EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

(del 6 al 28 de octubre de 1978)

Fecha	Duración	Tema	Profesor
6 de Oct.	17 a 21 h	CONCEPTOS DE INGENIERIA ECONOMICA	DR. VICTOR GEREZ G.
7 de Oct.	9 a 13 h y 14 a 17 h	CRITERIOS DE EVALUACION	" " " "
13 de Oct.	17 a 21 h	EJEMPLOS DE PROYECTOS DE DESARROLLO	ING. JESUS GALERA
14 de Oct.	9 a 13 h y 14 a 17 h	REPASO DE LA TEORIA DE PROBABILIDAD	M. en C. MARCIAL PORTILLA ROBERTSON
20 de Oct.	17 a 21 h	DECISIONES DE ACUERDO A LA TEORIA DEL VALOR	M. en C. RODOLFO FELIX FLORES
21 de Oct.	9 a 13 h	" " " " " " "	
21 de Oct.	14 a 17 h	LA TEORIA DE LA UTILIDAD	M. en C. LUIS PABLO GRIJALVA LOPEZ
27 de Oct.	17 a 21 h	" " " " "	
28 de Oct.	9 a 13 h 14 a 17 h	INTRODUCCION A LA TEORIA DE LOS JUEGOS	M. en C. CARLOS VALENCIA RODRIGUEZ
		CLAUSURA	

'edcs.

DIRECTORIO DE PROFESORES DEL CURSO

EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

LIC. JESUS GALERA LAMADRID
Gerencia de la División Comercial
Editorial Trillas, S.A.
Clzda. de la Viga No. 1162
México D.F.
Tel.657.91.88 Ext. 21

M. EN C. RODOLFO FELIX FLORES
GERENCIA DE INGENIERIA VIAL Y TRANSPORTE
Comisión de Vialidad y Transporte Urbano
Av. Juárez No. 42 Edif. B-2°
México 1, D.F.
Tel.: 585.10.11 Ext. 234

M. EN C. LUIS PABLO GRIJALVA LOPEZ
Investigador
División de Sistemas de Potencia
Instituto de Investigaciones Eléctricas
Shakesperare No. 6-3°
México 5, D.F.
Tel.: 525.64.52

DR. VICTOR GEREZ GREISER
Profesor
Ingeniería Mecánica y Eléctrica
Facultad de Ingeniería
U. N. A. M.
México 20, D.F.
Tel. 550.52.15 Ext. 4750

M. EN C. MARCIAL PORTILLA ROBERTSON
Jefe de la Sección de Computación
Edif. de Ing. Mec. y Eléctrica
Fac. de Ing. , UNAM
Tel. 550.52.15 Ext. 3746

M. EN C. CARLOS VALENCIA RODRIGUEZ
Jefe del Depto. de Información de Planes y Programas
S. C. T.
Dirección General de Planeación
Centro SCOP Cuerpo H-3°
Av. Universidad y Xola
Tel. 538.37.72

centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

EJEMPLOS

DR. VICTOR GEREZ GREISER

OCTUBRE, 1978.

4

Continuous Interest and Discounting

W. B. Hirschmann and J. R. Brauweiler

4.1 Logic for continuous interest

Interest can be compounded periodically, e.g., annually, semiannually, or even daily, or it can be compounded continuously. Annual discounting is appropriate for handling mortgages, bonds, and similar financial transactions, which require payments or receipts at discrete times. In most businesses, however, transactions occur throughout the year; and these circumstances suggest a continuous flow of money, for which continuous compounding and discounting is more realistic than annual compounding and discounting.

This chapter develops formulas for discounting and compounding cash flows on a continuous basis. It also illustrates how continuous discounting readily and simply copes with the variety of cash flows that might result from an investment over its life.

4.2 Continuous interest as an operator

If i is the nominal interest rate expressed as a decimal and compounding occurs p times per year, then

$$\left(1+\frac{i}{p}\right)^p$$

Table 4.2T1 Comparison of Compounding Factors

Period	Relationship	For $i = 0.06$	Factor for $i = 0.06$
Annually	$(1+i)^1$	1.06^1	1.06000
Semiannually	$\left(1+\frac{i}{2}\right)^2$	1.03^2	1.06090
Quarterly	$\left(1+\frac{i}{4}\right)^4$	1.015^4	1.0613635
Monthly	$\left(1+\frac{i}{12}\right)^{12}$	1.005^{12}	1.0616778
Daily	$\left(1+\frac{i}{365}\right)^{365}$	1.00016^{365}	1.0618305
Continuously	e^i	$e^{0.06}$	1.0618365

is the value of 1 at the end of 1 year, as developed in Sec. 2.1. Table 4.2T1 shows the effect of increasing the number of compounding periods in 1 year. Note that there is little difference between the factors for monthly and continuous compounding.

