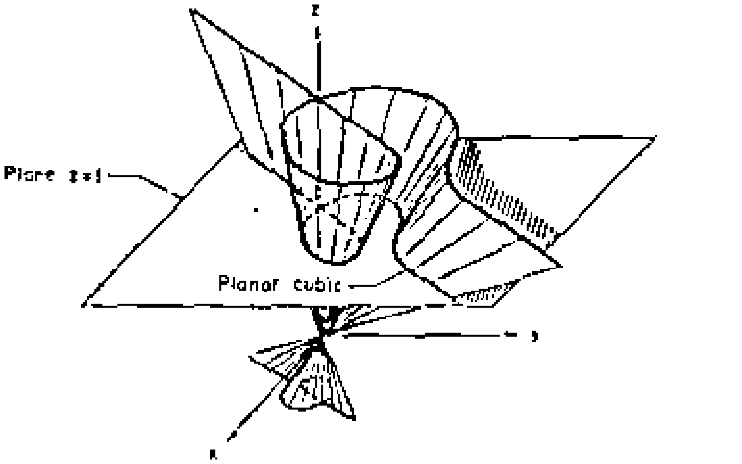


Fig. 11 First cubic cone

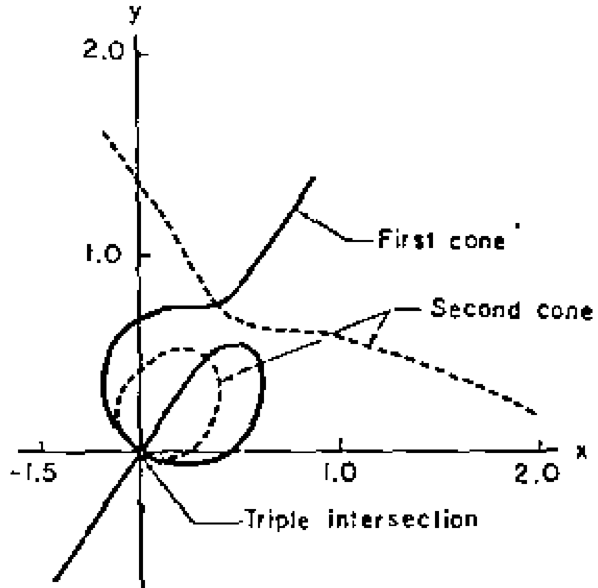


Fig. 12 Intersection of two cubic cones and plane $z = 1$

The first cubic cone, equation (55), is shown in Fig. 11. The intersection of the two cones and the plane $z = 1$ is shown in Fig. 12.

Summary

Design equations have been presented for binary links with all possible combinations of spherical, revolute, and cylindric pairs. The only nontrivial binary link with a prismatic pair, the slider-sphere link, was also treated. In addition, the slider-slider-sphere and revolute-slider-sphere dyads which one obtains by adding a second link to the prismatic pair of a "trivial" binary link combination were considered.

The effect of combining binary links and dyads into closed chains was considered, and the question of kinematic inversion was dealt with.

The particular illustrative numerical example was chosen because of its kinship to the classical Burmester-point problem of instantaneous planar kinematics. In direct analogy to previous finite position work [1], it is pointed out that in the case of planar motions the cones given by equations (35) and (36) become right cubic cylinders, and the curve labeled "first cone" in Fig 12 becomes the cubic of stationary curvature while the two intersections (other than the triple point) become the Burmester points. Most important, it should be understood that the cylindric-cylindric link axes for the spherical motion problem (defined in Table 3 provided $\mathbf{d}$ and all its higher derivatives are set to zero) have exactly the same directions as for the spatial problem. For spherical motion we let the pair axes pass through the origin (i.e., set $a = b = c = \alpha = \beta = \gamma = 0$); in which case the cylindric joints may be replaced by revolutes. Figs. 11 and 12 are identical for all spherical and spatial problems which have the same angular motions.

The application of the results of this paper to problems of kinematic synthesis has been previously treated [5]. Although [5] deals only with finite displacements, the ideas are equally valid for synthesis problems involving infinitesimally and mixed finitely and infinitesimally separated positions.

Acknowledgment

The financial support of the National Science Foundation (NSF GK-600) is gratefully acknowledged.

References

1 Chen, P., and Roth, B., "A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis," Paper No. 68—Mech-7, to be published in Trans. ASME.

2 Chen, P., "An Analytical Investigation of Infinitesimal Spatial Motion Theory and Its Application to Three-Dimensional Linkages," dissertation, Stanford University, 1968.

3 Muir, T., *A Treatise on the Theory of Determinants*, Dover Publications, New York, 1960.

4 Roth, B., "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 89, No. 1, Feb. 1967, pp. 102–110.

5 Roth, B., "Finite-Position Theory Applied to Mechanism Synthesis," *Journal of Applied Mechanics*, Vol. 34, No. 3, Trans. ASME, Vol. 89, Series E, Sept. 1967, pp. 599–605.

Printed in U. S. A.

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

1. The Design of Binary Cranks with Revolute, Cylindric, and Prismatic Joints

2. Design of Dyads with Helical, Cylindrical, Spherical Revolute and Prismatic Joints

3. A note on the Design of Revolute-Revolute Cranks

4. Design of triads using the Screw-Triangle Chain

5. Incompletely Specified Displacements: Geometry and Spatial Linkage Synthesis

6. A Transmission Index for Spatial Mechanisms

Dr.Bernard Roth

Junio, 1981

(6)

Jnl. Mechanisms Volume 3, pp.61–72/Pergamon Press 1968/Printed in Great Britain

The Design of Binary Cranks with Revolute, Cylindric, and Prismatic Joints

Prof. Bernard Roth*

Received 16 October 1967

Abstract
Using screw-axis geometry, the synthesis of cranks with prismatic joints is discussed and equations are derived for the design of spatial cranks with one revolute and one cylindric joint, or two revolute joints. These new formulations, coupled with previous work on the multi-position synthesis of cylindric–cylindric links [1], complete and unify the design theory for all binary cranks with joints that are either revolutes, sliders, or cylinders. All such binary cranks are designed to be compatible with a specified set of finitely separated positions of a rigid body which they couple to a fixed link. It has been previously shown [2] that results of this type are applicable to a large number of different multi-position spatial synthesis problems.

Zusammenfassung—Der Entwurf von binären Kurbeln mit Drehgelenken, Drehschubgelenken und Gleitgelenken.
Die Schrauben-Achsen Geometrie benutzend, wird die Synthese von Kurbeln mit prismatischen Verbindungen betrachtet und Gleichungen abgeleitet für den Entwurf von Raumkurbeln mit einem Drehpaar und einem Zylindergelenk oder zwei Drehpaaren. Diese neue Formulierung vereint mit früheren Arbeiten über die mehrlagen Synthese der Zylinder-Zylinder Glieder [1] vollendet und vereinheitlicht die Entwurfstheories für alle binären Kurbeln mit Paaren, die entweder Drehpaare, Gleitpaare oder Zylinder sind. Alle solche binären Kurbeln sind verträglich mit einer bestimmten Anzahl von endlich benachbarten Lagen eines festen Körpers, den sie mit einem Gestell verbinden. Es wurde früher gezeigt [2] dass Resultate von dieser Art anwendbar sind für die Synthese einer grossen Anzahl verschiedener Mehrlagen räumlicher Probleme.

Резюме—Проэктирование кривошипов с вращательными, цилиндрическими и поступательными парами—Б. Рот.
Применяя теорию винтовых осей, синтез кривошипов с призматическими парами разсмотрен и уравнения выведены для проэктирования пространственных кривошипов с одной вращательной парой и одной цилиндрической или двумя вращательными парами. Эта новая формулировка в соединении с предядущими работами по синтезу многократных положений цилиндрических-цилиндрических звеньев [1] завершает и объединяет теорию всех простых кривошипов с парами которые являются или вращательными, или поступательными или цилиндрическими. Все такие простые кривошипы совместимы с определенным рядом конечно разделеных положений твердого тела, которое они соединяют с неподвижным звеном. Раньше было показано [2] что результаты этого типа применимы к синтезу большого количества различных пространственных проблем с многократными положениями.

1. Introduction

As shown in [2], a large number of different synthesis problems are of the type where several finitely separated positions of a moving rigid-body are known. In such problems, the unknowns are the dimensions of one or more links which connect the moving body to a so-called fixed body.

* Associate Professor, Department of Mechanical Engineering, Stanford University, Stanford, California, U.S.A.

This paper deals with the design of binary connecting links which we call "cranks". In order to apply these results to an engineering problem, it is necessary that the kinematic chain formed by the fixed-body, crank, and moving-body be closed by adding at least one other joint or link between the moving body and the fixed one. In some cases (e.g. the spatial four bar with one revolute and three cylindric pairs) the closure may be effected by a second binary crank of the type considered in this paper. However, in general any connecting links or joints may be used provided: they are compatible with the several specified positions of the moving body, allow the linkage to move between design positions, and result in the entire chain having a mobility of one (and sometimes more). Readers interested in examples illustrating how to apply the results of this paper to actual design problems are referred to [2].

In previous papers the author has shown how to determine the dimensions of cylindric–cylindric cranks [1] and revolute-revolute cranks [2] compatible with a specified set of finitely separated positions of a rigid body. In this work equations are derived for the design of cranks with one revolute and one cylindric joint, and a new formulation of the revolute–revolute crank is given. By combining these new results with the previous ones we can treat all possible combinations of revolute and cylindric joints. In addition the design of cranks with prismatic joints is discussed.

Considering for now only revolute and cylindric joints, we have four possibilities for binary cranks (Fig. 1):

1. Cylinder–Cylinder (C–C)
2. Revolute–Cylinder (R–C)
3. Cylinder–Revolute (C–R)
4. Revolute–Revolute (R–R)

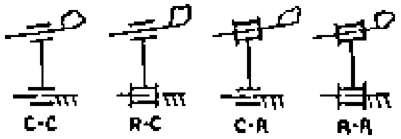


Figure 1. Binary cranks with cylindric and revolute joints.

The convention adopted here is that the joint which is mentioned first connects the binary crank to the body which we consider fixed, while the second named pairs the crank to the moving body. Since we are only concerned with relative motion, both bodies connected by the crank may in fact be moving. The finite displacement of the moving body (not the crank) from some position i to a new position j is characterized by a screw denoted by $\$_{ij}$, which is assumed known. The rotation about the screw axis will be denoted by θ_{ij} and the translation along it as d_{ij}. These screw parameters may be represented as a dual angle $\hat{\theta}_{ij}$, where $\hat{\theta}_{ij}=\theta_{ij}+\varepsilon d_{ij}$ and $\varepsilon^2=0$.

We now consider each of the above in turn.

2. C–C Cranks

The cylindric–cylindric crank has already been studied in some detail [1]. It is the most general of all the cranks considered here, since all others may be obtained as special cases. In fact, we shall design revolute joints by considering them as cylindrical joints with zero translation, and prismatic joints as cylindrical ones with zero rotation.

For our purposes, the following results from [1] are worth recalling:

For three positions any line whatsoever may be chosen as the fixed or moving axis. However, once one axis is chosen the other is uniquely determined by a certain (1, 1) correspondence.

For four positions there are a double infinity of possible C-C cranks, while for five positions there are at most six.

In what follows we seek to determine along which lines, when chosen as three position C–C crank axes, the crank displacements have either no rotation or translation.

3. R–C Cranks

The derivation we give relies on the following theorem from [1]:

> "The dual angle subtended at the fixed cylindric-cylindric crank axis by screws S_{ij} and S_{jk} is equal to one-half, or one-half the supplement, of the crank displacement from position i to k."

We will call this Theorem 1.

Realizing that an R-C crank is equivalent to a C–C crank with zero translation along the fixed axis, this theorem tells us that the fixed-joint axes of R–C cranks must be located so that the dual part of the dual angle, subtended by two screws with one common number in their subscripts, is zero. Which means that *the normals from each of the screw axes to any revolute axis must intersect the revolute axis at the same point.*

Three Positions

For three specified positions of the moving body, we have screws S_{12}, S_{13}, S_{23} (all embedded in the fixed body). Assuming the fixed axis of the crank to be an arbitrary line, with cosines (l, m, n), which passes through point (a, b, c), we would generally find the situation as shown in Fig. 2. Here, l_j $(j = 1, 2, 3)$ is the common normal directed from the joint axis to screw S_{jk}. Now, from the preceding paragraph, it is clear that the fixed axis cannot be taken arbitrarily but must instead be located so that l_1, l_2, and l_3 all intersect. (They are also, of course, co-planar.) We therefore undertake to derive a general expression for the distance D_{jk} between l_j and l_k. Having such a result we could then set $D_{12} = D_{13} = 0$ and obtain the necessary restrictions on our choice of revolute axis.

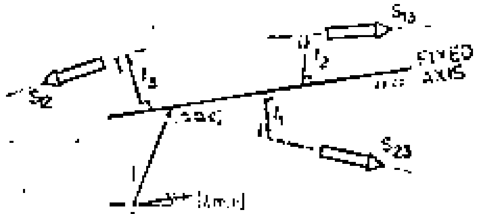


Figure 2. Fixed crank-axis, screws, and common normals.

It has been shown by Yang [3] that the unit line vector along l_j is given by the unit dual line-vector

$$\hat{l}_j = (l_j^*, m_j^*, n_j^*) + \epsilon(r_j^*, s_j^*, t_j^*)$$

where, if the screw axis $\mathbf{S}_{jk}$ is given by the unit dual line-vector $\widehat{\mathbf{S}}_{jk}=(l_j, m_j, n_j)+\varepsilon(r_j, s_j, t_j)$,

$$l_j^*=\frac{a_j}{J^{1/2}}$$

$$m_j^*=\frac{b_j}{J^{1/2}}$$

$$n_j^*=\frac{c_j}{J^{1/2}}$$

$$r_j^*=\frac{a_{oj}}{J^{1/2}}+\frac{d_j d_{oj} a_j}{J^{3/2}}$$

$$s_j^*=\frac{b_{oj}}{J^{1/2}}+\frac{d_j d_{oj} b_j}{J^{3/2}}$$

$$t_j^*=\frac{c_{oj}}{J^{1/2}}+\frac{d_j d_{oj} c_j}{J^{3/2}}$$

and

$$(a_j, b_j, c_j)=(l, m, n)\times(l_j, m_j, n_j)$$

$$(a_{oj}, b_{oj}, c_{oj})=(r, s, t)\times(l_j, m_j, n_j)-(r_j, s_j, t_j)\times(l, m, n)$$

$$d_j=(l, m, n)\cdot(l_j, m_j, n_j)$$

$$d_{oj}=(r, s, t)\cdot(l_j, m_j, n_j)+(r_j, s_j, t_j)\cdot(l, m, n)$$

$$J=a_j^2+b_j^2+c_j^2$$

$$(r, s, t)=(a, b, c)\times(l, m, n)$$

Hence we have an expression for $\mathbf{l}_j$ in terms of the screw axis $\mathbf{S}_{jk}$ and the (fixed) crank axis.

It is easy to prove that the normal from the origin of coordinates to $\mathbf{l}_j$ is the vector (A_j, B_j, C_j) where

$$A_j=\frac{c_{oj}b_j-b_{oj}a_j}{J}$$

$$B_j=\frac{a_{oj}c_j-c_{oj}a_j}{J} \tag{1}$$

$$C_j=\frac{b_{oj}a_j-a_{oj}b_j}{J}$$

The distance between $\mathbf{l}_j$ and $\mathbf{l}_k$, denoted by D_{jk}, is given by

$$D_{jk}=(A_j-A_k)l+(B_j-B_k)m+(C_j-C_k)n \tag{2}$$

In writing (2) we have used the fact that the fixed axis (l, m, n) is the common normal to $\mathbf{l}_j$ and $\mathbf{l}_k$.

Substituting from (1) into (2), setting $D_{jk}=0$, and expanding (a_{oj}, b_{oj}, c_{oj}) in terms of (r, s, t), i.e. the dual part of the unit vector along the crank axis, we get after some algebraic manipulation

$$rE_{jk}+sF_{jk}+tG_{jk}+H_{jk}=0. \tag{3}$$

Where, for simplicity we have set

$$E_{jk}=\frac{d_k a_k}{K}-\frac{d_j a_j}{J}$$

$$F_{jk}=\frac{d_k b_k}{K}-\frac{d_j b_j}{J}$$

$$G_{jk}=\frac{d_k c_k}{K}-\frac{d_j c_j}{J}$$

$$H_{jk}=(r_k a_k+s_k b_k+t_k c_k)/K+(r_j a_j+s_j b_j+t_j c_j)/J$$

It is important to note that E_{jk}, F_{jk}, G_{jk} and H_{jk} are only dependent upon the screw axes $\$_{ik}$ and $\$_{ji}$ and the inclination (l, m, n) of the crank axis.

Now since we require $D_{13}=D_{23}=0$ we get from (3)

$$rE_{23}+sF_{23}+tG_{23}+H_{23}=0$$
$$rE_{13}+sF_{13}+tG_{13}+H_{13}=0. \tag{4}$$

Combining these with the orthogonality condition

$$lr+ms+nt=0 \tag{5}$$

we obtain three linear equations in (r, s, t). Thus we are free to choose the direction of the crank axis (l, m, n) arbitrarily, and can expect to determine a unique vector (r, s, t) from (4) and (5). If we now take (a, b, c) as the normal to this crank axis (from the origin), we have

$$(a, b, c)=(l, m, n)\times(r, s, t) \tag{6}$$

from which we determine a unique crank location. Hence, *corresponding to any direction there is generally a unique location for a three-position revolute axis.*

Those directions for which (4) and (5) do not yield unique solutions admit an infinite number of co-planar parallel axes.

Alternatively, it is possible to specify any two of the coordinates of the point (a, b, c) through which the axis must pass and then solve (4) and (5) for the axis directions $(l/n, m/n)$ and the third coordinate of the point. In fact, it is possible to select arbitrarily any two of the five unknowns a, b, c, l/n, m/n. However, since (4) is nonlinear in (l, m, n) it is most convenient to specify the direction and solve for the location.

Once the fixed revolute axis is known, a unique moving cylindric axis may easily be determined using the (1,1) cylindric–cylindric correspondence given in [1]. Although the equations will not be repeated here, it is interesting to recall that: if we screw l_1 and l_2 about $\$_{13}$ and $\$_{12}$ by one half the corresponding screw motions, the common normal to these two new lines will be the moving crank-axis. (This is Theorem 2 given in Section 4 below.)

Four Positions

For four (or more positions) no such cranks generally exist. The reasoning is as follows: Introducing a fourth position results in a third equation in set (4), say, $D_{14}=0$. Now (4) and (5) contain four equations in the five unknowns r, s, t, l/n, m/n. But we must now not only satisfy the three position cylindric-cylindric constraints which (4) and (5) do automatically—but also the *four position* C–C constraint. This requires two additional equations, of the type given in [1], and results in a set of six equations with five unknowns. Hence *there are generally no four position R–C cranks.*

From a practical point of view the change in twist or length required of a crank (which does not satisfy the four-position C–C constraints) in going from position 3 to 4 may be very small. This means that if such a crank were used the fourth position of the moving body would be slightly in error. It is possible to restrict the error to the translational part of the motion by choosing (l, m, n) as one of the intersections of the screw cone {(15) in [1]} and

$$\begin{vmatrix} l & m & n & 0 \\ E_{13} & F_{13} & G_{13} & H_{13} \\ E_{23} & F_{23} & G_{23} & H_{23} \\ E_{34} & F_{34} & G_{34} & H_{34} \end{vmatrix} = 0$$

4. C–R Cranks

C–R cranks are inversions of R–C cranks and hence we already have a means of effecting their design: simply by interchanging fixed and moving bodies—i.e. kinematic inversion. However, it is interesting and useful to obtain the equations for the C–R crank directly for the given motion.

We proceed in approximately the same way as we did in the R–C crank derivation except now we must apply Theorem 1 to the moving body. In order to accomplish this, we start with all the elements shown in Fig. 2 and from these determine the normals between the moving crank-axis and the screws. This is done by applying another theorem from [1]:

> "Each pair of fixed and moving axes subtends a screw axis at a dual angle equal to one-half, or the supplement of one-half, the dual angle associated with that screw".

We call this theorem 2. If we denote the normal to screw $\mathbf{\$}_{ij}$ from the moving axis as $\mathbf{L}_j$, then this theorem tells us that $\mathbf{l}_j$ and $\mathbf{L}_j$ must both be normal to $\mathbf{\$}_{ij}$ and at dual angle equal one-half the screw motion, i.e. $(\hat{\theta}_{ij}/2)=(\theta_{ij}/2)+\varepsilon(d_{ij}/2)$, measured from $\mathbf{L}_j$ to $\mathbf{l}_j$.

Figure 3 shows $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$ and $\mathbf{L}'_3$ which are obtained by screwing $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$, and $\mathbf{l}_3$ about $\mathbf{\$}_{23}$, $\mathbf{\$}_{13}$, $\mathbf{\$}_{12}$, and $\mathbf{\$}_{12}$, respectively, by the dual angles $-(\hat{\theta}_{23}/2)$, $-(\hat{\theta}_{13}/2)$, $-(\hat{\theta}_{12}/2)$, and $(\hat{\theta}_{12}/2)$ respectively. The moving crank-axis, in position 1, must be the normal to $\mathbf{L}_3$ and $\mathbf{L}_2$. [It is also the line normal to $\mathbf{L}'_3$ and $\mathbf{L}_1$ when in position 2, and normal to $\mathbf{L}'_2$ and $\mathbf{L}'_1$ (not shown) when in position 3.]

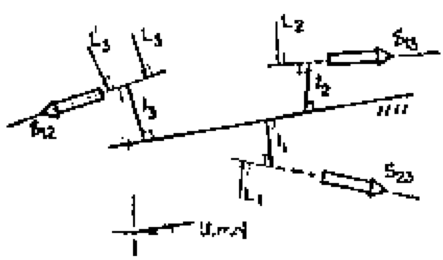


Figure 3. The L's are obtainable by screwing the l's about the screws.

Applying theorem 1 to the moving axis we conclude that the cylindric joint may be replaced by a revolute if L_2 and L_3 intersect (guaranteeing no translation from position 2 to 3) and also L_1 and L_3' intersect (guaranteeing no translation for positions 1 to 3). The conditions that these lines intersect restrict our choice of fixed axis:

Denoting the unit dual line vectors $\hat{L}_j$ and $\hat{L}_j'$ by their components:

$$\hat{L}_j = (L_j, M_j, N_j) + \varepsilon(R_j, S_j, T_j)$$

$$\hat{L}_j' = (L_j', M_j', N_j') + \varepsilon(R_j', S_j', T_j')$$

we can compute the moment $D_{jk} \sin \alpha_{jk}$ between L_j and L_k from

$$D_{jk} \sin \alpha_{jk} = L_j R_k + M_j S_k + N_j T_k + L_k R_j + M_k S_j + N_k T_j .$$

Then, the condition that L_2 and L_3 intersect is

$$L_2 R_3 + M_2 S_3 + N_2 T_3 + L_3 R_2 + M_3 S_2 + N_3 T_2 = 0 . \tag{7a}$$

Similarly, L_1 and L_3' intersect if

$$L_1 R_3' + M_1 S_3' + N_1 T_3' + L_3' R_1 + M_3' S_1 + N_3' T_1 = 0 . \tag{7b}$$

Making use of Theorem 2 we can express $\hat{L}_j$ in terms of $\hat{l}_j$ and the screw motion:

$$\hat{L}_j = \left[\cos\left(\frac{\hat{\theta}_{jk}}{2}\right) - \hat{S}_{jk} \sin\left(\frac{\hat{\theta}_{jk}}{2}\right) \right] \hat{l}_j \tag{8}$$

First we expand (8) and then substitute for the components of $\hat{L}_j$ into (7a). Finally proceeding as in the R-C derivation we separate those terms which depend on the direction of the fixed axis from those depending on the location, and obtain:

$$rU_{23} + sV_{23} + tW_{23} + X_{23} = 0 \tag{9}$$

where

$$U_{23} = J_{3_{11}} L_2^* + J_{2_{11}} L_3^* + J_{3_{21}} M_2^* + J_{2_{21}} M_3^* + J_{3_{31}} N_2^* + J_{2_{31}} N_3^*$$

$$-\left(\frac{l_3 d_3}{(a_3^2 + b_3^2 + c_3^2)} + \frac{l_2 d_2}{(a_2^2 + b_2^2 + c_2^2)} \right) (L_2^* L_3^* + M_2^* M_3^* + N_2^* N_3^*)$$

V_{23} and W_{23} are similar to U_{23} except that $J_{j_{k1}}$ is replaced by $J_{j_{k2}}$ and $J_{j_{k3}}$ respectively and l_j is replaced by m_j and n_j respectively. For X_{23} replace $J_{j_{k1}}$ with $J_{j_{k4}}$ and l_j with $lR_j + mS_j + nT_j$.

Here

$$L_j^* = a_j \cos\left(\frac{\theta_{ik}}{2}\right) - (m_j c_j - n_j b_j)\sin\left(\frac{\theta_{ik}}{2}\right)$$

$$M_j^* = b_j \cos\left(\frac{\theta_{ik}}{2}\right) - (n_j a_j - l_j c_j)\sin\left(\frac{\theta_{ik}}{2}\right)$$

$$N_j^* = c_j \cos\left(\frac{\theta_{ik}}{2}\right) - (l_j b_j - m_j a_j)\sin\left(\frac{\theta_{ik}}{2}\right)$$

$$J_{j11} = -(m_j^2 + n_j^2)\sin\left(\frac{\theta_{ik}}{2}\right)$$

$$J_{j12} = n_j \cos\left(\frac{\theta_{ik}}{2}\right) + m_j l_j \sin\left(\frac{\theta_{ik}}{2}\right)$$

$$J_{j13} = -m_j \cos\left(\frac{\theta_{ik}}{2}\right) + n_j l_j \sin\left(\frac{\theta_{ik}}{2}\right)$$

$$J_{j14} = (mt_j - ns_j)\cos\left(\frac{\theta_{ik}}{2}\right) - (s_j c_j - t_j b_j + l(m_j s_j + n_j t_j) + mm_j r_j + nn_j r_j)\sin\left(\frac{\theta_{ik}}{2}\right)$$

$$-\frac{d_{ik}}{2}\left[a_j \sin\left(\frac{\theta_{ik}}{2}\right) + (m_j c_j - n_j b_j)\cos\left(\frac{\theta_{ik}}{2}\right)\right]$$

and J_{j21} is obtained from J_{j11} by a single permutation of the components of (l_j, m_j, n_j), (a_j, b_j, c_j) and (r_j, s_j, t_j).
J_{j31} is obtained from a second permutation (i.e. $J_{j31} = -(l_j^2 + m_j^2)\sin \theta_{ik}/2$, etc.).
All other quantities are as defined in Section 3.
Similarly, from (7b) we obtain

$$rU'_{13} + sV'_{13} + tW'_{13} + X'_{13} = 0 \tag{10}$$

All the primed quantities in (10) may be derived from the corresponding unprimed ones in (9) by letting subscript 2 replace 1, and replacing θ_{12} and d_{12} with respectively, $-\theta_{12}$ and $-d_{12}$.

Equations (9) and (10) are exactly of the same form as the two members of (4). Hence combining (9), (10) and (5) we can again conclude that:

1. Any two of the five quantities l/n, m/n, a, b, c may be chosen arbitrarily.
2. Corresponding to any fixed crank-axis direction (l, m, n) there is in general one and only one C–R crank. The location of the fixed cylindrical axis follows from (9), (10), (5) and (6). While the moving (revolute) axis is obtained from the C–C correspondence. (Theorem 2 or any of the several other techniques discussed in [1]).

Using the same reasoning as before we conclude that there are not, in general, any C-R cranks corresponding to four arbitrary positions of a rigid body.

5. R–R Cranks

If we restrict both the moving and fixed axes to be revolutes it is required that (4), (5), (9) and (10) all be simultaneously satisfied. For this to be the case, it is necessary that

$$\begin{vmatrix} l & m & n & 0 \\ E_{23} & F_{23} & G_{23} & H_{23} \\ E_{13} & F_{13} & G_{13} & H_{13} \\ U_{23} & V_{23} & W_{23} & X_{23} \end{vmatrix} = 0 \tag{11}$$

and

$$\begin{vmatrix} l & m & n & 0 \\ E_{23} & F_{23} & G_{23} & H_{23} \\ E_{13} & F_{13} & G_{13} & H_{13} \\ U'_{13} & V'_{13} & W'_{13} & X'_{13} \end{vmatrix} = 0 \tag{12}$$

These two determinants contain (l, m, n) as the only unknowns.

The degree of the unknowns is three in H, four in E, F, and G, five in U, V, W and six in X. Since each term is homogeneous in (l, m, n) we divide the first row by n, the second and third by n^4, and the fourth by n^5. Then setting $x=(l/n)$, $y=(m/n)$, and $(x^2+y^2+1)=n^2$ we obtain two planar algebraic curves in x, y of order 15. However, $(x^2+y^2+1)^2$ and $(a_1^2+b_1^2+c_1^2)$ are common factors and the order reduces to 9. Since even after these reductions the resulting polynomials contain extraneous factors, it is more economical to use the results of [2]. In [2] the equations analogous to (11) and (12), called (c), are of sixth degree in x, y. This yields 36 possible solutions but 12 of these are spurious and there are generally at most 24 possible axis directions $(l/n, m/n)$. Once $(l/n, m/n)$ are known the design may be treated as either a R–C or a C–R synthesis. This results in one R–R crank for each set of (x, y). In general, there are therefore at most 24 three position R–R cranks.

6. Prismatic Joints

For a crank to have a three-position prismatic joint along the fixed axis, Theorem 1 tells us that $\mathbf{l}_1$, $\mathbf{l}_2$, and $\mathbf{l}_3$ must all be parallel. However, this is only possible when the three common normals between screws S_{12}, S_{13}, S_{23} are also parallel. Hence, there generally are no three position cranks with a fixed prismatic pair and a moving cylindric, revolute, or prismatic joint. By similar reasoning we can show that three position C–P and R–P cranks are also not possible. However, for two position problems, as we show in the next section, prismatic joints do exist.

7. Two Positions

The basic geometry relating the screw axis and the two crank axes is shown in Fig. 4. As before, $\mathbf{l}_3$ and $\mathbf{L}_3$ represent respectively the normals from the fixed and moving axes to the screw S_{12}. A and B denote the points where the normals intersect the crank axes. L is the common normal between fixed and moving axis. From Theorem 2 it follows that the dual angle measured from L_3 to l_3 is $(\hat{\theta}_{12}/2)$ (i.e. $(\theta_{12}/2)+\varepsilon(d_{12}/2)$). For our purposes, it is most important to note that the dual angle measured from L_3 to L is equal to one-half the dual displacement of the moving body relative to the crank, denoted by $(\hat{\theta}_m/2)$, and that the dual angle, $(\hat{\theta}_f/2)$, measured from L to $\mathbf{l}_3$ is one-half the crank displacement relative to the fixed body.

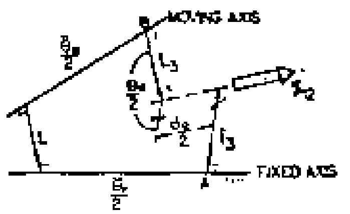


Figure 4. $\hat{\theta}_M/2$ is the dual angle between $\mathbf{L}_3$ and L. $\hat{\theta}_F/2$ is the dual angle between $\mathbf{l}_3$ and L.

The foregoing implies the following additional restrictions on the geometry of Fig. 4:

Type of joint	*Additional restriction*
C–C	None
R–C	L and $\mathbf{l}_3$ intersect
C–R	L and $\mathbf{L}_3$ intersect
R–R	L intersects both $\mathbf{l}_3$ and $\mathbf{L}_3$
P–C	L and $\mathbf{l}_3$ parallel
C–P	L and $\mathbf{L}_3$ parallel
P–P	L, $\mathbf{l}_3$ and $\mathbf{L}_3$ all parallel

The remaining paragraphs of this section described how to fulfill these additional restrictions.

Given $\mathbf{S}_{12}$, we can choose an arbitrary fixed axis and determine first $\mathbf{l}_3$ and then $\mathbf{L}_3$. Since the moving axis is the common normal between L and $\mathbf{L}_3$, the two position design may be completed as soon as we obtain L. L is determined as follows:

C–C crank .
L is any line (normal to the fixed axis).

R–C crank
L is any line through point A (normal to the fixed axis).

C–R crank
L is the line through any point on $\mathbf{L}_3$ (and normal to the fixed axis).

R–R crank
If B is the point where the line $\mathbf{L}_3$ pierces the plane which is normal to the fixed axis and contains point A, L is the line $\overline{AB}$.

P–C
L is any line parallel to $\mathbf{l}_3$ (and intersecting the fixed axis).

From the foregoing we note that after arbitrarily choosing the fixed axis we have a doubly infinite set of moving axes for C–C cranks, a singly infinite set of possible moving axes for R–C, C–R, and P–C cranks, and a unique R–R crank. It is easy to see that (except for P–C cranks) we get essentially the same result if we reverse the procedure and choose the moving axis arbitrarily.

For *C–P cranks* we cannot choose the fixed axis arbitrarily and generally find a line L which is both parallel to $\mathbf{L}_3$ and normal to the fixed axis. However, we may take any line as the moving axis and then determine $\mathbf{L}_3$ and $\mathbf{l}_3$. Now, L is any line through the moving axis parallel to $\mathbf{L}_3$. The fixed axis is the common normal to L and $\mathbf{l}_3$. By similar reasoning we could not have chosen the moving axis in the P–C crank design. Hence, in P–C or C–P cranks the P axis may be chosen at will but not the C axis.

Unless we have pure translational motion, *P-P cranks* are not possible. *P-R* and *R-P* cranks may be designed by first choosing the *P* axis and then determining the point where the normal (L_3 or l_3 respectively) between S_{12} and the *R* axis pierces the plane through the *P* axis and its common normal (l_3 or L_3 respectively) with S_{12}. The *R* axis passes through this point and is parallel to S_{12}.

8. Numerical Examples

The results of applying the foregoing equations to the design of R-C, C-R, and R-R cranks are presented in tables on the following page. The screws used are the same as in [1] where analogous results for C-C cranks have been presented.

Digital computer programs in FORTRAN IV for performing these computations on an IBM 7090 are available from the author.

9. Summary

For convenience we give a tabular summary of the foregoing results.

Type of crank	Maximum number of design positions	Number of arbitrary design choices	Maximum number of solutions
C-C	5	0	6
R-C	3	2	∞
C-R	3	2	∞
R-R	3	0	24
P-C	2	3*	∞
C-P	2	3	∞
P-R	2	2	∞
R-P	2	2	∞
P-P	1	4	∞

* In choosing a prismatic joint the number of arbitrary design parameters is two, since only the inclination of the axis is important.

Acknowledgement—The financial support of the National Science Foundation (NSF Gk-660) is gratefully acknowledged.

References

[1] Roth B. On the screw axes and other special lines associated with spatial displacements of a rigid body. *J. Engng Industry, Trans. ASME*, Series B, 89, 102–110 (1967).
[2] Roth B. Finite position theory applied to mechanism synthesis. *J. Appl. Mech., Trans. ASME*, Series E, 34, 599–605 (1967).
[3] Yang A. T. Application of quaternion algebra and dual numbers to the analysis of spatial mechanisms. Doctoral dissertation, Columbia University, New York, N.Y. (1963).

	Screws S_{ij}							
ij	Axis cosines			Normal to axis from origin			Screw rotation	Screw translation
12	0·3600*	0·0965	0·9280	0·9413	0·5214	−0·4194	133·2°	1·3844
13	0·1437	−0·0390	0·9886	−0·6103	0·8371	0·1230	70·62°	1·8981
23	−0·3036	−0·3618	−0·8814	1·1921	1·5650	−1·0331	66·45°	−0·9396

	Fixed axis						Moving axis								Crank displacement	
Type	Cosines			Normal			Cosines			Normal			Crank length	Crank twist	1–2	1–3
R–C	−0·9357	0·0411	0·3503	−0·2797	44·57	−3·971	0·4836	−0·3173	0·8158	3·9971	0·9238	−2·009	39·64	−79·64°	−47·88°	−31·18°
C–R	0·5935	0·3327	0·7329	1·954	1·269	−2·159	0·5677	−0·0207	0·8230	1·8211	−0·6232	−1·2719	0·2219	−21·07°	−82·53° 1·613	−89·45° 2·618
R–R	−0·4012	−0·3951	0·6963	−5·651	2·061	−1·495	0·4783	0·0466	0·8770	1·375	−0·0958	−0·7450	3·621	−66·98°	15·80°	25·78°

The sign of the crank twist is defined by the right-handed screw rule, with the screw pointing from the moving toward the fixed axis. Similarly, the crank displacement is taken with the positive direction defined by the fixed axis cosines.

* In Table 1 of [1] this number is given incorrectly.

Mechanism and Machine Theory, 1972, Vol. 7, pp. 85–102. Pergamon Press. Printed in Great Britain

Design of Dyads with Helical, Cylindrical, Spherical, Revolute and Prismatic Joints

L. W. Tsai

and

B. Roth*

Received 15 June 1971

Abstract

In this paper, an "equivalent screw triangle" is introduced from which the design equations can be determined for any combinations of helical, cylindrical, revolute, spherical, and prismatic joints for both finitely and infinitesimally separated position problems in kinematic synthesis. In addition to presenting a new and unified approach to previously solved problems, this paper presents an entirely new treatment of helical joints. The equations developed in this paper can be applied to the design of binary links for rigid body guidance, function generation and similar problems. A numerical example illustrating the design of H-H links is included.

Zusammenfassung – Der Entwurf von Gliedpaaren mit schraubenförmigen, zylindrischen, kugelförmigen, drehbaren und prismatischen Gelenken. L. W. Tsai, B. Roth

In diesem Beitrag wird ein "gleichwertiges Schraubendreieck" eingeführt, mit dessen Hilfe die Entwurfsgleichungen beliebiger Kombinationen von schraubenförmigen, zylindrischen, drehbaren, kugelförmigen und prismatischen Gelenken in Aufgaben der kinematischen Synthese sowohl für endlich als auch infinitesimal benachbarte Lagen bestimmt werden können. Zusätzlich zu einer neuen und einheitlichen Darstellung bereits gelöster Aufgaben wird in dieser Arbeit das schraubenförmige Gelenk in einer gänzlich neuen Weise behandelt. Die hier entwickelten Gleichungen können auf den Entwurf binärer Glieder für die Führung starrer Körper, die Erzeugung von Übertragungsfunktionen und ähnliche Aufgaben angewendet werden. Ein numerisches Beispiel illustriert den Entwurf eines Gliedpaares mit zwei Schraubengelenken.

Резюме — Расчет двойных механизмов с винтовыми цилиндрическими, сферическими, вращательными и призматическими шарнирами. Л. В. Тсаи и Б. Рот.

В этой работе введен «эквивалентный винтовой треугольник» с помощью которого могут быть определены расчетные уравнения для всех комбинаций винтовых, цилиндрических, вращательных, сферических и призматических шарниров для конечно и бесконечно близко разделенных положений в проблемах кинематического синтеза. Кроме представления нового и унифицированного подхода к прежде разрешенным проблемам эта работа представляет

*Dept. of Mechanical Engineering, Stanford University, Stanford, California, U.S.A.

совершенно новую трактовку винтовых шарниров. Развернутые уравнения в этой работе могут быть применены к расчету двойных звеньев для направления твердого тела, воспроизведения функций и в подобных проблемах. Расчет В-В звеньев иллюстрирован числительным примером.

Introduction

SYNTHESIS of binary links for both finitely and infinitesimally separated position problems have previously been extensively studied[1–5]. Those linkage elements which have been treated to date include all possible combinations of cylindric, revolute, spherical, and prismatic joints. However, links with helical (screw) joints have not been previously included in the general theory.

The term "dyad" used in the title of this paper refers to a two-link chain (a fixed link and a moving binary coupling link) which is used to guide a third member through several design positions. The objective is therefore the synthesis of an open three-link kinematic chain. Similarly in a companion work[7] a three-link chain (triad) is used to guide a fourth member through a set of design positions. In [7] it is shown how to extend the idea of the "equivalent screw triangle" to a "screw triangle chain" from which we obtain design equations of three-link chains (triads). All possible combinations of three-link chains with helical, cylindrical, spherical, prismatic and revolute joints are considered in [7].

The design equations given in this paper can be applied to the synthesis of binary links for rigid body guidance, function generation, and similar problems. The basic ideas on how to apply these results have been previously given by several authors [3, 8–11]. The major point to be recalled from these previous works is that these methods lend themselves to the synthesis of closed-loop chains, and may therefore be applied to the synthesis of either closed or open-loop spatial kinematic chains. Alternative approaches to the synthesis of spatial linkages have recently been published by several researchers[12–15]. Earlier works are cited in[4] and[6].

Design of Binary Links

Nomenclature

Figure 1 shows a binary link which kinematically connects a moving body, σ_1, to a "fixed" frame, Σ. The joint connecting the binary link and σ_1 is called the moving joint, and the one connecting it to Σ is called the fixed joint. The following quantities

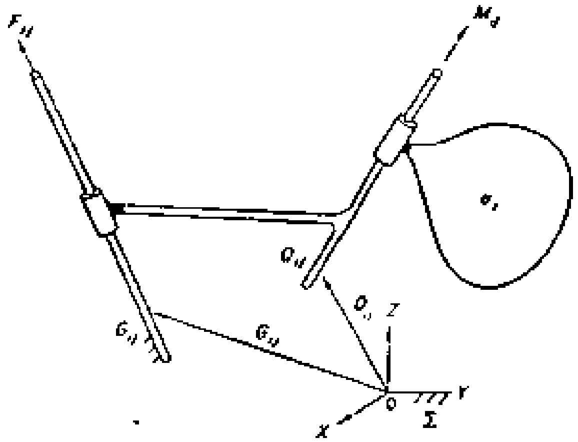


Figure 1. Binary link.

are defined in the figure: $\mathbf{M}_{ij}(M_{xij}, M_{yij}, M_{zij})$, a unit vector parallel to the ith position of the moving-joint axis; $Q_{ij}(Q_{xij}, Q_{yij}, Q_{zij})$, an arbitrary point on the moving-joint axis in its ith position; and $\mathbf{Q}_{ij}$, the position vector to Q_{ij}. Similarly, for the fixed-joint axis, we have the unit vector $\mathbf{F}_{ij}(F_{xij}, F_{yij}, F_{zij})$ parallel to the axis; an arbitrary point $G_{ij}(G_{xij}, G_{yij}, G_{zij})$ on the axis; and its position vector $\mathbf{G}_{ij}$. All of these vectors are measured in the "fixed" frame Σ.

The Equivalent Screw Triangle

A finite displacement of a rigid body σ, with respect to a reference frame Σ, from its ith to its jth position is completely described by a screw displacement: $\widehat{S}_{ij}$. However, when a binary link connects σ to Σ this same displacement is accomplished by rotations about and/or translations along the moving and the fixed-joint axes. Since the order of the displacements about these two axes is immaterial[3], we may regard the screw displacement, $\widehat{S}_{ij}$, as a product of two successive displacements, namely, a screw displacement, $\widehat{M}_{ij}$, about the moving-joint axis in its ith position followed by another screw displacement, $\widehat{F}_{ij}$, about the fixed-joint axis. These three screws, $\widehat{M}_{ij}$, $\widehat{F}_{ij}$, and $\widehat{S}_{ij}$, form an equivalent screw triangle[1], as shown in Fig. 2. We denote $\mathbf{S}_{ij}$ as a unit vector parallel to the screw axis $\widehat{S}_{ij}$, and A_{ij} as a point on that screw; (θ_{ij}, t_{ij}), (α_{ij}, u_{ij}), and (γ_{ij}, w_{ij}) are the screw parameters associated with $\widehat{S}_{ij}$, $\widehat{M}_{ij}$, and $\widehat{F}_{ij}$ respectively.

An investigation of the screw triangle geometry yields the following relations:

$$\tan\frac{\theta_{ij}}{2} = -\frac{\mathbf{F}_{ij}\cdot(\mathbf{S}_{ij}\times\mathbf{M}_{ij})}{(\mathbf{F}_{ij}\times\mathbf{S}_{ij})\cdot(\mathbf{S}_{ij}\times\mathbf{M}_{ij})} \tag{1}$$

$$\tan\frac{\alpha_{ij}}{2} = \frac{\mathbf{F}_{ij}\cdot(\mathbf{S}_{ij}\times\mathbf{M}_{ij})}{(\mathbf{S}_{ij}\times\mathbf{M}_{ij})\cdot(\mathbf{M}_{ij}\times\mathbf{F}_{ij})} \tag{2}$$

$$\tan\frac{\gamma_{ij}}{2} = \frac{\mathbf{F}_{ij}\cdot(\mathbf{S}_{ij}\times\mathbf{M}_{ij})}{(\mathbf{M}_{ij}\times\mathbf{F}_{ij})\cdot(\mathbf{F}_{ij}\times\mathbf{S}_{ij})} \tag{3}$$

$$\frac{t_{ij}}{2} = -\frac{\mathbf{S}_{ij}-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})\mathbf{M}_{ij}}{1-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})^2}\cdot(\mathbf{Q}_{ij}-\mathbf{A}_{ij}) + \frac{\mathbf{S}_{ij}-(\mathbf{S}_{ij}\cdot\mathbf{F}_{ij})\mathbf{F}_{ij}}{1-(\mathbf{S}_{ij}\cdot\mathbf{F}_{ij})^2}\cdot(\mathbf{G}_{ij}-\mathbf{A}_{ij}) \tag{4}$$

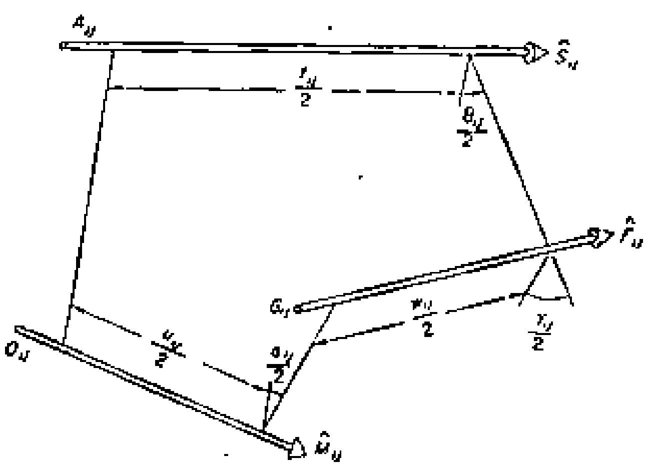


Figure 2. The equivalent screw triangle.

$$\frac{u_{ij}}{2}=\frac{\mathbf{M}_{ij}-(\mathbf{M}_{ij}\cdot\mathbf{F}_{ij})\mathbf{F}_{ij}}{1-(\mathbf{M}_{ij}\cdot\mathbf{F}_{ij})^2}\cdot(\mathbf{G}_{ij}-\mathbf{Q}_{ij})-\frac{\mathbf{M}_{ij}-(\mathbf{M}_{ij}\cdot\mathbf{S}_{ij})\mathbf{S}_{ij}}{1-(\mathbf{M}_{ij}\cdot\mathbf{S}_{ij})^2}\cdot(\mathbf{A}_{ij}-\mathbf{Q}_{ij}) \tag{5}$$

$$\frac{w_{ij}}{2}=\frac{\mathbf{F}_{ij}-(\mathbf{F}_{ij}\cdot\mathbf{S}_{ij})\mathbf{S}_{ij}}{1-(\mathbf{F}_{ij}\cdot\mathbf{S}_{ij})^2}\cdot(\mathbf{A}_{ij}-\mathbf{G}_{ij})-\frac{\mathbf{F}_{ij}-(\mathbf{F}_{ij}\cdot\mathbf{M}_{ij})\mathbf{M}_{ij}}{1-(\mathbf{F}_{ij}\cdot\mathbf{M}_{ij})^2}\cdot(\mathbf{Q}_{ij}-\mathbf{G}_{ij}). \tag{6}$$

The pitches, p_m and p_f, for the moving joint and the fixed joint are defined as follows:

$$u_{ij}=p_m\alpha_{ij} \tag{7}$$

$$w_{ij}=p_f\gamma_{ij}. \tag{8}$$

Different types of joints have special pitches, p, associated with them: For helical joints, p is a nonzero, finite constant; for revolute joints, $p=0$; for cylindric joints, p is a variable; for spherical joints, $p=0$; and for prismatic joints, $p=\infty$.

Before we discuss the linkage synthesis problem, the following general points should be understood by the reader:

(a) Equation (1) is bilinear in F_x, F_y, F_z, M_x, M_y, and M_z.

(b) Equations (1), (2) and (3) depend upon the screw directions only.

(c) Equations (4), (5), and (6) are linear in $Q_x, Q_y, Q_z, G_x, G_y, G_z, u_{ij}$, and w_{ij}.

(d) Since $\mathbf{F}_{ij}$ and $\mathbf{M}_{ij}$ are unit vectors, they must satisfy the following conditions:

$$F_{xij}^2+F_{yij}^2+F_{zij}^2=1 \tag{9}$$

$$M_{xij}^2+M_{yij}^2+M_{zij}^2=1. \tag{10}$$

(e) Since G_{ij} and Q_{ij} are arbitrary points on the fixed and moving-joint axes respectively, we may choose them as the points where the fixed and moving-joint axes intersect the x-y plane, i.e. set $G_{zij}=Q_{zij}=0$. Thereby we concern ourselves with only two parameters for each point (for spherical joints, we will define G_{ij} and Q_{ij} differently).

(f) The jth position of the binary link can be obtained by screwing the binary link (from its ith position) about the fixed-joint axis an amount (γ_{ij}, w_{ij}). Analytically, we have

$$\begin{pmatrix}M_{xjk}\\M_{yjk}\\M_{zjk}\end{pmatrix}=\begin{pmatrix}(a_{xij}+1) & b_{xij} & c_{xij}\\ a_{yij} & (b_{yij}+1) & c_{yij}\\ a_{zij} & b_{zij} & (c_{zij}+1)\end{pmatrix}\begin{pmatrix}M_{xij}\\M_{yij}\\M_{zij}\end{pmatrix} \tag{11}$$

$$\begin{pmatrix}Q_{xjk}\\Q_{yjk}\\Q_{zjk}\end{pmatrix}=\begin{pmatrix}(a_{xij}+1) & b_{xij} & c_{xij}\\ a_{yij} & (b_{yij}+1) & c_{yij}\\ a_{zij} & b_{zij} & (c_{zij}+1)\end{pmatrix}\begin{pmatrix}Q_{xij}\\Q_{yij}\\Q_{zij}\end{pmatrix}+\begin{pmatrix}d_{xij}\\d_{yij}\\d_{zij}\end{pmatrix} \tag{12}$$

where

$$\begin{aligned}a_{xij}&=(F_{xij}^2-1)(1-\cos\gamma_{ij})\\ b_{xij}&=F_{xij}F_{yij}(1-\cos\gamma_{ij})-F_{zij}\sin\gamma_{ij}\\ c_{xij}&=F_{xij}F_{zij}(1-\cos\gamma_{ij})+F_{yij}\sin\gamma_{ij}\\ d_{xij}&=w_{ij}F_{xij}-G_{xij}a_{xij}-G_{yij}b_{xij}-G_{zij}c_{xij}\end{aligned} \tag{13}$$

and so on[6].

(g) Since the line of the screw axis $\overline{F}_{ij}$ is fixed in the reference frame, except for spherical joints, both $\mathbf{F}_{ij}$ and $\mathbf{G}_{ij}$ do not depend upon i and j. Hence,

$$\mathbf{F}_{ij} = \mathbf{F}(F_x, F_y, F_z) \text{ for all } i \text{ and } j \tag{14}$$

$$\mathbf{G}_{ij} = \mathbf{G}(G_x, G_y, G_z) \text{ for all } i \text{ and } j \tag{15}$$

(h) If we formulate finitely separated position problems by screw displacements from the 1st position, $\widehat{S}_{1j}$, $j = 2, 3, 4, \ldots$, the vectors denoting the moving-joint axis in equations (1)–(6) will always be in their first position. This then allows us to simplify the notation:

$$\mathbf{M}_{ij} = \mathbf{M}_{1j} = \mathbf{M}(M_x, M_y, M_z) \text{ for all } j \tag{16}$$

$$\mathbf{Q}_{ij} = \mathbf{Q}_{1j} = \mathbf{Q}(Q_x, Q_y, Q_z) \text{ for all } j \tag{17}$$

(i) For a spherical joint, we specify $\mathbf{G}^*(G_x, G_y, G_z)$ [and/or $\mathbf{Q}^*(Q_x, Q_y, Q_z)$] to be the point at the center of the spherical joint. The axis direction, $\mathbf{F}_{ij}$, (and/or $\mathbf{M}_{ij}$) about which the system is rotated from position "i" to position "j" (see Figs. 3 and 4) is now not defined by the joint geometry. Hence, we need three parameters to specify $\mathbf{G}^*$ (and/or $\mathbf{Q}^*$); and we have different axis directions for each j. (The notation "*" is used simply to remind the reader that in this case the quantity so denoted depends upon three parameters.)

(j) In the case of two spherical joints connected together, it is convenient to neglect the component of $\overline{F}_{ij}$ along $\mathbf{G}^*\mathbf{Q}^*$ since this will not affect the displacement of σ (Fig. 5), and merely adds an extra degree-of-freedom which is trivial. Hence we will introduce the requirement that

$$(\mathbf{Q}^* - \mathbf{G}^*) \cdot \mathbf{F}_{ij} = 0. \tag{18}$$

(k) For each specified displacement, $\widehat{S}_{ij}$, equation (2) yields one equation in one

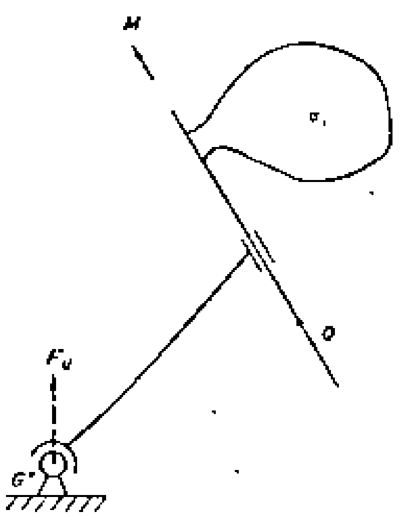


Figure 3. A fixed spherical joint.

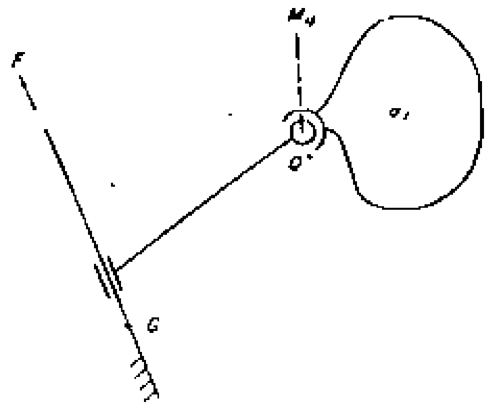


Figure 4. A moving spherical joint.

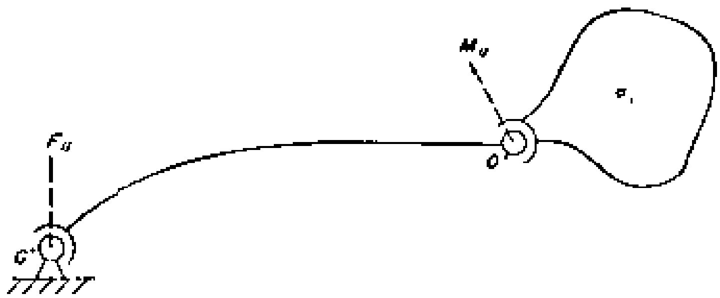


Figure 5. *S*-*S* crank.

unknown, α_{ij}, other than F and M. Hence, in solving synthesis problems (except for a helical joint where α_{ij} is related to u_{ij}) we can disregard these equations, since they simply give the rotation angles about the moving-joint axis. However, if we are interested in controlling these rotation angles, then we must include equation (2) in solving the synthesis problem.

(l) The same argument holds for equation (3).

Design of Binary Links

Equations (1)–(10) are the necessary conditions which must be satisfied by any binary link with combinations of helical, cylindrical, revolute, spherical, and prismatic joints. In applying these equations to the linkage synthesis problems, the screws S_{ij} (i.e. $\mathbf{S}_{ij}$, $\mathbf{A}_{ij}$, θ_{ij}, t_{ij}, $i=1$, $j=2,3,\ldots,n$) are known, and the link parameters (i.e., F, G, M, Q, α_{ij}, γ_{ij}, u_{ij}, w_{ij}) need to be determined. It is most convenient to first determine the axes directions and locations using as few equations as possible, and then obtain the rotational angles and translational distances from the remaining equations.

Once F, G, M, and Q are known the design dimensions of the binary link can be obtained. For example: The shortest distance between link axes is given by

$$\left|\frac{(\mathbf{Q}-\mathbf{G})\cdot(\mathbf{M}\times\mathbf{F})}{(\mathbf{M}\times\mathbf{F})}\right|$$

and twist in the link is

$$\tan^{-1}\left|\frac{(\mathbf{M}\times\mathbf{F})}{(\mathbf{M}\cdot\mathbf{F})}\right|$$

measured in the sense of M toward F about the directed line M × F. In the case of one or more spherical joints there is no twist in the binary link, and the shortest distance between the joints is given by $|(\mathbf{Q}-\mathbf{G}^*)\cdot\mathbf{MM}-(\mathbf{Q}-\mathbf{G}^*)|$ or $|(\mathbf{G}-\mathbf{Q}^*)\cdot\mathbf{FF}-(\mathbf{G}-\mathbf{Q}^*)|$ or $|\mathbf{G}^*-\mathbf{Q}^*|$ depending respectively upon whether the case is represented by Figs. 3, 4 or 5.

In what follows, we list the appropriate design equations and unknown parameters for each possible combination of joints. Since all binary links with cylindrical, revolute, spherical, and prismatic joints have been previously studied (see Refs. [1], [2], [4] and [5]), we give detailed discussions of only those cases where a helical joint is employed. In each case we consider only the maximum possible number of design positions. For fewer positions the method of solution is analogous, except that the designer is free to make arbitrary choices of some parameters.

1. *H-H cranks**

Design equations: (1)–(8).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y, p_f, p_m, w_{1j}, \gamma_{1j}, u_{1j}$, and α_{1j}.†

Maximum number of design positions: 3 ($j = 2, 3$).

Method of solution:

Equations (1)–(8) written twice yield 16 equations in 18 unknowns. Hence, we are at liberty to choose any two of the 18 unknowns. It is most convenient to choose the direction of either the moving-axis or the fixed-axis. Once the direction of the moving-axis (or the fixed-axis) is chosen, the direction of the fixed-axis (or the moving-axis), which is in (1,1) correspondence, can be determined from equations (1) (as well as the condition on a unit vector). Rotation angles, α_{1j} and γ_{1j}, can then be obtained from equations (2) and (3).

Substituting equations (7) into (5), (8) into (6), with $j = 2, 3$, and then eliminating p_m and p_f between the $j = 2$ and $j = 3$ equations yields:

$$\alpha_{12}\left\{\frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{F})\mathbf{F}}{1-(\mathbf{M}\cdot\mathbf{F})^2}\cdot(\mathbf{G}-\mathbf{Q})-\frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{S}_{13})\mathbf{S}_{13}}{1-(\mathbf{M}\cdot\mathbf{S}_{13})^2}\cdot(\mathbf{A}_{13}-\mathbf{Q})\right\}$$
$$=\alpha_{13}\left\{\frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{F})\mathbf{F}}{1-(\mathbf{M}\cdot\mathbf{F})^2}\cdot(\mathbf{G}-\mathbf{Q})-\frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{S}_{12})\mathbf{S}_{12}}{1-(\mathbf{M}\cdot\mathbf{S}_{12})^2}\cdot(\mathbf{A}_{12}-\mathbf{Q})\right\} \tag{19}$$

$$\gamma_{12}\left\{\frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{S}_{13})\mathbf{S}_{13}}{1-(\mathbf{F}\cdot\mathbf{S}_{13})^2}\cdot(\mathbf{A}_{13}-\mathbf{G})-\frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{M})\mathbf{M}}{1-(\mathbf{F}\cdot\mathbf{M})^2}\cdot(\mathbf{Q}-\mathbf{G})\right\}$$
$$=\gamma_{13}\left\{\frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{S}_{12})\mathbf{S}_{12}}{1-(\mathbf{F}\cdot\mathbf{S}_{12})^2}\cdot(\mathbf{A}_{12}-\mathbf{G})-\frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{M})\mathbf{M}}{1-(\mathbf{F}\cdot\mathbf{M})^2}\cdot(\mathbf{Q}-\mathbf{G})\right\}. \tag{20}$$

Equations (19) and (20) together with equations (4) are a set of 4 linear equations in the unknowns G_x, G_y, Q_x and Q_y. Hence, we can easily determine the axes locations

*Following the terminology used in [2] a binary link is called a crank if we assume one end jointed to the reference system. The first letter denotes the joint along F and the second, the type of joint along M.

†A parameter bearing a subscript j should be counted as j-1 unknowns. In equations (1)–(6) F (and M) will always be counted as two parameter vectors, since their third components are determined by equations (9) and (10) respectively.

(by, for example, employing Cramer's rule). The translational distances, u_{1j} and w_{1j}, then follow from equations (5) and (6) (since F, M, G and Q are known). Finally, the pitches, p_m and p_f, are obtained from equations (7) and (8).

Since the rotation angles are obtained from equations (2) and (3) by use of inverse trigonometric functions, corresponding to each set of axis direction there exist infinitely many values of α_{1j} and γ_{1j}. These, in turn, yield infinitely many different axis locations and pitches all of this satisfy the same three position problem. Note also that once the moving and fixed-axis are determined, any of the following pitches could theoretically be used instead of p_m and p_f:

$$p'_m = \frac{u_{1j}}{\alpha_{1j} + 2k\pi}, \qquad k = 1, 2, 3, \ldots \tag{21}$$

$$p'_f = \frac{w_{1j}}{\gamma_{1j} + 2k\pi}, \qquad k = 1, 2, 3, \ldots \tag{22}$$

Practically speaking, we usually deal with $|\alpha_{1j}| < 2\pi$, $|\gamma_{1j}| < 2\pi$ and $k = 0$. In general, there will be no *H-H* links which can be used to help guide a body through four arbitrarily specified positions.

2. *H-C cranks*

Design equations: (1), (3), (4), (6) and (8).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y, p_f, \gamma_{1j}$ and w_{1j}.

Additional information*: α_{1j} from equations (2); u_{1j} from equations (5).

Maximum number of design positions: 4 ($j = 2, 3, 4$)

Method of solution:

For three positions

Equations (1), (3), (4), (6) and (8) written twice yield 10 equations in 13 unknowns. Hence, there are three free parameters among the 13 unknowns. It is most convenient to choose F_x, F_y and p_f (or M_x, M_y, and p_f), and solve for the other parameters. The solution is then quite straightforward.

For four positions we require a numerical iteration.

3. *C-H cranks*

Design equations: (1), (2), (4), (5) and (7).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y, p_m, \alpha_{1j}$ and u_{1j}.

Additional information: γ_{1j} from equations (3); w_{1j} from equations (6).

Maximum number of design positions and methods of solution for *C-H* cranks are the same as for *H-C* cranks.

4. *H-S cranks*

Design equations: (1), (3), (4), (5), (6) and (8).

Design parameters: $F_x, F_y, M_{x1j}, M_{y1j}, G_x, G_y, Q_x^*, Q_y^*, Q_z^*, p_f, \gamma_{1j}, w_{1j}$, ($u_{1j} = 0$).

Maximum number of design positions: 5 ($j = 2, 3, 4, 5$)

Method of solution:

First we discuss the three position problem ($j = 2, 3$); equations (1), (3), (4), (5), (6) and (8) written twice yield 12 equations in 16 unknowns. Hence, there are 4 parameters

*Additional information refers to parameters which are not required to complete the synthesis, but may be of interest in analyzing a given design.

which may be specified arbitrarily. It is most convenient to choose the direction of the fixed axis (F_x, F_y) and the rotation angles $(\gamma_{12}, \gamma_{13})$ about it.

For each choice of $(F_x, F_y, \gamma_{12}, \gamma_{13})$, equations (1) and (3) [with conditions (9) and (10)] yield a unique set of moving-axis direction cosines. Equations (4) and (5), written twice, together with equation (20) yield 5 linear equations in Q_x^*, Q_y^*, Q_z^*, G_x and G_y. Hence, corresponding to each set of axis-directions (F, M_{ij}), there is a unique axis-location. Finally, the displacements along the fixed axis can be found from equation (6), and the pitch, p_f, is given by equation (7).

For the four-position problems $(j = 2, 3, 4)$, equations (1), (3), (4), (5), (6) and (8) written three times yield 18 equations in 20 unknowns. Hence, there are now only two free choices. The solution is now more complicated since equations (1) and (3) must be solved simultaneously with (4), (5), (6) and (8). This seems to require a numerical iteration. Five positions will also require a numerical iteration.

5. *S-H cranks*

Design equations: (1), (2), (4), (5), (6) and (7).

Design parameters: $F_{x1j}, F_{y1j}, M_x, M_y, G_x^*, G_y^*, G_z^*, Q_x, Q_y, p_m, \alpha_{1j}, u_{1j}$ $(w_{1j} = 0)$.

The design procedure is in complete analogy to that outlined for the *H-S* cranks.

6. *H-R cranks*

Design equations: (1), (3), (4), (5), (6) and (8).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y, p_f, \gamma_{1j}, w_{1j}$ $(u_{1j} = 0)$.

Maximum number of design positions: 3 $(j = 2, 3)$

Method of solution:

Equations (1), (3), (4), (5), (6) and (8) written twice yield 12 equations in 13 unknowns. Hence, there is one free choice among these 13 unknowns. The only direct methods of solution seem to require numerical iterations. However a simpler approach is possible if we use the fact that the design of *H-R* cranks may be obtained from *H-H* cranks with $p_m = 0$.

An additional simplified solution is possible using a spatial version of the well-known point-position reduction technique. If we select the (fixed) helical-joint screw $\hat{F}$ to be concurrent with $\hat{S}_{12}$ then (1), (3), (4) and (6) are satisfied by any $\hat{M}_{ij}$ for $i = 1$, $j = 2$, and (2) and (5) are identically zero. We are left with four equations for M and Q. [The equations are (1), (4), (5) and (3) combined with (6), using the condition that $p_f = t_{13}/\theta_{13}$, with $i = 1, j = 3$].

7. *R-H cranks*

Design equations: (1), (2), (4), (5), (6) and (7).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y, p_m, \alpha_{1j}, u_{1j}$ $(w_{1j} = 0)$.

The synthesis (and maximum number of design positions) of *R-H* cranks is analogous to that of *H-R* cranks, except that now $p_f = 0$ and $p_m \neq 0$.

8. *C-C cranks*

Design equations: (1) and (4).

Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x$ and Q_y.

Maximum number of design positions: 5 $(j = 2, 3, 4, 5)$

First we consider three positions $(j = 2, 3)$: equations (1) and (4) written twice yield 4 equations in 8 unknowns. Hence, we have four appropriate free choices for these 8 unknowns. If both the direction and location of the fixed axis are arbitrarily chosen, equation (1) written twice [together with equation (10)] yields a unique result for the moving-axis cosines; and equation (4) written twice yields a unique result for the axis location. As previously pointed out[1], there is a (1,1) correspondence between the moving and the fixed axes.

In this case, however, if we are interested in controlling the rotation as well as the translation distances, any two of the four rotations, α_{12}, α_{13}, γ_{12}, γ_{13}, as well as two of the four translations, u_{12}, u_{13}, w_{12}, w_{13}, can be arbitrarily chosen. The axis-direction and axis-location for both the moving and fixed axes are then obtained by solving equations (1)–(6).

For four specified positions ($j = 2, 3, 4$), equation (1) written three times yields three homogeneous linear equations in F_x, F_y and F_z. The condition for non-zero F_x, F_y and F_z to exist yields a third degree algebraic equation in M_x, M_y and M_z. This is the equation of the screw cone studied in Ref.[1]. Equation (4) written three times yields three linear equations in G_x, G_y, Q_x, and Q_y. Hence, corresponding to each direction defined by a generator of the screw cone, there is a single infinity of axis locations.

For five arbitrary positions ($j = 2, 3, 4, 5$), equations (1) and (4) written four times yield 8 equations in 8 unknowns. The unique set of solutions corresponds to those given in[1].

9. *C-R cranks*

Design equations: (1), (4) and (5).
Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x$ and Q_y ($u_{1j} = 0$).
Maximum number of design positions: 3 ($j = 2, 3$)

Equations (1), (4) and (5) written twice yield 6 equations in 8 unknowns. Hence, we have two arbitrary choices. Corresponding to each choice of moving-axis (or fixed-axis) direction cosines, equation (1) yields a unique result for the fixed-axis (or moving-axis) direction cosines, while equations (4) and (5) yield a unique result for the axis-location for both the moving and fixed axes.

10. *R-C cranks*

Design equations: (1), (4) and (6).
Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y$ ($w_{1j} = 0$).
The solution is analogous to that given for the *C-R* cranks.

11. *R-R cranks*

Design equations: (1), (4), (5) and (6).
Design parameters: $F_x, F_y, M_x, M_y, G_x, G_y, Q_x, Q_y$ ($u_{1j} = w_{1j} = 0$).
Maximum number of positions: 3 ($j = 2, 3$)

Equations (1), (4), (5) and (6) written twice yield 8 equations in 8 unknowns. Hence, as has been shown in[2] and[3] there is a finite set of *R-R* cranks for any three positions.

12. *S-C cranks*

Design equations: (1), (4) and (6).
Design parameters: $F_{x1j}, F_{y1j}, M_x, M_y, G_x^*, G_y^*, G_z^*, Q_x, Q_y$ ($w_{1j} = 0$).
Maximum number of design positions: 8 ($j = 2, 3, 4, 5, 6, 7, 8$)

For the eight position problem, equations (1), (4) and (6) written seven times yield 21 equations in 21 unknowns. Hence there are no free choices unless we consider seven or fewer positions.

13. *C-S cranks*

Design equations: (1), (4) and (5).
Design parameters: $F_x, F_y, M_{x1j}, M_{y1j}, G_x, G_y, Q_x^*, Q_y^*, Q_z^*$, ($u_{1j}=0$).
The synthesis of *C-S* cranks is similar to that of *S-C* cranks.

14. *S-S cranks*

Design equations: (1), (4), (5), (6) and (18).
Design parameters: $F_{x1j}, F_{y1j}, M_{x1j}, M_{y1j}, G_x^*, G_y^*, G_z^*, Q_x^*, Q_y^*, Q_z^*$, ($u_{1j}=w_{1j}=0$).
The maximum number of design positions: 7 ($j = 2, 3, 4, 5, 6, 7$).
For the seven-position problem, equations (1), (4), (5), (6) and (18) written six times, yield 30 equations in 30 unknowns.

15. *S-R cranks*

Design equations: (1), (4), (5) and (6).
Design parameters: $F_{x1j}, F_{y1j}, M_x, M_y, G_x^*, G_y^*, G_z^*, Q_x, Q_y$, ($u_{1j}=w_{1j}=0$).
The maximum number of design positions: 4 ($j = 2, 3, 4$).

For four-position problems, equations (1), (4), (5) and (6), written three times yield 12 equations in 13 unknowns. There is a single infinity of solutions for a four position problem.

16. *R-S cranks*

Design equations: (1), (4), (5) and (6).
Design parameters: $F_x, F_y, M_{x1j}, M_{y1j}, G_x, G_y, Q_x^*, Q_y^*, Q_z^*$, ($w_{1j}=u_{1j}=0$).
The solutions of *R-S* cranks are similar to those of *S-R* cranks.

17. *P-C cranks*

For a fixed prismatic joint, we require that $\gamma_{1j}=0$. This implies, from equation (3), that $\mathbf{F}\cdot(\mathbf{S}_{1j}\times\mathbf{M})=0$. Unless σ has a pure translational displacement θ_{1j} and α_{1j} are not generally zero. Hence, in order to have a non-trivial solution, $\mathbf{S}_{1j}\times\mathbf{M}$ must equal zero, i.e., $\mathbf{M}\|\mathbf{S}_{1j}$. In this case the equivalent screw triangle degenerates into the form shown in Fig. 6. We now have degenerate forms of equations (1)-(6). The corresponding degenerate form is given by the original equation number followed by the letter "a" as follows:

$$\tan\frac{\theta_{1j}}{1}=\frac{(\mathbf{A}_{1j}-\mathbf{Q}_{1j})\cdot[\mathbf{F}_{1j}-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})\mathbf{S}_{1j}]}{(\mathbf{A}_{1j}-\mathbf{Q}_{1j})\cdot(\mathbf{S}_{1j}\times\mathbf{F}_{1j})} \tag{1a}$$

$$\alpha_{1j}=\theta_{1j} \tag{2a}$$

$$\gamma_{1j}=0 \tag{3a}$$

$$\frac{t_{1j}}{2}=\frac{\mathbf{S}_{1j}\cdot\mathbf{G}_{1j}-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})\mathbf{F}_{1j}\cdot(\mathbf{G}_{1j}-\mathbf{A}_{1j})}{1-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})^2}-d_{1j} \tag{4a}$$

$$\frac{u_{1j}}{2}=\frac{\mathbf{S}_{1j}\cdot\mathbf{G}_{1j}-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})\mathbf{F}_{1j}\cdot(\mathbf{G}_{1j}-\mathbf{Q}_{1j})}{1-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})^2}-d_{1j} \tag{5a}$$

$$\frac{w_{1j}}{2}=\frac{\mathbf{F}_{1j}-(\mathbf{F}_{1j}\cdot\mathbf{S}_{1j})\mathbf{S}_{1j}}{1-(\mathbf{F}_{1j}\cdot\mathbf{S}_{1j})^2}\cdot(\mathbf{A}_{1j}-\mathbf{Q}_{1j}) \tag{6a}$$

where d_{1j} is an arbitrary constant. Combining equations (4a) and (5a), we obtain

$$\frac{t_{1j}}{2}=\frac{u_{1j}}{2}+\frac{(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})\mathbf{F}_{1j}\cdot(\mathbf{A}_{1j}-\mathbf{Q}_{1j})}{1-(\mathbf{S}_{1j}\cdot\mathbf{F}_{1j})^2}. \tag{A}$$

With these equations we can synthesize *P-C* cranks by using:
Design equations: (1a), (6a), (A).
Design parameters: $F_x, F_y, Q_x, Q_y, u_{1j}, w_{1j}, (\mathbf{M}=\mathbf{S}_{1j})$.
The maximum number of design positions: 2 ($j=2$).

Note that $\mathbf{G}_{1j}$ disappears from equations (1a), (6a) and (A). Hence in choosing a prismatic joint only the direction of the axis is important, since the location, $\mathbf{G}_{1j}$, is always arbitrary.

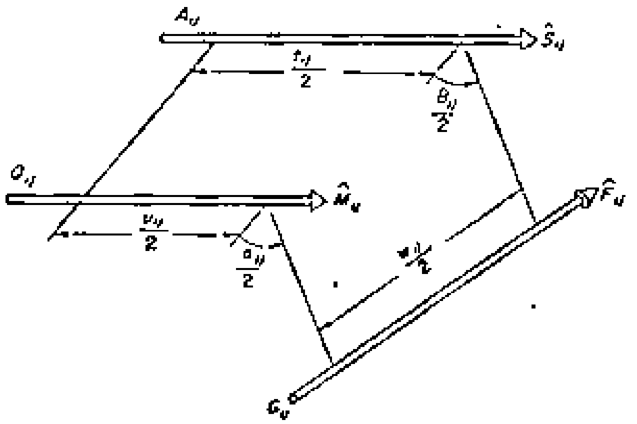


Figure 6. Equivalent screw triangle when the fixed joint is prismatic.

For two-position problems, we have three equations in six unknowns. Hence there are three free parameters. If we choose F_x, F_y, and w_{1j}, equations (1a) and (6a) yield two linear equations in Q_x and Q_y. Equation (A) provides the information for u_{1j}. There will generally be no solutions for three positions, since $\mathbf{M}$ cannot be parallel to both $\mathbf{S}_{12}$ and $\mathbf{S}_{13}$ which are generally skew.

18. *P-R cranks*

Design equations: (1a), (6a), (A).
Design parameters: $Q_x, Q_y, F_x, F_y, w_{1j}, (u_{1j}=0, \mathbf{M}_{1j}=\mathbf{S}_{1j})$.
The maximum number of design positions: 2 ($j=2$).

For two positions we have 3 equations in 5 unknowns. Hence there are two free choices.

19. *P-S cranks*

Design equations: (1a), (A).
Design parameters: $F_x, F_y, Q_x^*, Q_y^*, Q_z^*, (u_{1j}=0, \mathbf{M}_{1j}=\mathbf{S}_{1j})$.
The maximum number of design positions: 3 ($j=2, 3$).
Additional information: w_{1j} from (6a).

Equations (1a) and (A) written twice yield 4 equations in 5 unknowns. Hence there is a single infinity of dyads.

20. *P-H cranks*

Design equations: (1a), (A), and (7).
Design parameters: $F_x, F_y, Q_x, Q_y, u_{ij}, P_w$, $(\alpha_{ij} = \theta_{ij}, M = S_{ij})$.
The maximum number of design positions: 2 ($j = 2$).

For two positions we have 3 equations with 6 unknowns. Hence there are three free parameters.

21. *C-P, R-P, S-P or H-P cranks*

For a moving-prismatic joint, we require that $\alpha_{ij} = 0$, $\mathbf{F}_{ij} \| \mathbf{S}_{ij}$; the "equivalent screw-triangle" degenerates to the form shown in Fig. 7. The degenerate forms of equations (1)-(6) are given by the original equation numbers followed by the letter "*b*" as follows:

$$\tan\frac{\theta_{ij}}{2} = \frac{(\mathbf{A}_{ij}-\mathbf{G}_{ij})\cdot[-\mathbf{M}_{ij}+(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})\mathbf{S}_{ij}]}{(\mathbf{A}_{ij}-\mathbf{G}_{ij})\cdot(\mathbf{S}_{ij}\times\mathbf{M}_{ij})} \tag{1b}$$

$$\alpha_{ij} = 0 \tag{2b}$$

$$\gamma_{ij} = \theta_{ij} \tag{3b}$$

$$\frac{t_{ij}}{2} = -\frac{\mathbf{S}_{ij}\cdot\mathbf{Q}_{ij}-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})\mathbf{M}_{ij}\cdot(\mathbf{Q}_{ij}-\mathbf{A}_{ij})}{1-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})^2} + d_{ij} \tag{4b}$$

$$\frac{u_{ij}}{2} = \frac{\mathbf{M}_{ij}-(\mathbf{M}_{ij}\cdot\mathbf{S}_{ij})\mathbf{S}_{ij}}{1-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})^2}\cdot(\mathbf{G}_{ij}-\mathbf{A}_{ij}) \tag{5b}$$

$$\frac{w_{ij}}{2} = -\frac{\mathbf{S}_{ij}\cdot\mathbf{Q}_{ij}-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})\mathbf{M}_{ij}\cdot(\mathbf{Q}_{ij}-\mathbf{G}_{ij})}{1-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})^2} + d_{ij}. \tag{6b}$$

Combining equations (4b) and (6b), we obtain

$$\frac{t_{ij}}{2} = \frac{w_{ij}}{2} + \frac{(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})\mathbf{M}_{ij}\cdot(\mathbf{G}_{ij}-\mathbf{A}_{ij})}{1-(\mathbf{S}_{ij}\cdot\mathbf{M}_{ij})^2}. \tag{B}$$

Again, $\mathbf{Q}_{ij}$ disappears from equations (1b), (5b), and (B). Equations (1b), (5b) and (B)

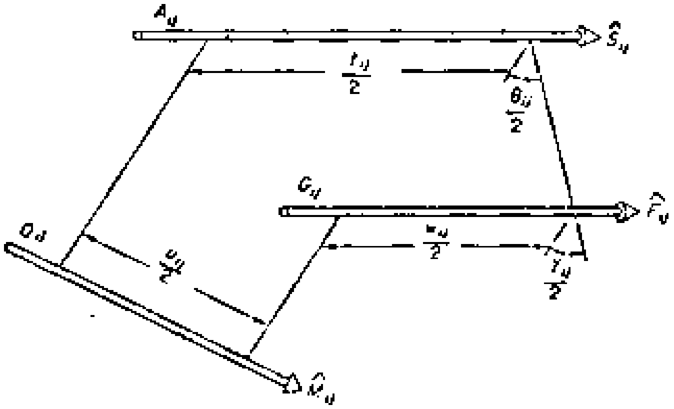


Figure 7. Equivalent screw triangle when the moving joint is prismatic.

are the necessary design equations for *C-P*, *R-P*, *S-P*, and *H-P* cranks. The syntheses of *C-P*, *R-P*, *S-P*, and *H-P* cranks are similar to those of *P-C*, *P-R*, *P-S*, and *P-H* cranks respectively.

Unless σ has a pure translational displacement, there will be no *P-P* crank for two design positions.

Infinitesimally Separated Position Problems

By equating the sum of the angular velocities along the moving and fixed axes to the angular velocity of the rigid body along the instantaneous screw axis, we obtain:

$$\frac{d\alpha}{d\tau}\mathbf{M}+\frac{d\gamma}{d\tau}\mathbf{F}=\frac{d\theta}{d\tau}\mathbf{S}. \tag{23}$$

where τ is the parameter representing time.

If we take θ as the motion parameter and divide (23) by $d\theta/d\tau$, using subscript "i" to denote the position from which the displacement occurs, we obtain:

$$\frac{d\alpha_i}{d\theta}\mathbf{M}_i+\frac{d\gamma_i}{d\theta}\mathbf{F}_i=\mathbf{S}_i. \tag{24}$$

Operating on (24) with $(\mathbf{F}_i\times\mathbf{M}_i)\cdot$ yields

$$\mathbf{S}_i\cdot(\mathbf{F}_i\times\mathbf{M}_i)=0. \tag{25}$$

Operating on (24) with $\mathbf{M}\cdot$ and $\mathbf{F}\cdot$, we can solve for $d\alpha_i/d\theta$ and $d\gamma_i/d\theta$:

$$\frac{d\gamma_i}{d\theta}=-\frac{(\mathbf{S}_i\times\mathbf{M}_i)\cdot(\mathbf{M}_i\times\mathbf{F}_i)}{(\mathbf{M}_i\times\mathbf{F}_i)^2}\text{ or more simply }\frac{d\gamma_i}{d\theta}=\frac{(\mathbf{S}_i\times\mathbf{M}_i)_k}{(\mathbf{F}_i\times\mathbf{M}_i)_k} \tag{26}$$

$$\frac{d\alpha_i}{d\theta}=-\frac{(\mathbf{S}_i\times\mathbf{F}_i)\cdot(\mathbf{F}_i\times\mathbf{M}_i)}{(\mathbf{M}_i\times\mathbf{F}_i)^2}\text{ or more simply }\frac{d\alpha_i}{d\theta}=\frac{-(\mathbf{S}_i\times\mathbf{F}_i)_k}{(\mathbf{F}_i\times\mathbf{M}_i)_k}, \tag{27}$$

where the subscript K denotes the x, y or z component of the vector cross product.

Similarly an expression for the velocity along $\mathbf{S}_i$ can be transformed into a kinematic relation with θ as the motion parameter by dividing by $d\theta/d\tau$:

$$\frac{dt_i}{d\theta}\mathbf{S}_i=\frac{dw_i}{d\theta}\mathbf{F}_i+\frac{d\gamma_i}{d\theta}\mathbf{F}_i\times(\mathbf{A}_i-\mathbf{G}_i)+\frac{du_i}{d\theta}\mathbf{M}_i+\frac{d\alpha_i}{d\theta}\mathbf{M}_i\times(\mathbf{A}_i-\mathbf{Q}_i). \tag{28}$$

Operating on (28) with $(\mathbf{F}_i\times\mathbf{M}_i)\cdot$, and eliminating $d\gamma_i/d\theta$ and $d\alpha_i/d\theta$, we obtain

$$(\mathbf{S}_i\times\mathbf{M}_i)\cdot(\mathbf{F}_i\times\mathbf{M}_i)(\mathbf{F}_i\times\mathbf{M}_i)\cdot[\mathbf{F}_i\times(\mathbf{A}_i-\mathbf{G}_i)]$$
$$+(\mathbf{S}_i\times\mathbf{F}_i)\cdot(\mathbf{M}_i\times\mathbf{F}_i)(\mathbf{F}_i\times\mathbf{M}_i)\cdot[\mathbf{M}_i\times(\mathbf{A}_i-\mathbf{Q}_i)]=0. \tag{29}$$

Operating on (28) with $\mathbf{M}_i\cdot$ and $\mathbf{F}_i\cdot$, and eliminating $d\alpha_i/d\theta$ and $d\gamma_i/d\theta$, we can solve for $du_i/d\theta$ and $dw_i/d\theta$:

$$\frac{du_i}{d\theta}=\frac{1}{(\mathbf{F}_i\times\mathbf{M}_i)^2}\Bigg\{-\frac{dt_i}{d\theta}(\mathbf{S}_i\times\mathbf{F}_i)\cdot(\mathbf{F}_i\times\mathbf{M}_i)-\frac{(\mathbf{S}_i\times\mathbf{M}_i)\cdot(\mathbf{F}_i\times\mathbf{M}_i)}{(\mathbf{F}_i\times\mathbf{M}_i)^2}\mathbf{M}_i$$
$$\cdot[\mathbf{F}_i\times(\mathbf{A}_i-\mathbf{G}_i)]+\frac{(\mathbf{S}_i\times\mathbf{F}_i)\cdot(\mathbf{M}_i\times\mathbf{F}_i)}{(\mathbf{F}_i\times\mathbf{M}_i)^2}(\mathbf{F}_i\cdot\mathbf{M}_i)\mathbf{F}_i\cdot\mathbf{M}_i\times(\mathbf{A}_i-\mathbf{Q}_i)\Bigg\} \tag{30}$$

$$\frac{dw_i}{d\theta} = \frac{1}{(\mathbf{F}_i \times \mathbf{M}_i)^2}\left\{-\frac{dt_i}{d\theta}(\mathbf{S}_i \times \mathbf{M}_i)\cdot(\mathbf{M}_i \times \mathbf{F}_i)\right.$$

$$+\frac{(\mathbf{S}_i \times \mathbf{M}_i)\cdot(\mathbf{F}_i \times \mathbf{M}_i)}{(\mathbf{F}_i \times \mathbf{M}_i)^2}\cdot(\mathbf{F}_i\cdot\mathbf{M}_i)\mathbf{M}_i\cdot[\mathbf{F}_i \times (\mathbf{A}_i - \mathbf{G}_i)]$$

$$\left.-\frac{(\mathbf{S}_i \times \mathbf{F}_i)\cdot(\mathbf{M}_i \times \mathbf{F}_i)}{(\mathbf{F}_i \times \mathbf{M}_i)^2}\mathbf{F}_i\cdot[\mathbf{M}_i \times (\mathbf{A}_i - \mathbf{Q}_i)]\right\}. \tag{31}$$

(Slightly less convenient, but lower order, forms of (29)–(31) can be obtained by using the simpler forms of (26) and (27).)

Equations (25)–(27), and (29)–(31) are analogous to (1)–(3), and (4)–(6) respectively. For helical joints, equations (7) and (8) become respectively:

$$\frac{du_i}{d\theta} = p_m\frac{d\alpha_i}{d\theta}, \tag{32}$$

$$\frac{dw_i}{d\theta} = p_i\frac{d\gamma_i}{d\theta} \tag{33}$$

For k_i infinitesimally separated positions from position "i", the design equations are (25)–(27), (29)–(31), (32), (33) and the $k_i - 1$ derivatives of these equations with respective to θ. For example, the design equation from (26) would be

$$\frac{d^n\gamma_i}{d\theta^n} = \frac{d^{n-1}}{d\theta^{n-1}}\left\{-\frac{(\mathbf{S}_i \times \mathbf{M}_i)\cdot(\mathbf{M}_i \times \mathbf{F}_i)}{(\mathbf{F}_i \times \mathbf{M}_i)^2}\right\}, \quad n = 1, 2, \ldots, k_i. \tag{34}$$

In the derivative equations we consider $d^{n-1}S_i/d\theta^{n-1}$, $d^{n-1}\mathbf{A}_i/d\theta^{n-1}$, $d^n t/d\theta^n$, $n = 1, 2, \ldots, k_i$, as specified quantities which describe the k_i infinitesimally separated position of σ_i. All other derivatives on the right-hand-side can be expressed in terms of $\mathbf{F}_i$, $\mathbf{M}_i$, $\mathbf{G}_i$ and $\mathbf{Q}_i$ using the following:

$$\frac{d\mathbf{M}_i}{d\theta} = \frac{\partial\mathbf{M}_i}{\partial\gamma}\left(\frac{d\gamma_i}{d\theta}\right) + \frac{\partial\mathbf{M}_i}{\partial\alpha}\left(\frac{d\alpha_i}{d\theta}\right) = (\mathbf{F}_i \times \mathbf{M}_i)\frac{d\gamma_i}{d\theta} + \frac{\partial\mathbf{M}_i}{\partial\alpha}\left(\frac{d\alpha_i}{d\theta}\right) \tag{35}$$

$$\frac{d^2\mathbf{M}_i}{d\theta^2} = \mathbf{F}_i \times (\mathbf{F}_i \times \mathbf{M}_i)\left(\frac{d\gamma_i}{d\theta}\right)^2 + \frac{d\mathbf{F}_i}{d\gamma} \times \mathbf{M}_i\left(\frac{d\gamma_i}{d\theta}\right)^2 + 2\mathbf{F}_i \times \frac{\partial\mathbf{M}_i}{\partial\alpha}\left(\frac{d\alpha_i}{d\theta}\right)\left(\frac{d\gamma_i}{d\theta}\right)$$

$$+\frac{\partial\mathbf{M}_i}{\partial\alpha}\frac{d^2\alpha_i}{d\theta^2} + \frac{\partial^2\mathbf{M}_i}{\partial\alpha^2}\left(\frac{d\alpha_i}{d\theta}\right)^2 + (\mathbf{F}_i \times \mathbf{M}_i)\frac{d^2\gamma_i}{d\theta^2} \tag{36}$$

etc. . . . .

$$\frac{d\mathbf{Q}_i}{d\theta} = \mathbf{F}_i \times (\mathbf{Q}_i - \mathbf{G}_i)\frac{d\gamma_i}{d\theta} + \frac{dw_i}{d\theta}\mathbf{F}_i \tag{37}$$

$$\frac{d^2\mathbf{Q}_i}{d\theta^2} = \mathbf{F}_i \times [\mathbf{F}_i \times (\mathbf{Q}_i - \mathbf{G}_i)]\left(\frac{d\gamma_i}{d\theta}\right)^2 + \frac{d^2w_i}{d\theta^2}\mathbf{F}_i + \frac{d\mathbf{F}_i}{d\gamma} \times (\mathbf{Q}_i - \mathbf{G}_i)\left(\frac{d\gamma_i}{d\theta}\right)^2$$

$$+\frac{dw_i}{d\theta}\left(\frac{d\gamma_i}{d\theta}\right)\frac{d\mathbf{F}_i}{d\gamma} + \mathbf{F}_i \times (\mathbf{Q}_i - \mathbf{G}_i)\frac{d^2\gamma}{d\theta^2} \tag{38}$$

etc. . . . .

where we consider $F_i = F_i(\gamma)$, $M_i = M_i(\alpha, \gamma)$, $w = w(\theta)$,

$$Q_i = Q_i(\gamma, w),\ G_i = \text{constant},\ \gamma_i = \gamma_i(\theta),\ \text{and}\ \alpha_i = \alpha_i(\theta).$$

If the fixed joint is not a spherical joint then $(d^nF_i/d\gamma^n) = 0$, $n = 1, 2, \ldots, k_i - 1$. When the fixed joint is a spherical joint we have $G_i = G_i^*$, but now F_i, $dF_i/d\gamma, \ldots d^{(k_i-1)}F_i/d\gamma^{(k_i-1)}$ are k_i independent vectors. Similarly, if the moving joint is not spherical $(\partial^nM_i/\partial\alpha^n) = 0$, but if it is M_i, and $\partial^nM_i/\partial\alpha^n$, $n = 1, 2, \ldots, k_i - 1$ are k_i independent vectors, and $Q_i = Q_i^*$ is a three parameter vector. When both joints are spherical we must also use the derivatives of equation (18).

For infinitesimally separated positions the results are completely analogous to those given for finite displacements. For mixed finitely and infinitesimally separated positions a suitable combination of equations (1)–(10), (18), and the derivative equations provide the necessary equations. The synthesis procedure and results are again analogous to the corresponding finite position problems.

In[7] an alternative formulation using the screw triangle and limit reasoning is presented. The same method could have been used in this paper. Similarly, the derivation given in this paper could also have been used to obtain the design equations associated with triads[7].

The choice of θ as motion parameter is arbitrary, but seems to be the most convenient. Any one of the other parameters could have been used as a motion parameter. In[7], γ and not θ was used. If we wished to use γ in this paper, we could, following exactly the method we used here except that we would divide by $d\gamma/d\tau$ instead of $d\theta/d\tau$. If we used γ then $(d^n\theta_i/d\gamma^n)$, $n = 1, 2, \ldots, k_i$ should be considered as unknowns and not as specified quantities (as was mistakenly stated in[7]). Similarly, θ could have been used in[7]. Whether θ, γ, or any other parameter is used as the motion parameter is unimportant, and any one may prove more convenient in a given application.

Numerical Example

In order to illustrate the theory we give a numerical example of the design of *H-H* cranks in three finitely separated positions. The method used has been outlined in

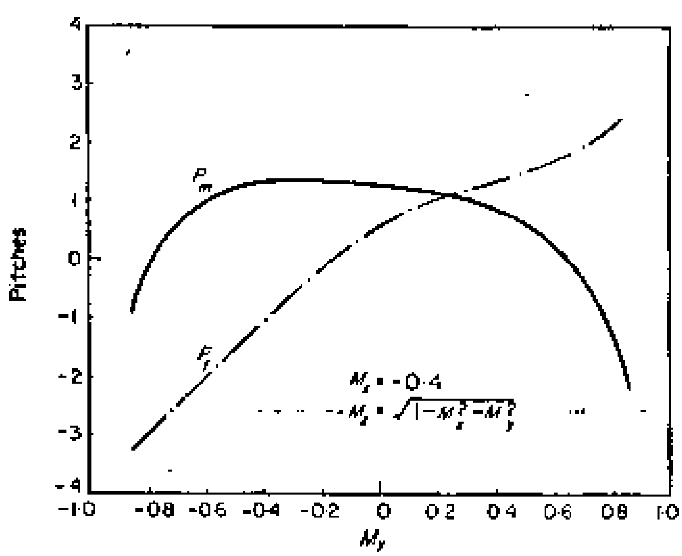


Figure 8. Pitches vs. M_y curve.

Table 1. Dimensions of the *H-H* cranks

Moving-Joint					Fixed-Joint								
Direction $M_z = -0.4$ $M_y = \sqrt{(1-M_z^2-M_x^2)}$	Location $(Q_z = 0)$		Displacements along the axis		Pitch	Direction			Location $(G_z = 0)$		Displacements along the axis		Pitch
M_x	Q_x	Q_y	u_{12}	u_{13}	P_m	F_x	F_y	F_z	G_x	G_y	w_{12}	w_{13}	P_f
−0·85	−12·02	1·017	−0·785	−0·530	−0·9580	−0·0719	−0·4321	−0·8990	−0·0744	2·233	7·593	3·834	−3·269
−0·70	−8·903	1·600	0·497	0·362	0·6224	−0·1303	−0·4558	−0·8805	0·4890	2·550	5·187	2·481	−2·440
−0·50	−6·540	2·311	0·873	0·725	1·209	−0·2267	−0·4312	−0·8733	1·182	2·452	3·066	1·364	−1·533
−0·30	−5·027	2·855	0·709	0·773	1·341	−0·3237	−0·3319	−0·8860	1·610	1·804	1·188	0·459	−0·5897
−0·10	−4·017	3·607	0·265	0·651	1·320	−0·3691	−0·1758	−0·9126	1·426	0·9129	−0·572	−0·222	0·2611
0·10	−3·303	4·728	−0·220	0·434	1·235	−0·3340	−0·0368	−0·9418	0·7466	0·3734	−2·042	−0·796	0·8334
0·30	−2·628	6·441	−0·482	0·183	1·023	−0·2496	0·0133	−0·9678	−0·0039	0·3607	−3·163	−1·306	1·190
0·50	−2·206	9·120	−0·394	0·009	0·618	−0·1549	0·0407	−0·9871	−0·6463	0·7196	−4·101	−1·843	1·497
0·70	−1·487	13·75	0·241	0·041	−0·336	−0·0646	0·0077	−0·9979	−0·9404	1·507	−5·245	−2·520	1·944
0·85	−4·072	25·51	1·598	0·458	−2·112	0·0083	−0·0410	−0·9991	−1·106	2·367	−6·631	−3·257	2·581

our discussion of H-H cranks. These results were obtained with the assistance of a FORTRAN IV (IBM 360) program, using the three positions specified in [2] (on page 72) as design positions. The dimensions of several synthesized H-H cranks are given in Table 1 and the variation of the pitches are plotted in Fig. 8. In this example the two arbitrary choices are the direction cosines of the moving axis. Here we hold M_x at $-0{\cdot}4$, and vary M_y from $-0{\cdot}85$ to $+0{\cdot}85$. All the H-H cranks listed will guide a body through the three specified design positions.

Conclusions

The advantage of the method given in this paper is that it allows for different types of joints to be synthesized from the same design equations. These include the helical pairs which are difficult to treat by other methods. This method can also be generalized to more complex linkage chains[7] and to problems whereby the positions of the guided body are not completely specified.

Acknowledgement — This research was supported in part by the NSF under grant NSF-GK-13930.

References

[1] ROTH B., On the screw axes and other special lines associated with spatial displacements of a rigid body, *Trans. ASME, J. Engng Ind.* 89B, 102–110 (1967).

[2] ROTH B., The design of binary cranks with revolute, cylindric, and prismatic joints, *J. Mechanisms* 3, 61–72 (1968).

[3] ROTH B., Finite-position theory applied to mechanism synthesis, *Trans. ASME, J. Appl. Mech.* 34E, 599–605 (1967).

[4] CHEN P. and ROTH B., A unified theory for the finitely and infinitesimally separated position problems of kinematic synthesis, *Trans. ASME, J. Engng Ind.* 91B, 203–208 (1969).

[5] CHEN P. and ROTH B., Design equations for the finitely and infinitesimally separated position synthesis of binary links and combined link chains, *Trans. ASME, J. Engng Ind.* 91B, 209–219 (1969).

[6] ROTH B., The kinematics of motion through finitely separated positions, *Trans. ASME, J. Appl. Mech.* 34, 591–598 (1967).

[7] TSAI L. W. and ROTH B., Design of triads using the screw triangle chain, *Proceedings of the III World Congress, IFToMM*, (Volume D, pp. 273–286), Dubrovnik, Yugoslavia, September (1971).

[8] WILSON J. T., Analytical kinematic synthesis by finite displacements, *Trans. ASME, J. Engng Ind.* 87B, 161–169 (1964)

[9] SUH C. H., Design of space mechanisms for rigid body guidance, *Trans. ASME, J. Engng Ind.* 89B, 215–222 (1967).

[10] SUH C. H., Design of space mechanisms for function generation, *ibid*, pp. 507–512.

[11] SANDOR G. N. and BISSHOPP K. E., On a general quaternion-operator method of spatial kinematic synthesis by means of a stretch rotation tensor, *Trans. ASME, J. Engng Ind.* 91B, 115–121 (1969).

[12] BRÁT V., Maticová metoda postupných oprav v syntéze prostorových mechanismu, *Strojnícky Časopis* XXI, C. 1, 43–56 (1969).

[13] GUPTA V. K., Optimum finite displacement synthesis of spatial linkages, Ph.D. Dissertation, U. C. Berkeley, June 1970, 100 pp

[14] BAGCI C., Minimum error synthesis of space mechanisms for the generation of constrained and underconstrained screws, *Trans. ASME, J. Engng Ind.* 93B, 165–175 (1971).

[15] LEVITSKII N. I. and SARKISIAN Y., On the special properties of Lagrange's multipliers in the least-square synthesis of mechanisms, *J. Mechanisms* 3, 3–10 (1968).

Mechanism and Machine Theory, 1973, Vol. 8, pp. 23–31. Pergamon Press. Printed in Great Britain

A Note on the Design of Revolute-Revolute Cranks

Lung-Wen Tsai*

and

Bernard Roth*

Received 15 September 1972

Abstract

The design equations for Revolute-Revolute cranks are reduced to a single cubic polynomial in one unknown. From this it is shown that there is generally one and only one pair of Revolute-Revolute cranks compatible with a set of three finitely separated, spatial, design positions.

Introduction

THE DESIGN of Revolute-Revolute cranks to help guide a rigid body through a series of finitely separated positions has been studied extensively[1, 2, 4–6]. It has been shown that, in general, the maximum number of design positions is three with no free choice of design parameters. Roth showed that for "three finitely separated positions" there could be no more than 24 Revolute-Revolute cranks[4, 5] while Suh discovered that the Revolute-Revolute cranks always exist in pairs. Moreover each pair of Revolute-Revolute cranks form a Bennet four-bar mechanism[2].

Veldkamp[3] considered the equivalent of three infinitesimally separated positions and showed there are, in general, at most two Revolute-Revolute cranks. He called the moving revolute axis an h-line, and showed that for every motion there exists to the second order a set of two h-lines. These two h-lines as well as their corresponding center axes form a Bennett isogram.

In this paper we use the idea of "equivalent screw triangle" introduced in [1] to study the design of Revolute-Revolute cranks for three finitely separated positions. By following an algebraic analysis similar to Veldkamp's[3] the problem is reduced to finding the roots of a single third degree polynomial. It is shown that there is, in general, one and only one pair of Revolute-Revolute cranks compatible with an arbitrary set of three finitely separated positions of a rigid body.

Design of R-R Cranks for Three Finitely Separated Positions

Nomenclature

A Revolute-Revolute crank connecting a moving body σ to ground is shown in Fig. 1. We call the axis attached to σ the moving axis and the axis attached to ground

*Department of Mechanical Engineering, Stanford University, Stanford, California, U.S.A.

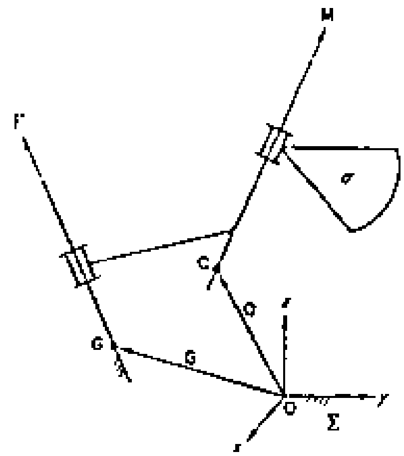


Figure 1. Revolute-Revolute crank notation.

the fixed axis. We denote the following:

M a unit vector parallel to the first position of the moving-joint axis,
Q an arbitrary point on the moving-joint axis in its first position,
Q the position vector from the origin to point Q,
F a unit vector parallel to the fixed-joint axis,
G an arbitrary point on the fixed-joint axis, and
G the position vector from the origin to point G.

It is important to note that the four vectors F, G, M and Q are defined so that they each are determined by two and not three scalars.

Design Equations

For simplicity, we formulate the problem in terms of screw displacements from the first position, i.e. using screws $\hat{S}_{1j}, j = 2, 3$. The design equations are as given in [1]:

$$\tan\frac{1}{2}\theta_{1j} = -\frac{\mathbf{F}\cdot(\mathbf{S}_{1j}\times\mathbf{M})}{(\mathbf{F}\times\mathbf{S}_{1j})\cdot(\mathbf{S}_{1j}\times\mathbf{M})} \tag{1}$$

$$\frac{t_{1j}}{2} = -\frac{\mathbf{S}_{1j}-(\mathbf{S}_{1j}\cdot\mathbf{M})\mathbf{M}}{1-(\mathbf{S}_{1j}\cdot\mathbf{M})^2}\cdot(\mathbf{Q}-\mathbf{A}_{1j}) + \frac{\mathbf{S}_{1j}-(\mathbf{S}_{1j}\cdot\mathbf{F})\mathbf{F}}{1-(\mathbf{S}_{1j}\cdot\mathbf{F})^2}\cdot(\mathbf{G}-\mathbf{A}_{1j}) \tag{2}$$

$$0 = \frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{F})\mathbf{F}}{1-(\mathbf{M}\cdot\mathbf{F})^2}\cdot(\mathbf{G}-\mathbf{Q}) - \frac{\mathbf{M}-(\mathbf{M}\cdot\mathbf{S}_{1j})\mathbf{S}_{1j}}{1-(\mathbf{M}\cdot\mathbf{S}_{1j})^2}\cdot(\mathbf{A}_{1j}-\mathbf{Q}) \tag{3}$$

$$0 = \frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{S}_{1j})\mathbf{S}_{1j}}{1-(\mathbf{F}\cdot\mathbf{S}_{1j})^2}\cdot(\mathbf{A}_{1j}-\mathbf{G}) - \frac{\mathbf{F}-(\mathbf{F}\cdot\mathbf{M})\mathbf{M}}{1-(\mathbf{F}\cdot\mathbf{M})^2}\cdot(\mathbf{Q}-\mathbf{G}) \tag{4}$$

where $\mathbf{S}_{1j}$ is a unit vector parallel to the axis of screw $\hat{S}_{1j}$; $\mathbf{A}_{1j}$ is a position vector to an arbitrary point, A_{1j}, on the axes of screw $\hat{S}_{1j}$; θ_{1j} and t_{1j} are the screw parameters representing the rotation and translation associated with $\hat{S}_{1j}$. From equations (1) to (4) with $j = 2, 3$ (i.e. for three positions) we obtain 8 design equations in the 8 unknown scalar design parameters. (There are two parameters in each of the following: F, G, M and Q.) Hence a finite set of solutions is expected.

It is convenient to choose points G and Q at the intersection of the axes with their common normal. We then require that

$$\mathbf{F} \cdot (\mathbf{Q} - \mathbf{G}) = 0, \tag{5}$$

$$\mathbf{M} \cdot (\mathbf{Q} - \mathbf{G}) = 0, \tag{6}$$

and consider both G and Q as three parameter vectors.

Using (5) and (6), allows equations (3) and (4) to be simplified to the following:

$$[\mathbf{M} - (\mathbf{M} \cdot \mathbf{S}_{ij})\mathbf{S}_{ij}] \cdot (\mathbf{Q} - \mathbf{A}_{ij}) = 0 \tag{7}$$

$$[\mathbf{F} - (\mathbf{F} \cdot \mathbf{S}_{ij})\mathbf{S}_{ij}] \cdot (\mathbf{G} - \mathbf{A}_{ij}) = 0. \tag{8}$$

Upon substitution of (7) and (8) into (2), we obtain

$$\frac{t_{ij}}{2} = \mathbf{S}_{ij} \cdot (\mathbf{G} - \mathbf{Q}). \tag{9}$$

We use (1) and (5)-(9) as the design equations for the synthesis of R-R cranks.

Solutions of Revolute-Revolute Cranks

For three positions, there are three screws, namely, $\hat{S}_{12}$, $\hat{S}_{23}$ and $\hat{S}_{13}$. Without losing generality, we can always choose the reference system in such a way that the z-axis is along the axis of screw $\hat{S}_{12}$, the y-axis is along the common normal of $\hat{S}_{12}$ and $\hat{S}_{13}$, and the x-axis is perpendicular to both y and z axes according to the right hand rule, as shown in Fig. 2.

For this system of reference, screws $\hat{S}_{12}$ and $\hat{S}_{13}$ may be expressed as follows:

$$\hat{S}_{12}: \mathbf{S}_{12} = (0, 0, 1), \mathbf{A}_{12} = (0, 0, 0), \theta_{12}, t_{12},$$

$$\hat{S}_{13}: \mathbf{S}_{13} = (u, 0, v), \mathbf{A}_{13} = (0, h, 0), \theta_{13}, t_{13}.$$

With these coordinates equation (1), written with $j = 2, 3$ yields respectively:

$$(M_x \tan \frac{1}{2}\theta_{12} + M_y)\, F_x + (M_y \tan \frac{1}{2}\theta_{12} - M_x)\, F_y = 0 \tag{10}$$

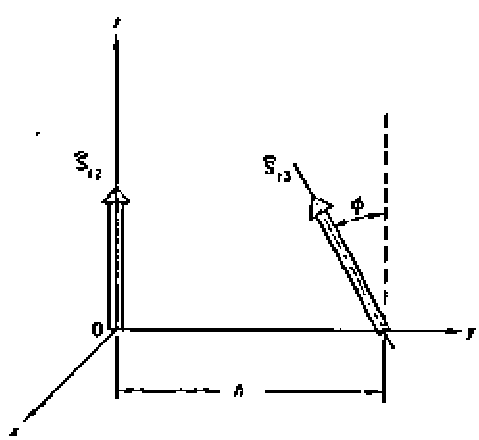


Figure 2. Coordinate system defined by screws.

$$[v(uM_z - vM_x)\tan\tfrac{1}{2}\theta_{13} - vM_y]F_x + (vM_x - uM_z - M_y\tan\tfrac{1}{2}\theta_{13})F_y$$

$$+u[M_y + (vM_x - uM_z)\tan\tfrac{1}{2}\theta_{13}]F_z = 0. \tag{11}$$

Solving (10) and (11) for F_x/F_z and F_y/F_z yields

$$\frac{F_x}{F_z} = \frac{\Delta_x}{\Delta} \tag{12}$$

$$\frac{F_y}{F_z} = \frac{\Delta_y}{\Delta} \tag{13}$$

where

$$\Delta = (M_y + M_x\tan\tfrac{1}{2}\theta_{12})(vM_x - uM_z - M_y\tan\tfrac{1}{2}\theta_{13})$$

$$+v(M_y\tan\tfrac{1}{2}\theta_{12} - M_x)[M_y + (vM_x - uM_z)\tan\tfrac{1}{2}\theta_{13}] \tag{14}$$

$$\Delta_x = u(M_y\tan\tfrac{1}{2}\theta_{12} - M_x)[M_y + (vM_x - uM_z)\tan\tfrac{1}{2}\theta_{13}] \tag{15}$$

$$\Delta_y = -u(M_y + M_x\tan\tfrac{1}{2}\theta_{12})[M_y + (vM_x - uM_z)] \tag{16}$$

$$u = \sin\Phi \tag{17}$$

$$v = \cos\Phi. \tag{18}$$

From (5) and (6), we solve for $\mathbf{G} - \mathbf{Q}$

$$\mathbf{G} - \mathbf{Q} = k(\mathbf{F} \times \mathbf{M}) \tag{19}$$

where k is a constant representing the distance between $\mathbf{G}$ and $\mathbf{Q}$ divided by the sine of the angle between $\mathbf{F}$ and $\mathbf{M}$.

Substituting (19) into (9) and eliminating k between the $j = 2$ and $j = 3$ equations, we obtain provided $u \neq 0$

$$t_{12}[-vM_yF_x + (vM_x - uM_z)F_y + uM_yF_z] = t_{13}(-M_yF_x + M_xF_y). \tag{20}$$

Substituting (12) and (13) into (20), we obtain

$$t_{12}(M_y + M_x\tan\tfrac{1}{2}\theta_{12})[(vM_x - uM_z)^2 + M_y^2]\tan\tfrac{1}{2}\theta_{13}$$

$$-t_{13}(M_x^2 + M_y^2)[M_y + (vM_x - uM_z)\tan\tfrac{1}{2}\theta_{13}]\tan\tfrac{1}{2}\theta_{12} = 0. \tag{21}$$

From (5) and (9), we have

$$\mathbf{F}\cdot\mathbf{G} = \mathbf{F}\cdot\mathbf{Q} \tag{22}$$

$$\mathbf{S}_{1j}\cdot\mathbf{G} = \frac{t_{1j}}{2} + \mathbf{S}_{1j}\cdot\mathbf{Q}. \tag{23}$$

Substituting (22) and (23) into (8), we obtain

$$[\mathbf{F}-(\mathbf{F}\cdot\mathbf{S}_{1j})\mathbf{S}_{1j}]\cdot(\mathbf{Q}-\mathbf{A}) = \frac{t_{1j}}{2}(\mathbf{F}\cdot\mathbf{S}_{1j}). \tag{24}$$

Both (7) and (24), taken twice, $j = 2, 3$, yield the following four linear equations in Q_x, Q_y and Q_z:

$$M_xQ_x + M_yQ_y = 0 \tag{25}$$

$$v(vM_x - uM_z)Q_x + M_yQ_y - u(vM_x - uM_z)Q_z = M_yh \tag{26}$$

$$F_xQ_x + F_yQ_y = \frac{t_{12}}{2}F_z \tag{27}$$

$$v(vF_x - uF_z)Q_x + F_yQ_y - u(vF_x - uF_z)Q_z = \frac{t_{13}}{2}(uF_x + vF_z) + hF_y. \tag{28}$$

The compatibility condition for these four equations is, after substituting for F_x, F_y and F_z from (12) and (13):

$$\begin{aligned}&\frac{t_{12}}{2}\Big\{\Big(M_y + M_x\tan\frac{1}{2}\theta_{12}\Big)(uM_x + vM_z)\Big(vM_x - uM_z - M_y\tan\frac{1}{2}\theta_{13}\Big)\\ &+ M_z\Big(M_y\tan\frac{1}{2}\theta_{12} - M_x\Big)\Big[M_y + (vM_x - uM_z)\tan\frac{1}{2}\theta_{13}\Big]\Big\}\\ &+ h(M_x^2 + M_y^2)\Big[M_y + (vM_x - uM_z)\tan\frac{1}{2}\theta_{13}\Big]\tan\frac{1}{2}\theta_{12} = 0,\end{aligned} \tag{29}$$

provided

$$u\Big(M_y + M_x\tan\frac{1}{2}\theta_{12}\Big)\Big[(vM_x - uM_z)^2 + M_y^2\Big]\tan\frac{1}{2}\theta_{13} \neq 0.$$

Since (21) and (29) are two cubic equations homogeneous in M_x, M_y, and M_z there are at most nine solutions for the ratios M_x/M_z and M_y/M_z. However solutions $M_x:M_y:M_z = 0:0:1$, $M_x:M_y:M_z = u:0:v$, and

$$M_x:M_y:M_z = -u\tan\frac{1}{2}\theta_{13} : u\tan\frac{1}{2}\theta_{12}\tan\frac{1}{2}\theta_{13} : \Big(\tan\frac{1}{2}\theta_{12} - v\tan\frac{1}{2}\theta_{13}\Big),$$

which correspond respectively to the directions of screws S_{12}, S_{13} and S_{23}^{1*}, are spurious. Hence we have at most six meaningful solutions for M_x/M_z and M_y/M_z.

We may rewrite (21) and (29) in the following form:

$$ax^2 + bx + c = 0 \tag{30}$$

$$a'x^2 + b'x + c' = 0 \tag{31}$$

where

$$x = vM_x - uM_z, \tag{32}$$

$$a = M_xf + M_y, \tag{33}$$

$$b = -d(M_x^2 + M_y^2)g. \tag{34}$$

*S_{23}^1 is the screw, in position 1, fixed in the moving system which will coincide with S_{23} when moving system is in position 2 or 3.

$$c = aM_y^2 - d(M_x^2 + M_y^2)M_y. \tag{35}$$

$$a' = (M_x - fM_y)g - va. \tag{36}$$

$$b' = a(M_x + vgM_y) - (vM_xg - M_y)(M_x - fM_y) + e(M_x^2 + M_y^2)g. \tag{37}$$

$$c' = -aM_xM_yg - vM_xM_y(M_x - fM_y) + e(M_x^2 + M_y^2)M_y. \tag{38}$$

$$d = (t_{13}f)/(t_{12}g). \tag{39}$$

$$e = (2huf)/t_{12}. \tag{40}$$

$$f = \tan\frac{1}{2}\theta_{12}. \tag{41}$$

$$g = \tan\frac{1}{2}\theta_{13}. \tag{42}$$

Eliminating x between (30) and (31), and factoring out the root

$$M_x \tan\frac{1}{2}\theta_{12} + M_y = 0,$$

we obtain the following equation*:

$$a_0y^6 + a_1y^5 + a_2y^4 + a_3y^3 + a_4y^2 + a_5y + a_6 = 0 \tag{43}$$

from which the factor $1 + g^2$ has been removed.

Here $y = M_x/M_y$. (44)

$$a_0 = -defg + vdfg - df^2 + d^2vfg - d^2g^2. \tag{45}$$

$$a_1 = v^2d^2f + d^2fg^2 - 2vdef + 2v^2df - 2vdf^2g - 4df + e^2f - 2vef + v^2f + f^3. \tag{46}$$

$$a_2 = v^2d^2 + 3vd^2fg - 2d^2g^2 + vdfg - 2vde - 3defg - 4v^2df^2 + 2dv^2 + df^2 - 4d + e^2 + 4vef^2 - 2ve - 4v^2f^2 + v^2 + 5f^2 \tag{47}$$

$$a_3 = 2v^2d^2f + 2d^2fg^2 - 4vdef - 4v^2df - 4vdf^2g + 4v^2f^3 - 6v^2f - 2f^3 + 8f + 2e^2f + 4vef. \tag{48}$$

$$a_4 = 2v^2d^2 + 3vd^2fg - d^2g^2 + df^2 - 4d + 8v^2f^2 - 2v^2 - 6f^2 + 4 - 4vde - 3defg - 4v^2df^2 - vdfg + 2e^2 + 4vef^2. \tag{49}$$

$$a_5 = v^2d^2f + d^2fg^2 - 2vdef - 6v^2df - 2vdf^2g + 4df + f^3 - 4f + e^2f + 6vef + 5v^2f. \tag{50}$$

$$a_6 = v^2d^2 + vd^2fg - defg - 2v^2d - vdfg - df^2 + v^2 + f^2 - 2vde + e^2 + 2ve. \tag{51}$$

By making the following change of variable

$$y = \frac{z + t}{-zt + 1}. \tag{52}$$

*The expansion of the 6th degree polynomial was done with the aid of an IBM 360 computer using the program REDUCE II written by A. Hearn it can be shown that the odd order terms in the transformed equation vanish. Thus, we have

$$c_0 z^6 + c_1 z^4 + c_2 z^2 + c_3 = 0 \tag{53}$$

or

$$c_0 \bar{z}^3 + c_1 \bar{z}^2 + c_2 \bar{z} + c_3 = 0 \tag{54}$$

$$\text{where } \bar{z} = z^2, \tag{55}$$

$$t = \tan \frac{1}{4} \theta_{12}, \tag{56}$$

$$c_0 = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4 - a_5 t^5 + a_6 t^6, \tag{57}$$

$$\begin{aligned} c_1 = {} & 15 a_0 t^2 + 5 a_1 t (1 - 2t^2) + a_2 (1 - 8t^2 + 6t^4) - 3 a_3 t (1 - 3t^2 + t^4) \\ & + a_4 t^2 (6 - 8t^2 + t^4) + 5 a_5 t^3 (-2 + t^4) + 15 a_6 t^4, \end{aligned} \tag{58}$$

$$\begin{aligned} c_2 = {} & 15 a_0 t^4 + 5 a_1 t^3 (2 - t^2) + a_2 t^2 (6 - 8t^2 + t^4) + 3 a_3 t (1 - 3t^2 + t^4) \\ & + a_4 (1 - 8t^2 + 6t^4) - 5 a_5 t (1 - 2t^2) + 15 a_6 t^2 \end{aligned} \tag{59}$$

$$c_3 = a_0 t^6 + a_1 t^5 + a_2 t^4 + a_3 t^3 + a_4 t^2 + a_5 t + a_6. \tag{60}$$

Expanding these coefficients in terms of the original design variables, and using the trigonometric identity $f = 2t/(1-t^2)$, results in a simplified form of the cubic (54). After factoring out $(-v^2)\,[(1+t^2)^3/(1-t^2)^3]$ we obtain:

$$k_3 \bar{z}^3 + k_2 \bar{z}^2 + k_1 \bar{z} + k_0 = 0 \tag{61}$$

where

$$k_3 = t^2 [t(t + K_1) + (1 - K_2)]^2 \tag{62}$$

$$\begin{aligned} k_2 = {} & 2t^6 - K_3 t^5 - (K_2(2 + K_2) - 2K_1^2 - 3) t^4 \\ & - 2(K_1(3K_2 + 1) + K_3) t^3 - (2K_2(1 - K_2) + K_1^2) t^2 \\ & - (2K_1 + K_3)\, t - 1 - 4\left(\frac{1}{v^2} - 1\right) t^2 (t^2 + 1)^2 \end{aligned} \tag{63}$$

$$\begin{aligned} k_1 = {} & t^6 - (2K_1 + K_3) t^5 + (2K_2(1 - K_2) + K_1^2) t^4 - 2(K_1(3K_2 + 1) + K_3) t^3 \\ & + (K_2(2 + K_2) - 2K_1^2 - 3) t^2 - K_3 t - 2 - 4\left(\frac{1}{v^2} - 1\right)(t^2 + 1)^2 \end{aligned} \tag{64}$$

$$k_0 = -[t^2(1 - K_2) + (1 - tK_1)]^2 \tag{65}$$

and

$$K_1 = 2\left[\frac{t_{13}}{t_{12} \tan \frac{1}{2} \theta_{13}} - \frac{2hu}{t_{12} v}\right]$$

$$K_2 = 2 \frac{t_{13}}{t_{12} v}$$

$$K_3 = \frac{-8}{v^2 \tan \frac{1}{2} \theta_{13}} \, \frac{t_{13}}{t_{12}}.$$

From (62) it follows that the sign of k_3 must always be positive, and from (65) we conclude that k_0 must always be a negative number. This means there must always be at least one positive real root of (61). Furthermore, since $-k_3+k_2-k_1+k_0=4(r^2+1)^2[(1/r^2)-1]$ is always positive, there will always be one real root between $\xi=0$ and $\xi=-1$ and one real root between $\xi=-1$ and $\xi=-\infty$. Hence all three roots of (61) will be real but only one is positive. The conclusion is then that (61) always has one and only one significant root. Corresponding to this positive real root are two real values of y (see equations (55) and (52)). This means that there will always be a pair of R-R cranks derived from the one positive real root of (61). Such a pair of R-R cranks can be used to form the cranks of a Bennett Mechanism[2, 3].

The case of perpendicular screws (i.e. $r=0$) is included in the above if we multiply (61) by r^2, thereby restoring a factor we have removed, and then set $r=0$. Similarly $t_{12}=0$ can be treated by first multiplying (61) by t_{12}^2. Equation (61) degenerates for those special displacements for which no Revolute-Revolute cranks exist in the ordinary sense; that is those cranks having an axis at infinity. In this way parallel screws ($u=0$) or pure translation screws ($\theta_{12}=0$, or $\theta_{13}=0$) are exceptions. Similarly we get special results for the case when both screws are pure rotations ($t_{12}=0$ and $t_{13}=0$): The screws may be used as the axes for the Revolute-Revolute cranks, and if in addition $u=0$ or $h=0$ we have respectively the well known cases of planar or spherical displacements.

Synthesis Procedure

The synthesis procedure is then as follows: We describe three finitely separated positions in terms of S_{12} and S_{13}, as for example, in Table 1.

Table 1. Screw S_{ij}*

	Direction			Location			Rotation	Translation
ij	S_x	S_y	S_z	A_x	A_y	A_z	θ_{ij}	t_{ij}
12	0·0000	0·0000	1·0000	0·0000	0·0000	0·0000	40·0°	0·80
13	sin 30°	0·0000	cos 30°	0·0000	1·0000	0·0000	70·0°	0·60

Table 2. R-R cranks

Moving joint					
Direction			Location		
0·3649	0·4520	0·8140	−0·9388	0·7580	0·7793
−0·5960	0·3593	−0·7181	0·8460	1·4034	−0·3793
Fixed joint					
Direction			Location		
0·5960	0·3593	0·7181	−0·8460	1·4034	0·3793
−0·3649	0·4520	−0·8140	0·9388	0·7580	−0·7793

Using this data, we calculate k_3–k_0 from equations (62) through (65). With these coefficients we solve the cubic polynomial (61) for ξ. Corresponding to the positive real root ξ, we then find two values of z from (55). Using these values in (52) together with (56) and recalling the definition of y from (44) yields one value of M_y/M_y for each Ξ. The ratio M_z/M_x can then be found from the common root of (30) and (31). From $M_z/$

*This data is taken from [2].

M_y and M_z/M_y we can determine M. By continuing this backward substitution, one F, Q and G can be determined corresponding to each M: from (12), (13) and the unit vector condition we get F; Q then follows from (25)–(27); finally G is obtained from (22), and (23) with $j = 2, 3$.

For the values in Table 1, the resultant pair of R-R cranks are listed in Table 2.

Conclusion

We have given a general computational procedure for synthesizing Revolute-Revolute cranks compatible with three finitely separated design positions. This method yields all possible solutions since it depends ultimately upon the roots of a cubic polynomial. It is proved that in general for any three positions there is one and only one pair of Revolute-Revolute cranks.

These results are in perfect agreement with those obtained by Veldkamp[3] for infinitesimally separated positions.

Acknowledgements — The financial support of the National Science Foundation and the supportive interest of Professor A. Leifer are most appreciatively acknowledged.

References

[1] TSAI L. W. and ROTH B., Design of dyads with helical, cylindrical, spherical, revolute, and prismatic joints. *Mechanism and Machine Theory* 7(1), 85–102 (Spring 1972).

[2] SUH C. H., On the duality in the existence of R-R links for three positions. *J. Engng Ind., Trans. ASME* 91B (1), 129–134 (1969).

[3] VELDKAMP G. R., Canonical systems and instantaneous invariants in spatial kinematics, *J. Mechanisms* 2(3), 329–388 (Autumn 1967).

[4] ROTH B., The design of binary cranks with revolute, cylindric, and prismatic joints, *J. Mechanisms* 3(2), 61–72 (Summer 1968).

[5] ROTH B., Finite position theory applied to mechanism synthesis, *J. Appl. Mech., Trans. ASME* 34E, 599–605 (1967).

[6] TSAI L. W., Design of open loop chains for rigid body guidance, Ph. D. Dissertation, Stanford University (1972).

Zur Berechnung von Raumkurbeln mit zwei Drehgelenken

L. W. Tsai und B. Roth

Kurzfassung — In diesem Beitrag werden die Entwurfsgleichungen vor binären Kurbeln mit zwei Drehpaaren gelöst. Die hier entwickelten Lösungen können auf den Entwurf binärer Kurbeln mit zwei Drehgelenken für die räumliche Führung starrer Körper durch drei zugeordnete endlich benachbarte Lagen angewandert werden. Wir haben gefunden daß dieses Problem sich auf die Berechnung der Wurzeln eines kubischen Polynoms reduzieren läßt. Die Gleichung (55) zeigt, daß nur die reellen positiven Wurzeln des kurischen Polynoms (61) zu wirklichen Kurbeln führen. Die koeffizienten (62)-(65) zeigen, daß die Gleichung (61) immer eine und nur eine positive Wurzel hat. Ferner folgt aus Gleichung (55) daß jede reelle positive Wurzel ein Paar binäre Kurbeln gibt. Es gilt also der Satz: Für drei endlich benachbarte Körperlagen eines starren räumlichen Systems gibt es immer ein paar und nur ein paar Kurbeln mit Drehpaaren. Die zwei Kurbeln in jedem Paar sind mit einander verbünden wie die Kurbeln eines Bennett Viergelenkgetriebes.

PROCEEDINGS
of
THE THIRD WORLD CONGRESS
for
THE THEORY OF MACHINES AND MECHANISMS
Kupari, Yugoslavia, September 13–20, 1971
Volume D Paper D-19, pp. 273-256

DESIGN OF TRIADS USING THE SCREW-TRIANGLE CHAIN

Lung-Wen TSAI, Research Assistant
Bernard ROTH, Professor,
Department of Mechanical Engineering,
Stanford University, Stanford, California, U.S.A.

1. Introduction

In a companion paper [1] the concept of an "equivalent screw-triangle" is introduced and applied to the finitely and/or infinitesimally separated position problems associated with the kinematic synthesis of binary cranks. Here, we extend the notion of an equivalent screw-triangle to a "screw-triangle chain", and show how to use it to derive the design equations for open-loop three-link chains (i.e., triads). All possible combinations of triads with helical, cylindrical, spherical, revolute, and prismatic joints are considered.

2. Nomenclature

Referring to Fig. 1, we define the following terminology and nomenclature. The link jointed directly to the reference frame is called the first moving link. The joint between the fixed system and the first moving link is called the fixed joint. The moving link jointed to the first moving link is called the second moving link. The joint between the first and second moving links is called the first moving joint. Similarly the joint between the second moving link and the link c is called the second moving joint. With the entire linkage in the i^{th} position the following vectors are defined:

$\overline{F}_{ij}$, $\overline{M}_{ij}$, and $\overline{N}_{ij}$ are unit vectors parallel respectively to the fixed joint axis, and the first and second moving joint axes. $\overline{G}_{ij}$, $\overline{Q}_{ij}$, and $\overline{R}_{ij}$ are position vectors to arbitrary points, $G_{ij}(G_{xij}, G_{yij}, G_{zij})$, $Q_{ij}(Q_{xij}, Q_{yij}, Q_{zij})$, and $R_{ij}(R_{xij}, R_{yij}, R_{zij})$, on respectively the fixed joint axis, and the first and second moving joint axis.

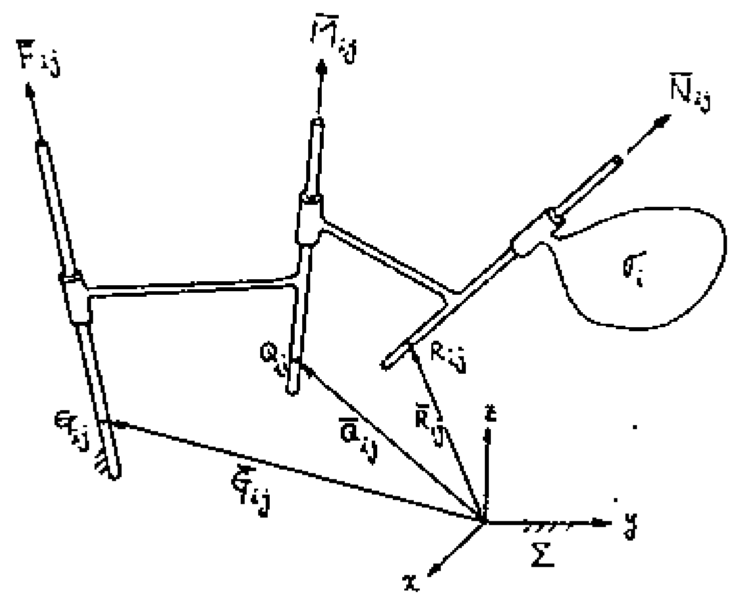


Fig. 1 - Three-Link Chain (Triad)

3. <u>The Screw-Triangle Chain</u>

Any displacement of σ may be characterized by a screw displacement $\widehat{S}_{ij}$ relative to the fixed system. When σ is constrained by a triad as in Fig. 1, we may always regard this screw displacement, $\widehat{S}_{ij}$, as the resultant of three successive screw displacements: first the rigid body σ is screwed about the second moving-axis, then about the first moving-axis, and finally about the fixed-axis. During such a transformation each joint axis is active as a screw axis while the joint is in the i^{th} position. Hence, all vectors will be referred to the i^{th} position of the link. Screw axes $\hat{N}_{ij}, \hat{M}_{ij}, \hat{F}_{ij}$ (taken along the correspondingly labeled joint axis) and $\hat{S}_{ij}$ form a screw quadrilateral (see Ref. [2]). Using the methods developed in [2] we can find a line $\hat{L}_{ij}$ which subtends the opposite sides of the screw quadrilateral at equal (or supplementary) dual angles. $\hat{L}_{ij}$ divides the screw quadrilateral into two screw triangles as shown in Fig. 2, where $(\theta_{ij}, t_{ij}), (\gamma_{ij}, v_{ij}), (\alpha_{ij}, u_{ij}), (\beta_{ij}, v_{ij})$, and (ϕ_{ij}, ρ_{ij}) are the screw parameters associated with $\hat{S}_{ij}$, $\hat{F}_{ij}, \hat{M}_{ij}, \hat{N}_{ij}$ and $\hat{L}_{ij}$ respectively. We call the configuration shown in Fig. 2 a <u>screw triangle chain</u>.

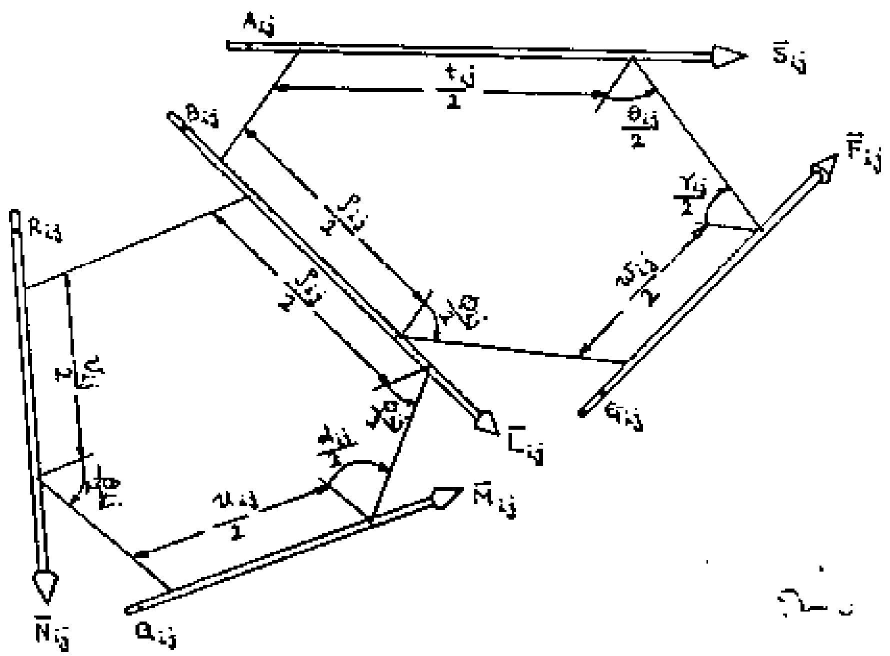


Fig. 2 - Screw-Triangle Chain

The geometric relations defined by Fig. 2 are:

$$\tan\frac{\theta_{ij}}{2} = -\frac{\hat{F}_{ij}\cdot\hat{S}_{ij}\times\hat{L}_{ij}}{(\hat{F}_{ij}\times\hat{S}_{ij})\cdot(\hat{S}_{ij}\times\hat{L}_{ij})} \tag{1}$$

$$\tan\frac{\gamma_{ij}}{2} = \frac{\hat{F}_{ij}\cdot\hat{S}_{ij}\times\hat{L}_{ij}}{(\hat{L}_{ij}\times\hat{F}_{ij})\cdot(\hat{F}_{ij}\times\hat{S}_{ij})} \tag{2}$$

$$\frac{\hat{F}_{ij}\cdot\hat{S}_{ij}\times\hat{L}_{ij}}{(\hat{S}_{ij}\times\hat{L}_{ij})\cdot(\hat{L}_{ij}\times\hat{F}_{ij})} = -\frac{\hat{M}_{ij}\cdot\hat{L}_{ij}\times\hat{N}_{ij}}{(\hat{M}_{ij}\times\hat{L}_{ij})\cdot(\hat{L}_{ij}\times\hat{N}_{ij})} = \tan\frac{\phi_{ij}}{2} \tag{3}$$

$$\tan\frac{\beta_{ij}}{2} = \frac{\hat{M}_{ij}\cdot\hat{L}_{ij}\times\hat{N}_{ij}}{(\hat{L}_{ij}\times\hat{N}_{ij})\cdot(\hat{N}_{ij}\times\hat{M}_{ij})} \tag{4}$$

$$\tan\frac{\alpha_{ij}}{2} = \frac{\hat{M}_{ij}\cdot\hat{L}_{ij}\times\hat{N}_{ij}}{(\hat{N}_{ij}\times\hat{M}_{ij})\cdot(\hat{M}_{ij}\times\hat{L}_{ij})} \tag{5}$$

$$\frac{t_{ij}}{2} = -\frac{\hat{S}_{ij}-(\hat{S}_{ij}\cdot\hat{L}_{ij})\hat{L}_{ij}}{1-(\hat{S}_{ij}\cdot\hat{L}_{ij})^2}\cdot(\bar{B}_{ij}-\bar{A}_{ij}) + \frac{\hat{S}_{ij}-(\hat{S}_{ij}\cdot\hat{F}_{ij})\hat{F}_{ij}}{1-(\hat{S}_{ij}\cdot\hat{F}_{ij})^2}\cdot(\bar{G}_{ij}-\bar{A}_{ij}) \tag{6}$$

$$\frac{w_{ij}}{2} = \frac{\hat{F}_{ij}-(\hat{F}_{ij}\cdot\hat{S}_{ij})\hat{S}_{ij}}{1-(\hat{F}_{ij}\cdot\hat{S}_{ij})^2}\cdot(\bar{A}_{ij}-\bar{G}_{ij}) - \frac{\hat{F}_{ij}-(\hat{F}_{ij}\cdot\hat{L}_{ij})\hat{L}_{ij}}{1-(\hat{F}_{ij}\cdot\hat{L}_{ij})^2}\cdot(\bar{B}_{ij}-\bar{G}_{ij}) \tag{7}$$

$$\frac{\hat{L}_{ij}-(\hat{L}_{ij}\cdot\hat{F}_{ij})\hat{F}_{ij}}{1-(\hat{L}_{ij}\cdot\hat{F}_{ij})^2}\cdot(\bar{G}_{ij}-\bar{B}_{ij}) - \frac{\hat{L}_{ij}-(\hat{L}_{ij}\cdot\hat{S}_{ij})\hat{S}_{ij}}{1-(\hat{L}_{ij}\cdot\hat{S}_{ij})^2}\cdot(\bar{A}_{ij}-\bar{B}_{ij})$$

$$= -\frac{\hat{L}_{ij}-(\hat{L}_{ij}\cdot\hat{N}_{ij})\hat{N}_{ij}}{1-(\hat{L}_{ij}\cdot\hat{N}_{ij})^2}\cdot(\bar{R}_{ij}-\bar{B}_{ij}) + \frac{\hat{L}_{ij}-(\hat{L}_{ij}\cdot\hat{M}_{ij})\hat{M}_{ij}}{1-(\hat{L}_{ij}\cdot\hat{M}_{ij})^2}\cdot(\bar{Q}_{ij}-\bar{B}_{ij}) = \frac{\rho_{ij}}{2} \tag{8}$$

$$\frac{v_{ij}}{2} = \frac{\hat{N}_{ij}-(\hat{N}_{ij}\cdot\hat{M}_{ij})\hat{M}_{ij}}{1-(\hat{N}_{ij}\cdot\hat{M}_{ij})^2}\cdot(\bar{Q}_{ij}-\bar{R}_{ij}) - \frac{\hat{N}_{ij}-(\hat{N}_{ij}\cdot\hat{L}_{ij})\hat{L}_{ij}}{1-(\hat{N}_{ij}\cdot\hat{L}_{ij})^2}\cdot(\bar{B}_{ij}-\bar{R}_{ij}) \tag{9}$$

$$\frac{u_{ij}}{2} = \frac{\hat{M}_{ij}-(\hat{M}_{ij}\cdot\hat{L}_{ij})\hat{L}_{ij}}{1-(\hat{M}_{ij}\cdot\hat{L}_{ij})^2}\cdot(\bar{B}_{ij}-\bar{Q}_{ij}) - \frac{\hat{M}_{ij}-(\hat{M}_{ij}\cdot\hat{N}_{ij})\hat{N}_{ij}}{1-(\hat{M}_{ij}\cdot\hat{N}_{ij})^2}\cdot(\bar{R}_{ij}-\bar{Q}_{ij}) \tag{10}$$

We also introduce P_f, P_m, and P_n for the pitches of the screws along the fixed joint, the first moving joint, and the second moving joint as follows:

$$w_{ij} = P_f\gamma_{ij} \tag{11}$$

$$u_{ij} = P_m\alpha_{ij} \tag{12}$$

$$v_{ij} = P_n\theta_{ij} \tag{13}$$

Note that G_{ij}, Q_{ij}, R_{ij}, and B_{ij} are arbitrary points on the lines of action of the screws $\hat{F}_{ij}$, $\hat{M}_{ij}$, $\hat{N}_{ij}$ and $\hat{L}_{ij}$ respectively; in order to specify any one of these points, only two parameters are needed (except in the case of a spherical joint where the point will be taken at the center of the joint).

Since $\overline{F}_{ij}, \overline{M}_{ij}, \overline{N}_{ij}$, and $\overline{L}_{ij}$ are defined to be unit vectors, the following conditions must be satisfied:

$$F^2_{xij} + F^2_{yij} + F^2_{zij} = 1 \quad (14)$$

$$M^2_{xij} + M^2_{yij} + M^2_{zij} = 1 \quad (15)$$

$$N^2_{xij} + N^2_{yij} + N^2_{zij} = 1 \quad (16)$$

$$L^2_{xij} + L^2_{yij} + L^2_{zij} = 1 \quad (17)$$

For simplicity, we will describe finitely separated positions of σ in terms of the screw displacements $\hat{S}_{1j}$, $j=2,3,\ldots,n$. Thereby, all the moving vectors in Equations (1) through (17) are referred to their first position**, i.e.,

$$\overline{M}_{1j} = \overline{M}(M_x, M_y, M_z) \quad (18)$$

$$\overline{N}_{1j} = \overline{N}(N_x, N_y, N_z) \quad (19)$$

$$\overline{Q}_{1j} = \overline{Q}(Q_x, Q_y, Q_z) \quad (20)$$

$$\overline{R}_{1j} = \overline{R}(R_x, R_y, R_z) \quad (21)$$

for $i=1$, $j=2,3,4,\ldots$

The fixed joint axis, except when we have a spherical joint, is fixed in the reference frame, thus

$$\overline{F}_{ij} = \overline{F}(F_x, F_y, F_z) \quad (22)$$

$$\overline{G}_{ij} = \overline{G}_T(G_x, G_y, G_z) \quad (23)$$

The line of action of $\hat{L}_{ij}$ is determined by the screw quadrilateral $(\hat{S}_{ij}\ \hat{F}_{ij}\ \hat{M}_{ij}\ \hat{N}_{ij})$. Hence it changes position with each displacement $\hat{S}_{ij}$.

For spherical joints, we use the symbols $\overline{G}^*_{ij}$, $\overline{Q}^*_{ij}$, and $\overline{R}^*_{ij}$ in place of $\overline{G}_{ij}, \overline{Q}_{ij}$, and $\overline{R}_{ij}$. The * simply reminds us that the quantity denoted is a vector with three as opposed to two unknowns (since it defines the center of the

---

**However, if the screws $\hat{S}_{ij}$ instead of $\hat{S}_{1j}$ are specified, we may either convert the screws to the $\hat{S}_{1j}$ or transfer all the moving vectors into the i^{th} position, $i=1,2,3,\ldots$ The i^{th} position of the first moving joint may be obtained from its k^{th} position by rotating it a dual angle γ_{ki} about the fixed joint axis; and the i^{th} position of the second moving joint axis may be obtained from its k^{th} position by rotating it a dual angle α_{ki} about the first moving joint in the k^{th} position followed by a screw displacement of dual angle γ_{ki} about the fixed joint axis. The mathematical expression for these screw transformations are well known. (See for example [2], or Eqn. (11) and (12) in [1].)

joint and not just an arbitrary point on the axis). In case two or three spherical joints are connected together, we add the following constraints in order to eliminate the extra degree-of-freedom due to rotations about the axis through the line of centers of the joints:

(a) S-S-X triads**(Fig.3)

$$(\bar{Q}^* - \bar{G}^*)\cdot \bar{F}_{ij} = 0 \qquad (24)$$

(b) X-S-S triads (Fig.4)

$$(\bar{R}^* - \bar{Q}^*)\cdot \bar{N}_{ij} = 0 \qquad (25)$$

(c) S-X-S triads (Fig.5)

$$(\bar{R}^* - \bar{G}^*)\cdot \bar{F}_{ij} = 0 \qquad (26)$$

(d) S-S-S triads (Fig.6)

Equations (74),(25) and (26) are needed.

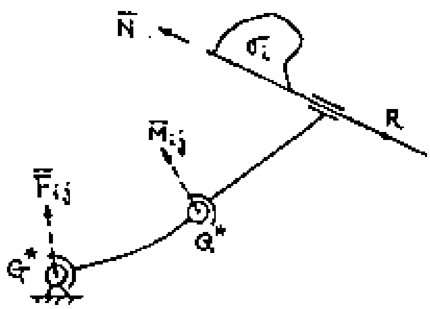


Fig. 3

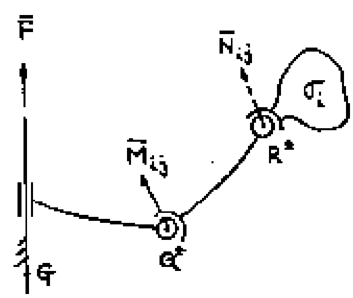


Fig. 4

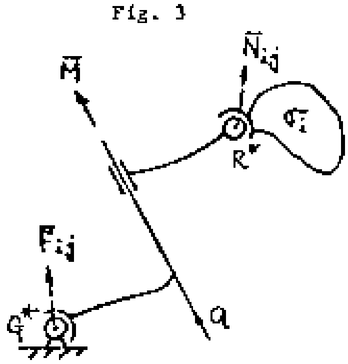


Fig. 5

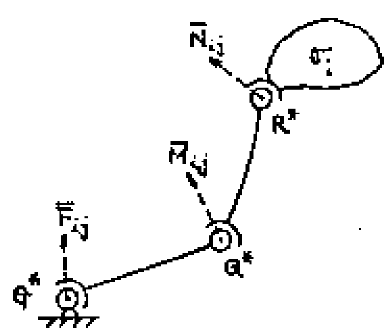


Fig. 6

**The first letter denotes the type of the fixed joint, the second the type of moving joint, and the third the second moving joint. Here we have respectively a spherical fixed joint, a spherical first moving joint, and an arbitrary second moving joint.

4. Design of Triads

Using Equations (1) - (26) we can synthesize all combinations of triads with helical, cylindrical, spherical, and revolute joints. In this section we explain the C-C-C case in detail by way of illustrating the general procedure. All other results are presented in tabular form (Table I). Prismatic joints are discussed in the Appendix.

For "ℓ" finitely separated positions, $\bar{S}_{1j}, \bar{I}_{1j}, \theta_{1j}$, and t_{1j} are specified, $i=1$; $j=2,3,4,\ldots,\ell$. We list the appropriate design equations, design parameters, and the number of possible solutions for every possible combination of three link-chains (triads) with lower pair joints in the table. In what follows, we always regard $\bar{F}_{1j}$ as a two parameter vector since the third parameter is determined by Equation (14). Similarly, $\bar{H}_{1j}, \bar{N}_{1j}$, and $\bar{L}_{1j}$ are two parameter vectors.

In each type of chain the maximum number of design positions is obtained by solving for "ℓ", from the expressions given for m and n in the table, when m is as large as possible under the restriction $m \leq n$. m and n are the number of design equations and the number of unknowns respectively. The number of free parameters is equal to the number of unknowns minus the number of equations, i.e., $n-m$.

For example, in the case of C-C-C triads, $n-m = [12+4(\ell-1)]-[4(\ell-1)] = 12$. Hence, for any number of prescribed design positions, we generally have ∞^{12} triads. We may arbitrarily choose the triad. If we choose $\bar{F}, \bar{H}, \bar{N}, \bar{C}, \bar{Q}$, and $\bar{M}$, we may solve the remaining equations for $\bar{L}_{1j}$ and $\bar{E}_{1j}$. Equations (2),(4),(5),(7),(9), and (10) then provide the information about the rotations and translations about each joint-axis. For each choice of $(\bar{F}, \bar{H}, \bar{N}, \bar{C}, \bar{Q}, \bar{M})$, since Equation (1) is linear in $(L_{x1j}, L_{y1j}, L_{z1j})$ while Equation (3) is cubic in $(L_{x1j}, L_{y1j}, L_{z1j})$ for each j, we have either one or three possible real solutions for $\bar{L}_{1j}$. Corresponding to each $\bar{L}_{1j}$, Equations (6) and (8) give $\bar{B}_{1j}$ uniquely. Hence, for each displacement, there exists either one or three possible configurations of the selected triad.

5. Infinitesimally Separated Position Problems

By substituting infinitesimals for finite displacements, and replacing the tangent of the angle by the angle, we can obtain a set of equations analogous to (1) through (5), but in terms of infinitesimals. If we then take τ as the independent motion parameter and divide each equation by $\frac{d\tau}{2} = \frac{\bar{F}_i \cdot \bar{S}_i \times \bar{L}_i}{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}$ we obtain design equations for infinitesimally

Table 1

Type of Chain	Design Equations (i=1, j=1,2,...,J)	Design Parameters (j=1,2,...,J)	No. of Des. Par. f	No. of Eqns. E	No. of Unknowns n	No. of Possible Choices
C-C-C	Eqns. 1,3,4,6	F,N,R,L_ij,G,Q,A,S_ij	[illegible]	4(J-1)	17+4(J-1)	[illegible]^{12}
[illegible]	Eqns. 1,3,4,7,8,9,10,24,25,26	[illegible]	[illegible]	10(J-1)	9+10(J-1)	[illegible]^{9}
[illegible]	Eqns. 1,3,4,7,8,10,24	[illegible]	[illegible]	7(J-1)	10+8(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,8,9,10,25	[illegible]	[illegible]	7(J-1)	10+8(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,7,8,9,26	[illegible]	[illegible]	7(J-1)	10+8(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,7,8,9,10,24	[illegible]	[illegible]	8(J-1)	10+8(J-1)	[illegible]^{10}
[illegible]	Eqns. 1,3,4,7,8,9,10,25	[illegible]	[illegible]	8(J-1)	10+8(J-1)	[illegible]^{10}
[illegible]	Eqns. 1,3,4,7,8,9,10,26	[illegible]	[illegible]	8(J-1)	10+8(J-1)	[illegible]^{10}
[illegible]	Eqns. 1,3,4,6,7,8,9,10,13,24	[illegible]	[illegible]	10(J-1)	11+10(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,2,3,6,7,8,9,10,11,25	[illegible]	[illegible]	10(J-1)	11+10(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,5,6,7,8,9,10,12,26	[illegible]	[illegible]	10(J-1)	11+10(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,7,8	[illegible]	[illegible]	5(J-1)	13+4(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,8,9	[illegible]	[illegible]	5(J-1)	11+4(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,8,10	[illegible]	[illegible]	5(J-1)	11+4(J-1)	[illegible]
[illegible]	Eqns. 1,3,4,7,8,10	[illegible]	[illegible]	6(J-1)	11+4(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,7,8,10	[illegible]	[illegible]	6(J-1)	11+4(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,8,9,10	[illegible]	[illegible]	6(J-1)	11+4(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,7,8,9	[illegible]	[illegible]	6(J-1)	11+4(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,7,8,9	[illegible]	[illegible]	6(J-1)	13+4(J-1)	[illegible]^{11}
[illegible]	Eqns. 1,3,4,8,9,10	[illegible]	[illegible]	6(J-1)	11+4(J-1)	[illegible]^{11}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	12+4(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	12+6(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	13+8(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	12+8(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	12+8(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	8(J-1)	14+8(J-1)	[illegible]^{12}
[illegible]	[illegible]	[illegible]	[illegible]	5(J-1)	13+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	5(J-1)	13+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	5(J-1)	13+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	13+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	13+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	15+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	9(J-1)	13+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	16+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	16+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	15+6(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	11(J-1)	15+10(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	11(J-1)	15+10(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	11(J-1)	17+10(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	10(J-1)	16+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	10(J-1)	16+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	10(J-1)	16+8(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	6(J-1)	15+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	6(J-1)	15+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	6(J-1)	15+4(J-1)	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	7(J-1)	17+6(J-1)	[illegible]
[illegible]	Eqns. 1 through 13	[illegible]	[illegible]	11(J-1)	15+10(J-1)	[illegible]

In this table [illegible] are all vectors.

separated positions. Using the single subscript i to denote the position from which the infinitesimal displacement occurs, we obtain:

$$\frac{d\theta_i}{d\gamma} = -\frac{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}{(\bar{F}_i \times \bar{S}_i)\cdot(\bar{S}_i \times \bar{L}_i)} \tag{27}$$

$$\frac{d\phi_i}{d\gamma} = \frac{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}{(\bar{S}_i \times \bar{L}_i)\cdot(\bar{L}_i \times \bar{F}_i)} \tag{28}$$

$$\bar{F}_i\cdot(\bar{S}_i \times \bar{L}_i) = 0\ . \tag{29a}$$

$$\bar{M}_i\cdot(\bar{L}_i \times \bar{N}_i) = 0 \tag{29b}$$

$$\frac{\bar{M}_i\cdot(\bar{L}_i \times \bar{N}_i)}{\bar{F}_i\cdot(\bar{S}_i \times \bar{L}_i)} = -\frac{(\bar{M}_i \times \bar{L}_i)\cdot(\bar{L}_i \times \bar{N}_i)}{(\bar{S}_i \times \bar{L}_i)\cdot(\bar{L}_i \times \bar{F}_i)} \triangleq \gamma_i \tag{29c}$$

$$\frac{d\beta_i}{d\gamma} = \frac{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}{(\bar{L}_i \times \bar{N}_i)\cdot(\bar{N}_i \times \bar{M}_i)}\,\gamma_i \tag{30}$$

$$\frac{d\alpha_i}{d\gamma} = -\frac{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}{(\bar{N}_i \times \bar{M}_i)\cdot(\bar{M}_i \times \bar{L}_i)}\,\gamma_i \tag{31}$$

For Equations (6)-(10), the analogous infinitesimal displacement equations are obtained from the kinematic relation for the screw velocity along $\hat{S}_i$:

$$\frac{dt_i}{d\gamma}\bar{S}_i = \frac{dw_i}{d\gamma}\bar{F}_i + \bar{F}_i \times (\bar{A}_i - \bar{G}_i) + \frac{d\rho_i}{d\gamma}\bar{L}_i + \frac{d\phi_i}{d\gamma}\bar{L}_i \times (\bar{A}_i - \bar{B}_i) \tag{32}$$

and the screw velocity along $\hat{L}_i$:

$$\frac{d\rho_i}{d\gamma}\bar{L}_i = \frac{du_i}{d\gamma}\bar{M}_i + \frac{d\alpha_i}{d\gamma}\bar{M}_i \times (\bar{B}_i - \bar{Q}_i) + \frac{dv_i}{d\gamma}\bar{N}_i + \frac{d\beta_i}{d\gamma}\bar{N}_i \times (\bar{B}_i - \bar{R}_i) \tag{33}$$

Operating on (32) with $\bar{F}\cdot$ and $\bar{L}\cdot$ we can solve for $\frac{dw_i}{d\gamma}$ and $\frac{d\rho_i}{d\gamma}$:

$$\frac{dw_i}{d\gamma} = -\left\{\frac{dt_i}{d\gamma}\left[(\bar{F}_i\cdot\bar{L}_i)(\bar{S}_i\cdot\bar{L}_i) - (\bar{S}_i\cdot\bar{F}_i)\right] + (\bar{F}_i\cdot\bar{L}_i)(\bar{F}_i \times \bar{L}_i)\cdot(\bar{A}_i - \bar{G}_i) + \frac{(\bar{F}_i \times \bar{S}_i)\cdot(\bar{L}_i \times \bar{F}_i)}{(\bar{S}_i \times \bar{L}_i)\cdot(\bar{L}_i \times \bar{F}_i)}(\bar{F}_i \times \bar{L}_i)\cdot(\bar{A}_i - \bar{B}_i)\right\}\frac{1}{(\bar{F}_i \times \bar{L}_i)^2} \tag{34a}$$

$$\frac{d\rho_i}{d\gamma} = \frac{dt_i}{d\gamma}(\bar{S}_i\cdot\bar{L}_i) - \frac{dw_i}{d\gamma}(\bar{F}_i\cdot\bar{L}_i) - (\bar{L}_i \times \bar{F}_i)\cdot(\bar{A}_i - \bar{G}_i) \tag{35a}$$

Operating on (32) with $(\bar{F} \times \bar{L})\cdot$ yields

$$\frac{d\phi_i}{d\gamma} = -\frac{(\bar{F}_i \times \bar{L}_i)\cdot[\bar{F}_i \times (\bar{A}_i - \bar{G}_i)]}{(\bar{F}_i \times \bar{L}_i)\cdot[\bar{L}_i \times (\bar{A}_i - \bar{B}_i)]} \tag{36a}$$

Similarly operating on (33) with $\bar{M}_i\cdot$, $\bar{N}_i\cdot$, and $(\bar{M}_i \times \bar{N}_i)\cdot$ yield Equations

(34b), (35b), and (36b) for $\frac{du_i}{d\gamma}$, $\frac{dv_i}{d\gamma}$, and $\frac{d\beta_i}{d\gamma}$ respectively. Equations (34), (35), and (36) are analogous to (6) - (10).

Equations (11), (12), and (13) now become respectively:

$$\frac{dw_i}{d\gamma} = P_f, \quad \frac{du_i}{d\gamma} = P_m \frac{d\alpha_i}{d\gamma}, \text{ and } \frac{dv_i}{d\gamma} = P_n \frac{d\beta_i}{d\gamma} \tag{37}$$

For k_i infinitesimally separated positions from position "i", the design equations are (27)-(31), (34)-(37) and the k_i-1 derivatives of these equations with respect to γ. For example, the design equation from (28) would be

$$\frac{d^n \phi_i}{d\gamma^n} = \frac{d^{n-1}}{d\gamma^{n-1}} \left[\frac{(\bar{L}_i \times \bar{F}_i)\cdot(\bar{F}_i \times \bar{S}_i)}{(\bar{S}_i \times \bar{L}_i)\cdot(\bar{L}_i \times \bar{F}_i)} \right], \quad n=1,2,\ldots,k_i$$

In the derivative equations we consider $\frac{d^m \bar{S}_i}{d\gamma^m}$, $\frac{d^m \bar{\lambda}_i}{d\gamma^m}$, $\frac{d^m \theta_i}{d\gamma^m}$, $\frac{d^m t_i}{d\gamma^m}$; m=n-1, n=1,2,...,$k_i$, as specified quantities which describe the k_i infinitesimally separated positions of body o. We treat $\frac{d^m \bar{L}_i}{d\gamma^m}$ and $\frac{d^m \bar{B}_i}{d\gamma^m}$ as two sets of $2k_i$ independent unknowns. All other derivatives on the right-hand-side can be expressed in terms of $\bar{F}_i, \bar{M}_i, \bar{N}_i, \bar{G}_i, \bar{Q}_i$, and $\bar{R}_i$ using the following:

$$\frac{d\bar{M}_i}{d\gamma} = \bar{F}_i \times \bar{M}_i + \frac{\partial \bar{M}_i}{\partial \alpha} \left(\frac{d\alpha_i}{d\gamma}\right)$$

$$\frac{d^2 \bar{M}_i}{d\gamma^2} = \bar{F}_i \times (\bar{F}_i \times \bar{M}_i) + \frac{d\bar{F}_i}{d\gamma} \times \bar{M}_i + 2\bar{F}_i \times \frac{\partial \bar{M}_i}{\partial \alpha} \left(\frac{d\alpha_i}{d\gamma}\right) + \left(\frac{d\alpha_i}{d\gamma}\right)^2 \frac{\partial^2 \bar{M}_i}{\partial \alpha^2} + \frac{\partial \bar{M}_i}{\partial \alpha}\left(\frac{d^2\alpha_i}{d\gamma^2}\right)$$

etc. - - - - - - - - - - - -

$$\frac{d\bar{N}_i}{d\gamma} = \bar{F}_i \times \bar{N}_i + \frac{d\alpha_i}{d\gamma} (\bar{M}_i \times \bar{N}_i) + \frac{\partial \bar{N}_i}{\partial \beta} \left(\frac{d\beta_i}{d\gamma}\right)$$

$$\begin{aligned}\frac{d^2 \bar{N}_i}{d\gamma^2} = {} & \bar{F}_i \times (\bar{F}_i \times \bar{N}_i) + 2\frac{d\alpha_i}{d\gamma} \bar{F}_i \times (\bar{M}_i \times \bar{N}_i) + \frac{d^2\alpha_i}{d\gamma^2} (\bar{M}_i \times \bar{N}_i) + \left(\frac{d\alpha_i}{d\gamma}\right)^2 \bar{M}_i \times (\bar{M}_i \times \bar{N}_i) + \frac{d\bar{F}_i}{d\gamma} \times \bar{N}_i \\ & + \bar{F}_i \times \frac{\partial \bar{N}_i}{\partial \beta}\left(\frac{d\beta_i}{d\gamma}\right) + \frac{\partial \bar{M}_i}{\partial \alpha} \times \bar{N}_i \left(\frac{d\alpha_i}{d\gamma}\right)^2 + \bar{M}_i \times \frac{\partial \bar{N}_i}{\partial \beta} \left(\frac{d\beta_i}{d\gamma}\right)\left(\frac{d\alpha_i}{d\gamma}\right) + \left(\frac{\partial \bar{N}_i}{\partial \beta}\right)\frac{d^2\beta_i}{d\gamma^2} + \frac{\partial^2 \bar{N}_i}{\partial \beta^2}\left(\frac{d\beta_i}{d\gamma}\right)^2 \\ & + \bar{F}_i \times \frac{\partial \bar{N}_i}{\partial \beta}\left(\frac{d\beta_i}{d\gamma}\right) + \bar{M}_i \times \frac{\partial \bar{N}_i}{\partial \beta}\left(\frac{d\alpha_i}{d\gamma}\right)\left(\frac{d\beta_i}{d\gamma}\right)\end{aligned}$$

etc. - - - - - - - - - - - -

$$\frac{d\bar{Q}_i}{d\gamma} = \bar{F}_i \times (\bar{Q}_i - \bar{G}_i) + \frac{dw_i}{d\gamma} \bar{F}_i$$

$$\frac{d^2 \bar{Q}_i}{d\gamma^2} = \bar{F}_i \times [\bar{F}_i \times (\bar{Q}_i - \bar{G}_i)] + \frac{d^2 w_i}{d\gamma^2} \bar{F}_i + \frac{d\bar{F}_i}{d\gamma} \left[\times (\bar{Q}_i - \bar{G}_i) + \frac{dw_i}{d\gamma}\right] - \bar{F}_i \times \frac{d\bar{G}_i}{d\gamma}$$

etc. - - - - - - - - - - - -

$$\frac{d\bar{R}_i}{d\gamma} = \bar{F}_i \times (\bar{R}_i - \bar{G}_i) + \frac{d\alpha_i}{d\gamma}\bar{M}_i \times (\bar{R}_i - \bar{Q}_i) + \frac{dw_i}{d\gamma}\bar{F}_i + \frac{du_i}{d\gamma}\bar{M}_i$$

$$\frac{d^2\bar{R}_i}{d\gamma^2} = \bar{F}_i \times [\bar{F}_i \times (\bar{R}_i - \bar{G}_i)] + \frac{d\alpha_i}{d\gamma}\left\{\bar{F}_i \times [\bar{M}_i \times (\bar{R}_i - \bar{Q}_i)] + (\bar{F}_i \times \bar{M}_i) \times (\bar{R}_i - \bar{Q}_i) + \bar{M}_i \times [\bar{F}_i \times (\bar{R}_i - \bar{Q}_i)]\right\}$$

$$+ \left(\frac{d\alpha_i}{d\gamma}\right)^2 \bar{M}_i \times [\bar{M}_i \times (\bar{R}_i - \bar{Q}_i)] + 2\frac{du_i}{d\gamma}\bar{F}_i \times \bar{M}_i + \frac{d^2\alpha_i}{d\gamma^2}\bar{M}_i \times (\bar{R}_i - \bar{Q}_i) + \frac{d^2u_i}{d\gamma^2}\bar{M}_i + \frac{d^2w_i}{d\gamma^2}\bar{F}_i$$

$$+ \frac{d\bar{F}}{d\gamma} \times (\bar{R}_i - \bar{G}_i) + \frac{dw_i}{d\gamma}\left(\frac{d\bar{F}_i}{d\gamma}\right) - \bar{F}_i \times \frac{d\bar{G}_i}{d\gamma} + \left(\frac{du_i}{d\gamma}\right)\left(\frac{d\alpha_i}{d\gamma}\right)\frac{\partial \bar{M}_i}{\partial \alpha} + \frac{\partial \bar{M}_i}{\partial \alpha} \times (\bar{R}_i - \bar{Q}_i)\left(\frac{d\alpha_i}{d\gamma}\right)^2$$

etc.

If the fixed joint is not a spherical joint then $\frac{d^n\bar{F}_i}{d\gamma^n} = 0$, and $\frac{d^n\bar{G}_i}{d\gamma^n} = 0$, $n = 1, 2, \ldots, k_i - 1$. When the fixed joint is a spherical joint we have $\bar{G}_i = \bar{G}^*$ and $\frac{d^n\bar{G}_i}{d\gamma^n} = 0$, but now $\bar{F}_i$, $\frac{d\bar{F}_i}{d\gamma}, \ldots, \frac{d^{(k_i-1)}\bar{F}_i}{d\gamma^{(k_i-1)}}$ are k_i independent vectors. Similarly, if the first moving joint is not spherical $\frac{\partial^n \bar{M}_i}{\partial \alpha^n} = 0$, but if it is $\frac{\partial^n \bar{M}_i}{\partial \alpha^n}$, $n = 1, 2, \ldots, k_i - 1$, are $k_i - 1$ independent vectors; and $\frac{\partial^n \bar{N}_i}{\partial \theta^n} = 0$ generally but $\frac{\partial^n \bar{N}_i}{\partial \theta^n}$ yield $k_i - 1$ independent vectors depending when the second moving joint is spherical. When two or more joints are spherical we must also use the derivatives of Equations (24)-(26).

For infinitesimally separated positions the results are completely analogous to those given for finite displacements. For mixed finitely and infinitesimally separated positions a suitable combination of Equations (1) through (13) and the derivative equations provide the necessary design equations.

6. Acknowledgment

The financial support of the National Science Foundation (NSF-GK-13930) is appreciatively acknowledged.

7. References

[1] TSAI L.W., ROTH B., Design of Dyads with Helical, Cylindrical, Spherical, Revolute, and Prismatic Joints, Journal of Mechanisms, submitted for publication.

[2] ROTH B., On the Screw Axes and Other Special Lines Associated with Spatial Displacements of a Rigid Body, Transaction A.S.M.E., Series B, 89, 1, 1967, pp. 102-110.

8. Appendix

On the following page we give the design equations for prismatic joints. The results for all nondegenerate triads with prismatic joints are listed in Table II.

For a prismatic joint, one of the "vector-triangles" in the "vector-triangle chain" will degenerate [1]. There are three cases depending upon which of the three joints is prismatic. We now have degenerated forms of Equations (1)-(10). A degenerated form is given the original equation number followed by the letter a, b, or c, corresponding to the case P-R-R, R-P-R, or R-R-P respectively.

(a) P-R-R triads ($\overline{E}_{ij} = \overline{S}_{ij}$)

$$\text{[illegible]} \quad (1a)$$

$$\text{[illegible]} \quad (2a)$$

$$\text{[illegible]} \quad (3a)$$

$$\text{[illegible]} \quad (6a)$$

$$\text{[illegible]} \quad (7a)$$

$$\text{[illegible]} \quad (8a)$$

where d_{ij} is an arbitrary constant. Combining Equations (6a) and (8a), we obtain

$$\text{[illegible]} \quad (A)$$

Note that $\overline{Z}$ disappears from Equations (7a) and (A). Hence in choosing a prismatic joint only the direction of the axis is important, since $\overline{Z}$ can always be arbitrary.

(b) R-P-R triads ($\overline{E}_{ij} = \overline{N}_{ij}$)

$$\text{[illegible]} \quad (1b)$$

$$\phi_{ij} = \psi_{ij} \quad (4b)$$

$$\alpha_{ij} = 0 \quad (5b)$$

$$\text{[illegible]} \quad (8b)$$

$$\text{[illegible]} \quad (9b)$$

$$\text{[illegible]} \quad (10b)$$

Similarly, we combine Equations (8b) and (9b) to

$$\text{[illegible]} \quad (B)$$

Again, $\overline{Z}_{ij}$ disappear from Equation (B).

(c) R-R-P triads ($\overline{E}_{ij} = \overline{N}_{ij}$)

$$\text{[illegible]} \quad (1c)$$

$$\phi_{ij} = 0 \quad (4c)$$

$$\alpha_{ij} = \psi_{ij} \quad (5c)$$

$$\text{[illegible]} \quad (8c)$$

$$\text{[illegible]} \quad (9c)$$

$$\text{[illegible]} \quad (10c)$$

Combining Equation (8c) with (10c), we obtain

$$\text{[illegible]} \quad (C)$$

We now list the appropriate design equations, design parameters, and the possible solutions of triads with a prismatic joint in Table II. Triads with two prismatic joints are not considered here since they are equivalent to dyads with a single prismatic joint, which have been studied in [1].

Table 11

Type of Chains	Design Equations i=1,i-2,3,...,i	Design Parameters j=2,3,...,i	No. of Des. Par. i	No. of Eqns. m	No. of Unknowns n	No. of Possible Chains
F-C-C	[illegible]	[illegible]	3-11	3(i-1)	10+2(i-1)	∞^{11-i}
C-F-C	[illegible]	[illegible]	3-11	3(i-1)	10+2(i-1)	∞^{11-i}
C-C-F	[illegible]	[illegible]	3-11	3(i-1)	10+2(i-1)	∞^{11-i}
F-B-F	[illegible]	[illegible]	3-∞	4(i-1)	8+4(i-1)	∞^{8}
B-F-B	[illegible]	[illegible]	3-∞	4(i-1)	8+4(i-1)	∞^{8}
B-B-B	[illegible]	[illegible]	3-∞	4(i-1)	8+4(i-1)	∞^{8}
F-B-B	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
F-B-B	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
B-F-B	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
B-F-B	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
B-B-F	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
B-B-F	[illegible]	[illegible]	3-10	5(i-1)	9+4(i-1)	∞^{10-i}
F-B-C	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
F-C-B	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
C-F-B	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
B-F-C	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
C-B-F	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
B-C-F	[illegible]	[illegible]	3-6	4(i-1)	10+2(i-1)	∞^{12-2i}
F-B-C	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
F-C-B	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
B-F-C	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
C-F-B	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
B-C-F	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
C-B-F	[illegible]	[illegible]	3-∞	4(i-1)	9+4(i-1)	∞^{9}
F-B-N	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
F-N-B	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
B-F-N	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
N-F-B	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
B-N-F	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
N-B-F	[illegible]	[illegible]	2-11	7(i-1)	10+6(i-1)	∞^{11-i}
F-N-C	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
F-C-N	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
C-F-N	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
N-F-C	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
C-N-F	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
N-C-F	[illegible]	[illegible]	3-6	6(i-1)	11+6(i-1)	∞^{13-2i}
F-N-B	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
F-B-N	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
F-B-N	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
B-F-N	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
B-N-F	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
N-B-F	[illegible]	[illegible]	3-6	7(i-1)	11+6(i-1)	∞^{14-3i}
F-N-N	[illegible]	[illegible]	3-5	9(i-1)	12+6(i-1)	∞^{15-3i}
N-F-N	[illegible]	[illegible]	3-5	9(i-1)	12+6(i-1)	∞^{15-3i}
N-N-F	[illegible]	[illegible]	3-5	9(i-1)	12+6(i-1)	∞^{15-3i}
F-N-N	[illegible]	[illegible]	2-6	5(i-1)	10+2(i-1)	∞^{13-3i}
N-F-N	[illegible]	[illegible]	2-6	5(i-1)	10+2(i-1)	∞^{13-3i}
N-N-F	[illegible]	[illegible]	2-6	5(i-1)	10+2(i-1)	∞^{13-3i}

In this table F,M,N,L_{ij},F_{ij},R_{ij},N_{ij},C,Q,R,C*,Q*,R*, and a_{ij} are all vectors.

TSAI L.W., ROTH B.

DESIGN OF TRIADS USING THE SCREW TRIANGLE CHAIN

Summary

Using screw-axis geometry, equations are derived for the design of spatial three-link chains with helical, revolute, cylindrical, spherical, and prismatic joints. All such chains are designed to be compatible with a specific set of finitely separated or infinitesimally separated positions of a rigid body which they couple to a fixed link. Equations giving the maximum number of design positions are presented.

ENTWURF VON DREIGELENKKETTEN MITTELS DIE SCHRAUBENDREIECKKETTE

Zusammenfassung

Die Schrauben-Achsen Geometrie benutzend, wird Gleichungen abgeleitet für den Entwurf von dreigliedriger Raumketten mit Schraubegelenken, Drehgelenken, Zylindergelenken, Sphäregelenken, und Gleitgelenken. Alle solche Ketten sind verträglich mit einer bestimmten Anzahl von endlich benachbarte und unendlich benachbarte Lagen eines festen Körpers, den sie mit einem Gestell verbinden. Gleichungen für die maximal Anzahl der Entwurfslagen sind angegeben.

Incompletely Specified Displacements: Geometry and Spatial Linkage Synthesis

LUNG-WEN TSAI
Research Assistant.

BERNARD ROTH
Professor.
Mem. ASME

Department of Mechanical Engineering,
Stanford University,
Stanford, Calif.

The screw axis geometry associated with displacements of points and lines is studied. Analytical expressions are developed for rigid body screw displacements which have one or more free parameters. It is shown how to apply these results to the synthesis of spatial linkages. The theory is illustrated by numerical examples in which Cylindric-Cylindric cranks are designed to guide two points in a rigid body through five and then nine specified positions.

Introduction

In this paper we consider rigid body displacements which are described by less than six independent parameters. We call such displacements *incompletely specified* as opposed to *completely specified* displacements which are described by six parameters. If b parameters are specified, there remain 6-b unspecified parameters, and therefore $\infty^{(6-b)}$ possible displacements. It is well known that a completely specified displacement is associated with a unique screw displacement. It then follows that an incompletely specified displacement is associated with $\infty^{(6-b)}$ screw displacements. In this paper we determine the $\infty^{(6-b)}$ screw displacements associated with two positions of the following elements:

$b = 5$: two points

$b = 4$: a line

$b = 3$: a point

$b = 2$: a point on a line

$b = 1$: a point on a plane

In the first part of this paper we consider each of the above in turn and seek to establish canonical descriptions of the screw displacements from a geometric point of view. Then we develop analytical expressions for the associated screws. The description of finitely-separated incompletely-specified displacements is followed by a similar development for infinitesimally separated displacements.

The results of the first part of this paper may be applied directly to the synthesis of spatial linkages. In previous works [5, 8] it has been shown how to synthesize binary-link chains which can be used to guide rigid bodies through specified screw displacements. In order to extend these previous results to include incompletely specified displacements it is only necessary that the completely specified screws referred to in [5, 8] be replaced by the incompletely specified screws described herein. This procedure is discussed and illustrated in the latter part of this paper. As examples, we show how to synthesize Cylindric-Cylindric cranks to guide a rigid body through first five and then nine incompletely specified design positions.

Although there have been many publications dealing with the synthesis of spatial linkages [1–19][1] relatively few of these have considered problems in which the precision positions are incompletely specified [1, 2, 3, 19]. It is hoped that this study will help to stimulate interest in this branch of kinematic synthesis.

Incompletely Specified Displacements

In the design of machines it is often necessary that some element of a link, such as a line or a point, be in specified positions but, except for this element, the position of the link is arbitrary within certain practical limits. We use the term *incompletely specified* positions to denote such situations. For spatial displacements it requires six independent parameters to uniquely define a single displacement of a link. Incompletely specified displacements denote those displacements for which some of those six parameters may be chosen arbitrarily. There are many ways to describe incompletely specified displacements. In this work we will use screw axis geometry. The reasons for this choice are that we believe such descriptions to be simple and physically appealing. Moreover, the use of screw axes allows these new results to be most easily incorporated into formulations of linkage synthesis problems previously published by the authors.

The five cases of incompletely specified displacements listed in the introduction seem to be the ones of greatest practical interest. There are of course other possibilities not treated herein. We

Contributed by the Design Engineering Division and presented at the Mechanisms Conference, San Francisco, Calif., October 8–12, 1972, of The American Society of Mechanical Engineers. Manuscript received at ASME Headquarters, June 6, 1972. Paper No. 72-Mech-13.

[1] Numbers in brackets designate References at end of paper.

have limited ourselves to the cases involving linear displacements of linear elements. In each case the development follows the same pattern: we first try to identify a screw which is uniquely determined by the given parameters, then we identify an incompletely determined screw which when compounded with the unique one gives the required incompletely specified displacement. These two screws together with their resultant form a screw triangle. From the screw triangle geometry we then derive an analytical description for the resultant screw. In what follows we consider each of the five cases in turn, first we deal with finitely separated displacements and then with infinitesimally separated displacements.