By Eq. (2.1#4)

$$S = P\left(1+\frac{i}{p}\right)^{np}$$

where S is the future amount of a present amount P after n years with nominal decimal interest rate per year i compounded p times per year. In the limit with p equal to infinity for continuous compounding

$$1+\frac{i}{p} = e^i$$

$$S = Pe^{in} \qquad (4.2\#1)$$

where e is the naperian constant 2.71828 . . . . Also solving for P in terms of S,

$$P = Se^{-in} \qquad (4.2\#2)$$

Thus the factor e^{in} is an operator that moves $1 n years with the calendar at a nominal decimal rate per year i. Similarly, the factor e^{-in} is an operator that moves $1 n years against the calendar at a nominal decimal rate per year i.

Generally there is no confusion between periodic and continuous interest inasmuch as the two are never used together. However, in this book a bar over a letter will be used when necessary to emphasize that continuous interest or continuous flow is intended. Thus in keeping with the terminology of Chap. 2,

$$F_{PS\bar{i},n} = e^{\bar{i}n} \qquad (4.2\#3)$$

$$F_{SP\bar{i},n} = e^{-\bar{i}n} \qquad (4.2\#4)$$

The factor $F_{PS,i,n}$ converts a single amount P to a future amount S, with continuous interest at nominal decimal rate i per year, n years with the calendar. The factor is tabulated in Appendix 2, Table 1. Similarly, the factor $F_{SP,i,n}$ converts S to P, is a present-worth factor for continuous compounding, and is tabulated in Appendix 2, Table 2.

A useful characteristic of continuous compounding and discounting factors is evident from the tabulations. Note that in Appendix 2, Tables 1 and 2, i and n appear as a product in. Because of this circumstance, a continuous-discount function has the same value for each combination of interest rate and time period which has the same product. Consequently, continuous discounting requires only one table of factors, based on the product in, while annual discounting requires many tables, one for each interest rate.

This mathematical characteristic also permits continuous factors to be placed on a discounted cash-flow slide rule so that present-worth and other calculations can be made even more simply because of not having to refer to tables of factors.[1] The tables for continuous discounting are much more compact than those for periodic discounting. In addition, the continuous form combines readily with several functions describing common cash-flow patterns so that the summation (integral) of the present worth is easily found from one or two tabulated factors. This convenience is shown by the simple formulas and procedures developed in this chapter.

It should be noted that in this chapter i is a decimal annual rate. The discounting or compounding interval determines what the effective annual rate will be. The relationship between effective interest rate and nominal interest rate was given by Eq. (2.1#5) and for continuous compounding becomes

$$\text{Effective interest rate} = e^i - 1 \qquad (4.2\#5)$$

The converse relationship is

$$i_{\text{nom}} = 2.303 \log (1 + i_{\text{eff}}) \qquad (4.2\#6)$$

In Eq. (4.2#6) the reference is to common logarithms. Equations (4.2#5) and (4.2#6) permit conversions from nominal to effective rates and vice versa. They can be useful because at times a problem arises in terms of annual interest but discount factor tables may be available only for continuous interest, or the converse.

Figure 4.2F1 shows a comparison of equivalent annual and continuous rates. Thus by the figure

5% annual = 4.9% continuous
10% annual = 9.5% continuous
20% annual = 18.2% continuous
30% annual = 26.2% continuous

[1] Available from the Graphic Calculator Co., Barrington, Ill.

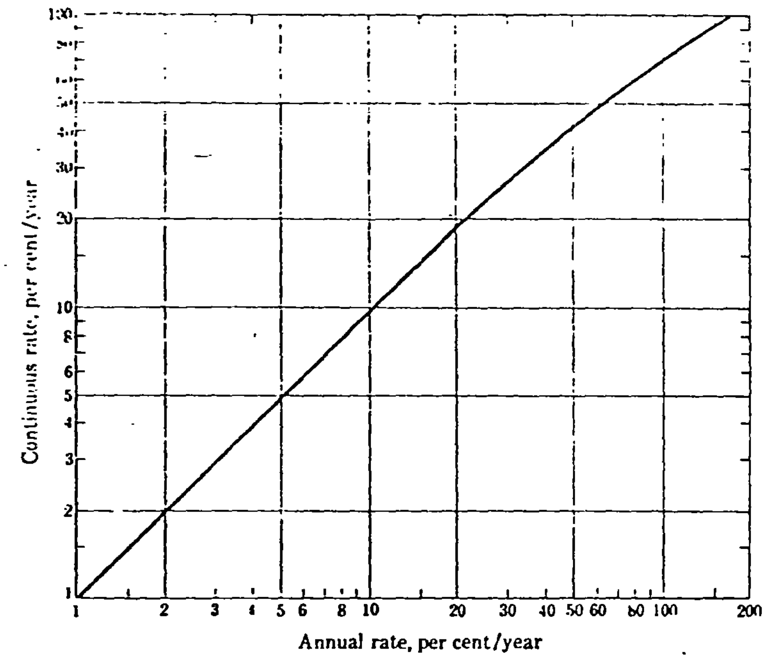


Fig. 4.2F1 Equivalent annual and continuous rates.

Tables 1 and 2 of Appendix 2 can be extended because of the properties of exponentials. Thus, from Appendix 2, Table 1, knowing further that $e^{0.003} = 1.0030$,

$$e^{10.613} = e^5 e^5 e^{0.61} e^{0.003}$$
$$= 148.41(148.41)(1.8404)(1.0030) = 40{,}666$$

Example 4.2E1 If the discount rate is 10% per year, what is the present worth of \$2,500 to be received as a single payment 20 years hence?

By Eq. (4.2#2) and Appendix 2, Table 2,

$$P = Se^{-in} = 2{,}500e^{-(0.10)(20)} = 2{,}500(0.1353) = \$338.25$$

Example 4.2E2 If \$5 is received now, what will it amount to in 20 years at 30% per year interest?

By Eq. (4.2#1) and Appendix 2, Table 1,

$$S = Pe^{in} = 5e^{(0.30)(20)} = 5(400.4) = \$2{,}002$$

4.3 Uniform flow

In the previous section, compounding and discounting were performed on a single amount. In this section the operations will be performed on a continuous flow. Suppose that an amount flows at the rate $\bar{R}$ per year for n years. Consider a small interval of time dX starting X years from now, as in Fig. 4.3F1. The flow during this interval is given by rate multiplied by time and is $\bar{R}\,dX$. The present worth for this small element of time from Eq. (4.2#2) is

$$P_{\text{elem}} = \bar{R}\,dX\,e^{-iX}$$

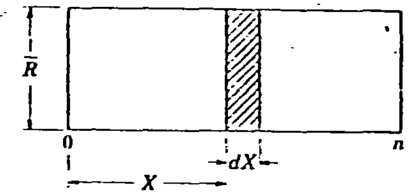


Fig. 4.3F1 Discounting a uniform flow.

and for all the elements

$$P = \bar{R}\int_0^n e^{-iX}\,dX = \bar{R}\left[\frac{e^{-iX}}{-i}\right]_0^n = \bar{R}\,\frac{1 - e^{-in}}{i} \qquad (4.3\#1)$$

If the relationship above is multiplied and divided by n, it becomes

$$P = n\bar{R}\left[\frac{1 - e^{-in}}{in}\right] \qquad (4.3\#2)$$

The value $n\bar{R}$ is the *total flow* for the period. The factor within the brackets now appears as a function of *in* only and can be tabulated compactly. In the terminology of this book

$$F_{\bar{R}P,i,n} = \frac{1 - e^{-in}}{in} \qquad (4.3\#3)$$

and

$$F_{P\bar{R},i,n} = \frac{in}{1 - e^{-in}} \qquad (4.3\#4)$$

The factor $F_{\bar{R}P,i,n}$, which converts $n\bar{R}$ to P, is tabulated in Appendix 2, Table 3, as the evaluation of $(1 - e^{-x})/x$, where $x = in$.

Example 4.3E1 A mine is expected to yield a cash income after taxes of \$20,000 per year continuously for each of the next 15 years. If the minimum acceptable rate of return on investment is 12% per year, find the maximum amount that can be economically justified for buying the mine.

By Eq. (4.3#2) and Appendix 2, Table 3,

$$P = (n\bar{R})F_{\bar{R}P,0.12,15} = 15(20{,}000)(0.4637) = \$139{,}110$$

Example 4.3E2 If \$1 per day is invested as received at 8% per year interest, what will the sum be in 15 years?