Screw Axis Geometry—Finitely Separated Positions

Two points. Let r and s be two points of a rigid body σ, and $\mathbf{r}$, $\mathbf{s}$ the position vectors of r and s. When two positions of r and s are specified, there exists a given pure-rotation screw $\hat{T}_{ij}$ such that, no matter how the motion of σ with respect to a reference frame Σ actually occurs, a rotation of the rigid body will bring both points r and s from their ith to their jth position. As shown in Fig. 1, we denote the two given positions of r and s as (r_i, s_i) and (r_j, s_j), the plane which is perpendicular to and bisects s_is_j as π_s; and the plane which bisects r_ir_j perpendicularly as π_r. The axis of the rotation screw $\hat{T}_{ij}$ is the line of intersection of planes π_r and π_s; and the rotation angle ϕ_{ij} is the angle between the normals from r_i and r_j (as well as s_i and s_j) to the axis of $\hat{T}_{ij}$. Let $\mathbf{l}_{ij}$ be a unit vector parallel to the screw axis and $\mathbf{b}_{ij}$ be the position vector of a point on $\hat{T}_{ij}$ so that they satisfy

$$\mathbf{l}_{ij} \cdot \mathbf{l}_{ij} = 1 \quad (1)$$

$$\mathbf{l}_{ij} \cdot \mathbf{b}_{ij} = 0 \quad (2)$$

Analytically, $\mathbf{l}_{ij}$ and $\mathbf{b}_{ij}$ have to satisfy the following conditions:

$$\mathbf{l}_{ij} \cdot (\mathbf{r}_j - \mathbf{r}_i) = 0 \quad (3)$$

$$\mathbf{l}_{ij} \cdot (\mathbf{s}_j - \mathbf{s}_i) = 0 \quad (4)$$

$$r_j^2 - r_i^2 - 2(\mathbf{r}_j - \mathbf{r}_i)\cdot\mathbf{b}_{ij} = 0 \quad (5)$$

$$s_j^2 - s_i^2 - 2(\mathbf{s}_j - \mathbf{s}_i)\cdot\mathbf{b}_{ij} = 0 \quad (6)$$

Solving equations (1) through (6) for $\mathbf{l}_{ij}$ and $\mathbf{b}_{ij}$ we obtain

$$\mathbf{l}_{ij} = \frac{(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{s}_j - \mathbf{s}_i)}{|(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{s}_j - \mathbf{s}_i)|} \quad (7)$$

$$\mathbf{b}_{ij} = \frac{1}{2\mathbf{l}_{ij}\cdot(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{s}_j - \mathbf{s}_i)} [(s_j^2 - s_i^2)\mathbf{l}_{ij} \times (\mathbf{r}_j - \mathbf{r}_i) + (r_j^2 - r_i^2)(\mathbf{s}_j - \mathbf{s}_i) \times \mathbf{l}_{ij}] \quad (8)$$

where $|(\quad)|$ represents the absolute value of $(\quad)$. The rotation angle ϕ_{ij} is given by

$$\tan \phi_{ij} = \frac{\mathbf{l}_{ij}\cdot[(\mathbf{s}_j - \mathbf{r}_j) \times (\mathbf{s}_i - \mathbf{r}_i)]}{(\mathbf{s}_j - \mathbf{r}_j)\cdot(\mathbf{s}_i - \mathbf{r}_i) - [\mathbf{l}_{ij}\cdot(\mathbf{s}_j - \mathbf{r}_j)][\mathbf{l}_{ij}\cdot(\mathbf{s}_i - \mathbf{r}_i)]} \quad (9)$$

Since specifying the position of two points is equivalent to specifying the position of one point plus a direction $\mathbf{n}$ determined by two points, we may rewrite equations (7), (8), and (9) in terms of $\mathbf{r}$ and the direction $\mathbf{n}$ as follows

$$\mathbf{l}_{ij} = \frac{(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{n}_j - \mathbf{n}_i)}{|(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{n}_j - \mathbf{n}_i)|} \quad (10)$$

$$\mathbf{b}_{ij} = \frac{1}{|(\mathbf{r}_j - \mathbf{r}_i) \times (\mathbf{n}_j - \mathbf{n}_i)|}\left[(\mathbf{r}_j\cdot\mathbf{n}_j - \mathbf{r}_i\cdot\mathbf{n}_i)\mathbf{l}_{ij} \times (\mathbf{r}_j - \mathbf{r}_i) + \frac{1}{2}(r_j^2 - r_i^2)(\mathbf{n}_j - \mathbf{n}_i) \times \mathbf{l}_{ij}\right] \quad (11)$$

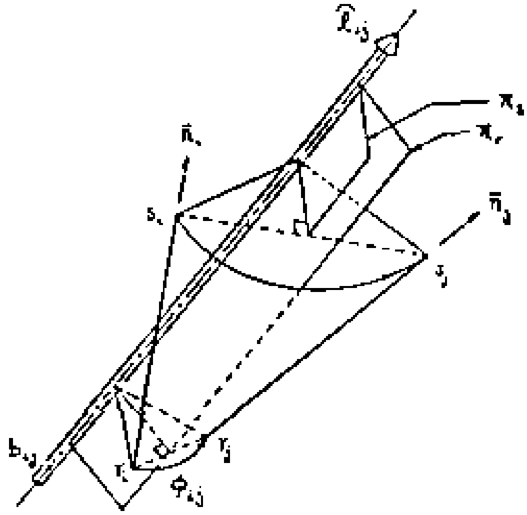


Fig. 1 Rotational screw $\hat{T}_{ij}$ associated with two positions of points r and s

$$\tan \phi_{ij} = \frac{\mathbf{l}_{ij}\cdot(\mathbf{n}_i \times \mathbf{n}_j)}{\mathbf{n}_i\cdot\mathbf{n}_j - (\mathbf{n}_i\cdot\mathbf{l}_{ij})(\mathbf{n}_j\cdot\mathbf{l}_{ij})} \quad (12)$$

where

$$\mathbf{n}_i = (\mathbf{s}_i - \mathbf{r}_i)/|(\mathbf{s}_i - \mathbf{r}_i)|, \qquad \mathbf{n}_j = (\mathbf{s}_j - \mathbf{r}_j)/|(\mathbf{s}_j - \mathbf{r}_j)| \quad (13)$$

The Screw Cylindroid

We have shown that for two positions of points r and s there exists a pure-rotation screw $\hat{T}_{ij}$. However, this is not the only possible screw displacement which can effect the displacement through the two given positions of r and s. In what follows, we find all of the possible screws which satisfy the same specifications of r and s.

We use $\hat{n}_i$ (or $\hat{n}_{ij}$)[1] to denote a pure-rotation screw for the rigid body σ about the line n_i which passes through points r_i and s_i. Since rotation of the rigid body about the axis of $\hat{n}_{ij}$ does not affect the displacement of r and s, the resultant of screws $\hat{n}_{ij}$ and $\hat{T}_{ij}$ is a screw which will bring r and s from their ith to their jth position. As shown in Fig. 2, we have the screw $\hat{S}_{ij}$ as the resultant of a rotation of angle ψ_{ij} about the axis of $\hat{n}_{ij}$, followed by a rotation of ϕ_{ij} about the axis of $\hat{T}_{ij}$. Screws $\hat{n}_{ij}$, $\hat{T}_{ij}$, and $\hat{S}_{ij}$ form a spatial screw triangle [5, 6, 7, 8] as shown in Fig. 2.

$\hat{S}_{ij}$ is given in terms of $\hat{n}_{ij}$ and $\hat{T}_{ij}$ as follows:

$$\mathbf{S}_{ij} = k_{ij}\left(\mathbf{n}_{ij} \tan\frac{\psi_{ij}}{2} + \mathbf{l}_{ij} \tan\frac{\phi_{ij}}{2} + \mathbf{l}_{ij} \times \mathbf{n}_{ij} \tan\frac{\psi_{ij}}{2} \tan\frac{\phi_{ij}}{2}\right) \quad (14)$$

$$\tan\frac{\theta_{ij}}{2} = \frac{1}{k_{ij}\left[1 - (\mathbf{n}_{ij}\cdot\mathbf{l}_{ij}) \tan\frac{\psi_{ij}}{2} \tan\frac{\phi_{ij}}{2}\right]} \quad (15)$$

$$\mathbf{A}_{ij} = \frac{1}{\mathbf{S}_{ij}\cdot(\mathbf{l}_{ij} \times \mathbf{n}_{ij})} [c_{ij}(\mathbf{S}_{ij} \times \mathbf{l}_{ij}) + d_{ij}(\mathbf{n}_{ij} \times \mathbf{S}_{ij})] \quad (16)$$

$$\frac{t_{ij}}{2} = k_{ij}(\mathbf{r}_{ij} - \mathbf{b}_{ij})\cdot(\mathbf{n}_{ij} \times \mathbf{l}_{ij}) \tan\frac{\psi_{ij}}{2} \tan\frac{\phi_{ij}}{2} \quad (17)$$

where

---

[1] A single subscript is used to denote the position of the system. When double subscripts are used, the 1st subscript "i" denotes that the system is in position "i," and the second subscript "j" denotes that the system will go to the jth position after the displacement.

$$c_{ij} = r_{ij}\cdot[\mathbf{S}_{ij} \times (\mathbf{n}_{ij} \times \mathbf{S}_{ij})] + \frac{(\mathbf{n}_{ij} \times \mathbf{S}_{ij})^2}{(\mathbf{n}_{ij} \times \mathbf{l}_{ij})^2}\,[\mathbf{l}_{ij} \times (\mathbf{n}_{ij} \times \mathbf{l}_{ij})]\cdot(\mathbf{b}_{ij} - \mathbf{r}_{ij}) \quad (18)$$

$$d_{ij} = \mathbf{b}_{ij}\cdot[\mathbf{S}_{ij} \times (\mathbf{l}_{ij} \times \mathbf{S}_{ij})] + \frac{(\mathbf{l}_{ij} \times \mathbf{S}_{ij})^2}{(\mathbf{n}_{ij} \times \mathbf{l}_{ij})^2}\,[\mathbf{n}_{ij} \times (\mathbf{l}_{ij} \times \mathbf{n}_{ij})]\cdot(\mathbf{r}_{ij} - \mathbf{b}_{ij}) \quad (19)$$

$$\frac{1}{k_{ij}^2} = \tan^2\frac{\psi_{ij}}{2} + \tan^2\frac{\phi_{ij}}{2} + 2(\mathbf{n}_{ij}\cdot\mathbf{l}_{ij})\tan\frac{\psi_{ij}}{2}\tan\frac{\phi_{ij}}{2} + (\mathbf{n}_{ij}\times\mathbf{l}_{ij})^2\tan^2\frac{\psi_{ij}}{2}\tan^2\frac{\phi_{ij}}{2} \quad (20)$$

Given two positions of pointer r and s, screw $\widehat{\mathrm{T}}_{ij}$ is uniquely defined and so is the screw axis for $\widehat{\mathrm{n}}_{ij}$, only the rotation angle ψ_{ij} is arbitrary. Hence corresponding to each ψ_{ij} there is a screw $\widehat{\mathrm{S}}_{ij}$. The screw axis of $\widehat{\mathrm{S}}_{ij}$ generates a ruled surface as ψ_{ij} varies from 0 deg to 360 deg. It can be shown that this ruled surface is a cylindroid.

A line. Let n be the line of interest in σ, and e the line in Σ with which n coincides. For two positions, say the ith and jth positions, we have two lines in Σ which we will denote as e^i and e^j. It is then given that in the ith position line n (i.e., n_i) is on e^i and in the jth position line n (n_j) is on e^j. We define the line n by its direction $\mathbf{n}$ and the position vector to one of its points $\mathbf{r}$, and the line e by its direction $\mathbf{e}$ and the position vector of one of its points $\mathbf{c}$. We may take $\mathbf{c}^i = \mathbf{r}_i$ and $\mathbf{e}^i = \mathbf{n}_i$. Since n_j is on e^j, $\mathbf{e}^j = \mathbf{n}_j$, and the position vector $\mathbf{r}_j$ can be expressed as $\mathbf{r}_j = \mathbf{c}^j + \lambda_{ij}\mathbf{e}^j = \mathbf{c}^j + \lambda_{ij}\mathbf{n}_j$, where λ_{ij} is an arbitrary number.

In what follows we seek all screws such that the jth position of n can be obtained by screwing the rigid body, from its ith position. First we note there is a special screw $\widehat{\mathrm{T}}_{ij}$ with parameters ϕ_{ij}, μ_{ij} such that it is uniquely determined by the given two lines. The axis of $\widehat{\mathrm{T}}_{ij}$ being the common normal of the two given lines, e^i and e^j, defined by two positions of n. ϕ_{ij} is the angle of twist between the two directed lines e^i and e^j, μ_{ij} is the shortest distance between e^i and e^j. Denote $\mathbf{l}_{ij}$ as a unit vector parallel to the common normal of e^i and e^j, and $\mathbf{b}_{ij}$ as the position vector to the point in the common normal where $\mathbf{b}_{ij}\cdot\mathbf{l}_{ij} = 0$. Then the screw $\widehat{\mathrm{T}}_{ij}$ is analytically given by

$$\mathbf{l}_{ij} = (\mathbf{n}_i \times \mathbf{n}_j)/|\mathbf{n}_i \times \mathbf{n}_j| \quad (21)$$

$$\mathbf{b}_{ij} = |\mathbf{n}_i \times \mathbf{n}_j|\{[\mathbf{c}^j\cdot(\mathbf{l}_{ij} \times \mathbf{n}_j)]\mathbf{n}_i - [\mathbf{c}^i\cdot(\mathbf{l}_{ij} \times \mathbf{n}_i)]\mathbf{n}_j\} \quad (22)$$

$$\tan\phi_{ij} = 1/[(\mathbf{n}_i\cdot\mathbf{n}_j)|\mathbf{n}_i \times \mathbf{n}_j|] \quad (23)$$

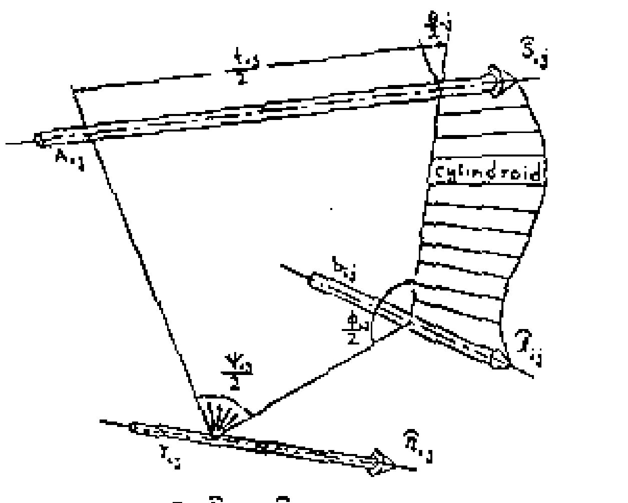


Fig. 2 Screw triangle $\widehat{n}_{ij}$, $\widehat{T}_{ij}$ and $\widehat{S}_{ij}$ is defined by two positions of r and s. As ψ_{ij} varies $\widehat{S}_{ij}$ generates a screw cylindroid.

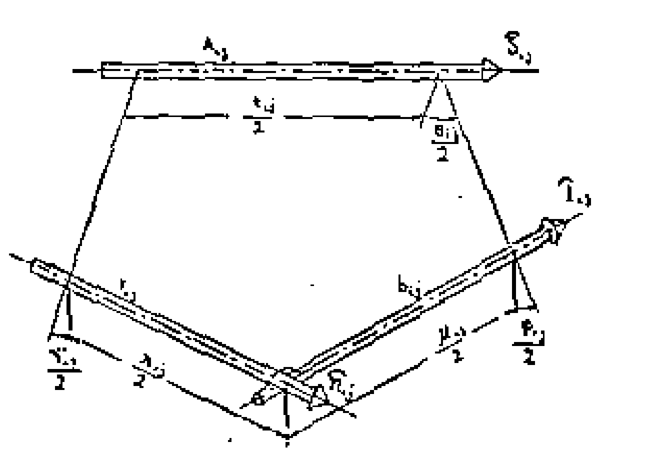


Fig. 3 Screw triangle defined by two positions of a line. As ψ_{ij} and λ_{ij} vary, $\widehat{S}_{ij}$ generates a screw congruence

$$\mu_{ij} = (\mathbf{c}^j - \mathbf{c}^i)\cdot\mathbf{l}_{ij} \quad (24)$$

We now proceed to find all possible screws which satisfy the same specifications. Denote a screw displacement of σ about the line n_i as $\widehat{\mathrm{n}}_{ij}$ and its screw parameters as ψ_{ij} and λ_{ij}. Since any displacement about the line n does not affect the displacement of the line itself, the resultant of screws $\widehat{\mathrm{n}}_{ij}$ and $\widehat{\mathrm{T}}_{ij}$ is a screw which will yield the required displacement of the line. If we consider the resultant screw, which will be denoted as $\widehat{\mathrm{S}}_{ij}$, as the product of a screw displacement $\widehat{\mathrm{n}}_{ij}$ followed by the screw displacement $\widehat{\mathrm{T}}_{ij}$, the screws $\widehat{\mathrm{n}}_{ij}$, $\widehat{\mathrm{T}}_{ij}$, and $\widehat{\mathrm{S}}_{ij}$ form a spatial screw triangle as shown in Fig. 3. Note that, in this case, the axis of screw $\widehat{\mathrm{T}}_{ij}$ intersects the axis of $\widehat{\mathrm{n}}_{ij}$ perpendicularly. The analytic expressions for screw $\widehat{\mathrm{S}}_{ij}$ are as follows:

$$\mathbf{S}_{ij} = k_{ij}\left(\mathbf{n}_{ij}\tan\frac{\psi_{ij}}{2} + \mathbf{l}_{ij}\tan\frac{\phi_{ij}}{2} + \mathbf{l}_{ij} \times \mathbf{n}_{ij}\tan\frac{\psi_{ij}}{2}\tan\frac{\phi_{ij}}{2}\right) \quad (25)$$

$$\mathbf{A}_{ij} = \frac{1}{\mathbf{S}_{ij}\cdot(\mathbf{l}_{ij} \times \mathbf{n}_{ij})}\,[c_{ij}(\mathbf{S}_{ij} \times \mathbf{l}_{ij}) + d_{ij}(\mathbf{n}_{ij} \times \mathbf{S}_{ij})] \quad (26)$$

$$\tan\frac{\theta_{ij}}{2} = \frac{1}{k_{ij}} \quad (27)$$

$$\frac{t_{ij}}{2} = \frac{1}{2}\,k_{ij}\left(\lambda_{ij}\tan\frac{\psi_{ij}}{2} + \mu_{ij}\tan\frac{\phi_{ij}}{2}\right) \quad (28)$$

where

$$\left(\frac{1}{k_{ij}}\right)^2 = \tan^2\frac{\psi_{ij}}{2} + \tan^2\frac{\phi_{ij}}{2} + \tan^2\frac{\psi_{ij}}{2}\tan^2\frac{\phi_{ij}}{2} \quad (29)$$

$$c_{ij} = (\mathbf{n}_{ij}\cdot\mathbf{r}_{ij}) - (\mathbf{n}_{ij}\cdot\mathbf{S}_{ij})(\mathbf{S}_{ij}\cdot\mathbf{r}_{ij}) + (\mathbf{n}_{ij} \times \mathbf{S}_{ij})^2\left[\mathbf{n}_{ij}\cdot(\mathbf{b}_{ij} - \mathbf{r}_{ij}) - \frac{\lambda_{ij}}{2}\right] \quad (30)$$

$$d_{ij} = -(\mathbf{l}_{ij}\cdot\mathbf{S}_{ij})(\mathbf{S}_{ij}\cdot\mathbf{b}_{ij}) + (\mathbf{l}_{ij} \times \mathbf{S}_{ij})^2\left(\mathbf{l}_{ij}\cdot\mathbf{r}_{ij} + \frac{\mu_{ij}}{2}\right) \quad (31)$$

Screw $\widehat{\mathrm{T}}_{ij}$ is unique as is also the axis of screw $\widehat{\mathrm{n}}_{ij}$. It follows then that since both ψ_{ij} and λ_{ij} are arbitrary, we have ∞^2 screws. If we hold λ_{ij} constant and vary ψ_{ij} from 0 deg to 360 deg, we obtain a screw cylindroid similar to the one given for two positions of two points. In this case, there are a single infinity of cylindroids on which the ∞^2 possible screw axes must lie.

A point. Let r be the point of interest on a rigid body σ. For

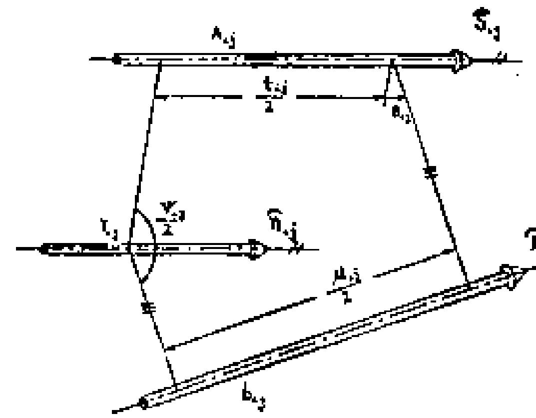


Fig. 4 Screw triangle defined by two positions of a point. As n_{ij} and ψ_{ij} vary, $\hat{S}_{ij}$ generates a screw complex.

two positions of r, we can always find a direction $\mathbf{l}_{ij}$ such that a translation of the rigid body in the direction of $\mathbf{l}_{ij}$ will bring the point r from its ith to jth position.[3] The direction $\mathbf{l}_{ij}$ is parallel to the vector determined by $\mathbf{r}_j - \mathbf{r}_i$, and the magnitude of the translation μ_{ij} is equal to the distance between points r_i and r_j, i.e.,

$$\mathbf{l}_{ij} = (\mathbf{r}_j - \mathbf{r}_i)/\mu_{ij} \tag{32}$$

$$\mu_{ij} = |\mathbf{r}_j - \mathbf{r}_i| \tag{33}$$

Since any rotation of the rigid body about the point r does not affect the position of r itself, all possible screw displacements which yield the same two positions of r can be obtained as the resultant of the translation given by screw $\hat{T}_{ij}$ and an arbitrary rotation of ψ_{ij} about the point r_i. Denoting the rotation screw about point r_i as $\hat{n}_{ij}$, and the resultant screw of $\hat{n}_{ij}$ and $\hat{T}_{ij}$ as $\hat{S}_{ij}$, we have the screw triangle configuration shown in Fig. 4. The resultant screw $\hat{S}_{ij}$ is given as follows (see reference [19] for derivation).

$$\mathbf{S}_{ij} = \mathbf{n}_{ij} \text{ (arbitrary)} \tag{34}$$

$$\theta_{ij} = \psi_{ij} \text{ (arbitrary)} \tag{35}$$

$$\mathbf{A}_{ij} = \frac{1}{(\mathbf{n}_{ij} \times \mathbf{l}_{ij})^2}\{c_{ij}(\mathbf{n}_{ij} \times \mathbf{l}_{ij}) + d_{ij}[\mathbf{l}_{ij} - (\mathbf{l}_{ij}\cdot\mathbf{n}_{ij})\mathbf{n}_{ij}]\} \tag{36}$$

$$\frac{t_{ij}}{2} = \frac{\mu_{ij}}{2}(\mathbf{l}_{ij}\cdot\mathbf{n}_{ij}) - \frac{(\mathbf{l}_{ij}\cdot\mathbf{n}_{ij})^2}{(\mathbf{l}_{ij} \times \mathbf{n}_{ij})^2}(\mathbf{n}_{ij}\cdot\mathbf{r}_{ij}) \tag{37}$$

where

$$c_{ij} = \mathbf{r}_{ij}\cdot\mathbf{n}_{ij} \times \mathbf{l}_{ij} + \frac{\mu_{ij}}{2}(\mathbf{l}_{ij} \times \mathbf{n}_{ij})^2/\tan\frac{\psi_{ij}}{2} \tag{38}$$

$$d_{ij} = (\mathbf{l}_{ij} \times \mathbf{n}_{ij})^2\frac{\mu_{ij}}{2} + \mathbf{r}_{ij}\cdot[\mathbf{l}_{ij} - (\mathbf{l}_{ij}\cdot\mathbf{n}_{ij})\mathbf{n}_{ij}] \tag{39}$$

Since $\mathbf{n}_{ij}$ and ψ_{ij} are arbitrary, $\mathbf{S}_{ij}$ and θ_{ij} are arbitrary. Hence for two positions of a point, the direction and the rotation angle of the displacement screw can be chosen arbitrarily. However, once the direction and the rotation angle of the screw are chosen, its location and translational displacement are determined by equations (36) and (37). There are ∞^2 possible screws in this case.

A point anywhere on a given line. Let r be the point of interest in σ, and e a given line in Σ on which point r is to be located as shown in Fig. 5. For two positions, say the ith and jth positions, we are given two lines in Σ which we will denote as e^i and e^j. It is then required that r_i be somewhere on e^i and r_j be somewhere on e^j. We may then assume that the ith position of r is known and the jth position is indeterminate along e^j. Since r_j has to lie on the line e^j, its coordinates can be expressed as

$$\mathbf{r}_j = \mathbf{c}^j + \lambda_j\mathbf{e}^j \tag{40}$$

where c^j is any given point on the line e^j and $\mathbf{e}^j$ is a unit vector parallel to the line e^j. Using this convention for $\mathbf{r}_j$, all of the arguments used in the previous section are again valid. Hence we have

$$\mathbf{l}_{ij} = (\mathbf{c}^j + \lambda_j\mathbf{e}^j - \mathbf{r}_i)/\mu_{ij} \tag{41}$$

$$\mu_{ij} = |\mathbf{c}^j + \lambda_j\mathbf{e}^j - \mathbf{r}_i| \tag{42}$$

where λ_j is an arbitrary number which provides an additional degree of freedom for point r. All of the possible screws which yield the given two positions of a point r anywhere on the given lines e are determined by equations (34) through (39), with the condition that equations (41) and (42) be used for $\mathbf{l}_{ij}$ and μ_{ij} instead of (32) and (33). Here we have ∞^3 screws.

A point anywhere on a given plane. Let r be the point of interest in σ, and π a given plane in Σ on which r is to be located. We define π in terms of one of its points c and its unit normal u as shown in Fig. 6.

For two such positions of r we have two different planes in Σ on which r must lie. For convenience we denote these different planes as π^i and π^j. We may consider the first position, i, of r as known and express the second position, j, as follows

$$\mathbf{r}_j = \mathbf{c}^j + \lambda_j\mathbf{e}^j \tag{43}$$

where $\mathbf{e}^j$ is a unit vector in the plane π^j, i.e., $\mathbf{e}^j\cdot\mathbf{u}^j = 0$ and $\mathbf{e}^j\cdot\mathbf{e}^j = 1$. Thus we have one undetermined parameter for $\mathbf{e}^j$ and two undetermined parameters for r_j since λ_j is also arbitrary. Using equation (43) for $\mathbf{r}_j$, we have

$$\mathbf{l}_{ij} = (\mathbf{c}^j + \lambda_j\mathbf{e}^j - \mathbf{r}_i)/\mu_{ij} \tag{44}$$

[3] For a pure translation the location of the screw axis is arbitrary and need not be considered in our discussion.

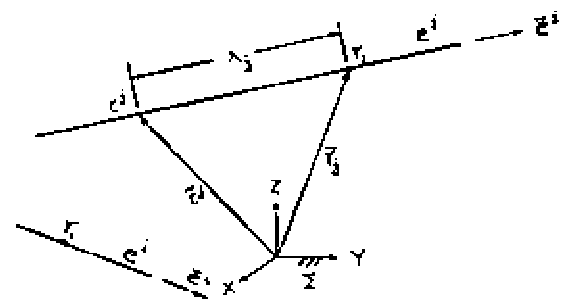


Fig. 5 Two positions of a point which must lie on given lines

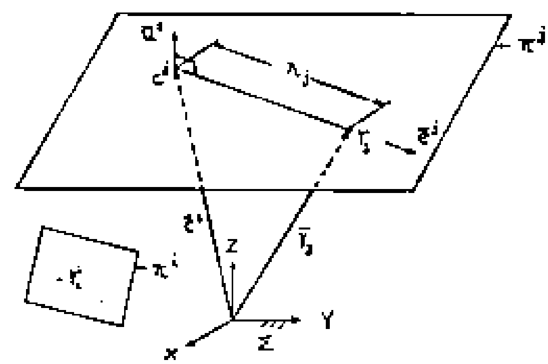


Fig. 6 Two positions of a point which must lie on given planes

$$\mu_{ij} = |c^j + \lambda_i s^j - r_i| \quad (45)$$

With this change equations (34) through (39) may then be used to obtain the resultant screw $\hat{S}_{ij}$. In this case there are ∞^4 possible screws. In addition it is noted that since there are two free parameters for l_{ij}, l_{ij} can be arbitrarily chosen.

Screw Axis Geometry—Infinitesimally Separated Positions

We will now extend the preceding study to include the screw axis geometry associated with two infinitesimally-separated incompletely-specified positions. For infinitesimally-separated incompletely-specified position the gross displacement of the rigid body σ is infinitesimally small. All of the notations used in this section will be the same as those used for finitely separated positions.

Two points. Let the jth position of points r and s be infinitesimally separated from the ith position. We may write r_j and s_j in terms of r_i, s_i and their infinitesimal changes dr_i, ds_i as follows:

$$r_j = r_i + \frac{dr_i}{d\tau} d\tau \quad (46)$$

$$s_j = s_i + \frac{ds_i}{d\tau} d\tau \quad (47)$$

where τ is the motion parameter.

Upon substitution of (46) and (47) into (7), (8), and (9), neglecting all of the higher order terms, we obtain the infinitesimal pure rotation screw $\hat{I}_i$ as follows:

$$l_i = \left(\frac{dr_i}{d\tau} \times \frac{ds_i}{d\tau}\right) \Big/ \left|\frac{dr_i}{d\tau} \times \frac{ds_i}{d\tau}\right| \quad (48)$$

$$b_i = \left[\left(s_i \cdot \frac{ds_i}{d\tau}\right)\left(l_i \times \frac{dr_i}{d\tau}\right) + \left(r_i \cdot \frac{dr_i}{d\tau}\right)\left(\frac{ds_i}{d\tau} \times l_i\right)\right] \Big/ \left|\frac{dr_i}{d\tau} \times \frac{ds_i}{d\tau}\right| \quad (49)$$

$$\frac{d\phi_i}{d\tau} = \frac{l_i \cdot \left[\left(\frac{ds_i}{d\tau} - \frac{dr_i}{d\tau}\right) \times (s_i - r_i)\right]}{(s_i - r_i)^2 - [l_i \cdot (s_i - r_i)]^2} \quad (50)$$

In terms of r_i and n_i, we have

$$l_i = \left(\frac{dr_i}{d\tau} \times \frac{dn_i}{d\tau}\right) \Big/ \left|\frac{dr_i}{d\tau} \times \frac{dn_i}{d\tau}\right| \quad (51)$$

$$b_i = \left[\left(n_i \cdot \frac{dr_i}{d\tau} + r_i \cdot \frac{dn_i}{d\tau}\right)\left(l_i \times \frac{dr_i}{d\tau}\right) + \left(r_i \cdot \frac{dr_i}{d\tau}\right)\left(\frac{dn_i}{d\tau} \times l_i\right)\right] \Big/ \left|\frac{dr_i}{d\tau} \times \frac{dn_i}{d\tau}\right| \quad (52)$$

$$\frac{d\phi_i}{d\tau} = \left[l_i \cdot \left(n_i \times \frac{dn_i}{d\tau}\right)\right] \Big/ (n_i \times l_i)^2 \quad (53)$$

where

$$n_i = (s_i - r_i)/|s_i - r_i| \quad (54)$$

Instantaneous Screw Cylindroid

Similar to the case of finitely separated positions, an infinitesimal rotation of the rigid body σ about the line n_i passing through r_i and s_i does not affect the specification of r and s. Denote the infinitesimal screw displacement about the line n_i as $\hat{n}_i(n_i, r_i, d\psi_i, 0)$. The resultant of $\hat{n}_i$ and $\hat{I}_i$ is an infinitesimal screw displacement which carries r_i and s_i into r_j and s_j. Taking $\hat{S}_i(S_i, A_i, d\theta_i, dt_i)$ as the resultant, we have the following kinematic relations:

$$\frac{d\theta_i}{d\tau} S_i = \frac{d\psi_i}{d\tau} n_i + \frac{d\phi_i}{d\tau} l_i \quad (55)$$

$$\frac{dt_i}{d\tau} S_i = \frac{d\psi_i}{d\tau} n_i \times (A_i - r_i) + \frac{d\phi_i}{d\tau} l_i \times (A_i - b_i) \quad (56)$$

Solving (55) for S_i and $\frac{d\theta_i}{d\tau}$, yields

$$S_i = k_i \left(\frac{d\psi_i}{d\tau} n_i + \frac{d\phi_i}{d\tau} l_i\right) \quad (57)$$

$$\frac{d\theta_i}{d\tau} = \frac{1}{k_i} \quad (58)$$

where

$$\left(\frac{1}{k_i}\right)^2 = \left(\frac{d\psi_i}{d\tau}\right)^2 + \left(\frac{d\phi_i}{d\tau}\right)^2 + 2(n_i \cdot l_i)\left(\frac{d\psi_i}{d\tau}\right)\left(\frac{d\phi_i}{d\tau}\right) \quad (59)$$

Operating on (56) with $S_i \cdot$ we obtain

$$\frac{dt_i}{d\tau} = k_i \left(\frac{d\psi_i}{d\tau}\right)\left(\frac{d\phi_i}{d\tau}\right)(r_i - b_i) \cdot (n_i \times l_i) \quad (60)$$

Operating on (56) with $n_i \cdot$ and $(n_i \times l_i) \cdot$, yields

$$A_i \cdot (n_i \times l_i) = b_i \cdot (n_i \times l_i) + \frac{dt_i}{d\tau} (S_i \cdot n_i) \Big/ \left(\frac{d\phi_i}{d\tau}\right) \quad (61)$$

$$A_i \cdot \left[(n_i \times l_i) \times n_i \frac{d\psi_i}{d\tau} + (n_i \times l_i) \times l_i \frac{d\phi_i}{d\tau}\right] = r_i \cdot [(n_i \times l_i) \times n_i] \frac{d\psi_i}{d\tau} + b_i \cdot [(n_i \times l_i) \times l_i] \frac{d\phi_i}{d\tau} \quad (62)$$

Since A_i is an arbitrary point on the screw axis of $\hat{S}_i$, we may choose it at the position where

$$A_i \cdot S_i = 0 \quad (63)$$

Solving (61) through (63) for A_i, we obtain

$$A_i = \frac{1}{(n_i \times l_i)^2} \left\{c_i k_i^2 \left[(n_i \times l_i) \times n_i \frac{d\psi_i}{d\tau} + (n_i \times l_i) \times l_i \frac{d\phi_i}{d\tau}\right] + d_i (n_i \times l_i)\right\} \quad (64)$$

where

$$c_i = r_i \cdot [(n_i \times l_i) \times n_i] \frac{d\psi_i}{d\tau} + b_i \cdot [(n_i \times l_i) \times l_i] \frac{d\phi_i}{d\tau} \quad (65)$$

$$d_i = b_i \cdot (n_i \times l_i) + (S_i \cdot n_i)\left(\frac{dt_i}{d\tau}\right) \Big/ \left(\frac{d\phi_i}{d\tau}\right) \quad (66)$$

Equations (57), (58), (60), and (64) represent all of the infinitesimal screw displacements which satisfy the same infinitesimally separated positions of r and s. Since $\frac{d\psi_i}{d\tau}$ is arbitrary, the screw axis of $\hat{S}_i$ generates an (infinitesimal) screw cylindroid as $\frac{d\psi_i}{d\tau}$ varies from $-\infty$ to $+\infty$.

A line. For infinitesimally separated positions, we define the jth position of e in terms of e^i and its infinitesimal changes, de^i and dc^i, as follows:

$$e^j = e^i + \frac{de^i}{d\tau} d\tau, \quad \text{or} \quad n_j = n_i + \frac{dn_i}{d\tau} d\tau \quad (67)$$

$$c^j = c^i + \frac{dc^i}{d\tau} d\tau \quad (68)$$

Then the corresponding equations of (21) through (24) for screw $\hat{\mathbf{l}}_i$ are:

$$\mathbf{l}_i = \left(\mathbf{n}_i \times \frac{d\mathbf{n}_i}{d\tau}\right) \bigg/ \left|\mathbf{n}_i \times \frac{d\mathbf{n}_i}{d\tau}\right| \tag{69}$$

$$\mathbf{b}_i = \left[\left(\frac{d\mathbf{n}_i}{d\tau}\right)^2 (\mathbf{c}^i \cdot \mathbf{n}_i) + \left(\frac{d\mathbf{c}^i}{d\tau}\right)\left(\frac{d\mathbf{n}_i}{d\tau}\right)\right]\mathbf{n}_i + \left(\mathbf{c}^i \frac{d\mathbf{n}_i}{d\tau}\right)\frac{d\mathbf{n}_i}{d\tau} \tag{70}$$

$$\frac{d\phi_i}{d\tau} = 1 \bigg/ \left|\mathbf{n}_i \times \frac{d\mathbf{n}_i}{d\tau}\right| \tag{71}$$

$$\frac{d\mu_i}{d\tau} = \frac{d\mathbf{c}^i}{d\tau} \cdot \mathbf{l}_i \tag{72}$$

The rigid body, in this case, can have an arbitrary infinitesimal rotation $d\psi_i$ as well as an infinitesimal translation $d\lambda_i$ along the line $\mathbf{n}_i$ without affecting the specification of the line. All of the possible screws, which are the resultants of $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{l}}_i$, are given by:

$$\mathbf{S}_i = k_i\left(\frac{d\psi_i}{d\tau}\mathbf{n}_i + \frac{d\phi_i}{d\tau}\mathbf{l}_i\right) \tag{73}$$

$$\frac{d\theta_i}{d\tau} = \frac{1}{k_i} \tag{74}$$

$$\mathbf{A}_i = \left[c_i\left(-\frac{d\psi_i}{d\tau}\mathbf{l}_i + \frac{d\phi_i}{d\tau}\mathbf{n}_i\right) + d_i(\mathbf{n}_i \times \mathbf{l}_i)\right] \bigg/ (\mathbf{n}_i \times \mathbf{l}_i)^2 \tag{75}$$

$$\frac{dt_i}{d\tau} = k_i\left[\left(\frac{d\lambda_i}{d\tau}\right)\left(\frac{d\psi_i}{d\tau}\right) + \left(\frac{d\mu_i}{d\tau}\right)\left(\frac{d\phi_i}{d\tau}\right) + \left(\frac{d\psi_i}{d\tau}\right)\left(\frac{d\phi_i}{d\tau}\right)(\mathbf{b}_i - \mathbf{r}_i)\cdot(\mathbf{n}_i \times \mathbf{l}_i)\right] \tag{76}$$

where

$$\left(\frac{1}{k_i}\right)^2 = \left(\frac{d\psi_i}{d\tau}\right)^2 + \left(\frac{d\phi_i}{d\tau}\right)^2 \tag{77}$$

$$c_i = k_i^2(\mathbf{b}_i \cdot \mathbf{n}_i - \mathbf{r}_i \cdot \mathbf{l}_i) \tag{78}$$

$$d_i = \left[k_i\left(\frac{d\psi_i}{d\tau}\right)\left(\frac{d\mu_i}{d\tau}\right) - \frac{d\lambda_i}{d\tau} + \mathbf{b}_i\cdot(\mathbf{n}_i \times \mathbf{l}_i)\frac{d\phi_i}{d\tau}\right] \bigg/ \left(\frac{d\phi_i}{d\tau}\right) \tag{79}$$

A point. For two infinitesimally separated positions of point r, the corresponding screw $\hat{\mathbf{l}}_i$ is given by

$$\mathbf{l}_i = \left(\frac{d\mathbf{r}_i}{d\tau}\right) \bigg/ \left(\frac{d\mu_i}{d\tau}\right) \tag{80}$$

$$\frac{d\mu_i}{d\tau} = \left|\frac{d\mathbf{r}_i}{d\tau}\right| \tag{81}$$

$$\frac{d\phi_i}{d\tau} = 0, \quad \text{and} \quad \mathbf{b}_i \text{ is arbitrary.}$$

The rigid body σ can have an arbitrary infinitesimal rotation $d\psi_i$ about any axis $\mathbf{n}_i$, which passes through the point r_i, without affecting the displacement of r. Denote this pure rotational screw as $\hat{\mathbf{n}}_i$. Then the resultant screw displacement $\hat{\mathbf{S}}_i$ is related to $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{l}}_i$ as follows:

$$\frac{d\theta_i}{d\tau}\mathbf{S}_i = \frac{d\psi_i}{d\tau}\mathbf{n}_i \tag{82}$$

$$\frac{dt_i}{d\tau}\mathbf{S}_i = \frac{d\mu_i}{d\tau}\mathbf{l}_i + \frac{d\psi_i}{d\tau}\mathbf{n}_i \times (\mathbf{A}_i - \mathbf{r}_i) \tag{83}$$

From (82) and (83), we can solve for $\hat{\mathbf{S}}_i$:

$$\mathbf{S}_i = \mathbf{n}_i \text{ (arbitrary)} \tag{84}$$

$$\frac{d\theta_i}{d\tau} = \frac{d\psi_i}{d\tau} \tag{85}$$

$$\mathbf{A}_i = [c_i\mathbf{n}_i \times (\mathbf{l}_i \times \mathbf{n}_i) + d_i(\mathbf{n}_i \times \mathbf{l}_i)]/(\mathbf{n}_i \times \mathbf{l}_i)^2 \tag{86}$$

$$\frac{dt_i}{d\tau} = \frac{d\mu_i}{d\tau}(\mathbf{n}_i \cdot \mathbf{l}_i) \tag{87}$$

where

$$c_i = \mathbf{l}_i \cdot \mathbf{r}_i + (\mathbf{n}_i \cdot \mathbf{r}_i)(\mathbf{n}_i \cdot \mathbf{l}_i) \tag{88}$$

$$d_i = \mathbf{r}_i \cdot (\mathbf{l}_i \times \mathbf{n}_i) + (\mathbf{n}_i \times \mathbf{l}_i)^2 \frac{d\mu_i}{d\tau} \bigg/ \left(\frac{d\psi_i}{d\tau}\right) \tag{89}$$

Equations (84) through (87) represent all of the possible screw displacements defined by two infinitesimally separated positions of point r.

A point anywhere on a given line. For two infinitesimally separated positions, we may assume the ith position of point r is known, and the jth position of r lies in the vicinity of a point $\mathbf{r}_j^*$, where $\mathbf{r}_j^*$ is a given point on e^j whose position can be expressed in terms of $\mathbf{r}_i$ and its infinitesimal change $d\mathbf{r}_i$ as follows:

$$\mathbf{r}_j^* = \mathbf{r}_i + \frac{d\mathbf{r}_i}{d\tau}d\tau \tag{90}$$

Thus we may express the position of $\mathbf{r}_j$ as a function of an infinitesimal parameter $d\lambda_i$ which is measured along e^j:

$$\mathbf{r}_j = \mathbf{r}_i + \frac{d\mathbf{r}_i}{d\tau}d\tau + \frac{d\lambda_i}{d\tau}d\tau\,\mathbf{e}^j \tag{91}$$

where we regard $\frac{d\mathbf{r}_i}{d\tau}$ as a specified vector.

Using this convention, we have the infinitesimal translation screw $\hat{\mathbf{l}}_i$ as follows:

$$\mathbf{l}_i = \left(\frac{d\mathbf{r}_i}{d\tau} + \frac{d\lambda_i}{d\tau}\mathbf{e}^j\right) \bigg/ \left(\frac{d\mu_i}{d\tau}\right) \tag{92}$$

$$\frac{d\mu_i}{d\tau} = \left|\frac{d\mathbf{r}_i}{d\tau} + \frac{d\lambda_i}{d\tau}\mathbf{e}^j\right| \tag{93}$$

$\frac{d\phi_i}{d\tau} = 0$, and $\mathbf{b}_i$ is arbitrary.

All of the possible infinitesimal screw displacements which yield the same specifications are given by equations (84) through (89) with the condition that (92) and (93) be used for $\hat{\mathbf{l}}_i$ instead of (80) and (81).

A point anywhere on a given plane. For two infinitesimally separated positions, we have two infinitesimally separated planes, $\boldsymbol{\pi}^i$ and $\boldsymbol{\pi}^j$, in Σ on which point r must lie. We define $\boldsymbol{\pi}^j$ in terms $\boldsymbol{\pi}^i$ and its infinitesimal changes as follows:

$$\mathbf{c}^j = \mathbf{c}^i + \frac{d\mathbf{c}^i}{d\tau}d\tau \tag{94}$$

$$\mathbf{u}^j = \mathbf{u}^i + \frac{d\mathbf{u}^i}{d\tau}d\tau \tag{95}$$

We may assume the ith position of r coincides with point $\mathbf{c}^i$ and express the jth position of r as

$$\mathbf{r}_i = \mathbf{c}^j + d\lambda_i\mathbf{e}^j \tag{96}$$

where $d\lambda_i$ is an arbitrary infinitesimal quantity and $\mathbf{e}^j$ is an arbitrary unit vector on the plane $\boldsymbol{\pi}^j$. Substituting (94) into (96) yields

$$\mathbf{r}_j = \mathbf{r}_i + \frac{d\mathbf{c}^i}{d\tau}d\tau + \frac{d\lambda_i}{d\tau}d\tau\,\mathbf{e}^j \tag{97}$$

Using this convention for $\mathbf{r}_j$, we obtain the infinitesimal translation screw $\hat{\mathbf{l}}_i$ as follows:

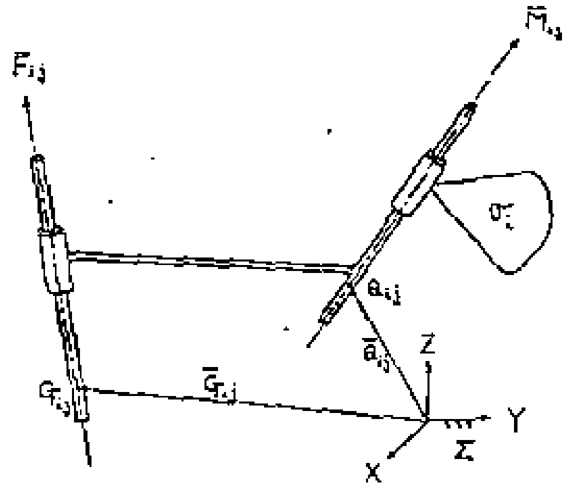


Fig. 7 Cylindric-Cylindric crank

$$l_i = \left(\frac{dc^i}{d\tau} + \frac{d\lambda_i}{d\tau} s^i\right) \Big/ \left(\frac{d\mu_i}{d\tau}\right) \qquad (98)$$

$$\frac{d\mu_i}{d\tau} = \left|\frac{dc^i}{d\tau} + \frac{d\lambda_i}{d\tau} s^i\right| \qquad (99)$$

$$\frac{d\phi_i}{d\tau} = 0, \quad \text{and} \quad b_i \text{ is arbitrary.}$$

All of the possible infinitesimal screw displacements which yield the same specifications are again given by equations (84) through (89) with the condition that (98) and (99) be used for $\bar{l}_i$ instead of (80) and (81).

The Design of Binary Links with Incompletely Specified Precision Positions

With the above mentioned expressions for the screw displacement, $\hat{S}_{ij}$, we can design any binary links with combinations of spherical, pin-in-sphere, cylindric, revolute, helical, and prismatic joints by using the appropriate design equations listed in references [5, 8, 10]. In this case, however, the screw $\hat{S}_{ij}$ is not completely known. The unknown parameters associated with each screw displacement $\hat{S}_{ij}$ need to be determined, and hence should be added into the set of *design parameters* given for completely specified precision positions. In Table 1 we list the additional design parameters associated with each of the above discussed cases of incompletely specified finitely and infinitesimally separated displacements. In Table 1, l is the number of design positions and $m = j - 1$.

The synthesis of binary links for incompletely specified precision positions can be divided into two cases.

(1) When the number of design positions are equal to or less than the maximum number of design positions given for the corresponding completely specified precision position problems, we are at liberty to choose the unknown screw parameters listed in Table 1. Once the unknown screw parameters are chosen, the problem becomes exactly the same as that of the corresponding completely specified precision position problems. Alternatively, we may use the additional degrees of freedom to choose some of the link parameters, and then determine the unknown screw parameters.

(2) When the design positions exceed the maximum number of design positions given for the corresponding completely specified precision position problem, we can no longer arbitrarily choose all (or part) of the unknown screw parameters. Instead

Table 1

Displacement specified in terms of:	Additional design parameters, $j=2,3,4,\cdots,l$	
	Finitely separated positions	Infinitesimally separated positions
Two points	ψ_{ij}	$\frac{d^m\psi_i}{d\tau^m}$
A line	ψ_{ij} and λ_{ij}	$\frac{d^m\psi_i}{d\tau^m}$, $\frac{d^m\lambda_i}{d\tau^m}$
A point	ψ_{ij} and $\bar{a}_{ij}$	$\frac{d^m\psi_i}{d\tau^m}$, $\frac{d^{m-1}\bar{a}_i}{d\tau^{m-1}}$
A point anywhere on a given line	ψ_{ij}, λ_{ij}, and $\bar{a}_{ij}$	$\frac{d^m\psi_i}{d\tau^m}$, $\frac{d^m\lambda_i}{d\tau^m}$, $\frac{d^{m-1}\bar{a}_i}{d\tau^{m-1}}$
A point anywhere on a given plane	ψ_{ij}, λ_{ij}, $\bar{a}_{ij}$, and one component of $\bar{r}^j$	$\frac{d^m\psi_i}{d\tau^m}$, $\frac{d^m\lambda_i}{d\tau^m}$, $\frac{d^{m-1}\bar{a}_i}{d\tau^{m-1}}$, one component of $\frac{d^{m-1}\bar{r}^i}{d\tau^{m-1}}$

they are determined by the constraints imposed on the particular link chain we are synthesizing. To illustrate the technique, we give an example of synthesizing Cylindric-Cylindric cranks when the positions of two points, r and s, of σ are specified.

Design of C-C Cranks for Finitely Separated Positions

Following the notations used in reference [5], denote M as a unit vector parallel to the first position of the moving-joint axis; Q an arbitrary point on the first position of the moving-joint axis; F a unit vector parallel to the fixed-joint axis; and G an arbitrary point on the fixed-joint axis as shown in Fig. 7.

For simplicity, we will formulate the problem by screw displacements from the first position, $\hat{S}_{ij}$, $i = 1, j = 2, 3, 4, \ldots$, so that the vectors F_{ij}, G_{ij}, M_{ij}, Q_{ij}, n_{ij}, and r_{ij} are all in their first positions, and we will omit the subscripts for these vectors.

The design equations are as given in [5]:

$$\tan\frac{\theta_{ij}}{2} = -\frac{F\cdot(S_{ij}\times M)}{(F\times S_{ij})\cdot(S_{ij}\times M)}, \qquad j = 2, 3, \ldots, l \quad (100)$$

$$\frac{t_{ij}}{2} = -\frac{S_{ij}-(S_{ij}\cdot M)M}{1-(S_{ij}\cdot M)^2}\cdot(Q - A_{ij}) + \frac{S_{ij}-(S_{ij}\cdot F)F}{1-(S_{ij}\cdot F)^2}\cdot(G - A_{ij}), \qquad j = 2, 3, \ldots, l \quad (101)$$

where S_{ij}, A_{ij}, θ_{ij}, and t_{ij}, $j = 2, 3, \ldots, l$, are given by equations (14) through (20). From (100) and (101) it follows that there are $2(l-1)$ design equations. Since the design parameters are F, M, G, Q, and ψ_{ij}, $j = 2, 3, \ldots, l$ there are $8 + (l-1)$ design parameters. Hence, the number of possible binary links is ∞^{9-l}, and the number of design positions, l, may range from 2 to 9.

For five or less than five specified design positions, we may arbitrarily choose the unknown screw parameters, ψ_{ij}, $j = 2, 3, \ldots, 5$, and then determine the screws, $\hat{S}_{ij}$, $j = 2, 3, \ldots, 5$, from equations (14) through (20). After the screws are determined, the design procedure is then exactly the same as that outlined for completely specified precision positions in [5].

On the other hand, if we substitute S_{ij}, A_{ij}, θ_{ij}, and t_{ij} from equations (14) through (20) into (100) and (101), we obtain the following two equations:

$$a_{ij}'x_{ij}^2 + b_{ij}'x_{ij} + c_{ij}' = 0 \qquad (102)$$

$$a_{ij}'x_{ij}^5 + b_{ij}'x_{ij}^4 + c_{ij}'x_{ij}^3 + d_{ij}'x_{ij}^2 + e_{ij}'x_{ij} + f_{ij}' = 0 \qquad (103)$$

where

$$x_{ij} = \tan\frac{\psi_{ij}}{2} \qquad (104)$$

$$a_{ij}' = \{[F \times (n \times l_{ij})]\cdot[(n \times l_{ij}) \times M] + (n\cdot l_{ij})F\cdot[(n \times l_{ij}) \times M]\} \tan^2 \frac{\phi_{ij}}{2} - \{(n\cdot l_{ij})F\cdot n \times M - (F \times n)\cdot[(n \times l_{ij}) \times M] - (n \times M)\cdot F \times (n \times l_{ij})\} \tan \frac{\phi_{ij}}{2} + (F \times n)\cdot(n \times M) \quad (105)$$

$$b_{ij}' = -\{(n\cdot l_{ij})F\cdot l_{ij} \times M + (F \times l_{ij})\cdot[(n \times l_{ij}) \times M] + (l_{ij} \times M)\cdot(F \times (n \times l_{ij}))\} \tan^2 \frac{\phi_{ij}}{2} + \{(F \times n)\cdot(l_{ij} \times M) + (F \times l_{ij})\cdot(n \times M) - F\cdot[(n \times l_{ij}) \times M]\} \tan \frac{\phi_{ij}}{2} + F\cdot n \times M \quad (106)$$

$$c_{ij}' = (F \times l_{ij})\cdot(l_{ij} \times M) \tan^2 \frac{\phi_{ij}}{2} + F\cdot(l_{ij} \times M) \tan \frac{\phi_{ij}}{2} \quad (107)$$

and a_{ij}', b_{ij}', c_{ij}', d_{ij}', e_{ij}', and f_{ij}' are fifth degree polynomials in F, M, G, and Q (second degree in F, 2nd degree in M, and first degree in G and Q). We may use Sylvester's dyalitic method to eliminate x_{ij} $\left(\text{i.e., } \tan \frac{\psi_{ij}}{2}\right)$ between (102) and (103). Normally this would yield a 20th degree polynomial in F, M, G, and Q (9th degree in F, 9th degree in M, and 2nd degree in both G and Q) for each j.

Hence, for five positions, $j = 2, 3, 4, 5$, we have four 20th degree polynomials in the eight unknowns F, M, G, and Q (each vector is counted as two unknowns). There are four appropriate free choices. It is most convenient to choose both the direction of the moving-joint axis and the direction of the fixed-joint axis, i.e., F and M. Once F and M are chosen we are left with four second degree polynomials in G and Q. Hence there are at most sixteen possible axis locations, all of which may be easily determined.

Alternatively, we may substitute arbitrary values for F and M into (102) and solve for x_{ij} which will then yield two possible values for ψ_{ij}. Substituting one set of x_{ij} into (103) yields a linear set of four equations in G and Q. Taking all possible combinations of x_{ij} we obtain sixteen sets of *C-C* cranks. This method is the one used in Example 1 which follows.

For nine positions (the maximum number of design positions), we have eight 20th degree polynomials in eight unknowns. This implies at most 20^8 Cylindric-Cylindric cranks, and seems to require a numerical iteration. Alternatively, we can iterate (102) and (103) directly if we consider the x_{ij} as eight additional unknowns. This is the method used in Example 2. The reader should note that $F \| S_{ij}$ or $M \| S_{ij}$ with $j = 2$, or 3, ..., or 9 are all spurious solutions to this problem.

Numerical Examples

We now give two numerical examples. In each we treat the design of *C-C* cranks when two points of the rigid body σ are specified. The given incompletely specified finitely separated positions are in terms of a point r and a direction n as follows:

Positions	r_x	r_y	r_z	n_x	n_y	n_z
1	0.0000	0.0000	0.0000	1.0000	0.0000	0.0000
2	1.1117	0.7674	0.6300	0.9014	0.3659	−0.1065
3	1.9035	2.0877	1.6715	0.7008	0.6563	−0.2706
4	2.0854	2.9447	2.2133	0.3332	0.8977	−0.2884
5	1.7450	4.9021	3.5109	−0.2090	0.9537	−0.0685
6	1.2840	5.8619	5.0169	−0.5136	0.8235	0.1617
7	0.6335	6.1987	7.4780	−0.6975	0.6022	0.3709
8	−0.0075	8.6772	9.5527	−0.7497	0.3816	0.5470
9	−1.0176	6.9228	15.3181	−0.4302	−0.0956	0.8674

The corresponding pure rotation screw, $\hat{T}_{ij}$, are found by equations (10) through (12) to be as follows:

ij	l_x	l_y	l_z	L_x	L_y	L_z	ϕ_{ij} (deg.)
12	−0.6977	0.2372	0.7658	−0.2042	2.5136	−0.9390	32.18
13	−0.6589	0.0191	0.7511	−0.3037	3.1955	−0.3482	61.96
14	−0.5518	−0.1803	0.7909	−0.3032	2.9248	0.4426	90.77
15	−0.3905	−0.4744	0.7890	−0.7023	3.2480	1.6040	121.94
16	−0.2404	−0.6004	0.7627	−1.1160	3.4446	2.3595	129.67
17	−0.0745	−0.6689	0.7396	−1.9785	4.3118	3.7002	134.99
18	0.0459	−0.7394	0.6717	−2.4260	4.2587	4.8515	138.81
19	0.3560	−0.7941	0.4926	−4.2101	3.1310	8.0944	131.35

Example 1: Five Positions. For five positions, we use positions 1, 3, 5, 7, and 9 given above as our design positions, and choose the directions of the moving- and fixed-joint axes as follows:

$$F = (0.2500, 0.4330, 0.8660),$$

$$M = (0.7500, 0.4330, 0.5000).$$

The rotations ψ_{ij}, of the rigid body σ about the line c_i which passes through the two given points r_i and s_i are found from (102) as follows:

	ij	13	15	17	19
ψ_{ij}	1st solution	68.70	108.78	114.84	144.45
(deg.)	2nd solution	71.53	147.82	−176.06	−119.12

The sixteen possible axis locations are found from (103) in which we use all possible combinations of the above ψ_{ij}:

G_x	G_y	G_z	Q_x	Q_y	Q_z
−1.0821	1.7636	0	−0.0263	−4.9128	0
−9.1594	5.5163	0	25.7950	24.0038	0
13.3557	−4.3859	0	−35.5927	−28.6641	0
−11.2023	6.4137	0	31.3419	28.7940	0
5.2768	0.5554	0	−16.0063	−11.9527	0
−0.6306	2.8731	0	1.9227	5.1826	0
0.6396	0.1371	0	−5.2133	3.5327	0
−0.2325	0.3155	0	2.3910	2.3857	0
−0.8427	1.6053	0	−2.0194	−4.6399	0
−5.3772	3.6609	0	12.1501	12.8165	0
−26.7153	12.0432	0	52.2319	44.5895	0
0.9520	1.1191	0	−0.8826	1.2220	0
−17.8167	3.6887	0	31.4494	18.0600	0
−2.8216	3.3197	0	7.0154	9.0557	0
−27.2350	11.7834	0	53.1125	44.7225	0
1.0000	1.0000	0	−1.0000	1.0000	0

For each binary link found above the rotations about the joint axes, α_{ij} and γ_{ij}, and the translations along the joint axes, u_{ij} and w_{ij}, can then be computed by using equations (2), (3), (5), and (6) in reference [5].

Example 2: Nine Positions. For nine positions we require a numerical iteration of equations (102) and (103). One solution for the above specified nine positions is as follows:

	Fixed-joint axis	Moving-joint axis
Direction	0.2500 0.4330 0.8660	0.7500 0.4330 0.5000
Location	1.0000 1.0000 0	−1.0000 1.0000 0

The screw displacements associated with this binary link are found to be:

ij	S_x	S_y	S_z	A_x	A_y	A_z	t_{ij}	θ_{ij} (deg.)
12	0.4682	0.4919	0.7209	−0.2119	0.9831	−0.5077	1.3744	29.48
13	0.5036	0.5226	0.6879	−0.4016	0.7116	−0.2468	3.2350	53.19
14	0.4366	0.5524	0.7101	−0.3730	0.6449	−0.2724	4.1068	79.85
15	0.4192	0.6180	0.6651	−0.6531	0.3916	0.0664	6.2956	124.90
16	0.4190	0.6637	0.6197	−0.9765	0.2588	0.3631	7.5365	150.74
17	−0.3655	−0.7031	−0.6951	−1.6913	0.3359	0.6975	−10.4736	−173.61
18	−0.3413	−0.7394	−0.5604	−2.1461	0.0853	1.1546	−11.9572	168.30
19	−0.2187	−0.6630	−0.4543	−4.5970	−0.6809	3.5071	−14.3513	121.53

Again the rotations about the joints axes, α_{ij} and γ_{ij}, and the translations along the joint axes, u_i, and w_{ij}, can be computed by using equations (2), (3), (5), and (6) in reference [5].

Conclusions

We have introduced the concept of incompletely specified design positions and endeavored to study the displacement of linear elements in a systematic manner. In addition we have indicated how to apply these incompletely specified screw displacements to the design of binary links. In references [5] and [8], we presented a method of using equivalent screw triangles and screw triangle chains for the synthesis of dyads and triads. This present work adds new categories of possible problems which may be treated by the methods of [5] and [8]. In order to tie all of these together we present formulas which establish the multiplicity of possible synthesis solutions, and the maximum number of design positions associated with different types of chains. These formula include both completely and incompletely specified design position problems. The number of possible design solutions for an open loop chain compatible with the given displacements is ∞^q where:

$$q = p - (b - f)(n - 1)$$

- p is the number of independent linkage parameters needed to specify the dimensions of the binary links making up the desired link chain.
- b is an integer between 1 and 6;
  - For completely specified precision position, $b = 6$;
  - For specification of two points, $b = 5$;
  - For specification of a line, $b = 4$;
  - For specification of a point, $b = 3$;
  - For specification of a point on a given line, $b = 2$;
  - For specification of a point on a given plane, $b = 1$
- f is the sum of the independent degrees of freedom in the joints of the open-link chain (for two spherical joints we reduce f by one.)
- n is the number of design positions.

When $q = 0$ there are a finite number of design solutions.

The maximum number of design positions is obtained by solving the following inequality for the positive integer n when the left-hand-side is as close to zero as possible:

$$p - (b - f)(n - 1) \geq 0$$

For example, in the design of a *C-C* crank to guide a rigid body when only the positions of two points are specified, we have: $p = 8$, $b = 5$, and $f = 4$. If we want to design a *C-C* crank for five such positions, then $n = 5$, and $q = p - (b - f)(n - 1) = 8 - (5 - 4)(5 - 1) = 4$. Hence there are four appropriate free choices and we have ∞^4 solutions. To determine the maximum number of design positions we solve the following inequality for n:

$$8 - (5 - 4)(n - 1) \geq 0$$

The maximum n which satisfies the above inequality is 9 and this is the maximum number of design positions.

Note that whenever two spherical joints exist in the open-link chain the degrees of freedom, f, should be one less than the sum of the degrees of freedom associated with each joint thereby eliminating the extra freedom about the center line of the spheres. When m spherical joints exist in the open-link chain, the degrees of freedom should be $m(m - 1)/2$ less than the sum of the degrees of freedom associated with each joint.

The values of p associated with many commonly used kinematic chains are tabulated in [5] and [8]. p is the number of unsubscripted design parameters associated with each chain. The reader should note that special caution is indicated in determining the maximum number of design positions for chains with prismatic joints. If in doubt [5] and [8] should be consulted.

Acknowledgement

The financial support of the National Science Foundation is gratefully acknowledged.

References

1 Roth, B., and Freudenstein, F., "Synthesis of Path-Generating Mechanisms by Numerical Methods," JOURNAL OF ENGINEERING FOR INDUSTRY, Trans. ASME, Series B, Vol. 85, No. 3, Aug. 1963, pp. 298–306.

2 Suh, C. H., "Synthesis and Analysis of Space Mechanisms with Use of Displacement Matrix," Dissertation, University of California, Berkeley, California, 1966.

3 Roth, B., "Finite-Position Theory Applied to Mechanism Synthesis," *Journal of Applied Mechanics*, Trans. of ASME, Series E, Vol. 34, No. 3, Sept. 1967, pp. 599–605.

4 Roth, B., "The Kinematics of Motion through Finitely Separated Positions," *Journal of Applied Mechanics*, Vol. 34, No. 3, Sept. 1967, pp. 591–598.

5 Tsai, L. W., and Roth, B., "Design of Dyads with Helical, Cylindrical, Spherical, Revolute, and Prismatic Joints," *Mechanism and Machine Theory* (formerly *Journal of Mechanisms*), Spring 1972, Vol. 7, No. 1.

6 Roth, B., "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body," JOURNAL OF ENGINEERING FOR INDUSTRY, ASME, Series B, Vol. 89, No. 1, Feb. 1967, pp. 102–110.

7 Roth, B., "The Design of Binary Cranks with Revolute, Cylindric, and Prismatic Joints," *Journal of Mechanisms* (Summer 1968), Vol. 3, No. 2, pp. 61–72.

8 Tsai, L. W., and Roth, B., "Design of Triads Using the Screw Triangle Chain," *Proceedings of the III World Congress, IFToMM*, Dubrovnik, Yugoslavia, Sept. 1971. (Volume D, pp. 273–286).

9 Chen, P., and Roth, B., "A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis," JOURNAL OF ENGINEERING FOR INDUSTRY, Trans. ASME, Series B, Vol. 91, No. 1, Feb. 1969, pp. 203–208.

10 Chen, P., and Roth, B., "Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains," JOURNAL OF ENGINEERING FOR INDUSTRY, Trans. ASME, Series B, Vol. 91, No. 1, Feb. 1969, pp. 209–219.

11 Wilson, J. T., "Analytical Kinematic Synthesis by Finite Displacements," JOURNAL OF ENGINEERING FOR INDUSTRY, *Trans. ASME*, Series B, Vol. 87, No. 1, Feb. 1965, pp. 161–169.

12 Suh, C. H., "Design of Space Mechanisms for Rigid-Body Guidance," JOURNAL OF ENGINEERING FOR INDUSTRY, *Trans. ASME*, Series B, Vol. 89, No. 2, May 1967, pp. 215–222.

13 Suh, C. H., "Design of Space Mechanisms for Function Generation," ibid., pp. 507–512.

14 Sandor, G. N., and Bisshop, K. E., "On a General Quaternion-Operator Method of Spatial Kinematic Synthesis by Means of a Stretch Rotation Tensor," JOURNAL OF ENGINEERING FOR INDUSTRY, *Trans. ASME* Series B, Vol. 91, No. 1, Feb. 1969, pp. 115–121.

15 Brát, V., "Maticová metoda postupných úprav v syntéze prostorových mechanismů," *Strojnícky Časopis* XXI, Č.1, 1969, pp. 43–56.

16 Gupta, V. K., "Optimum Finite Displacement Synthesis of Spatial Linkages," PhD Dissertation, U. C. Berkeley, June 1970, 100 pp.

17 Bagci, C., "Minimum Error Synthesis of Space Mechanisms for the Generation of Constrained and Underconstrained Screws," JOURNAL OF ENGINEERING FOR INDUSTRY, *Trans. ASME*, Series B, Vol. 93, No. 1, Feb. 1971, pp. 165–175.

18 Levitskii, N. I., and Sarkisian, Y., "On the Special Properties of Lagrange's Multipliers in the Least-Square Synthesis of Mechanisms," *Journal of Mechanisms*, Vol. 3, No. 1, 1968, pp. 3–10.

19 Tsai, L. W., "Design of Open Loop Chains for Rigid Body Guidance," PhD Dissertation, Stanford University, 1972.

Printed in U.S.A.

Paper No. 72-Mech-38

an ASME publication

$3.00 PER COPY $1.00 TO ASME MEMBERS

(11)

G. SUTHERLAND
Research Assistant.

B. ROTH
Professor,
Mem. ASME

Department of Mechanical Engineering,
Stanford University,
Stanford, Calif.

A Transmission Index for Spatial Mechanisms

A general index of the quality of motion transmission for spatial mechanisms is developed using the theory of screws. This index is shown to be related to the mechanical error possible in a linkage. A method for synthesizing spatial linkages with desirable motion transmission and mechanical error characteristics is developed and examples are shown for the RGGR linkage.

Introduction

Since its introduction by Alt [1],[1] the transmission angle has served as an indicator of the quality of motion transmission in planar mechanisms. Hain [2] defines the transmission angle for a number of planar mechanisms including linkages, cams, and compound mechanisms. Recent works [3, 4, 5, 6] have dealt with including the transmission angle in the synthesis of planar four-bar linkages.

For spatial linkages, Bagci [7] defines four force transmission indices related to the linkage loadings. Yuan et al [15] considers Ball's virtual coefficient [9] as a transmission factor. However, there does not seem to have been developed a spatial transmission index which not only is independent of the linkages loading, but also is constrained to have finite values for all possible mechanism configurations. A single index of this type would be valuable for comparing the motion characteristics of spatial linkages of different dimensions and type. Such a transmission index (TI) is developed in this paper.

Denavit and Hartenberg [8] (pp. 315–320) show how the transmission angle for the planar four-bar linkage is related to the mechanical error in the linkage due to statistical deviations in the link lengths. The transmission index developed in this paper has a similar relationship to the possible mechanical error in spatial linkages.

If a number of linkages of different dimensions are available which perform the task of path generation, function generation, or rigid body guidance with satisfactory structural error, then there is need for criteria for selecting the best linkages. A good linkage design must fit within the space allotted, have good motion transmission characteristics, and be relatively insensitive to deviations of the link parameters from their ideal values.

In this paper the problem of determining a RGGR function generator with optimum motion transmission and error sensitivity characteristics is dealt with. Computationally efficient procedures can be developed because TI is a single-valued function of the input crank angle for the RGGR linkage. However the overall method used is applicable to all spatial linkages with a total of five degrees of freedom in the link joints not connected to the frame link (hereafter called the "floating joints").

Wrenches, Velocity Screws, and Reciprocity

The force distribution at any joint in a linkage can be represented by its equivalent wrench, where a wrench is defined as a single force combined with a couple in a plane perpendicular to the force [9]. This equivalent wrench is termed the "joint reaction wrench."

The instantaneous velocity distribution in a rigid body relative to some reference frame can be represented by a velocity screw, where a velocity screw is defined as an angular velocity around a straight line combined with a translation along that line. The straight line is called the instantaneous screw axis (ISA) for the relative motion of the body considered.

A screw is a straight line with which a scalar quantity called the "pitch" is associated. Five scalar quantities are required to completely define a screw. A screw can be represented by the unit motor [10]

$$(1 + \epsilon p)\hat{S}$$

where ϵ is the Clifford screw operator, p is the pitch, and $\hat{S}$ is a unit line vector [11] lying on the screw axis.

A wrench can be considered as a screw with which a scalar quantity called the intensity is associated. The intensity is the resultant, F, of the force system considered. A wrench can thus be represented by

$$F(1 + \epsilon p')\hat{s} \qquad (1)$$

where p', the pitch, is the ratio of the magnitude of the wrench couple to the force resultant, and $\hat{s}$ is a unit line vector on the axis of the wrench.

A velocity screw is a screw with which a scalar quantity called

[1] Numbers in brackets designate References at end of paper.

Contributed by the Design Engineering Division for presentation at the Mechanisms Conference, San Francisco, Calif., October 8-12, 1972, of The American Society of Mechanical Engineers. Manuscript received at ASME Headquarters, July 6, 1972. Paper No. 72-Mech-38.

Copies will be available until July, 1973.

Discussion on this paper will be accepted at ASME Headquarters until November 13, 1972

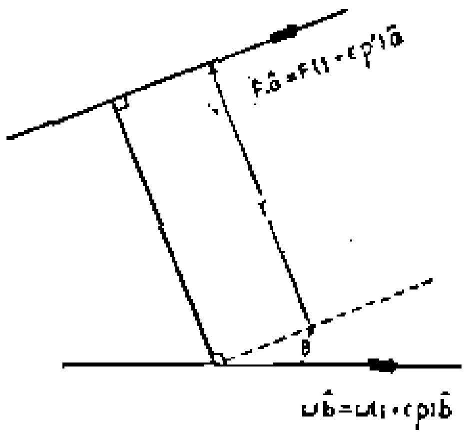


Fig. 1 Reciprocity relation parameters

the amplitude is associated. The amplitude is the angular velocity, ω, of the body considered. A velocity screw can thus be represented by

$$\omega(1 + \epsilon p)\hat{b} \tag{2}$$

where p, the pitch, is the ratio of the magnitude of the sliding velocity of a body fixed point on the ISA to the magnitude of the angular velocity of the body. The unit line vector $\hat{b}$ lies on the ISA of the velocity screw.

Ball defines [9] two screws as being "reciprocal" when a wrench acting on one screw can do no work on a body constrained to move on the other screw. The work per unit time done on a body, which twists about a screw $\hat{b}$, by a wrench on a screw $\hat{a}$ is (Fig. 1)

$$F\omega[(p + p')\cos\theta - r\sin\theta] \tag{3}$$

where r is the distance from $\hat{b}$ to $\hat{a}$ and θ is the angle from $\hat{b}$ to $\hat{a}$ in the direction of a right-handed screw moving along r. The "virtual coefficient" [9] between $\hat{b}$ and $\hat{a}$ is defined to be

$$\hat{\omega}_{ba} = \tfrac{1}{2}[(p + p')\cos\theta - r\sin\theta] \tag{4}$$

Thus two screws are reciprocal when $\hat{\omega}_{ba}$ equals zero.

The relative ISA for two bodies joined by revolute (R), helical (H), cylindrical (C), or ball and socket (G) joints passes through the physical center of the joint. In fact, for R, H, and C joints the ISA has a fixed position relative to either coupled link. For a prismatic (P) joint the location of the relative ISA between the two coupled links is indeterminant and thus may be considered to lie anywhere providing it has the correct direction (i.e., line vector $\hat{b}$ degenerates to a free vector). For the planar (F) joint, the ISA is perpendicular to the containing plane of the joint and moves relative to both coupled links.

Each joint type is characterized by both its reaction wrench and its relative velocity screw. These two quantities are related by the fact that the screw upon which the joint reaction wrench acts is reciprocal to the screw about which the joint relative velocity screw twists. In other words, the joint reaction wrench can effect no relative motion of the two connected bodies.

Table 1 gives a summary of the conditions implied by R, H, P, C, G, and F joints. The force distribution at each joint is resolved into force and moment components (F_x, F_y, F_z, M_x, M_y, M_z). The z axis is taken along the joint relative ISA and the x and y axes lie in any plane perpendicular to the ISA. The G and C joint information in Table 1 requires further explanation. Two collinear screws of arbitrary but different pitch lying on the ISA for a C-joint completely describe the screw system which characterizes that joint. In order for a general screw to be reciprocal to both these collinear screws it must intersect them orthogonally. Hence the joint reaction wrench for a C-joint must intersect that

Table 1 Joint characteristics

Joint	Force Distribution	Condition for ISA and Wrench Intersection ($r = 0$)	Constant Screw Pitches	Reciprocal Condition $\hat{\omega}_{ba} = 0$
R	$M_z = 0$	$F_x M_x + F_y M_y = 0$	$p = 0$	$p' \cos\theta - r\sin\theta = 0$
P	$F_z = 0$	indeterminant	$p = \infty$	[illegible]
H	$\frac{M_z}{F_z} = -\frac{L}{2\pi}$	$p(F_x^2 + F_y^2) + 2\pi(F_x M_x + F_y M_y) = 0$	p = const	$(p + p')\cos\theta - r\sin\theta = 0$
C	$M_z = F_z = 0$	none required	none	$\cos\theta = 0$
G	$M_x = M_y = M_z = 0$	none required	$p = 0$, $p' = 0$	none required
F	$F_x = F_y = M_z = 0$	$M_x = M_y = 0$	$p = 0$, $p' = 0$	$\sin\theta = 0$

joint's relative ISA orthogonally. Three concurrent noncoplanar zero pitch screws completely describe the screw system for the relative ISA of a G-joint. The reaction wrench for a G-joint is of zero pitch and intersects these three screws at their point of concurrency. Thus p, p', and r are equal to zero, satisfying the reciprocity conditions identically for a G-joint.

Transmission Wrench Screw (TWS)

In many mechanism designs the prime function is to transmit motion from the input (driving) member to the output (driven) member. Since the driven member always offers some resistance to motion, power must be supplied to the driven member through the driving member in order to effect the desired motion. This transfer of power can be accomplished either with a direct contact or a lower-pair jointed mechanism. For the direct contact mechanisms the force transmission wrench is simply the contact force between the driven and driving members. For the lower-pair jointed mechanism (a linkage) the nature of the force transmission wrench is less obvious.

The only way force can be transmitted from one link to another is through their connecting joints. Hence the only forces related to the driving link which can be transmitted to the driven link are the joint reaction wrenches exerted on it by the adjacent moving link and the frame link. Since the reaction wrench exerted by the frame link must be reciprocal to the output link velocity screw, the reaction wrench exerted by the moving link is the only input related wrench which can do useful work on the output link. In single loop chains of binary links, if inertial and external forces on the floating links are neglected, each floating (nonframe connected) link is a body in "two-wrench equilibrium."

Equilibrium conditions on each floating link require the two joint reaction wrenches to be of equal magnitude and line of action but of opposite sense. Hence there is only one screw which characterizes the joint reaction wrenches of the floating links of a linkage. This is the screw on which the static force transmission wrench for the linkage acts.

In this paper the intensity of the transmission wrench is of no interest (because it is not geometrical in nature) so that the only concern is the screw on which the transmission wrench acts, the TWS. Five scalar quantities are required to determine any screw; hence, five scalar conditions are required for a TWS. "f" independent screws are required to describe the screw system [9] which characterizes a joint with f degrees of freedom. The reaction wrench at each joint must be reciprocal to all f independent screws characterizing the joint freedom. Hence for the general spatial problem, the summation of the degrees of freedom in the nonframe connected (floating) joints must be five for TWS to be completely determined by reciprocity conditions. Since the reciprocity conditions are purely geometric, the TWS determined by them is independent of the magnitude and form of the loading on the driving and driven links.

Table 2 (with the symbols illustrated in Fig. 2) gives the form of the transmission wrench for a number of typical spatial linkages with five degrees of freedom in their floating joints. The RCCC and RRGG linkages are examples of linkages which do not have a total of five degrees of freedom in their floating joints. For the RCCC linkage the line of action of the TWS is along the common perpendicular of the two floating cylindric ISA's. However the pitch of the TWS is dependent on the form of the loading on the driving and driven links. For the RRGG linkage the TWS pitch is zero and the TWS line of action intersects the floating R-joint ISA and passes through the center of the G-joint. However, the point of intersection of the TWS and the R-joint ISA depends on the driving and driven link loading.

Appendix A contains a calculation of the transmission wrench for a RCRCR linkage using screw algebra and dual quantities.

Transmission Index Function (TI)

The input link causes motion of the output link through the transmission wrench. The output link is instantaneously constrained to twist about its ISA relative to the frame link. Thus the input link tends to move the output link only when the TWS is not reciprocal to the output link velocity screw. The larger the virtual coefficient between the TWS and the velocity screw, the larger the power transfer to the output link for a given transmission wrench intensity and output link velocity screw amplitude.

The virtual coefficient can take on values from minus infinity to plus infinity, so in itself it is not a convenient transmission index. Also, if either p' or p is infinite, which is a definite possibility, the virtual coefficient is meaningless as a transmission index.

However, a variation of the virtual coefficient which does not have the problems that the virtual coefficient has, can be defined. Fig. 3 shows the output link of a linkage with TWS in a general position. ISA_m is the relative ISA for the moving joint connected to the output link, and ISA_f is the ISA for the output link as determined by the joint which attaches the output link to the frame. "C" is the characteristic point for this configuration; it is defined as the point on the TWS where the perpendicular from ISA_m to TWS intersects TWS. "ρ" is the distance from ISA_f to C.

If the transmission wrench is constrained to pass through C, but is otherwise allowed freedom of orientation, then there is some orientation of the TWS such that the virtual coefficient is maximized. The magnitude of the virtual coefficient corresponding to this optimum orientation is

$$\hat{\omega}_{tw,max} = \pm \tfrac{1}{2}[(p + p')^2 + \rho^2]^{1/2} \qquad (5)$$

The transmission index (TI) for a linkage is defined as

$$\mathrm{TI} = \left|\frac{\hat{\omega}_{tw}}{\hat{\omega}_{tw,max}}\right| = \left|\frac{(p + p')\cos\theta - r\sin\theta}{[(p + p')^2 + \rho^2]^{1/2}}\right| \qquad (6)$$

Often the absolute value sign can be avoided by using

$$\mathrm{TI}^2 = \frac{[(p + p')\cos\theta - r\sin\theta]^2}{(p + p')^2 + \rho^2} \qquad (7)$$

Equation (6) is useful as an index even when p and/or p' is infinite. In such cases it reduces to $\mathrm{TI} = |\cos\theta|$ for finite r and ρ. The range of values for TI is from zero to one (hence the range of values for TI^2 is from zero to one). A linkage configuration which has a TI value of one would have optimum static force and motion transmission characteristics. A linkage configuration which has a TI value of zero would have its output link in a dead center position.

If TI is evaluated for a planar four-bar linkage it becomes $|\sin\tau|$ where τ is the transmission angle [1].

Table 3 gives the TI expressions for a number of spatial linkages in terms of the link parameters and pair variables shown in Fig. 2. Appendix B contains the calculations required to determine TI for the RCRCR linkage.

TI can be given a geometric interpretation. Let

$$\mathbf{A} = (p + p')\mathbf{i} - r\mathbf{j} + (\rho^2 - r^2)^{1/2}\mathbf{k}$$

and (8)

$$\mathbf{B} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

Then TI can be written

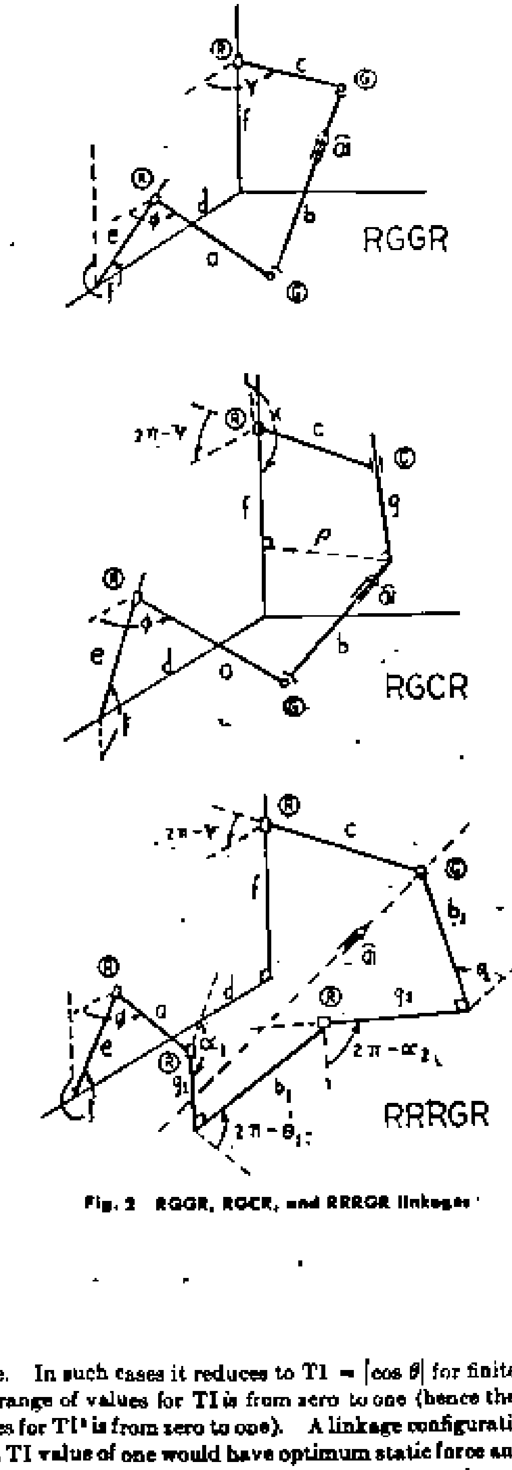


Fig. 2 RGGR, RGCR, and RRRGR linkages

Table 2 Transmission wrench screws $(1 + \epsilon p')\hat{s}$

Linkage	p'	$\hat{s}$
[illegible]	0	passes through each [illegible]-joint
[illegible]	0	passes through G-joint; intersects R-axis orthogonally
[illegible]	0	passes through G-joint; intersects 2 floating R-axes
RCRCR	see Appendix A	intersects 2 floating C-axes orthogonally

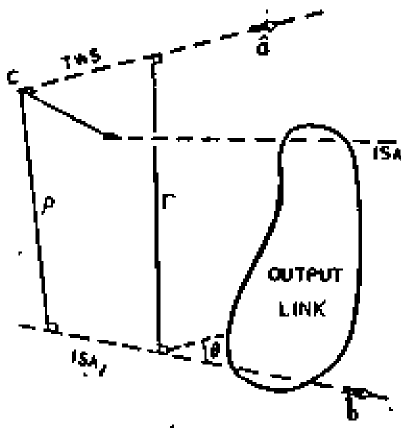


Fig. 3 Parameters for TI expression

$$\mathrm{TI} = \frac{|\mathbf{A}\cdot\mathbf{B}|}{|\mathbf{A}||\mathbf{B}|} = |\cos K| \qquad (9)$$

If K is considered to be the complement of angle τ, then the generalized transmission angle τ for a given linkage is defined to be the complement of the smallest angle between vectors $\mathbf{A}$ and $\mathbf{B}$ defined by (8).

TI and Mechanical Error

When a linkage is used as a function generator a structural error, E_s, exists in the linkage because even a perfect linkage can only approximately generate most functions over a finite range. However, no linkage can be built exactly as specified; thus a mechanical error, E_m, due to link deflections, joint clearances, and manufacturing tolerances must exist.

If the input and output links each have one degree of motion represented by ϕ and ψ respectively, then the linkage mechanical error $\Delta\psi$ for a given ϕ is [8]

$$\Delta\psi = -\sum_{k=1}^{n} \frac{\partial F/\partial q_k}{\partial F/\partial \psi}\,\Delta q_k \qquad (10)$$

where $F(q_1, q_2, \ldots, q_n, \phi, \psi) = 0$ is the linkage closure equation and q_i are the "n" linkage parameters which describe the linkage. The denominator $\partial F/\partial\psi$, which is the same for all Δq_i's, is called the mechanical error denominator (MED). Maximizing the minimum value of MED for the desired range of linkage motion will have the effect of lowering the mechanical error possible in the linkage.

It is shown in reference [8] that

$$\frac{1}{\mathrm{MED}} = \frac{1}{\pm 2bc \sin\tau}$$

for the planar four-bar linkage. ($\sin\tau = \mathrm{TI}$ for the planar four-bar.) This same result, namely

$$\Delta\psi = E_m \,\alpha\, \frac{1}{\mathrm{TI}} \qquad (11)$$

seems to also hold for spatial linkages. This indicates that as TI approaches zero small errors due to manufacturing tolerances or link elasticity can have large effects on the accuracy of the linkage as a function generator.

Appendix C contains a proof of relation (11) for the RGCR linkage. Although no general proof of (11) has been devised which is independent of the type of linkage considered, it has been verified for the RGGR, RGCR, RCGR, and RRRCR linkages.

Position Synthesis Equations for RGGR Function Generation

The closure equation for the RGGR linkage (Fig. 2) can be written as

$$c\phi_i c\psi_i = Q_1 c\psi_i + Q_2 s\psi_i + Q_3 c\phi_i + Q_4 s\phi_i + Q_5 s\phi_i s\psi_i + Q_6 s\phi_i c\psi_i + Q_7 c\phi_i s\psi_i + Q_8 \qquad (12)$$

where

$$Q_1 = \frac{2c}{K}(es\xi s\psi_0 - dc\psi_0)$$

$$Q_2 = \frac{2c}{K}(ds\psi_0 + es\xi c\psi_0)$$

$$Q_3 = \frac{2a}{K}(dc\phi_0 - fs\xi s\phi_0)$$

$$Q_4 = \frac{-2a}{K}(ds\phi_0 + fs\xi c\phi_0)$$

$$Q_5 = \frac{-2ac}{K}(s\psi_0 s\phi_0 + c\xi c\psi_0 c\phi_0) \qquad (13)$$

$$Q_6 = \frac{2ac}{K}(c\psi_0 s\phi_0 - c\xi s\psi_0 c\phi_0)$$

$$Q_7 = \frac{2ac}{K}(s\psi_0 c\phi_0 - c\xi c\psi_0 s\phi_0)$$

$$Q_8 = \frac{1}{K}(a^2 - b^2 + c^2 + d^2 + e^2 + f^2 - 2efc\xi)$$

and

$$K = 2ac(c\psi_0 c\phi_0 + c\xi s\psi_0 s\phi_0)$$

The letters c, s, or t directly in front of an angle denote respectively the cosine, sine, or tangent of that angle.

The inversion formulae for the link parameters in terms of the Q_i's are (with $a = 1$, since only the ratios of the link lengths are important)

$$t\phi_0 = H \pm (1 + H^2)^{1/2}$$

$$H = \frac{(Q_6^2 + Q_5^2 - Q_7^2 - 1)}{2(Q_7 - Q_5Q_6)}$$

$$t\psi_0 = \frac{Q_7 - t\phi_0 Q_5}{1 + t\phi_0 Q_6}$$

$$c\xi = \frac{t\phi_0 - Q_6}{Q_7 - t\phi_0 Q_5}$$

$$c = \frac{c\psi_0(Q_2 t\psi_0 - Q_1)}{c\phi_0(Q_3 - Q_4 t\phi_0)} \qquad (14)$$

$$d = \frac{c\phi_0(Q_2 t\psi_0 - Q_1)}{(1 + t\psi_0 Q_1)}$$

$$e = \frac{c\phi_0(Q_1 t\psi_0 + Q_2)}{s\xi(1 + t\psi_0 Q_7)}$$

$$f = \frac{d(Q_4 + Q_3 t\phi_0)}{s\xi(Q_3 t\phi_0 - Q_4)}$$

$$b = (1 + c^2 + d^2 + e^2 + f^2 - 2efc\xi - KQ_8)^{1/2}$$

There are four groups of link parameters determined by the inversion formulas. (This is in accordance with Levitskii's solution [12] determined using a different coordinate system.) The solutions are in two pairs which behave similarly to the roots of two quadratics. The first pair corresponds to the plus sign in the quadratic for $t\phi_0$; the second pair to the minus sign. For each member of a pair the sign of $s\xi$ differs. Thus the two solutions in each pair differ only in the signs of ξ, e, and f, while ϕ_0, ψ_0, a, b, c, and d are identical. The two solutions in each pair are in fact only mirror images of each other reflected in the plane of motion of either the crank or follower links. Their motion transmission, error sensitivity, and function generating characteristics are identical. Thus only one member (the positive sine case) of each pair need be considered in a synthesis procedure.

Motion Transmission Expressions for RGGR

For the RGGR linkage to have desirable motion characteristics throughout its range of motion, the minimum TI should be as large as possible. An effective criterion for keeping TI large is to minimize

$$\int_{\phi_0}^{\phi_0+\Delta\phi} (1 - \mathrm{TI}^2)d\phi \qquad (15)$$

where TI^2 is expressed as a function of ϕ, the input crank angle, ϕ_0 is the crank angle corresponding to the start of the crank range of rotation, and $\Delta\phi$ is the desired range of crank rotation. [$(1-\mathrm{TI}^2)$ can be thought of as $\cos^2 \tau$ where τ is the transmission angle.]

For the RGGR linkage with

$$\mathrm{TI}^2 = \frac{1}{4b^2c^2}\{4c^2(cs\xi - ac\xi s\phi)^2 + 4c^2(d + ac\phi)^2 - (A + 2adc\phi - 2afs\xi s\phi)^2\} \qquad (16)$$

where

$$A = a^2 - b^2 + c^2 + d^2 + e^2 + f^2 - 2efc\xi$$

by performing the required integrations relation (15) becomes

$$\int_{\phi_0}^{\phi_0+\Delta\phi} (1 - \mathrm{TI}^2)d\phi = \frac{1}{b^2c^2}\Bigg\{\left[c^2(b^2 - d^2 - e^2s^2\xi) + \frac{A^2}{4}\right]\Delta\phi$$
$$+ a^2(d^2 - c^2)\left[\frac{\Delta\phi}{2} + \frac{1}{4}\left(s[2(\phi_0 + \Delta\phi)] - s(2\phi_0)\right)\right]$$
$$+ a^2(f^2s^2\xi - c^2c^2\xi)\left[\frac{\Delta\phi}{2} - \frac{1}{4}\left(s[2(\phi_0 + \Delta\phi)] - s(2\phi_0)\right)\right]$$
$$+ ad(A - 2c^2)[s(\phi_0 + \Delta\phi) - s\phi_0]$$
$$+ as\xi(Af - 2c^2ec\xi)[c(\phi_0 + \Delta\phi) - c\phi_0]$$
$$+ a^2dfs\xi[s^2\phi_0 - s^2(\phi_0 + \Delta\phi)]\Bigg\} \qquad (17)$$

If the desired range of motion of the crank is 360 deg, then (17) reduces to

$$\int_0^{2\pi} (1 - \mathrm{TI}^2)d\phi = \pi\Bigg\{2 - \frac{K^2}{4b^2c^2}(1 + 2Q_1^2 + 2Q_1^2 - Q_i^2 - Q_i^2 + Q_i^2 + Q_i^2 + Q_i^2 - 2Q_i^2)\Bigg\} \qquad (18)$$

where K and the Q_i are defined in equations (13).

Error Sensitivity Expressions for RGGR

The sensitivity of a linkage to deviations in the link parameters can be minimized by maximizing the minimum value of the mechanical error denominator (MED) for the desired range of linkage motion. A suitable criterion for achieving this goal is to maximize

$$\int_{\phi_0}^{\phi_0+\Delta\phi} \mathrm{MED}^2 d\phi \qquad (19)$$

For the RGGR linkage,

$$\mathrm{MED} = \pm 2bc\mathrm{TI} \qquad (20)$$

Therefore expression (19) has expanded forms similar to expressions (17) and (18) for partial and full cycle ranges, respectively.

Dimensional Synthesis Using TI and MED

If equation (12) is written for seven precision positions, then Q_i, $i = 1, \ldots, 7$, can be expressed as linear functions of Q_8. The link parameters in (14) could then also be expressed in terms of Q_8. Thus (although it is not convenient to do so) expressions (15) and (19) could also be written explicitly in terms of Q_8. In the development which follows expressions (15) and (19) are considered as implicit functions of Q_8.

Since only one variable, Q_8, is involved for a given seven position dimensional synthesis problem, expressions (15) and (19) can be plotted as functions of the independent variable Q_8. The local maxima of (19) and minima of (15) can then be determined from the plots. The optimum linkage design may correspond to a minimum of (15) or a maximum of (19). (Both extremums occurring simultaneously is unlikely.) It is, however, possible that the optimum linkage design may be at a point on the two curves which does not correspond to an extremum of either expression. This is especially so when other factors, such as the ratio of the longest to the shortest link length, link interference, etc. enter into the decision of which design to use. However, the plots are excellent indicators of the quality of motion transmission and of the error sensitivity and should provide invaluable aid in picking the best linkage for a particular problem.

Examples

Problem No. 1. Generate the function $y = \sin x$ for $-90 \text{ deg} \leq x \leq 90$ deg using a RGGR linkage with $\Delta\phi = 180$ deg and $\Delta\psi = 90$ deg.

For seven positions with Chebychev spacing expressions (15) and (19) plot as shown in Fig. 4. The solution corresponding to $Q_8 = 1.5$ is chosen as the "optimum" solution and its characteristics are compared to the eight precision point (Chebychev spacing) solution in Fig. 5. Eight precision points are the maximum possible for the RGGR linkage as a function generator and yield a unique set of Q_i's.

The mechanical error shown in Fig. 5(a) is determined by calculating the output (follower) angle error [using equation (10)] resulting from a 0.001 error in the length of each of the three moving links (a, b, and c). The magnitudes of the rotation error due to each link's deviation are scalarly added to form the total mechanical error which is plotted. In Fig. 5(b) TI is plotted as a function of the crank angle for the eight position and the

Table 3 Transmission indices for spatial linkages

Type	TI
RSSR	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	see Appendix B

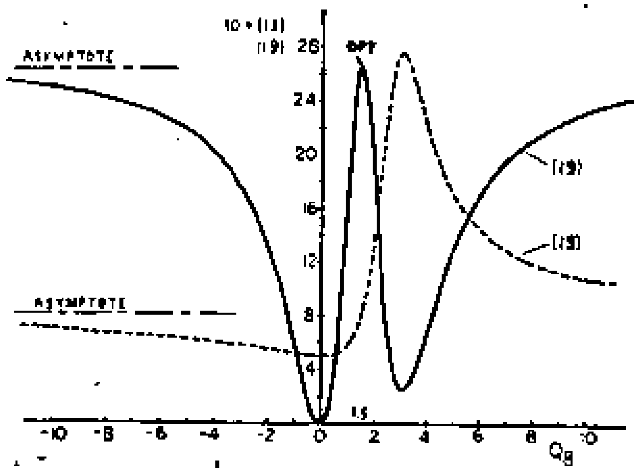


Fig. 4 Plots of expressions (15) and (19) for y = sin x

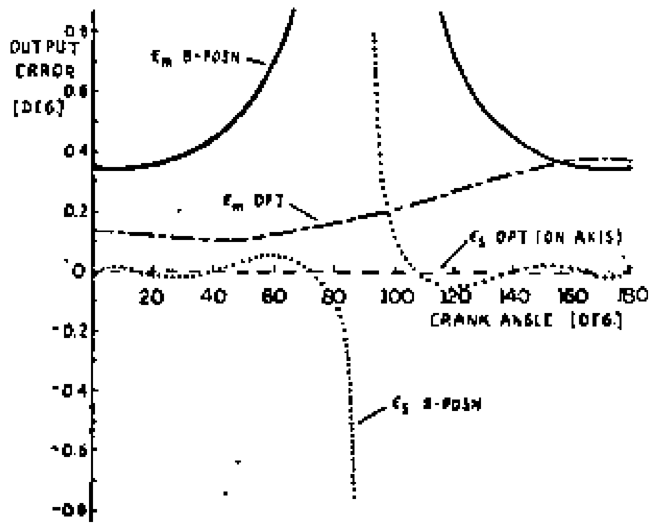


Fig. 5(a) Mechanical and structural error for y = sin x

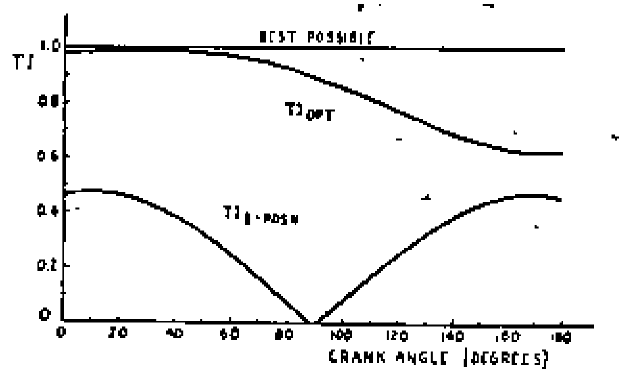


Fig. 5(b) TI for y = sin x

Fig. 5

"optimum" seven position solutions. It is notable that the eight position solution is defective because mobility between the fourth and fifth precision points is not possible. This points out the importance of accounting for TI.

The link parameters calculated for the eight-position and seven-position "optimum" solutions are shown in Table 4 along with the two "optimum" solutions for problem number 2.

Problem No. 2. Generate the function $y = x^2$ for $0 \leq x \leq 1$, with $\Delta\phi = 90$ deg and $\Delta\psi = 90$ deg using a RGGR linkage.

The solution to this problem illustrates that there can be two or more reasonable solutions which have different characteristics. Which solution is best depends on the relative importance of expressions (15) and (19). The plots of (15) and (19) are shown in Fig. 6. Two solutions ($Q_s = 2.05$ and $Q_s = 49.$) are compared in Figs. 7. $Q_s = 49.$ corresponds to a minimum of expression (15) and does not show up in Fig. 6. $Q_s = 49.$ gives excellent motion transmission; however, $Q_s = 2.05$ gives a 1/3 reduction in the maximum mechanical error at the expense of poorer motion transmission.

Linkage Class

When TI = 0 (or $TI^2 = 0$) the linkage output link is in a dead center position. Output dead center positions can only occur in double-rocker type linkages. If TI is expressed as a function of the input variable, then real roots of TI = 0 indicate that a particular linkage is of the double-rocker type. No real roots of TI = 0 indicate either that linkage closure is impossible or that the linkage has full cycle crank mobility (crank-rocker or double crank).

For the RGGR linkage TI^2 is given by equation (16). If this expression is set equal to zero, it leads to a quartic in $\tan\frac{\phi}{2}$ (or $\cos\phi$). Careful examination of the sign and type of the roots of the quartic indicate the linkage class. The quartic obtained by setting $TI^2 = 0$ for the RGGR linkage is identical to that used by Nolle [13]; hence the results are identical and are not repeated here.

Table 4 Link parameters

Prob. No.	Prec. pp.	Q_g	a	b	c	d	e	f	ξ	ζ	η
1	8	--	1.440	5.715	0.754	[illegible]	-2.107	-0.161	71.50°	[illegible]	65.00°
1	7	3.5	1.000	2.977	0.545	-0.761	-2.968	-0.199	-83.00°	-4.86°	-7.60°
2	7	2.05	1.000	3.118	-0.773	0.619	-2.120	0.401	110.59°	3.61°	-75.50°
2	7	49.	1.000	1.468	0.161	0.160	-0.579	-0.093	43.60°	-4.33°	84.95°

Physical Significance of TI Extremums

The expression (18) for TI^2 for the RGGR linkage reaches an extremum with respect to ϕ when $\frac{dTI}{d\phi} = 0$, or

$$A(\phi)s\xi c\phi + B(\phi)(fs\xi c\phi + ds\phi) = 0$$

where

$$A(\phi) = 2c^2(f - as\xi s\phi - ec\xi)$$

and

$$B(\phi) = a^2 - b^2 - c^2 + d^2 + e^2 + f^2 - 2efc\xi + 2adc\phi - 2afs\xi s\phi$$

Therefore

$$\tan\phi_{TI_{min}} = -\frac{s\xi[A(\phi) + fB(\phi)]}{dB(\phi)} \qquad (20)$$

Equation (20) can be directly related to screw system theory. Three concurrent noncoplanar zero pitch screws define the three-system characteristic of a spheric joint. For the output spheric joint (Fig. 8) these three screws can be defined as lying on the line vectors $\widehat{b}$, $\widehat{bex}$, and $\widehat{n} = \widehat{bex} \times \widehat{b}$. The screw on $\widehat{b}$ relates to the motion of the coupler link about its own axis; this motion is of no importance to function generating and is therefore ignored.

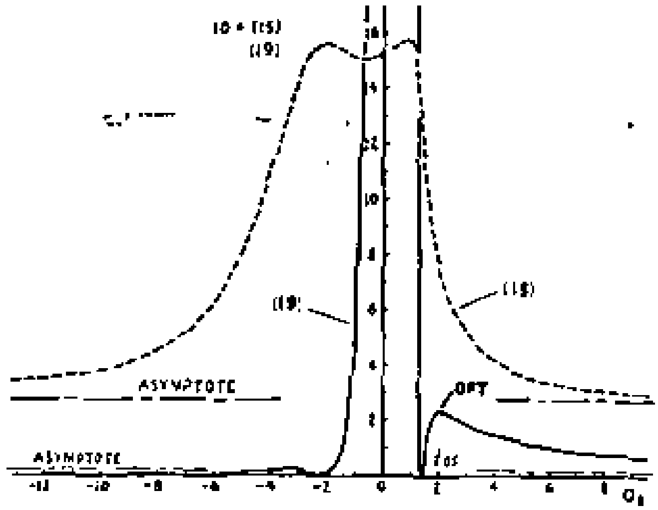


Fig. 6 Plots of expressions (15) and (19) for $y = x^2$

Motion about $\hat{n}$ causes changes in the transmission angle τ. $\widehat{bc\tau}$ serves as a convenient third screw axis.

There are six relative motion screws in the linkage exclusive of the one along $\hat{n}$ (and $\hat{b}$). These screws are all of zero pitch and in general are independent (i.e., they belong to a sixth order screw system [9]). However, for some special orientation of the linkage these six screws may become dependent and belong to a screw system of the fifth or lower order.

The condition that three concurrent noncoplanar zero pitch screws (associated with the input spheric joint) belong to the same fifth order system requires that the principal screw (the negative of the screw to which all screws in the five system are reciprocal) of the fifth order system pass through the point of concurrency and have zero pitch. The other three zero pitch screws (associated with the input and output link revolute joints and $\widehat{bc\tau}$) must then also intersect this principal screw. Thus the six screws in question will belong to a screw system of the fifth order only when a line passing through the input link spheric joint and the two revolute axes intersects $\widehat{bc\tau}$. (The point of intersection is on the output link revolute axis.) Some simple calculations in analytic geometry indicate that this intersection can occur only when equation (20) is satisfied. This result, namely that TI reaches its extremum values only when certain screws which describe the linkage motion belong to a screw system of order one less than that which normally occurs, is in accordance with the general theory of transitorily inactive freedoms proposed by K. Hunt in his published discussion of reference [14].

Summary

The transmission index theory developed using the theory of screws has been applied to linkage synthesis. Expressions for evaluating the quality of motion transmission and linkage sensitivity to link parameter errors have been developed and applied to the synthesis of the RGGR linkage as a function generator. These expressions lead to plots which aid in the selection of an optimum set of link parameters. The TI has been shown to be useful in determining the class of the linkage synthesized. Also, the physical significance of TI extremums has been interpreted in terms of screw system theory.

Acknowledgments

The financial support of the NSF is gratefully acknowledged.

References

1 Alt, H., "Der Übertragungswinkel und seine Bedeutung für das Konstruieren periodischer Getriebe," *Werkstattstechnik* 26, 1932 S. 61/64.

2 Hain, K., *Applied Kinematics* (English translation of German), McGraw-Hill, 1967.

3 Roth, B., Freudenstein, F., and Sandor, G., "Synthesis of Four-link Path Generating Mechanisms with Optimum Transmission Characteristics," *Transactions of the 7th Conference on Mechanisms*, Purdue University, 1962, pp. 44–48.

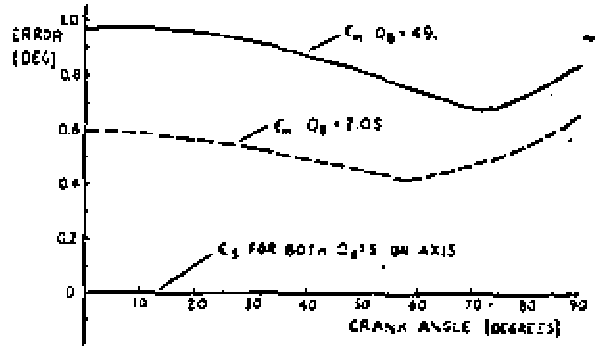


Fig. 7(a) Mechanical and structural error for $y = x^2$

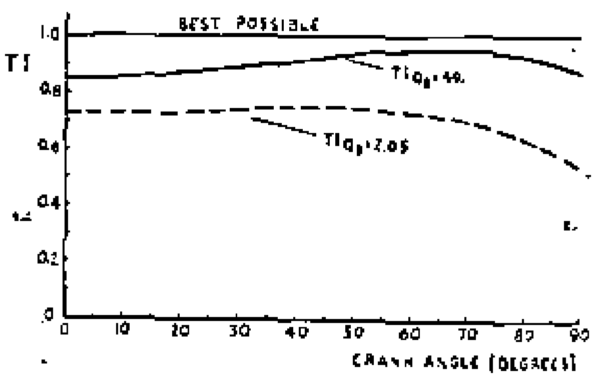


Fig. 7(b) TI for $y = x^2$

Fig. 7

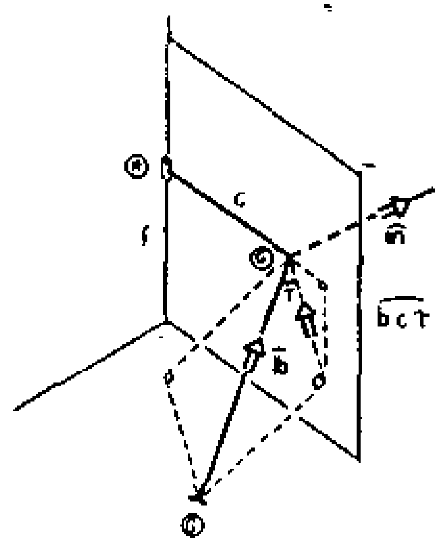


Fig. 8 Screw axes for RGGR output G-joint

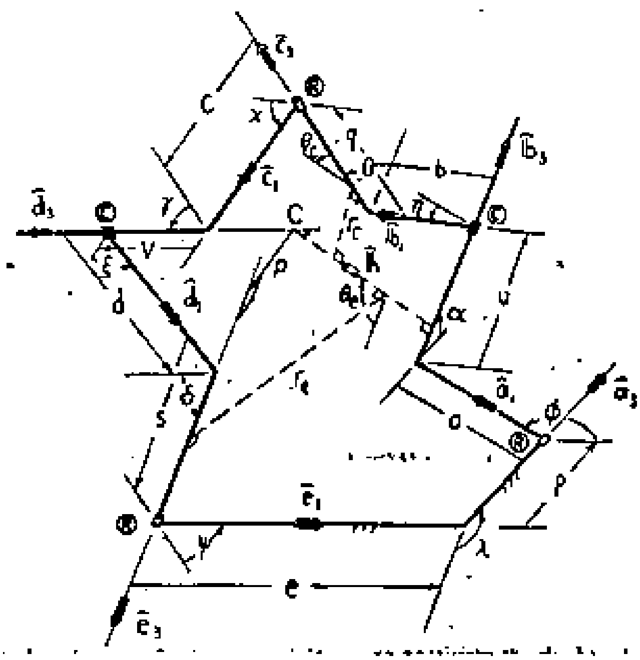


Fig. 9 RCRCR linkage

4 Eschenbach, P., and Tesar, D., "Optimization of Four-bar Linkages Satisfying Four Generalized Coplanar Positions," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 91, No. 1, Feb. 1969, pp. 75–82.

5 Eschenbach, P., Tesar, D., "Link Length Bounds on the Four-bar Chain," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 93, No. 1, Feb. 1971, pp. 287–293.

6 Chen, F. Y., "An Analytical Method for Synthesizing the Four-bar Crank-rocker Mechanism," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 91, No. 1, Feb. 1969, pp. 45–54.

7 Bagci, C., "Static Force and Torque Analysis Using 3 × 3 Screw Matrix, and Transmission Criteria for Space Mechanisms," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 93, No. 1, Feb. 1971, pp. 94–101.

8 Hartenberg, T., Denavit, J., *Kinematic Synthesis of Linkages*, McGraw-Hill, 1964.

9 Ball, Sir R. S., *A Treatise on the Theory of Screws*, Cambridge University Press, 1900.

10 Dimentberg, F. M., *The Screw Calculus and Its Applications in Mechanics* (English translation of Russian text available through Clearinghouse, Department of Commerce), 1966.

11 Yang, A. T., *Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms*, Doctoral dissertation, Columbia University, 1963 (No. 64-2830, University Microfilms, Ann Arbor, Mich.).

12 Levitskii, N. I., and Shakvasian, K., "Synthesis of Four-Element Spatial Mechanisms with Lower Pairs," (English translation of Russian paper) *Int. J. Mech. Sci.*, Vol. 2, 1960, pp. 76–92.

13 Nolle, H., "Ranges of Motion Transfer by the R-G-G-R Linkage," *Journal of Mechanisms*, Vol. 4, No. 2, 1969, pp. 145–157.

14 Waldron, K., "Symmetric Overconstrained Linkages," Journal of Engineering for Industry, trans. ASME, Series B, Vol. 91, No. 1, Feb. 1969, pp. 158–164.

15 Yuan, M. S. C., Freudenstein, F., Woo, L. S., "Kinematic Analysis of Spatial Mechanisms by Means of Screw Coordinates, Part 2—Analysis of Spatial Mechanisms," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 93, No. 1, Feb. 1971, pp. 67–73.

APPENDIX A

Derivation of the TWS for a RCRCR Linkage

The joint variables and link parameters associated with the RCRCR linkage are shown in Fig. 9. The axis for TWS lies along $\hat{h}$. The dual angle between $\hat{e}_1$ and $\hat{h}$ is $\hat{\theta}_e = \theta_e + \epsilon r_e$. The distance from the characteristic point C to $\hat{e}_1$ is ρ. The dual angle between $\hat{h}$ and $\hat{e}_2$ is $\hat{\theta}_r = \theta + \epsilon r_r$.

$\hat{h}$ is obtained in coordinate system $\hat{e}_1; \hat{e}_2; \hat{e}_3$ from the line vector product (11).

$$\hat{b}_2 \times \hat{d}_2 = s\hat{\sigma}\hat{h}$$

where $\hat{\sigma}$ is the dual angle between $\hat{b}_2$ and $\hat{d}_2$.

$$\hat{h} = \frac{\hat{b}_1 \times \hat{d}_2}{|\hat{b}_1 \times \hat{d}_2|}$$

$$= \frac{[(s\hat{\beta}c\hat{\chi}c\hat{\gamma} + s\hat{\gamma}c\hat{\beta})\hat{e}_1 - s\hat{\beta}s\hat{\chi}c\hat{\gamma}\hat{e}_2 - s\hat{\beta}s\hat{\chi}s\hat{\gamma}\hat{e}_3]}{[(s\hat{\beta}c\hat{\chi}c\hat{\gamma} + s\hat{\gamma}c\hat{\beta})^2 + s^2\hat{\beta}s^2\hat{\chi}]^{1/2}} \quad (21)$$

The reciprocity condition between the revolute velocity screw on $\hat{e}_2$ and the TWS on $\hat{h}$ gives the TWS pitch

$$p' = r_e \sin \theta_e / \cos \theta_e \quad (22)$$

$r_e \sin \theta_e$ and $\cos \theta_e$ are obtained from the line scalar product between $\hat{e}_2$ and h in reference frame $\hat{e}_1; \hat{e}_2; \hat{e}_3$; namely

$$\hat{e}_2 \cdot \hat{h} = c\hat{\theta}_e = c\theta_e - \epsilon r_e s\theta_e$$

$$= \frac{-s\hat{\beta}s\hat{\chi}s\hat{\gamma}}{[(s\hat{\beta}c\hat{\chi}c\hat{\gamma} + s\hat{\gamma}c\hat{\beta})^2 + s^2\hat{\beta}s^2\hat{\chi}]^{1/2}} \quad (23)$$

Reduction of equation (23) into scalar form gives

$$p' = -\frac{D(\hat{e}_2 \cdot \hat{h})}{P(\hat{e}_2 \cdot \hat{h})}$$

$$= [A(BC + G) - F(B^2 + E^2)]/A(B^2 + E^2) \quad (24)$$

where D and P denote respectively the dual and proper parts of the dual quantity,

$$A = s\beta s\chi s\gamma,$$

$$B = s\beta c\chi c\gamma + s\gamma c\beta,$$

$$C = bc\beta c\chi c\gamma - qs\beta s\chi c\gamma - cs\beta c\chi s\gamma + cc\gamma c\beta - bs\gamma s\beta,$$

$$G = bs\beta c\beta s\chi + qs\beta s\chi c\chi,$$

$$E = s\beta s\chi,$$

and

$$F = bc\beta s\chi s\gamma + qs\beta c\chi s\gamma + cs\beta s\chi c\gamma$$

The line vector $\hat{h}$ given by equation (21) and the pitch p' given by equation (24) are sufficient to describe the TWS

$$(1 + \epsilon p')\hat{h}$$

for a RCRCR linkage.

APPENDIX B

Derivation of TI for a RCRCR Linkage

For the RCRCR linkage the TI expression (equation (6)) reduces to

$$\mathrm{TI} = \frac{|p' \cos \theta_r - r_r \sin \theta_r|}{(p^2 + \rho^2)^{1/2}} \quad (25)$$

p' is given by expression (24) in Appendix A. The numerator of (25) is equal to one half the magnitude of the virtual coefficient between unit line vectors $\hat{e}_3$ and $\hat{h}$. As indicated by Dimentberg [10], this quantity may be determined by taking the magnitude of the dual part of the line scalar product between $\hat{e}_3$ and $(1 + p')\hat{h}$. Therefore using reference frame $\hat{h}; \hat{d}_2 \times \hat{h}; \hat{d}_2$

$$|p' \cos \theta_r - r_r \sin \theta_r| = |D[\hat{e}_3 \cdot (1 + \epsilon p')\hat{h}]| \quad (26)$$

where

$$\hat{e}_3 \cdot (1 + \epsilon p')\hat{h} = \{s\hat{\delta}[s\hat{\xi}(s\hat{\beta}c\hat{\chi}c\hat{\gamma} + s\hat{\gamma}c\hat{\beta}) + c\hat{\xi}s\hat{\beta}s\hat{\chi}]$$

$$\times (1 + \epsilon p')\}/[(s\hat{\beta}c\hat{\chi}c\hat{\gamma} + s\hat{\gamma}c\hat{\beta})^2 + s^2\hat{\beta}s^2\hat{\chi}]^{1/2}$$

ρ is determined by considering $\hat{e}_3$ in reference frame $\hat{h}; \hat{d}_2 \times \hat{h}; \hat{d}_2$ with origin "C." In this case

$$\rho = P(\hat{e}_3) \times D(\hat{e}_3) \quad (27)$$

where

$$\dot{\hat{a}}_1 = \{s\delta[s\xi(s\beta c\chi c\gamma + s\gamma c\beta) + c\xi s\beta s\chi]\hat{h} + s\delta[s\xi s\beta s\chi - c\xi(s\beta c\chi c\gamma + s\gamma c\beta)]\hat{a}_1 \times \hat{h} + c\delta\hat{a}_1\}/\{(s\beta c\chi c\gamma + s\gamma c\beta)^2 + s^2\beta s^2\chi\}^{1/2}$$

Therefore substituting expressions (24), (26), and (27) in (25) will yield the TI expression for a RCRCR linkage.

APPENDIX C

MED for the RGCR Linkage

The joint variables and link parameters associated with the RGCR linkage are shown in Fig. 2.

A relationship between $\partial F/\partial\psi$ (MED) in equation (10) and TI (equation (6)) is desired. "$F = 0$" is the linkage closure equation which for the RGCR linkage is

$$F = As\psi + Bc\psi + C = 0 \tag{28}$$

where

$$A = 2(ecs\xi + dgs\kappa - acc\xi s\phi + ags\kappa c\phi),$$

$$B = 2(ed - ags\xi s\kappa + agc\xi s\kappa s\phi + acc\phi),$$

and

$$C = a^2 - b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + 2[fgc\kappa + efc\xi + egc\xi c\kappa + as\xi(f + gc\kappa)s\phi + adc\phi]$$

If g is considered as a function of ϕ only, then

$$\frac{\partial F}{\partial\psi} = -Bs\psi + Ac\psi \tag{29}$$

The TI expression for the RGCR linkage in Table 3 does not resemble equation (29). However, if α is expressed as a function of ψ and ϕ, then the TI expression becomes

$$\text{TI} = \frac{-Bs\psi + AC\psi}{2\rho b}$$

where

$$\rho = (c^2 + g^2s^2\kappa)^{1/2}$$

Therefore

$$\frac{\partial F}{\partial\psi} = \text{MED} = \pm 2\rho b\text{TI} \tag{30}$$

Since ρ for the planar four-bar is equal to c (the follower length), (30) is identical to the mechanical error denominator for the planar four-bar linkage.

Printed in U. S. A.

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

1. The Kinematics of Motion Through Finitely Separated Positions
2. Finite-Position Theory Applied to Mechanism Synthesis
3. On the Screw Axes and Other Special Lines Associated with Spatial Displacements of a Rigid Body
4. A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis
5. Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains

Dr Bernard Roth

Junio, 1981

BERNARD ROTH

Associate Professor,
Mechanical Engineering,
Stanford University,
Stanford, Calif.
Mem. ASME

The Kinematics of Motion Through Finitely Separated Positions

A rigid body is studied in a series of finitely separated positions, in order to determine those points which lie on a special locus (a sphere, circle, plane, line, or cylinder). Equations governing these special points are derived and their numerical evaluation is discussed. Several numerical examples are presented. In a companion paper [21],[1] these results are applied to the synthesis of spatial linkages, and special motions (e.g., planar and spherical) are incorporated into the general theory presented herein.

Introduction

There are two basic methods which have been successfully applied to the analytical synthesis of planar linkages. One involves writing the synthesis equations so that they explicitly include the unknown linkage dimensions [1, 2, 3, and so on], while the other method, which stems from the earlier graphical theory, requires determining those points in the moving plane which lie on curves which can be readily mechanized [1, 5, and so on]. To date, most spatial-linkage synthesis work has followed the first method [6, 7, 8, 9, 10 11], while the study of points with easily mechanized motions—which is the subject of this paper—seems to have been largely ignored.

Several recent works [13, 14, 15] have treated isolated aspects of this second method, but the only previous work in the same vein as this present study is Schoenflies' [12] classical text. Unfortunately, Schoenflies' development is entirely descriptive (i.e., synthetic as opposed to analytic) and does not lend itself to the solution of practical problems.

In this paper we study points which lie on spheres, circles, planes, lines, or cylinders. Here we consider only general spatial displacements, while a companion paper [21] deals with important special displacements (e.g., planar). Applications to linkage synthesis are discussed in [21]. The main contribution of this present work is that it combines algebraic geometry and computer techniques to yield quantitative results, thereby, for the first time, making it possible to apply the theory to engineering practice. All the derivations are new. In addition, the discussion about points on a cylinder is entirely new, as is the unified treatment of the loci of points with two, three, four, and five positions on a circle.

Kinematic Preliminaries

We refer to two systems of points which for convenience we call the moving system, Σ, and the fixed system, Σ'. However, since our concern is only with their relative positions, both Σ and Σ' may actually be moving. The distance between points in any one system does not vary and hence we may consider Σ and Σ' as rigid bodies, each of which contain all the points in three-dimensional space. The position of Σ is uniquely defined, relative to a coordinate system fixed in Σ', by the coordinates of any three noncollinear points.

[1] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division for presentation at the Applied Mechanics Conference, Pasadena, Calif., June 26-28, 1967, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until one month after final publication of the paper itself in the Journal of Applied Mechanics. Manuscript received by ASME Applied Mechanics Division, November 12, 1965; final draft, September 26, 1966. Paper No. 67—APM—2.

Linear Transformations and Screw Motion

In considering two finitely separated positions of Σ it is convenient to call one the reference position, Σ_1, and describe the other, Σ_j, as the jth. It is possible to move from position one to position "j" in an infinite number of ways, but for our purposes it will be convenient to regard this motion as a screw (i.e., a rotation about and a translation along the same axis). The screw is denoted by S_{1j}.

If we are given two positions (i.e., two sets of three noncollinear points) then the screw may be determined from a generalized form of Rodrigue's equation. Alternatively, if we are given the screw we obtain the new position, (x_j, y_j, z_j), of some point (x_1, y_1, z_1) from the following well-known linear transformation.

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} (a_{xj}+1) & b_{xj} & c_{xj} \\ a_{yj} & (b_{yj}+1) & c_{yj} \\ a_{zj} & b_{zj} & (c_{zj}+1) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} d_{xj} \\ d_{yj} \\ d_{zj} \end{bmatrix} \quad (1)$$

where

$$a_{xj} = (u_{xj}^2 - 1)(1 - \cos\theta_j)$$
$$b_{xj} = u_{xj}u_{yj}(1 - \cos\theta_j) - u_{zj}\sin\theta_j$$
$$c_{xj} = u_{xj}u_{zj}(1 - \cos\theta_j) + u_{yj}\sin\theta_j$$
$$d_{xj} = d_j u_{xj} - a_j a_{xj} - b_j b_{xj} - c_j c_{xj}$$
$$a_{yj} = u_{yj}u_{xj}(1 - \cos\theta_j) + u_{zj}\sin\theta_j$$
$$b_{yj} = (u_{yj}^2 - 1)(1 - \cos\theta_j)$$
$$c_{yj} = u_{yj}u_{zj}(1 - \cos\theta_j) - u_{xj}\sin\theta_j$$
$$d_{yj} = d_j u_{yj} - a_j a_{yj} - b_j b_{yj} - c_j c_{yj}$$
$$a_{zj} = u_{zj}u_{xj}(1 - \cos\theta_j) - u_{yj}\sin\theta_j$$
$$b_{zj} = u_{zj}u_{yj}(1 - \cos\theta_j) + u_{xj}\sin\theta_j$$
$$c_{zj} = (u_{zj}^2 - 1)(1 - \cos\theta_j)$$
$$d_{zj} = d_j u_{zj} - a_j a_{zj} - b_j b_{zj} - c_j c_{zj}$$

In the foregoing we have taken the displacement from position 1 to j as equivalent to a translation d_j along, and a rotation θ_j about an axis parallel to the unit vector (u_{xj}, u_{yj}, u_{zj}) which passes through the point (a_j, b_j, c_j). The terms θ_j and d_j are referred to as the screw parameters, their positive senses being defined by the right-handed screw rule. Equation (1) emphasizes that the rotational and translational aspects of the motion may be regarded as occurring separately and their effects superimposed.

We now present a theorem which will be very useful in studying points with special motions:

The locus of all points in Σ_1 which under a general displacement remain a fixed distance from a given point (in Σ') is a plane. The

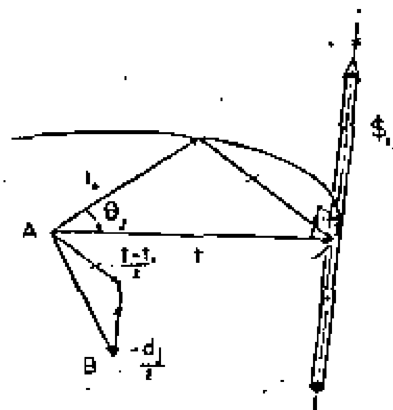


Fig. 1 Normal to plane given by equation (2) is determined by fixed point A and screw $\$_{1j}$. Vector AB, the required normal, is sum of $\frac{t - t_\theta}{2}$ and $\frac{-d_j}{2}$.

proof is as follows. Let **A** be the vector from the origin to the point A in Σ^*, and r_1 the vector to some point in Σ_1. For motion from Σ_1 to Σ_j the constant distance condition requires:

$$|r_1 - \mathbf{A}| = |r_j - \mathbf{A}|$$

which is equivalent to

$$\frac{r_1^2 - r_j^2}{2} + \mathbf{A}\cdot(r_j - r_1) = 0 \qquad (2)$$

Now from (1) we know that r_j is a linear function of r_1 and it remains only to show that $r_1^2 - r_j^2$ is linear.

Substituting $r_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$, $r_j = x_j\mathbf{i} + y_j\mathbf{j} + z_j\mathbf{k}$, and making use of the usual orthogonality conditions for the rotational part of the rigid-body transformation, we obtain:

$$\begin{aligned}\frac{r_j^2 - r_1^2}{2} = {} & (d_j u_{1j} + a_j a_{xj} + b_j a_{yj} + c_j a_{zj})x_1 \\ & + (d_j u_{2j} + a_j b_{xj} + b_j b_{yj} + c_j b_{zj})y_1 \\ & + (d_j u_{3j} + a_j c_{xj} + b_j c_{yj} + c_j c_{zj})z_1 \\ & + \frac{d_j^2}{2} + (a_j^2 + b_j^2 + c_j^2)(1 - \cos\theta_j) \qquad (3)\end{aligned}$$

Hence (2) is linear in (x_1, y_1, z_1) and the theorem is proved.

We note that, if the screw passes through the origin, (3) simplifies to

$$\frac{r_j^2 - r_1^2}{2} = \frac{d_j^2}{2} + d_j(u_{1j}x_1 + u_{2j}y_1 + u_{3j}z_1)$$

and if in addition the motion is a pure rotation: $r_j^2 - r_1^2 = 0$.

The plane given by equation (2) is a distance of $\left[\left(\frac{d_j}{2}\right)^2 + t^2\sin^2\left(\frac{\theta_j}{2}\right)\right]^{1/2}$ from A and is normal to the vector $\frac{1}{2}(\mathbf{t} - \mathbf{t}_\theta - \mathbf{d}_j)$. Here $\mathbf{t}$ is the normal vector from the fixed point A to the screw axis $\$_{1j}$, and $\mathbf{t}_\theta$ is the vector obtained by rotating $\mathbf{t}$ by $-\theta_j$ about an axis through A parallel to $\$_{1j}$, Fig. 1. The term $\mathbf{d}_j$ is the translation vector $d_j(u_{1j}\mathbf{i} + u_{2j}\mathbf{j} + u_{3j}\mathbf{k})$. If the screw axis passes through the fixed point, the plane given by (2) becomes normal to the screw axis and intersects it a point $-\frac{\mathbf{d}_j}{2}$ from the fixed point.

Equation (2) is also useful if we regard r_1 and r_j as known and seek the locus of points in Σ^* which are at an equal distance from the known points in Σ_1 and Σ_j. Under this interpretation (2) becomes the locus of A and is, of course, the perpendicular bisector plane of the cord connecting (x_1, y_1, z_1) and (x_j, y_j, z_j).

Correspondence—The Cubic Transformation

Consider four finitely separated positions of Σ. Associated with each point, p_1, in Σ_1 is the set of four homologous points p_1, p_2, p_3, p_4 given by the four positions of p. Since it is possible to pass a sphere through any four points, the four positions of each point define a sphere. Associated with each such set of homologous points in Σ is the point in the fixed system which is at the center of the sphere. Since the relationship between points in Σ and Σ^* is important to what follows, we undertake to describe this correspodence.

We require that (in four positions) a point in the moving system remain a fixed distance from a point (A_x, A_y, A_z) in the fixed system. This is given by Equation (2) taken three times:

$$\begin{aligned}(x_2 - x_1)A_x + (y_2 - y_1)A_y + (z_2 - z_1)A_z + \left(\frac{r_1^2 - r_2^2}{2}\right) &= 0 \\ (x_3 - x_1)A_x + (y_3 - y_1)A_y + (z_3 - z_1)A_z + \left(\frac{r_1^2 - r_3^2}{2}\right) &= 0 \qquad (4) \\ (x_4 - x_1)A_x + (y_4 - y_1)A_y + (z_4 - z_1)A_z + \left(\frac{r_1^2 - r_4^2}{2}\right) &= 0\end{aligned}$$

Now, given screws $\$_{12}$, $\$_{13}$, $\$_{14}$ (or any equivalent set, e.g., $\$_{12}$, $\$_{23}$, $\$_{34}$) and any point (x_1, y_1, z_1) we are able to generally solve the three linear nonhomogeneous equations (4) for a unique center point (A_x, A_y, A_z). Similarly, if the screws and the center of the sphere are given, we know from (3) and (1) that the equations (4) become a linear set of three nonhomogeneous equations in (x_1, y_1, z_1). Hence, a unique (x_1, y_1, z_1) may generally be determined. Thus we generally have a (1, 1) correspondence between points in Σ and points in Σ^*. This correspondence is invariant to kinematic inversion, since under an inversion the center becomes the moving point and the moving point becomes the center. This transformation is called *the cubic transformation* [19].

Singularities in this correspondence occur when the rank of the augmented matrix of system (4) is less than three. The necessary and sufficient conditions for the rank to be two are that the coefficient matrix

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1)\end{vmatrix} = 0, \qquad (5)$$

and one other 3 × 3 be singular, say,

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (r_1^2 - r_2^2) \\ (x_3 - x_1) & (y_3 - y_1) & (r_1^2 - r_3^2) \\ (x_4 - x_1) & (y_4 - y_1) & (r_1^2 - r_4^2)\end{vmatrix} = 0, \qquad (6)$$

provided the common 2 × 2's of (5) and (6) do not all vanish. If the rank of the system is two, a point in the moving system corresponds to a line in the fixed system, and under kinematic inversion a point in Σ^* corresponds to a line in Σ. As we shall see later this leads us to points whose four positions fall on circles.

If the rank of the system is one, a point in the moving body Σ corresponds to a plane in Σ^*. This requires that all of the 2 × 2's of the augmented matrix vanish which is impossible under general motions.

Having laid the groundwork, we now proceed to determine those points in Σ whose several positions lie on special loci.

Points on Special Loci

Points Which Lie on a Sphere

A general point will not have more than four positions on a sphere. Those points with five positions on one sphere will satisfy equation (2) written four times (i.e., j = 2, 3, 4, 5). The condition for the four nonhomogeneous linear equations in the three unknowns A_x, A_y, A_z to be compatible is:

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) & (r_1^2 - r_2^2) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) & (r_1^2 - r_3^2) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) & (r_1^2 - r_4^2) \\ (x_5 - x_1) & (y_5 - y_1) & (z_5 - z_1) & (r_1^2 - r_5^2)\end{vmatrix} = 0 \qquad (7)$$

Table 1 Input defining seven positions of a body in terms of six screws

Position	Axis orientation			Axis location			Parameters	
i to j	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1 to 7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 2 Equation of the locus of points in a body that lie on a sphere when the body is in positions 1, 2, 3, 4, and 5

E^4_{12345} = [illegible] = 0

As shown previously, substituting (1) and (3) explicitly makes all of these elements linear in (x_1, y_1, z_1). Hence (7), which is the locus of all points in Σ_1 with five positions on a sphere, is of fourth degree. It will be convenient to refer to (7) as $E_{12345}^4 = 0$ and sometimes simply as E^4. Geometrically, the locus E^4 is a fourth-order algebraic surface embedded in Σ_1.

In order to expand (7) and obtain explicit expressions for the 35 coefficients of E^4 in terms of the motion parameters, we have to expand $4^4(4 \times 4)$ determinants and sort out 6144 terms which each consist of a product of four parameters. This prodigious amount of algebra is simplified by the symmetry in x_1, y_1, z_1, but still the task is formidable. We have elected to always determine these coefficients of E^4 numerically, and have found that a very small amount of programming and a trivial amount of computational time are required for this development.[3] Table 1 gives the screws which define a set of finitely separated positions. Table 2 lists a set of coefficients of E^4 corresponding to the screws of Table 1.

By, for example, arbitrarily choosing x_1 and y_1 and then solving the resulting quartic for (at most) four possible z_1's we may compute the coordinates of points on E^4. The center point corresponding to any point on E^4 is obtained from the cubic transformation, equation (4). Alternatively, we could have started by regarding x_1, y_1, z_1 as the unknowns and instead of (7) obtained a (4×4) determinant in A_x, A_y, A_z. This would yield a fourth-order algebraic surface embedded in Σ' which is the locus of all sphere centers corresponding to a given set of five positions of Σ. For any point on this surface we could compute the corresponding moving point from (4).

Now considering a sixth position of Σ, we use any four of the original five positions (for example, the first four) and the sixth position in equation (7), and obtain a second equation, say, (7^1). (7^1) represents a second fourth-order algebraic surface $E_{12346}^4 = 0$ which is embedded in Σ_1. Four positions will generally uniquely determine a sphere, and since the two fourth-order surfaces share four positions, we conclude that the locus of all points with six positions on a sphere is included in the intersection of $E_{12345}^4 = 0$ with $E_{12346}^4 = 0$. Algebraically, these two fourth-degree equations (7) and (7^1) are the compatibility conditions for the five nonhomogeneous linear equations obtained by writing (2) five times.

The intersection may be written as $E_{12345}^4 \times E_{12346}^4$. It consists of two components: One is a tenth-order space curve k_{123456}^{10} or simply k^{10}, of genus eleven, and the other is a sixth-order curve k_{1234}^6, or simply k^6, of genus three. k^{10} contains all the points with six positions on a sphere. As will be shown in the next section, k^6 is the locus of all points which lie on a circle for the four positions 1, 2, 3, 4. Physically, the reason k^6 does not contain points with six positions on a sphere is that the four common positions of the two surfaces do not define a unique sphere. Hence the fifth and sixth positions (of any point on k^6) will, when taken in combination with the circle, define two different spheres.

Analytically, k^6 corresponds to the singular case of the cubic transformation given by equations (5) and (6). Under these circumstances equations (7) and (7^1) are no longer sufficient to guarantee the compatibility of equation (2) written five times (with j = 2, 3, 4, 5, 6) and, therefore, points on k^6 will not satisfy all five equations.

Table 3 lists points on k^{10} corresponding to the motion given in Table 1. The corresponding center points are computed from (4). Alternatively, we could, as described previously, obtain two fourth-order surfaces imbedded in Σ', find their intersections and compute the corresponding points in Σ_1 from (4).

For seven positions on a sphere the reasoning is analogous to the foregoing. We substitute the subscript 7 for 5 in equation (7) and obtain a new equation (7^{11}) which yields a third surface $E_{12347}^4 = 0$. Intersecting these three surfaces one finds that there are at most 20 points with seven positions on a sphere,[4] Appendix 1.

Analytically, equations (7), (7^1), (7^{11}) represent the compatibility conditions for (2) written six times, and the 44 spurious solutions correspond to the singular case given by (5) and (6).

As in the case of points with five and six positions on a sphere, the corresponding center points are obtained from (4). Alternatively, by suitable relabeling or by kinematic inversion we could obtain three surfaces in Σ' whose intersection would contain the "seven-position" centers.

If we introduce an eighth position, we require the common intersections of four surfaces. Since these will not generally

[3] This and all other programs referred to in this paper may be obtained by writing to the author. The program language is FORTRAN II.

[4] The original proof due to Schoenflies [12] depends upon surfaces which are computationally not as convenient as the foregoing. In Appendix 1 we give a new proof which is due to E. J. F. Primrose.

Table 3 Points on a sphere

Points with j positions on a sphere	Initial position x_1	y_1	z_1	Sphere center A_x	A_y	A_z	Radius of sphere
j = 5	[illegible]	1.0000	0.0000	[illegible]	[illegible]	[illegible]	[illegible]
	1.1461	1.5000	1.0000	0.3442	[illegible]	1.2276	1.0601
	-1.0510	2.5000	0.5000	2.1276	1.7456	1.7199	[illegible]
j = 6	1.0000	-1.4566	-1.4980	-0.5010	1.0003	-1.6476	2.7726
	1.5000	-4.2012	-1.7064	-0.3971	0.9996	-1.2967	3.5520
	1.0000	-4.8011	-1.3341	-0.5141	0.9991	-1.2663	[illegible]
j = 7	0.0000	1.0000	0.0000	0.0000	0.0000	0.0000	1.0000
	0.2144	-0.4414	-2.4477	-1.0175	1.0467	-2.6232	2.0711
	0.0510	1.0256	[illegible]	0.9435	0.0087	0.2340	1.0186
	-0.4751	0.1415	-1.2615	-1.2100	0.9600	-1.2173	[illegible]
	-2.4455	0.4199	[illegible]	-0.8654	0.1900	-0.3164	5.4619

exist, it follows that under general motion there are no points with more than seven positions on one sphere.

Table 3 contains several points which under the motion given in Table 1 have seven positions on a sphere. As a result of the prohibitive size of the eliminants required to develop explicit expressions for the intersections, iterative techniques are employed to determine points common to the three surfaces.

Points Which Lie on a Circle

When dealing with points on a circle it is advantageous to formulate equations which explicitly give the correspondence between the points (fixed in Σ) and their axes (fixed in Σ'). A circle axis is specified by its direction cosines (l, m, n) and the coordinates of one of its points (A_x, A_y, A_z).

All points whose several positions fall on a circle must satisfy two conditions: (*a*) Their distance from any (and all) points on the axis is the same in each position; (*b*) in the several positions, they lie in a plane which is normal to the axis. These conditions may be expressed analytically as follows:

$$(x_j - x_1)A_x + (y_j - y_1)A_y + (z_j - z_1)A_z + \left(\frac{r_1^2 - r_j^2}{2}\right) = 0 \qquad (8)$$

$$(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n = 0 \qquad (9)$$

We have previously given a geometrical interpretation of (8), which is in fact equation (2), and we now give a similar interpretation to (9). From equation (9) it follows that *the locus of all points which have two positions on a line (or in a plane) normal to a given direction is a plane.* By substituting (1), we write (9) explicitly in terms of the screw $\$_{1j}$:

$$[(1 - \cos\theta_j)(u_{1j}\cos\alpha_j - l) + \sin\theta_j(mu_{3j} - nu_{2j})](x_1 - a_j)$$
$$+ [(1 - \cos\theta_j)(u_{2j}\cos\alpha_j - m) + \sin\theta_j(nu_{1j} - lu_{3j})](y_1 - b_j)$$
$$+ [(1 - \cos\theta_j)(u_{3j}\cos\alpha_j - n) + \sin\theta_j(lu_{2j} - mu_{1j})](z_1 - c_j)$$
$$+ d_j\cos\alpha_j = 0 \qquad (10)$$

Here α_j is the angle between the screw axis (u_{1j}, u_{2j}, u_{3j}) and circle axis (l, m, n). Dividing the aforementioned by 2 sin $\left(\frac{\theta_j}{2}\right)$ sin α_j puts the equation into normal form. For convenience, we call this plane P_j.

Now from (10) it may easily be shown that the distance from P_j to the screw axis is $\frac{d_j}{2}$ cot (α_j) csc $\left(\frac{\theta_j}{2}\right)$ and that the dihedral angle between P_j and a plane parallel to the screw axis and the circle axis is $\left(\frac{\theta_j}{2}\right)$. Alternatively, this last result asserts that $\left(\frac{\theta_j}{2}\right)$ is the angle between the common normal to the two axes and the normal to P_j. It should be noted that the inclination of P_j depends only on the direction of the screw axis and the magnitude of the rotation; it is independent of the location of the screw and the translation.

If the screw axis and the circle axis are parallel, P_j becomes the plane at infinity. However, if in addition to the axes being parallel, the motion is a pure rotation, every point in Σ_1 identically satisfies equation (9). If the motion is a pure translation, regardless of axis orientation, P_j again becomes the plane at infinity.

Finally, we note that since P_j is parallel to the screw axis, P_j cannot meet the (screw) axis in a finite point unless $\alpha_j = 90$ deg or the motion is a pure rotation in which case P_j contains the screw.

For any specified axis, equations (8) and (9) yield two planes which intersect in a line which is embedded in Σ_1. Hence, *the locus of all points which have two positions on a circle with a specified axis is a line.* If we take a third position we require the intersection of two skew lines and hence for an arbitrary choice of axis there are generally no points which lie on a circle for three positions.

Now we take as specified a point through which the circle axis passes, and leave the direction of the axis unspecified. For three positions, we write (8) twice ($j = 2, 3$). These equations with (A_x, A_y, A_z) known, represent two planes whose line of intersection satisfies (8). Equation (9) written twice will generally yield unique values for $\frac{l}{n}, \frac{m}{n}$ for any points in Σ_1 and in particular for the line determined from (8) with $j = 2, 3$. Hence *the locus of all points having three positions on a circle whose axis passes through a specified point is a line.*

Considering a fourth position, equation (8) with $j = 2, 3, 4$ yields a unique point, but equation (9) written with $j = 2, 3, 4$ gives three homogeneous equations in (l, m, n). Hence, we do not have the possibility of finding a solution unless (x_1, y_1, z_1) are such that:

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix} = 0$$

This is the same cubic surface as given by equation (5). We have a solution only if (A_x, A_y, A_z) are chosen so that the corresponding point (x_1, y_1, z_1) lies on the previous cubic. If (A_x, A_y, A_z) are finite, this second requirement can only be met if all the (3×3) determinants formed from the coefficients of (8) (written three times) are zero. This requires that, in addition to (5), equation (6) also be satisfied. [As we have shown previously, these are the necessary conditions for the system matrix (4) to be of rank two. Physically, we require that the three perpendicular bisectors given by equation (2), or (8), with $j = 2, 3, 4$, intersect in a common line.] Thus, for four positions, we are not at liberty to arbitrarily specify (A_x, A_y, A_z).

Equations (5) and (6) are third-degree algebraic polynomials

Table 4 Points on circles, lines, planes, and cylinders

	Initial position			Center			Radius	Direction cosines of axis		
	x_1	y_1	z_1	x_c	y_c	z_c		l	m	n
Points with Four Positions on a Circle	1.2000	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1.3632	[illegible]	[illegible]	[illegible]
	-0.7500	2.0[illegible]	-1.4[illegible]	0.6145	0.7536	-1.0251	1.5666	-0.3109	-0.4454	-0.8158
	1.4500	2.1534	3.2200	1.2193	1.2904	2.4610	2.4104	0.9411	0.2431	0.8743
Points with Three Positions on a Straight Line	-4.1700	0.1451	-5.0000							
	-0.2109	0.4466	-5.0000							
	3.0900	-0.7124	-5.0000							
Points with Six Positions on a Plane	0.9960	1.7300	1.3257							
	-0.0260	1.3455	-1.3036							
	-4.7365	3.3441	-5.8633							
Points with Six Positions on a Cylinder	0.4461	1.0626	-0.5351	0.7432	0.0023	0.7072	0.6059	-0.6710	-0.3351	0.6610
	1.9992	3.6303	1.5143	0.0606	-0.1675	2.2151	1.3150	"	"	"
	0.3606	0.9726	4.0005	0.2550	0.5549	0.3663	0.2757	"	"	"

which represent the two cubic surfaces F^3 and G^3, respectively. The intersection of $F^3 \times G^3$ (or more analytically, the common roots of (5) and (6)) contains all the points in Σ_1 which have four positions on a circle. The intersection consists of k^6 which is a sixth-order space curve of genus three, and a cubic j^3. The "residual" part of the intersection, j^3, is the space cubic which is the intersection of the two hyperboloids

$$\begin{vmatrix} (x_3 - x_1) & (y_3 - y_1) \\ (x_2 - x_1) & (y_2 - y_1) \end{vmatrix} = 0$$

$$\begin{vmatrix} (x_3 - x_1) & (y_3 - y_1) \\ (x_4 - x_1) & (y_4 - y_1) \end{vmatrix} = 0 \qquad (11)$$

The line $(x_2 - x_1) = 0$, $(y_2 - y_1) = 0$ common to these hyperboloids is not included in j^3. Analytically, equations (11) give the conditions under which (5) and (6) are no longer sufficient to guarantee a rank of two for the system matrix (4). For all points on k^6 the rank of (4) is two and, therefore, *the locus of all points which have four positions on a circle is a space curve of order six.*[4]

We return to the study of three positions, but now we specify the direction of the axis and leave its location arbitrary. Equation (8) with $j = 2, 3$ gives two nonhomogeneous equations in the unknowns (A_x, A_y, A_z). Therefore, for any point (x_1, y_1, z_1) there is a singly infinite set of solutions corresponding to the points on an axis. Equation (9), with $j = 2, 3$ and (l, m, n) known, yields two planes whose line of intersection gives us permissible values for (x_1, y_1, z_1) which when substituted into (8) (with $j = 2, 3$) yield the location of the axis. Hence, *the locus of all points which have three positions on a circle of a specified inclination is a line.* This same line is also the locus of all points which have three positions in a plane whose normal is (l, m, n).

For four positions with the axis direction specified, equation (9) yields three planes and therefore generally a unique point (x_1, y_1, z_1). At first, it might seem that (8) taken three times ($j = 2, 3, 4$) would always yield a unique (A_x, A_y, A_z). This is not the case. For any (l, m, n) the values of (x_1, y_1, z_1) computed from (9) must be such that they automatically satisfy (5), and if (A_x, A_y, A_z) are finite the rank of (8), with $j = 2, 3, 4$, is at most two. In this case (8) yields a line which, of course, defines the axis location. However, the center will generally be at infinity since (A_x, A_y, A_z) are only finite when (x_1, y_1, z_1) also satisfy (6). Hence, *for any given axis inclination there is a unique point whose four positions lie in a plane.* Since this point is determined by the intersection of three planes, this last result affords a rather simple means of obtaining points on the surface F^3 (which is discussed in a later section). For certain axis inclinations the corresponding point also falls on G^3 and, therefore, on k^6.

[4] A curve such as k^6, whose points correspond to curves (lines in this case), is singular in regard to the given (1, 1) correspondence and is called the *fundamental curve* of the transformation.

There generally are no points having five positions on a circle. Analytically, we would require two additional equations (5') and (6') which are like (5) and (6) except that they have the subscript 5 instead of 4. The four equations (5), (6), (5'), (6') generally would not be compatible. Geometrically, two space curves such as $k_{34}{}^6$ and $k_{35}{}^6$ will generally not intersect[5] even when they are contained on the same surface E^3. There are, however, some important special motions (e.g., planar and spherical) for which these four equations are compatible. We shall discuss these cases elsewhere [21].

Computationally, points on k^6 are obtained by numerically determining the coefficients of the cubics (5) and (6). These equations are then "solved" simultaneously for common roots. The roots are then sorted according to whether they belong to k^6 or j^3, and the corresponding axes are determined from (8) (with $j = 2, 3$). In Table 4 we list several points on the curve k^6 which correspond to the motion given in Table 1.

Points Which Lie on a Straight Line

Points with three positions on a line are given by the condition that their cords have parallel perpendicular-bisector planes, or by the condition that they lie on a three-point circle with axis at infinity. In either case we require:

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix} = 0$$

$$\begin{vmatrix} (x_2 - x_1) & (z_2 - z_1) \\ (x_3 - x_1) & (z_3 - z_1) \end{vmatrix} = 0 \qquad (12)$$

These two second-degree polynomials are equations of hyperboloids (of one sheet) which intersect in a cubic space curve i^3 and the common generator $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. The curve i^3, which is embedded in Σ_1, is the locus of all points with three positions on a line. Since i^3 is a cubical hyperbola[6] it may also be obtained as the intersection of two hyperbolic cylinders.

Generally there are no points with four positions on a straight

[5] The real points at infinity along the screw axes have two positions which coincide and, hence, having only three distinct positions, such points always lie on a circle for four positions. The curve $k_{34}{}^6$ contains the points where $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{23}{}'$, $\$_{24}{}'$, $\$_{34}{}'$ pierce the plane at infinity and $k_{35}{}^6$ contains the points at infinity along $\$_{12}$, $\$_{13}$, $\$_{15}$, $\$_{23}{}'$, $\$_{25}{}'$, $\$_{35}{}'$. Since k^6 can be intersected by a plane at most six times, we conclude from the foregoing that the six screw axes give the directions of the only asymptotes. The intersection $k_{34}{}^6 \times k_{35}{}^6$ contains the point at infinity along $\$_{12}$, $\$_{13}$, $\$_{23}{}'$, but these three points do not fall on five-point circles. Schoenflies [12] has shown that these curves have four imaginary intersections. (The screw axis $\$_{ij}{}'$ is taken as that line in Σ_1 which coincides with $\$_{ij}$ when Σ is in position j or i.)

[6] The points at infinity along the screw axes are stationary points for two positions and are, therefore, always on i^3. Hence $\$_{12}$, $\$_{13}$, and $\$_{23}{}'$ are the directions of the asymptotes and, since they generally cut the plane at infinity in distinct points, i^3 is a cubical hyperbola.

line. Four points on a straight line would require that the rank of the system matrix of equation (4) be two and that its coefficient matrix be of rank one. In other words, we require all (2 × 2)'s of the determinant (5) to vanish. If, say, $(x_2 - x_1) \neq 0$, this requirement is met by equations (12) and a similar set, say, (12^1) with the subscript 3 replaced by 4. These four equations will generally not be compatible. Geometrically, four quadratics do not generally have a common intersection, nor do the two space cubics $i_{23}{}^3$ and $i_{24}{}^3$.[7] This leads us to conclude that, unlike the well-known planar case, ***there are generally no "four-point" circles with infinite radii.***

As we shall see later [21], there are, however, important special cases (e.g., planar and spherical motion) when (12) and (12^1) are compatible.

Computationally, equations (12) are simple enough to allow (after substituting from (1)) one of the unknowns, say, x_1, to be explicitly eliminated. Considering one of the remaining unknowns, say, y_1, as a parameter, the eliminant may be regarded as a quartic in z_1. For any given value of y_1, one obtains at most four values of z_1, three of which belong to i^3. The extraneous root is a point on the line $(z_2 - z_1) = 0$, $(x_2 - x_1) = 0$. Alternatively, we could, after substituting from (1), eliminate x_1 between $(z_2 - z_1) = 0$, $(x_2 - x_1) = 0$. Then for any value of y_1 compute the spurious z_1 and eliminate it from the quartic by synthetic division. Table 4 lists several points on i^3 corresponding to the motion given in Table 1.

Points Which Lie on a Plane

For three positions all points lie on planes. The condition that a point lie on a plane for four positions is obtained from the cubic transformation by requiring that the center of the sphere be at infinity:

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) \end{vmatrix} = 0$$

This is of course the surface we have called F^3 which is embedded in Σ_1 and is identical to equation (5). (Assuming no other (3 × 3) determinants of (4) vanish and that all the (2 × 2)'s of (5) are not zero, this condition leaves the rank of the system matrix of (4) unaltered while reducing the rank of the coefficient matrix to two.) Hence, ***the locus of all points having four positions on a plane is a third-order algebraic surface.*** From physical reasoning (as well as from the foregoing) it is obvious that this surface contains k^6 and i^3.

Introducing a fifth position, we follow the procedure described in the discussion on points on a sphere and require $F_{234}{}^3 \times F_{235}{}^3$. This intersection contains c^6 a space curve of order six and genus three which is the locus of all points with five positions on a plane; curve i^3 is also common to both surfaces but it does not contain the required points. (For points on i^3, the rank of the coefficient matrix of (4) is one.) An alternative approach, which is computationally inferior, is to require $F_{234}{}^3 \times F_{345}{}^3$ in which case the intersection is composed of c^6 and k^6 with k^6 being spurious.

Introducing a sixth position we require $F_{234}{}^3 \times F_{235}{}^3 \times F_{236}{}^3$. Since these three surfaces have i^3 as a common component, there are at most ten points in Σ_1 with six positions on a plane.[8]

This result also follows if we consider $k^{10} \times F_{236}{}^3$.[9] In either case, this means that there is no upper limit to the radii of the "six-position" spheres and that ten points on k^{10} fall on spheres of infinite radii. In addition since four surfaces such as F^3 will generally not have a common intersection point, ***there are (generally) no points which have seven positions on a plane*** and, hence, all 20 "seven-position" spheres will (generally) be finite.

Computationally, curve c^6 may be developed as a function of a single parameter: Eliminating, say, x_1 between $F_{234}{}^3$ and $F_{235}{}^3$ yields a ninth-degree polynomial in which we regard, say, y_1 as a parameter and z_1 as an unknown. Then using i^3 (after eliminating x_1 as described previously) corresponding to a given value of the "parameter" y_1, we synthetically divide the ninth-degree polynomial and obtain a sextic in z_1 whose roots lie on c^6. This procedure has the advantage of being systematic and is useful for plotting the entire curve. However, for obtaining arbitrary points on c^6 and points with six positions on a plane, simple iteration techniques generally suffice. For the motion defined in Table 1, points with six positions on a plane are listed in Table 4.

From the discussion of four positions on a circle, it follows that to each set of directions, (l, m, n), of the plane there corresponds a point on F^3. Hence, (8) taken three times may be used to parametrically generate F^3. Similarly, c^6 may be generated parametrically by writing (8) four times (j = 2, 3, 4, 5) and eliminating, say, m. The resulting three quadratics, containing $\left(\frac{n}{l}\right)$ as a single parameter, may be solved for (x_1, y_1, z_1).

Points Which Lie on a Cylinder

We specify the axis of a right-circular cylinder by its direction cosines (l, m, n) and the coordinates of one of its points (a, b, c), and let the normal vector r_A from the origin terminate at (a, b, c). We take r_j as the vector from the origin to a point in Σ_j, and define the vector $\mathbf{D}_j = \mathbf{r}_j - \mathbf{r}_A$. In Fig. 2, the directed distance between planes normal to the axis and through the tips of r_1 (i.e., r_j with $j = 1$) and r_j is taken as $\mathbf{c}$. The vector $\mathbf{d}_j$ is taken from (a, b, c) to the point obtained by projecting the tip of r_j (parallel to the cylinder axis) into the plane containing the tip of r_1. From the figure it follows that $\mathbf{D}_j = \mathbf{d}_j + \mathbf{c}$ and, hence, $\mathbf{d}_j = \mathbf{r}_j - \mathbf{r}_A - \mathbf{c}$.

If the point (x_j, y_j, z_j), i.e., the tip of r_j, is to lie on a cylinder about the given axis, we require $|\mathbf{D}_1| = |\mathbf{d}_j|$. Therefore, $d_j{}^2 - D_1{}^2 = 0$ or using the foregoing

$$(\mathbf{r}_j - \mathbf{r}_A - \mathbf{c})\cdot(\mathbf{r}_j - \mathbf{r}_A - \mathbf{c}) - (\mathbf{r}_1 - \mathbf{r}_A)\cdot(\mathbf{r}_1 - \mathbf{r}_A) = 0$$

which when expanded and simplified yields

$$(x_j - x_1)a + (y_j - y_1)b + (z_j - z_1)c + \frac{r_1{}^2 - r_j{}^2}{2} + \frac{1}{2}\left[(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n\right]\left[(x_j + x_1)l + (y_j + y_1)m + (z_j + z_1)n\right] = 0 \quad (13)$$

If the axis is specified, then equation (13) (with (l, m, n) and

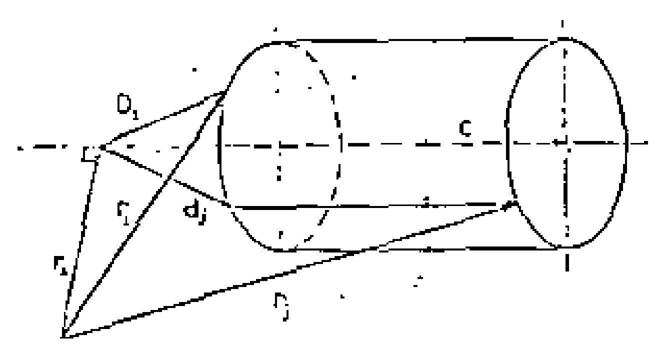


Fig. 2 Geometry for point with two positions on a right-circular cylinder. C is cylinder axis; r_j and r_1 are position vectors for required point in positions j and 1, respectively.

[7] The point at infinity along S_{12} will be common to both cubics, but it does not lie on a four-point circle.

[8] Schoenflies' original proof [12] deals with somewhat different surfaces, so we require a new proof: For $F_{234}{}^3 \times F_{235}{}^3$ we have $N_1 = N_2 = 3$. Using $i^3(n = 3, p = 0)$, equation (15) yields 8 as the number of intersections of curves c^6 and i^3. Now c^6 intersects $F_{236}{}^3$ in 18 points, but since $F_{236}{}^3$ contains i^3, 8 of these fall on i^3. Hence, 18 − 8 = 10 points have six positions on a plane.

[9] The curve k^{10} intersects $F_{236}{}^3$ in 30 points and, as shown in Appendix 1, k^{10} intersects k^6 in 20 points. Now since $F_{236}{}^3$ contains k^6, there are only 30 − 20 = 10 points of $k^{10} \times F_{236}{}^3$ which do not lie on k^6.

(a, b, c) fixed) represents a quadratic surface embedded in Σ_1 which is the locus of all points with two positions on a cylinder, about the given axis. If the cylinder axis coincides with screw axis $\$_{1j}$, equation (13) is identically satisfied by every point in Σ_1.[10]

For three positions we take (13) with $j = 2, 3$, and find that *the locus of all points with three positions on a cylinder is a quartic space curve.*

For four positions we have equation (13) taken three times, $j = 2, 3, 4$, with the result that *there are at most eight points with four positions on a cylinder with a specified axis.*

For five positions we are not at liberty to choose an arbitrary axis. If only the direction (l, m, n) is specified, then the orthogonality condition $al + bm + cn = 0$ must be explicitly satisfied. With $j = 2, 3, 4$ in equation (13) and the orthogonality condition one obtains a set of four nonhomogeneous linear equations in (a, b, c). The compatibility condition is that the determinant of the system matrix equal zero. Hence we require:

$$\begin{vmatrix} l & m & n & 0 \\ (x_2 - x_1) & (y_2 - y_1) & (z_2 - z_1) & f_2^2 \\ (x_3 - x_1) & (y_3 - y_1) & (z_3 - z_1) & f_3^2 \\ (x_4 - x_1) & (y_4 - y_1) & (z_4 - z_1) & f_4^2 \end{vmatrix} = 0 \quad (14)$$

where

$$f_j^2 = \frac{r_j^2 - r_1^2}{2} + \tfrac{1}{2}\,[(x_j - x_1)l + (y_j - y_1)m + (z_j - z_1)n] \times [(x_j + x_1)l + (y_j + y_1)m + (z_j + z_1)n]$$

Equation (14) represents a quartic surface embedded in Σ_1. This surface which we call H_{234}^4, or simply H^4, is the locus of all points with four positions on right-circular cylinders with axes parallel to (l, m, n). For a specified location (a, b, c), the intersection of H^4 with the aforementioned quartic ((13) taken twice) yields the eight points previously described.[11]

Resuming consideration of five positions, we conclude that the intersection $H_{234}^4 \times H_{235}^4$ will contain the required locus. This intersection is composed of a thirteenth-order space curve, k^{13}, which is the locus of all points with five positions on a cylinder with specified axis orientation, and a residual of three lines which are the loci of all points with two of their first three positions on one generator; that is, a line parallel to (l, m, n).[12]

For six positions we have the intersections of $H_{234}^4 \times H_{235}^4 \times H_{236}^4$ which yield 34 points with six positions on a cylinder for each specified axis orientation.[13]

If we do not specify the inclination, we may instead specify the normal to the axis (a, b, c). However, in this case and in the case of both (l, m, n) and (a, b, c) left unspecified, the nonlinearity of (l, m, n) in the f_j^2 terms makes the analysis more complicated.

Reference Positions and Kinematic Inversion

In the foregoing we have dealt with Σ_1 as the reference position and determined those loci which are embedded in it. It is felt that this choice affords the simplest means of explanation, derivation, and computation. However, it should be noted that analogous loci exist in each position Σ_j. These loci can be computed either directly, by a suitable change in the subscripts which denote the position, or indirectly, by remembering that all loci embedded in Σ_1 move with the rigid-body motion of the system and can, therefore, be transformed from Σ_1 to Σ_j by screw $\$_{1j}$.

We also point out that for some of the previous cases, one could work instead with loci which are embedded in Σ'. This is especially true in the case of the sphere in which points in Σ_1 and Σ' are in (1, 1) correspondence. Here we note the analogy with the well-known planar center and circle-point curves which also contain points in (1, 1) correspondence. The loci in Σ' are obtained either directly, by interchanging "known" and "unknown" variables in the derivations (as described in the section on points on a sphere), or indirectly, by inversion. In an inversion we fix Σ in position Σ_1 and move Σ' under that set of screws (embedded in Σ_1) which have the same axes as the original screws but opposite senses.[14] If we invert the motion, then the loci in Σ_1 and Σ' are interchanged. Actually, both Σ and Σ' may be in motion, since the results of the preceding sections depend only on the relative positions of the two systems.

Conclusions

We have considered points whose several positions fall on special surfaces or curves. Equations for the loci of points which lie on spheres, planes, circles, cylinders, or lines were derived. In addition, it has been shown how to compute points on these loci. Computer programs have been written to perform these computations and several numerical examples have been presented.

In addition to their intrinsic interest, these results have direct application to problems of mechanism design. The application of these theories is discussed in a companion paper [21]. Special motions (e.g., planar and spherical) are also studied, and it is shown, in [21], that some of these motions yield the conditions under which equations (5), (6), (5') and (6'), as well as (12) and (12'), become compatible.

Acknowledgments

The author wishes to express his appreciation to Mr. Pictiaw Chen for doing the bulk of the computer programming, and Mr. Kenneth Waldron for programming assistance. The author is also indebted to Prof. F. Freudenstein for having introduced him to the problems discussed herein and for his continued interest in this work; to Dr. E. J. F. Primrose for his interest and willing assistance; to the National Science Foundation whose grant NSF-GP 2005 made this work possible.

APPENDIX 1

Seven Positions on a Sphere

In general, if F_1 and F_2 are surfaces of order N_1 and N_2 which pass through a nonsingular curve of order n and genus p, they will contain another curve of order n^1, and genus p^1. The number of intersections of these curves is I, which is given by [20]

$$I = n(N_1 + N_2 - 4) - (2p - 2). \qquad (15)$$

The following condition connects the genera

$$2(p^1 - p) = (n^1 - n)(N_1 + N_2 - 4) \qquad (16)$$

Consider first the four-point circle condition $F^4 \times G^3$ which we know yields k^6 and l^6. Now for l^6 $n = 3$ and $p = 0$, hence with $n^1 = 6$, and $N_1 = N_2 = 3$, (16) yields $p^1 = 3$ as the genus of k^6.

For $E_{234}^4 \times E_{235}^4$ we use (15) to determine the number of points of intersection of k^6 with k^{10}. Substituting $N_1 = N_2 = 4$, $n = 6$, $p = 3$, one obtains $I = 20$. Now k^{10} intersects E_{236}^4 in 40 points, but since E_{236}^4 contains k^6, 20 of these fall on k^6. Hence, there are $40 - 20 = 20$ points with seven positions on a sphere.

---

[10] For any general orientation the screw axis $\$_{1j}$, considered as a line in Σ_1, meets the quadratic at two points, one of which is at infinity.

[11] There are actually 16 points, but half of them fall on the lines described in the next paragraph and therefore do not have the required axis location. These 8 redundant points lie 4 each on the lines associated with positions 1, 2 and 1, 3.

[12] These three lines are easily obtained from the intersection of planes $(x_j - x_1)m - (y_j - y_1)l = 0$ and $(x_j - x_1)n - (z_j - z_1)l = 0$, $j = 2, 3$, and from plane $[(x_3 - x_1) - (x_2 - x_1)]m - [(y_3 - y_1) - (y_2 - y_1)]l = 0$ with $[(x_3 - x_1) - (x_2 - x_1)]n - [(z_3 - z_1) - (z_2 - z_1)]l = 0$.

[13] There are 52 points common to k^{13} and H_{236}^4; however, 18 of them fall on the three lines previously described. Of the remaining 34 three are at infinity along the axes $\$_{12}$, $\$_{13}$, $\$_{14}$.

[14] That is, instead of $\$_{1j}$ we screw about $\$_{1j}^{-1}$.

References

1 F. Freudenstein, "Approximate Synthesis of Four-Bar Linkages," Trans. ASME, vol. 77, 1955, pp. 853–859.

2 F. Freudenstein and G. N. Sandor, "Synthesis of Path-Generating Mechanisms by Means of a Programmed Digital Computer," *Journal of Engineering for Industry*, Trans. ASME, vol. 81, Series B, 1959, pp. 159–168.

3 B. Roth and F. Freudenstein, "Synthesis of Path-Generating Mechanisms by Numerical Methods," *Journal of Engineering for Industry*, Trans. ASME, vol. 85, Series B, 1963, pp. 298–306.

4 F. Freudenstein and G. N. Sandor, "On the Burmester Points of a Plane," Journal of Applied Mechanics, vol. 28, Trans. ASME, vol. 83, Series E, 1961, pp. 41–49.

5 G. N. Sandor, "On the Existence of a Cycloidal Burmester Theory in Planar Kinematics," Journal of Applied Mechanics, vol. 31, Trans. ASME, vol. 86, Series E, 1964, pp. 694–728.

6 J. Denavit and R. S. Hartenberg, "Approximate Synthesis of Spatial Linkages," Journal of Applied Mechanics, vol. 27, Trans. ASME, vol. 82, Series E, 1960, pp. 201–206.

7 R. S. Hartenberg and J. Denavit, *Kinematic Synthesis of Linkages*, McGraw-Hill Book Company, Inc., New York, N. Y. 1964, 435 pages.

8 N. I. Levitskii and K. K. Shakvazian, "Synthesis of Four-Element Spatial Mechanisms With Lower Pairs," *International Journal of Mechanical Sciences*, vol. 2, pp. 76–92. Translation by F. Freudenstein of a paper which was originally published in 1954.

9 E. P. Novodvorskii, "A Method of Synthesis of Mechanisms," in Russian, *Trudi. Semin. po Theor. Mash. Mekh.*, Akademii Nauk, SSSR, vol. 42, 1951.

10 F. M. Dimentberg, "A General Method for the Investigation of Finite Displacements of Spatial Mechanisms and Certain Cases of Passive Constraints," in Russian, *Trudi. Semin. po. Teor. Mash. Mekh.*, Akademii Nauk, SSSR, vol. 5, no. 17, 1948, pp. 5–39.

11 K. K. Shakvazian, "Analytical Method for the Design of Spatial Link Mechanisms With Lower Pairs," in Russian, *Trudi Inst. Mash.*, vol. 20, 1960, pp. 5–9.

12 A. Schoenflies, *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, Germany, 1886, 194 pages.

13 M. L. Keler, "Analyse und Synthese der Raumkurbelgetriebe mittels Raumliniengeometrie und dualer Grössen," *Forschung Ing. Wes.*, vol. 25, 1959, pp. 27–32 and 55–63.

14 G. Hein, "Analytische und Zeichnerische Unterlagen für zwei und drei Lagen eines sphärisch bewegten Getriebegliedes als Hilfsmittel zur Synthese sphärischer Kurbelgetriebe," *Forsch Ing. Wes.*, vol. 25, 1959, pp. 92–96 and 115–126.

15 J. T. Wilson, III, "Analytical Kinematic Synthesis by Finite Displacements," *Journal of Engineering for Industry*, Trans. ASME, vol. 87, Series B, 1965, pp. 161–169.

16 R. S. Ball, *Theory of Screws*, Cambridge University Press, Cambridge, England, 1900.

17 A. T. Yang, "Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms," doctoral dissertation, Columbia University, New York, N. Y., 1963, 241 pages.

18 E. Study, *Geometrie der Dynamen*, B. G. Teubner, Leipzig, Germany, 1903, 603 pages.

19 M. Nöther, "Ueber die eindeutigen Raumtransformationen insbesondere in ihrer Anwendung auf die Abbildung algebraischer Flachen," *Mathematische Annalen*, vol. 3, 1871, pp. 547–580.

20 J. G. Semple and L. Roth, *Introduction to Algebraic Geometry*, Oxford University Press, England, 1949, 446 pages.

21 B. Roth, "Finite Position Theory Applied to Mechanism Synthesis," ASME Paper No. 67—APM-30.

Printed in U. S. A.

Finite-Position Theory Applied to Mechanism Synthesis

B. ROTH
Associate Professor,
Mechanical Engineering Department,
Stanford University, Stanford, Calif.
Mem. ASME

The well-known finite-position planar theory of kinematic synthesis (the so-called Burmester theory) and the corresponding spherical theory are derived from the results of the general spatial theory which has been given in a companion paper [1].[1] Other special displacements studied are those for which the author has coined the names "similarity transformation," "pseudoplanar," and "pseudospherical." These results, as well as those obtained in [1], are shown to be applicated to the synthesis of spatial, spherical, and planar linkages. Several numerical examples are presented.

Introduction

IN A PREVIOUS paper [1], we developed equations governing the loci of points which, under a series of rigid-body displacements, have several positions on spheres, planes, circles, cylinders, or lines. In this paper, we particularize these general results to three types of special displacements having important practical applications. Since the author is unaware of any previous discussion of the first special case, he has (with misgiving) coined the name similarity transformation. The other two special cases—intersecting screws and parallel screws—lead to the well-known spherical and planar motion finite-position theories which may now be viewed in a more general context. In addition, these latter two special cases lead, respectively, to the intriguing, and seemingly new, concept of pseudospherical and pseudoplanar mechanisms.

Applications of the general theory, as well as the foregoing special cases, are discussed. It is shown that these results are applicable to various types of design problems, and several mechanism syntheses are presented. The function-generation problem has been treated previously elsewhere ([3, 5] and especially [6]), but most of the other syntheses seem to be new.

Nomenclature

We describe a finite displacement of a rigid body, from position i to position j, as a screw displacement $\$_{ij}$. The system of points in the moving system is called Σ, in general, and Σ_i when we wish to emphasize that the system is in the ith position. The system of points in the reference system (also referred to as the fixed system) is called Σ'. The screw axis is considered as a line in Σ' unless it contains a superscript, as in $\$_{ij}{}^i$, in which case it is taken as that line in Σ_i which coincides with $\$_{ij}$ when Σ is in position i.

Following the notation developed in [1]: E^4 is the fourth-order surface which contains all the points in Σ_1 which have five positions on a sphere; k^6 is a sixth-order space curve which contains all the points in Σ_1 which lie on a circle for four positions, and i^3 is a third-order space curve which contains all the points in Σ_1 which have three positions on a line. For the sake of clarity, it is sometimes necessary to use subscripts (e.g., $k_{24}{}^6$) to denote which positions are being considered. However, in what follows, position 1 is generally omitted; since it is taken as the reference position, it is always understood to be the first position.

Finally, all numbered equations refer to equations derived in [1]. In this context, it is useful to know that:

(*a*) Equation (1) defines the linear transformation under which point (x_1, y_1, z_1) is "screwed" to (x_j, y_j, z_j) by $\$_{1j}$:

$$x_j = (a_{x_j} + 1)x_1 + b_{x_j}y_1 + c_{x_j}z_1 + d_{x_j}$$
$$y_j = a_{y_j}x_1 + (b_{y_j} + 1)y_1 + c_{y_j}z_1 + d_{y_j}$$
$$z_j = a_{z_j}x + b_{z_j}y_1 + (c_{z_j} + 1)z_1 + d_{z_j}$$

(*b*) The square of the distance from the origin is given by:

$$r_j{}^2 = x_j{}^2 + y_j{}^2 + z_j{}^2$$

(*c*) Equations (5) and (6) are the equations which define $k_{234}{}^6$, while (5′) and (6′) define $k_{235}{}^6$. These two curves generally do not have any common finite points.

(*d*) Equation (9) gives the locus of all points with positions 1 and j in a plane normal to some given direction (l, m, n).

(*e*) Equation (8) gives the locus of all points (x_1, y_1, z_1) which in positions 1 and j have the same distance from a point (A_x, A_y, A_z) fixed in Σ'.

(*f*) Equation (12) is a set of two equations which defines $i_{23}{}^3$, while (12′) defines $i_{24}{}^3$. These two curves generally do not have a common finite point.

Special Motions

We now undertake the study of three types of screw motions which are commonly used in mechanical linkwork. These motions yield special cases of the general theory which are of theoretical and practical interest.

Similarity Transformations

We consider the case when the screw is determined by pure rotations about two axes, $\$_A$ and $\$_B$, which are embedded, respectively, in Σ' and Σ (Fig. 1). Further, the distance and inclination between $\$_A$ and $\$_B$ are fixed.

Let A and B be the transformation matrices, for motion of points in Σ about $\$_A$ and $\$_B$, respectively, such that the total transformation due to a rotation θ about $\$_B$ followed by a rotation Φ about $\$_A$ is given by the matrix product AB. Now, if we reverse the order of the two rotations so that we first rotate Σ by Φ about $\$_A$, we note that axis $\$_B$ has a new position (in Σ'), say, $\$_{B_2}$, and we must determine the transformation matrix, say, B_2, which corresponds to a rotation θ about $\$_{B_2}$. Since the only change between $\$_B$ and $\$_{B_2}$ is the rotation defined by A, the new transformation B_2 is given by the so-called similarity transformation ABA^{-1}. Hence the total transformation is $(ABA^{-1})A =$

[1] Numbers in brackets designate References at end of paper.

Contributed by the Applied Mechanics Division for presentation at the Applied Mechanics Conference, Pasadena, Calif., June 26–28, 1967, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until one month after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, November 12, 1965; final draft, September 26, 1966. Paper No. 67—APM-39.

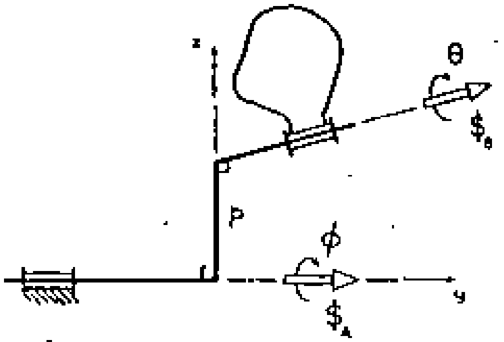


Fig. 1 "Similarity-transformation" motion is defined by rotations about a fixed axis ($\$_A$) and an axis moving with the body ($\$_B$). The distance, p, and angle between the axes are constant.

AB. We conclude that the order of the transformations about $\$_A$ and $\$_B$ is immaterial, and that, in the special case of one axis fixed in the moving body, finite rotations (as well as general screw displacements) do commute.

We may simplify the derivations without losing any generality by choosing, say, the y-axis along $\$_A$ and the z-axis along the common normal from $\$_A$ to $\$_B$. The length of the common normal is p, and the direction cosines of $\$_B$ (in position Σ_1) are $(u_1, u_2, 0)$. There is a unique screw which is equivalent to the two rotation screws $\$_A$, $\$_B$; we compute its parameters (d_{eq}, θ_{eq}) and its direction cosines $(u_{1eq}, u_{2eq}, u_{3eq})$:

$$d_{eq} = 2pu_1^2 \frac{\sin\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$\cos\left(\frac{\theta_{eq}}{2}\right) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\Phi}{2}\right) - u_2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)$$

$$u_{1eq} = u_1 \frac{\cos\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$u_{2eq} = \frac{u_2\cos\left(\frac{\Phi}{2}\right)\sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

$$u_{3eq} = -u_1 \frac{\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\Phi}{2}\right)}{\sin\left(\frac{\theta_{eq}}{2}\right)}$$

With these results, the transformation coefficients of equation (1) become:

$$a_{x_j} = -u_2[\sin\theta_j\sin\Phi_j + u_2\cos\Phi_j(1-\cos\theta_j)] - (1-\cos\Phi_j)$$

$$b_{x_j} = u_1[\sin\theta_j\sin\Phi_j + u_2\cos\Phi_j(1-\cos\theta_j)]$$

$$c_{x_j} = u_2\sin\theta_j\cos\Phi_j + \cos\theta_j\sin\Phi_j$$

$$d_{x_j} = -p[u_2\sin\theta_j\cos\Phi_j + (\cos\theta_j - 1)\sin\Phi_j]$$

$$a_{y_j} = u_1u_2(1-\cos\theta_j)$$

$$b_{y_j} = -u_1^2(1-\cos\theta_j)$$

$$c_{y_j} = -u_1\sin\theta_j$$

$$d_{y_j} = pu_1\sin\theta_j$$

$$a_{z_j} = -u_2[\sin\theta_j\cos\Phi_j - u_2\sin\Phi_j(1-\cos\theta_j)] - \sin\Phi_j$$

$$b_{z_j} = u_1[\sin\theta_j\cos\Phi_j - u_2\sin\Phi_j(1-\cos\theta_j)]$$

$$c_{z_j} = -u_2\sin\theta_j\sin\Phi_j + \cos\theta_j\cos\Phi_j - 1$$

$$d_{z_j} = p[u_2\sin\Phi_j\sin\theta_j + \cos\Phi_j(1-\cos\theta_j)]$$

Further, it may be shown that

$$\frac{r_j^2 - r_1^2}{2} = u_2\sin\theta_j x_1 - u_1\sin\theta_j y_1 + (1-\cos\theta_j)(z_1 - p).$$

Under these conditions, E^4 degenerates into two quadratic surfaces, and, as shown in Appendix 1, we may write $E^4 = u_1F^2G^2$. Since $F^2 = [u_2x_1 - u_1y_1]^2 + [z_1 - p]^2$, it is independent of the rotations. The only real points on F^2 are given by the intersection of planes $u_2x_1 - u_1y_1 = 0$ and $z_1 - p = 0$, which is in fact the moving axis $\$_B$ (in Σ_1). Physically, line $\$_B$ satisfies E^4 identically because all of its points always lie on circles.

If $u_1 = 0$, E^4 is identically zero. This corresponds to the case of $\$_A$ and $\$_B$ parallel which restricts all points to planar motion and, hence, to spheres of infinite radii.

The surface G^2 is generally a hyperboloid. Its equation is derived in Appendix 1.

Obviously, the same degeneracies must occur for arbitrary orientation of the coordinate axes: One of the quadratics is imaginary except for an isolated line which is always $\$_B$. The equation of this quartic is, in fact, $(f^2 + g^2) = 0$. Where $f = 0$ is the plane through $\$_B$ perpendicular to the common normal between $\$_A$ and $\$_B$, and $g = 0$ is the plane through $\$_B$ and the common normal (i.e., it is normal to $\$_A$), the equations of both planes are taken in normal form. The other quadratic is generally a real hyperboloid and, of course, $E^4 = 0$ is identically satisfied if the axes are parallel.

Assuming the rotation axes are not parallel, the points of interest lie on G^2. For six positions on a sphere, we consider $G_{234}^2 \times G_{256}^2$, which yields a space cubic, and a line l_1 which is spurious. For seven positions, $G_{234}^2 \times G_{256}^2 \times G_{267}^2$ yields (at most) four points.[2] Since the surfaces are only of order two, these computations may be carried out without recourse to iterative techniques.