First find the present worth of the uniform flow of \$365 per year by Eq. (4.3#2) and Appendix 2, Table 3,

$$P = 15(365)F_{\bar{R}P,0.08,15} = 15(365)(0.5823) = 3{,}188$$

Next convert to a future worth by Eq (4.2#1) and Appendix 2, Table 1,

$$S = 3{,}188e^{(0.08)(15)} = 3{,}188(3.3201) = \$10{,}584$$

Example 4.3E3 The parents of a baby plan to save enough to send it through college. How much must they invest monthly in 4% per year continuous-interest bonds to accumulate the \$12,000 they figure will be needed 17 years hence?

Present worth of the \$12,000 needed is by Eq. (4.2#2) and Appendix 2, Table 2,

$$P = 12{,}000e^{-(0.04)(17)} = 12{,}000(0.5066) = 6{,}079.20$$

which in turn can be converted to a yearly uniform flow by Eq. (4.3#2) and Appendix 2 Table 3,

$$6{,}079.20 = 17\bar{R}F_{\bar{R}P,0.04,17} = 17\bar{R}(0.7256)$$

$$\bar{R} = \$492.83 \text{ per year}$$

or

$$\frac{492.83}{12} = \$41.07 \text{ per month}$$

4.4 Flow changing at an exponential rate

It is appropriate at this point to highlight an aspect that is often unrecognized or overlooked: the most important, and perhaps the most difficult part of an economic analysis is making a realistic estimate of what the future cash flows will prove to be. At times, this step seems elementary, but the simplicity can be deceptive. Consider a homeowner who has a 25-year 5% mortgage requiring payments every month. This seems like a straightforward cash flow: money loaned in a lump sum and repaid regularly. But what about the initial fee for writing the mortgage? What happens if the homeowner loses his job and income or dies? What about the refinance charges if he sells his home and moves to a different one?

Changes such as these are more likely to happen than not with the cash flows of any investment. Recognizing that such changes occur is more appropriate than assuming that there will be no change; but projecting them realistically is a challenge.

During initial scoping studies, it can be convenient to assume level performance—to ignore change or assume there will be none—in order to simplify the analysis. It is important to recognize, though, that this assumption is being made and to interpret the results accordingly. In real life or for a definitive analysis, such an assumption can rarely be made with safety; it may not only be misleading but disastrous.

This chapter will not dwell on methods or techniques of projecting cash flows. As a point of departure, though, it is convenient to recognize that changes which seem erratic over the short term often move in regular trends over the long term. Wage rates seem to increase continuously, and competition continually erodes profit margins. Such trends can be readily discounted or compounded by the continuous method.

If an initial flow R_0 dollars per year, increases continuously at a rate g per year expressed as a decimal, the rate of flow at any time X by analogy with Eq. (4.2#1) is

$$R_X = R_0e^{gX}$$

Consider a small interval of time dX starting X years from now, as in Fig. 4.4F1. The flow for this interval is $R_0e^{gX}\,dX$. The present worth for this small element of

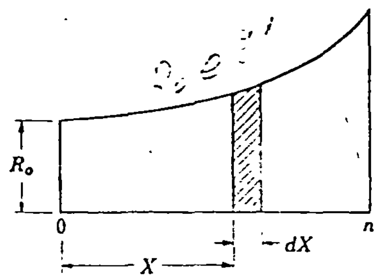


Fig. 4.4F1 Discounting a flow changing at an exponential rate.

flow from Eq. (4.2#2) is

$$P_{\text{elem}} = R_0 e^{gX}\, dX\, e^{-iX} = R_0 e^{(g-i)X}\, dX$$

and for all the elements

$$P = R_0 \int_0^n e^{(g-i)X}\, dX = R_0 \frac{e^{(g-i)n} - 1}{g - i}$$

$$= nR_0 \frac{1 - e^{-(i-g)n}}{(i-g)n} = (nR_0)F_{RP,\overline{i-g},n} \qquad (4.4\#1)$$

Thus, the present worth is easily calculated from a knowledge of the initial flow rate and from the factors tabulated in Appendix 2, Table 3.

Equation (4.4#1) holds if g is negative, i.e., the flow is decreasing at a rate g per year, provided of course that g is introduced as a negative number.

Example 4.4E1 Repeat Example 4.3E1 but with a forecast that inflation will raise prices 3% per year continuously.

By Eq. (4.4#1) and Appendix 2, Table 3,

$P = nR_0F_{RP,0.12-0.03,15} = 15(20{,}000)(0.5487) = \$164{,}610$

Example 4.4E2 Repeat 4.3E1 with the condition that the mine will become gradually depleted so that its net income declines at the rate of 5% per year.