For this motion, equation (6) factors into a plane P^* and the foregoing "line quadratic" explicitly (6) yields:

$$u_1F^2P^* = 0$$

where

$$P^* = \sum_{\substack{i=2\\j=2\\k=2}}^{4} (r_i - r_1)\sin\theta_j(1-\cos\theta_k)\epsilon_{ijk}$$

Here ϵ_{ijk} is +1 or −1 if (i, j, k) is, respectively, an even or odd permutation of (2, 3, 4), and is zero if two of the subscripts are equal.

Equation (5) remains a cubic, but it may be put into the form:

$$u_1[(x_1u_2 - y_1u_1)F_A^{**} - (z_1 - p)F_B^{**}] = 0$$

where F_A^{**} and F_B^{**} are quadratic functions of x_1, y_1, z_1.

From the aforementioned, we may easily verify that all points on the moving rotation axis satisfy (5) and (6). Further, the intersection of P^* with (5) is a quadratic curve, which is the residual part of the intersection of (5) and (6), and a line l_1. Hence *the loci of points in Σ with four (or more) positions on a circle are the lines $\$_B$ and l_1.*

Parallel Screws

In the case of three parallel, pure-rotation ($\$_{12}, \$_{13}, \$_{14}$) screws and a fourth arbitrary screw ($\$_{15}$), the quartic surface E^4 degenerates to a plane and a cubic cylinder. Taking (l, m, n) as the direction cosines of the three parallel screw axes, we may write:

$$E^4 = PK^3$$

where

[2] The space cubic cuts G_{267}^2 in six points; however, two of these are on l_1. The three surfaces contain the line l_1 as a common component.

Table 1 Five-position circles for three-parallel pure-rotation screws and one-skew screw

SCREW $_{1j}$ j	DIRECTION COSINES OF SCREW AXIS l_j	m_j	n_j	POINT ON SCREW x	y	z	ROTATION ϕ_{1j} (RADIANS)	TRANSLATION s_{1j}
2	0.0000	0.0000	1.0000	3.4594	-1.2942	0.0000	0.3533	0.0000
3	"	"	"	3.3880	-2.4833	0.0000	0.3648	0.0000
4	"	"	"	2.7423	-3.4354	0.0000	0.3120	0.0000
5	-0.1888	-0.5706	-0.8?51	0.2623	-0.6?04	-0.5104	0.6499	-1.6674

CIRCLE NUMBER	INITIAL POSITION x_1	y_1	z_1	CENTER x_c	y_c	z_c	RADIUS	DIRECTION COSINES OF AXIS l	m	n
1	-0.2553	0.1655	-0.4073	0.5164	0.0456	-0.0072	0.9910	0.0000	0.0000	-1.0000
2	2.8476	1.4309	-0.4153	3.1462	-0.0405	-0.0302	1.5925	"	"	"
3	48.6331	27.5968	1.8483	-26.8711	-17.4856	1.8353	136.3646	"	"	"
4	-0.5784	1.8594	-2.0529	0.5683	0.5466	-2.0507	1.3654	"	"	"
5	1.4421	1.0541	-2.3435	3.5412	0.4417	-2.2875	1.4457	"	"	"
6	22.7466	0.1512	20.1556	41.5904	5.5410	20.5554	73.6501	"	"	"

$$P = (x_5 - x_1)l + (y_5 - y_1)m + (z_5 - z_1)n$$

and

$$E^3 = \frac{1}{2\pi}\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1) & (r_1^2 - r_2^2)\\(x_3 - x_1) & (y_3 - y_1) & (r_1^2 - r_3^2)\\(x_4 - x_1) & (y_4 - y_1) & (r_1^2 - r_4^2)\end{vmatrix}$$

The cubic cylinder E^3 has generators which are all parallel to (l, m, n) (six of which are the axes $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{23}{}^1$, $\$_{24}{}^1$, $\$_{34}{}^1$). Any plane section normal to the generators yields the circle-point curve, associated with the motion in positions 1, 2, 3, 4, which is, in fact, planar.

The plane P is the locus of all points whose positions 1 and 5 fall on a plane normal to the three parallel screws and hence have five positions on an infinitely large sphere. Since P is basically the same plane we studied in connection with circular motion, equation (9), the previous geometrical interpretation is immediately applicable (with the understanding that (l, m, n) now represents the direction of the three parallel screws and not the direction of a circle axis).

For this motion, (5) is identically zero and (6) is essentially sE^3. Therefore, the cubic cylinder E^3 is the locus of all points with four positions on a circle.

Points with five positions on a circle will generally exist because (5) is identically zero. Such points are determined by the intersection of (6), (5^1) and (6^1). In this case (5^1) (which is the same as (5) except that subscript 4 is changed to 5), may be shown to degenerate to $P \times F_r{}^{*2}$, but, since all points with five positions on a circle obviously fall in one plane, equation (5^1) will always be satisfied by points on P and we may ignore the quadratic $F_r{}^{*2}$. Denoting the cubic surface (6^1) as $G_{35}{}^3$, the required points are given by $P \times G_{35}{}^3 \times E^3$. This yields (at most) six real points with five positions on a circle.[3] In Table 1, we list the results of one such computation.[4]

If $\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{15}$ are all parallel, pure-rotation screws, the term E^4 is identically zero since P is indeterminate and is satisfied by every point in Σ_1. Both (5) and (5^1) are identically zero, and (6) and (6^1) become (circular) cubic cylinders; $k_{34}{}^3$ and $k_{35}{}^3$, with parallel generators. The normal sections of these cubics yield the circle-point curves of the planar theory. Of the nine lines of the intersection, at most seven are real and three are the screws $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$. This leaves four lines which correspond to the well-known Burmester points. If, in addition, the screws are defined by a similarity transformation, one of these four lines is always the (moving) axis $\$_{g}$.

If the four screws ($\$_{12}$, $\$_{13}$, $\$_{14}$, $\$_{15}$) are parallel, but one, say, $\$_{15}$, has pure translation while the others have pure rotation, P becomes the plane at infinity and there are no finite points with five positions on a circle.

[3] Since the points lie on P they are all coplanar. Three of the nine intersections have been discounted since they fall on $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$, and will not generally yield a five-point circle.

[4] The computations described in this paper have been programmed in FORTRAN II. Programs may be obtained by writing to the author.

For parallel screws with rotation and translation, E^4 will generally not degenerate. However, points with four and five positions on right, circular cylinders with axes parallel to the screws may be obtained by merely neglecting the translation and determining points which lie on circles.

Two parallel, pure-rotation screws generally do not degenerate E^4. In addition, since (6) does not vanish, there are, generally, no points with five positions on a circle. However, since (5) is of the form $P \times F_r{}^{*2}$, all points with four positions on a circle are given by $P \times G_{34}{}^3$ and are therefore coplanar.

Shoenflies has shown that any two parallel screws will cause P to decompose and that, conversely, if P decomposes, the screws must be parallel. For this case, the locus, of all points with three positions on a line, is a line. This same result follows directly from equations (12): For screws parallel to, say, the z-axis, the hyperboloids become, respectively, a right, circular cylinder (with generators parallel to the screws) and a plane parallel to the screws. The intersection, apart from the residual line $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$, is a line common to the cylinder and plane (see Appendix 2).

We now restrict the parallel screws to pure rotation and, for convenience, again take the screws along the z-direction. For pure rotation, the first equation in set (12) represents the same right, circular cylinder as previously mentioned but the second equation of (12) vanishes identically. Hence the locus of all points with three positions on a line becomes a right, circular cylinder embedded in Σ_1. This cylinder obviously contains the screws $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$. Any normal section yields a planar, image-pole triangle and its circumscribing circle. For three parallel, pure-rotation screws, equations (12^1) also have one member identically equal to zero. Hence points with four positions on a line are given by the intersection of the aforementioned cylinder and

$$\begin{vmatrix}(x_2 - x_1) & (y_2 - y_1)\\(x_4 - x_1) & (y_4 - y_1)\end{vmatrix} = 0$$

These two parallel cylinders intersect in two lines. This corresponds with the well-known planar theory since the line $(x_2 - x_1) = 0$, $(y_2 - y_1) = 0$, which is in fact the axis for $\$_{12}$, is the residual part of the intersection.

Intersecting Screws

For screws that intersect in a common point to lead to degenerate cases, they must generally be restricted to pure rotations.

For four positions corresponding to intersecting, pure-rotation screws, E^4 is identically satisfied since all points are undergoing spherical motion about the point of intersection of the screws. If, instead, one of the motions, for example, $\$_{14}$, is arbitrary, $E^4 = P_A C^3$. P_A is the plane given by equation (2) with $\$_j$ equal to $\$_{14}$ and (A_x, A_y, A_z) taken as the coordinates of the point of intersection of the screws. C^3 is formally the cubic surface given by equation (5) (i.e., F^3); however, due to the pure-rotation intersecting screws ($\$_{12}$, $\$_{13}$, $\$_{23}$), the surface becomes a cubic cone with apex at the intersection point of the screws. It can be easily shown that for this motion equation (6) yields either C^3 or is identically zero.[5] In either case, C^3 is the locus of all points with four positions on a circle. Hence, in the case of $\$_{15}$ skew, points with five positions on a sphere are either points which lie on a circle for the four positions associated with the intersecting pure rotation screws, or points whose distances from the intersection point (of $\$_{12}$, $\$_{13}$, $\$_{14}$) are not altered by the skew screw $\$_{15}$.

The intersection of cone C^3 with any sphere whose center is at the apex of C^3 is a spherical curve of order six. This curve is composed of two symmetrical and equal portions, either of which may be regarded as the spherical analog of a planar cubic. These spherical curves are of course analogous to the circle-point curves of the planar theory.

If $\$_{15}$ is also taken as a pure-rotation, intersecting screw, P_{A_5} is identically satisfied by every point in the body, and hence E^4 is identically zero. For five-position spherical motion, equations (5) and (5[1]), that is, $C_{234}{}^3 \times C_{235}{}^3$, give the locus of all points with five positions on a circle.[6] Now since the *surfaces* C^3 are cones with common vertexes, they will intersect in (at most) nine lines which are elements of the cones. $\$_{12}$, $\$_{13}$, $\$_{1^*}$ are three of these lines, the remaining six lines are the loci of points which have five positions on a circle. If we define a sphere on which the spherical motion is taking place, for example, the unit sphere, then there are at most 12 real points in which these lines pierce the sphere. Only half of these are independent since each pair is symmetric about the sphere center; hence we conclude *there are (at most) six independent points which, under spherical motion, have five positions on a circle*. These points are analogous to those studied by Burmester for the case of planar motion.[7] If, in addition, the screws are defined by a similarity transformation, one of these points is always on the (moving) axis $\$_p$.

Computationally the intersections $C_{234}{}^3 \times C_{235}{}^3$ can be obtained by "intersecting" each of these surfaces with an arbitrary plane. Eliminating one variable, say, z_1, between the plane and each C^3 yields two planar cubics from which a second variable, say, y_1, can then be eliminated. The roots of the resulting ninth-degree polynomial in x_1 give one of the required coordinates; the corresponding y_1 and z_1 are then obtained by back-substituting x_1 into the aforementioned. Since we know the location of the apex of the cone, we now have two points on each line, which means we have determined the intersection of $C_{234}{}^3$ and $C_{235}{}^3$. Table 2 lists the results of one such computation.[8]

Alternatively, since the cubics are homogeneous, we could divide through by, say, $(x_1)^3$ and reduce the cones to planar cubics with variables (y_1/x_1) and (z_1/x_1). (If we take $x_1 = 1$, the two methods become identical.)

Returning to the case where screw $\$_{15}$ is skew, we note that the locus of all points with four positions on a great circle of their five-position sphere is the planar cubic $P_A \times C^3$. All other points on C^3 lie on small circles of their sphere.

Introducing a sixth position defined by an arbitrary screw $\$_{16}$, we conclude that, since $E_{235}{}^4$ and $E_{236}{}^4$ have a common component in C^3, the locus of all points with six positions on a sphere is given by the line $P_{A_5} \times P_{A_6}$. This line cuts C^3 in (at most) three points; these points have four positions on a great circle. Similarly, introducing a seventh position defined by skew screw $\$_{17}$, we find that only the point given by $P_{A_5} \times P_{A_6} \times P_{A_7}$ has seven positions on a sphere.

[5] If we choose the origin at the point of intersection, the terms $r_j{}^2 - r_1{}^2 = 0$ for $j = 2, 3, 4$, and hence (6) is identically zero. The fact that a plane intersects a sphere in a circle is a physical reason for the conditions for points on a plane (5) and points on a circle (5 and 6) to be identical under spherical motion.

[6] As shown previously, (6) and therefore also (6[1]) are either identically zero or equivalent to (5) and (5[1]).

[7] For the case of infinitesimal, spherical motion, Dobrovolskii [2] has pointed out the existence of six points which have five-point contact with a circle in the tangent plane.

[8] In determining centers for points on the moving lines, we merely determine the axis for any point on the line. For three positions of a body in planar or spherical motion, there is obviously a (1, 1) correspondence between lines in Σ^1 and Σ. This correspondence is called the quadratic transformation and replaces the cubic transformation which has no significance in spherical or planar motion.

Table 2 Spherical motion; points with five positions on a circle

SCREWS $\$_{1j}$

j	DIRECTION COSINES u_{1j}	v_{1j}	w_{1j}	ROTATION θ_{1j} (deg.)
2	-0.6064	-0.6619	-0.5041	-31.0500
3	0.0641	-0.7825	0.6188	- 3.6800
4	0.5116	0.0520	0.8572	14.2000
5	0.2056	0.7758	0.5937	27.5400

All the screws have zero translation and intersect at the origin.

Root Number	MOVING POINTS $y_1 = xx_1$, $z_1 = zz_1$: x	z	FIXED POINTS $y = x_0 x$, $z = z_0 x$: x_0	z_0
1	0.6306	-0.1523	0.6523	-0.3649
2	-0.8426	-0.3468	-60.7336	-5.4335
3	0.5435	0.4621	0.5145	0.7938
4	5.3426	6.8096	5.1630	3.5691
5, 6	IMAGINARY			

Table 3 Synthesis of two-revolute, two-spheric-pair spatial four bar

PROBLEM STATEMENT

Function: $y = \cos x$ $0 \le x \le 180°$

Range: Input crank (ϕ) = 150°
Output crank (ψ) = 100°

INPUT TO COMPUTER

Precision positions (using Chebyshev spacing)

j	1	2	3	4	5	6	7
$\Delta\phi_{1j}$	0.0	3.6920	14.4145	49.5300	85.1755	96.4400	99.5400
$\Delta\psi_{1j}$	0.0	17.9347	33.9663	70.6347	107.9167	127.9167	140.[illegible]

Angle between axes = 90°

RESULTS OF COMPUTATION

	x	y	z
Point on a sphere	0.9833	0.1659	-0.4561
Center of sphere	0.7891	0.7049	0.7771
Radius of sphere	1.5845		

MECHANISM PARAMETERS

Link lengths: Input crank = 0.5686
Coupler = 1.5895
Output crank = 0.6177

Angle between axes = 90°
Distance between axes = 1
Distance to common normal between crank axes
from plane of:

a) input-crank circle = 0.9433
b) output-crank circle = 0.3049

ANALYSIS OF SOLUTIONS

Maximum error = .0020 which corresponds to 0.10% of the range of y.

For $0 \le x \le 177°$, maximum error is 0.0007 which corresponds to 0.035% of the range of y.

In the case of intersecting, pure-rotation screws, equation (12) yields two quadratic cones with a common apex. These cones intersect in $\$_{12}$, $\$_{13}$, $\$_{23}{}^1$ and the spurious line $(x_2 - x_1) = 0$, $(x_3 - x_1) = 0$. Hence F degenerates into three lines. There are no nontrivial points with three positions on a line, since a line will not pierce a sphere in more than two points.

Application to Mechanism Synthesis

Examples. In the following paragraphs, we illustrate various ways in which the foregoing results can be applied to the synthesis of linkwork. Since it is the versatility of the method which we seek to illustrate, we limit ourselves to one or two examples of a given type and do not consider all the structural variants which may be synthesized by essentially the same procedure.

Motion of a coupler link relative to a fixed link is the most straightforward application of the theory. We illustrate the procedure with a type of seven-bar linkage which has been used in automotive suspensions and elsewhere. The linkage (Fig. 5) has a coupler, a fixed link, and five binary cranks in parallel between the coupler and fixed link. All 10 joints are spherical. For any specified seven positions of the coupler, we may find, at most, 20 points with spherical motion. These points and their corresponding centers define the cranks. Taken five at a time, these 20 points yield 15,504 seven bars whose couplers pass through the seven specified positions. On the other hand, there will generally be fewer, and maybe no, solutions since, for a given set of positions, there may be less than 20, or even five, real points with seven positions on a sphere. For less than seven prescribed positions, we generally have an infinite number of solutions. The points given in Table 3 of [1] are, in effect, the results of such a synthesis.

The relative motion of two moving links can be treated by inversion. This is illustrated by a procedure described by Wilson [3] for the function-generation synthesis of a 2R2S four bar. In this linkage, the input and output crank are joined to the frame by turning pairs (revolutes) which have skew axes. Each crank connects to the coupler with a spherical pair. In synthesizing a function, say, $\Phi = f(\Psi)$, we choose sets of angle changes $\Delta\Phi_{1j}$ and $\Delta\Psi_{1j}$ which satisfy this functional relationship, and require that the two cranks rotate through these angles from some unspecified initial position Φ_1, Ψ_1.

The procedure is as follows: We arbitrarily choose the skew, rotation axes[1] and then invert the mechanism by holding either the input or output crank fixed. If we fix the input crank, the two sets of rotation angles are $-\Delta\Psi_{1j}$ (about the fixed axis) and $\Delta\Phi_{1j}$ (about the moving axis). Alternatively, fixing the output crank we have, respectively, $-\Delta\Phi_{1j}$ and $\Delta\Psi_{1j}$. In either case, the motion of what is now the coupler link is defined under the special case of a similarity transformation and the order in which the rotations are taken is immaterial. The synthesis is completed by determining one point, in what is now the coupler, which lies on a sphere in the several design positions. This point and its corresponding center are taken as the spherical joints in the original coupler. For five angle changes, we use any point on G^s. For six specified angle changes, we use any point given by $G_{234}{}^s \times G_{345}{}^s$; and for seven sets of $\Delta\Phi_{1j}$, $\Delta\Psi_{1j}$, we choose one of the points given by $G_{234}{}^s \times G_{345}{}^s \times G_{456}{}^s$. One such synthesis is described in Table 3.

The same procedure may be applied, for example, to the four bar, shown in Fig. 2, except that here we hold the revolute-cylinder link fixed and seek a point which lies on a cylinder.

In order to move a body according to a given timing cycle, it is desirable to be able to specify the corresponding input motion. Consider, for example, the four bar with two turning, one cylindrical, and one spherical pair, shown in Fig. 2. Here we arbitrarily choose dimensions of the crank with the cylindrical pair, and specify up to four positions. For these design positions, we define the motion of the coupler by specifying a rotation about, and a translation along, the cylindrical joint axis. The synthesis is completed by choosing any point on I^s as the location of the spherical joint and the corresponding axis as the fixed revolute.

Similar procedures may be applied, for example, to a three bar with a revolute-cylinder input crank whose third joint is a sphere-in-cylinder pair, Fig. 3. Here we are limited to three positions and seek a point on I^s.

Inversions of the same linkage are treated differently. For example, if, instead of the aforementioned, the revolute-cylinder crank of the four bar in Fig. 2 is fixed, one could either specify three arbitrary positions of the coupler and synthesize the linkage by determining a revolute-revolute crank (this is discussed later) and one point with three positions on a cylinder (any point will do, since all points have three positions on a cylinder), or choose revolute-revolute crank and consider only similarity-transformation coupler motions. In this latter case, the synthesis is solved by any point which falls on a cylinder in the several design positions.

In certain linkages, it is required that a point simultaneously lie on two special loci. Graphical constructions for such syntheses have been given by Altmann [1]. The methods of this paper may be used to study such problems. The most direct approach is to choose the appropriate equations from the ones developed in [1] and then numerically determine their simultaneous solutions.

Dwell mechanisms and other linkwork can be designed by using points in the body with special motions. For example, to design a dwell mechanism we take any convenient spatial linkage, and from its known "coupler" motion determine which points on the coupler have, say, seven positions on a sphere. To one such point, we attach a dyad of the type shown in Fig. 4. The dimensions of the floating link of the dyad are completely determined by a point which lies on a sphere and its corresponding center. The dimensions of the rotating dyad crank (which is the link with the approximate dwell) are arbitrary.

Spherical and planar linkages are synthesized exactly as above. Therefore, we may solve function generation problems and the like; the only difference is that, instead of a general screw, the motion must be defined by, respectively, intersecting and parallel pure-rotation screws. The results in Table 2 may, of course, be interpreted as a five-position synthesis of a spherical four bar.

[1] The distance between skew axes and their absolute orientation does not affect the relative rotation between input and output. However, the angle between the skew axes does affect the final solution. This procedure is extremely useful in designing linkages which act like noncircular gears connecting skew shafts at a specified angle.

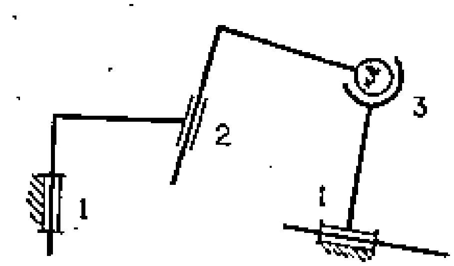


Fig. 2 Four bar with one cylinder, one spheric, and two revolute joints. The numbers denote the freedom in the joints.

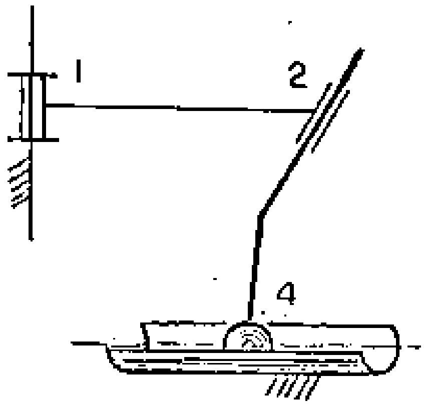


Fig. 3 Three bar with revolute, cylinder, and ball-in-cylinder joint. The numbers denote the freedom in the joints.

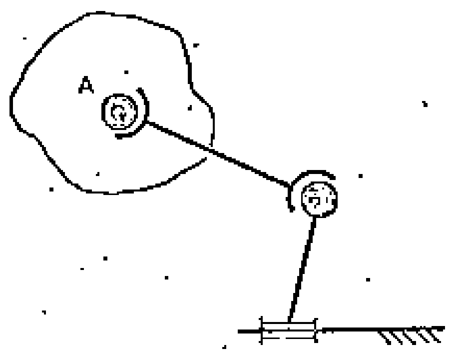


Fig. 4 Spatial dwell linkage. The rotating-crank dwells if the center of spheric joint A moves on a sphere whose center is at the other spheric joint.

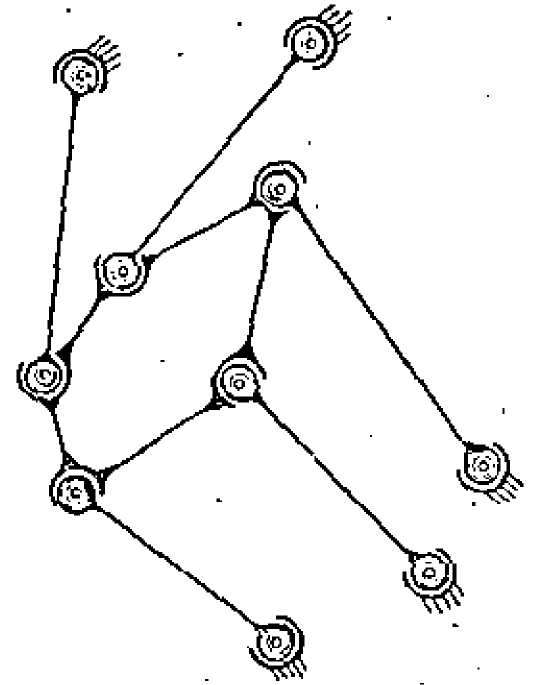


Fig. 5 Seven bar with 10 spheric joints. This linkage has five binary cranks, each of which is free to rotate about its own center line.

When *kinematic pairs other than those included* in our treatment are present, the synthesis will require some arbitrary choices (or an extension of the foregoing development to these pairs). These choices reduce the number of precision conditions. However, they do have the advantage of allowing an element of control over the final solution.

Pseudoplanar and pseudospherical linkages are names we coin for linkages which have several of their design positions given by, respectively, parallel or intersecting pure-rotation screws. Such linkages are spatial but may be designed to have several coplanar or cospherical precision positions. Their synthesis is accomplished in the usual way, except that we now make use of the degenerate cases discussed in the sections on special motions.

New Kinematic Pairs—The Revolute-Revolute Crank. In this section, we illustrate how the results of [1] may be adapted to the study of new pairs. The design of a link with two turning pairs (a so-called revolute-revolute crank) requires the determination of a moving rotation axis (in Σ) and a fixed rotation axis (in Σ^1) for which the relative distance and angle (between these axes) remains constant. Such a link may be synthesized if we can find two points in Σ which lie on circles about the same axis, since all the points on the line connecting these two points will also lie on circles about this same axis.

From (9), we have the condition that a point has two positions on a plane normal to a given direction. This may be written:

$$e_j x_1 + f_j y_1 + g_j z_1 + h_j = 0$$

where

$$e_j = a_{x_j}(l/n) + a_{y_j}(m/n) + a_{z_j}$$

$$f_j = b_{x_j}(l/n) + b_{y_j}(m/n) + b_{z_j}$$

$$g_j = c_{x_j}(l/n) + c_{y_j}(m/n) + c_{z_j}$$

$$h_j = d_{x_j}(l/n) + d_{y_j}(m/n) + d_{z_j}$$

Here (l, m, n) are the direction cosines of the fixed axis, and (x_1, y_1, z_1) is a point on the moving axis. The $a_{x_j}, b_{x_j}, c_{x_j}, d_{x_j}, \ldots, d_{z_j}$ are the coefficients of the matrix which defines the motion (see equation (1)).

Further, it has been shown in [1] that the locus of all points with three positions on a circle, of a specified inclination, is a line. Hence, if we regard (l, m, n) as known, the previous equation written twice $(j = 2, 3)$ yields the equation of the moving axis:

$$x_1 = (s/v)y_1 + r/v \qquad (a)$$

$$z_1 = (t/v)y_1 + u/v$$

where

$$s = f_3 g_2 - f_2 g_3$$

$$r = g_2 h_3 - g_3 h_2$$

$$v = e_3 g_2 - e_2 g_3$$

$$t = e_3 f_2 - e_2 f_3$$

$$u = e_2 h_3 - e_3 h_2$$

Substituting these results for x_1 and z_1 into (8) [which is the condition that point (x_1, y_1, z_1) remain at a constant distance from a point (A_x, A_y, A_z) in Σ^1] yields:

$$A_j A_x + B_j A_y + C_j A_z + D_j = 0 \qquad (b)$$

where (A_x, A_y, A_z) is any point on the fixed axis, and

$$A_j = (a_{x_j}s + b_{x_j}v + c_{x_j}t)y_1 + (a_{x_j}r + c_{x_j}u + d_{x_j}v),$$

$$B_j = (a_{y_j}s + b_{y_j}v + c_{y_j}t)y_1 + (a_{y_j}r + c_{y_j}u + d_{y_j}v),$$

$$C_j = (a_{z_j}s + b_{z_j}v + c_{z_j}t)y_1 + (a_{z_j}r + c_{z_j}u + d_{z_j}v),$$

$$D_j = ((r_j^2)_x s + (r_j^2)_y v + (r_j^2)_z t)y_1 + ((r_j^2)_x r + (r_j^2)_z u + (r_j^2)_d v)$$

In D_j, the coefficients of the x, y, z and constant terms of $(r_1^2 - r_j^2)/2$ have been written as $(r_j^2)_x$, $(r_j^2)_y$, $(r_j^2)_z$, $(r_j^2)_d$, respectively.

If (l, m, n) and y_1 were known, the foregoing equation taken twice $(j = 2, 3)$ would yield the fixed axis. Now, clearly, the choice of y_1 should in no way affect the location of the fixed axis. Similarly, we are at liberty to arbitrarily choose one of the coordinates (A_x, A_y, A_z) of the fixed axis. For simplicity, we take $A_z = 0$ and $y_1 = 0$ in (b):

$$A_2^0 A_x + B_2^0 A_y + D_2^0 = 0$$

$$A_3^0 A_x + B_3^0 A_y + D_3^0 = 0$$

(The superscript denotes the value of y_1.)

Similarly, if we take $A_z = 0$ and $y_1 = 1$:

$$A_2^1 A_x + B_2^1 A_y + D_2^1 = 0$$

$$A_3^1 A_x + B_3^1 A_y + D_3^1 = 0$$

Now, these two sets of equations will yield the same A_x and A_y if the following two compatibility equations are satisfied:

$$\begin{vmatrix} A_2^0 & B_2^0 & D_2^0 \\ A_3^0 & B_3^0 & D_3^0 \\ A_2^1 & B_2^1 & D_2^1 \end{vmatrix} = 0 \qquad (c)$$

$$\begin{vmatrix} A_2^0 & B_2^0 & D_2^0 \\ A_3^0 & B_3^0 & D_3^0 \\ A_3^1 & B_3^1 & D_3^1 \end{vmatrix} = 0$$

When expanded, these determinants give two sixth-degree polynomials in (l/n) and (m/n). For each valid set of roots determined from (c), we have a unique revolute-revolute crank. The moving axis is computed from (a), and the fixed axis from (b) with $j = 2, 3$ and y_1 arbitrary. Table 4 lists the results of one such computation for the motion defined in Table 1 of [1].

Table 4 Axes of a revolute-revolute crank corresponding to positions 1, 2, 3 of Table 1 in [1]

MOVING AXIS

$x_1 = -0.5843y_1 - 3.3546$

$z_1 = -0.2892y_1 - 2.9623$

FIXED AXIS

$x = 0.3921y - 1.0092$

$z = 4.5116y - 2.6542$

Conclusions

The similarity transformation leads to a simplification of the screw-transformation coefficients. These, in turn, considerably simplify E^4, which, in effect, degenerates into a second-order surface. It is important to note that the similarity transformation provides a unified approach to function generation problems for spatial, spherical, and planar mechanisms.

The intersecting pure-rotation screws bring spherical motion into the general theory. Obviously, the loci of interest become conics, or curves and points on conics. The study of parallel pure-rotation screws shows how the well-known planar theory is derived from more general considerations. In this case, the surfaces of interest become cylinders—i.e., cones with apexes at infinity. The inclusion of "skew-screws" with planar and spherical motion allows the design of mechanisms with pseudoplanar and pseudospherical motion. Such linkages are spatial, but most of their design positions are, respectively, planar and spherical.

This paper and [1] deal almost exclusively with special points. The study of special lines may be pursued by imposing simultaneous conditions on two points, as has been done in the revolute-revolute crank synthesis described in the preceding section. Alternatively, it is possible to study special lines directly. This latter approach is utilized in a subsequent study [7] to effect the synthesis of cylindric-cylindric cranks.

These results and those presented in [1] and [7] are suited to various types of syntheses, and may be regarded as generalizations of the classical finite-position Burmester theory. The results given in these papers are applicable to function-generation, path-generation, and coupler-plane-motion syntheses of spatial, spherical, and planar linkages.

Acknowledgments

The author wishes to express his appreciation to Pietaw Chen for doing the bulk of the computer programming, and to the National Science Foundation whose Grant NSF-GP 2005 made this work possible.

References

1 Roth, B., "The Kinematics of Motion Through Finitely Separated Positions," ASME Paper No. 67—APM-2.

2 Dobrovolskii, V. V., "Burmester's Points in Spherical Motion" (in Russian), *Prikladnaya Matematika i Mekhanika*, Vol. 9, 1945, pp. 159–401.

3 Wilson, J. T., III, "Analytical Kinematic Synthesis by Finite Displacements," *Journal of Engineering for Industry*, TRANS. ASME, Series B, Vol. 87, No. 2, May 1965, pp. 161–169.

4 Altmann, F. G., "Zur Masssynthese der Raumgetriebe," *Machinenbautechnik*, Vol. 6, 1957, pp. 93–100.

5 Denavit, J., and Hartenberg, R. S., "Approximate Synthesis of Spatial Linkages," JOURNAL OF APPLIED MECHANICS, Vol. 27, No. 1, TRANS. ASME, Vol. 82, Series E, Mar. 1960, pp. 201–206.

6 Levitskii, N. I., and Shakvasian, K. K., "Synthesis of Four-Element Spatial Mechanisms With Lower Pairs," *International Journal of Mechanical Sciences*, Vol. 2, pp. 76–92. (This is a translation by F. Freudenstein of a paper which was originally published in 1954.)

7 Roth, B., "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body," *Journal of Engineering for Industry*, TRANS. ASME, Series B, Vol. 89, No. 1, Feb. 1967, pp. 102–110.

APPENDIX 1

Proof That Under a Similarity Transformation $E^4 = u_1 \times F^2 \times G^2$. We write (7):

$$E^4 = \frac{1}{2}\begin{vmatrix} (r_2^2 - r_1^2) & (y_2 - y_1) & (z_2 - z_1) & (z_3 - z_1) \\ (r_3^2 - r_1^2) & (y_3 - y_1) & (z_3 - z_1) & (z_4 - z_1) \\ (r_4^2 - r_1^2) & (y_4 - y_1) & (z_4 - z_1) & (z_1 - z_1) \\ (r_5^2 - r_1^2) & (y_5 - y_1) & (z_5 - z_1) & (z_5 - z_1) \end{vmatrix} = 0$$

Expanding this determinant according to the Laplace development yields:

$$E^4 = \sum_{j=2}^{4}\sum_{k=3}^{5}(-1)^{(j+k+l+m)}\left[\left(\frac{r_j^2 - r_1^2}{2}\right)(y_k - y_1) - \left(\frac{r_k^2 - r_1^2}{2}\right)(y_j - y_1)\right]\{(z_l - z_1)(z_m - z_1) - (z_m - z_1)(z_l - z_1)\} = 0$$

Here l is 2, 3, or 4 and m is 2, 3, or 4 as determined from the restrictions that $j \neq k \neq l \neq m$, $j < k$ and $l < m$.

Substituting motion parameters, one finds that the first square bracket may be written:

$$u_1[(u_2x_1 - u_1y_1)^2 + (z_1 - p)^2][\sin\theta_j(1 - \cos\theta_k) - \sin\theta_k(1 - \cos\theta_j)]$$

and hence $u_1[(u_2x_1 - u_1y_1)^2 + (z_1 - p)^2]$, which we write as u_1F^2, may be brought outside of the summation signs. Denoting the terms in the summation as G^2 (since they are quadratic), we have $E^4 = u_1F^2G^2$.

APPENDIX 2

The Curve I^2 Under Parallel Screws. We write equation (12):

$$\begin{vmatrix} (x_2 - x_1) & (y_2 - y_1) \\ (x_3 - x_1) & (y_3 - y_1) \end{vmatrix} = 0 \qquad (1)$$

$$\begin{vmatrix} (x_2 - x_1) & (z_2 - z_1) \\ (x_3 - x_1) & (z_3 - z_1) \end{vmatrix} = 0 \qquad (2)$$

For simplicity, we take the screws, $\$_{12}$ and $\$_{13}$, parallel to the z-axis. Expanding (1) and substituting the motion parameters yields:

$$[(x_1 - a_2)(x_1 - a_3) + (y_1 - b_2)(y_1 - b_3)]\left[\sin\left(\frac{\theta_3 - \theta_2}{2}\right)\right] + [x_1(b_2 - b_3) + y_1(a_3 - a_2) + (a_2b_3 - a_3b_2)] \times \left[\cos\left(\frac{\theta_3 - \theta_2}{2}\right)\right] = 0 \quad (3)$$

This is obviously the equation of right, circular cylinder. We note that the translations (d_2 and d_3) do not enter into this result. Further, since $(a_2, b_2, 0)$ and $(a_3, b_3, 0)$ are, respectively, the coordinates of the ft of the perpendiculars from the origin to $\$_{12}$ and $\$_{13}$, this cylinder contains $\$_{12}$ and $\$_{13}$ (and also $\$_{23}'$). θ_2 and θ_3 are the rotations corresponding to $\$_{12}$ and $\$_{13}$, respectively.

Expanding (2) yields:

$$(x_1 - a_3)(1 - \cos\theta_3)d_2 - (x_1 - a_2)(1 - \cos\theta_2)d_3 + (y_1 - b_3)(\sin\theta_3)d_2 - (y_1 - b_2)(\sin\theta_2)d_3 = 0 \quad (4)$$

which is a plane parallel to the z-axis.

The plane (4) cuts the cylinder (3) in two lines—one of which $(x_2 - x_1 = 0, x_3 - x_1 = 0)$ is spurious. *Hence, for parallel screws, I^2 becomes a line.* If $d_2 = d_3 = 0$, the motion is pure rotation and (4) vanishes. This leaves (3); any normal section yields the circle of the classical planar theory. If the screws are collinear, (3) becomes a "line" cylinder, and I^2 coincides with the screw axis.

Printed in U. S. A.

B. ROTH
Assistant Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif.
Mem. ASME

On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body

For a rigid body in two, three, four, and five finitely separated positions, the loci of the screw axes and other special lines are derived. It is shown that the classical planar theory is a special case of a more general theory which includes planar, spherical, and spatial displacements.

Introduction

In this paper, we study the kinematics of a rigid body in a series of finitely separated positions, and seek to determine spatial equivalents for the basic quantities of the classical planar theory. Of special interest are those spatial geometries which are analogous to the pole triangles, pole curves, center and circle-point curves, and Burmester-point pairs.

These studies are in the spirit of two of this author's previous works [1, 2].[1] However, here we are primarily concerned with lines in the body, while [1] and [2] deal with points.

Several other works deal with extensions of the planar theory but they are concerned with generalized planar concepts [3, 4] or with spherical motion [5, 6, 7]. Although Keler [8] has used line geometry and dual numbers to consider general spatial motion for two and three positions, and Schoenflies [9] and more recently others [10, 1] have considered points which lie on special loci, this author is unaware of any previous work in which line congruences are taken as the spatial analogs of the pole, center-point, and circle-point curves.

Screw Axis Geometry

Nomenclature

In this paper we are concerned with the relative position of two rigid bodies. It is convenient to refer to one as a moving body and to the other as a fixed body. The moving system is denoted by Σ and the fixed system by Σ'. We number the various positions of the moving system and use subscripts to indicate which position of Σ we are referring to.

It is well known that, regardless of how a motion actually occurs, the displacement may always be regarded as a rotation about a given axis and a translation along the same axis. Such a motion is called a screw displacement. As shown in Fig. 1, we denote the screw which takes Σ from the ith to the jth position as $\$_{ij}$. The corresponding rotation is taken as θ_{ij} and the translation is d_{ij}. Further, the unit vector $\mathbf{U}_{ij}$ parallel to this screw axis has components $(u_{1ij}, u_{2ij}, u_{3ij})$, and the screw passes through the point (a_{ij}, b_{ij}, c_{ij}); these six components are measured along coordinate axes fixed in Σ'.

θ_{ij} and d_{ij} are called the screw parameters, and their ratio d_{ij}/θ_{ij} is the pitch of the screw. The notion of pitch implies that the rotation and translation occur simultaneously. However, it is equally important to remember that these motions may be regarded as occurring separately, and that their effects may be superimposed.

For plane motion $d_{ij} = 0$, and it is customary to take $u_{3ij} = 1$, $c_{ij} = 0$.

Two Positions

We now describe an alternative characterization of a finite displacement, which, although less well known than the screw, will prove to be equally important in what follows.

It is known [11] that any finite displacement may be considered as the result of successive reflections about two fixed lines.[2] The two lines must be normal to and intersect the screw axis. In addition, the distance and angle between them must be, respectively, $d_{ij}/2$ and $\theta_{ij}/2$ (measured from the first line to the second in the positive sense of $\$_{ij}$). Otherwise, the two lines are arbitrary.

The foregoing is illustrated and proved, with the aid of Fig. 2, as follows. L_i is any line normal to $\$_{ij}$, and L_j is the line obtained by operating on L_i with a screw coincident to $\$_{ij}$ but with

[1] Numbers in brackets designate References at end of paper.

[2] A reflection about a line is equivalent to a 180 deg rotation about the line.

Contributed by the Machine Design Division for presentation at the Mechanisms Conference, Lafayette, Ind., October 10–12, 1966, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received at ASME Headquarters, June 2, 1966. Paper No. 66—Mech-7.

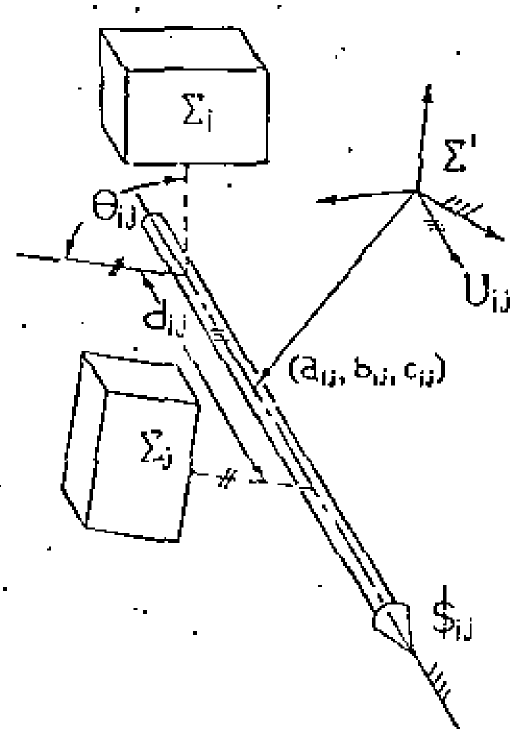


Fig. 1 Screw displacement $\$_{ij}$ and the associated nomenclature

Discussion on this paper will be accepted at ASME Headquarters until November 14, 1966

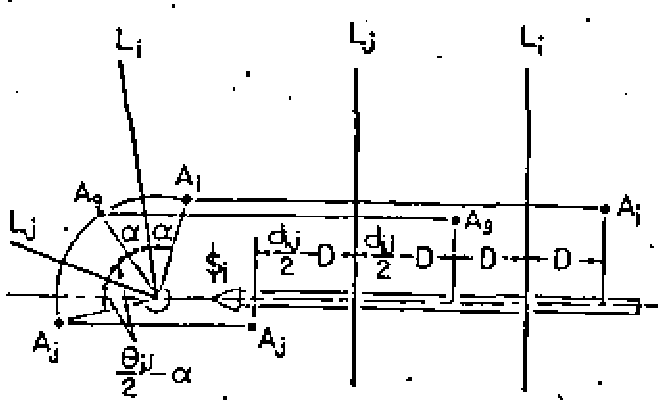


Fig. 2 Two reflections are equivalent to a screw displacement. Point A_1 is reflected about L_i to position A_g, and A_g is reflected about L_j to position A_j. The figure shows side and end views of the screw $\$_{ij}$.

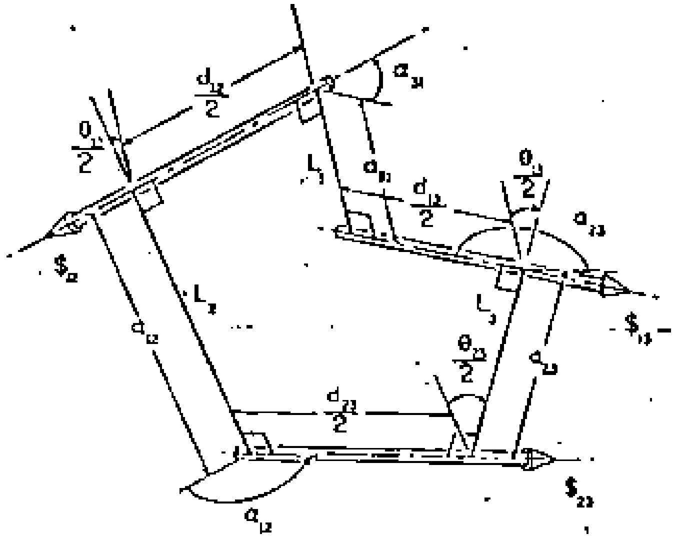


Fig. 3 Screw triangle geometry showing lengths of sides $d_{12}/2$, $d_{23}/2$, $d_{31}/2$; angles between sides α_{12}, α_{23}, α_{31}; lengths of normals between sides a_{12}, a_{23}, a_{31}; angles between normals $\theta_{12}/2$, $\theta_{23}/2$, $\theta_{31}/2$

parameters $\theta_{ij}/2$, $d_{ij}/2$. Take A_i as a generic point in Σ_i. Then, reflecting A_i about L_i brings it to A_g and reflecting A_g about L_j brings the point to position A_j. From the figure, which shows an end and side view of the screw, it follows that the relative position of A_i and L_i, which is given by D and α, does not affect the displacement. Explicitly, the displacement along the screw axis is given by

$$D + D + 2\left(\frac{d_{ij}}{2} - D\right) = d_{ij},$$

and the angular displacement, of the radial line to A_i, about the screw axis is

$$\alpha + \alpha + 2\left(\frac{\theta_{ij}}{2} - \alpha\right) = \theta_{ij}.$$

Hence, the two reflections effect a displacement identical to the one defined by $\$_{ij}$, and therefore, we conclude that the position of L_i (along $\$_{ij}$) is indeed arbitrary.

Three Positions

For three finitely separated positions, Σ_1, Σ_2, Σ_3, there are three screw axes, $\$_{12}$, $\$_{23}$, $\$_{31}$, but only two are independent. The geometry relating these screws, as shown in Fig. 3, may be easily derived by replacing each screw by two reflections. Given $\$_{12}$ and $\$_{23}$ we call their common perpendicular L_2, and define L_1 as the line L_2 screwed about $\$_{12}$ by an amount $-d_{12}/2$, $-\theta_{12}/2$. L_3 is the line normal to $\$_{23}$ which is $d_{23}/2$, $\theta_{23}/2$ from L_2. The third screw axis $\$_{31}$ is then given as the line normal to L_3 and L_1. The validity of this can be seen by considering an arbitrary point A_1. We reflect about L_1 and then L_2, the resulting position A_2, as we have seen above, is the same as that after screw $\$_{12}$. Reflecting A_2 about L_2 and then L_3 results in the point being in position A_3. Thus A_1 has moved to A_3 by undergoing four reflections, but two successive reflections are about the same line, L_2, and hence may be ignored. Therefore, the same results may be obtained from a reflection about L_1 followed by a reflection about L_3. Hence the normal to L_1 and L_3 defines the screw axis for $\$_{13}$ and the parameters $\theta_{13}/2$, $d_{13}/2$ are given, respectively, by the angle and the distance between L_1 and L_3. These quantities are always measured from L_1 to L_3.

Since any three lines taken two at a time generally have three unique normals, it is apparent that any set of three axes defines a unique set of lines L_1, L_2, L_3 and therefore a set of parameters d_{12}, d_{23}, d_{31}, θ_{12}, θ_{23}, θ_{31}. Hence *there is a set of screw motions determined by any configuration of three axes.*

The geometrical configuration formed by three lines and their common normals has been called by Yang [12] a "spatial triangle." The configuration of three screw axes and their normals will be called a "screw triangle" in analogy to the pole triangle of planar motion. The pole triangle is a special case of the "screw triangle." By investigating the geometry of the screw triangle shown in Fig. 3, we obtain analytical expressions for the screw $\$_{31}$[3] in terms of $\$_{12}$ and $\$_{23}$.

Taking projections along the screw axes yields:

$$\sin\left(\frac{\theta_{31}}{2}\right) = \frac{\sin \alpha_{12}}{\sin \alpha_{31}} \sin\left(\frac{\theta_{12}}{2}\right)$$

$$\cos\left(\frac{\theta_{31}}{2}\right) = \cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right) - \cos\alpha_{23}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)$$

$$\sin\alpha_{31} = -\frac{1}{\cos\left(\frac{\theta_{12}}{2}\right)}\left[\sin\alpha_{12}\cos\alpha_{23} + \cos\alpha_{12}\sin\alpha_{23}\cos\left(\frac{\theta_{23}}{2}\right)\right]$$

$$\cos\alpha_{31} = -\frac{1}{\sin\left(\frac{\theta_{31}}{2}\right)}\left[\sin\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{23}}{2}\right) + \cos\alpha_{23}\cos\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right]$$

$$d_{31} = \frac{2}{\sin\left(\frac{\theta_{31}}{2}\right)}\left[\frac{d_{23}}{2}\left(\sin\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{12}}{2}\right) + \cos\alpha_{23}\cos\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right) + \frac{d_{12}}{2}\left(\cos\left(\frac{\theta_{23}}{2}\right)\sin\left(\frac{\theta_{12}}{2}\right) + \cos\alpha_{23}\sin\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{12}}{2}\right)\right) - a_{23}\sin\alpha_{23}\sin\left(\frac{\theta_{12}}{2}\right)\sin\left(\frac{\theta_{23}}{2}\right)\right]$$

$$a_{31} = \frac{1}{\sin^2\left(\frac{\theta_{31}}{2}\right)}\left[a_{23}\left(\cos\left(\frac{\theta_{12}}{2}\right)\cos\left(\frac{\theta_{31}}{2}\right) - \cos\left(\frac{\theta_{23}}{2}\right)\right) + \sin\left(\frac{\alpha_{23}}{2}\right)\left(d_{12}\sin\left(\frac{\theta_{23}}{2}\right)\cos\left(\frac{\theta_{31}}{2}\right) - d_{23}\sin\left(\frac{\theta_{12}}{2}\right)\right)\right]$$

α_{12} and a_{12} are, respectively, the angle and distance between $\$_{12}$ and $\$_{23}$ taken in the sense of screwing $\$_{12}$ about L_2 into $\$_{23}$. Similarly, for (α_{23}, a_{23}) and (α_{31}, a_{31}) we measure from $\$_{23}$ to $\$_{31}$ and from $\$_{31}$ to $\$_{12}$, respectively.

---

[3] $\$_{13}$ is a screw along the same axis and with the same magnitude as $\$_{31}$; however, the sense (of rotation and translation) is opposite in both screws.

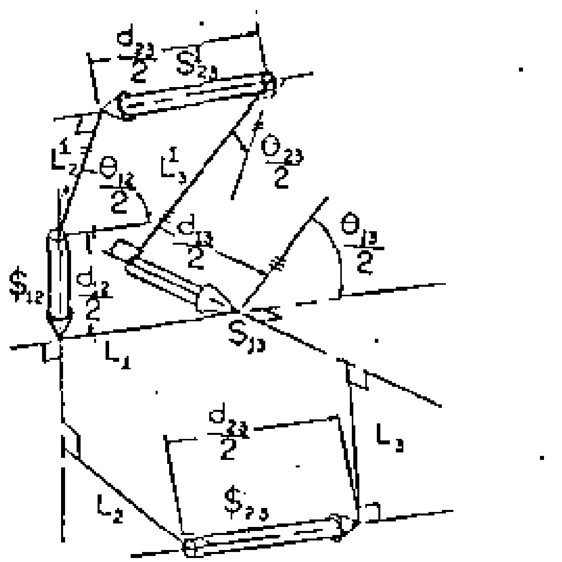


Fig. 4 Image screw triangle $\$_{12}$, $\$_{23}$, ${}^1\$_{13}$ and screw triangle $\$_{12}$, $\$_{23}$, $\$_{13}$

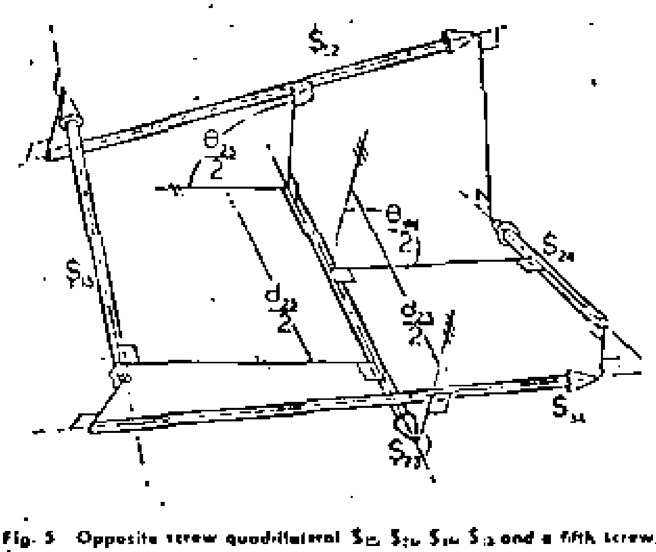


Fig. 5 Opposite screw quadrilateral $\$_{12}$, $\$_{23}$, $\$_{34}$, $\$_{14}$ and a fifth screw, $\$_{24}$, which subtends equal dual angles at opposite sides of the quadrilateral

We have chosen to work with $\$_{31}$ instead of $\$_{13}$ to preserve the symmetry, hence all the above expressions are valid under cyclic permutation of indices. (These equations depend exclusively on the geometry of the screw triangle and may be viewed as special cases of the equations obtained from the so-called spatial-triangle sine and cosine laws as given by Yang [12].) A host of similar relationships may be obtained.

For planar motion $\alpha_{ij} = 0$ or 180 deg, and the second equation yields the well-known summation rule $\theta_{13} = \theta_{12} + \theta_{23}$.

The Ground (or Cardinal[4]) Point

Just as in planar work, there is a unique point A_g which can be reflected about the "sides" of the screw triangle to give the three positions of any point A. The "sides" of the screw triangle are the lines L_1, L_2, L_3 (i.e., the common normals to the screw axes). If we call the three positions of point A, A_1, A_2, A_3, respectively, then A_g may be found by reflecting A_1 about L_1, or A_2 about L_2, or A_3 about L_3. Conversely, by reflecting A_g about L_1 we obtain A_1, etc., for A_2 and A_3. The proof of this follows directly from the equivalence of two reflections to a screw as outlined above.

Similarly, it follows that to every geometric entity there corresponds an equivalent cardinal member. For example, given a line l_1, a reflection about L_1 yields l_g which may be reflected about L_2 and L_3 to yield, respectively, l_2 and l_3. The same is of course true for planes.

Image Screw Triangles

We now consider the screw axes as fixed in the moving system Σ. In position one, Σ_1, the three axes are $\$_{12}{}^1$, $\$_{13}{}^1$, $\$_{23}{}^1$. $\$_{12}{}^1$ and $\$_{13}{}^1$ are of course identical to $\$_{12}$ and $\$_{13}$ in the fixed system Σ'. The axis $\$_{23}{}^1$ is obtained by screwing that line in Σ_2 which is coincident with $\$_{23}$ (in Σ') about $\$_{12}$ or screwing that line in Σ_3 which is coincident with $\$_{23}$ about $\$_{13}$.[5] Alternatively, $\$_{23}{}^1$ may be obtained by reflecting axis $\$_{23}$ about L_2 or L_3 (depending upon if we wish to regard the line in Σ coincident with the line $\$_{23}$ in Σ' as being in Σ_2 or Σ_3),[5] and then reflecting it about L_1. Since L_2 and L_3 both intersect (and are normal to) $\$_{23}$, it follows that the first reflection only reverses the "head and tail" of the screw and that the line of action of $\$_{23}{}^1$ depends exclusively on the reflection about L_1. See Fig. 4.

Similarly, $\$_{13}{}^2$ is obtained by reversing the "head and tail" of $\$_{13}$ and then reflecting it about L_2. $\$_{12}{}^3$ is obtained by reflecting $\$_{12}$ about L_3 after having reversed its sense. Fig. 4 shows the "image screw triangle" $\$_{12}$, $\$_{23}{}^1$, $\$_{13}$ with sides L_1, $L_2{}^1$, $L_3{}^1$ and some pertinent dimensions. The well-known planar case, which is a special case of the above, follows if we take all the axes as parallel, pure rotation screws.

Four Positions

There are now six screws associated with the motion: $\$_{12}$, $\$_{23}$, $\$_{34}$, $\$_{13}$, $\$_{24}$, $\$_{14}$. We assume knowledge of any four which, in analogy to the planar opposite-pole quadrilateral, form an "opposite screw quadrilateral"[6] and inquire as to the locus of the other two screws.

Referring to Fig. 5 and recalling the relationship between the normals and the screw parameters of a screw triangle, we conclude that given, for example, $\$_{12}$, $\$_{23}$, $\$_{34}$, $\$_{14}$, the locus of $\$_{24}$ must be such that:

(*a*) The angle which the perpendicular between $\$_{24}$ and $\$_{12}$ makes with the perpendicular between $\$_{12}$ and $\$_{23}$ must equal the angle which the perpendicular between $\$_{24}$ and $\$_{34}$ makes with the perpendicular between $\$_{23}$ and $\$_{34}$ (since in each case the angle is $\theta_{24}/2$).

(*b*) The distance between the points on $\$_{23}$ where it is met by its common perpendiculars with $\$_{12}$ and $\$_{24}$ is equal to the distance between the points on $\$_{23}$ where the normals from $\$_{34}$ and $\$_{24}$ fall (since in each case the distance is $d_{24}/2$).

The distance and angle between two lines may be combined to form the so-called dual angle. (For example, in the foregoing, dual angle $\hat{\theta}_{24}$ would be given by $\hat{\theta}_{24} = \theta_{24} + \epsilon d_{24}$, where $\epsilon^2 \equiv 0$.) We define the dual angle which any two lines subtend at a third line as the dual angle between the normals from the two lines to the third. With this definition (*a*) and (*b*) may be combined into the following: The dual angle subtended by $\$_{12}$ and $\$_{23}$ at $\$_{24}$ equals the dual angle subtended by $\$_{34}$ and $\$_{14}$ at $\$_{24}$.

If we call two adjacent screws a "side" of the quadrilateral, and call two sides which do not have a common screw "opposite sides," then (*a*) and (*b*) require that one set of opposite sides subtend equal dual angles at $\$_{24}$. From the figure it follows that if one pair of opposite sides subtends equal dual angles, the other pair must do likewise. Generalizing the foregoing, and taking into account the possible ±180 deg variation in the angle, we have: *The position of all screw axes must be such that they form*

[4] Like other terms in kinematics, the various names are due to different translations of the original German words. Here the problem is whether "grund" in "grundpunkt" means ground or basic (i.e., cardinal).

[5] If we regard the screw axis as a sliding line vector, then there is no need to distinguish between the line being in Σ_2 or Σ_3. However, if we consider the axis as a line fixed in the body, the distinction between Σ_2 and Σ_3 is important since corresponding points on the axis are dis-part.

[6] An "opposite screw quadrilateral" is a spatial polygon bounded by four screw axes and the four normals between adjacent axes. (Axes are adjacent if they have one subscript in common.)

equal (or supplementary[4]) dual angles with opposite sides of an opposite screw quadrilateral.[5]

In the foregoing we have been working with the screw axes in Σ^1. If we invert the motion so that, say, Σ_1 is fixed, it becomes obvious the positions of the screw axes in the moving plane are also governed by the above. For example, the locus of $\$_{23}^1$ is such that it must "see" opposite sides of the opposite screw quadrilateral $\$_{12}$, $\$_{24}^1$, $\$_{34}^1$, $\$_{13}$ at equal (or supplementary) dual angles.

Screw Congruences

Since, in the aforementioned example, $\$_{23}$ must satisfy two conditions, the freedom in its location is reduced from four to two, and the locus of $\$_{23}$ is a two-dimensional assemblage of lines called a congruence. Similarly, $\$_{14}$ will have to fulfill conditions analogous to (*a*) and (*b*), and, by symmetry, so will all the other screws.

Equations governing these geometric loci are derived in Appendix 1 and also in the section on the cylindric-cylindric crank. These results may be summarized as follows:

(i) The six screw axes associated with four positions of a body are parallel to six generators of a cubic cone. The cone is completely defined by the directions of any four "opposite" screws, and the remaining two screws may be parallel to any two elements of this cone. We refer to this cone as a "screw cone."

(ii) Each generator of the screw cone defines a unique direction, *and to each such direction there corresponds a singly infinite set of possible screw axis positions.* (Such a one-dimensional assemblage of lines is known as a line series.) The members of a given line series are all subtended by the same (or supplementary) angles at the opposite sides of an opposite screw quadrilateral. However, the distance subtended (and hence the dual angle) is different for each member of the series.

All the members of a single line series are coplanar and parallel.

The first result, (i), is completely independent of the translation along the screws and the location of the screws. Therefore, in questions dealing exclusively with inclination—as opposed to location—one may consider all the screws as intersecting, pure rotation screws. This leads immediately to the conclusion that the screw cone for spherical motion is identical to the cone for corresponding general spatial motion. Further, if the apex is taken at infinity the cone degenerates into a (circular) cubic cylinder, and any normal section yields a planar pole curve.

The screw cone associated with the motion defined in Table 1 is shown in Fig. 6. We note that the cone is completely defined by

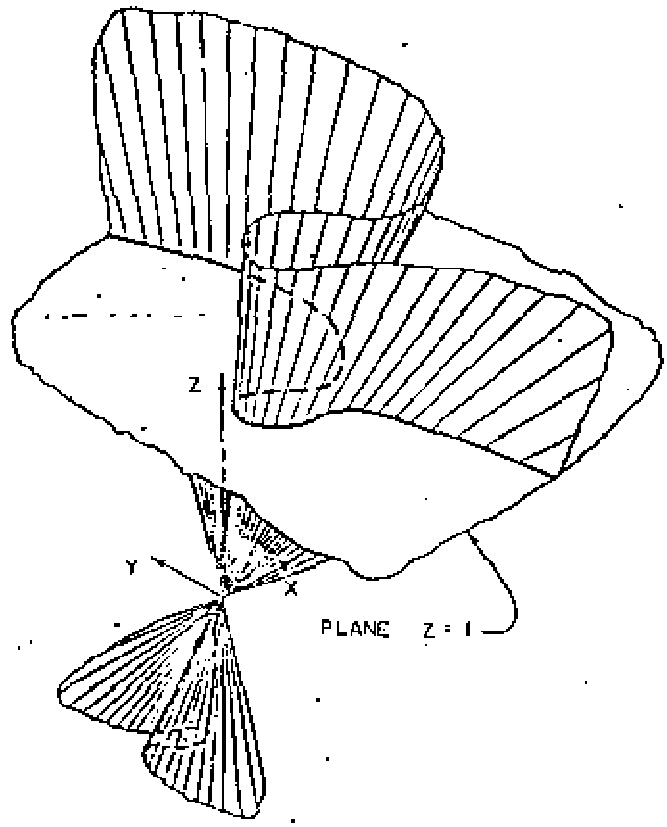


Fig. 6 Cubic cone, and planar cubic falling in plane $Z = 1$. This cone contains the directions of the "moving" screw axes and the moving cylindric-cylindric crank axes associated with the first four positions of the motion given in Table 1. The X, Y, Z coordinates correspond, respectively, to the cosines L_1, M_1, N_1.

the planar cubic which is its curve of intersection with any plane not passing through the origin. The curve cut by the plane $Z = 1$ is shown in the figure. Alternatively, one could—as is customary in spherical-motion studies—intersect the cone with a unit-sphere; however, this seems much less convenient.

Using the notion that the screw axes may be considered as intersecting, pure rotation screws, one can rephrase condition (*a*):

(*a'*) The dihedral angle between the plane $\$_{12} \times \$_{13}$ and the plane $\$_{13} \times \$_{23}$ is equal (or supplementary) to the dihedral angle between planes $\$_{24} \times \$_{23}$ and $\$_{34} \times \$_{24}$.

Conditions (*a*) and (*a'*) are, of course, equally valid for spatial, spherical, or planar motion, while (*b*) is trivial if there is no translation.

By inversion, one immediately obtains analogous results for the loci of the screw axes fixed in the moving body.

[4] Two dual angles are supplementary if their primary parts differ by 180 deg.

[5] The rules governing the sense of these angles are exactly the same as in the planar case [13]. The sense of a dual angle is defined as the sense of the right-handed screw required to bring the line *from which* the angle is being measured into coincidence with the line *to which* the angle is being measured.

Table 1 The screws for five arbitrarily defined positions are listed. Corresponding to the first four, we give the equation of the screw cone taken in the moving body in Position 1, and the plane containing all the moving axes parallel to one of the generators of this screw cone. Fig. 6 shows this same screw cone.

Position	Axis cosines			Axis location			Screw parameters	
i to j	l_{ij}	m_{ij}	n_{ij}	a_{ij}	b_{ij}	c_{ij}	θ_{ij} deg	d_{ij}
1 to 2	0.3510	0.1065	0.9280	0.9414	0.7214	−0.4194	131.2	1.384
1 to 3	0.1457	−0.4690	0.9886	−0.6103	0.8471	0.1230	70.6	1.809
1 to 4	0.4267	−0.2464	0.8702	0.7327	0.6889	−0.1643	87.9	−1.117
1 to 5	0.4027	−0.0251	−0.9150	5.903	0.1040	2.595	25.2	2.559

Equation of Screw Cone (corresponding to Positions 1, 2, 3, and 4)

$$-0.3461L_1^3 - 0.0519M_1^3 + 0.0213N_1^3 + 0.0518L_1^2M_1 + 0.5197L_1^2N_1 - 0.3805L_1M_1^2 + 0.3310M_1^2N_1 - 0.2168L_1N_1^2 - 0.0175M_1N_1^2 + 0.0071L_1M_1N_1 = 0$$

All axes parallel to the generator $L_1 = -0.1489$, $M_1 = +0.0502$, $N_1 = 0.9876$ lie in the plane: $0.0285A_1 - 0.0889B_1 - 0.0889C_1 + 0.0848 = 0$

The Cylindric-Cylindric Crank

Introduction

It is well known that, in the planar case, the pole curve is also the locus of the fixed centers of all "four-point" circles. In this context, the pole curve is called the center-point curve. Similarly, the pole curve fixed in the moving body is also the locus of points whose four positions fall on a circle, i.e., the so-called circle-point curve. *These dualisms are special cases of more general ones associated with spatial motion.*

In space, the entity analogous to a circle point is a line in the moving body whose four positions lie at the same distance and same angle from a line in the fixed body. This fixed line is analogous to the corresponding center point. Each such set of fixed and moving lines define the axes of a crank with two cylindrical joints. Fig. 7 shows a cylindric-cylindric crank in positions associated with four finitely separated positions of Σ.

In the context of a general theory, the revolute-revolute crank defined by a center and circle point should, of course, be viewed as a special case of the cylindric-cylindric crank. By adopting this viewpoint we may state the following general results:

The loci of screw axes defined by the congruence of* (i) *and* (ii) *are identical to the loci of the fixed axes of all four-position cylindric-cylindric cranks. Correspondingly, the loci of the moving axes for all four-position cylindric-cylindric cranks coincide with the loci of screw axes in the moving system.

If, and only if, the screws are intersecting pure rotation screws,[7] there will be no translation in the cylindrical joints, and the cylindric-cylindric cranks become, in effect, revolute-revolute cranks.

Nomenclature

The following quantities are introduced to describe a cylindric-cylindric crank (see Fig. 8).

$\mathbf{L}_j(L_j, M_j, N_j)$, a unit vector parallel to the jth position of the moving axis; $\mathbf{A}_j(A_j, B_j, C_j)$, the normal vector to the moving axis; and a vector (R_j, S_j, T_j) defined as $\mathbf{A}_j \times \mathbf{L}_j$. Similarly, for the fixed axis we have the unit vector along the axis (λ, μ, ν); the normal vector from the origin (α, β, γ); and their cross-product (ρ, σ, τ). All of these quantities are measured in the fixed coordinate system.

Using Φ_j and D_j to denote, respectively, the angle and distance between the moving and fixed axes, we write expressions for the cosine of the angle

$$\cos \Phi_j = L_j\lambda + M_j\mu + N_j\nu \qquad (1)$$

and the moment

$$D_j \sin \Phi_j = L_j\rho + M_j\sigma + N_j\tau + \lambda R_j + \mu S_j + \nu T_j \qquad (2)$$

between the axes.

Correspondence (Three Positions)

We now show that for three positions there is a (1,1) correspondence between the moving and fixed axes of a cylindric-cylindric crank.

The "twist" in the crank does not vary with the motion, and hence in positions l, m, and n[8] it is necessary that

$$\cos \Phi_l = \cos \Phi_m = \cos \Phi_n$$

Substituting from (1) we find that

$$\left.\begin{aligned} \lambda(L_m - L_l) + \mu(M_m - M_l) + \nu(N_m - N_l) &= 0 \\ \lambda(L_n - L_l) + \mu(M_n - M_l) + \nu(N_n - N_l) &= 0 \end{aligned}\right\} \qquad (3)$$

Actually, (3) requires $\mathbf{L}_l$, $\mathbf{L}_m$, $\mathbf{L}_n$ to be the generators and (λ, μ, ν) the axis of a right circular cone.

If the moving axis is arbitrarily chosen, the L's, M's, and N's are all known. Equation (3) may be solved for, say, λ/ν and μ/ν, which together with $\lambda^2 + \mu^2 + \nu^2 = 1$ yield a unique result for the fixed-axis cosines.

If (λ, μ, ν) are specified, and we substitute the linear transformation

$$L_m - L_l = a_{1lm}L_l + b_{1lm}M_l + c_{1lm}N_l$$

$$M_m - M_l = a_{2lm}L_l + b_{2lm}M_l + c_{2lm}N_l$$

and so on.[11]

Equation (3) becomes a linear homogeneous set in L_l, M_l, N_l which yields, say, the ratios L_l/N_l, M_l/N_l. Finally, calling upon $L_l^2 + M_l^2 + N_l^2 = 1$ one determines a unique set of moving-axis direction cosines corresponding to the given fixed axis.

It may easily be shown that the screw axes are singular lines in

[7] This motion is either planar or spherical depending upon whether the screw axes intersect in an infinite or finite point.

[8] Usually, $l = 1$, $m = 2$, and $n = 3$.

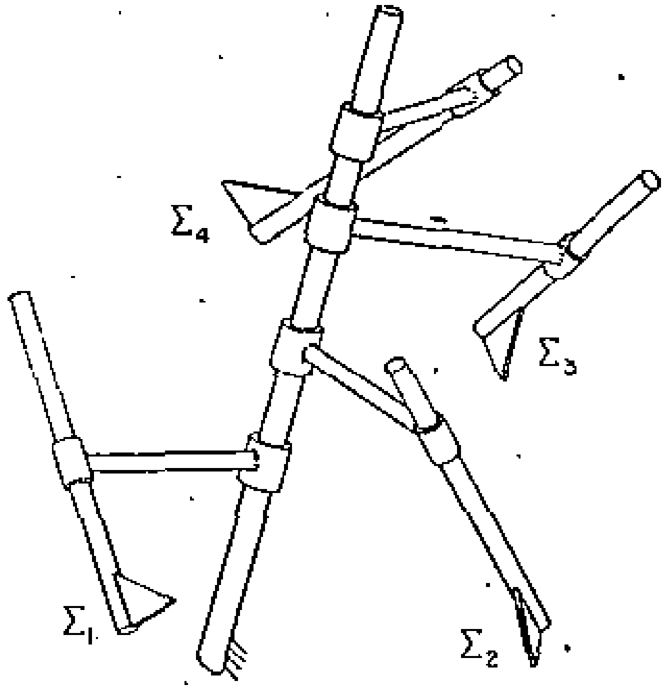


Fig. 7 Cylindric-cylindric crank for the four positions Σ_1, Σ_2, Σ_3, Σ_4

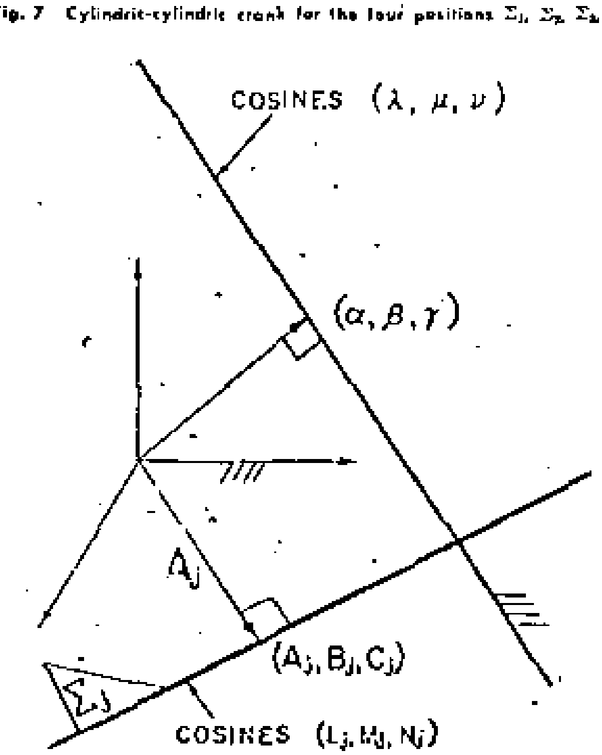


Fig. 8 Nomenclature used in the cylindric-cylindric crank derivations

[11] The a's, b's, and c's are function of the known screw-axis cosines and the angle of rotation. See Appendix 2 for explicit expressions.

regard to the quadratic correspondence defined by equation (3).

For any given crank, the distance and "twist" between the axes are fixed, hence we require that $D_l \sin\Phi_l = D_m \sin\Phi_m = D_n \sin\Phi_n$, which, when we substitute (2), yields

$$
\begin{aligned}
\rho(L_m - L_l) + \sigma(M_m - M_l) + \tau(N_m - N_l) \\
= \lambda(R_l - R_m) + \mu(S_l - S_m) + \nu(T_l - T_m) \\
\rho(L_n - L_l) + \sigma(M_n - M_l) + \tau(N_n - N_l) \\
= \lambda(R_l - R_n) + \mu(S_l - S_n) + \nu(T_l - T_n)
\end{aligned}
\qquad (4)
$$

Since, from the above, the direction cosines of both axes are known, (4) yields two linear, nonhomogeneous equations in either (ρ, σ, τ) or (R_l, S_l, T_l). When the moving axis is specified, the R's, S's, and T's are known and (ρ, σ, τ) is restricted to the line of intersection of the two planes (4). Finally, using

$$\rho = \beta\nu - \gamma\mu, \quad \sigma = \gamma\lambda - \alpha\nu, \quad \tau = \alpha\mu - \beta\lambda,$$

and adding the restriction $\alpha\lambda + \beta\mu + \gamma\nu = 0$, one obtains a unique normal vector (α, β, γ).

If, instead, we specify the fixed axis, it is required that we first substitute

$$
\begin{aligned}
R_j &= B_j N_j - C_j M_j \\
S_j &= C_j L_j - A_j N_j \\
T_j &= A_j M_j - B_j L_j \quad (j = l, m, n) \text{ into (4),}
\end{aligned}
$$

and then apply the linear transformation of Appendix 2.[13]

Substituting (18) explicitly converts the right-hand sides of (4) into linear functions of (A_l, B_l, C_l), and these two equations determine a line. Finally, by requiring that $A_l L_l + B_l M_l + C_l N_l = 0$ a unique normal vector is obtained, and the moving axis corresponding to the given fixed axis is uniquely determined.

In the above we used the specified motion throughout. It is, of course, also possible to determine this correspondence from (3) and (4) by use of inversion.

We have shown that for three positions of the moving body we are at liberty to choose any line in Σ' as the fixed axis, or any line in Σ as the moving axis. However, *once either axis is specified the other is uniquely defined by the above (1,1) correspondence.*

It can be shown that this same correspondence is given by two applications of the following theorem:

Each pair of fixed and moving axes subtends a screw axis at a dual angle equal to one-half, or the supplement of one-half the dual angle associated with that screw. For example, with the moving axis in position 1, the dual angle measured along $\$_{12}$ is $\hat{\theta}_{12}/2$ (± 180 deg). The one along $\$_{13}$ is $\hat{\theta}_{13}/2$ (± 180 deg), and the one along $\$_{23}{}'$ equals $\hat{\theta}_{23}/2$ (± 180 deg). The sense of the dual angle is from the moving to the fixed axis.

Finally, once one set of fixed and moving axes are known, the following theorem could be used to establish the same correspondence: *Corresponding sets of two moving and two fixed axes appear from either $\$_{lm}$ or $\$_{ln}$ as well as from $\$'_{mn}$ to subtend two equal or two supplementary dual angles.*[14] (In this we have taken the moving axis in position l.)

Four Positions

Axis Direction

For four finitely separated positions, l, m, n, p[15], we add a third equation to (3) and obtain:

$$
\begin{aligned}
\lambda(L_m - L_l) + \mu(M_m - M_l) + \nu(N_m - N_l) &= 0 \\
\lambda(L_n - L_l) + \mu(M_n - M_l) + \nu(N_n - N_l) &= 0 \qquad (5) \\
\lambda(L_p - L_l) + \mu(M_p - M_l) + \nu(N_p - N_l) &= 0
\end{aligned}
$$

Regarding (λ, μ, ν) as the unknowns, the compatibility condition for (5) becomes:

$$
\begin{vmatrix}
(L_m - L_l) & (M_m - M_l) & (N_m - N_l) \\
(L_n - L_l) & (M_n - M_l) & (N_n - N_l) \\
(L_p - L_l) & (M_p - M_l) & (N_p - N_l)
\end{vmatrix} = 0 \qquad (6)
$$

By remembering that $L_j - L_l = a_{lj}L_l + b_{lj}M_l + c_{lj}N_l$, and so on, it is possible to write (6) as a homogeneous polynomial in (L_l, M_l, N_l), hence (6) represents a cone of third order. In fact, *this cone is the screw cone* studied earlier.

The generators of this cone define the directions in which there exist lines in Σ_l which maintain the same angle with a fixed line for all four positions. The directions of the corresponding fixed lines may be obtained from the (1,1) correspondence described above.

If we change from general to spherical motion, equation (6) remains unaltered. In fact, (6) is identical to the condition obtained, by the author [2], for the locus of moving lines whose points all fall on circles for four-position spherical motion.

It is computationally convenient to reduce the cone to a planar cubic. Dividing (6) by, say, $(N_l)^3$ yields a third-order planar curve in the variables (L_l/N_l), (M_l/N_l). Fig. 6 shows the cone and planar curve associated with the four positions given in Table 1.

Axis Location

If, in addition to the constant angle, a moving line remains at a constant distance from some fixed line, the moving and fixed lines may be used as four-position cylindric-cylindric crank axes.

For a direction corresponding to any generator (L_l, M_l, N_l) the axes locations are obtained as follows:

For four positions, the equation

$$
\rho(L_p - L_l) + \sigma(M_p - M_l) + \tau(N_p - N_l) = \lambda(R_l - R_p) + \mu(S_l - S_p) + \nu(T_l - T_p)
$$

added to (4) yields a set of three equations. In addition, the condition that (ρ, σ, τ) is normal to (λ, μ, ν) yields

$$\rho\lambda + \sigma\mu + \tau\nu = 0.$$

Regarding (ρ, σ, τ) as the unknowns in these four equations, the compatibility conditions require that[16]:

$$
\begin{vmatrix}
\lambda & \mu & \nu & 0 \\
(L_m - L_l) & (M_m - M_l) & (N_m - N_l) & \lambda(R_l - R_m) + \mu(S_l - S_m) + \nu(T_l - T_m) \\
(L_n - L_l) & (M_n - M_l) & (N_n - N_l) & \lambda(R_l - R_n) + \mu(S_l - S_n) + \nu(T_l - T_n) \\
(L_p - L_l) & (M_p - M_l) & (N_p - N_l) & \lambda(R_l - R_p) + \mu(S_l - S_p) + \nu(T_l - T_p)
\end{vmatrix} = 0 \qquad (7)
$$

After substituting (18), (7) is converted to a linear equation in (A_l, B_l, C_l). This, coupled with the second linear condition,

$$A_l L_l + B_l M_l + C_l N_l = 0 \qquad (8)$$

yields a line as the locus of the tip of the normal vector (A_l, B_l, C_l) in the moving axis. The moving axis may be chosen through

[13] Similarly, instead of (2) and (4) we could have used $D_l = D_m = D_n$ where $D_j =$ [illegible]. Here $\xi_j = (M_j\nu - N_j\mu)$, $\zeta_j = (N_j\lambda - L_j\nu)$, $\eta_j = (L_j\mu - M_j\lambda)$, are all known. This could then be carried forward to the four and five-position derivations to yield equations analogous to (6) and (7). Such a formulation has the advantage of using the position vectors explicitly.

[14] These angles are measured in the same sense (just as in the equivalent planar theorem). This theorem is also valid if one takes the circle [illegible] lines instead of the moving axes. However, in this case the sense is reversed.

[15] Usually one takes $l = 1$, $m = 2$, $n = 3$, $p = 4$.

[16] Since our choice of (L_l, M_l, N_l), in effect, determines (λ, μ, ν), we regard all the cosines as known.

any point on this line. Hence, *corresponding to each direction defined by an element of the cubic cone, there is a singly infinite array of moving axes.* All the axes of a given array are (*a*) parallel to a single element of the cone, and (*b*) coplanar.

Once the moving axis is selected, the fixed axis is uniquely determined (by using any three positions) from the (1,1) correspondence.

Five Positions

We extend the four-position analysis to include a fifth position.

Axis Direction

We now write (5) as a set of four equations by adding

$$\lambda(L_r - L_l) + \mu(M_r - M_l) + \nu(N_r - N_l) = 0.$$

$$(r \neq p \neq m \neq n \neq l)^{14}$$

Using the same argument as before, the compatibility condition now requires that any two (3 × 3) determinants vanish. It is convenient to use (6) and the cubic cone obtained by changing one of the sub-cripts in (6), say p, to r. Since the two cubic cones have the same apex, there are at most nine real directions (L_l, M_l, N_l) for which the compatibility conditions may be satisfied. However, three of these directions are spurious since they correspond to the screw axes $\$_{lm}$, $\$_{ln}$, $\$_{nm}$[15] and *there are generally at most six moving-axis directions for any set of five spatial displacements.*

Again, these nine directions are identical to the ones previously determined [2] for the case of spherical motion.

By using any three positions in the (1,1) correspondence a corresponding fixed-axis direction may be determined for each of the six directions.

Axis Location

The lines corresponding to these six directions are located as follows:

We add a fourth equation to (4),

$$\rho(L_r - L_l) + \sigma(M_r - M_l) + \tau(N_r - N_l)$$
$$= \lambda(R_l - R_r) + \mu(S_l - S_r) + \nu(T_l - T_r)$$

and proceed as before. The compatibility condition now requires that two (4 × 4) determinants vanish. It is convenient to use (7) and a similar determinant (7') which is the same as (7) except for the subscript in the last row, which is changed from p to r. Substituting (18) into (7') yields a linear equation in (A_l, B_l, C_l). The simultaneous solution of (7), (7'), and (8) yields a unique point (A_l, B_l, C_l) for each axis direction.

Therefore, *to each of the at most six nontrivial directions common to the two cubic cones, there corresponds a unique moving axis.* Again, by using any three positions the corresponding fixed axis may be determined.

It follows that, associated with any five positions, there are either six, four, two, or zero cylindric-cylindric cranks. If the motion is planar, two of these cranks become imaginary, while the remaining four (at most) become the revolute-revolute axes corresponding to the well-known Burmester points.

The solutions to a five-position problem are listed in Table 2.

Inversion

In the foregoing we first computed the moving axis and then determined the fixed one from the (three-position) correspondence. In an analogous manner we could have treated the fixed axis first, and then determined the moving axis from the correspondence. Alternatively, if we inverted the motion, the fixed axes could have been computed from exactly those equations developed for the moving axis.

Parametric Developments

The two basic sets of parameters associated with the cylindric-cylindric cranks are:

1 *The twist* (Φ) *and the length* (*D*) *of the crank.* In order to determine which, and how many, lines of the congruence correspond to cranks with a given twist and/or length, it is necessary to derive loci analogous to the planar R_m and R^t curves. (It is anticipated that these questions will be discussed in a subsequent publication.)

2 *The displacements of the crank along the fixed or moving axis.* The fixed-axis congruence may be developed as a function of the displacement along the fixed axis by a procedure based on the following theorem: *The dual angle subtended at the fixed cylindric-cylindric crank axis by screws* $\$_{lj}$ *and* $\$_{lk}$ *is equal to one-half, or one-half the supplement, of the crank displacement from position j to k.*

Accordingly, any desired crank rotation is substituted for θ_{lj} and θ_{lm} in (14) and (14'), respectively. The unit vectors, $\bar{S}_l$, common to these two equations determine the required fixed-axis directions. The loci given by (14) and (14') are completely analogous to the families of intersecting circles used to parametrically construct the planar pole curves.

By using any one of these directions and any desired crank translation for d_{lj} and d_{lm} in (10), one obtains two linear equations which determine a fixed axis for which the corresponding crank displacement from position j to k is as specified.

By inversion, the above may be used to parametrically develop the moving-axis congruence. The parameters now correspond to displacements of the crank relative to the moving body, Σ.

[14] Usually $r = 5$.

[15] For the directions of these three screw axes the compatibility conditions are no longer sufficient, since the rank of the system becomes one instead of two.

Table 2 For the five positions specified in Table 1, the cubic cones have seven (real) common generators, therefore, there are four cylindric-cylindric cranks. For each of these cranks we list the axes for the fixed and moving cylindrical joints and also the corresponding crank dimensions. The sign on the twist, Φ, is defined by the right-handed screw rule, with the screw pointing from the moving toward the fixed axis.

Solution Number	Fixed Axis cosines λ	μ	ν	location α	β	γ
1	0.4058	−0.2511	0.8788	0.5934	2.547	0.4538
2	0.1451	−0.0401	0.9886	2.181	−0.4675	−0.3392
3	0.0894	0.3113	0.9357	3.159	−1.210	0.1393
4	−0.4672	0.0077	0.8787	1.176	−5.920	1.253

Moving Axis cosines L_1	M_1	N_1	locations A_1	B_1	C_1	Crank twist Φ (deg)	length D
0.1489	−0.0502	0.9876	2.016	1.022	−0.2214	−19.8	0.1525
0.3658	0.0022	0.9261	−0.4225	−1.010	0.3270	15.2	0.3518
0.4718	0.7008	0.5230	−0.6259	−2.360	3.768	39.3	1.0265
0.5040	−0.4012	0.7302	−6.560	−2.478	2.963	−68.8	0.7600

Conclusion

The strong analogy between these results and the classical planar theory makes it seem appropriate to regard the planar theory as but one special case of a more general spatial-motion theory.

The basic geometry associated with spatial motion is spatial. However, since the inclinations of lines and axes are independent of the screw locations and translations, questions of inclination may be resolved without considering the location. Hence, *as far as inclination is concerned, spherical motion and spatial motion are identical*, and the fundamental geometric entities are made up of intersecting lines. For planar motion, the point of intersection is at infinity, and the lines become parallel. Thus, *questions of inclination in spatial and spherical motion are analogous to questions of location for the planar case.*

For spatial motion, in order to determine location, it is necessary to "dualize" the inclination by taking into account the translation and screw location.

The equations derived for the spatial equivalents of the Burmester points, pole curves, center-point and circle-point curves have been programmed in FORTRAN IV. Copies of this program may be obtained by writing to the author.

In addition to being of theoretical interest, these results should be useful in the actual design of spatial linkages. The synthesis techniques described elsewhere [2] are directly applicable to mechanisms with cylindric-cylindric links.

Acknowledgments

The author wishes to express his appreciation to Mr. Daniel Silva for assisting with the derivations and figures, and to the National Science Foundation whose grants NSF-GP 2005 and NSF-GK 660 made this work possible.

References

1 B. Roth, "The Kinematics of Motion Through Finitely Separated Positions," submitted to the *Journal of Applied Mechanics*, Trans. ASME, Series E.

2 B. Roth, "Finite Position Theory Applied to Mechanism Synthesis," submitted to the *Journal of Applied Mechanics*, Trans. ASME, Series E.

3 G. N. Sandor, "On the Existence of a Cycloidal Burmester Theory in Planar Kinematics," *Journal of Applied Mechanics*, vol. 31, Trans. ASME, Series E, vol. 86, 1964, pp. 694–699.

4 B. Roth and F. Freudenstein, "Synthesis of Path-Generating Mechanisms by Numerical Methods," Journal of Engineering for Industry, Trans. ASME, Series B, vol. 85, 1963, pp. 298–306.

5 G. Hein, "Analytische und Zeichnerische Unterlagen für zwei und drei Lagen eines Sphärisch bewegten Getriebegliedes als Hilfsmittel zur Synthese sphärischer Kurbelgetriebe," *Forsch. Ing. Wes.*, vol. 25, 1959, pp. 92–96, 115–126.

6 G. Dittrich, "Über die momentane Bewegungsgeometrie eines sphärisch bewegten starren Systems," doctoral dissertation, Technical University, Aachen, Germany, 1964.

7 V. V. Dobrovolskii, "Burmester's Points in Spherical Motion" (in Russian), *Prikl. Mat. Mekh.*, vol. 9, 1945, pp. 489–491.

8 M. L. Keler, "Analyse und Synthese der Raumkurbelgetriebe mittels Raumliniengeometrie und dualer Grössen," *Forsch. Ing. Wes.*, vol. 25, 1959, pp. 27–32, 55–63.

9 A. Schoenflies, *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, 1886, 194 pp.

10 J. T. Wilson, III, "Analytical Kinematic Synthesis by Finite Displacements," Journal of Engineering for Industry, Trans. ASME, Series B, vol. 87, 1965, pp. 161–169.

11 J. S. Ames and F. D. Murnaghan, *Theoretical Mechanics*, Ginn and Co., 1929. (Dover Publications, 1958.)

12 A. T. Yang, "Application of Quaternion Algebra and Dual Numbers to the Analysis of Spatial Mechanisms," doctoral dissertation, Columbia University, New York, N. Y., 1963, 241 pp.

13 R. S. Hartenberg and J. Denavit, *Kinematic Synthesis of Linkages*, McGraw-Hill Book Company, New York, N. Y., 1964, 435 pp.

14 E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, 1937. (Dover Publications, 1944.)

15 E. Study, *Geometrie der Dynamen*, Liepzig, 1903, 603 pp.

APPENDIX 1

The Screw Congruence

In this section we derive analytical expressions for the line complexes which are spatial analogs of the planar pole curves. As we showed earlier, the basic requirements are that all six screw axes, associated with four finitely separated positions, are subtended at equal, or supplementary, dual angles by opposite sides of opposite-screw quadrilaterals.

Let $\hat{S}_j$, $\hat{S}_k$, $\hat{S}_l$, $\hat{S}_m$ represent unit line vectors along any four screws which form an opposite-screw quadrilateral, and let $\hat{S}_1$ represent a unit line vector along one of the remaining screws.

Choosing any position vector (A_j, B_j, C_j) from the origin of coordinates to line $\hat{S}_j$, and the unit vector (l_j, m_j, n_j) parallel to $\hat{S}_j$, we define (r_j, s_j, t_j) as the cross product $(A_j, B_j, C_j) \times (l_j, m_j, n_j)$. The unit line vector $\hat{S}_j$ can now be expressed as the dual vector $\hat{S}_j = l_j\mathbf{i} + m_j\mathbf{j} + n_j\mathbf{k} + \epsilon(r_j\mathbf{i} + s_j\mathbf{j} + t_j\mathbf{k})$, where $\epsilon^2 \equiv 0$. We express $\hat{S}_1$ similarly.

The normal from line $\hat{S}_1$ to $\hat{S}_j$ is one of the quantities given by the product $\hat{S}_1\hat{S}_j$. Denoting the unit line vector along this normal by $\hat{I}_j$, we obtain:

$$\hat{S}_1\hat{S}_j = -\cos\hat{\alpha}_{1j} + \hat{I}_j \sin\hat{\alpha}_{1j}.$$

Here $\hat{\alpha}_{1j}$ is the dual angle between $\hat{S}_1$ and $\hat{S}_j$, and as has been shown by Yang [12],

$$\hat{I}_j = \frac{a_j\mathbf{i} + b_j\mathbf{j} + c_j\mathbf{k}}{(a_j^2 + b_j^2 + c_j^2)^{1/2}} + \epsilon\left[\frac{a_{0j}\mathbf{i} + b_{0j}\mathbf{j} + c_{0j}\mathbf{k}}{(a_j^2 + b_j^2 + c_j^2)^{1/2}} + d_j d_{0j}\frac{(a_j\mathbf{i} + b_j\mathbf{j} + c_j\mathbf{k})}{(a_j^2 + b_j^2 + c_j^2)^{3/2}}\right]$$

where

$$\begin{aligned}
d_j &= -(l_1 l_j + m_1 m_j + n_1 n_j)\\
d_{0j} &= -(l_1 r_j + l_j r_1 + m_1 s_j + m_j s_1 + n_1 t_j + n_j t_1)\\
a_j &= m_1 n_j - n_1 m_j\\
a_{0j} &= n_j s_1 + m_1 t_j - n_1 s_j - m_j t_1\\
b_j &= n_1 l_j - l_1 n_j\\
b_{0j} &= l_j t_1 + n_1 r_j - l_1 t_j - n_j r_1\\
c_j &= l_1 m_j - m_1 l_j\\
c_{0j} &= l_1 s_j + m_j r_1 - m_1 r_j - l_j s_1
\end{aligned}$$

By altering the subscripts one obtains similar expressions for $\hat{I}_k$, $\hat{I}_l$, and $\hat{I}_m$, the unit line vectors along the normals from $\hat{S}_1$ to, respectively, $\hat{S}_k$, $\hat{S}_l$, and $\hat{S}_m$.

Now if we take $\hat{S}_j$ and $\hat{S}_k$ as one "side" of the opposite screw quadrilateral, and $\hat{S}_l$ and $\hat{S}_m$ as the other, the required condition is that the dual angle, $\hat{\theta}_{jk}/2$, between $\hat{I}_j$ and $\hat{I}_k$ be equal, or supplementary, to the dual angle, $\hat{\theta}_{lm}/2$, between $\hat{I}_l$ and $\hat{I}_m$. The product $\hat{I}_j\hat{I}_k$ yields

$$\hat{I}_j\hat{I}_k = -\cos\left(\frac{\hat{\theta}_{jk}}{2}\right) + \hat{S}_1\sin\left(\frac{\hat{\theta}_{jk}}{2}\right)$$

from which it follows [12] that

$$\cos\left(\frac{\theta_{jk}}{2}\right) = \frac{a_j a_k + b_j b_k + c_j c_k}{(a_j^2 + b_j^2 + c_j^2)^{1/2}(a_k^2 + b_k^2 + c_k^2)^{1/2}} \qquad (9)$$

and

$$\frac{-d_{jk}}{2}\sin\left(\frac{\theta_{jk}}{2}\right) = \left\{(a_j a_{0k} + b_j b_{0k} + c_j c_{0k}) + (a_k a_{0j} + b_k b_{0j} + c_k c_{0j}) + \left[\frac{d_k d_{0k}}{(a_k^2 + b_k^2 + c_k^2)} + \frac{d_j d_{0j}}{(a_j^2 + b_j^2 + c_j^2)}\right]\right.$$

$$\times (a_j a_k + b_j b_k + c_j c_k)\Big\} \Big/ [(a_j^2 + b_j^2 + c_j^2)^{1/2} \times (a_k^2 + b_k^2 + c_k^2)^{1/2}] \quad (10)$$

$\theta_{jk}/2$ and $d_{jk}/2$, the angle and distance between l_j and l_k, are the principal and dual parts of $\hat{\theta}_{jk}/2$, i.e.,

$$\frac{\hat{\theta}_{jk}}{2} = \frac{\theta_{jk}}{2} + \frac{\epsilon d_{jk}}{2}$$

The equal angle condition requires that

$$\cos\left(\frac{\theta_{jk}}{2}\right) = \cos\left(\frac{\theta_{lm}}{2}\right)$$

Substituting from (9), and squaring in order to remove the radicals, one obtains:

$$(a_j a_k + b_j b_k + c_j c_k)^2(a_l^2 + b_l^2 + c_l^2)(a_m^2 + b_m^2 + c_m^2) - (a_l a_m + b_l b_m + c_l c_m)^2(a_j^2 + b_j^2 + c_j^2)(a_k^2 + b_k^2 + c_k^2) = 0 \quad (11)$$

This is a homogeneous algebraic polynomial of degree eight in the unknown directions (l_1, m_1, n_1). When expanded, the common factor $(l_1^2 + m_1^2 + n_1^2)$ may be removed, and (11) reduced to a sextic. This sextic is made up of two cubic cones: One contains the directions (l_1, m_1, n_1) for which $\theta_{jk}/2 = \theta_{lm}/2$ (± 180 deg), and the other contains the directions for which $\theta_{jk}/2 = -\theta_{lm}/2$ (± 180 deg). We are interested in only the one for which $\theta_{jk}/2 = \theta_{lm}/2$ (± 180 deg); we refer to this cone as the screw cone.

We also require that

$$\frac{d_{jk}}{2}\sin\left(\frac{\theta_{jk}}{2}\right) = \pm\frac{d_{lm}}{2}\sin\left(\frac{\theta_{lm}}{2}\right) \quad (12)$$

where the sign is chosen plus or minus according to whether the angles are equal or supplementary.

Substituting from (10) and considering (l_1, m_1, n_1) as known, one obtains a linear equation in the a_s's, b_s's, c_s's, and d_s's, which are linear functions of (r_1, s_1, t_1). Finally, since the position vector (A_j, B_j, C_j) is a linear function of (r_1, s_1, t_1), we have the result that *parallel to each generator of the screw cone there is an infinite set of coplanar lines. This doubly infinite array of lines is the locus of the six screws.*

The foregoing derivation shows analytically that the directions of the screws are independent of their positions. Using this fact we now obtain the screw-cone equation free of extraneous factors:

Taking the unit vectors $\hat{S}_j, \hat{S}_k, \hat{S}_l, \hat{S}_m, \hat{S}_1$ as intersecting lines, the dihedral angle, $\theta_{jk}/2$, between the planes containing $(\hat{S}_j, \hat{S}_1)$ and $(\hat{S}_k, \hat{S}_1)$ is given by:

$$\frac{(\hat{S}_j \times \hat{S}_1) \times (\hat{S}_k \times \hat{S}_1)}{(\hat{S}_j \times \hat{S}_1)\cdot(\hat{S}_k \times \hat{S}_1)} = \hat{S}_1 \tan\left(\frac{\theta_{jk}}{2}\right) \quad (13)$$

Expanding and simplifying yields

$$\frac{(\hat{S}_j \times \hat{S}_k)\cdot\hat{S}_1}{(\hat{S}_j\cdot\hat{S}_k) - (\hat{S}_j\cdot\hat{S}_1)(\hat{S}_k\cdot\hat{S}_1)} = \tan\left(\frac{\theta_{jk}}{2}\right) \quad (14)$$

The dihedral angle, $\theta_{lm}/2$, between planes $(\hat{S}_l, \hat{S}_1)$ and $(\hat{S}_m, \hat{S}_1)$ is given by equation (14'), which is obtained from (14) by replacing j and k with l and m.

From the section on screw congruences, we apply condition a', and require that $\tan(\theta_{jk}/2) = \tan(\theta_{lm}/2)$. This yields

$$(\hat{S}_j \times \hat{S}_k).\hat{S}_1[(\hat{S}_l.\hat{S}_m) - (\hat{S}_l.\hat{S}_1)(\hat{S}_m.\hat{S}_1)] - (\hat{S}_l \times \hat{S}_m).\hat{S}_1[(\hat{S}_j.\hat{S}_k) - (\hat{S}_j.\hat{S}_1)(\hat{S}_k.\hat{S}_1)] = 0 \quad (15)$$

which is the equation of the screw cone.

It is possible, but not necessary, to also phrase the moment condition in terms of the tangent of the half angle [by dividing (10) by (9)].

This same derivation yields the planar pole curve, if in (13) one substitutes the vector differences $(\hat{S}_j - \hat{S}_1)$ and $(\hat{S}_k - \hat{S}_1)$ for the corresponding cross products, and replaces the unit vectors by the vectors from the origin to the poles.

APPENDIX 2

Linear Transformations

The following well-known [14] linear transformation gives the coordinates of a point (A_m, B_m, C_m) in terms of its coordinates in the lth position and the screw $\$_{lm}$. The point is fixed in the moving system but all coordinates are measured along axes in the fixed system.

$$\begin{bmatrix} A_m \\ B_m \\ C_m \end{bmatrix} = \begin{bmatrix} (a_{rlm}+1) & b_{rlm} & c_{rlm} \\ a_{slm} & (b_{slm}+1) & c_{slm} \\ a_{tlm} & b_{tlm} & (c_{tlm}+1) \end{bmatrix} \begin{bmatrix} A_l \\ B_l \\ C_l \end{bmatrix} + \begin{bmatrix} d_{rlm} \\ d_{slm} \\ d_{tlm} \end{bmatrix} \quad (16)$$

where $a_{rlm} = (u_{1lm}^2 - 1)(1 - \cos\theta_{lm})$

$b_{rlm} = u_{1lm}u_{2lm}(1 - \cos\theta_{lm}) - u_{3lm}\sin\theta_{lm}$

$c_{rlm} = u_{1lm}u_{3lm}(1 - \cos\theta_{lm}) + u_{2lm}\sin\theta_{lm}$

$d_{rlm} = d_{lm}u_{1lm} - a_{lm}a_{rlm} - b_{lm}b_{rlm} - c_{lm}c_{rlm}$

$a_{slm} = u_{2lm}u_{1lm}(1 - \cos\theta_{lm}) + u_{3lm}\sin\theta_{lm}$

and so on.[12]

Similarly, the transformation of a set of direction cosines (L, M, N) is given by

$$\begin{bmatrix} L_m \\ M_m \\ N_m \end{bmatrix} = \begin{bmatrix} (a_{rlm}+1) & b_{rlm} & c_{rlm} \\ a_{slm} & (b_{slm}+1) & c_{slm} \\ a_{tlm} & b_{tlm} & (c_{tlm}+1) \end{bmatrix} \begin{bmatrix} L_l \\ M_l \\ N_l \end{bmatrix} \quad (17)$$

Now for the quantities (R, S, T) defined by the cross product of a position vector (A, B, C) and the unit vector (L, M, N) we use

$$R_m = B_m N_m - C_m M_m$$
$$S_m = C_m L_m - A_m N_m$$
$$T_m = A_m M_m - B_m L_m$$

similarly for the subscript l. Then substituting (16) we determine that

$$\begin{aligned}
\lambda(R_m - R_l) &+ \mu(S_m - S_l) + \nu(T_m - T_l) \\
&= A_l[\lambda(N_m a_{slm} - M_m a_{tlm}) \\
&\quad + \mu(L_m a_{tlm} - N_m a_{rlm} + N_l - N_m) \\
&\quad + \nu(M_m a_{rlm} - L_m a_{slm} + M_m - M_l)] \\
&\quad + B_l[\mu(L_m b_{tlm} - N_m b_{rlm}) \\
&\quad + \lambda(N_m b_{slm} - M_m b_{tlm} + N_m - N_l) \\
&\quad + \nu(M_m b_{rlm} - L_m b_{slm} + L_l - L_m)] \\
&\quad + C_l[\lambda(N_m c_{slm} - M_m c_{tlm} + M_l - M_m) \\
&\quad + \mu(L_m c_{tlm} - N_m c_{rlm} + L_m - L_l) \\
&\quad + \nu(M_m c_{rlm} - L_m c_{slm})] \\
&\quad + \lambda(N_m d_{slm} - M_m d_{tlm}) + \mu(L_m d_{tlm} - N_m d_{rlm}) \\
&\quad + \nu(M_m d_{rlm} - L_m d_{slm})
\end{aligned} \quad (18)$$

In our application of (18) (L_l, M_l, N_l) are known and (L_m, M_m, N_m) are determined from (17).

[12] The a's, d's, θ's, etc., are defined according to Fig. 1.

Printed in U. S. A.

P. CHEN
Assistant Agricultural Engineer,
University of California,
Davis, Calif.

B. ROTH
Associate Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif. Mem. ASME

A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis

A rigid body is studied in a series of different positions. These positions can be finitely separated, infinitesimally separated, or a combination of the two. A general method for determining the locations of points or lines (in the rigid body) which have their different multiple positions satisfying the constraints of binary links or combined link chains is developed. In a companion paper [10][1] equations governing the locations of these special points and lines are derived.

Introduction

In dimensional synthesis of spatial mechanisms the central problem is to determine the dimensions of a selected type of linkage which constrains the relative motion of a moving body σ and a fixed body Σ in a specified manner.[2] The "motion" of σ is specified by a given series of positions of σ relative to Σ. These several positions are either finitely separated, infinitesimally separated, or mixed finitely and infinitesimally separated.

Finitely separated position problems have previously been extensively studied. (See, for example, Schoenflies [1], Roth [2–4], Suh [5], Sandor [14], and Wilson [6].) However, comparatively little research has been done on infinitesimally separated position problems [1, 7, 8, 14]. The primary objective of this paper is to recast the existing finite position work and combine it with new results to form a general theory which unifies the analytical study of finite, infinitesimal, and mixed displacement problems.

In this paper a general theory is developed for determining the number and locus of the points or lines in σ which have their several positions satisfying the constraints of binary links or combined link chains. In a companion paper [10], we use these general results to obtain explicit formulations for chains of practical interest. It has been previously shown [3] that such results may be applied to a variety of different synthesis problems.

Linear Transformations

Central to what follows are the relationships between the coordinates of points or lines before and after a displacement. Such relationships are known as transformations. For rigid-body displacements these transformations are linear. Hence, the coordinates, as measured in the fixed system, of the jth position of a point may be expressed as a linear function of the coordinates of the initial (or any reference) position. In this section, we first introduce nomenclature which analytically describes displacements and then obtain explicit expressions for the linear transformations.

Finite Displacements. We select (x, y, z) as the Cartesian coordinates of a point in a moving system σ and (X, Y, Z) as its coordinates in a fixed system Σ. Let σ_j denote the jth position of σ, and (X_j, Y_j, Z_j) the coordinates in Σ of the jth position of the point (x, y, z) in σ. Knowing the position of σ_j relative to Σ, we can express (X_j, Y_j, Z_j) in terms of (x, y, z):

$$\begin{bmatrix} X_j \\ Y_j \\ Z_j \end{bmatrix} = \begin{bmatrix} a_{1j} & b_{1j} & c_{1j} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3j} & b_{3j} & c_{3j} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} d_{1j} \\ d_{2j} \\ d_{3j} \end{bmatrix} \quad (1)$$

where the a's, b's, c's, and d's are functions of the parameters governing the relative position of σ_j and Σ. If we let the original position of σ coincide with Σ, the displacement to σ_j may be described as a screw displacement which is equivalent to a translation d_j along, and a rotation θ_j about an axis which is parallel to the unit vector $\mathbf{u}_j(u_j, v_j, w_j)$ and passes through the point (a_j, b_j, c_j) in Σ. In this case we have

$$\begin{aligned} a_{1j} &= (u_j^2 - 1)(1 - \cos\theta_j) + 1 \\ b_{1j} &= u_j v_j(1 - \cos\theta_j) - w_j \sin\theta_j \\ c_{1j} &= u_j w_j(1 - \cos\theta_j) + v_j \sin\theta_j \\ d_{1j} &= d_j u_j - a_j(a_{1j} - 1) - b_j b_{1j} - c_j c_{1j} \\ a_{2j} &= v_j u_j(1 - \cos\theta_j) + w_j \sin\theta_j \end{aligned} \quad (2)$$

etc. (See, for example, [2]).

If (l, m, n) are the components of a vector fixed to σ_j and (L_j, M_j, N_j) are the components of the same vector when measured in Σ, (L_j, M_j, N_j) can be expressed as linear functions of (l, m, n):

$$\begin{bmatrix} L_j \\ M_j \\ N_j \end{bmatrix} = \begin{bmatrix} a_{1j} & b_{1j} & c_{1j} \\ a_{2j} & b_{2j} & c_{2j} \\ a_{3j} & b_{3j} & c_{3j} \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} \quad (3)$$

The screw displacement leads to one of the most convenient forms of the linear transformation. However it is not the only possibility. For example, one could describe the displacement in regard to two axes as has been done by Hartenberg and Denavit [13], or by the displacement of some arbitrary reference point and the rotation about it, or in any number of other ways.

Infinitesimal Displacements. As in the case of finite displacement, infinitesimal motion of a rigid body can be described in many different ways. For example, it can be described by a series of successive infinitesimal screw displacements or by the displacement of a point in the body and the rotation of the body. In any case we can take one parameter as the reference parameter of the motion, designated by ϕ, and express all other parameters as functions of ϕ. Let σ_1 be the position of σ at $\phi = \phi_1$, and $d\phi$ an infinitesimal change in ϕ. The second infinitesimally separated position of σ (infinitesimally separated from σ_1) can be defined as the position of σ at $\phi = \phi_2 = (\phi_1 + d\phi)$, and the third infinitesimally separated position as the position of σ at $\phi = \phi_3 = (\phi_2 + d\phi)$, and so on. In general, we can specify n infinitesimally separated positions of σ in terms of the first $(n - 1)$ derivatives (with respect to ϕ) of the displacement parameters of σ.

We consider the motion of the moving body as a series of consecutive infinitesimal screw displacements. If we take the rotation θ about the screw as the independent parameter of the motion and express the other screw parameters (i.e., the direc-

[1] Numbers in brackets designate References at end of paper.

[2] The moving and fixed bodies are represented by two Cartesian-coordinate systems σ and Σ, respectively.

Contributed by the Design Engineering Division for presentation at the Mechanisms Conference, Atlanta, Ga., October 6–9, 1968, of The American Society of Mechanical Engineers. Manuscript received by the ASME Mechanisms Division, June 18, 1968. Paper No. 68—Mech-7.

tion of the screw axis s, the translation d along the axis, and the position vector s of the axis) as function of θ, the position vector $X(X, Y, Z)$ (relative to Σ) of a point (x, y, z) in σ would depend on θ. Let X_0 be the position vector of the point (x, y, z) at $\theta = 0$, X_n be the position vector of the same point at $\theta = n\Delta\theta$, where $\Delta\theta$ represents a small change in θ. Denoting as $X_0^{(n)}$ the nth derivative of X_0 with respect to θ, we can express the derivatives of X_0 with respect to θ by taking the limit as $\Delta\theta \to 0$ of the successive forward differences as follows:

$$
\begin{aligned}
X_0^{(1)} &= \lim_{\Delta\theta\to 0} \frac{X_1 - X_0}{\Delta\theta} \\
X_0^{(2)} &= \lim_{\Delta\theta\to 0} \frac{X_2 - 2X_1 + X_0}{(\Delta\theta)^2} \\
X_0^{(3)} &= \lim_{\Delta\theta\to 0} \frac{X_3 - 3X_2 + 3X_1 - X_0}{(\Delta\theta)^3} \\
X_0^{(4)} &= \lim_{\Delta\theta\to 0} \frac{X_4 - 4X_3 + 6X_2 - 4X_1 + X_0}{(\Delta\theta)^4} \\
X_0^{(5)} &= \lim_{\Delta\theta\to 0} \frac{X_5 - 5X_4 + 10X_3 - 10X_2 + 5X_1 - X_0}{(\Delta\theta)^5}
\end{aligned} \qquad (4)
$$

and so forth.

If we know X_i $(i = 0, 1, 2, \ldots, p)$ we can calculate $X_0^{(p)}$ from the foregoing equations. The problem now is to find the X_i. We first introduce two column matrices

$$
\underline{X}_i = \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

where (X_i, Y_i, Z_i) and (x, y, z) are, respectively, the components of X_i and x, and the screw transformation matrix

$$
\underline{A}_j = \begin{bmatrix} a_{1j} & b_{1j} & c_{1j} & d_{1j} \\ a_{2j} & b_{2j} & c_{2j} & d_{2j} \\ a_{3j} & b_{3j} & c_{3j} & d_{3j} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

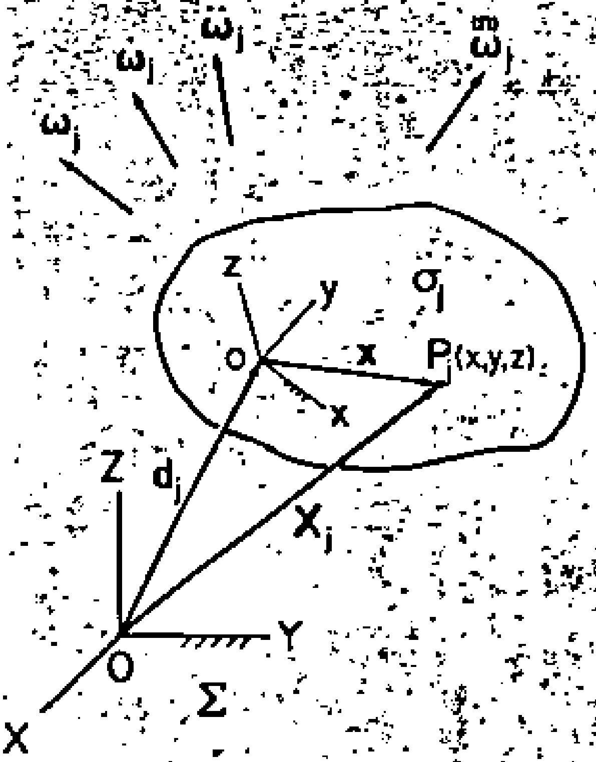


Fig. 1

where the elements in $\underline{A}_j$ are the same as those in (2), but here

$$
\begin{aligned}
u_1 &= u \\
u_2 &= u + u^{(1)}d\theta \\
u_3 &= u + 2u^{(1)}d\theta + u^{(2)}d\theta^2 \\
u_4 &= u + 3u^{(1)}d\theta + 3u^{(2)}d\theta^2 + u^{(3)}d\theta^3 \\
u_5 &= u + 4u^{(1)}d\theta + 6u^{(2)}d\theta^2 + 4u^{(3)}d\theta^3 + u^{(4)}d\theta^4, \text{ etc.}
\end{aligned}
$$

and similarly for v_j, w_j, a_j, b_j and c_j. Also

$$
\begin{aligned}
d_1 &= d^{(1)}d\theta \\
d_2 &= d^{(1)}d\theta + d^{(2)}d\theta^2 \\
d_3 &= d^{(1)}d\theta + 2d^{(2)}d\theta^2 + d^{(3)}d\theta^3 \\
d_4 &= d^{(1)}d\theta + 3d^{(2)}d\theta^2 + 3d^{(3)}d\theta^3 + d^{(4)}d\theta^4 \\
d_5 &= d^{(1)}d\theta + 4d^{(2)}d\theta^2 + 6d^{(3)}d\theta^3 + 4d^{(4)}d\theta^4 + d^{(5)}d\theta^5
\end{aligned}
$$

and so forth, where $[\ ]^{(n)} \equiv \dfrac{d^n[\ ]}{d\theta^n}$.

Therefore, from the screw displacement (1) and (2), we have

$$
\begin{aligned}
\underline{X}_0 &= \underline{x} \\
\underline{X}_1 &= \underline{A}_1\underline{x} \\
\underline{X}_2 &= \underline{A}_2\underline{X}_1 = \underline{A}_2\underline{A}_1\underline{x} \\
&\ \ \vdots \\
\underline{X}_n &= \underline{A}_n\underline{X}_{n-1} = \underline{A}_n\underline{A}_{n-1}\ldots\underline{A}_2\underline{A}_1\underline{x} = \underline{A}^{(n)}\underline{x}
\end{aligned} \qquad (5)
$$

where $\underline{A}^{(n)} = \underline{A}_n\underline{A}_{n-1}\ldots\underline{A}_1$.

Hence, from the equations (4) and (5), we can express the $(X_0^{(i)}, Y_0^{(i)}, Z_0^{(i)})$ as linear functions of (x, y, z). These expressions are shown in the Appendix.

Instead of a series of screw motions, the motion of a body can be described in terms of the motion of any one point in the body and the rotation of the body. For example, with time as the independent parameter, if we know the time-derivatives of the displacement vector of a point in the body and also of the rotation vector of the body, we can find the time-derivatives of the position vector of any point in the body.

Considering the motion of σ_j, we let d_j be the position vector of the origin of σ_j, ω_j be the angular velocity of σ_j, x be the vector from the origin of σ_j to a point P_j in σ_j, x_j be the same vector measured in Σ, and X_j be the position vector of P_j relative to the origin of Σ, as shown in Fig. 1. Using dots to denote differentiation with respect to time, we can express the time-derivatives of X_j as follows:[4]

$$
\begin{aligned}
\dot{X}_j &= \dot{d}_j + \dot{x}_j = \dot{d}_j + \omega_j \times x_j \\
\ddot{X}_j &= \ddot{d}_j + \ddot{x}_j = \ddot{d}_j + \dot{\omega}_j \times x_j + \omega_j \times \dot{x}_j \\
\dddot{X}_j &= \dddot{d}_j + \dddot{x}_j = \dddot{d}_j + \ddot{\omega}_j \times x_j + 2\dot{\omega}_j \times \dot{x}_j + \omega_j \times \ddot{x}_j \qquad (6a) \\
\overset{(4)}{X}_j &= \overset{(4)}{d}_j + \overset{(4)}{x}_j = \overset{(4)}{d}_j + \dddot{\omega}_j \times x_j + 3\ddot{\omega}_j \times \dot{x}_j + 3\dot{\omega}_j \times \ddot{x}_j + \omega_j \times \dddot{x}_j \\
\overset{(5)}{X}_j &= \overset{(5)}{d}_j + \overset{(5)}{x}_j = \overset{(5)}{d}_j + \overset{(4)}{\omega}_j \times x_j + 4\dddot{\omega}_j \times \dot{x}_j + 6\ddot{\omega}_j \times \ddot{x}_j + 4\dot{\omega}_j \times \dddot{x}_j + \omega_j \times \overset{(4)}{x}_j, \text{ etc.}
\end{aligned}
$$

[4] $\overset{(i)}{X}_j$ denotes the ith time-derivative of X_j, similarly for any quantity $\overset{(i)}{[\ ]}$ we use $\overset{(i)}{[\ ]} \equiv \dfrac{d^i[\ ]}{dt^i}$.

Since $\overset{(i)}{x}_j$ is linear in x_j, $\overset{(i)}{X}_j$ is also linear in x_j. Therefore we can express $\overset{(i)}{X}_j$ as given in (6a) as follows:

$$\begin{bmatrix} \overset{(i)}{X}_j \\ \overset{(i)}{Y}_j \\ \overset{(i)}{Z}_j \end{bmatrix} = \overset{(i)}{A}_j \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} + \begin{bmatrix} \overset{(i)}{d}_{x_j} \\ \overset{(i)}{d}_{y_j} \\ \overset{(i)}{d}_{z_j} \end{bmatrix}$$

where $\overset{(i)}{A}_j$ is a 3×3 matrix whose elements are functions of the components of the time-derivatives of the rotation of σ_j.

But

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A_j is the 3×3 matrix in (3). Therefore

$$\begin{bmatrix} \overset{(i)}{X}_j \\ \overset{(i)}{Y}_j \\ \overset{(i)}{Z}_j \end{bmatrix} = \overset{(i)}{A}_j A_j \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} \overset{(i)}{d}_{x_j} \\ \overset{(i)}{d}_{y_j} \\ \overset{(i)}{d}_{z_j} \end{bmatrix} \qquad (6b)$$

Equation (6b) explicitly shows $\overset{(i)}{X}_j$ is a linear function of x.

Conditions for Several Positions of a Point in σ to Satisfy the Constraint of a Given Link

Finitely Separated Positions. When the motion of a point P in σ is constrained by a link, the constraint which the link imposes on P can be represented by one or two constraint equations of the form

$$F(X, Y, Z; \; a_1, a_2, \ldots, a_n) = 0 \qquad (7)$$

where (X, Y, Z) are the coordinates in Σ of P, and $a_1, a_2, \ldots, a_n$ are the independent parameters of the constraint. These constraint parameters define the dimensions of the link.

When σ assumes m finitely separated positions $\sigma_1, \sigma_2, \ldots, \sigma_m$, a point P in σ which has its m positions satisfying the constraint described by (7) must satisfy

$$F(X_j, Y_j, Z_j; \; a_1, a_2, \ldots, a_n) = 0 \qquad j = 1, 2, \ldots, m \qquad (8)$$

where (X_j, Y_j, Z_j) are the coordinates in Σ of the jth position of P.

Substituting the linear expressions for X_j, Y_j, and Z_j from (1) into (8), we can write (8) in terms of x, y, and z.

Infinitesimally Separated Positions. At the initial position ($\phi = 0$) we let σ coincide with Σ. A point P in σ whose initial position satisfies the constraint of a given link, equation (7), must satisfy

$$F(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) = 0 \qquad (9)$$

where (X_0, Y_0, Z_0) are the coordinates (in Σ) of the initial position of P. After an infinitesimal displacement of σ, the coordinates of P become $(X_0 + dX_0, Y_0 + dY_0, Z_0 + dZ_0)$. The new position of P will satisfy the constraint $F = 0$ only if

$$F(X_0 + dX_0, \; Y_0 + dY_0, \; Z_0 + dZ_0; \; a_1, a_2, \ldots, a_n) = 0 \qquad (10)$$

which is

$$F(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) + \frac{d}{d\phi}\{F(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n)\}\, d\phi = 0 \qquad (11)$$

If $d\phi \neq 0$, the substitution of (9) yields

$$F'(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) = 0 \qquad (12)$$

where the prime denotes differentiation with respect to ϕ. Similarly, after a second infinitesimal displacement of σ, the new position of P which again satisfies the constraint $F = 0$ must satisfy

$$F(X_0 + dX_0, \; Y_0 + dY_0, \; Z_0 + dZ_0; \; a_1, a_2, \ldots, a_n) + F'(X_0 + dX_0, \; Y_0 + dY_0, \; Z_0 + dZ_0; \; a_1, a_2, \ldots, a_n)d\phi = 0 \qquad (13)$$

which is

$$F(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) + F'(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n)d\phi + F'(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n)d\phi + F''(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n)(d\phi)^2 = 0 \qquad (14)$$

If $d\phi \neq 0$, the substitution of (9) and (12) yields

$$F''(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) = 0 \qquad (15)$$

By induction, we can show that the necessary conditions for all k infinitesimally separated positions of P to satisfy the constraint $F = 0$ are given by

$$F^{(i)}(X_0, Y_0, Z_0; \; a_1, a_2, \ldots, a_n) = 0 \qquad i = 0, 1, 2, \ldots, k - 1 \qquad (16)$$

where $F^{(i)}$ denotes the ith derivative of F with respect to the independent motion parameter ϕ.

After the differentiation is carried out, each $F^{(i)}$ becomes a new function of $X_0^{(j)}$, $Y_0^{(j)}$, and $Z_0^{(j)}$, where j, which denotes the order of the derivative, can be any integer less than or equal to i. Hence (16) can be written in the form

$$F_i(X_0^{(j)}, Y_0^{(j)}, Z_0^{(j)}; \; a_1, a_2, \ldots, a_n) = 0 \qquad i = 0, 1, 2, \ldots, k - 1 \qquad (17)$$

Mixed Finite and Infinitesimal Displacements. In the foregoing we have discussed finite and infinitesimal displacement problems separately. We shall now investigate the possibility of combining the two.

We have shown earlier that if (X_j, Y_j, Z_j) are the coordinates in Σ of P_j (the jth position of a point P in σ), the condition for the point P to have m finitely separated positions satisfying the link constraint (7) is that it must satisfy (8). Furthermore, we have shown that if (X_0, Y_0, Z_0) are the coordinates in Σ of the initial position of P, the condition for P to have its initial and next $k - 1$ infinitesimally separated positions satisfying the same constraint is that it must satisfy (16). Now if we consider P_j as the initial position, the condition for the next k infinitesimally separated positions of P to satisfy (7) is

$$F^{(l)}(X_j, Y_j, Z_j; \; a_1, a_2, \ldots, a_n) = 0 \qquad l = 1, 2, \ldots, k \qquad (18)$$

If the ith position is one of the m finitely separated position considered in (8), the combination of (8) and (18) gives the necessary and sufficient condition for a point P in σ to have $m + k$ positions (m finitely separated and k infinitesimally separated from the ith finitely separated position) satisfying the constraint (7).

From this we can draw a general rule for combined finite and infinitesimal displacements as follows:

If σ assumes $m + k_1 + k_2 + \ldots + k_m$ positions such that m positions are finitely separated and each group of k_j positions are infinitesimally separated from the jth finitely separated position, then the condition for a point P in σ to have all these $m + \sum_{j=1}^{m} k$ positions satisfying (7) is that it must satisfy

$$F^{(l_j)}(X_j, Y_j, Z_j; \; a_1, a_2, \ldots, a_n) = 0; \qquad l_j = 0, 1, 2, \ldots, k_j; \; j = 1, 2, \ldots, m \qquad (19)$$

Maximum Number of Design Positions. The number of design positions, designated by p, is the number of positions of σ which one specifies for the design of a given link. The maximum number of design positions depends on the number of equations N_e and the number of the total unknowns N_u. In order to be compatible, N_e must be less than or equal to N_u. For a given link

$$N_e = p \times N_c \qquad (20)$$

where N_c is the number of constraint equations associated with the given link. The number of unknowns is

$$N_u = n + 3, \qquad (21)$$

where n is the total number of parameters in the constraint equation(s), and the number 3 is equal to the number of coordinates (x, y, z) of the point being constrained. From the compatibility condition we require that

$$N_e \leq N_u$$

or

$$p \times N_c \leq n + 3 \qquad (22)$$

$$p \leq (n + 3)/N_c$$

Therefore, the maximum p is equal to the largest integer contained in $(n + 3)/N_c$.

General Procedure for Determining Locations of Special Points. We have shown that all special points whose different positions satisfy a given link constraint (7) must satisfy (19). In what follows we will discuss the procedure for determining the coordinates (x, y, z) of such special points. After carrying out the differentiation, we can rewrite (19) in the following form:

$$F_{l_j}(X_j^{(l)}, Y_j^{(l)}, Z_j^{(l)}; a_1, a_2, \ldots, a_n) = 0;$$
$$l_j = 0, 1, 2, \ldots, k_j; \; j = 1, 2, \ldots, m; \; i \leq l_j \qquad (23)$$

Using (1) and (6), we can express $(X_j^{(l)}, Y_j^{(l)}, Z_j^{(l)})$ linearly in terms of (x, y, z). Hence (23) can be written as

$$F_r(x, y, z; a_1, a_2, \ldots, a_n) = 0 \qquad r = 1, 2, \ldots, p \qquad (24)$$

where $p = m + \sum_{j=1}^{m} k_j$.

Theoretically, when $N_e \leq N_u$ (here $N_e = p$, $N_u = n + 3$), we can solve (24) for all the unknowns. In the case where $N_e < N_u$ we can arbitrarily specify any $(N_u - N_e)$ unknowns and solve for the rest.

Fortunately, in many cases, the constraint equations are linear in the a's. In such cases we can write (24) in the following form:

$$f_{r1}a_1 + f_{r2}a_2 + \ldots + f_{rn}a_n + f_{r(n+1)} = 0 \qquad r = 1, 2, \ldots, p \qquad (25)$$

where f_{rj} are functions of (x, y, z) and the motion parameters. When $p = n$, given any point (x, y, z) in σ, we can solve (25) for at least one set of a's. This indicates that a generic point in σ has n positions satisfying (7). For $p > n$, the necessary and sufficient condition for a point (x, y, z) in σ to satisfy all p-equations in (25) is that the rank of the augmented matrix of the system (regarding the a's as unknowns) must be equal to or less than n. In other words, the p-equations must be compatible. The compatibility condition requires that the determinants of all the $(n + 1)$ by $(n + 1)$ matrices of the system must vanish. This requirement can be met if all

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & f_{1(n+1)} \\ f_{21} & f_{22} & \cdots & f_{2n} & f_{2(n+1)} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & f_{n(n+1)} \\ f_{(n+k)1} & f_{(n+k)2} & \cdots & f_{(n+k)n} & f_{(n+k)(n+1)} \end{vmatrix} = 0 \qquad (26)$$

$$k = 1, 2, \ldots, p - n$$

and the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} & f_{1(n+1)} \\ f_{21} & f_{22} & \cdots & f_{2n} & f_{2(n+1)} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & f_{n(n+1)} \end{bmatrix} \qquad (27)$$

is n. Points at which the rank of (27) is less than n will be called "residual points."

Since all f_{ij} are functions of (x, y, z), each of the equations in (26) can be expanded into a polynomial in (x, y, z) which describes a surface embedded in σ. This surface which we name the "compatibility surface" will be designated by S_k. It should be noted that all the residual points also satisfy (26). Therefore every S_k contains all the residual points. The surface S_k is the locus of all points whose $(n + k)$th position satisfies the constraint (7) defined by the points' first n positions. However, the residual points distinguish themselves in that their first n positions do not define a unique member of the family $F = 0$. Hence, all points in σ, other than the residual points, that satisfy (26) will have their p positions on the locus given by equation (7). The residual points satisfy

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} & f_{2l} \\ \vdots & & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{n(n-1)} & f_{nl} \end{vmatrix} = 0 \qquad (28)$$

$$l = n, n + 1$$

provided that the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1(n-1)} \\ f_{21} & f_{22} & \cdots & f_{2(n-1)} \\ \vdots & & & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{n(n-1)} \end{bmatrix} \qquad (29)$$

is $(n - 1)$.

The equations in (28) represent two surfaces whose intersection contains the locus of all residual points and a "subresidual curve" which is the locus of points which do not satisfy the rank condition (29). If we call the intersection of the two surfaces in (28) C_n, the residual curve R_n, and the subresidual curve R_{n-1}, then symbolically

$$C_n = R_n + R_{n-1}$$

The next step is to determine R_{n-1}. Proceeding in the same way, we find that

$$R_{n-1} = C_{n-1} - R_{n-2}$$

or

$$C_{n-1} = R_{n-1} + R_{n-2}$$

where C_{n-1} is the intersection of the two surfaces

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(n-2)} & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2(n-2)} & f_{2l} \\ \vdots & & & & \vdots \\ f_{(n-2)1} & f_{(n-2)2} & \cdots & f_{(n-2)(n-2)} & f_{(n-2)l} \\ f_{n1} & f_{n2} & \cdots & f_{n(n-2)} & f_{nl} \end{vmatrix} = 0 \qquad (30)$$

$$l = (n - 1), n$$

and R_{n-2} is the curve at which the rank of

$$\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1(n-2)} \\ f_{21} & f_{22} & \cdots & f_{2(n-2)} \\ \vdots & & & \vdots \\ f_{(n-2)1} & f_{(n-2)2} & \cdots & f_{(n-2)(n-2)} \end{bmatrix} \qquad (31)$$

is less than $(n - 2)$.

The continuation of this procedure leads to the following:

$$C_i = R_i + R_{i-1} \quad i = 2, 3, \ldots, n$$

$$C_1 = R_1$$

where C_i is the intersection of

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2l} \\ \cdot & \cdot & & \cdot \\ f_{(i-1)1} & f_{(i-1)2} & \cdots & f_{(i-1)l} \\ f_{k1} & f_{k2} & \cdots & f_{kl} \end{vmatrix} = 0 \qquad (32)$$

$$l = i, \ldots, i + 1$$

if $(n - i)$ is odd, or the intersection of

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1(i-1)} & f_{1l} \\ f_{21} & f_{22} & \cdots & f_{2(i-1)} & f_{2l} \\ \cdot & \cdot & & & \cdot \\ f_{n1} & f_{n2} & \cdots & f_{n(i-1)} & f_{nl} \end{vmatrix} = 0 \qquad (33)$$

$$l = i, \ldots, i + 1$$

if $(n - i)$ is even. Therefore

$$R_n = \sum_{i=1}^{n} (-1)^{(i+\delta)} C_i$$

$$\delta = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

Knowing the degree of each f_{rk} we can calculate the order of C_i and thereby determine the order of R_{n}. In this context the C's and R's may be given numerical values equal to the orders of the curves.

When $p = n + 1$, the compatibility surface S_1 represents the locus of all points in σ which have $n + 1$ positions satisfying $F = 0$. For $p = n + 2$, the intersection of S_1 and S_2 yields a space curve K_{12} whose order equals the product of the orders of S_1 and S_2. It can easily be shown that the order of K_{12} is greater than that of R_n. Since both S_1 and S_2 have R_n in common, K_{12} must be degenerate and contain R_n and another curve k_{12} which is the locus of all points in σ which have their $n + 2$ positions satisfying $F = 0$. For $p = n + 3$, we have three compatibility surfaces S_1, S_2, and S_3. These surfaces intersect one another in the curve R_n and a finite set of points. Since points on R_n do not satisfy the first $n + 3$ equations of (25), only those points belonging to the finite set have $n + 3$ positions satisfying $F = 0$. The number of such points can be calculated by a method given by Semple and Roth [9]. The same method is shown in reference [11, Appendix 3].

In some cases the constraint equations are nonlinear in the unknown constraint parameters (i.e., the a's); no general method for solving such equations exists. Although elimination techniques and numerical methods may be successfully employed, it does not seem feasible to attempt a general discussion.

Special Lines. Although the foregoing is concerned with the determination of the special points in σ, the same analysis can be applied to problems relating to those special lines in σ which have their multiple positions satisfying the constraint of a given link. In the case of lines the constraint equations can be written in the following form:

$$F(X, Y, Z, L, M, N; a_1, a_2, \ldots, a_n) = 0, \qquad (34)$$

where (X, Y, Z) are the coordinates in Σ of a point on the line and (L, M, N) are the components of a vector (usually of unit length) parallel to the line. The six quantities X, Y, Z, L, M, and N will be called the "coordinates of the line." Since a line has only four degrees of freedom, clearly only four of the six coordinates are independent. For example, we can arbitrarily specify one of the point coordinates (X, Y, Z) and one of the vector components (L, M, N).

Replacing equation (7) by (34), and following the procedure used in the previous sections, we find that equation (19) becomes

$$F^{(t_j)}(X_j, Y_j, Z_j, L_j, M_j, N_j; a_1, a_2, \ldots, a_n) = 0;$$

$$t_j = 0, 1, 2, \ldots, k_j; \quad j = 1, 2, \ldots, m \qquad (35)$$

which, after substituting (1), (3), and (6), can be transformed to

$$F_r(x, y, z, l, m, n; a_1, a_2, \ldots, a_n) = 0 \quad r = 1, 2, \ldots, p \qquad (36)$$

Here (x, y, z, l, m, n) are the coordinates of the line in σ. These coordinates are measured in the σ system which is taken as initially coincident with Σ.

Since the number of independent coordinates of a line is four

$$N_\sigma = n + 4,$$

the maximum number of design positions p is equal to the largest integer contained in $(n + 4)/N_\sigma$.

The procedure for determining lines which satisfy (36) is essentially the same as that for determining points which satisfy (24). However, in the case of lines, the results will be in the form of line loci (i.e., complexes, congruences, reguli, or finite sets of lines which are represented, respectively, by one, two, three, or four equations in x, y, z, l, m, and n).

Conclusion

A general method for solving multiple-position problems in spatial kinematic synthesis has been presented. This method unifies the treatment of finitely and infinitesimally separated positions, and is applicable to the dimensional design of essentially all types of binary links and combined link chains. For each link one must first derive the constraint equation(s). Once the constraint equations are obtained, we can predict the maximum design positions and determine the locations of those special points or lines which define the dimensions of the desired link. The procedure is straightforward, and all computations can be performed on a digital computer. The application of the method to various types of links is discussed in a companion paper [10].

Acknowledgment

The financial support of the National Science Foundation (NSF GK-660) is gratefully acknowledged.

APPENDIX

Expressions for $(X^{(1)}, Y^{(1)}, Z^{(1)})$ in Terms of (x,y,z) and Screw Parameters Governing Motion of σ (see equations (4) and (5))

After performing the algebraic computations in equations (4) and (5), we obtain the following expressions for $(X^{(1)}, Y^{(1)}, Z^{(1)})$:

$$\begin{bmatrix} X^{(1)} \\ Y^{(1)} \\ Z^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} bw - cv + ud^{(1)} \\ cu - aw + vd^{(1)} \\ av - bu + wd^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} X^{(2)} \\ Y^{(2)} \\ Z^{(2)} \end{bmatrix} = \begin{bmatrix} u^2 - 1 & uv - w^{(2)} & uw + v^{(2)} \\ uv + w^{(2)} & v^2 - 1 & vw - u^{(2)} \\ uw - v^{(2)} & vw + u^{(2)} & w^2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} D_{x2} \\ D_{y2} \\ D_{z2} \end{bmatrix}$$

where

$$D_{x2} = -a(u^2 - 1) - buv - cuw + [bw - cv + ud^{(1)}]^{(1)}$$

$$D_{y2} = -auv - b(v^2 - 1) - cvw + [cu - aw + vd^{(1)}]^{(1)}$$

$$D_{z2} = -auw - bvw - c(w^2 - 1) + [av - bu + wd^{(1)}]^{(1)}$$

$$\begin{bmatrix} X^{(1)} \\ Y^{(1)} \\ Z^{(1)} \end{bmatrix} = \begin{bmatrix} -vv^{(1)} - ww^{(1)} + [u^2 - 1]^{(1)} & w + uv^{(1)} + [uv - w^{(1)}]^{(1)} & -v + uw^{(1)} + [uw + v^{(1)}]^{(1)} \\ -w + vu^{(1)} + [uv + w^{(1)}]^{(1)} & -uu^{(1)} - ww^{(1)} + [v^2 - 1]^{(1)} & u + vw^{(1)} + [(vw - u^{(1)}]^{(1)} \\ v + wu^{(1)} + [uw - v^{(1)}]^{(1)} & -u + wv^{(1)} + [vw - u^{(1)}]^{(1)} & -uu^{(1)} - vv^{(1)} + [w^2 - 1]^{(1)} \end{bmatrix} + \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

where

$$D_x = a(vv^{(1)} + ww^{(1)}) - b(w + uv^{(1)}) - c(-v + uw^{(1)}) + (uu^{(1)} - vv^{(1)})\theta^{(1)} + D_x{}^{(1)}$$

$$D_y = -a(-w + vu^{(1)}) + b(uu^{(1)} + ww^{(1)}) - c(u + vw^{(1)}) + (uw^{(1)} - wu^{(1)})\theta^{(1)} + D_y{}^{(1)}$$

$$D_z = -a(v + wu^{(1)}) - b(-u + wv^{(1)}) + c(uu^{(1)} + vv^{(1)}) + (vu^{(1)} - uv^{(1)})\theta^{(1)} + D_z{}^{(1)}$$

The algebraic manipulations were performed on a digital computer using a program called "REDUCE" [12]. Higher derivatives can be obtained in a similar manner.

References

1 Schoenflies, A., *Geometrie der Bewegung in Synthetischer Darstellung*, Leipzig, 1886.

2 Roth, B., "The Kinematics of Motion Through Finitely Separated Positions," *Journal of Applied Mechanics*, Vol. 34, No. 3, Trans. ASME, Series E, Vol. 89, Sept. 1967, pp. 591–598.

3 Roth, B., "Finite Position Theory Applied to Mechanism Synthesis," *Journal of Applied Mechanics*, Vol. 34, No. 3, Trans. ASME, Vol. 89, Sept. 1967, pp. 599–605.

4 Roth, B., "On the Screw Axis and Other Special Lines Associated With Spatial Displacements of a Rigid Body," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 89, No. 1, 1967, pp. 102–110.

5 Suh, C. H., "Synthesis and Analysis of Space Mechanisms With Use of the Displacement Matrix," dissertation, University of California, Berkeley, 1966.

6 Wilson, J. T., III, "Analytical Kinematic Synthesis by Finite Displacements," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 87, No. 2, 1965, pp. 161–169.

7 Disteli, M., "Über das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes," *Zeitschrift für Mathematik und Physik*, Vol. 62, 1914, pp. 261–309.

8 Skreiner, M., "A Study of the Geometry and the Kinematics of Instantaneous Spatial Motion," *Journal of Mechanisms*, Vol. 1, No. 2, 1966, pp. 115–143.

9 Semple, J. G., and Roth, L., *Introduction to Algebraic Geometry*, Oxford University Press, England, 1949.

10 Chen, P., and Roth, B., "Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains." Paper no. 68-mech-8.

11 Chen, P., "An Analytical Investigation of Infinitesimal Spatial Motion Theory and Its Application to Three-Dimensional Linkages," dissertation, Stanford University, 1968.

12 Hearn, A. C., "Reduce User's Manual," Memo No. 50, Stanford Artificial Intelligence Project, Feb. 1967.

13 Hartenberg, R. S., and Denavit, J., *Kinematic Synthesis of Linkages*, McGraw-Hill, New York, 1964.

14 Sandor, G. N., "Principles of a General Quaternion-Operator Method of Spatial Kinematic Synthesis," *Journal of Applied Mechanics*, Vol. 35, No. 1, Trans. ASME, Vol. 90, Series E, Mar. 1968, pp. 40–46.

Printed in U. S. A.

Paper No.
68—Mech-8

P. CHEN
Assistant Agricultural Engineer,
University of California,
Davis, Calif.

B. ROTH
Associate Professor,
Department of Mechanical Engineering,
Stanford University,
Stanford, Calif. Mem. ASME

Design Equations for the Finitely and Infinitesimally Separated Position Synthesis of Binary Links and Combined Link Chains

A general method for synthesizing spatial linkages (derived in a previous paper [1][1]) is applied to the design of various types of binary links and combined link chains. Equations governing those special points and lines which define the link dimensions are derived. Types and order of loci of such special points and lines and the maximum number of design positions for each link are given. Numerical method for computing these special points and lines are discussed, and a numerical example is presented.

Introduction

In spatial kinematic synthesis problems it is often necessary to determine the loci of those special points and lines which have multiple positions satisfying the constraints of certain links and link chains. In this paper we use a method described elsewhere [1] to determine such loci for various types of links and link chains. (Since the Nomenclature and methods used in this paper are as described in [1], the reader is advised to first familiarize himself with that work.)

The moving and fixed bodies are represented by a moving Cartesian coordinate system σ and a fixed system Σ, respectively. The coordinates of a point in σ are given by (x, y, z) and by (X, Y, Z) when measured in Σ. In what follows we consider the moving body, σ, in p general multiple positions. These p positions consist of $m + k_1 + k_2 + \ldots + k_m$ positions $\left(p = m + \sum_{j=1}^{m} k_j\right)$ such that m positions are finitely separated and each group of k_j positions are infinitesimally separated from the jth finitely separated position. If all the positions of σ are finitely separated, $k_j = 0$ for all j. On the other hand, when all the positions are infinitesimally separated, $m = 1$.

If $P(x, y, z)$ is a point in σ, the coordinates in Σ of P_j (the jth finitely separated position of P) will be denoted by (X_j, Y_j, Z_j), and the ith derivative of the position vector of P_j will be denoted by $(X_j^{(i)}, Y_j^{(i)}, Z_j^{(i)})$.

Application to Link Design

Sphere-Sphere Binary Link. A sphere-sphere binary link, shown in Fig. 1, constrains the moving pivot P in σ to remain on a sphere whose center is at the fixed pivot C. The constraint equation is

$$(X - X_c)^2 + (Y - Y_c)^2 + (Z - Z_c)^2 - R^2 = 0 \qquad (1)$$

where (X, Y, Z) and (X_c, Y_c, Z_c) are, respectively, the coordinates of P and C in Σ; R is the distance from P to C. Equation (1) can be written in the form

$$X^2 + Y^2 + Z^2 + a_1X + a_2Y + a_3Z + a_4 = 0 \qquad (2)$$

where

$$a_1 = -2X_c, \quad a_2 = -2Y_c, \quad a_3 = -2Z_c$$
$$\text{and} \quad a_4 = X_c^2 + Y_c^2 + Z_c^2 - R^2.$$

According to [1], the condition for a point $P(x, y, z)$ in σ to have p multiple positions on the sphere (2) is that it must satisfy

$$(X_j^2 + Y_j^2 + Z_j^2)^{(l_j)} + a_1X_j^{(l_j)} + a_2Y_j^{(l_j)} + a_3Z_j^{(l_j)} + a_4^{(l_j)} = 0$$
$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$
$$p = m + \sum_{j=1}^{m} k_j \qquad (3)$$

where (X_j, Y_j, Z_j) are the coordinates in Σ of the jth finitely separated position of P and the superscript (l_j) denotes the l_jth derivative with respect to the reference parameter of the motion. For $l_j = 0$ we have

$$X_j^2 + Y_j^2 + Z_j^2 + a_1X_j + a_2Y_j + a_3Z_j + a_4 = 0$$
$$j = 1, 2, \ldots, m \qquad (4)$$

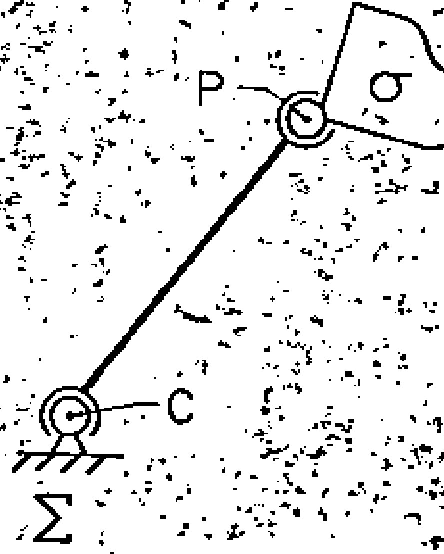


Fig. 1 Sphere-sphere binary link

[1] Numbers in brackets designate References at end of paper.

Contributed by the Design Engineering Division for presentation at the Mechanisms Conference, Atlanta, Ga., October 6-9, 1968, of The American Society of Mechanical Engineers. Manuscript received by the ASME Mechanisms Division, June 18, 1966. Paper No. 68—Mech-8.

Subtracting the first equation ($j = 1$) from the others, we get

$$r_j^2 - r_1^2 + a_1(X_j - X_1) + a_2(Y_j - Y_1) + a_3(Z_j - Z_1) = 0 \qquad j = 2, 3, \ldots, m \qquad (5)$$

where $r_j^2 = X_j^2 + Y_j^2 + Z_j^2$.

For $l_j > 0$ we have

$$(r_j^2)^{(l_j)} + a_1X_j^{(l_j)} + a_2Y_j^{(l_j)} + a_3Z_j^{(l_j)} = 0 \qquad (6)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

It has been shown [1] that X_j, Y_j, Z_j, $X_j^{(l_j)}$, $Y_j^{(l_j)}$, and $Z_j^{(l_j)}$ are linear in x, y, and z (the coordinates of P in σ), and it can be shown (Appendix 1 and 2 of [2]) that $(r_j^2 - r_1^2)$ and $(r_j^2)^{(l_j)}$ are also linear in x, y, and z.

Therefore both (5) and (6) can be written as

$$f_{r1}a_1 + f_{r2}a_2 + f_{r3}a_3 + f_{r4} = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (7)$$

where f_{ri} are linear functions of x, y, and z. Equations (7) will be referred to as design equations.

For four positions ($p = 4$) we have three equations with the three unknowns a_1, a_2, and a_3. Therefore, given any point in σ, we can solve the three equations for the three unknowns and thereby determine the coordinates (X_a, Y_a, Z_a) of the center of the sphere. The radius of the sphere can be calculated from the first equation of (4).

For p greater than four, the number of equations is greater than the number of unknowns. The compatibility condition requires that the rank of the system be equal to the number of the unknowns, which is three. This requirement can be met if

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{k1} & f_{k2} & f_{k3} & f_{k4} \end{vmatrix} = 0, \qquad (8)$$

$$k = 4, 5, \ldots, p - 1$$

and the rank of

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} \qquad (9)$$

is three.

Since f_{ri} are linear in x, y, and z, each equation of (8) represents a fourth-order surface embedded in σ. This surface will be referred to as E_{123k}^4.

The points at which the rank of (9) is less than three have been called [1] the residual points. Following the procedure given in [1], we find that these points lie on a residual curve k^6 which is a sixth-order space curve. As will be shown later, k^6 is the locus of all points which have the first four positions on a circle.

Therefore, according to [1], E_{1234}^4 is the locus of all points which have five multiple positions satisfying the constraint of a sphere-sphere link. The locus of all points which have six positions on a sphere is the tenth-order space curve k^{10} which is the intersection of E_{1234}^4 and E_{1235}^4 excluding k^6. The intersection of E_{1234}^4, E_{1235}^4, and E_{1236}^4 yields k^6 and 20 discrete points which have their seven positions on a sphere. (The number of the discrete points can be calculated from equations given in Appendix 3 of [2].)

The foregoing is an example showing the detailed procedure for analyzing a multiple-position design problem of a sphere-sphere link. For other types of links the procedures are essentially the same as the foregoing. Therefore, for the remaining links and link chains, we will only derive the design equations and present the results in a tabulated form.

Slider-Slider-Sphere Dyad. A slider-slider-sphere dyad, as shown in Fig. 2, constrains the moving pivot P to move on a plane which is parallel to the sliding directions of the two sliders. The constraint equation is given by

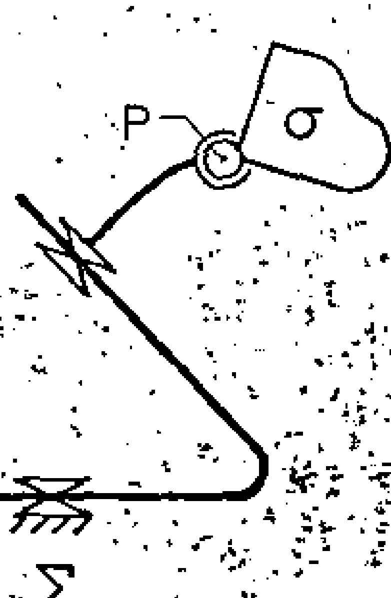


Fig. 2 Slider-slider-sphere dyad

$$(X - X_0)L + (Y - Y_0)M + (Z - Z_0)N = 0 \qquad (10)$$

where (L, M, N) are the direction cosines of the normal to the plane, and (X_0, Y_0, Z_0) the coordinates of a point on the plane. Equation (10) can be written as

$$X + b_1Y + b_2Z + b_3 = 0 \qquad (11)$$

where

$$b_1 = \frac{M}{L}, \quad b_2 = \frac{N}{L}, \quad \text{and} \quad b_3 = -(X_0 + Y_0b_1 + Z_0b_2).$$

The design equations for p multiple positions are

$$X_j^{(l_j)} + Y_j^{(l_j)}b_1 + Z_j^{(l_j)}b_2 + b_3^{(l_j)} = 0 \qquad (12)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

which can be reduced to

$$f_{r1} + f_{r2}b_1 + f_{r3}b_2 = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (13)$$

where f_{ri} are as defined in (7).

Slider-Sphere Binary Link. A slider-sphere binary link constrains point P (Fig. 3) to move on a straight line parallel to the sliding axis. The constraint equations are

$$X - \frac{L}{N}Z - X_0 = 0$$

$$Y - \frac{M}{N}Z - Y_0 = 0 \qquad (14)$$

where (L, M, N) are the direction cosines of the line and (X_0, Y_0) are the coordinates of the point where the line intersects the XY-plane. The design equations can be written

$$X_j^{(l_j)} + a_1Z_j^{(l_j)} + a_2^{(l_j)} = 0$$

$$Y_j^{(l_j)} + b_1Z_j^{(l_j)} + b_2^{(l_j)} = 0 \qquad (15)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

where

$$a_1 = -\frac{L}{N}, \quad a_2 = -X_0, \quad b_1 = -\frac{M}{N}, \quad \text{and} \quad b_2 = -Y_0$$

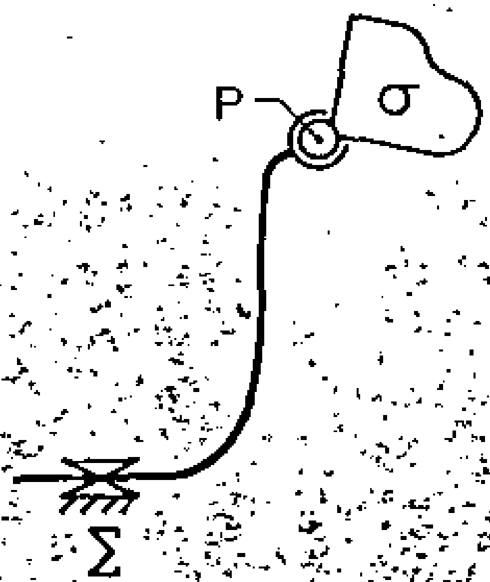


Fig. 3 Slider-sphere binary link

Equations (15) can be reduced to

$$
\begin{aligned}
f_{r4} + f_{r5}a_4 &= 0 \\
f_{r6} + f_{r7}b_3 &= 0 \\
r &= 1, 2, \ldots, p - 1
\end{aligned}
\tag{16}
$$

Revolute-Sphere Binary Link. A revolute-sphere binary link has the same constraint on the moving pivot P in σ as does a sphere-sphere-slider-slider four bar (Fig. 4). The constraint locus is a circle which may be determined as the intersection of a sphere

$$X^2 + Y^2 + Z^2 + a_1X + a_2Y + a_3Z + a_4 = 0 \tag{17}$$

and a plane

$$X + b_1Y + b_2Z + b_3 = 0 \tag{18}$$

where the a's and b's are as defined in (2) and (11).

Equations (17) and (18) are identical to (2) and (11), respectively; consequently, the design equations for p multiple positions are

$$
\begin{aligned}
f_{r1}a_1 + f_{r2}a_2 + f_{r3}a_3 + f_{r4} &= 0 \\
f_{r5} + f_{r6}b_1 + f_{r7}b_2 &= 0 \\
r &= 1, 2, \ldots, p - 1
\end{aligned}
\tag{19}
$$

Revolute-Slider-Sphere Dyad. A revolute-slider-sphere dyad, shown in Fig. 3, constrains a point P in σ to move on a one-sheet hyperboloid of revolution whose axis is coincident with that of the revolute joint, and whose generator through P is parallel to the sliding direction. The constraint equation for point P is

$$
\begin{aligned}
&(X - a)^2 + (Y - b)^2 + (Z - c)^2 \\
&\quad - [l(X - a) + m(Y - b) + n(Z - c)]^2 \left[1 + \frac{\alpha^2}{\beta^2}\right] \\
&\quad - \{k^2 - 2k[l(X - a) + m(Y - b) + n(Z - c)]\} \frac{\alpha^2}{\beta^2} = \alpha^2
\end{aligned}
\tag{20}
$$

where (X, Y, Z) are the coordinates of P in Σ; (l, m, n) are the direction cosines of the hyperboloid axis; (a, b, c) are the coordinates of a point A on this axis; k is the distance from A to the center of the hyperboloid; α and $\frac{\alpha}{\beta}$ are, respectively, the distance and the tangent of the angle between the axis and a generator.

If we select A such that

$$al + bm + cn = 0, \tag{21}$$

then (20) can be reduced to

$$
\begin{aligned}
&aX + bY + cZ - k(lX + mY + nZ)\frac{\alpha^2}{\beta^2} \\
&\quad - \frac{1}{2}(X^2 + Y^2 + Z^2) + \frac{1}{2}(lX + mY + nZ)^2\left(1 + \frac{\alpha^2}{\beta^2}\right) \\
&\quad = \frac{1}{2}\left[a^2 + b^2 + c^2 - \left(k\frac{\alpha}{\beta}\right)^2\right] - \frac{\alpha^2}{2}
\end{aligned}
\tag{22}
$$

Hence, the design equations for p multiple positions are

$$
\begin{aligned}
&aX_j^{(l_j)} + bY_j^{(l_j)} + cZ_j^{(l_j)} \\
&\quad - k(lX_j + mY_j + nZ_j)^{(l_j)}\frac{\alpha^2}{\beta^2} - \frac{1}{2}(X_j^2 + Y_j^2 + Z_j^2)^{(l_j)} \\
&\quad + \frac{1}{2}[(lX_j + mY_j + nZ_j)^2]^{(l_j)}\left(1 + \frac{\alpha^2}{\beta^2}\right) \\
&\quad = \frac{1}{2}\left[a^2 + b^2 + c^2 - \left(k\frac{\alpha}{\beta}\right)^2\right]^{(l_j)} - \left(\frac{\alpha^2}{2}\right)^{(l_j)} \\
&j = 1, 2, \ldots, m \\
&l_j = 0, 1, \ldots, k_j
\end{aligned}
\tag{23}
$$

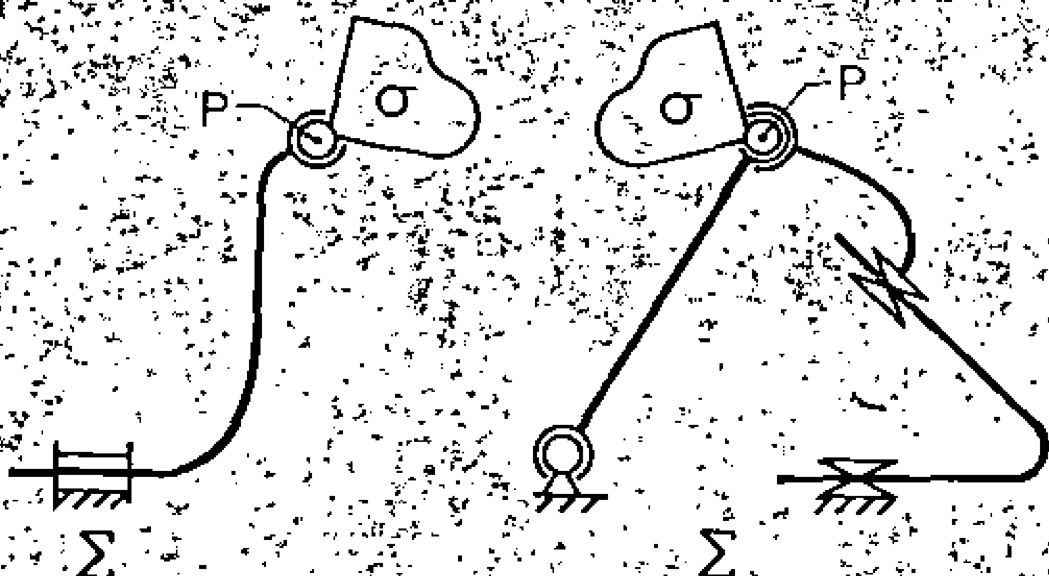


Fig. 4(a) Fig. 4(b)

Fig. 4 Revolute-sphere binary link and sphere-sphere-slider-slider four bar

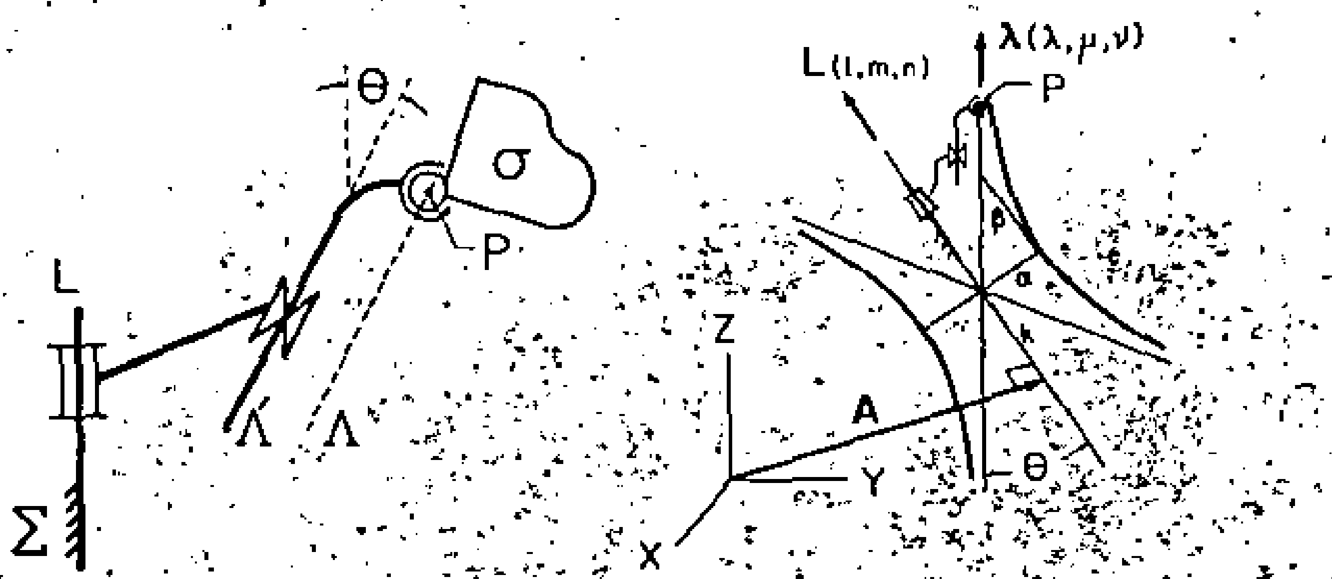


Fig. 5 Revolute-slider-sphere dyad

Equations (23) can be transformed into

$$f_{r1}a + f_{r2}b + f_{r3}c + s_r k\frac{\alpha^2}{\beta^2} + t_r^2\left(1 + \frac{\alpha^2}{\beta^2}\right) - \frac{1}{2}f_{r4} = 0 \quad r = 1, 2, \ldots, p-1 \quad (24)$$

where

$$s_r = \begin{cases} -(lX_j + mY_j + nZ_j) \times (lX_1 + mY_1 + nZ_1) & \text{if } l_j = 0 \\ -(lX_j^{(l_j)} + mY_j^{(l_j)} + nZ_j^{(l_j)}) & \text{if } l_j > 0 \end{cases}$$

$$t_r^2 = \begin{cases} \frac{1}{2}[(lX_j + mY_j + nZ_j)^2 - (lX_1 + mY_1 + nZ_1)^2] & \text{if } l_j = 0 \\ \frac{1}{2}[(lX_j + mY_j + nZ_j)^2]^{(l_j)} & \text{if } l_j > 0 \end{cases}$$

It can be shown that s_r and t_r^2 are, respectively, linear and quadratic in x, y, and z.

Since equations (24) are nonlinear in the parameters, the methods of [1] are not directly applicable. However, if we specify (l, m, n) and k $\left(\text{or } \frac{\alpha^2}{\beta^2}\right)$, equations (24) are linear in the remaining parameters. Thus we can use the method shown in [1] to determine the loci of points in σ which have several positions on a hyperboloid with a specified axis direction and k $\left(\text{or } \frac{\alpha^2}{\beta^2}\right)$. Once such points are known, the direction cosines of the sliding direction (λ, μ, ν), which are not contained in (24), can be determined from the following:

(a) The cosine of the angle θ is

$$l\lambda + m\mu + n\nu = \frac{1}{\left[1 + \left(\frac{\alpha}{\beta}\right)^2\right]^{1/2}} \quad (25)$$

(b) The distance a is

$$a = \frac{1}{\sin\theta}[l(Y\nu - Z\mu) + m(Z\lambda - X\nu) + n(X\mu - Y\lambda) + \lambda(bn - cm) + \mu(cl - an) + \nu(am - bl)]$$

and hence

$$\lambda[m(Z - c) - n(Y - b)] + \mu[n(X - a) - l(Z - c)] + \nu[l(Y - b) - m(X - a)] = \frac{a}{\left[\left(\frac{\beta}{\alpha}\right)^2 + 1\right]^{1/2}} \quad (26)$$

(c)

$$\lambda^2 + \mu^2 + \nu^2 = 1 \quad (27)$$

Equations (25), (26), and (27) yield two sets of (λ, μ, ν). This is due to the fact that there are always two generators passing through a point on a hyperboloid.

Cylinder-Sphere Binary Link. A cylinder-sphere binary link (Fig 6) is equivalent to the special case of a revolute-slider-sphere dyad where the angle between L and Λ' is zero $\left(\frac{\alpha}{\beta} = 0\right)$; see Fig. 5. In this case the constraint equation (22) becomes

$$aX + bY + cZ - \frac{1}{2}[(X^2 + Y^2 + Z^2) - (lX + mY + nZ)^2 + (a^2 + b^2 + c^2)] = -\frac{\alpha^2}{2} \quad (28)$$

and the design equations (24) reduce to

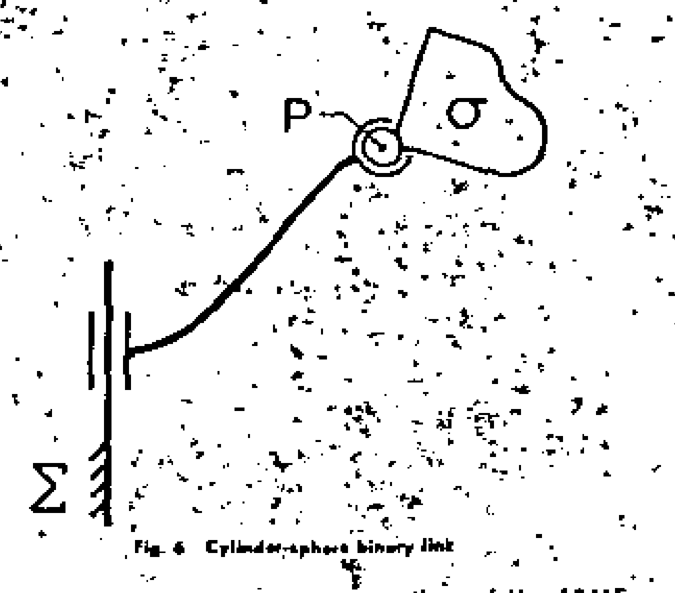


Fig. 6 Cylinder-sphere binary link

$$f_{r1}a + f_{r2}b + f_{r3}c + l_r^2 - \frac{1}{2} f_{r4} = 0 \qquad r = 1, 2, \ldots, p - 1 \quad (29)$$

Cylinder-Cylinder Binary Link. A cylinder-cylinder binary link (Fig. 7) constrains a line L in σ such that the angle θ and the distance D between L and the fixed axis Λ are constant. If we let $\mathbf{L}(l, m, n)$ and $\boldsymbol{\lambda}(\lambda, \mu, \nu)$ be two unit vectors parallel to L and Λ, respectively, $\mathbf{A}(a, b, c)$ and $\boldsymbol{\alpha}(\alpha, \beta, \gamma)$ be, respectively, the position vectors of a point on L and a point on Λ, the constraint equations, in vector notation, for a cylinder-cylinder link are

$$\boldsymbol{\lambda} \cdot \mathbf{L} = \cos\theta = \text{const} \quad (30)$$

$$(\boldsymbol{\lambda} \times \mathbf{L}) \cdot (\mathbf{A} - \boldsymbol{\alpha}) = D \sin\theta = \text{const} \quad (31)$$

A line in σ which has p multiple positions satisfying the constraint of a cylinder-cylinder link must satisfy

$$(\boldsymbol{\lambda} \cdot \mathbf{L}_j)^{(l_j)} = \begin{cases} \text{const} & \text{if } l_j = 0 \\ 0 & \text{if } l_j > 0 \end{cases} \quad (32)$$

$$[(\boldsymbol{\lambda} \times \mathbf{L}_j) \cdot (\mathbf{A}_j - \boldsymbol{\alpha})]^{(l_j)} = \begin{cases} \text{const} & \text{if } l_j = 0 \\ 0 & \text{if } l_j > 0 \end{cases} \quad (33)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 0, 1, \ldots, k_j$$

Thus the design equations for p multiple positions can be written as

$$\boldsymbol{\lambda} \cdot (\mathbf{L}_j - \mathbf{L}_1) = 0 \qquad j = 2, 3, \ldots, m \quad (34a)$$

$$(\boldsymbol{\lambda} \cdot \mathbf{L}_j)^{(l_j)} = 0 \quad (34b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

$$(\boldsymbol{\lambda} \times \mathbf{L}_j) \cdot (\mathbf{A}_j - \boldsymbol{\alpha}) - (\boldsymbol{\lambda} \times \mathbf{L}_1) \cdot (\mathbf{A}_1 - \boldsymbol{\alpha}) = 0$$
$$j = 2, 3, \ldots, m \quad (35a)$$

$$[(\boldsymbol{\lambda} \times \mathbf{L}_j) \cdot (\mathbf{A}_j - \boldsymbol{\alpha})]^{(l_j)} = 0 \quad (35b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

Equations (35) are linear in $\mathbf{A}_j^{(l_j)}$ and $\boldsymbol{\alpha}$. Since $\mathbf{A}_j^{(l_j)}$ is linear in a_1, b_1, and c_1, equations (35) are linear in $a_1, b_1, c_1, \alpha, \beta$, and γ.

Cylinder-Revolute Binary Link. A cylinder-revolute link (Fig. 8) is simply a cylinder-cylinder link with an additional constraint of no sliding along the moving axis. This additional constraint is governed by the equation

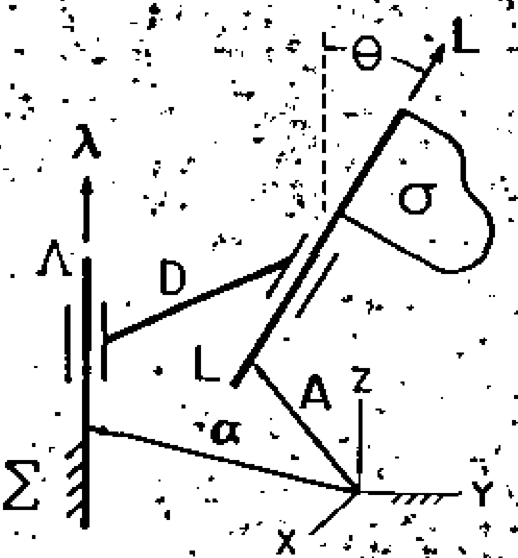


Fig. 7 Cylinder-cylinder binary link

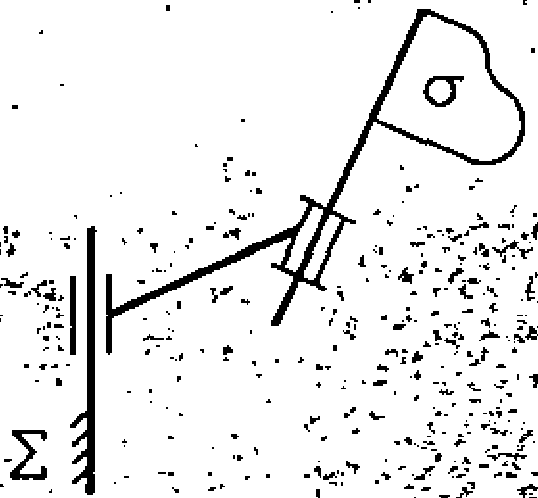


Fig. 8 Cylinder-revolute binary link

$$[(\mathbf{A} - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}] \cdot (\mathbf{L} \times \boldsymbol{\lambda}) = \text{const}$$

Therefore the design equations for p multiple positions are (34a), (34b), (35a), (35b),

$$[(\mathbf{A}_j - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}] \cdot (\mathbf{L}_j \times \boldsymbol{\lambda})$$
$$- [(\mathbf{A}_1 - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}] \cdot (\mathbf{L}_1 \times \boldsymbol{\lambda}) = 0 \qquad j = 2, 3, \ldots, m \quad (36a)$$

and

$$[[(\mathbf{A}_j - \boldsymbol{\alpha}) \times \boldsymbol{\lambda}] \cdot (\mathbf{L}_j \times \boldsymbol{\lambda})]^{(l_j)} = 0 \quad (36b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

Revolute-Revolute Binary Link. A revolute-revolute binary link (Fig. 9) is another special case of a cylinder-cylinder link. Such a link does not allow sliding on both the moving and fixed axes. The constraint equations for this condition are given by

$$(\mathbf{A} - \boldsymbol{\alpha}) \cdot \boldsymbol{\lambda} = \text{const} \quad (37)$$

and

$$(\mathbf{A} - \boldsymbol{\alpha}) \cdot \mathbf{L} = \text{const} \quad (38)$$

Hence the design equations are (34a), (34b), (35a), (35b),

$$(\mathbf{A}_j - \mathbf{A}_1) \cdot \boldsymbol{\lambda} = 0, \qquad j = 2, 3, \ldots, m \quad (39a)$$

$$\mathbf{A}_j^{(l_j)} \cdot \boldsymbol{\lambda} = 0, \quad (39b)$$

$$j = 1, 2, \ldots, m$$
$$l_j = 1, 2, \ldots, k_j$$

$$(\mathbf{A}_j \cdot \mathbf{L}_j - \mathbf{A}_1 \cdot \mathbf{L}_1) - \boldsymbol{\alpha} \cdot (\mathbf{L}_j - \mathbf{L}_1) = 0, \qquad j = 2, 3, \ldots, m \quad (40a)$$

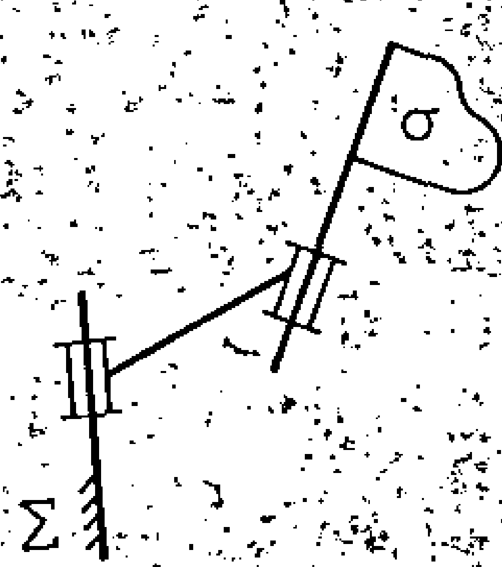


Fig. 9 Revolute-revolute binary link

Table 1

Link or dyad	No. of design positions	Locus (or number) of points that satisfy link constraint	Symbol for locus
Slider-slider-sphere	3	All points	
	4	3rd-order surface	(F^3)
	5	6th-order curve	(c^6)
	6	10 points	
Slider-sphere	2	All points	
	3	3rd-order curve	(i^3)
Sphere-sphere	4	All points	
	5	4th-order surface	(E^4)
	6	10th-order curve	(k^{10})
	7	20 points	
Revolute-sphere	3	All points	
	4	6th-order curve	(h^6)
Revolute-slider-sphere (with l,m,n, and k specified)	4	All points	
	5	5th-order surface	(H^5)
	6	16th-order curve	(h^{16})
	7	42 points	
Revolute-slider-sphere (with l,m,n, and α/β specified	4	All points	
	5	5th-order surface	(HH^5)
	6	16th-order curve	(hh^{16})
	7	42 points	
Cylinder-sphere (with l,m, and n specified)	3	All points	
	4	4th-order surface	(H^4)
	5	11th-order curve	(h^{11})
	6	26 points	
Cylinder-cylinder	3	All lines	
	4	Line congruence (which contains all lines which are parallel to the generators of a cubic cone and pass through an infinite number of straight lines each of which corresponds to a generator of the cone)	
	5	6 lines	
Cylinder-revolute (with l,m, and n specified)	3	One unique line	
Revolute-revolute	3	24 lines	

and

$$[(A_j - a)\cdot L_j]^{(l_j)} = 0 \qquad (40b)$$

$$j = 1, 2, \ldots, m$$

$$l_j = 1, 2, \ldots, k_j$$

Results. Starting with the design equations derived for each link, we follow the procedure illustrated in the foregoing discussion of the sphere-sphere link. The results obtained are found in Table 1. (Detailed discussions of these derivations are given in [2].)

Inversion. In Fig. 1–Fig. 9 we have chosen the inversion which leads to the simplest design equations. The design of a chain which is the kinematic inversion of any of these can be accomplished by either kinematic inversion, which requires inverting the motion and interchanging the fixed and moving bodies, or by the direct application of the general method, which requires the derivation of new design equations. The following is an example illustrating the derivation of design equations for an inverted chain.

We consider the inversion of a revolute-slider-sphere dyad; namely, a sphere-slider-revolute dyad as shown in Fig. 10. This dyad constrains a line L in σ such that the line always satisfies the equation

$$(X_c - a)^2 + (Y_c - b)^2 + (Z_c - c)^2$$
$$- [l(X_c - a) + m(Y_c - b) + n(Z_c - c)]^2 \left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- [k^2 - 2k[l(X_c - a) + m(Y_c - b) + n(Z_c - c)]] \frac{\alpha^2}{\beta^2} = \alpha^2 \qquad (41)$$

where (X_c, Y_c, Z_c) are the coordinates of the center of the spherical joint P_c; (l, m, n) are the direction cosines of line L; (a, b, c) are the coordinates of point A on L; Λ is a line passing through P_c parallel to the sliding axis Λ'; α is the shortest distance from Λ to L; $\frac{\alpha}{\beta}$ is the tangent of θ, the angle between L and Λ; k is the distance from A to A' (A' is the point on L at which the common normal from Λ terminates).

The design equations for p multiple positions are

$$[(X_c - a_j)^2 + (Y_c - b_j)^2 + (Z_c - c_j)^2]^{(l_j)}$$
$$- \{[l_j(X_c - a_j) + m_j(Y_c - b_j) + n_j(Z_c - c_j)]^2\}^{(l_j)} \left(1 + \frac{\alpha^2}{\beta^2}\right)$$
$$- \{k^2 - 2k[l_j(X_c - a_j) + m_j(Y_c - b_j)$$
$$+ n_j(Z_c - c_j)]^{(l_j)}\} \frac{\alpha^2}{\beta^2} = (\alpha^2)^{(l_j)} \qquad (42)$$

$$j = 1, 2, \ldots, m$$

$$l_j = 0, 1, \ldots, k_j$$

For $l_j = 0$, the subtraction of the first equation from all the others of (42) yields

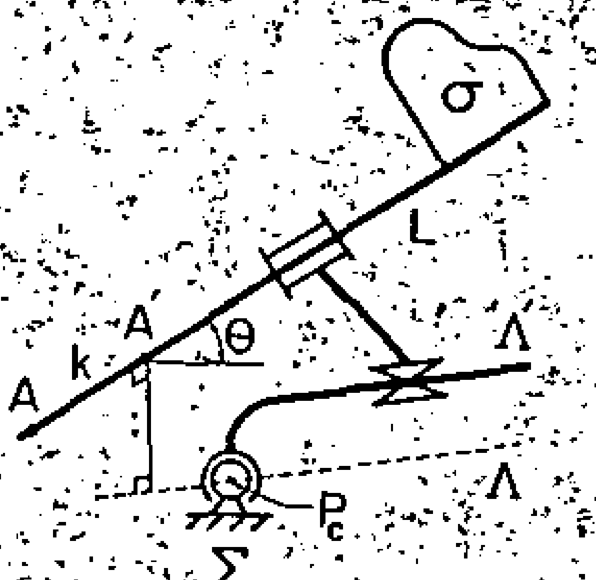


Fig. 10 Sphere-slider-revolute dyad

$$
\begin{aligned}
&-2[X_s(a_j - a_1) + Y_s(b_j - b_1) + Z_s(c_j - c_1)] \\
&\quad + (a_j^2 + b_j^2 + c_j^2) - (a_1^2 + b_1^2 + c_1^2) \\
&\quad - \{X_s(l_j - l_1) + Y_s(m_j - m_1) + Z_s(n_j - n_1) \\
&\quad - (l_j a_j + m_j b_j + n_j c_j) + (l_1 a_1 + m_1 b_1 + n_1 c_1)\}\{X_s(l_j + l_1) \\
&\quad + Y_s(m_j + m_1) + Z_s(n_j + n_1) - (l_j a_j + m_j b_j + n_j c_j) \\
&\quad - (l_1 a_1 + m_1 b_1 + n_1 c_1)\} \left(1 + \frac{\alpha^2}{\beta^2}\right) + 2k[l_j(X_s - a_j) \\
&\quad + m_j(Y_s - b_j) + n_j(Z_s - c_j) - l_1(X_s - a_1) - m_1(Y_s - b_1) \\
&\quad - n_1(Z_s - c_1)] \frac{\alpha^2}{\beta^2} = 0 \qquad j = 2, 3, \ldots, m \qquad (43)
\end{aligned}
$$

But, as shown in Appendix 1 of [2], $(l_j a_j + m_j b_j + n_j c_j)$ is equal to $(C_{Lj} + l_1 a_1 + m_1 b_1 + n_1 c_1)$, where C_{Lj} is a linear function of (l, m, n). If we choose point A such that

$$l_1 a_1 + m_1 b_1 + n_1 c_1 = 0$$

then (43) becomes

$$
\begin{aligned}
&X_s(a_j - a_1) + Y_s(b_j - b_1) + Z_s(c_j - c_1) - \frac{1}{2}(r_{Aj}^2 - r_{A1}^2) \\
&\quad + \frac{1}{2}[X_s(l_j - l_1) + Y_s(m_j - m_1) + Z_s(n_j - n_1) \\
&\quad - C_{Lj}][X_s(l_j + l_1) + Y_s(m_j + m_1) \\
&\quad + Z_s(n_j + n_1) - C_{Lj}] \left(1 + \frac{\alpha^2}{\beta^2}\right) - k[X_s(l_j - l_1) \\
&\quad + Y_s(m_j - m_1) + Z_s(n_j - n_1) - C_{Lj}] \frac{\alpha^2}{\beta^2} = 0 \qquad (44) \\
&\qquad j = 2, 3, \ldots, m
\end{aligned}
$$

where $\quad r_{Aj}^2 = a_j^2 + b_j^2 + c_j^2$.