Here $i = 0.12$ and $g = -0.05$. Thus $i - g = 0.12 - (-0.05) = 0.17$. By Eq. (4.4#1)

$P = 15(20{,}000)F_{RP,0.17,15} = 15(20{,}000)(0.3615) = \$108{,}450$

Example 4.4E3 Repeat 4.3E1 subject to both an inflation rate of 3% per year and a depletion rate of 5% per year.

Here $i - g = 0.12 - 0.03 - (-0.05) = 0.14$, and Eq. (4.4#1) becomes

$P = 15(20{,}000)F_{RP,0.14,15} = 15(20{,}000)(0.4179) = \$125{,}370$

4.5 Flow declining in a straight line to zero

Consider Fig. 4.5F1, in which an initial flow, R_0 dollars per year, declines to zero by a straight-line relationship in n years. At time X the flow is R_x, and by similar triangles

$$\frac{R_x}{n - X} = \frac{R_0}{n}$$

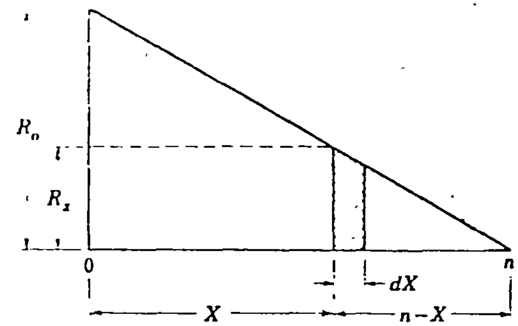


Fig. 4.5F1 Discounting a flow declining in a straight line to zero.

or

$$R_x = R_0\left(1 - \frac{X}{n}\right) \qquad (4.5\#1)$$

In a small interval of time dX starting X years from now, the flow for the interval is

$$R_x\, dX = R_0\left(1 - \frac{X}{n}\right) dX$$

and the present worth for this small element of flow is, from Eq. (4.2#2),

$$P_{\text{elem}} = R_0\left(1 - \frac{X}{n}\right) dX\, e^{-iX}$$

For all the elements

$$P = R_0 \int_0^n \left(1 - \frac{X}{n}\right) e^{-iX}\, dX = R_0 \int_0^n e^{-iX}\, dX - \frac{R_0}{n} \int_0^n X e^{-iX}\, dX \qquad (4.5\#2)$$

The first integral on the right has already been evaluated and is

$$R_0 \frac{1 - e^{-in}}{i}$$

Tables of integrals show

$$\int X e^{-aX}\, dX = -\frac{e^{-aX}}{a^2}(aX + 1)$$

so that the second integral on the right of Eq. (4.5#2) is

$$-\frac{R_0}{n}\left[-\frac{e^{-iX}}{i^2}(iX + 1)\right]_0^n = \frac{R_0}{i}\left(e^{-in} + \frac{e^{-in}}{in} - \frac{1}{in}\right)$$

The combined integrals on the right of Eq. (4.5#2) become

$$P = \frac{R_0}{i}\left(1 - e^{-in} + e^{-in} + \frac{e^{-in}}{in} - \frac{1}{in}\right) = \frac{R_0}{i}\left(1 - \frac{1 - e^{-in}}{in}\right)$$

The latter can be written

$$P = \frac{nR_0}{2}\frac{2}{in}\left(1 - \frac{1 - e^{-in}}{in}\right) \tag{4.5\#3}$$

The total flow Q is the area of Fig. 4.5F1 and is $nR_0/2$. Finally, Eq. (4.5#3) becomes

$$P = Q\left[\frac{2}{in}\left(1 - \frac{1 - e^{-in}}{in}\right)\right] \tag{4.5\#4}$$

A table of discount factors for such a flow is the evaluation of the bracketed terms on the right, i.e.,

$$\frac{2}{x}\left(1 - \frac{1 - e^{-x}}{x}\right) \qquad \text{with} \qquad x = in$$

and is tabulated in Appendix 2, Table 4. This type of flow approximates sum-of-digits (SD) depreciation and in symbols is

$$F_{SDP,i,n} = \frac{2}{in}\left(1 - \frac{1 - e^{-in}}{in}\right) \tag{4.5\#5}$$

Appendix 2, Table 4, is commonly referred to as the *years-digits table*.

Example 4.5E1 A machine costs $150,000 and can be depreciated over 20 years by sum-of-digits method of depreciation. Find the present worth of the depreciation, before taxes, if the discount rate is 16% per year.

By Eqs (4.5#4) and (4.5#5) and Appendix 2, Table 4,

$$P = QF_{SDP,\bar{i},n} = 150{,}000F_{SDP,0.16,20} = 150{,}000(0.4376)$$
$$= \$65{,}640$$

Internal Revenue Service regulations effectively require depreciation charges to begin at the middle of a calendar year. Consequently, if a plant begins operation just before the end of a calendar year, discounting of the actual depreciation cash flow is more closely approximated by substituting $n - \frac{1}{2}$ for n in the above function and $n + \frac{1}{2}$ if the plant goes on-stream just after the beginning of a year.

Example 4.5E2 Find the present worth of the machine in Example 4.5E1 if it is expected to begin operation in December.

The previous calculation becomes

$$P = 150{,}000F_{SDP,0.16,20-0.5} = 150{,}000F_{SDP,3.12} = 150{,}000(0.4446) = \$66{,}690$$

4.6 Discounting with improving performance—learning

Experience shows that practice makes perfect—that a thing can always be done better, not only the second time, but each succeeding time by trying. This experience is often reflected in a progressive increase in output or performance of a plant through increased skill of workers, advances in technology, resourcefulness of

management, bottleneck removal, and a general striving to do things better. Such expected improvement can be reflected by achievement or learning as developed more fully in Chap. 9. For the presentation here assume that the learning factor for a plant unit is manifested as an increase in profit margin M. Assume an exponential relationship such that

$$M_T = M_0(2 - e^{-kT}) \tag{4.6\#1}$$

where M_T = profit margin at time T

M_0 = initial profit margin

k = empirical constant

The present worth at i interest rate on such flow over T years is

$$P = \int_0^T M_0(2 - e^{-kT})e^{-iT}\,dT$$
$$= 2M_0T\frac{1 - e^{-iT}}{iT} - M_0T\frac{1 - e^{-(i+k)T}}{(i+k)T}$$
$$= 2(M_0T)F_{\bar{R}P,\bar{i},n} - (M_0T)F_{\bar{R}P,\overline{i+k},n} \tag{4.6\#2}$$

Equation (4.6#2) can be combined with a flow changing at an exponential rate. Suppose the selling price of each production unit changes such that the profit margin changes continuously at a rate g per year, then corresponding to Eq. (4.6#1) the relationship is

$$M_T = M_0e^{gT}(2 - e^{-kT}) \tag{4.6\#3}$$

which by an analogous procedure leads to

$$P = 2(M_0T)F_{\bar{R}P,\overline{i-g},n} - (M_0T)F_{\bar{R}P,\overline{i+k-g},n} \tag{4.6\#4}$$

In practice g is usually negative and in such cases must be introduced as a negative number.

Example 4.6E1 A plant is expected to have an initial profit margin of $100,000 per year. Find the present worth at 8% per year discount rate of this margin for 20 years of operation if:

(*a*) Profit margin and plant performance stay level.

(*b*) Performance traces an achievement curve such that

$$M_T = M_0(2 - e^{-0.10T})$$

(*c*) Performance traces the same curve, but margin shrinks 3% per year.

The following factors are available from Appendix 2, Table 3.

$$F_{\bar{R}P,0.08,20} = 0.4988$$
$$F_{\bar{R}P,0.08-0.10,20} = 0.2702$$
$$F_{\bar{R}P,0.08+0.03,20} = 0.4042$$
$$F_{\bar{R}P,0.08+0.10+0.03,20} = 0.2345$$

Part (*a*) is given by Eq. (4.3#2)

$$P = 100{,}000(20)(0.4988) = \$997{,}600$$

Part (*b*) is given by Eq. (4.6#2)

$$P = 2(100{,}000)(20)(0.4988) - 100{,}000(20)(0.2702) = \$1{,}454{,}800$$

Part (*c*) is given by Eq. (4.6#4)

$$P = 2(100{,}000)(20)(0.4042) - 100{,}000(20)(0.2345) = \$1{,}147{,}800$$

4.7 Unaflow—capital-recovery factor

In Sec. 4.3 a uniform flow was converted to a present worth or present value. The inverse of that procedure, the conversion of a present value to a uniform flow, will be considered in this section. Solving Eq. (4.3#2) for $\bar{R}$ gives

$$\bar{R} = \frac{P}{n}\frac{1}{(1 - e^{-in})/in} \qquad (4.7\#1)$$

which by Eq. (4.3#3) becomes

$$\bar{R} = \frac{P}{n}\frac{1}{F_{\bar{R}P,i,n}} \qquad (4.7\#2)$$

Equations (4.7#1) and (4.7#2) are important. They permit transforming a present value P having n years duration to a uniform flow. $\bar{R}$ will be referred to as *unaflow* and is analogous to unacost in periodic compounding. $\bar{R}$ could also be called the continuous capital-recovery amount.