For $l_j > 0$, equations (42) can be written as

$$
\begin{aligned}
&(r_{Aj}^2)^{(l_j)} - 2(a_j X_s + b_j Y_s + c_j Z_s)^{(l_j)} \\
&\quad - \{[(l_j X_s + m_j Y_s + n_j Z_s) \\
&\quad - (l_j a_j + m_j b_j + n_j c_j)]^2\}^{(l_j)} \left(1 + \frac{\alpha^2}{\beta^2}\right) \\
&\quad + 2k[l_j(X_s - a_j) + m_j(Y_s - b_j) \\
&\quad + n_j(Z_s - c_j)]^{(l_j)} \frac{\alpha^2}{\beta^2} = 0 \qquad (45) \\
&\qquad j = 1, 2, \ldots, m \\
&\qquad l_j = 1, 2, \ldots, k_j
\end{aligned}
$$

Since a_j, b_j, c_j, r_{Aj}^2, $a_j^{(l_j)}$, $b_j^{(l_j)}$, $c_j^{(l_j)}$, and $(r_{Aj}^2)^{(l_j)}$ are linear in a_1, b_1, and c_1, we can write (44) and (45) as:

$$
\begin{aligned}
&f_{r1} a_1 + f_{r2} b_1 + f_{r3} c_1 + g_r k \frac{\alpha^2}{\beta^2} + t_r^2 \left(1 + \frac{\alpha^2}{\beta^2}\right) \\
&\qquad - \frac{1}{2} f_{r4} = 0 \qquad r = 1, 2, \ldots, p - 1 \qquad (46)
\end{aligned}
$$

where f_{r1}, f_{r2}, f_{r3}, and f_{r4} are linear functions of X_s, Y_s, and Z_s.

$$
g_r = \begin{cases} -[X_s(l_j - l_1) + Y_s(m_j - m_1) + Z_s(n_j - n_1) - C_{Lj}] & \text{if } l_j = 0 \\ -[l_j X_s + m_j Y_s + n_j Z_s - (l_j a_j + m_j b_j + n_j c_j)]^{(l_j)} & \text{if } l_j > 0 \end{cases}
$$

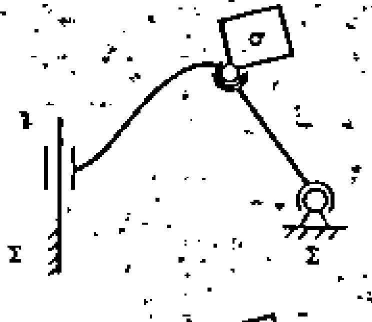
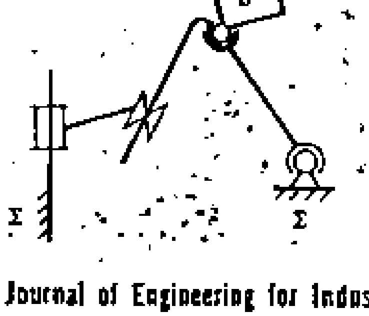
$$
t_r^2 = \begin{cases} \frac{1}{2}[X_s(l_j - l_1) + Y_s(m_j - m_1) + Z_s(n_j - n_1) - C_{Lj}] \\ \quad \times [X_s(l_j + l_1) + Y_s(m_j + m_1) + Z_s(n_j + n_1) - C_{Lj}] & \text{if } l_j = 0 \\ \frac{1}{2}\{[(l_j X_s + m_j Y_s + n_j Z_s) - (l_j a_j + m_j b_j + n_j c_j)]^2\}^{(l_j)} & \text{if } l_j > 0 \end{cases}
$$

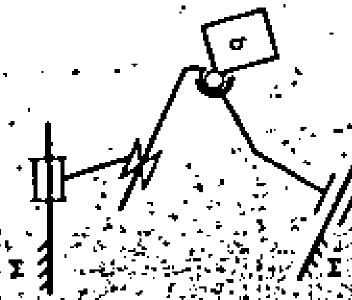
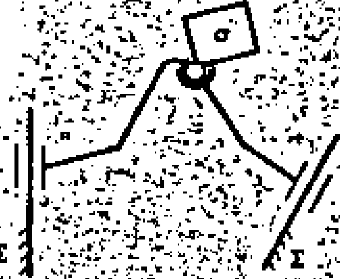
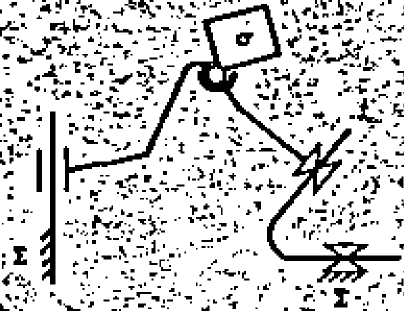
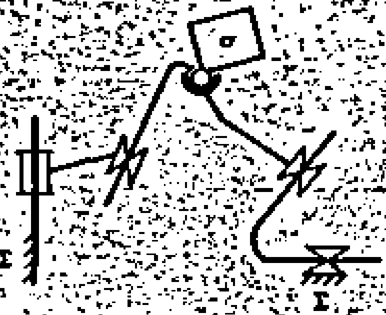
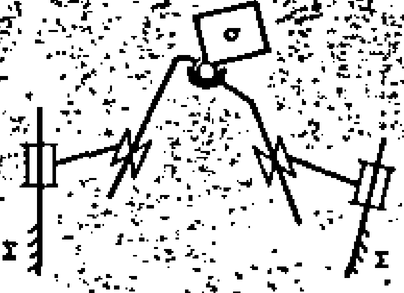
It can be shown (Appendix 1 and 2 of [2]) that $(l_j a_j + m_j b_j + n_j c_j)$ and $(l_j a_j + m_j b_j + n_j c_j)^{(l_j)}$ are independent of a_1, b_1, and c_1; therefore g_r and t_r^2 are independent of a_1, b_1, and c_1 and are, respectively, linear and quadratic in X_s, Y_s, and Z_s.

Equation (46) is exactly of the same form as (24). The functions f_{ri}, g_r, and t_r^2 are in terms of the coördinates of the pivot which in this case is fixed to Σ. Therefore, if we specify (l_1, m_1, n_1) and k $\left(\text{or } \frac{\alpha}{\beta}\right)$ and follow the procedure given in [1], we will obtain results similar to those for a revolute-slider-sphere dyad probem. However, in this case the final equations represent loci which are embedded in Σ instead of in σ.

Design equations for other inverted chains are given in [2].

Combined Link Chains. Many links discussed in the foregoing are links which constrain a point P in σ to move on a special surface. By combining any two such links so that both links share a common pivot at P, we can construct a link chain which constrains P so that it can only move along the intersection of the two surfaces associated with the two links. The following is a list of such combined link chains:

Linkage	Link combination	Geometrical locus of point P
	Cylinder-sphere link and sphere-sphere link (C-S-S three-bar)	Intersection of a sphere and a cylinder
	Revolute-slider-sphere dyad and sphere-sphere link (R-P-S-S- four-bar)	Intersection of a sphere and a hyperboloid

	Revolute-slider-sphere dyad and cylinder sphere link (*R-P-S-C* four-bar)	Intersection of a cylinder and a hyperboloid
	Two cylinder-sphere links (*C-S-C* three-bar)	Intersection of two cylinders with different axis inclinations
	Cylinder-sphere link and slider-slider-sphere dyad (*C-S-P-P* four-bar)	Intersection of a cylinder and a plane (an ellipse)
	Revolute-slider-sphere dyad and slider-slider-sphere dyad (*R-P-S-P-P* five bar)	Intersection of a plane and a hyperboloid
	Two revolute-slider-sphere dyads (*R-P-S-P-R* five bar)	Intersection of two hyperboloids

Points in σ that have n positions satisfying the constraint of such a linkage must satisfy the constraints of both chains which form the linkage. These points can be determined by "intersecting" the locus of points which have n positions satisfying the constraint of the first chain with that of the second chain. For example, the locus of all points in σ that have four multiple position on an ellipse, defined by the intersection of a plane and a cylinder with specified axis inclination, may be found by intersecting F^3 with H^4 which will give a twelfth-order space curve. Since there are six unknown parameters in the equations of a plane and a cylinder with specified axis inclination, according to equation (22) in [1], there are no points in σ which have five positions on such an ellipse.

In the case of a *C-S-S* three-bar, the procedure is slightly different. For four positions only points on H^4 have four pos on a cylinder with specified axis inclination, but every point in σ has four positions on a sphere. The locus of all points that have four positions on the intersection of a sphere and a cylinder is, therefore, identical to H^4.

For five positions the locus of five-position cylinder points is an eleventh-order space curve h^{11}, and the locus of five-position sphere points is a fourth-order surface F^4. The intersection of h^{11}

and E^4 yields 44 points which have five positions satisfying the constraint of a C-S-S three-bar.

Following the foregoing procedure, one can easily determine points in σ which have multiple positions satisfying the constraints of the remaining combined linkages. All such results can be found in Table 2.

Table 2

Linkage	No. of positions	Locus (or number) of points that satisfy linkage constraint
C-S-S three-bar	4	Fourth-order surface
	5	44 points
R-P-S-S four-bar	4	All points in σ
	5	Twentieth-order space curve
R-P-S-C four-bar	4	Fourth-order surface
	5	55 points
C-S-C three-bar	3	All points in σ
	4	Sixteenth-order space curve
C-S-P-P four-bar	3	All points in σ
	4	Twelfth-order space curve
R-P-S-P-P five-bar	4	Third-order surface
	5	30 points
R-P-S-P-R five bar	4	All points in σ
	5	Twenty-fifth-order space curve

Numerical Procedure for Computing Special Points and Lines. The equations which govern the loci of the special points or lines are obtained from the compatibility conditions which generally are in the form of determinants. Theoretically, it is possible to expand these determinants and obtain explicit expressions for all the coefficients in terms of the specified motion parameters. The expansion of these determinants, however, requires prodigious amounts of algebra which would result in impossibly lengthy expressions. Therefore, no attempt has been made to expand these equations algebraically. Instead, with the aid of a digital computer, both the coefficients and the "solutions" of these equations are determined numerically. The following is a general description of the computational procedure:

Starting with a set of specified motion parameters, we compute the elements of the determinant which represents the desired equation. In the case of finitely separated positions these elements can easily be obtained from the linear relation shown in equation (1) of [1]. In the case of infinitesimal displacements, if the motion is described by series of consecutive infinitesimal screw displacements, the elements can be computed from the expressions shown in equation (4) of [1]. On the other hand, when the motion is described by kinematic parameters, the elements are computed from the equations (6a) of [1].

Knowing all the elements, we can proceed to expand the determinant. First, we arrange the elements such that all like variables lie in the same subcolumns. For example, the elements in the equation of the cubic cone (51) can be arranged into the following form:

$$\begin{vmatrix} a_{11}l + b_{11}m + c_{11}n & a_{12}l + b_{12}m + c_{12}n & a_{13}l + b_{13}m + c_{13}n \\ a_{21}l + b_{21}m + c_{21}n & a_{22}l + b_{22}m + c_{22}n & a_{23}l + b_{23}m + c_{23}n \\ a_{31}l + b_{31}m + c_{31}n & a_{32}l + b_{32}m + c_{32}n & a_{33}l + b_{33}m + c_{33}n \end{vmatrix}$$

where (l, m, n) are the variables and the a's, b's, and c's are known coefficients. The foregoing determinant can be expanded into a series of three-by-three subdeterminants with the same variable in each column. An example of such a subdeterminant is

$$\begin{vmatrix} a_{11}l & b_{12}m & b_{13}m \\ a_{21}l & b_{22}m & b_{23}m \\ a_{31}l & b_{32}m & b_{33}m \end{vmatrix}$$

from which lm^2 may be factored:

$$\begin{vmatrix} a_{11} & b_{12} & b_{13} \\ a_{21} & b_{22} & b_{23} \\ a_{31} & b_{32} & b_{33} \end{vmatrix} lm^2$$

After evaluating all such subdeterminants the final result is obtained simply by summing all coefficients of like terms. Having the equations, the next step is to determine their solutions. The most general case involves three polynomials in three unknowns. Theoretically, we can always eliminate one unknown from two independent equations and obtain a resulting equation known as the eliminant [3]. Therefore, with three equations, we can first eliminate one unknown and obtain two eliminants in the two remaining unknowns. Then we can eliminate a second unknown between the two eliminants to obtain a single polynomial, the roots of which can be determined numerically. By back substituting each of these roots into the previous equations the corresponding two other unknowns can be determined. This process of elimination, however, is not practical for equations of high degree because each step of the elimination introduces extraneous roots and increases the degree of the equations. If the equations are all of the same degree in the three unknowns then each elimination increases the degree by the square. For instance, the elimination of two unknowns from three third-degree equations would result in an 81st degree single-unknown polynomial. For this reason the roots of high-degree equations are determined by (Newton's method of) iteration. It is desirable to use the method of elimination whenever possible since it gives all solutions whereas iteration does not.

Numerical Example. The following is an illustrative example in which, for a given motion, we determine the lines which have five infinitesimally separated positions satisfying the constraint of a cylinder-cylinder crank. From (34b) and (35b), the design equations for five infinitesimally separated positions ($m = 1$, $k_1 = 4$) are

$$(\lambda \cdot L_1)^{(k)} = 0 \qquad (47)$$

and

$$[(\lambda \times L_1) \cdot (A_1 - a)]^{(k)} = 0 \qquad k = 1, 2, 3, 4 \qquad (48)$$

In scalar form (47) and (48) can be written as[5]

$$l^{(i)}\lambda + m^{(i)}\mu + n^{(i)}\nu = 0 \qquad (49)$$

and

$$l^{(i)}\rho + m^{(i)}\sigma + n^{(i)}\tau + \lambda R^{(i)} + \mu S^{(i)} + \nu T^{(i)} = 0 \qquad (50)$$

$$i = 1, 2, 3, 4$$

where (ρ, σ, τ) and $(R^{(i)}, S^{(i)}, T^{(i)})$ are the components of $a \times \lambda$ and $(A \times L)^{(i)}$, respectively.

The compatibility condition for (49) yields

$$\begin{vmatrix} l^{(1)} & m^{(1)} & n^{(1)} \\ l^{(2)} & m^{(2)} & n^{(2)} \\ l^{(k)} & m^{(k)} & n^{(k)} \end{vmatrix} = 0 \qquad (51)$$

$$k = 3, 4$$

providing that the rank of

$$\begin{bmatrix} l^{(1)} & m^{(1)} & n^{(1)} \\ l^{(2)} & m^{(2)} & n^{(2)} \end{bmatrix} \qquad (52)$$

is two.

The four equations (50) are linear in a, b, c, α, β, and γ, because (ρ, σ, τ) are, by definition, linear in (α, β, γ), and $(R^{(i)}, S^{(i)}, T^{(i)})$ are linear in (a, b, c). By choosing A and a such that

$$al + bm + cn = 0 \qquad (53)$$

and

$$\alpha\lambda + \beta\mu + \gamma\nu = 0 \qquad (54)$$

we obtain a total of six linear equations in a, b, c, α, β, and γ.

Since $(l^{(i)}, m^{(i)}, n^{(i)})$ are linear and homogeneous in (l, m, n),

[5] For convenience the subscript "1" is omitted.

the two equations (51) represent two cubic cones having a common apex (the origin). These two cubic cones intersect each other in (a maximum of) nine lines, six of which represent the directions of the, at most, six possible moving axes. The other three are spurious since they correspond to the lines at which the rank of (52) is less than two. Further investigation shows that these three lines are coincident (a triple intersection, see Fig. 12) and are parallel to the instantaneous screw axis. Corresponding to each of the moving axis directions, we can determine a unique cylinder-cylinder crank by first solving any two equations of (49) for the fixed axis direction (λ, μ, ν) and then solving (50), (53), and (54) for the moving and fixed axis positions (a, b, c) and (α, β, γ), respectively.

Table 3 Specified motion of the moving body σ (five infinitesimally separated positions)

	X-component	Y-component	Z-component
$\mathbf{d}$	−4.01920	−7.52880	−12.21129
$\dot{\mathbf{d}}$	71.43568	−47.92848	1.58135
$\ddot{\mathbf{d}}$	79.23863	94.10794	476.1699
$\dddot{\mathbf{d}}$	−2202.585	1527.748	192.3612
ω	0.00000	0.00000	8.67932
$\dot{\omega}$	−31.68956	20.51287	−0.85769
$\ddot{\omega}$	−16.00582	−39.98313	−164.1540
$\dddot{\omega}$	641.3854	−409.7535	67.80389

where $\mathbf{d}$ is the velocity of the origin of σ and ω the angular velocity of σ (see Fig. 1 in [1]).

For the motion given in Table 3, the equations of the two cubic cones (51) are for $k = 3$:

$$(l^2 + m^2)(l - 0.63971m) - 0.43510n(l^2 - m^2) - 0.39053lmn = 0 \quad (55)$$

for $k = 4$:

$$0.34675l^3 + 0.56442m^3 + 0.62204l^2m - 0.88638l^2n + 0.72127m^2l + m^2n + 0.20591n^2l + 0.31810n^2m - 0.35901lmn = 0 \quad (56)$$

The real intersections of the foregoing two cones are:

Intersection	l	m	n	
1	0.00000	0.00000	1.00000	trivial
2	0.00000	0.00000	1.00000	trivial
3	0.00000	0.00000	1.00000	trivial
4	0.27990	0.57608	0.76798	
5	0.27652	0.41478	0.86689	

The moving and fixed axes corresponding to the two nontrivia (the 4th and 5th) intersections are:

Direction

Moving axis l	m	n	Fixed axis λ	μ	ν
0.27991	0.57608	0.76798	−0.33935	−0.69843	0.63011
0.27652	0.41478	0.86689	−0.53727	−0.80591	0.24868

Location

Moving axis a	b	c	Fixed axis α	β	γ
−0.32512	−0.05782	0.16187	1.9963	−0.79623	0.19233
0.00000	0.00000	0.00000	1.6410	−1.0385	0.18008

After replacing l, m, and n by x, y, and z, respectively, equations (55) and (56) represent two cubic cones in the moving system σ.

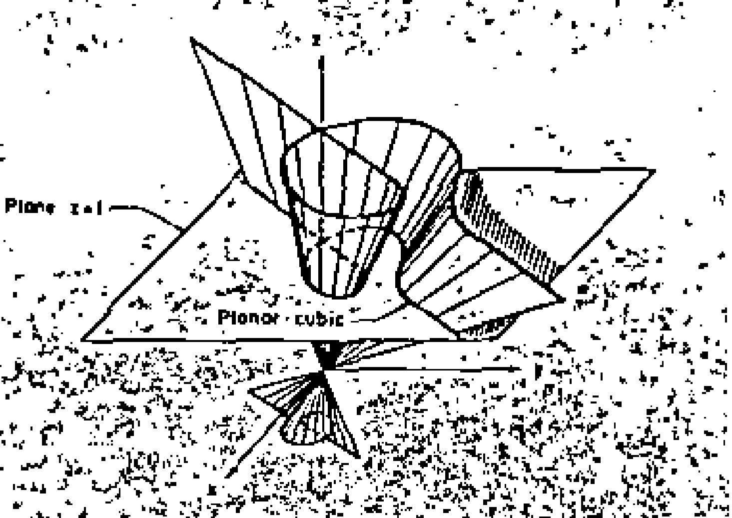


Fig. 11 First cubic cone

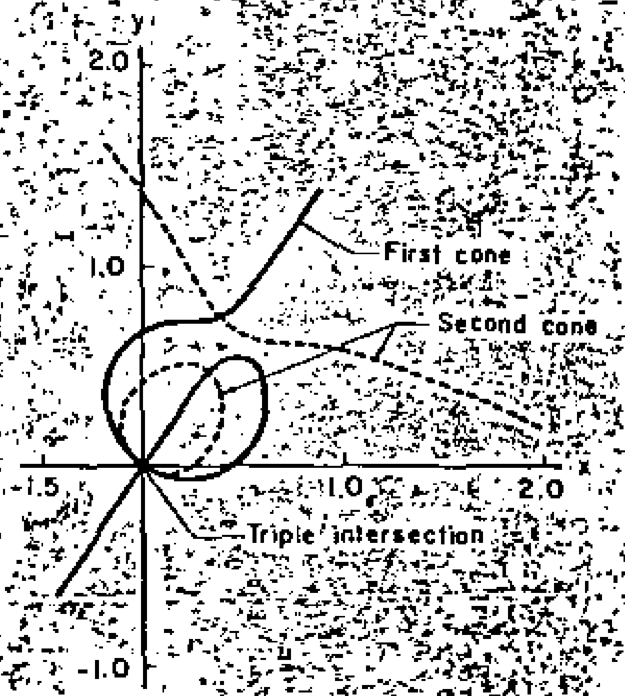


Fig. 12 Intersection of two cubic cones and plane $z = 1$

The first cubic cone, equation (55), is shown in Fig. 11. The intersection of the two cones and the plane $z = 1$ is shown in Fig. 12.

Summary

Design equations have been presented for binary links with all possible combinations of spherical, revolute, and cylindric pairs. The only nontrivial binary link with a prismatic pair, the slider-sphere link, was also treated. In addition, the slider-slider-sphere and revolute-slider-sphere dyads which one obtains by adding a second link to the prismatic pair of a "trivial" binary link combination were considered.

The effect of combining binary links and dyads into closed chains was considered, and the question of kinematic inversion was dealt with.

The particular illustrative numerical example was chosen because of its kinship to the classical Burmester-point problem of instantaneous planar kinematics. In direct analogy to previous finite position work [4], it is pointed out that in the case of planar motions the cones given by equations (55) and (56) become right cubic cylinders, and the curve labeled "first cone" in Fig 12 becomes the cubic of stationary curvature while the two intersections (other than the triple point) become the Burmester points. Most important, it should be understood that the cylindric-cylindric link axes for the spherical motion problem (defined in Table 3 provided $\dot{d}$ and all its higher derivatives are set to zero) have exactly the same directions as for the spatial problem. For spherical motion we let the pair axes pass through the origin (i.e., set $a = b = c = \alpha = \beta = \gamma = 0$); in which case the cylindric joints may be replaced by revolutes. Figs. 11 and 12 are identical for all spherical and spatial problems which have the same angular motions.

The application of the results of this paper to problems of kinematic synthesis has been previously treated [5]. Although [5] deals only with finite displacements, the ideas are equally valid for synthesis problems involving infinitesimally and mixed finitely and infinitesimally separated positions.

Acknowledgment

The financial support of the National Science Foundation (NSF GK-660) is gratefully acknowledged.

References

1 Chen, P., and Roth, B., "A Unified Theory for the Finitely and Infinitesimally Separated Position Problems of Kinematic Synthesis," Paper No. 68—Mech-7, to be published in Trans. ASME.

2 Chen, P., "An Analytical Investigation of Infinitesimal Spatial Motion Theory and Its Application to Three-Dimensional Linkages," dissertation, Stanford University, 1968.

3 Muir, T., *A Treatise on the Theory of Determinants*, Dover Publications, New York, 1960.

4 Roth, B., "On the Screw Axes and Other Special Lines Associated With Spatial Displacements of a Rigid Body," Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 89, No. 1, Feb. 1967, pp. 102–110.

5 Roth, B., "Finite-Position Theory Applied to Mechanism Synthesis," *Journal of Applied Mechanics*, Vol. 34, No. 3, Trans. ASME, Vol. 89, Series E, Sept. 1967, pp. 599–605.

Printed in U. S. A.

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y LOS MECANISMOS

CLASIFICACION DE LOS MECANISMOS

Dr. Jacques Marie Hervé

Junio 1981.

Palacio de Minería Calle de Tacuba 5 primer piso México 1, D. F. Tel: 521-40-20 Apdo. Postal M-2285

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

A N E X O S

Dr. Jorge Angeles Alvarez

Junio 1981

Palacio de Minería Calle de Tacuba 5 primer piso México 1, D. F. Tel: 521-40-20 Apdo. Postal M-2285

NOTACION:

$\underset{\sim}{x}, \underset{\sim}{r}$: vector de dimensión 3

$\underset{\sim}{A}$: matriz de 3 x 3

x_i, r_i: iª componente de $\underset{\sim}{x}$, de $\underset{\sim}{r}$

a_{ij}: elemento i,j de $\underset{\sim}{A}$

Así:

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \underset{\sim}{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Tr}\, \underset{\sim}{A} \equiv a_{ii} = a_{11} + a_{22} + a_{33}$$

$$\det \underset{\sim}{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$\underset{\sim}{A}^{-1}$: inversa de $\underset{\sim}{A}$, si $\det \underset{\sim}{A} \neq 0$

$$\underset{\sim}{A}\, \underset{\sim}{A}^{-1} = \underset{\sim}{A}^{-1} \underset{\sim}{A} = \underset{\sim}{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underset{\sim}{A}^T \equiv$ transpuesta de $\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$\underset{\sim}{A}$ es ortogonal si: $\underset{\sim}{A}\,\underset{\sim}{A}^T = \underset{\sim}{A}^T\underset{\sim}{A} = \underset{\sim}{I}$

$|\det \underset{\sim}{A}| = 1$

$\det \underset{\sim}{A} = \begin{cases} +1 \Rightarrow \underset{\sim}{A} \text{ es ortogonal propia (rotaciones)} \\ -1 \Rightarrow \underset{\sim}{A} \text{ es ortogonal impropia (reflexiones)} \end{cases}$

Si $\underset{\sim}{A}$ y $\underset{\sim}{B}$ son o.p., $\underset{\sim}{A}\underset{\sim}{B}$ también es o.p.

Si $\underset{\sim}{A}$ y $\underset{\sim}{B}$ son o.i., $\underset{\sim}{A}\underset{\sim}{B}$ es o.p.

<u>Valores y vectores característicos de $\underset{\sim}{A}$</u>

Dados un vector $\underset{\sim}{x}$ y una matriz $\underset{\sim}{A}$, $\underset{\sim}{A}\underset{\sim}{x}$ es la <u>imagen de $\underset{\sim}{x}$ bajo $\underset{\sim}{A}$</u>. En general $\underset{\sim}{x}$ y $\underset{\sim}{A}\underset{\sim}{x}$ no son paralelos; pero si lo son, $\underset{\sim}{x}$ es un vector característico de $\underset{\sim}{A}$. Normalizando $\underset{\sim}{x}$, lo llamamos $\underset{\sim}{e}$, esto es:

Def.: $\|\underset{\sim}{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ = magnitud de $\underset{\sim}{x}$

Entonces: $\underset{\sim}{e} = \frac{\underset{\sim}{x}}{\|\underset{\sim}{x}\|}$

y

$$\underset{\sim}{A}\,\underset{\sim}{e} = \lambda \underset{\sim}{e}, \quad \underset{\sim}{e} \neq \underset{\sim}{0}$$

$$\Rightarrow (\underset{\sim}{A} - \lambda \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{0} \Rightarrow \det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0,$$

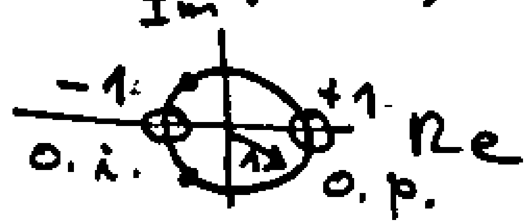
Que es la <u>ecuación característica</u> de $\underset{\sim}{A}$. Es un polinomio de grado 3 en λ:

$$P(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3$$

Tiene 3 raíces que son los valores λ_1, λ_2 y λ_3, llamados <u>valores característicos de A</u>. A cada valor característico λ_i corresponde <u>por lo menos</u> un vector característico $\underset{\sim}{e}_i$*

Si $\underset{\sim}{A} = \underset{\sim}{A}^T$, λ_1, λ_2 y λ_3 son reales y $\underset{\sim}{e}_1, \underset{\sim}{e}_2$ y $\underset{\sim}{e}_3$ son ortogonales

Si $\underset{\sim}{A}$ es ortogonal, $|\lambda_i| = 1$



---

* En todo caso $\underset{\sim}{A}$ no puede tener más de 3 vectores característico linealmente independientes. (V r p. 4)

Toda matriz ortogonal, $\underset{\sim}{Q}$, tiene por lo menos un valor característico de valor absoluto igual a 1.

Vale +1 si $\underset{\sim}{Q}$ es o.p.

Vale -1 si $\underset{\sim}{Q}$ es o.i.

Independencia lineal. Un conjunto de vectores $\mathcal{C} = \{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$ es l.i. si la combinación lineal

$$c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + \dots + c_n \underset{\sim}{x}_n$$

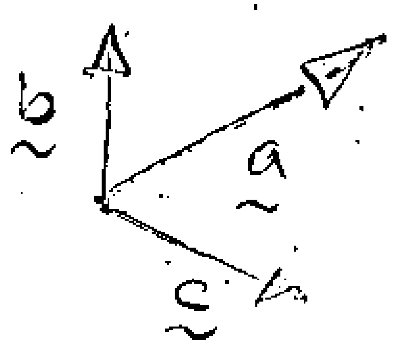
se anula si y sólo si $c_1 = c_2 = \dots = c_n = 0$.

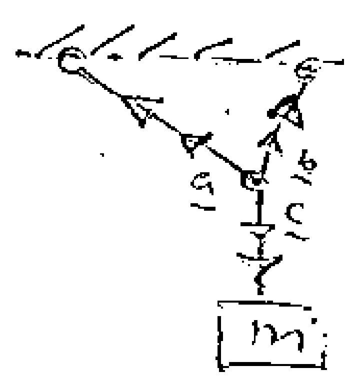
Si $\mathcal{C}$ no es l.i., es l.d.

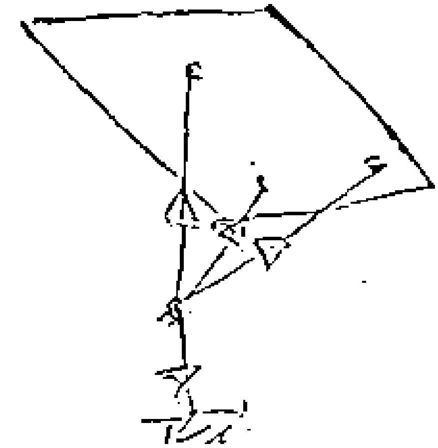
En E^3 no puede haber conjuntos de más de 3 vectores l.i.

$$\alpha \underset{\sim}{a} + \beta \underset{\sim}{b} = \underset{\sim}{0}$$

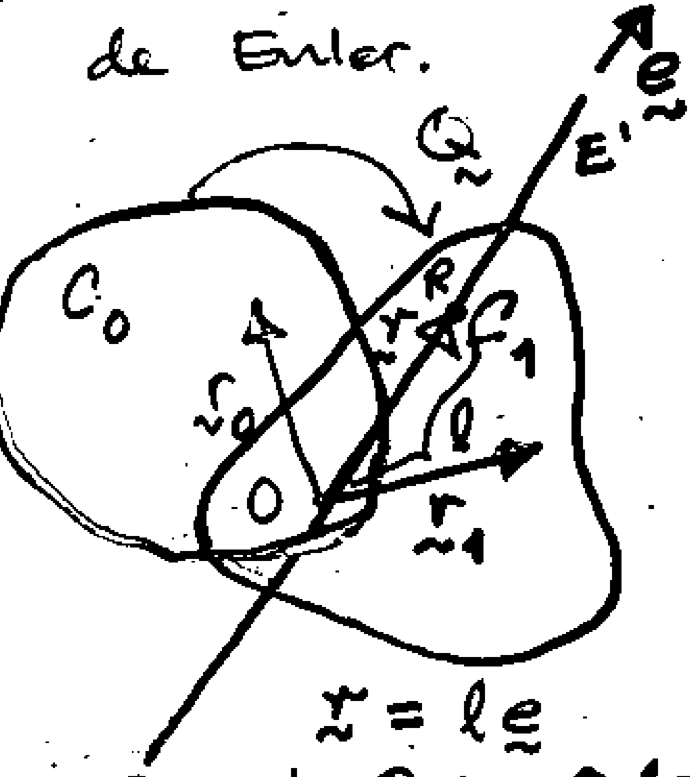
$$\alpha \underset{\sim}{a} + \beta \underset{\sim}{b} + \gamma \underset{\sim}{c} = \underset{\sim}{0}$$







1.1 Rotación de cuerpo rígido. Teorema de Euler.



$$r_1 = Q\, r_0$$

$$Q\, Q^T = I$$

$$Q\, e = e$$

$$r = l\, e$$

$$r' = Q\, r = Q\, l e = l\, Q e = l\, e = r$$

Teorema de Euler: Si un cuerpo rígido sufre un movimiento que lo lleve de una configuración C_0 a una configuración C_1, de manera un punto del cuerpo, O, permanece fijo, entonces existe un conjunto de puntos alojados sobre una línea EE' que pasa por O y es paralela a e, el vector característico real de Q, que corresponde al valor característico $\lambda = 1$.

6

rotation about point O, whose existence is guarenteed by Euler's Theorem. Moreover, let θ be the corresponding angle of rotation, as indicated in Fig 2.3.4, and $\underline{e}$ a unit vector parallel to L.

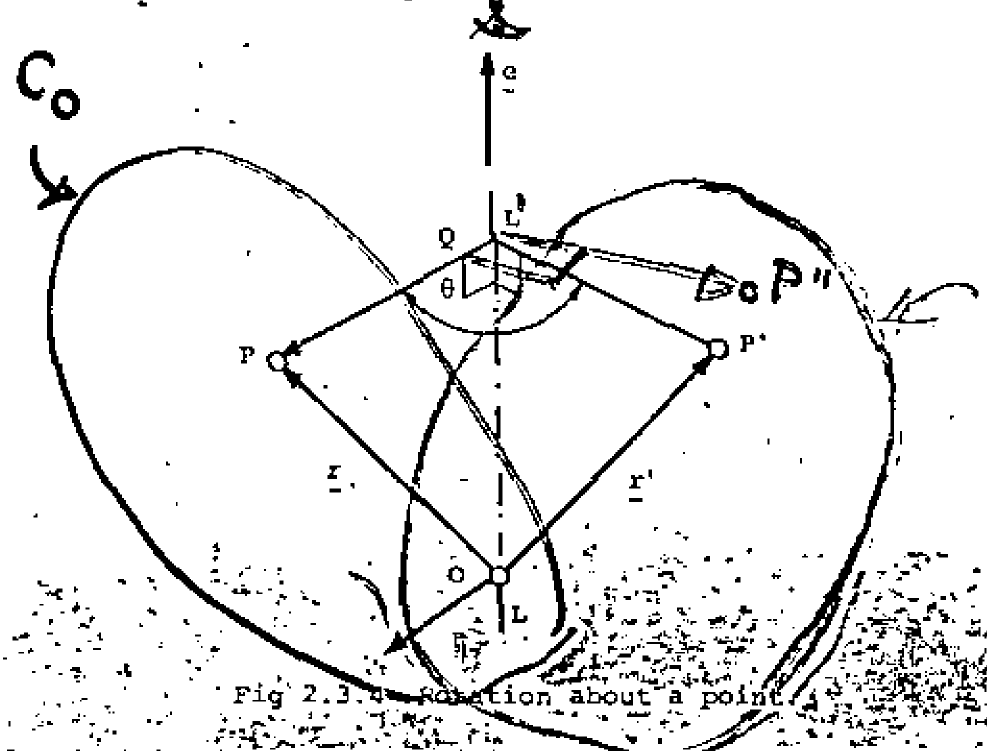
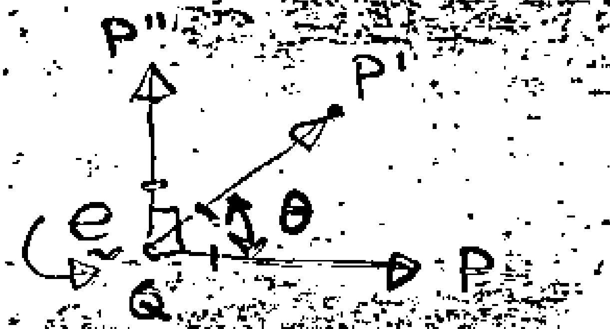


Fig 2.3.4 Rotation about a point

In Fig 2.3.4 P' is the rotated position of point P. If PQ is perpendicular to L, so is P'Q, because rotations preserve angles of rigid bodies. Thus points P, P' and Q determine a plane perpendicular to L, on which the angle of rotation, θ, is measured. From that figure



$$\underline{r}' = \overrightarrow{OQ} + \overrightarrow{QP'}$$

and

$$\overrightarrow{OQ} = \underline{r} - \overrightarrow{QP}$$

Hence

$$\underline{r}' = \underline{r} - \overrightarrow{QP} + \overrightarrow{QP'} \qquad (2.3.9)$$

Let QP" be a line contained in plane PP'Q, at right angles with line PQ and of length equal to that of QP. Thus, vector $\overrightarrow{QP'}$ can be expressed as a linear combination of vectors $\overrightarrow{QP}$ and $\overrightarrow{QP''}$. But

$$\overrightarrow{QP''} = \underline{e} \times \underline{r} \qquad (2.3.10)$$

whereas

$$\vec{QP} = -\underset{\sim}{e} \times \vec{QP}'' = \underset{\sim}{e} \times (\underset{\sim}{e} \times \underset{\sim}{r}) \qquad (2.3.11)$$

which can readily be proved. Besides, $\vec{QP}'$ can be expressed as

$$\vec{QP}' = \vec{QP} \cos\theta + \vec{QP}'' \sin\theta$$

which, in view of eqs. (2.3.10) and (2.3.11), yields

$$\vec{QP}' = \cos\theta\, \underset{\sim}{e} \times (\underset{\sim}{e} \times \underset{\sim}{r}) + \sin\theta\, \underset{\sim}{e} \times \underset{\sim}{r} \qquad (2.3.12)$$

Substituting eqs. (2.3.11) and (2.3.12) into eq. (2.3.9) leads to

$$\underset{\sim}{r}' = \underset{\sim}{r} + \underset{\sim}{e} \times (\underset{\sim}{e} \times \underset{\sim}{r}) - \cos\theta\, \underset{\sim}{e} \times (\underset{\sim}{e} \times \underset{\sim}{r}) + \sin\theta\, \underset{\sim}{e} \times \underset{\sim}{r} \qquad (2.3.13)$$

But

$$\underset{\sim}{e} \times (\underset{\sim}{e} \times \underset{\sim}{r}) = (\underset{\sim}{e} \cdot \underset{\sim}{r})\underset{\sim}{e} - (\underset{\sim}{e} \cdot \underset{\sim}{e})\underset{\sim}{r} = (\underset{\sim}{e}\underset{\sim}{e} - \underset{\sim}{1}) \cdot \underset{\sim}{r} \qquad (2.3.14)$$

where $\underset{\sim}{1}$ is the identity dyadic, i.e. a dyadic that is isomorphic to the identity matrix. Furthermore

$$\underset{\sim}{e} \times \underset{\sim}{r} = \underset{\sim}{1} \cdot \underset{\sim}{e} \times \underset{\sim}{r} = \underset{\sim}{1} \times \underset{\sim}{e} \cdot \underset{\sim}{r} \qquad (2.3.15)$$

where the dot and the point have been exchanged, what is possible to do by virtue of the algebra of cartesian vectors. Substituting eqs. (2.3.14) and (2.3.15) into eq. (2.3.13) one obtains

$$\underset{\sim}{r}' = \underset{\sim}{r} + (1-\cos\theta)(\underset{\sim}{e}\underset{\sim}{e} - \underset{\sim}{1}) \cdot \underset{\sim}{r} + \sin\theta\, \underset{\sim}{1} \times \underset{\sim}{e} \cdot \underset{\sim}{r} =$$

$$= \left((1-\cos\theta)\underset{\sim}{e}\underset{\sim}{e} + \cos\theta\, \underset{\sim}{1} + \sin\theta\, \underset{\sim}{1} \times \underset{\sim}{e}\right) \cdot \underset{\sim}{r} =$$

$$= \underset{\sim}{Q} \cdot \underset{\sim}{r} \qquad (2.3.16)$$

i.e. $\underset{\sim}{r}'$ has been expressed as a linear transformation of vector $\underset{\sim}{r}$. The dyadic $\underset{\sim}{Q}$ is, then, isomorphic to the rotation matrix defined in Section 2.2. That is

$$\underset{\sim}{Q} = \underset{\sim}{e}\underset{\sim}{e} + (\underset{\sim}{1} - \underset{\sim}{e}\underset{\sim}{e})\cos\theta + \sin\theta\, \underset{\sim}{1} \times \underset{\sim}{e} \qquad (2.3.17)$$

One can now prove the following

<u>THEOREM 2.3.2</u> *Let a rigid body undergo a pure rotation about a fixed point O and let $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ be the initial and the final position vectors of a point of the body (measured from O) not lying on the axis of rotation*

Construcción de la matriz de rotación $\mathbf{Q}$

$$(2.3.16) \Rightarrow \mathbf{Q} = (1-\cos\theta)\,\mathbf{e}\,\mathbf{e} + \cos\theta\,\mathbf{1} + \operatorname{sen}\theta\,\mathbf{1}\times\mathbf{e}$$

$$\mathbf{e} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \Rightarrow \mathbf{e}\,\mathbf{e}^T = \begin{bmatrix} u \\ v \\ w \end{bmatrix}[u, v, w] = \begin{bmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{bmatrix}$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{1} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$$

$$\begin{aligned}
\mathbf{1}\cdot\mathbf{r} &= (\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k})\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \\
&= \mathbf{i}\,(x\,\mathbf{i}\cdot\mathbf{i} + y\,\mathbf{i}\cdot\mathbf{j} + z\,\mathbf{i}\cdot\mathbf{k}) + \\
&+ \mathbf{j}\,(x\,\mathbf{j}\cdot\mathbf{i} + y\,\mathbf{j}\cdot\mathbf{j} + z\,\mathbf{j}\cdot\mathbf{k}) + \\
&+ \mathbf{k}\,(x\,\mathbf{k}\cdot\mathbf{i} + y\,\mathbf{k}\cdot\mathbf{j} + z\,\mathbf{k}\cdot\mathbf{k}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{r}
\end{aligned}$$

$$\begin{aligned}
\mathbf{1}\times\mathbf{e} &= (\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k})\times(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = \\
&= \mathbf{i}\,(u\,\mathbf{i}\times\mathbf{i} + v\,\mathbf{i}\times\mathbf{j} + w\,\mathbf{i}\times\mathbf{k}) + \\
&+ \mathbf{j}\,(u\,\mathbf{j}\times\mathbf{i} + v\,\mathbf{j}\times\mathbf{j} + w\,\mathbf{j}\times\mathbf{k}) + \\
&+ \mathbf{k}\,(u\,\mathbf{k}\times\mathbf{i} + v\,\mathbf{k}\times\mathbf{j} + w\,\mathbf{k}\times\mathbf{k}) =
\end{aligned}$$

$$\begin{aligned}
&= -w\,\mathbf{i}\,\mathbf{j} + v\,\mathbf{i}\,\mathbf{k} + \\
&+ w\,\mathbf{j}\,\mathbf{i} - u\,\mathbf{j}\,\mathbf{k} \qquad \Rightarrow \mathbf{1}\times\mathbf{e} = \begin{bmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{bmatrix} \\
&- v\,\mathbf{k}\,\mathbf{i} + u\,\mathbf{k}\,\mathbf{j}
\end{aligned}$$

$$\Rightarrow \underset{\sim}{Q} = \begin{bmatrix} (1-c\theta)u^2 + c\theta & (1-c\theta)uv - s\theta w & (1-c\theta)uw + vs\theta \\ (1-c\theta)uv + s\theta w & (1-c\theta)v^2 + c\theta & (1-c\theta)vw - us\theta \\ (1-c\theta)uw - vs\theta & (1-c\theta)vw + us\theta & (1-c\theta)w^2 + c\theta \end{bmatrix}$$

$$Tr\,\underset{\sim}{Q} = (1-c\theta)u^2 + c\theta + (1-c\theta)v^2 + c\theta + (1-c\theta)w^2 + c\theta =$$

$$= (1-c\theta)(u^2+v^2+w^2) + 3c\theta = 1 + 2c\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}\left[\frac{1}{2}(Tr\,\underset{\sim}{Q} - 1)\right]$$

Ejemplo 2.3.1

$$\underset{\sim}{Q} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix} \Leftarrow \text{Definición } 1.2.1$$

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \frac{1}{9}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \underset{\sim}{I}$$

$$\det \underset{\sim}{Q} = \frac{1}{3^3}\left[2\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} + 2\begin{vmatrix} -2 & 2 \\ -1 & -2 \end{vmatrix}\right] =$$

$$= \frac{1}{3^3}[12 + 3 + 12] = \frac{27}{27} = +1 \Rightarrow \text{o.p.}$$

$$\underset{\sim}{e} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} ; \quad \underset{\sim}{Q}\,\underset{\sim}{e} = \underset{\sim}{e}$$

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \Rightarrow (\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{0}$$

$$\Rightarrow \begin{matrix} 2u + v + 2w = 3u \Rightarrow -u + v = -2w \\ -2u + 2v + w = 3v \Rightarrow -2u - v = -w \end{matrix}$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}}_{\underset{\sim}{A}}\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{x}} = \underbrace{\begin{bmatrix} -2 \\ -1 \end{bmatrix} w}_{\underset{\sim}{b}} \Rightarrow \underset{\sim}{x} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$$

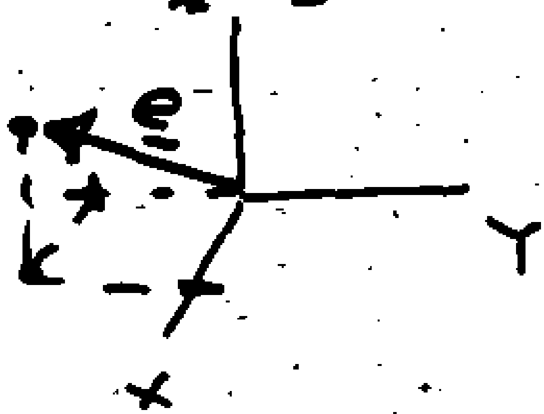
$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ -1 \end{bmatrix} w = \frac{1}{3}\begin{bmatrix} 3 \\ -3 \end{bmatrix} w = \begin{bmatrix} w \\ -w \end{bmatrix}$$

$u = w,\ v = -w,\ w = w$

$u^2 + v^2 + w^2 = 1 \Rightarrow w^2 + w^2 + w^2 = 1$

$\Rightarrow w^2 = \frac{1}{3} \Rightarrow w = \textcircled{+} \frac{\sqrt{3}}{3}$

$\underline{e} = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$



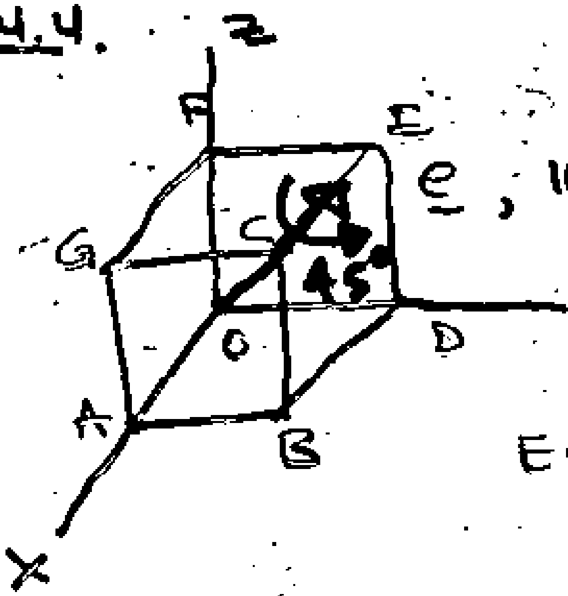
$\mathrm{Tr}\,\underline{Q} = 1 + 2\cos\theta$

$\Rightarrow 2 = 1 + 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = -60^\circ$

<u>Ejercicio 2.4.4.</u>

$\underline{e} = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$



$\underline{e},\ \|\underline{e}\| = 1$

$u = v = w = \frac{\sqrt{3}}{3}$

$\theta = 45^\circ$

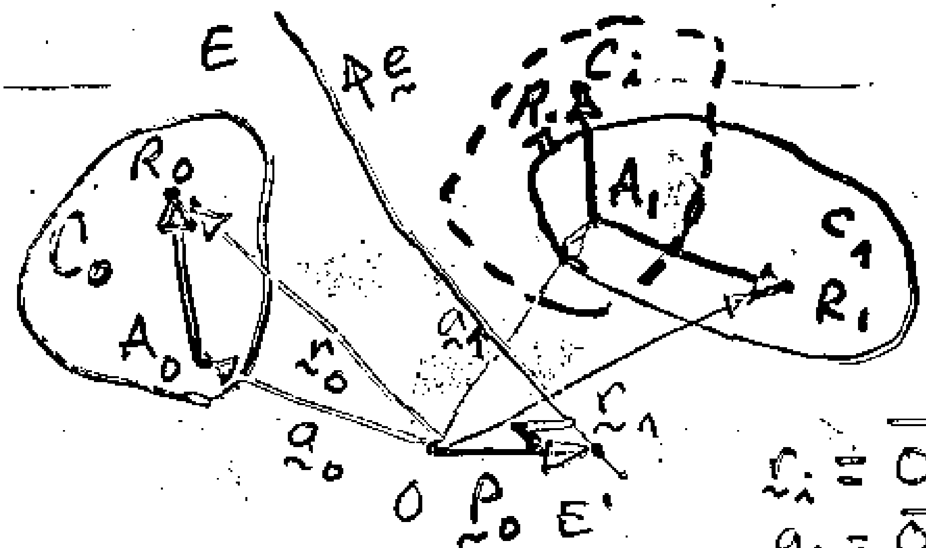
$\Rightarrow c\theta = s\theta = \frac{\sqrt{2}}{2}$

Ecs. (2.5.4) y (2.5.5)

⇓

$$Q = \begin{bmatrix} \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{2}}{2} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{3} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{3} \\ \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{6}}{6} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{2}}{2} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} - \frac{\sqrt{6}}{6} \\ \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} - \frac{\sqrt{6}}{6} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{6}}{6} & \left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{3} + \frac{\sqrt{2}}{2} \end{bmatrix}$$

1.2 Movimiento general de cuerpo rígido. Teorema de Chasles.



$$r_i = \vec{OR_i}$$
$$a_i = \vec{OA_i} = \vec{OA_1} = a_1$$

$$r_1 - a_1 = Q(r_i - a_i);\quad r_i - a_i = r_0 - a_0$$

$$\Rightarrow \boxed{r_1 = a_1 + Q(r_0 - a_0)} \qquad (8.1)$$

Desplazamiento de R: $u_R = r_1 - r_0$

$$u = a_1 + Q(r_0 - a_0) - r_0 =$$
$$= (Q - I) r_0 + a_1 - Q a_0$$

$$\varphi = \|u\|^2 = u^T u = \|(Q - I) r_0\|^2 + \|a_1 - Q a_0\|^2$$
$$+ 2(a_1 - Q a_0)^T (Q - I) r_0$$

Teorema de Chasles: Dado un cuerpo rígido que se mueve de una configuración C_0 a una configuración C_1, en movimiento general, existe un conjunto de puntos de C_0, alojados sobre una recta EE' paralela a e, el vector característico real de Q, y que pasa por un punto cuyo vector de posición es ρ_0.

Determinación de $\underset{\sim}{\rho}_0$:

$$\phi = \|\underset{\sim}{u}\|^2 \rightarrow \underset{\underset{\sim}{r}_0}{\mathrm{Min}}$$

$$\frac{\partial \phi}{\partial \underset{\sim}{r}_0} = \left[\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{r}_0}\right]^T \frac{\partial \phi}{\partial \underset{\sim}{u}} \tag{1}$$

$$\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{r}_0} = \underset{\sim}{Q} - \underset{\sim}{I}, \quad \frac{\partial \phi}{\partial \underset{\sim}{u}} = 2\underset{\sim}{u} \tag{2}$$

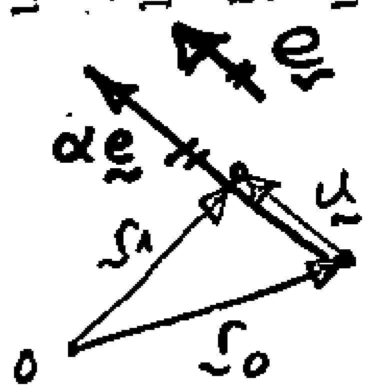
(2) en (1) = 0:

$$(\underset{\sim}{Q} - \underset{\sim}{I})^T \underset{\sim}{u} = \underset{\sim}{0} \Rightarrow \underset{\sim}{Q}^T \underset{\sim}{u} = \underset{\sim}{u} \Rightarrow \underset{\sim}{u} = \underset{\sim}{Q}\underset{\sim}{u} \tag{3}$$

$\Rightarrow$ $\underset{\sim}{u}$ de magnitud mínima es paralelo a $\underset{\sim}{e}$: $\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e}$, esto es, al eje de la rotación involucrada.

Nota: $(\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{0}$. Así, sustituyendo $\underset{\sim}{u}$ en (3):

$$(\underset{\sim}{Q} - \underset{\sim}{I})^T \underset{\sim}{u} = (\underset{\sim}{Q} - \underset{\sim}{I})^T(\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{r}_0 + (\underset{\sim}{Q} - \underset{\sim}{I})^T(\underset{\sim}{a}_1 - \underset{\sim}{Q}\underset{\sim}{a}_0) = \underset{\sim}{0}$$



$$\underset{\sim}{u}' \equiv \underset{\sim}{u}(\underset{\sim}{\rho}_0 + \alpha\underset{\sim}{e}) = (\underset{\sim}{Q} - \underset{\sim}{I})(\underset{\sim}{\rho}_0 + \alpha\underset{\sim}{e}) + \underset{\sim}{a}_1 - \underset{\sim}{Q}\underset{\sim}{a}_0 =$$

$$= (\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{\rho}_0 + \alpha\underbrace{(\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{e}}_{\underset{\sim}{0}} + \underset{\sim}{a}_1 - \underset{\sim}{Q}\underset{\sim}{a}_0 =$$

$$= (\underset{\sim}{Q} - \underset{\sim}{I})\underset{\sim}{\rho}_0 + \underset{\sim}{a}_1 - \underset{\sim}{Q}\underset{\sim}{a}_0 = \underset{\sim}{u}(\underset{\sim}{\rho}_0)$$

Si un punto R_0 tiene un desplazamiento $\underset{\sim}{u}$ de magnitud mínima, entonces todos los puntos de una recta paralela al eje de rotación y que pase por R_0 también tienen un desplazamiento de magnitud mínima.

Sea $\underset{\sim}{\rho}$ el vector de posición del punto de EE' que se encuentre más próximo al origen. Entonces,

$$\underset{\sim}{\rho}_0^T \underset{\sim}{e} = 0 \qquad (3')$$

Sea $\underset{\sim}{u}_0$ el desplazamiento de magnitud mínima. Entonces existe un escalar α para el cual

$$\underset{\sim}{u}_0 \equiv \underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_0 = \alpha \underset{\sim}{e} \qquad (4)$$

Fórmula de Rodrigues

Sea una rotación de ángulo θ alrededor de un eje paralelo al vector $\underset{\sim}{e}$: si $\underset{\sim}{\rho}_1$ es la imagen de $\underset{\sim}{\rho}_0$ bajo esta rotación,

$$\underset{\sim}{\rho}_1 - \underset{\sim}{\rho}_0 = \tan\frac{\theta}{2}\, \underset{\sim}{e} \times (\underset{\sim}{\rho}_1 + \underset{\sim}{\rho}_0) \qquad (FR)$$

Aplicando la fórmula de Rodrigues a un vector $\mathbf{a}_0$ y su imagen $\mathbf{a}_1$,

$$\mathbf{a}_1-\mathbf{a}_0=\tan\frac{\theta}{2}\,\mathbf{e}\times(\mathbf{a}_1+\mathbf{a}_0) \qquad (5)$$

$$(5)-\text{FR} \Rightarrow \mathbf{a}_1-\boldsymbol{\rho}_1-(\mathbf{a}_0-\boldsymbol{\rho}_0)=\tan\frac{\theta}{2}\,\mathbf{e}\times[(\mathbf{a}_1-\boldsymbol{\rho}_1)+(\mathbf{a}_0-\boldsymbol{\rho}_0)] \qquad (6)$$

$$(4) \Rightarrow \mathbf{e}\times(\boldsymbol{\rho}_1-\boldsymbol{\rho}_0)=\mathbf{0} \qquad (7)$$

$$(4),(7) \text{ en } (6) \Rightarrow \mathbf{a}_1-\mathbf{a}_0-\alpha\mathbf{e}=\tan\frac{\theta}{2}\,\mathbf{e}\times(\mathbf{a}_1+\mathbf{a}_0)-2\tan\frac{\theta}{2}\,\mathbf{e}\times\boldsymbol{\rho}_0 \qquad (8)$$

$$\cot\frac{\theta}{2}\,\mathbf{e}\times(8) \Rightarrow \cot\frac{\theta}{2}\,\mathbf{e}\times(\mathbf{a}_1-\mathbf{a}_0)=\mathbf{e}\times[\mathbf{e}\times(\mathbf{a}_1+\mathbf{a}_0)]-2\,\mathbf{e}\times(\mathbf{e}\times\boldsymbol{\rho}_0)=$$

$$=\mathbf{e}\times[\mathbf{e}\times(\mathbf{a}_1+\mathbf{a}_0)]-2\underbrace{(\mathbf{e}\cdot\boldsymbol{\rho}_0)}_{0}\mathbf{e}+2\underbrace{(\mathbf{e}\cdot\mathbf{e})}_{1}\boldsymbol{\rho}_0 \qquad (9)$$

(3') en (9) $\Rightarrow$

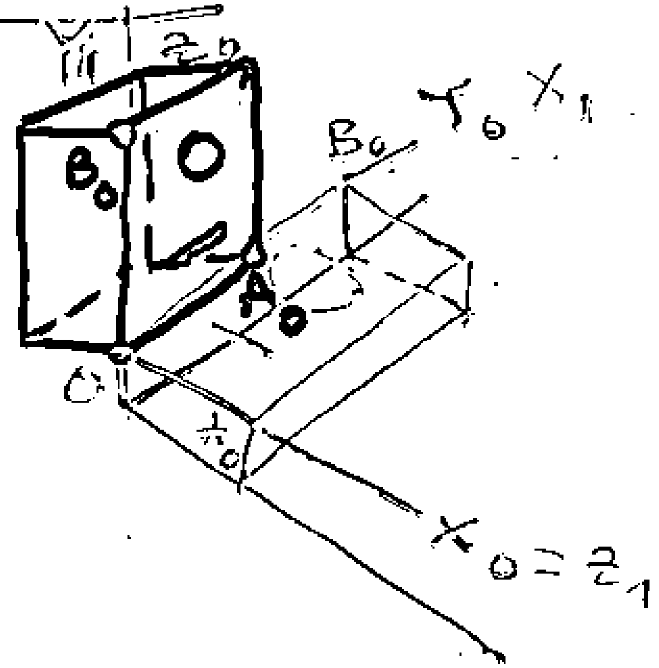
$$\boldsymbol{\rho}_0=\frac{1}{2}\cot\frac{\theta}{2}\,\mathbf{e}\times(\mathbf{a}_1-\mathbf{a}_0)-\frac{1}{2}\,\mathbf{e}\times[\mathbf{e}\times(\mathbf{a}_1+\mathbf{a}_0)]$$

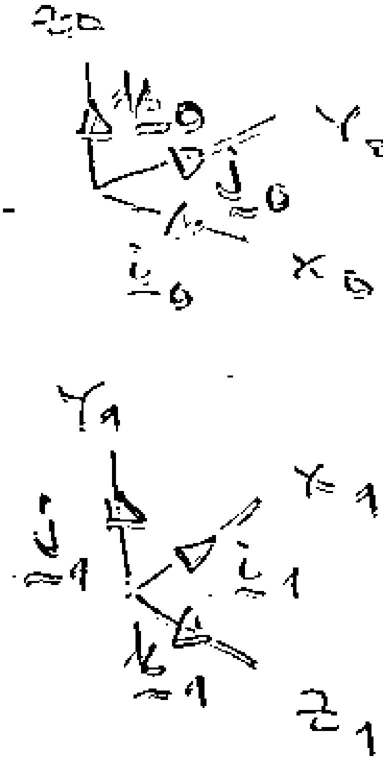
Parámetros del movimiento de cuerpo rígido:

1. θ
2. $\mathbf{e}$, $\|\mathbf{e}\|=1$
3. $\boldsymbol{\rho}_0$; $\boldsymbol{\rho}_0\cdot\mathbf{e}=0$
4. $\mathbf{u}\cdot\mathbf{e}=\alpha$

$\Rightarrow$ ocho parámetros escalares sujetos a ~~una~~ dos restricción $\Rightarrow$ 6 parámetros independientes.

Ejercicio 2.6.8





$\underset{\sim}{i}_1 = \underset{\sim}{j}_0 \;,\; \underset{\sim}{j}_1 = \underset{\sim}{k}_0 \;,\; \underset{\sim}{k}_1 = \underset{\sim}{i}_0$

$$[\underset{\sim}{Q}]_0 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \;,\; [\underset{\sim}{e}]_0 = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underset{\sim}{Q}\,\underset{\sim}{e} = \underset{\sim}{e} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\Rightarrow \left. \begin{matrix} w = u \\ u = v \end{matrix} \right\} \Rightarrow u = w,\; v = w,\; w \text{ arb.}$$

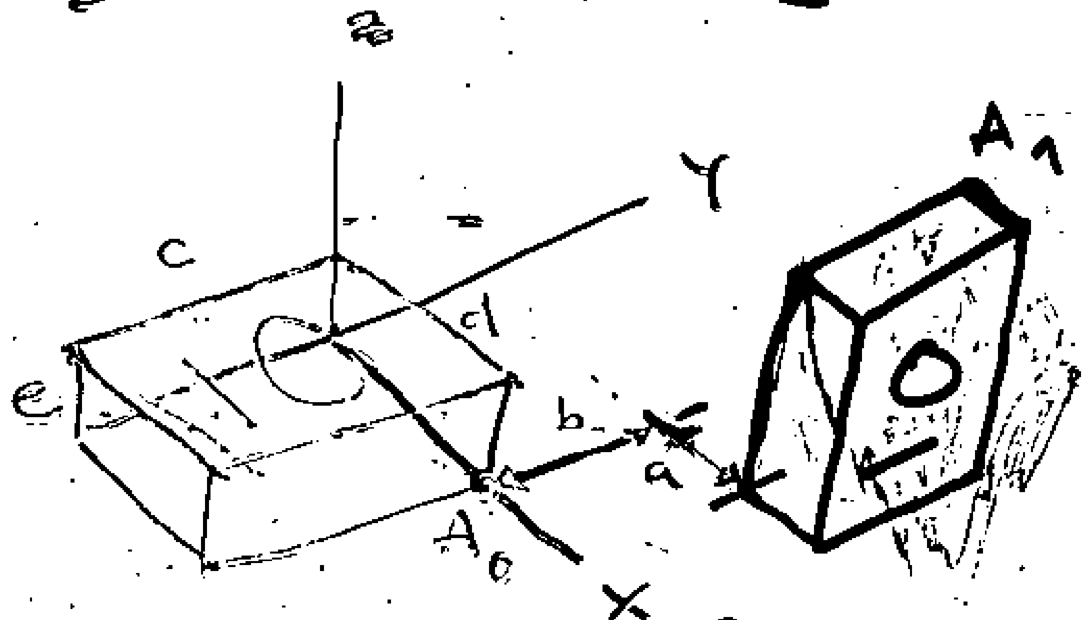
$u^2 + v^2 + w^2 = 1 \Rightarrow w^2 + w^2 + w^2 = 1 \Rightarrow w = \frac{\sqrt{3}}{3}$

$\underset{\sim}{e} = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\text{Tr}\, Q = 1 + 2c\theta$
$\Rightarrow 0 = 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$

$\theta = \pm 120°$ Ecs. (2.5.15) $\Rightarrow \operatorname{sen}\theta = \frac{1}{2e_1}(q_{32} - q_{23})$

(2.6.15): $\underset{\sim}{p}_0 = \frac{1}{2}\cot\frac{\theta}{2}\,\underline{e}\times(\underset{\sim}{a}_1 - \underset{\sim}{a}_0) - \frac{1}{2}\underline{e}\times[\underline{e}\times(\underset{\sim}{a}_1 + \underset{\sim}{a}_0)]$

$\cot\frac{\theta}{2} = \cot 60^\circ = \tan 30^\circ = \frac{\sqrt{3}}{3}$



$\underset{\sim}{a}_0 = \begin{bmatrix} d \\ l \\ 0 \end{bmatrix}, \quad \underset{\sim}{a}_1 = \begin{bmatrix} d+a \\ b+d \\ c \end{bmatrix}$

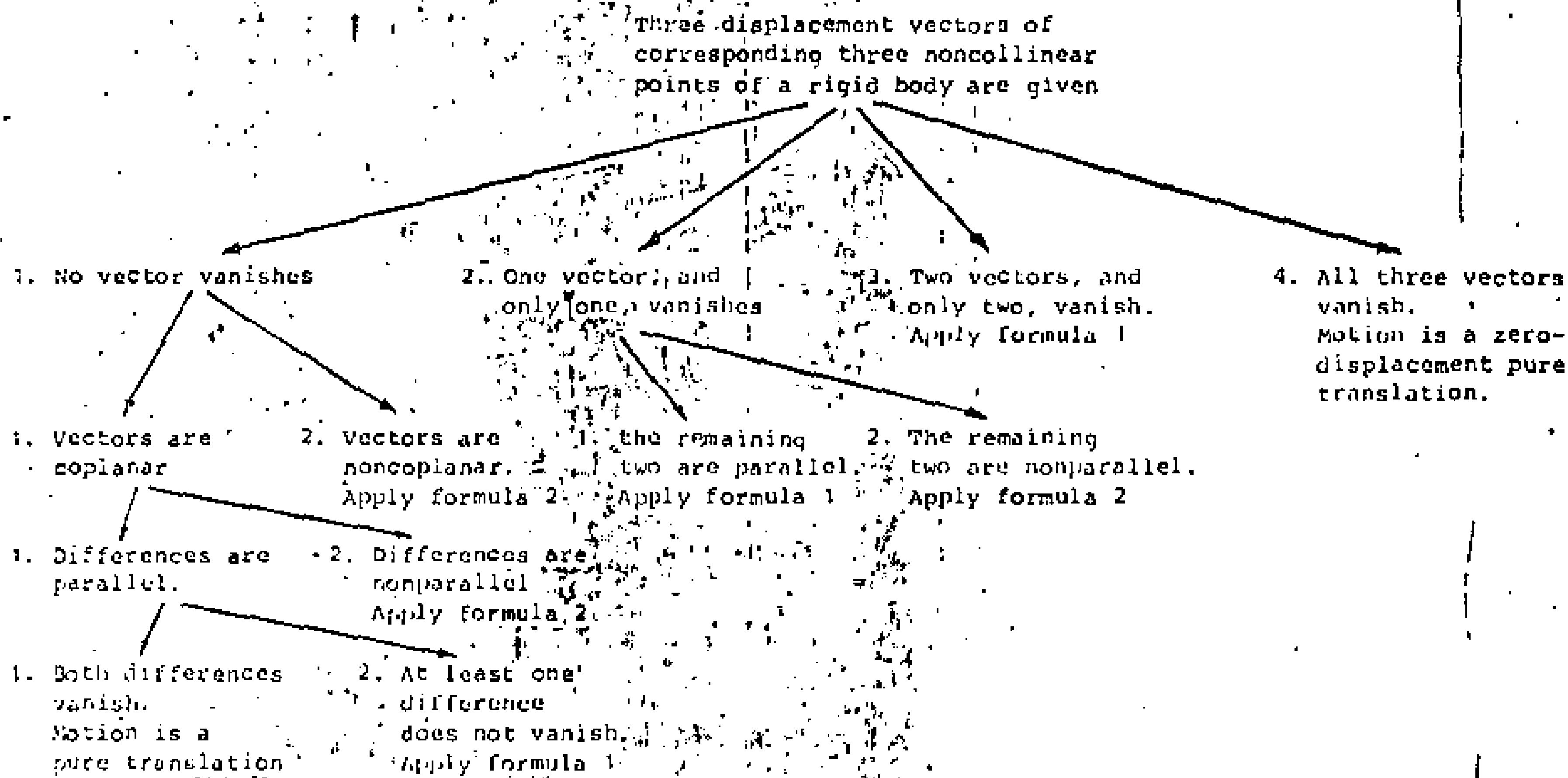
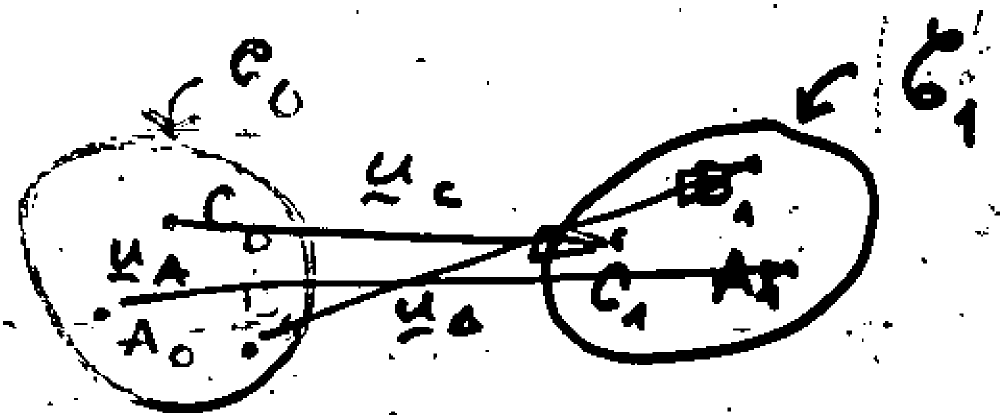
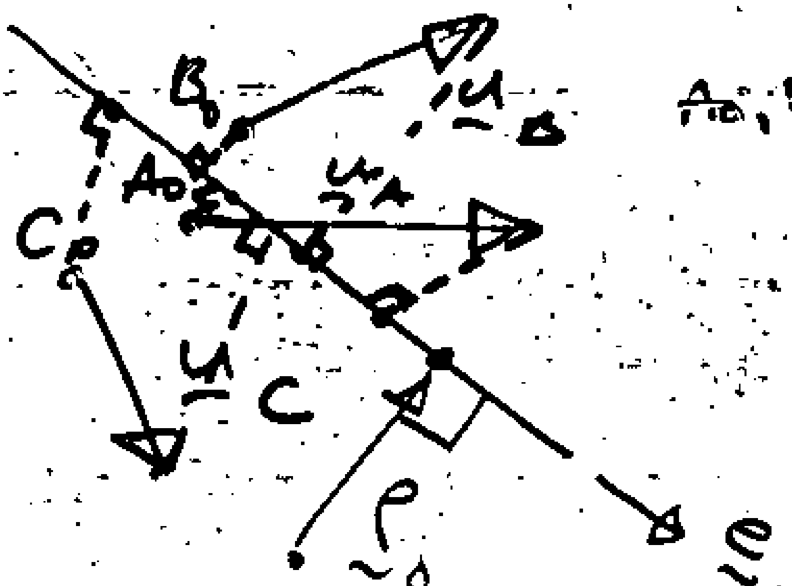


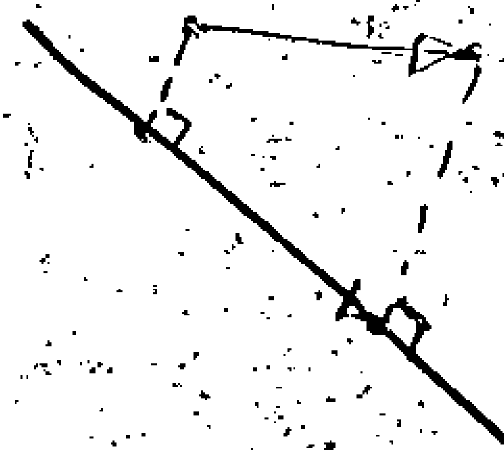
Fig 1. Tree diagram showing the different possible relationships amongst the displacements of three noncollinear points defining a rigid-body motion.

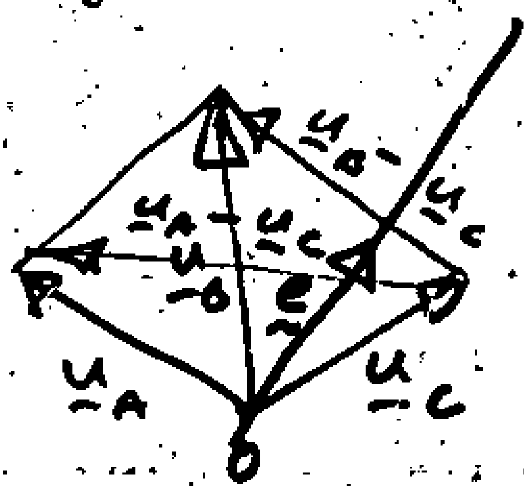
12





A_0, B_0, C_0 no son colineales

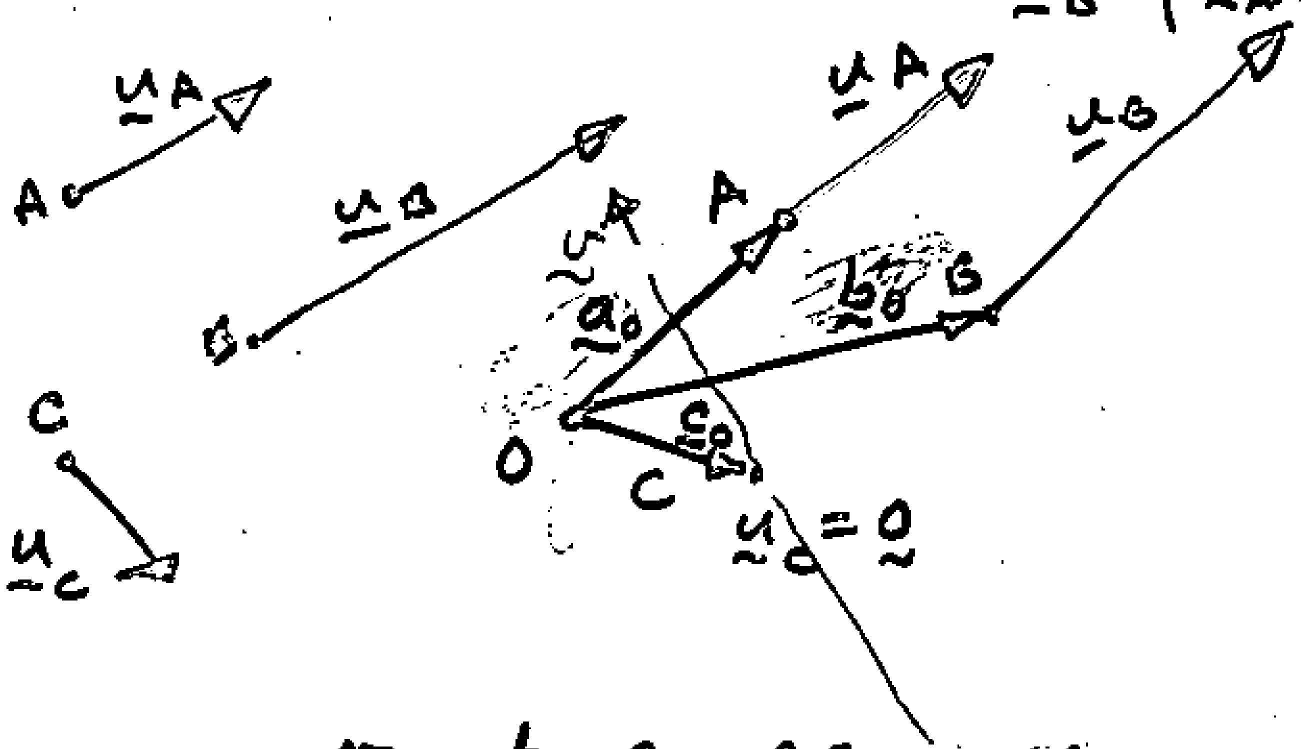




$$\underline{v} = (\underline{u}_B - \underline{u}_C) \times (\underline{u}_A - \underline{u}_C)$$

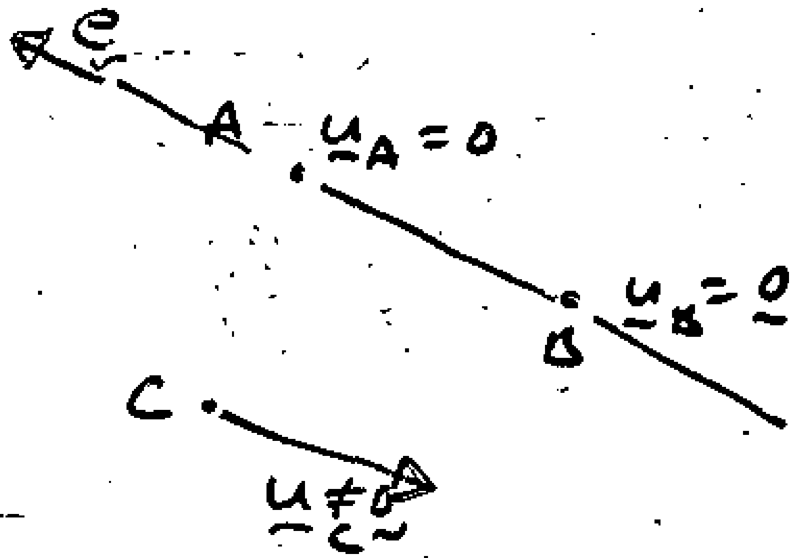
$$\underline{e} = \frac{\underline{v}}{\|\underline{v}\|}$$

$\underset{\sim}{u}_B = \beta \underset{\sim}{u}_A$

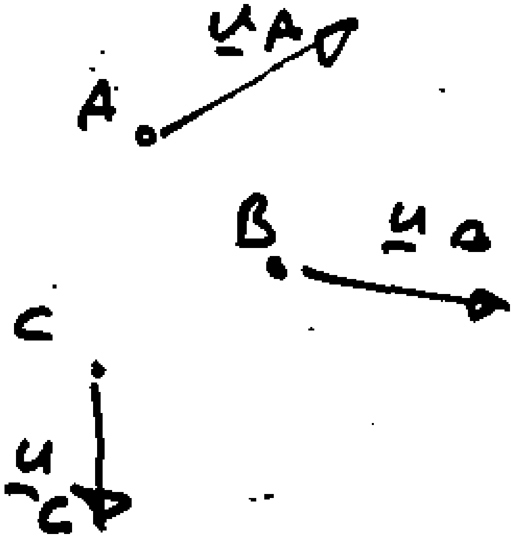


$$\underset{\sim}{v} = \underset{\sim}{b}_0 - \underset{\sim}{c}_0 - \beta(\underset{\sim}{a}_0 - \underset{\sim}{c}_0)$$

$$\underset{\sim}{e} = \frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|}$$

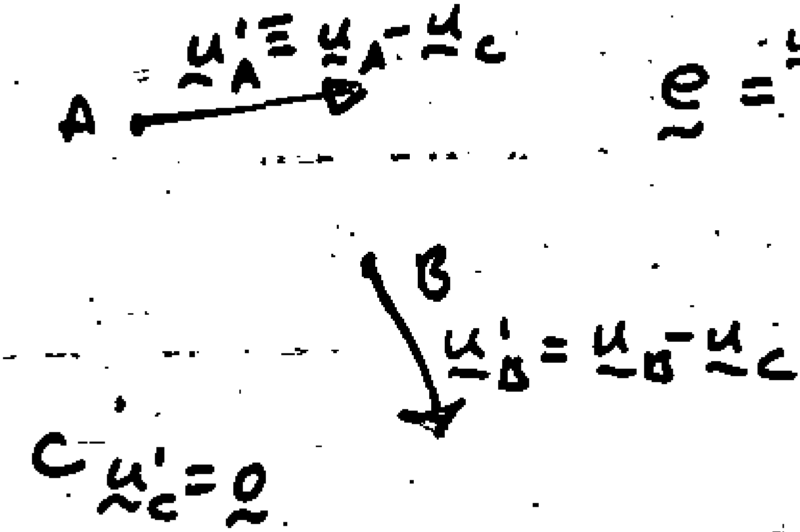


$$\underset{\sim}{e} = \frac{\underset{\sim}{a} - \underset{\sim}{b}}{\|\underset{\sim}{a} - \underset{\sim}{b}\|}$$

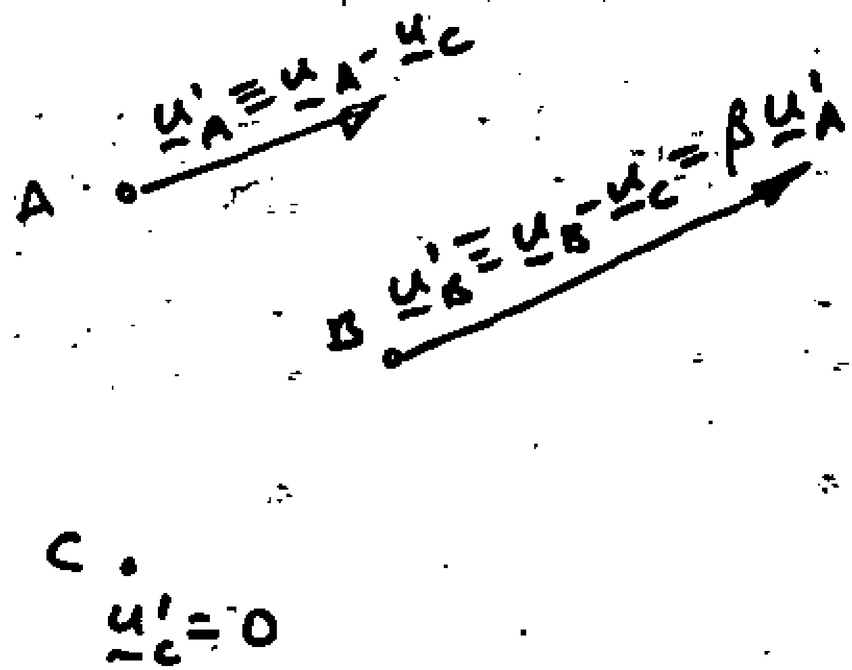


$$\underset{\sim}{u}_A \times \underset{\sim}{u}_B \cdot \underset{\sim}{u}_C \neq 0 \quad ✓$$

$$(\underset{\sim}{u}_A - \underset{\sim}{u}_C) \times (\underset{\sim}{u}_B - \underset{\sim}{u}_C) \neq \underset{\sim}{0}$$

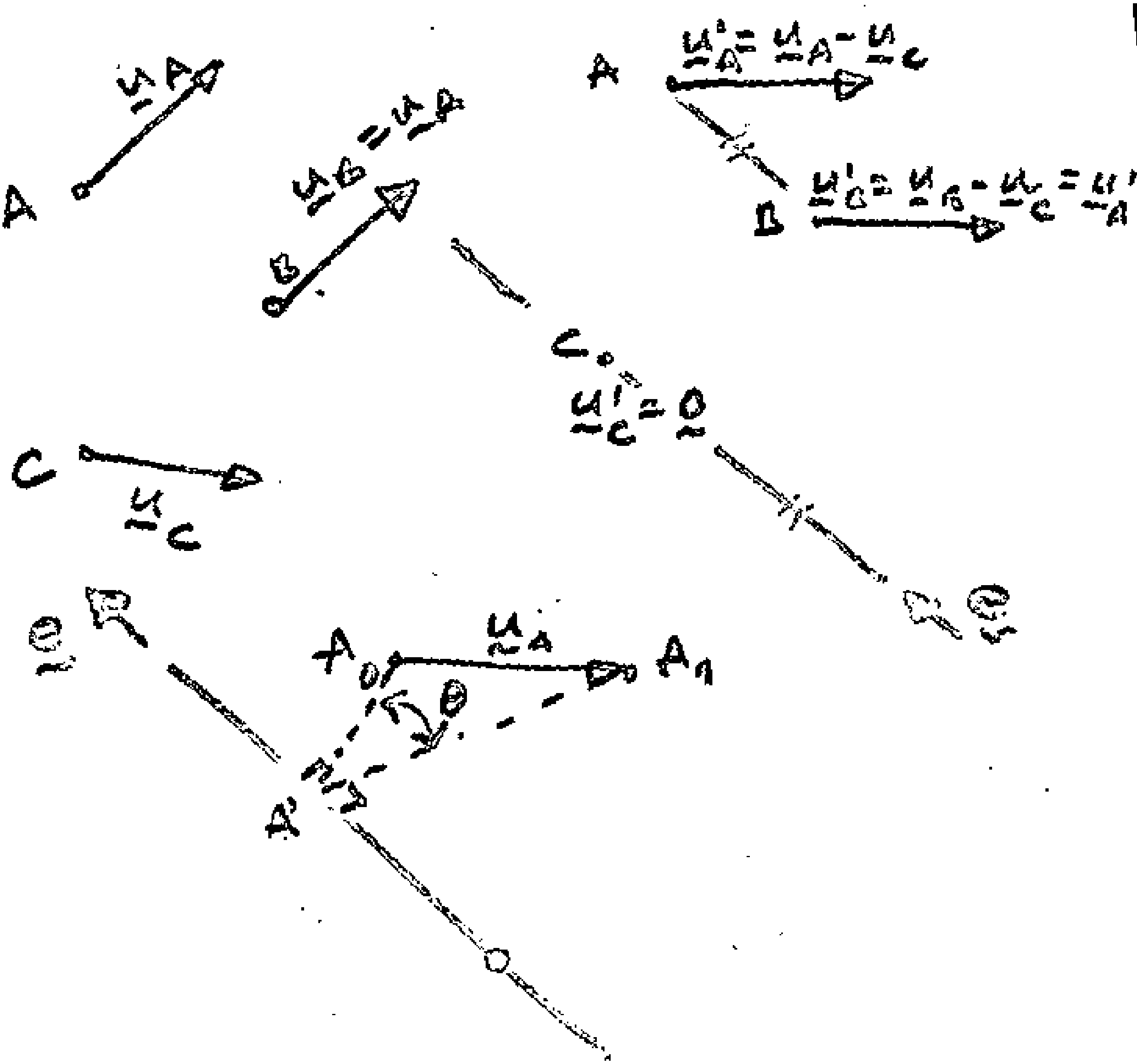


$$\underset{\sim}{e} = \frac{\underset{\sim}{u}'_A \times \underset{\sim}{u}'_B}{\| \underset{\sim}{u}'_A \times \underset{\sim}{u}'_B \|}$$



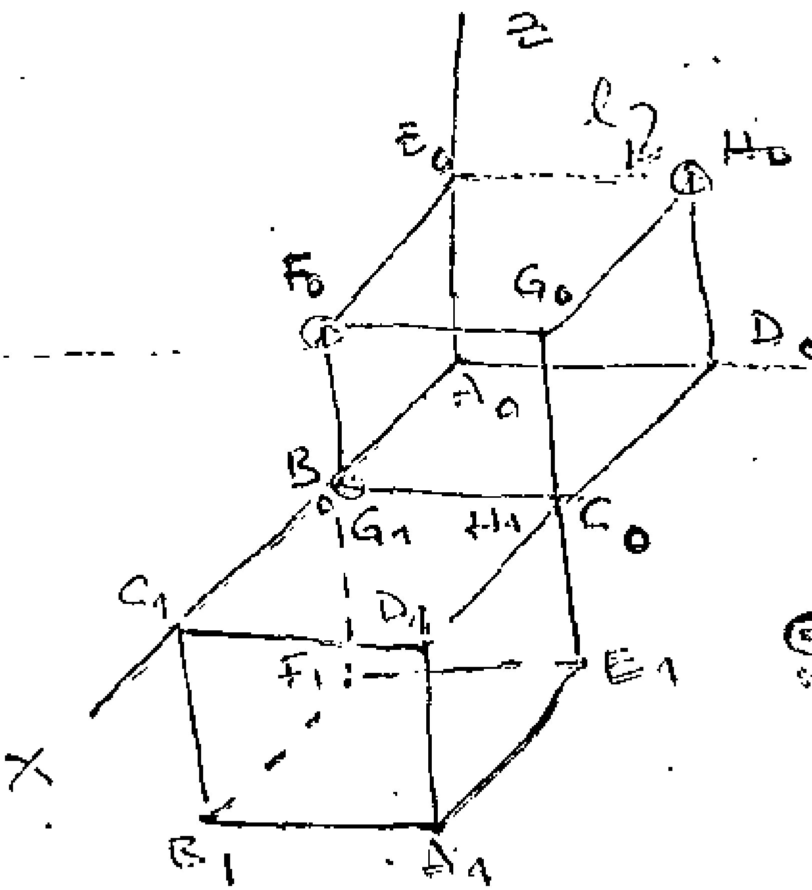
$$\underset{\sim}{u}_A \times \underset{\sim}{u}_B \cdot \underset{\sim}{u}_C = 0$$

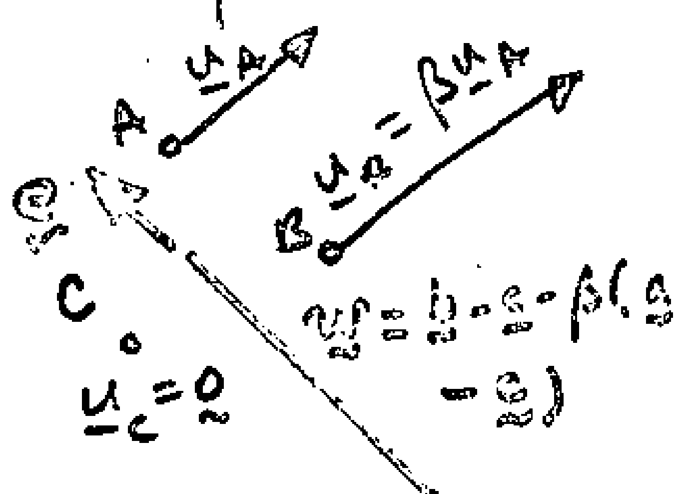
$$(\underset{\sim}{u}_A - \underset{\sim}{u}_C) \times (\underset{\sim}{u}_B - \underset{\sim}{u}_C) = \underset{\sim}{0}$$



$$a = u_A \cdot e = u_B \cdot e = u_C \cdot e$$

(E5)





$\underline{u}_A$, $\underline{u}_B = \beta \underline{u}_A$

$\underline{u}_C = \underline{0}$

$\underline{u}_B = \underline{b} - \underline{a} - \beta(\underline{a} - \underline{a})$

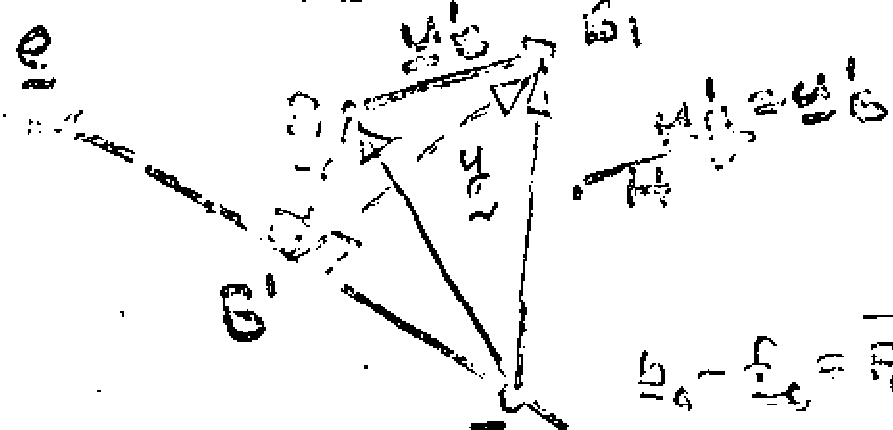
$\underline{u}_B = l(\underline{i} - \underline{k})$, $\underline{u}_F = -2l\underline{k}$, $\underline{u}_H = l(\underline{i} - \underline{k})$

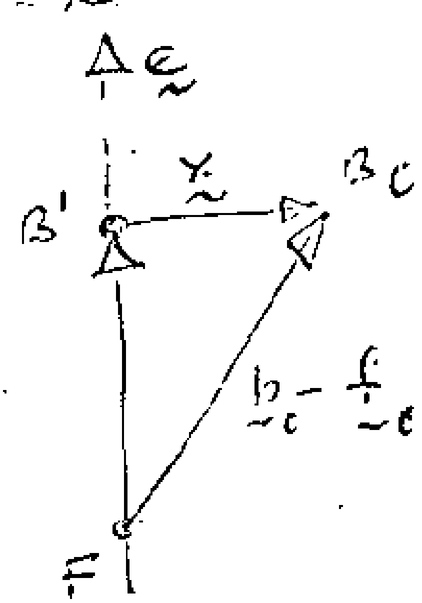
$\underline{u}'_B = \underline{u}_B - \underline{u}_F = l(\underline{i} + \underline{k})$, $\underline{u}'_F = \underline{0}$, $\underline{u}'_H = \underline{u}_H - \underline{u}_F = l(\underline{i} + \underline{k})$

$\underline{w} = \underline{h} - \underline{f} - 1(\underline{b} - \underline{f}) = \underline{h} - \underline{b} = l(\underline{i} - \underline{j} - \underline{k})$

$= l(-\underline{i} + \underline{j} + \underline{k})$, $\|\underline{w}\| = \sqrt{3}\, l$

$\underline{e} = \frac{\underline{w}}{\|\underline{w}\|} = \frac{\sqrt{3}}{3}(-\underline{i} + \underline{j} + \underline{k})$





$\underline{b}_C - \underline{f}_C = \overrightarrow{F_C B'} + \underline{x}$

$\vec{F}_0 B' = (\underline{b}_0 - \underline{f}_0) \cdot \underline{e}\,\underline{e}$ EG

$\Rightarrow \underline{x} = \underline{b}_0 - \underline{f}_0 - (\underline{b}_0 - \underline{f}_0) \cdot \underline{e}\,\underline{e}$

$\underline{y} = \underline{b}_1 - \underline{f}_0 - (\underline{b}_1 - \underline{f}_0) \cdot \underline{e}\,\underline{e}$

$$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\|^2}$$

$\underline{x} = \frac{\ell}{3}(\underline{i} - \underline{j} - 4\underline{k}), \; \underline{y} = \ell(\underline{j} - \underline{k})$

$\cos\theta = \frac{\ell^2}{2\ell^2} = \frac{1}{2}$, Teo. 2.3.2 $\Rightarrow \theta = -60^\circ$

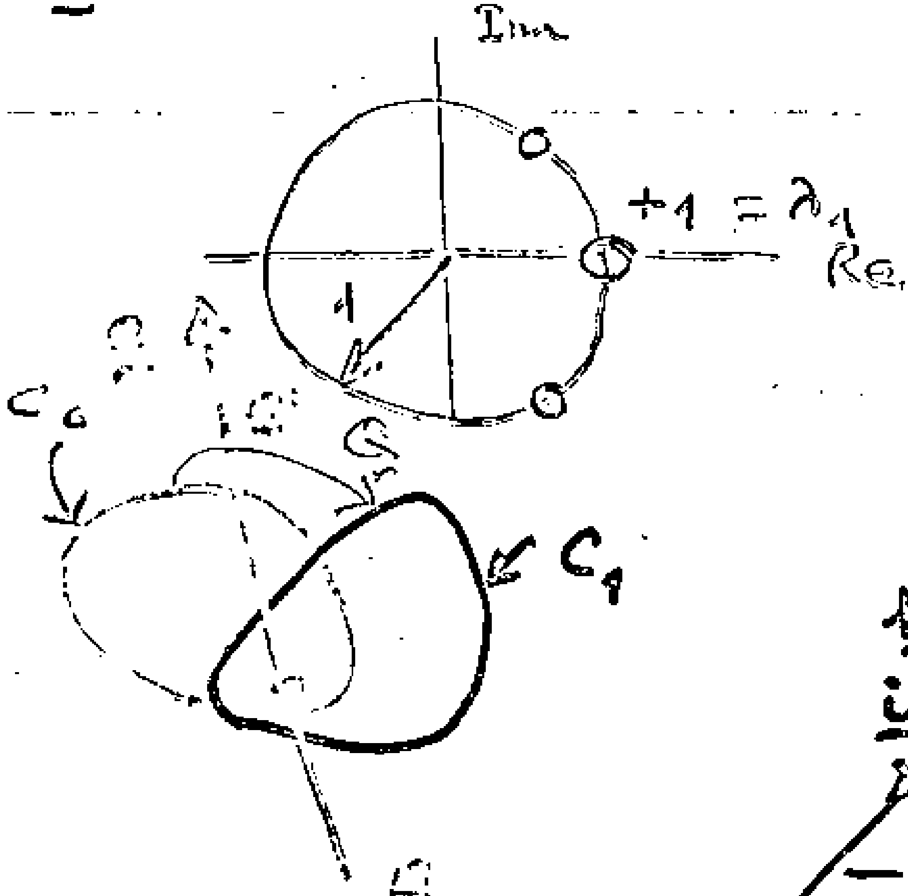
$\underline{\rho}_0 = \frac{1}{2}\cot\frac{\theta}{2}\,\underline{e} \times (\underline{f}_1 - \underline{f}_0) - \frac{1}{2}\,\underline{e} \times [\underline{e} \times (\underline{f}_1 + \underline{f}_0)] =$

$= \frac{1}{3}\ell\,(3\underline{i} + 2\underline{j} + \underline{k})$

$s = \underline{u}_F \cdot \underline{e} = -\frac{2\sqrt{3}}{3}\ell$

Repaso de vectores y valores característicos

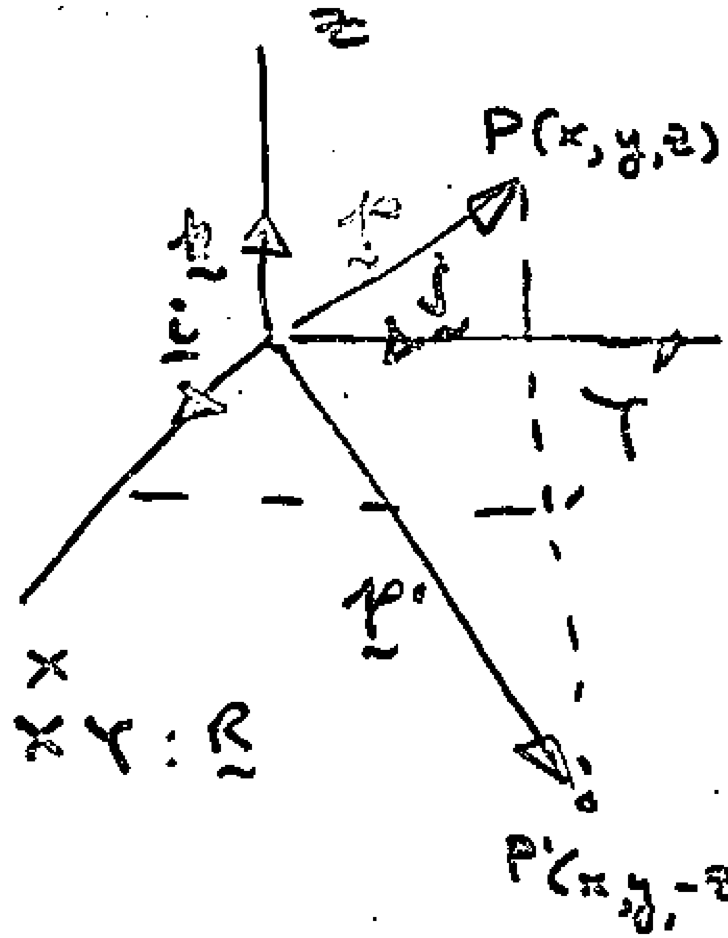
$\underset{\sim}{Q}$: matriz ortogonal propia



$\underset{\sim}{Q}\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$\lambda = +1 \Rightarrow$

$\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e}$



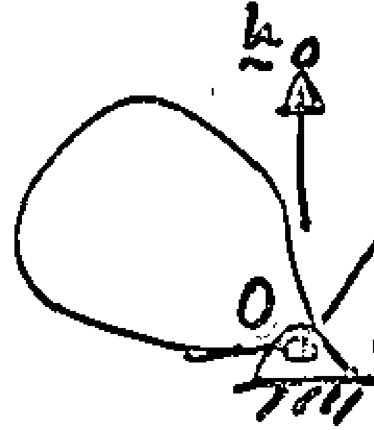
P' es la reflexión de P c.r. al plano XY : $\underset{\sim}{R}$

$\underset{\sim}{p}' = \underset{\sim}{R}\,\underset{\sim}{p}$

Vectores característicos de $\underset{\sim}{R}$:

$\underset{\sim}{R}\,\underset{\sim}{e} = \lambda \underset{\sim}{e}$

$\underset{\sim}{R}\,\underset{\sim}{k} = -\underset{\sim}{k}$



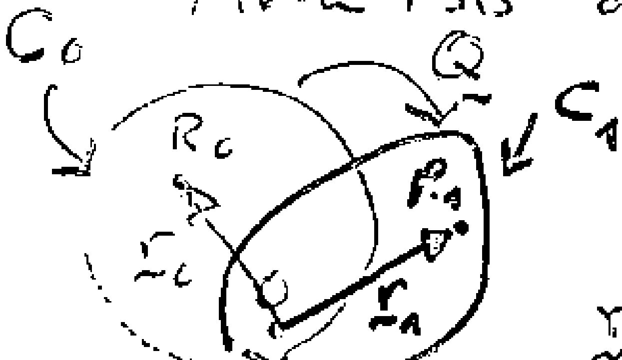
Mom. de inercia c.r. a O : $\mathbf{I}_0$

momentum angular del cuerpo c.r. a O

$$\mathbf{h}_0 = \mathbf{I}_0 \cdot \boldsymbol{\omega}$$

En particular, $\mathbf{h}_0 \times \boldsymbol{\omega} = \mathbf{0}$ si $\mathbf{h}_0$ es un vector característico de $\mathbf{I}_0$.