Unaflow is important because like unacost it can be made the basis for comparing articles or systems having different service lives. It reduces all service lives to a common denominator, equivalent uniform flow.

Example 4.7E1 A firm has the option of getting a patent license by a single payment of \$50,000 or royalty payments of \$5,000 per year for the 17-year life of the patent. If the payments can be expensed in either case, and if the firm earns 15% per year before taxes, which is the more attractive choice?

Unaflow for royalty payments is \$5,000 per year, as given. Unaflow for purchase of patent by Eq. (4.7#2) and Appendix 2, Table 3, is

$$\bar{R} = \frac{50{,}000}{17}\frac{1}{F_{\bar{R}P,0.15,17}} \frac{50{,}000}{17}\frac{1}{0.3615} = \$8{,}136$$

The annual royalties of \$5,000 per year are thus cheaper for this firm. The ratio of costs, purchase to lease, is 8,136/5,000 = 1.6272

The calculation assumes that in both cases the tax depreciation will be taken at a [illegible] rate and will cancel out as affecting both alternatives equally.

Example 4.7E2 A \$15,000 mortgage is to be repaid over 20 years at 6% per year intere Find the monthly payments.

By Eq. (4.7#2) and Appendix 2, Table 3, unaflow is

$$R = \frac{15{,}000}{20}\frac{1}{F_{\bar{R}P,0.06,20}} = \frac{15{,}000}{20}\frac{1}{0.5823} = \$1{,}288$$

That is, the flow must be \$1,288 per year, or

$$\frac{\$1{,}288}{12} = \$107.33 \text{ per month}$$

4.8 Capitalized cost

Capitalized cost, like unaflow, can be used to compare articles or systems havin different service lives. It reduces all service lives to a common denominator, i.e present value on the basis, for mathematical purposes, of service forever.

Consider an article that has an initial cost C and lasts n years. The presen worth of supplying service forever is

$$P_\infty = Ce^{-i0} + Ce^{-in} + Ce^{-2in} + Ce^{-3in} + \cdots$$

which is an infinite geometrical series with first term C and ratio e^{-in}. The sun is given by Eq. (2.2#2), and letting $P_\infty = K$,

$$K = \left[\frac{1 - (e^{-in})^\infty}{1 - e^{-in}}\right]C = \frac{1}{1 - e^{-in}}C \qquad (4.8\#1)$$

The bracketed term on the right converts a present worth of n years duration to a capitalized cost; i.e.,

$$K = P_n \frac{1}{1 - e^{-in}} \qquad (4.8\#2)$$

or

$$K = P_n \frac{e^{in}}{e^{in} - 1} \qquad (4.8\#3)$$

where the symbol P_n emphasizes that P is a present worth representing n years duration.

Equations (4.8#2) and (4.8#3) are important because they are the basis for using the capitalized-cost concept with continuous interest. Equation (4.8#3) is the more convenient form if Tables for e^{in} are available, as in the book, Appendix 2, Table 1. The reader is referred to Chap. 2 for a more complete discussion of capitalized cost.

A relationship between capitalized cost K and unaflow $\bar{R}$ is easily derived. The present worth of a unaflow $\bar{R}$ for n years is, by Eq. (4.3#2),

$$P_n = n\bar{R}\,\frac{1 - e^{in}}{in}$$

and the capitalized cost of this present worth becomes, by Eq. (4.8#2),

$$K = n\bar{R}\,\frac{1 - e^{-in}}{in}\,\frac{1}{1 - e^{-in}} = \frac{\bar{R}}{i}$$

that is,

$$\bar{R} = iK \tag{4.8\#4}$$

Equation (4.8#4) for continuous interest and unaflow is analogous to the corresponding relationship $R = iK$, Eq. (2.7#6), for periodic interest and unacost.

Example 4.8E1 Repeat Example 4.7E1 on the basis of capitalized cost. Capitalized cost of the royalty payments, by Eq. (4.8#4), is

$$K = \frac{\bar{R}}{i} = \frac{5{,}000}{0.15} = 33{,}333$$

Capitalized cost of purchase is given by Eq. (4.8#3), which, using Appendix 2, Table 1, becomes

$$K = 50{,}000\,\frac{e^{(0.15)(17)}}{e^{(0.15)(17)} - 1} = 50{,}000\,\frac{12.807}{12.807 - 1} = 54{,}235$$

It is cheaper to pay the royalties. The ratio of costs, purchase to lease, is 54,235/33,333 = 1.6271. This checks the calculation by unaflow in Example 4.7E1.