Análisis de velocidad



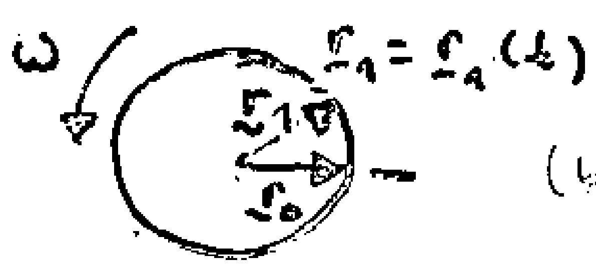
$\mathbf{Q} : C_0 \to C_1$

$C_1 = C_1(t)$

$$\mathbf{r}_1 = \mathbf{Q}\,\mathbf{r}_0 \qquad (*)$$

$\mathbf{r}_1 = \mathbf{r}_1(t)$, $\mathbf{Q} = \mathbf{Q}(t)$, $\mathbf{r}_0$ es cte.

$$\mathbf{r}_1(t) = \mathbf{Q}(t)\,\mathbf{r}_0 \Rightarrow \dot{\mathbf{r}}_1(t) = \dot{\mathbf{Q}}(t)\,\mathbf{r}_0 = \mathbf{v}(t) \quad (a$$



De $(*) \Rightarrow \mathbf{r}_0 = \mathbf{Q}^T \mathbf{r}_1 \qquad (b)$

(b) en (a) $\Rightarrow \mathbf{v}(t) = \dot{\mathbf{Q}}\,\mathbf{Q}^T \mathbf{r}_1$

$$= \mathbf{\Omega}\,\mathbf{r}_1 \quad (c$$

$\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}} \underset{\sim}{Q}^T$: matriz de vel. angular

$\underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T$ ← Propiedad

Demostr.: $\underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{I} \Rightarrow \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T + \underset{\sim}{Q}\dot{\underset{\sim}{Q}}^T = \underset{\sim}{0}$

$\Rightarrow \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T = -\underset{\sim}{Q}\dot{\underset{\sim}{Q}}^T \Rightarrow \underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T$, q.e.d.

Hecho: La forma cuadrática $\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}$ asociada a una matriz antimétrica $\underset{\sim}{A}$ es nula: $\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} = 0$ si $\underset{\sim}{A} = -\underset{\sim}{A}^T$

$$\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} = [x_1, x_2, \ldots, x_n] \begin{bmatrix} & & \\ & \underset{\sim}{A} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Energía potencial de un sist. vibratorio de grados de libertad n: $V = \frac{1}{2} \underset{\sim}{x}^T \underset{\sim}{K} \underset{\sim}{x}$

En ese mismo sistema vibratorio Energía cinética, $T = \frac{1}{2} \dot{\underset{\sim}{x}} \underset{\sim}{M} \dot{\underset{\sim}{x}}$

Sea $f = \underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x}$ un escalar

$$f = f^T = (\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x})^T = \underset{\sim}{x}^T \underset{\sim}{A}^T \underset{\sim}{x} = -\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} = -f$$

$\Rightarrow f = 0$

$\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} = 0$, para $\underset{\sim}{x} \neq \underset{\sim}{0}$

$\underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} = 0$ si $\underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{0}$

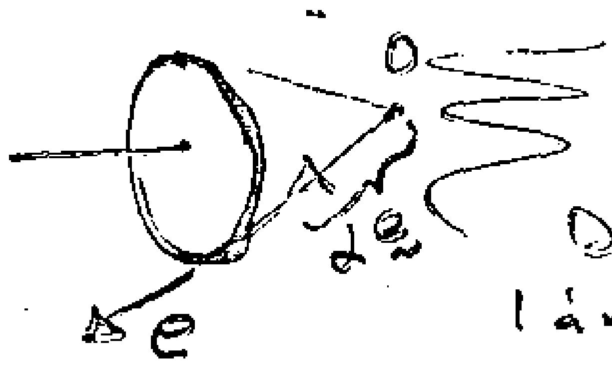
$$\underline{A}\,\underline{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \lambda_3 x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x \neq \underline{0}$. ~~La única~~ Una posibilidad es

$$\lambda_1, \lambda_2 \neq 0, \quad \underline{\lambda_3 = 0}, \quad x_1 = x_2 = 0, \quad x_3 \neq 0$$

Hecho: $\underset{\sim}{\Omega}$ tiene un valor característico nulo. Existe entonces un vector $\underline{e}$ tal que

$$\underset{\sim}{\Omega}\,\underline{e} = \underline{0}$$



Si $\underline{r} = \alpha\,\underline{e}$

$$\underset{\sim}{v} = \alpha\,\underset{\sim}{\Omega}\,\underline{e} = \underline{0}$$

De la ec. (8.1) de la lámina 8:

$$\underset{\sim}{r}_1 = \underline{a}_1 + \underset{\sim}{Q}(\underset{\sim}{r}_0 - \underline{a}_0) \qquad (8.1)$$

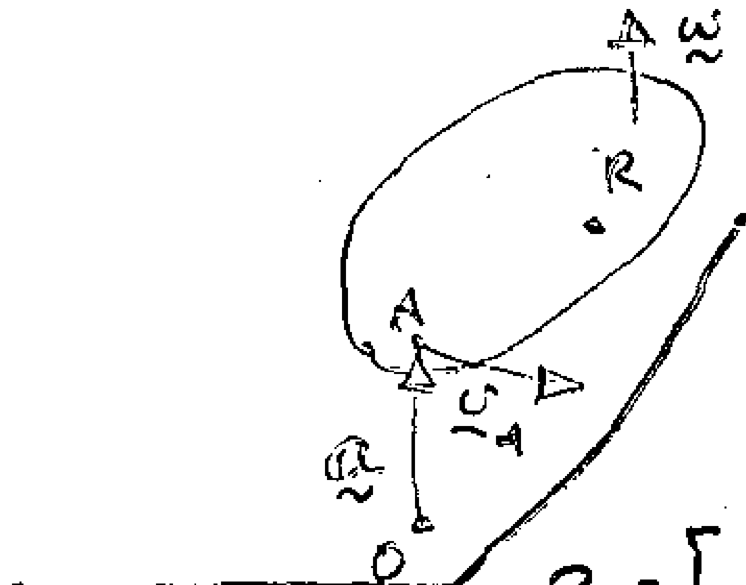
$$\dot{\underset{\sim}{r}}_1 = \underset{\sim}{v} = \underbrace{\dot{\underline{a}}_1}_{\underline{v}_A} + \dot{\underline{Q}}(\underline{r}_0 - \underline{a}_0) \qquad (d)$$

$$(8.1) \Rightarrow \underline{r}_1 - \underline{a}_1 = \underline{Q}(\underline{r}_0 - \underline{a}_0) \Rightarrow \underline{r}_0 - \underline{a}_0 = \underline{Q}^T(\underline{r}_1 - \underline{a}_1) \quad (e)$$

(e) en (d) $\Rightarrow$ $v = \underset{\sim}{v}_A + \dot{\underline{Q}}\,\underline{Q}^T(\underline{r}_1 - \underline{a}_1) =$

$$\underbrace{\underset{\sim}{v}_{R/A}}_{\underline{v} - \underline{v}_A = \underline{\Omega}(\underline{r}_1 - \underline{a}_1)} \qquad \underbrace{v_{R/A}}_{} = \underline{v}_A + \underset{\sim}{\Omega}(\underline{r}_1 - \underline{a}_1) \qquad (f)$$

Teorema: Dado un cuerpo rígido en movimiento general, existe un conjunto de puntos del cuerpo cuya velocidad es de magnitud mínima. Ese conjunto se encuentra sobre una recta paralela al eje instantáneo de rotación y la velocidad de magnitud mínima es también paralela a ese eje.



$\underset{\sim}{\omega} \parallel \underline{e} : \underset{\sim}{\Omega}\underline{e} = \underline{0}$

$\underline{v}_R = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{r} - \underline{a})$

Dado $\underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T$, existe un vector $\underset{\sim}{\omega}$ tal que

$$\underset{\sim}{\Omega}\underline{r} = \underset{\sim}{\omega} \times \underline{r} :$$

$$\underset{\sim}{\Omega} = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix}, \quad \underset{\sim}{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\underset{\sim}{\Omega}\underline{r} = \begin{bmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Omega_{12}y + \Omega_{13}z \\ -\Omega_{12}x + \Omega_{23}z \\ -\Omega_{13}x - \Omega_{23}y \end{bmatrix}$$

$$\underset{\sim}{\omega} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \underline{i}(\omega_2 z - \omega_3 y) + \underline{j}(\omega_3 x - \omega_1 z) + \underline{k}(\omega_1 y - \omega_2 x)$$

$$\Rightarrow \omega_2 = \Omega_{13}, \quad \omega_3 = -\Omega_{12}, \quad \omega_1 = -\Omega_{23}$$

$\underset{\sim}{v} \equiv \underset{\sim}{v}_R = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{r} - \underset{\sim}{a})$, $\underset{\sim}{r} = \underset{\sim}{r}(t)$, $\underset{\sim}{a} = \underset{\sim}{a}(t)$

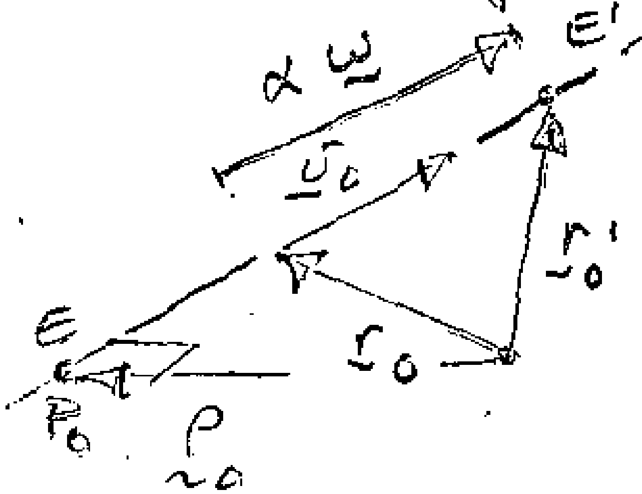
$\psi = \underset{\sim}{v}^T \underset{\sim}{v} = \|\underset{\sim}{v}\|^2$

ψ es estacionaria cuando $\psi'(\underset{\sim}{r}) = \dfrac{\partial \psi}{\partial \underset{\sim}{r}} = \underset{\sim}{0}$

$$\frac{\partial \psi}{\partial \underset{\sim}{r}} = \underbrace{\left[\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{r}}\right]^T}_{\underset{\sim}{\Omega}} \underbrace{\frac{\partial \psi}{\partial \underset{\sim}{v}}}_{2\underset{\sim}{v}} \Rightarrow \underset{\sim}{\Omega}^T \underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \underset{\sim}{\Omega}\, \underset{\sim}{v} = \underset{\sim}{0}$$

$\Rightarrow \underset{\sim}{v}$ es $\parallel$ a $\underset{\sim}{e}$

La velocidad $\underset{\sim}{v}$ de magnitud mínima es $\parallel$ al eje i. r. Llámese $\underset{\sim}{v}_0$ a esa vel.



$\underset{\sim}{\omega} \parallel \underset{\sim}{e}$

$\underset{\sim}{r}_0' \equiv \underset{\sim}{r}_0 + \alpha \underset{\sim}{\omega}$

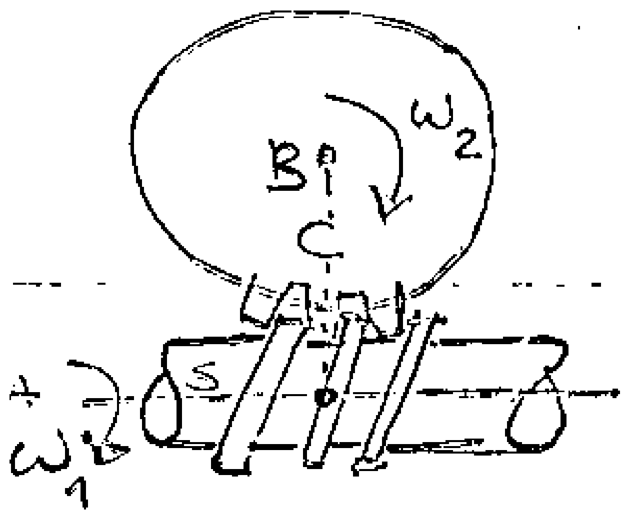
$$\underset{\sim}{v}_0' \equiv \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{r}_0 + \alpha \underset{\sim}{\omega} - \underset{\sim}{a}) = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{r}_0 - \underset{\sim}{a}) + \alpha \underset{\sim}{\Omega}\underset{\sim}{\omega} = \underset{\sim}{v}_0$$

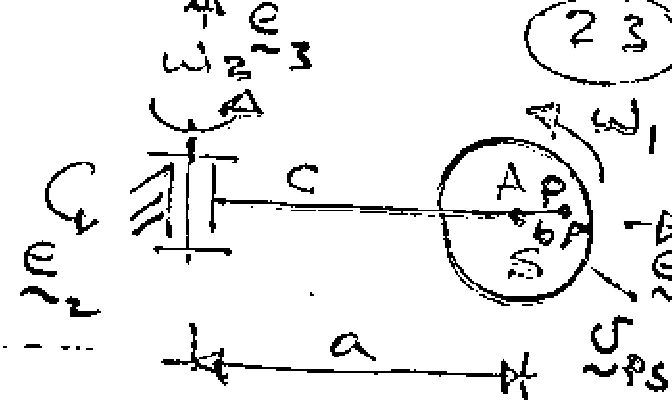
$$\underset{\sim}{v}_0 = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{\rho}_0 - \underset{\sim}{a}) = \underset{\sim}{v}_A + \underset{\sim}{\omega} \times (\underset{\sim}{\rho}_0 - \underset{\sim}{a})$$

$$\Rightarrow \underset{\sim}{\omega} \times \underset{\sim}{v}_0 = \underset{\sim}{\omega} \times \underset{\sim}{v}_A + \underbrace{\underset{\sim}{\omega} \times [\underset{\sim}{\omega} \times (\underset{\sim}{\rho}_0 - \underset{\sim}{a})]}_{[\underset{\sim}{\omega} \cdot (\underset{\sim}{\rho}_0 - \underset{\sim}{a})]\underset{\sim}{\omega} - \omega^2(\underset{\sim}{\rho}_0 - \underset{\sim}{a})}$$

$$\underset{\sim}{\omega} \cdot \underset{\sim}{\rho}_0 = 0 \Rightarrow \underset{\sim}{\omega} \times (\underset{\sim}{v}_0 - \underset{\sim}{v}_A) = (-\underset{\sim}{\omega} \cdot \underset{\sim}{a})\underset{\sim}{\omega} - \omega^2 \underset{\sim}{\rho}_0 + \omega^2 \underset{\sim}{a}$$

$$\Rightarrow \boxed{\underset{\sim}{\rho}_0 = \underset{\sim}{a} - \frac{1}{\omega^2}\left[\underset{\sim}{\omega} \times (\underset{\sim}{v}_0 - \underset{\sim}{v}_A) + (\underset{\sim}{\omega} \cdot \underset{\sim}{a})\, \underset{\sim}{\omega}\right]}$$



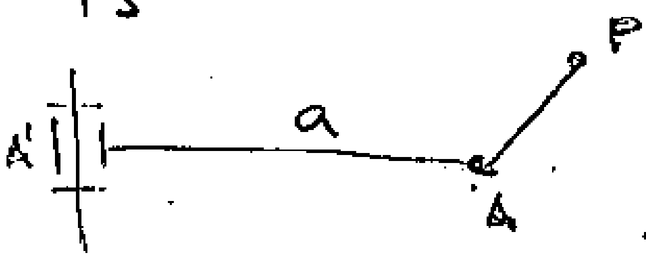


$$\underset{\sim}{v}_{PS} = \underset{\sim}{v}_{PC} + \underset{\sim}{v}_{PS/C}$$

$$\underset{\sim}{v}_{PC} = \omega_2 (a+b)\, \underset{\sim}{e}_2$$

$$\underset{\sim}{v}_{PS/C} = \omega_1 b\, \underset{\sim}{e}_3$$

$$\underset{\sim}{v}_{PS} = \omega_2 (a+b)\, \underset{\sim}{e}_2 + \omega_1 b\, \underset{\sim}{e}_3$$



$$\underset{\sim}{v}_{PS} = \underset{\sim}{v}_{AS} + \underset{\sim}{v}_{PS/AS}$$

$$\underset{\sim}{v}_{AS} = \omega_2 a\, \underset{\sim}{e}_2$$

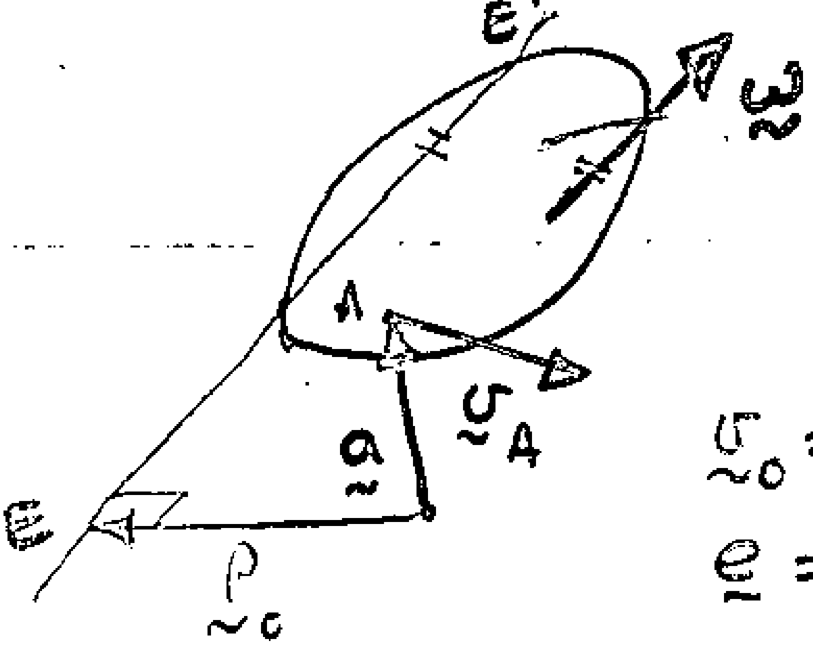
$$\underset{\sim}{v}_{PS/AS} = \underset{\sim}{\omega}_S \times \overrightarrow{AP} = (\omega_1 \underset{\sim}{e}_2 + \omega_2 \underset{\sim}{e}_3) \times \overrightarrow{AP}$$

$$\Rightarrow \underset{\sim}{v}_{PS} = \omega_2 a\, \underset{\sim}{e}_2 + (\omega_1 \underset{\sim}{e}_2 + \omega_2 \underset{\sim}{e}_3) \times \overrightarrow{AP} =$$

$$= \omega_2 a\, \underset{\sim}{e}_2 + \omega_1 \underbrace{\underset{\sim}{e}_2 \times \overrightarrow{AP}}_{\perp\, \underset{\sim}{e}_2} + \omega_2 \underbrace{\underset{\sim}{e}_3 \times \overrightarrow{AP}}_{\parallel\, \underset{\sim}{e}_2} =$$

$$= \underbrace{\omega_2 (a\, \underset{\sim}{e}_2 + \underset{\sim}{e}_3 \times \overrightarrow{AP})}_{\parallel\, \underset{\sim}{e}_2} + \omega_1 \underbrace{\underset{\sim}{e}_2 \times \overrightarrow{AP}}_{\perp\, \underset{\sim}{e}_2}$$

Parámetros del T. I.

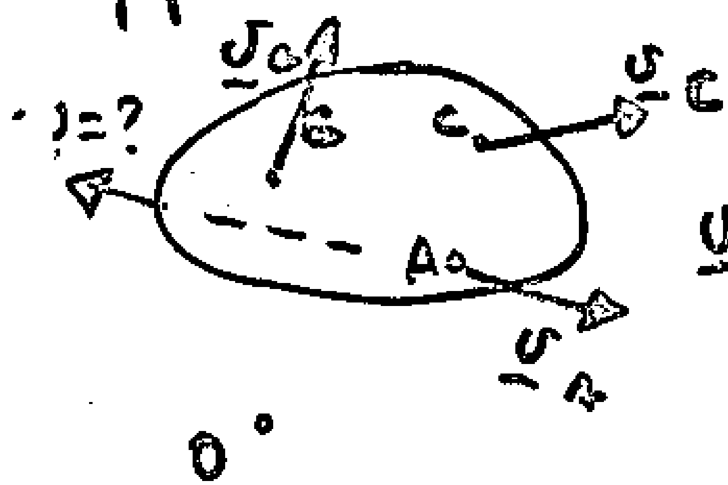


1. $\underset{\sim}{\omega}$
2. $\underset{\sim}{\rho}_0$
3. $\measuredangle\, \underset{\sim}{v}_0$

$$\underset{\sim}{v}_0 = (\underset{\sim}{v}_A \cdot \underset{\sim}{e})\, \underset{\sim}{e}$$

$$\underset{\sim}{e} = \frac{\underset{\sim}{\omega}}{\|\underset{\sim}{\omega}\|}$$

pp. 101 - 102 :



A, B y C no colin.

$$\underset{\sim}{v}_{C/A} = \underset{\sim}{\omega} \times (\underset{\sim}{c} - \underset{\sim}{a})$$

$$\underset{\sim}{v}_{B/A} \times \underset{\sim}{v}_{C/A} = \underset{\sim}{v}_{B/A} \times [\underset{\sim}{\omega} \times (\underset{\sim}{c} - \underset{\sim}{a})]$$

$$= [\underset{\sim}{v}_{B/A} \cdot (\underset{\sim}{c} - \underset{\sim}{a})]\, \underset{\sim}{\omega} - \underbrace{[\underset{\sim}{v}_{B/A} \cdot \underset{\sim}{\omega}]}_{\underset{\sim}{\omega} \times (\underset{\sim}{b} - \underset{\sim}{a}) \cdot \underset{\sim}{\omega}}{}^{\to 0} (\underset{\sim}{c} - \underset{\sim}{a})$$

$$\boxed{\underset{\sim}{\omega} = \frac{\underset{\sim}{v}_{B/A} \times \underset{\sim}{v}_{C/A}}{\underset{\sim}{v}_{B/A} \cdot (\underset{\sim}{c} - \underset{\sim}{a})}}$$

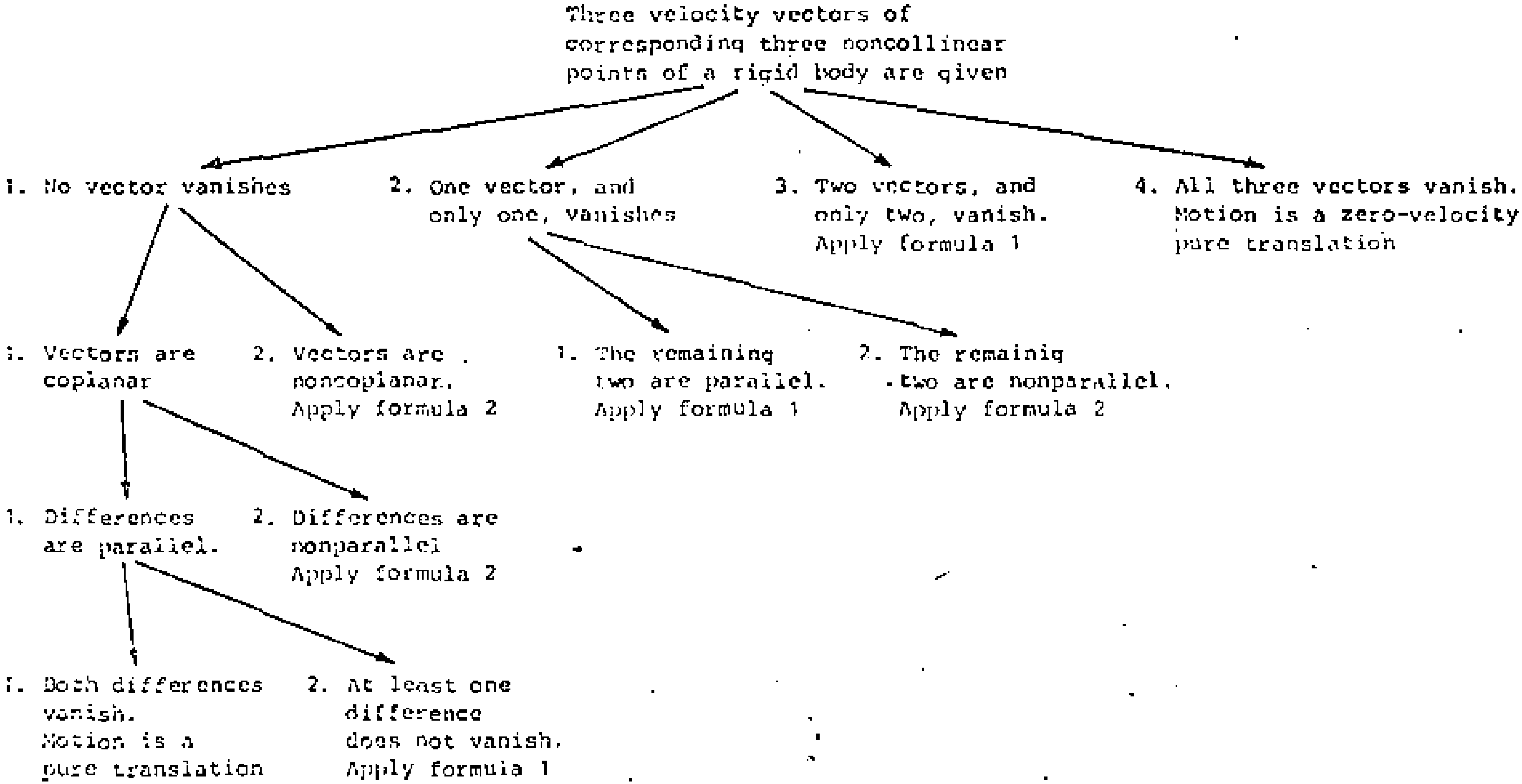
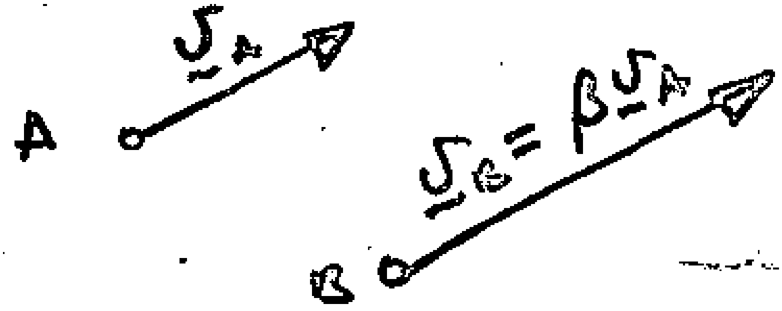


Fig 1. Tree diagram showing the different possible relationships amongst the velocities of three noncollinear points defining a rigid-body motion.



$\underline{\omega} \parallel \underline{w}$

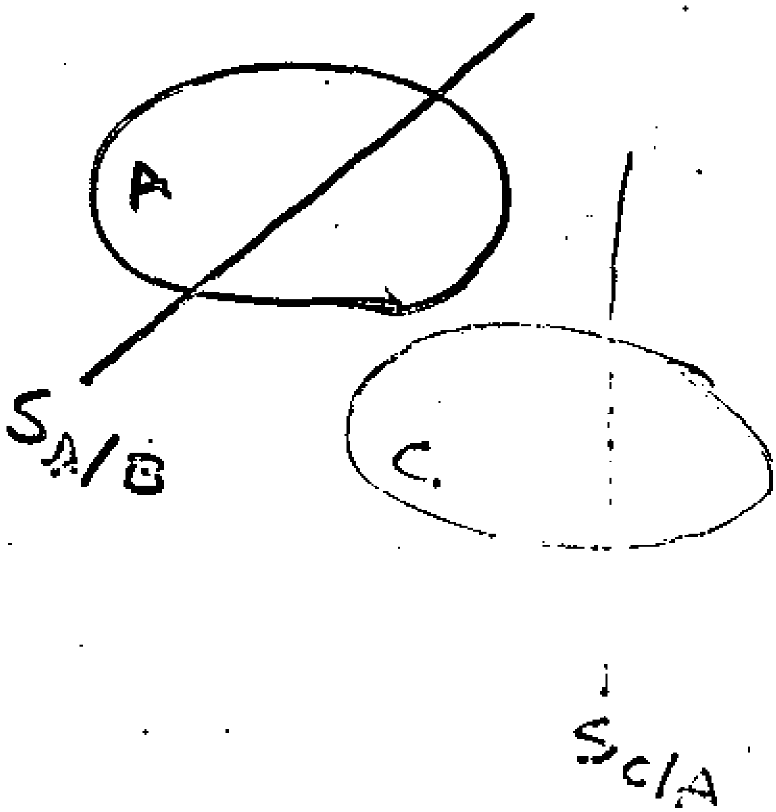
c $\underline{v}_c = 0$ $\underline{w} = \underline{b} - \underline{c} - \beta(\underline{a} - \underline{c})$

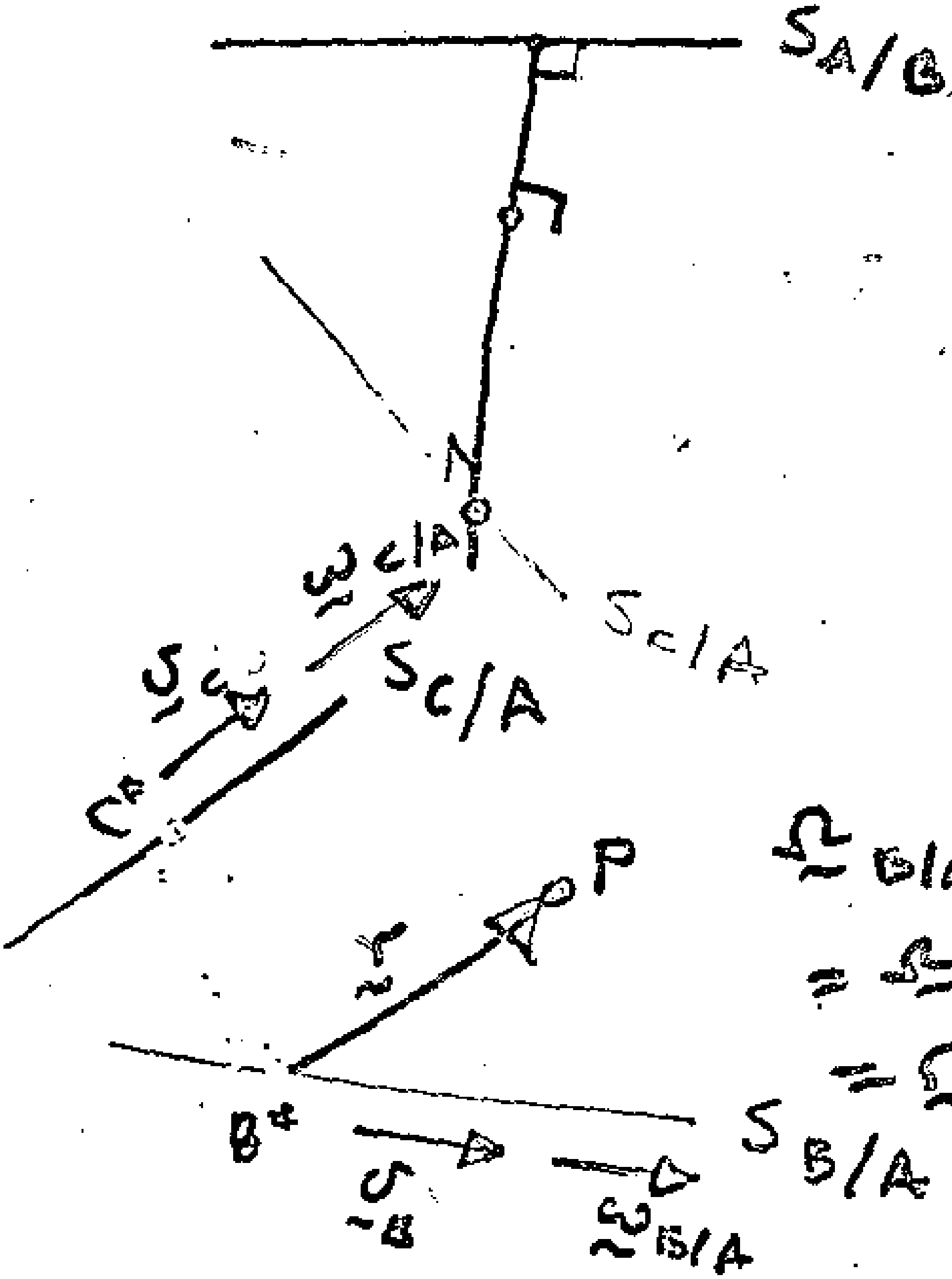
Corolario (   ) en publicaciones por separado.

$\underline{\omega} = \alpha \underline{w}$

$\underline{v}_A = \alpha \underline{w} \times (\underline{a} - \underline{c}) \Rightarrow \alpha$

El Teorema de Aronhold-Kennedy





$$\underset{\sim}{\Omega}_{B/A} - \underset{\sim}{\Omega}_{C/A} =$$
$$= \underset{\sim}{\Omega}_{B} - \underset{\sim}{\Omega}_{A} - (\underset{\sim}{\Omega}_{C} - \underset{\sim}{\Omega}_{A})$$
$$= \underset{\sim}{\Omega}_{B} - \underset{\sim}{\Omega}_{C} = \underset{\sim}{\Omega}_{B}$$

$$\underset{\sim}{v} = \underset{\sim}{v}_{PB} - \underset{\sim}{v}_{PC}$$

$$\underset{\sim}{v}_{PB} = \underset{\sim}{v}_{B} + \underset{\sim}{\Omega}_{B/A}\underset{\sim}{r}, \quad \underset{\sim}{v}_{PC} = \underset{\sim}{v}_{C} + \underset{\sim}{\Omega}_{C/A}(\underset{\sim}{r} - \underset{\sim}{c})$$

$$\underset{\sim}{v} = \underset{\sim}{v}_{B} + \underset{\sim}{\Omega}_{B/A}\underset{\sim}{r} - [\underset{\sim}{v}_{C} + \underset{\sim}{\Omega}_{C/A}(\underset{\sim}{r} - \underset{\sim}{c})] =$$
$$= \underset{\sim}{v}_{B/C} + \underset{\sim}{\Omega}_{B/C}\underset{\sim}{r} + \underset{\sim}{\Omega}_{C}\underset{\sim}{c}_{/A}$$

$$\phi = \underset{\sim}{v}^{T}\underset{\sim}{v} \longrightarrow \min$$

$$\frac{\partial \varphi}{\partial \underset{\sim}{r}} = \underset{\sim}{0}$$

$$\frac{\partial \varphi}{\partial \underset{\sim}{r}} = \underbrace{\left[\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{r}}\right]^T}_{\underset{\sim}{\Omega}_{B/c}} \underbrace{\frac{\partial \varphi}{\partial \underset{\sim}{v}}}_{2\underset{\sim}{v}} \Rightarrow \underset{\sim}{\Omega}^T_{B/c}\underset{\sim}{v} = \underset{\sim}{0} \Rightarrow \underset{\sim}{\omega}_{B/c}{}^{\times}\underset{\sim}{v} = \underset{\sim}{0}$$

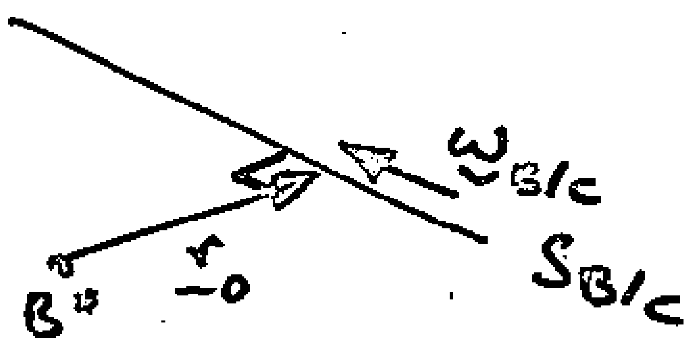
La $\underset{\sim}{v}$ de magnitud mínima es paralela a $\underset{\sim}{\omega}_{B/c}$

$$\underset{\sim}{\Omega}_{B/c}(\underset{\sim}{v}_{B/c} + \underset{\sim}{\Omega}_c\underset{\sim}{c} + \underset{\sim}{\Omega}_{B/c}\underset{\sim}{r}) = \underset{\sim}{0}$$

$$\Rightarrow \underset{\sim}{\omega}_{B/c}{}^{\times}[\underset{\sim}{v}_{B/c} + \underset{\sim}{\omega}_c{}^{\times}\underset{\sim}{c} + \underset{\sim}{\omega}_{B/c}{}^{\times}\underset{\sim}{r}] = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{B/c}{}^{\times}\underset{\sim}{v}_{B/c} + \underset{\sim}{\omega}_{B/c}{}^{\times}(\underset{\sim}{\omega}_c{}^{\times}\underset{\sim}{c}) + \underbrace{\underset{\sim}{\omega}_{B/c}{}^{\times}(\underset{\sim}{\omega}_{B/c}{}^{\times}\underset{\sim}{r})}_{(\underset{\sim}{\omega}_{B/c}\cdot\underset{\sim}{r})\underset{\sim}{\omega}_{B/c} - \omega^2_{B/c}\underset{\sim}{r}} = \underset{\sim}{0}$$

Si: $\underset{\sim}{\omega}_{B/c}\cdot\underset{\sim}{r}_0 = 0$



$$\omega^2_{B/c}\underset{\sim}{r}_0 = \underset{\sim}{\omega}_{B/c}{}^{\times}[\underset{\sim}{v}_{B/c} + \underset{\sim}{\omega}_c{}^{\times}\underset{\sim}{c}]$$

$$\Rightarrow \underset{\sim}{r}_0 = \frac{1}{\omega^2_{B/c}}\underset{\sim}{\omega}_{B/c}{}^{\times}[\underset{\sim}{v}_{B/c} + \underset{\sim}{\omega}_c{}^{\times}\underset{\sim}{c}]$$

$$\underset{\sim}{r}_0 \times \underset{\sim}{c} = \underset{\sim}{0}$$

$$\underset{\sim}{r}_0 \times \underset{\sim}{c} = \frac{1}{\omega_{B/C}^2}\left[\underset{\sim}{\omega}_{B/C} \times \left(\underset{\sim}{v}_{B/C} + \underset{\sim}{\omega}_C \times \underset{\sim}{c}\right)\right] \times \underset{\sim}{c} =$$

$$= \frac{1}{\omega_{B/C}^2}\left\{(\underset{\sim}{\omega}_{B/C} \times \underset{\sim}{v}_{B/C}) \times \underset{\sim}{c} + \left[\underset{\sim}{\omega}_{B/C} \times (\underset{\sim}{\omega}_C \times \underset{\sim}{c})\right] \times \underset{\sim}{c}\right\}$$

$$\underset{\sim}{\omega}_{B/C} \times \underset{\sim}{v}_{B/C} = (\underset{\sim}{\omega}_B - \underset{\sim}{\omega}_C) \times (\underset{\sim}{v}_B - \underset{\sim}{v}_C) =$$
$$= -\underset{\sim}{\omega}_B \times \underset{\sim}{v}_C - \underset{\sim}{\omega}_C \times \underset{\sim}{v}_B$$

$$(\underset{\sim}{\omega}_{B/C} \times \underset{\sim}{v}_{B/C}) \times \underset{\sim}{c} = -(\underset{\sim}{\omega}_B \times \underset{\sim}{v}_C) \times \underset{\sim}{c} - (\underset{\sim}{\omega}_C \times \underset{\sim}{v}_B) \times \underset{\sim}{c}$$
$$= -(\underset{\sim}{\omega}_B \cdot \underset{\sim}{c})^{0}\,\underset{\sim}{v}_C + (\underset{\sim}{v}_C \cdot \underset{\sim}{c})^{0}\,\underset{\sim}{\omega}_B - (\underset{\sim}{\omega}_C \cdot \underset{\sim}{c})^{0}\,\underset{\sim}{v}_B + (\underset{\sim}{v}_B \cdot \underset{\sim}{c})^{0}\,\underset{\sim}{\omega}_C$$
$$= \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{B/C} \times (\underset{\sim}{\omega}_C \times \underset{\sim}{c}) = (\underset{\sim}{\omega}_{B/C} \cdot \underset{\sim}{c})^{0}\,\underset{\sim}{\omega}_C - (\underset{\sim}{\omega}_{B/C} \cdot \underset{\sim}{\omega}_C)\,\underset{\sim}{c}$$

$$\underset{\sim}{\omega}_{B/C} \cdot \underset{\sim}{c} = (\underset{\sim}{\omega}_B - \underset{\sim}{\omega}_C) \cdot \underset{\sim}{c} = \underset{\sim}{\omega}_B \cdot \underset{\sim}{c}^{\,0} - \underset{\sim}{\omega}_C \cdot \underset{\sim}{c}^{\,0} = 0$$

$$\underset{\sim}{\omega}_{B/C} \cdot \underset{\sim}{\omega}_C = (\underset{\sim}{\omega}_B - \underset{\sim}{\omega}_C) \cdot \underset{\sim}{\omega}_C = \omega_B \cdot \underset{\sim}{\omega}_C - \underset{\sim}{\omega}_C^2$$

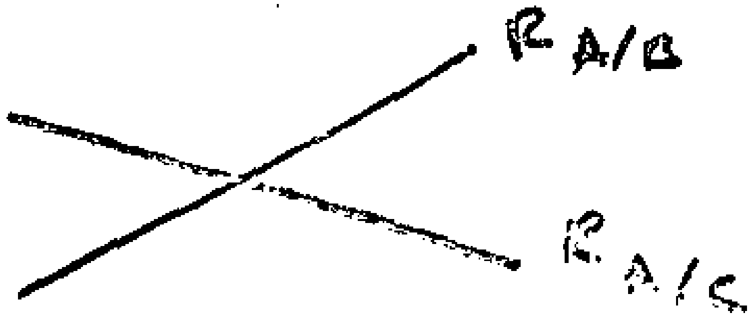
$$\left[\underset{\sim}{\omega}_{B/C} \times (\underset{\sim}{\omega}_C \times \underset{\sim}{c})\right] \times \underset{\sim}{c} = -(\underset{\sim}{\omega}_{B/C} \cdot \underset{\sim}{\omega}_C)\,\underset{\sim}{c} \times \underset{\sim}{c} = \underset{\sim}{0}$$

$\Rightarrow \underset{\sim}{r}_0 \times \underset{\sim}{c} = \underset{\sim}{0}$, q. e. d.

$\underset{\sim}{\omega}_A$, $\underset{\sim}{\omega}_B$ y $\underset{\sim}{\omega}_C$

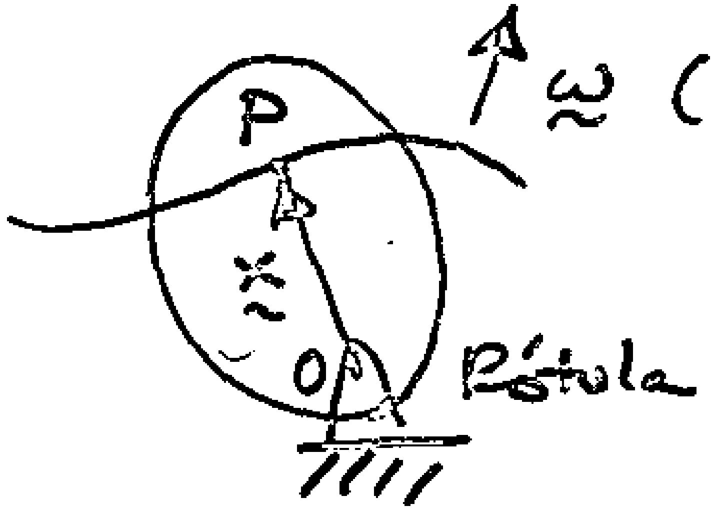
$\underset{\sim}{\omega}_B - \underset{\sim}{\omega}_A$, $\underset{\sim}{\omega}_C - \underset{\sim}{\omega}_A$, $\underset{\sim}{\omega}_B - \underset{\sim}{\omega}_C$ son l. d.

Si A y B tienen mov. de rotación pura, sea $R_{A/B}$ el eje i.r.p.



Si además A y C tienen mov. de r.p., sea $R_{A/C}$ el eje i.r.p.

⟹ B y C también tienen mov. de r.p. si y sólo si $R_{A/B}$ y $R_{A/C}$ se intersecan. En este caso, $R_{B/C}$ pasa por esa intersección y los tres ejes son coplanares

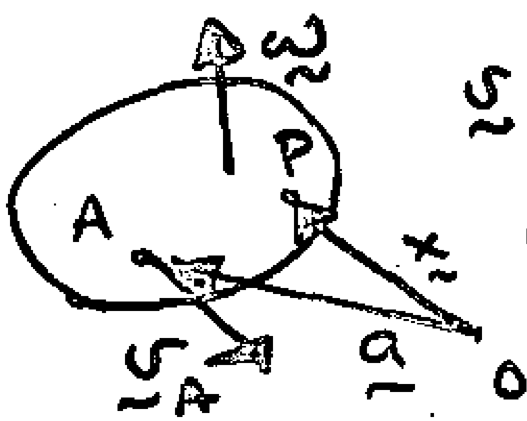


$$v(t) = \Omega x = \omega \times x$$

$$a(t) \cong \dot{v}(t) = \dot{\Omega} x + \Omega \underbrace{\dot{x}}_{v = \Omega x}$$

matriz de acel. ang.

$$a(t) = \overbrace{(\dot{\Omega} + \Omega^2)}^{\text{matriz de acel. ang.}} x = \dot{\omega} \times x + \omega \times (\omega \times x)$$



$$v = v_A + \Omega (x - a) \qquad (31.1)$$

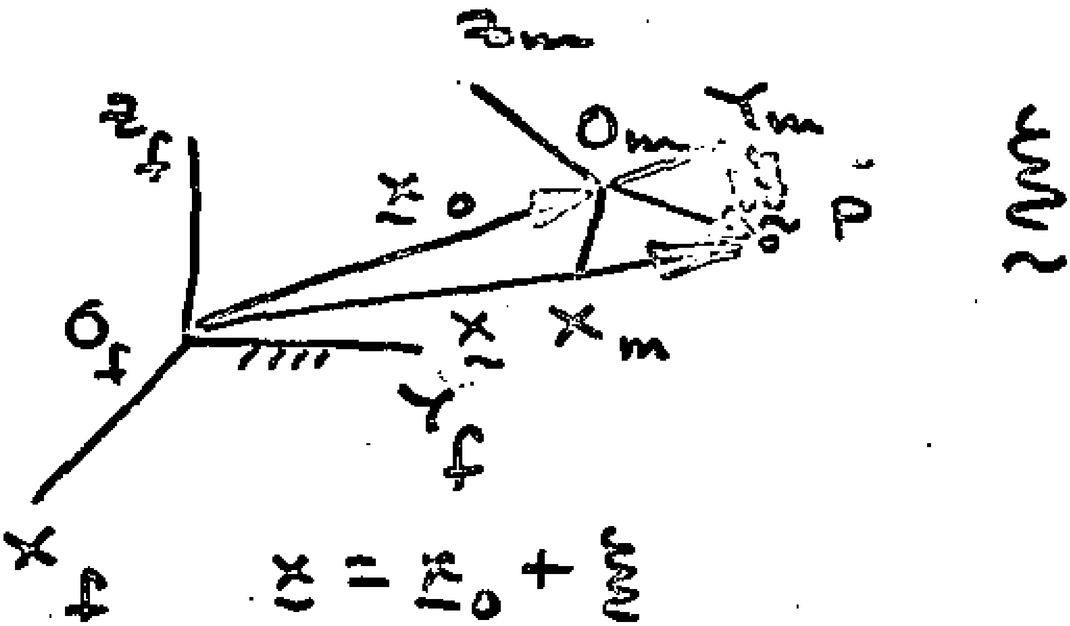
$$a \cong \dot{v} = \overbrace{\dot{v}_A}^{= a_A} + \dot{\Omega}(x - a) + \Omega(\underbrace{\dot{x}}_{v} - \underbrace{\dot{a}}_{v_A})$$

$$a = a_A + \dot{\Omega}(x - a) + \Omega \overbrace{(v - v_A)}^{\Omega(x - a)} \qquad (31.2)$$

(31.1) en (31.2) $\Rightarrow$

$$a = a_A + (\dot{\Omega} + \Omega^2)(x - a)$$

$$a = 0 \Rightarrow (\dot{\Omega} + \Omega^2)(x - a) = - a_A$$



$$\mathbf{x} = \mathbf{x}_0 + \boldsymbol{\xi}$$

$$[\mathbf{x}]_f = [\mathbf{x}_0]_f + [\boldsymbol{\xi}]_f$$

$$[\boldsymbol{\xi}]_f = [\mathbf{Q}]_f [\boldsymbol{\xi}]_m$$

$$[\mathbf{x}]_f = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow \mathbf{x} = a\,\mathbf{i}_f + b\,\mathbf{j}_f + c\,\mathbf{k}_f$$

$$[\mathbf{x}]_m = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} \Leftrightarrow \mathbf{x} = a'\,\mathbf{i}_m + b'\,\mathbf{j}_m + c'\,\mathbf{k}_m$$

$$[\mathbf{x}]_f = [\mathbf{x}_0]_f + [\mathbf{Q}]_f [\boldsymbol{\xi}]_m \quad \leftarrow$$

$$[\dot{\mathbf{x}}]_f = [\dot{\mathbf{x}}_0]_f + [\dot{\mathbf{Q}}]_f^{\downarrow} [\boldsymbol{\xi}]_m + [\mathbf{Q}]_f [\dot{\boldsymbol{\xi}}]_m$$

$$[\dot{\mathbf{x}}]_f = \dot{a}\,\mathbf{i}_f + \dot{b}\,\mathbf{j}_f + \dot{c}\,\mathbf{k}_f$$

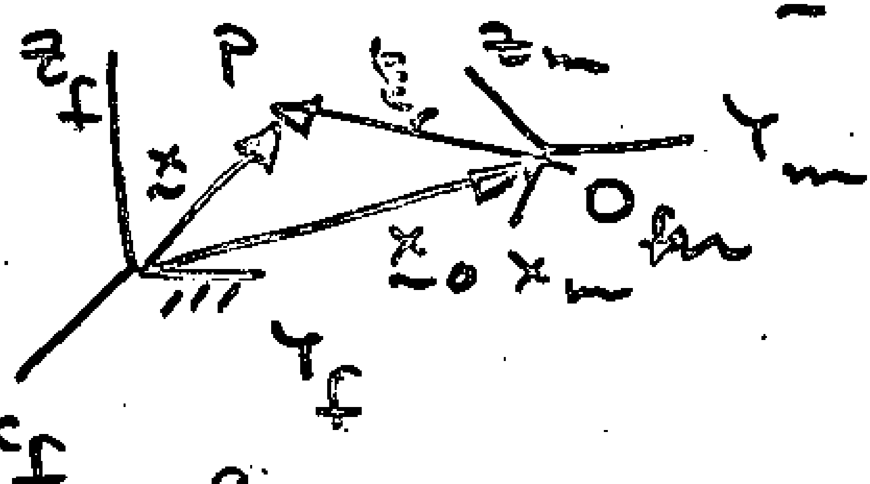
$$[\dot{\boldsymbol{\xi}}]_m = \dot{a}'\,\mathbf{i}_m + \dot{b}'\,\mathbf{j}_m + \dot{c}'\,\mathbf{k}_m$$

$$\downarrow = [\mathbf{Q}^T]_f [\mathbf{Q}]_f = \mathbf{I}$$

$$[\underset{\sim}{v}]_f = [v_0]_f + \underbrace{[\dot{\underset{\sim}{Q}}]_f[\underset{\sim}{Q}^T]_f}_{[\underset{\sim}{\Omega}]_f}\underbrace{[\underset{\sim}{Q}]_f[\underset{\sim}{\xi}]_m}_{[\underset{\sim}{\xi}]_f} + [\underset{\sim}{Q}]_f[\dot{\underset{\sim}{\xi}}]_m$$

O_{fm} ↓

$$[\underset{\sim}{v}]_f = [\underset{\sim}{v}_0]_f + [\underset{\sim}{\Omega}]_f\overbrace{[\underset{\sim}{Q}]_f[\underset{\sim}{\xi}]_m}^{[\underset{\sim}{\xi}]_f} + [\underset{\sim}{Q}]_f[\dot{\underset{\sim}{\xi}}]_m$$



$$[\underset{\sim}{a}]_f = [\dot{\underset{\sim}{v}}]_f = [\overset{a_0}{\dot{\underset{\sim}{v}}_0}]_f + [\dot{\underset{\sim}{\Omega}}]_f[\underset{\sim}{Q}]_f[\underset{\sim}{\xi}]_m +$$

$$+ [\underset{\sim}{\Omega}]_f[\dot{\underset{\sim}{Q}}]_f[\underset{\sim}{\xi}]_m + [\underset{\sim}{\Omega}]_f[\underset{\sim}{Q}]_f[\dot{\underset{\sim}{\xi}}]_m$$

$$+ [\dot{\underset{\sim}{Q}}]_f[\dot{\underset{\sim}{\xi}}]_m + [\underset{\sim}{Q}]_f[\ddot{\underset{\sim}{\xi}}]_m$$

$$\downarrow = [\underset{\sim}{Q}^T]_f[\underset{\sim}{Q}]_f = \underset{\sim}{I}$$

$$[\underset{\sim}{a}]_f = [\underset{\sim}{a}_0]_f + [\dot{\underset{\sim}{\Omega}}]_f[\underset{\sim}{Q}]_f[\underset{\sim}{\xi}]_m +$$

$$+ \overline{[\underset{\sim}{\Omega}]_f[\dot{\underset{\sim}{Q}}]_f[\underset{\sim}{Q}^T]_f}[\underset{\sim}{Q}]_f[\underset{\sim}{\xi}]_m + [\underset{\sim}{\Omega}]_f[\underset{\sim}{Q}]_f[\dot{\underset{\sim}{\xi}}]_m$$

$$+ [\dot{\underset{\sim}{Q}}]_f[\underset{\sim}{Q}^T]_f[\dot{\underset{\sim}{\xi}}]_m + [\underset{\sim}{Q}]_f[\ddot{\underset{\sim}{\xi}}]_m =$$

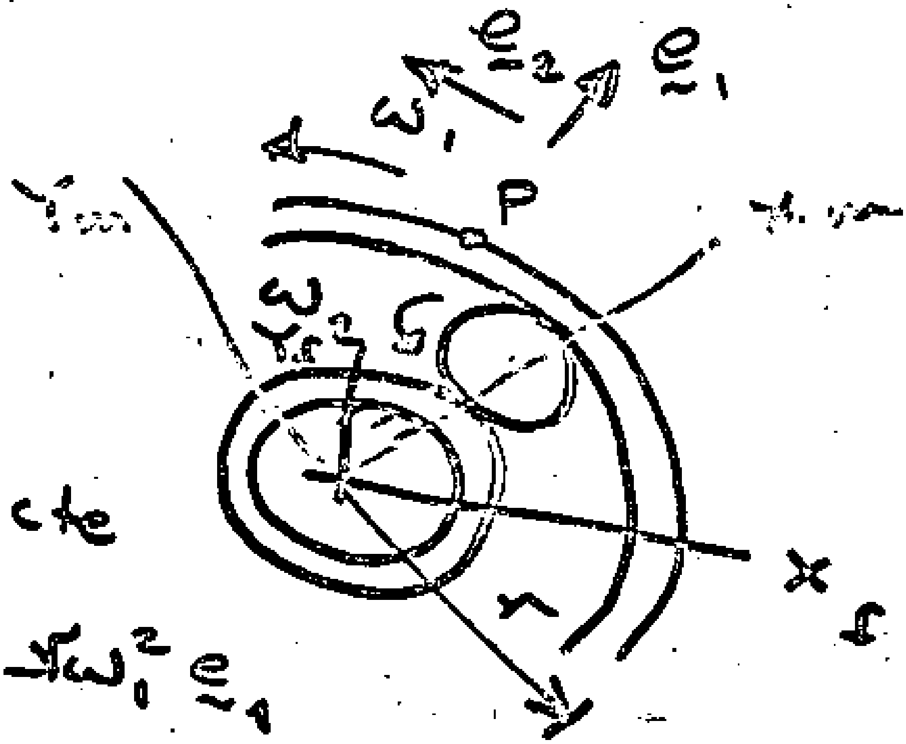
$[\underset{\sim}{Q}]_f$

$$[a]_f = [a_0]_f + [\dot{\Omega}]_f [Q]_f [\xi]_m + [\Omega^2]_f [Q]_f [\xi]_m + 2 [\Omega]_f [Q]_f [\dot{\xi}]_m + [Q]_f [\ddot{\xi}]_m$$

(34)

acel. Coriolis

Teorema de Coriolis



ω_1 es cte

$$a_P = -r\omega_1^2 e_1$$

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

-Anexos-

Dr Jorge Angeles Alvarez

Junio, 1981

Palacio de Minería Calle de Tacuba 5 primer piso México 1, D. F. Tel: 521-40-20 Apdo. Postal M-2285

(a)

Ec. (2.8.6) $\Rightarrow [\underset{\sim}{v}_P]_f = [\underset{\sim}{v}_O]_f + [\underset{\sim}{\Omega}]_f [\underset{\sim}{Q}]_f [\underset{\sim}{\xi}]_m +$

$$+ [\underset{\sim}{Q}]_f [\dot{\underset{\sim}{\xi}}]_m$$

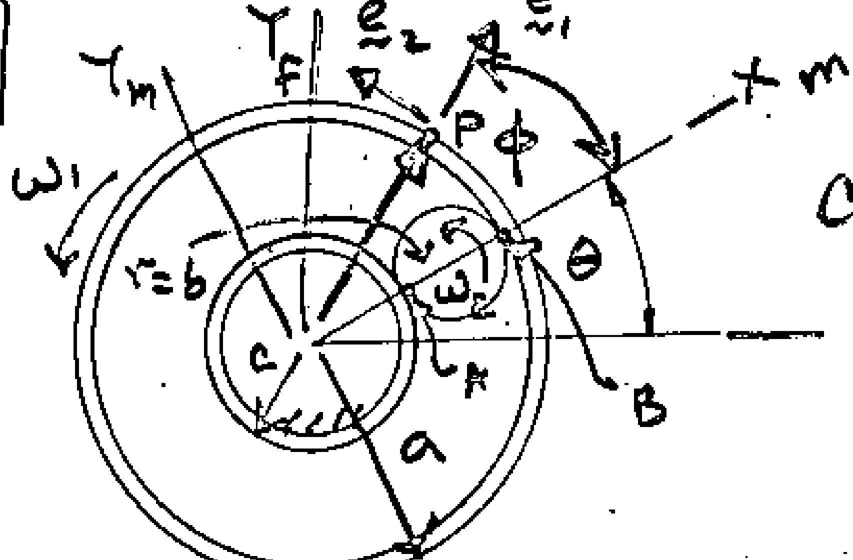
$$[\underset{\sim}{v}_O]_f = \underset{\sim}{0}, \quad [\underset{\sim}{Q}]_f = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}, [\underset{\sim}{\Omega}]_f = [\dot{\underset{\sim}{Q}}]_f [\underset{\sim}{Q}^T]_f$$

$$[\dot{\underset{\sim}{Q}}]_f = \dot{\theta} \begin{bmatrix} -s\theta & -c\theta \\ c\theta & -s\theta \end{bmatrix}, [\underset{\sim}{Q}^T]_f = \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}, \Rightarrow [\underset{\sim}{\Omega}]_f = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \dot{\theta}$$

$[\underset{\sim}{\xi}]_m = a \begin{bmatrix} c\varphi \\ s\varphi \end{bmatrix}$

$[\dot{\underset{\sim}{\xi}}]_m = a\dot{\varphi} \begin{bmatrix} -s\varphi \\ c\varphi \end{bmatrix}$

$\omega_1 = $ cte.



C: centro de la esfera

$$\Rightarrow [\underset{\sim}{v}_P]_f = a\dot{\theta} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} c\varphi \\ s\varphi \end{bmatrix}}_{\begin{bmatrix} c\theta c\varphi - s\theta s\varphi \\ s\theta c\varphi + c\theta s\varphi \end{bmatrix}} + a\dot{\varphi} \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} -s\varphi \\ c\varphi \end{bmatrix}}_{\begin{bmatrix} -c\theta s\varphi - s\theta s\varphi \\ -s\theta s\varphi + c\theta c\varphi \end{bmatrix}} =$$

$$= a\dot{\theta} \begin{bmatrix} -s(\theta+\varphi) \\ +c(\theta+\varphi) \end{bmatrix} + a\dot{\varphi} \begin{bmatrix} -s(\theta+\varphi) \\ +c(\theta+\varphi) \end{bmatrix} = a(\dot{\theta}+\dot{\varphi}) \begin{bmatrix} -s(\theta+\varphi) \\ c(\theta+\varphi) \end{bmatrix}$$

Pero $\dot{\theta} + \dot{\varphi} = \omega_1$

$\Rightarrow [\underset{\sim}{v}_P]_f = a\omega_1 \underset{\sim}{e}_2$ OK

(2.13.7) ⇒

$$[a_P]_f = [a_0]_f + [\dot{\Omega} + \Omega^2]_f [Q]_f [\xi]_m + [Q]_f [\ddot{\xi}]_m + 2[\Omega]_f [Q]_f [\dot{\xi}]_m$$

$$[a_0]_f = 0, \quad [\dot{\Omega}]_f = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ddot{\theta}$$

Pero de las condiciones de rodamiento puro: $v_A = 0$ y $v_B = 2\omega_2 b j_m = (c+2b)\omega_1 j_m$, y $\omega_2 b j_m = \dot{\theta}(c+b) j_m$ se tiene

$$\dot{\theta} = \frac{\omega_2 b}{c+b}, \quad \omega_2 = \frac{c+2b}{2b}\omega_1 \Rightarrow \dot{\theta} = \frac{c+2b}{2(c+b)}\omega_1$$

$$\Rightarrow \ddot{\theta} = 0 \Rightarrow [\dot{\Omega}]_f = 0$$

$$[\Omega^2]_f = \dot{\theta}^2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\dot{\theta}^2 I$$

$$[\ddot{\xi}]_m = a\ddot{\varphi} \begin{bmatrix} -s\varphi \\ c\varphi \end{bmatrix} + a\dot{\varphi}^2 \begin{bmatrix} -c\varphi \\ -s\varphi \end{bmatrix}$$

Pero $\dot{\varphi} = \omega_1 - \dot{\theta} \Rightarrow \ddot{\varphi} = \dot{\omega}_1 - \ddot{\theta} = 0$

$$\Rightarrow [a_P]_f = -\dot{\theta}^2 I a \begin{bmatrix} c(\theta+\varphi) \\ s(\theta+\varphi) \end{bmatrix} + 2a\dot{\theta}\dot{\varphi} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} -s\varphi \\ c\varphi \end{bmatrix}}_{\begin{bmatrix} -c\theta s\varphi - s\theta c\varphi \\ -s\theta s\varphi + c\theta c\varphi \end{bmatrix}}$$

$$+ a\dot{\varphi}^2 \underbrace{\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} -c\varphi \\ -s\varphi \end{bmatrix}}_{-\begin{bmatrix} c\theta c\varphi - s\theta s\varphi \\ s\theta c\varphi + c\theta s\varphi \end{bmatrix}}$$

(C)

$$\Rightarrow [\underline{a}_P]_f = -a\dot{\theta}^2 \begin{bmatrix} c(\theta+\varphi) \\ s(\theta+\varphi) \end{bmatrix} + 2a\dot{\theta}\dot{\varphi} \begin{bmatrix} -c(\theta+\varphi) \\ -s(\theta+\varphi) \end{bmatrix}$$

$$- a\dot{\varphi}^2 \begin{bmatrix} c(\theta+\varphi) \\ s(\theta+\varphi) \end{bmatrix} = -a\underbrace{(\dot{\theta}^2 + 2\dot{\theta}\dot{\varphi} + \dot{\varphi}^2)}_{\omega_1^2} \begin{bmatrix} c(\theta+\varphi) \\ s(\theta+\varphi) \end{bmatrix}$$

$$= -a\omega_1^2 \underset{\sim}{e}_1$$

DIVISION DE EDUCACION CONTINUA
FACULTAD DE INGENIERIA U.N.A.M.

FUNDAMENTOS CINEMATICOS PARA EL DISEÑO DE LAS MAQUINAS Y MECANISMOS

APENDICE

Dr. Jorge Angeles Alvarez

Junio, 1981.

APPENDIX 1

ALGEBRA OF DYADICS

Let U and V be m- and n- dimensional vector spaces over the same field F. Let $U=\{\underline{u}_1,\underline{u}_2,\ldots,\underline{u}_m\}$ and $V=\{\underline{v}_1,\underline{v}_2,\ldots,\underline{v}_n\}$ be bases for each of these spaces. The tensor product of U and V, in this order, is a space W represented as $U\otimes V$, a basis of which is defined as

$$W=\{\underline{u}_1\otimes\underline{v}_1,\underline{u}_1\otimes\underline{v}_2,\ldots,\underline{u}_1\otimes\underline{v}_n,\underline{u}_2\otimes\underline{v}_1,\ldots,\underline{u}_2\otimes\underline{v}_n,\ldots,\underline{u}_m\otimes\underline{v}_1,\ldots,\underline{u}_m\otimes\underline{v}_n\}$$

Each of $\underline{u}_i\otimes\underline{v}_j$ or, in general, any expression of the form $\underline{w}=\underline{a}\otimes\underline{b}$, where $\underline{a}\in U$ and $\underline{b}\in V$ is refered to as a dyadic. From the above definition, then any dyadic such as $\underline{w}$ can be expressed as a linear combination of the dyadics of W, i.e. as

$$\begin{aligned}\underline{w}= & w_{11}\underline{u}_1\otimes\underline{v}_1+\ldots+w_{1n}\underline{u}_1\otimes\underline{v}_n+\\ & +w_{21}\underline{u}_2\otimes\underline{v}_1+\ldots+w_{2n}\underline{u}_2\otimes\underline{v}_n+\\ & +w_{m1}\underline{u}_m\otimes\underline{v}_1+\ldots+w_{mn}\underline{u}_m\otimes\underline{v}_n\end{aligned} \qquad (A1.1)$$

Just in the same way as vectors can be expressed as one-dimensional arrays of numbers (of the field over which the corresponding vector space is defined), these numbers being the vector components referred to a given basis, dyadics are also expressed as arrays (in this case, two-dimensional) of their components. Thus, the above dyadic $\underline{w}$ is represented as

$$\begin{matrix} w_{11} & w_{12} & \ldots & w_{1n} \\ w_{21} & w_{22} & \ldots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{m1} & w_{m2} & \ldots & w_{mn} \end{matrix} \qquad (A1.2)$$

Hence, dyadics or second-rank tensors, as they are also referred to, are isomorphic to matrices. In particular, dyadics belonging to the space W as defined above, are isomorphic to $m\times n$ matrices, that is to say, those operations defined for matrices find their counterparts in the case of dyadics. For instance, given two dyadics, $\underline{p}$ and $\underline{q}$, both of W, they are equal if, and only if

$$p_{ij}=q_{ij} \ ; \ i=1,\ldots,m; \ j=1,\ldots,n$$

where subindices i, j refer to the (i,j)th component of the corresponding dyadic. The operations of addition of dyadics and multiplication by a scalar are defined in a similar fashion. Sometimes, for short, a dyadic $\underline{a}\otimes\underline{b}$ is represented simply as $\underline{ab}$. The latter practice is followed throughout. The usual multiplication of a matrix $\underline{A}$ by a vector $\underline{c}$ corresponds to the dot multiplication of a dyadic $\underline{ab}$ times the same vector $\underline{c}$ where the components of $\underline{ab}$ are identical to the corresponding entries of $\underline{A}$. This product is represented as

$$\underline{ab}\cdot\underline{c}=(\underline{b}\cdot\underline{c})\underline{a}$$

i.e. when a dyadic is dot-multiplied from the right by a vector, the result is a new vector whose value equals that of the left vector of the dyadic times the dot product of its right vector and the vector multiplying the dyadic. Similarly, the left dot multiplication of a dyadic times a vector is defined as

$$\underline{c}\cdot\underline{ab}=(\underline{c}\cdot\underline{a})\underline{b}$$

Notice, however, that if $\underline{a}$ and $\underline{b}$ are vectors of dimensions m and n, respectively, then $\underline{c}$ is n-dimensional in the first case, whereas m-dimensional in the latter.

Exercise A1.1 Letting U and V be equal to the three-dimensional Euclidean space, in the foregoing discussion, define the right- and the left- cross product of a dyadic by a vector.

Given two dyadics, $\underline{ab}$ and $\underline{cd}$, both of the same space W, their product is the dyadic $\underline{E}$ defined as

$$E = ab.dc = (b.d)ac$$

Under the above definition, if W is an mxn dimensional space, dyadic E is in an mxn dimensional space, thereby paralleling the definition of matrix multiplication. Corresponding to the transpose of a matrix, the transpose or conjugate of a dyadic $D=ab$ is defined as $D_c=ba$. Self-conjugate and antiself-conjugate dyadics are the counterparts of symmetric and antisymmetric matrices. Clearly, square matrices correspond to dyadics arising from the tensor products of one vector space times itself. The trace of such a dyadic is defined in a similar fashion to that of a square matrix. Letting this dyadic be ab, its trace is then

$$Tr(ab) = a_i b_i = a.b.$$

Similarly to the way the trace of a dyadic is defined, the vector of a 3x3 dyadic is given as

$$v = a \times b.$$

Exercise A1.2 Show that the vector of a nonzero 3x3 dyadic vanishes if the dyadic is self-conjugate.

Exercise A1.3 Define a suitable inner product of dyadic that allows you to define the angle between two dyadics. Hence show that self-conjugate and antiself-conjugate dyadics are orthogonal.

Exercise A1.4 How could you define the magnitude of a dyadic?

In just the same way as dyadics were defined, n-adics can be defined as the tensor product of n vector spaces. The study of such algebraic entities falls into the realm of multilinear algebra. The topic is widely discussed in reference A1.1

REFERENCES

A1.1 Bowen R.M. and C.C. Wang, Introduction to Vectors and Tensors, vol. I, Plenum Press, N. York, 1976.

Directorio de alumnos del Curso: Fundamentos Cinemáticos para el Diseño de las Máquinas y Mecanismos Junio de 1981.

Marco Antonio Alarcón Ramírez
Facultad de Estudios Superiores
Cuautitlán Izcalli
Edo. de México

Norte 92 No. 6507
San Pedro el Chico
México 14, D.F.
760 54 95

2. Pilar Colina Beltrán
C F E
Augusto Rodín

Camino a Sta. Teresa 890 8 303
Sta. Teresa
México 20, D.F.
568 65 33

3. Teodoro U. Andrade Maldonado
Inst. Tecnológico Regional de Veracruz
Catedrático
Veracruz, Ver.
4 41 01

Revillagigedo 1743
Veracruz, Ver.
7 33 15

4. Víctor Coronado Hortal
C F E
Río Atoyac 97
México 5, D.F.
553 71 33 Ext. 2561

Rancho Grande 82
Sta. Cecilia
México 21, D.F.
671 15 19

5. Eloy Cruz Gómez
Instituto Tecnológico Regional de Oaxaca
Asesor del Lab. de Ing. Méc.
Calzada Tecnológico y Riberas del Atoyac s/n
Oxcaca, Oaxaca
6 17 22

1ra. Cerrada de Alcalá
Oaxaca, Oaxaca
6 15 33

Juan Carlos Ferreyra Orozco
Servicios y Suministros Siderúrgicos, S.A.
Ingeniero en Diseño
Florencia 65 4 y 5° Piso
México 6, D.F.
528 99 81

Obrero Mundial 932 Int. 5
México 13, D.F.
530 34 48

7. Antonio Carlos Flores Ruiz
Facultad de Ingeniería
Profesor
U N A M
México 20, D.F.
550 52 15 Ext. 3737

Calle A 40 # 32
México 21, D.F.
677 91 25

8. José García González
Instituto de Ingeniería
Ayudante de Investigador
UNAM
México 20, D.F.
550 52 15 Ext. 4610

Pleyades Pte. 17
Prados Coyoacán
México 22, D.F.
677 50 69

9. Leandro Garza de Jonghe
CENETI
Profesor de Tiempo Completo
Av. de las Granjas 682
México 16, D.F.
561 80 11

Blvd. Capri 116
Lomas Estrella
México 13, D.F.
581 43 49

10. José David C. Hernández Hernández
Secc. de Graduados
ESIME
I P N
7 Edif. 3° Piso
México 14, D.F.
586 27 49

Av. de las Torres 20
Vallejo la Patera
México 14, D.F.

11. Luis Morales Flores
DECFI
UNAM
Tacuba No. 5-1° Piso
México 1, D.F.
521 40 20

Av. Caribe 30
Las Américas
Naucalpan de Juárez
Estado de México

12. Jaime Ignacio Nieto Cater
Instituto de Ingeniería
UNAM
México 20, D.F.
550 52 15 Ext. 4610

Av. Real de los Reyes 87 Edif. D 1
México 21, D.F.
548 77 44

13. Alejandro González Vidrio
Centro de Diseño Mecánico
Ayudante de Profesor
UNAM
México 20, D.F.
550 00 41

Sur 101 A NO. 439
Héroes de Churubusco
México 13, D.F.
582 99 81

14. Carlos Orozco Ferreyra
Fac. de Estudios Superiores
Cuautitlán Izcalli
Estado de México

Tercer Sol No. 49
Cuautitlán Izcalli
Estado de México

15. Candido Palacios Montufar
ESIME
Coordinador Académico
Av. Sta. Anna 1000
Sn. Fco. Culhuacán
México 13, D.F.
581 87 23

Mar Mediterráneo 145-6
México 17, D.F.
399 37 88

16. José Alfonso Rámanes García
Instituto Politécnico Nacional
Profesor e Investigador
Santa Anna 1000
Culhuacán
México 13, D.F.
581 87 23 Ext. 35

Reforma Note. 630-602
Unidad Tlatelolco.
México 3, D.F.
529 90 80 Ext. 602

17. Manuel Pamplona Rangel
    Centro Regional de Enseñanza Técnica Industrial
    Profesor Tiempo Completo
    El Chaco 3223
    Guadalajara, Jal.

    Punguato 132
    Col. Independencia
    Guadalajara, Jal.
    18 86 29

18. Ricardo Pineda Larios
    Universidad de Colima
    Director
    Av. Universidad 393
    Colima, Col.

    Antonio Caso 495
    Colima, Col.

    2 62 84

19. José MA. Rico Martínez
    Inst. Téc. Regional de Celaya
    Maestro Titula "B"
    Av. Tecnológico e Irrigación
    Celaya, Gto.
    2 20 23

    Benito Juárez 636
    Celaya, Gto.
    2 08 16

20. Eduardo Salas Córdova
    F E S Cuautitlán
    U N A M
    Carr. Cuautitlán Teoloyucan Km. 3
    Cuautitlán Izcalli, Edo. de Méx.
    2 03 45

    Diana 52
    Col. Ensueños
    Cuautitlán Izcalli, Edo. de Méx.

21. Luis Antonio Sánchez Caballero
    Cannon Mills S.A.
    Dr. Gustavo Baz 10
    Naucalpan, Edo. de Méx.
    560 01 33 Ext. 153

    Mar Bco. 10
    Lomas Lindas
    Atizapán de Zaragoza
    Edo. de México

22. Efrén Sánchez Tapia
    Inst. Téc. Regional de Acapulco
    Catedrático
    Km. 6.5 Carr. Pto.Marquez Crucero Cayacos
    La Sabana, Acapulco, Gro.

    Rinconada Velero Edif. D Depto.9
    U. FOVISSSTE
    Acapulco, Gro.

23. Javier Téllez Ortiz
    Serv. y Suministros Siderúrgicos S.A.
    Ingeniero en Diseño
    Florencia 65-5°
    México 6, D.F.
    528 99 81

    Sur 121 # 813
    Escuadrón 201
    México 13, D.F.
    581 45 85

24. J. Isidro Torres Navarro
    Inst. Téc. Regional No. 20
    Catedrático en el Departamento de Ingeniería Méc.
    Km. 87 Carr. México Pachuca
    Pachuca, Hgo.
    330 73

    Peña y Peña No. 403
    Pachuca, Hgo.

25. Mario Armin Zumbardo Aranda
    Inst. Téc. de Mérida
    Km. 5 Carr. Mérida Progreso
    Mérida, Yuc.

    16 No. 222 x 31
    García Gineres
    Mérida, Yuc.
    1 86 12