Example 4.8E2 In a given exposure, a paint job lasts 4 years and costs \$0.20 per square foot. A supplier offers a new coating which is claimed to last 20 years but costs \$0.60 per square foot. Is it economically attractive to change to the coating which lasts five times as long and costs only three times as much, if money is worth 10%? Neglect taxes.

Capitalized costs can be calculated from Eq. (4.8#3) and Appendix 2, Table 3, and are for the 4- and 20-year jobs, respectively,

$$K_4 = \$0.20\,\frac{e^{(0.10)(4)}}{e^{(0.10)(4)} - 1} = 0.20\,\frac{1.4918}{0.4918} = 0.6067$$

$$K_{20} = \$0.60\,\frac{e^{(0.10)(20)}}{e^{(0.10)(20)} - 1} = 0.60\,\frac{7.3891}{6.3891} = 0.6939$$

The 4-year coating is the more economical. The savings as flow per year per square foot can be obtained from Eq. (4.8#4) and is

$$\bar{R} = i(K_{20} - K_4) = 0.10(0.6939 - 0.6067) = 0.00872$$

That is, use of the 4-year coating saves \$0.00872 per year per square foot in comparison with the 20-year coating.

4.9 Income tax

The reader is referred to Chap. 3 for a detailed development of the effect of income tax using periodic interest. This section is concerned with the inclusion of income tax with continuous interest. Basically nothing new is involved. Suppose an item has a depreciable first cost C_d, that it lasts n years and can be written off in n years for tax purposes, that discounting will be on a continuous basis at a decimal rate r per year after taxes, and that the decimal tax rate is t. Then Eq. (3.13#4) can be written

$$P_n = C_d(1 - t\psi) \tag{4.9\#1}$$

where P_n is the present value for n years and ψ is the present value of \$1 of depreciation discounted continuously at rate r. The value of ψ depends upon the method of depreciation used and can be calculated for any method. For this chapter only two methods are considered, straight line and sum of digits.

Straight-line depreciation (SL) is treated as a uniform flow, i.e., as uniform continuous depreciation. For a total flow of unity, recalling that ψ is a present worth, Eq. (4.3#2) gives, with n' the life for tax purposes,

$$\psi_{SL} = \frac{1 - e^{-in'}}{in'} = F_{RP,i,n'} \tag{4.9\#2}$$

and Appendix 2, Table 3, can be used.

Sum-of-digits depreciation (SD) is approximated by a flow declining in a straight line to zero as developed in Sec. 4.5. For a total flow of unity, Eq. (4.5#5) becomes

$$\psi_{SD} = F_{SDP,i,n'} = \frac{2}{in'}\left(1 - \frac{1 - e^{-in'}}{in'}\right) \tag{4.9\#3}$$

where the right side is tabulated in Appendix 2, Table 4.

If an expenditure or receipt becomes eligible for tax credit at once, such as a maintenance expense, then for such items having no capitalization, for income tax purposes

$$\psi = 1 \tag{4.9\#4}$$

regardless of the depreciation method.

With these considerations it becomes possible to use continuous interest on an after-tax basis with the same ease as for computations with periodic interest. All items of expenditure and receipt are considered on an after-tax basis with proper regard to the timing of tax credits.

Example 4.9E1 A \$1,000 investment has an expected life of 20 years and is to be depreciated over a 15-year life at a 52% tax rate using sum-of-digits depreciation; money is worth 10% per year after taxes. Find (*a*) the present worth of the capital charges after taxes and (*b*) unaflow.

By Eqs. (4.9#1) and (4.9#3), and Appendix 2, Table 4,

$$P = 1{,}000(1 - 0.52F_{SDP,0.10,15}) = 1{,}000[1 - 0.52(0.6428)]$$

$$P = \$678.60 \quad ans.\ (a) \tag{4.9\#5}$$

If $\bar{R}$ is the unaflow before taxes, then $\psi = 1$ for this item by Eq. (4.9#4), and the unaflow after taxes is

$$\bar{R}(1 - 0.52) = 0.48\bar{R} \tag{4.9\#6}$$

Equations (4.9#5) and (4.9#6) are both on an after-tax basis and must be equivalent. Using Eq. (4.7#2) with $0.48\bar{R}$ in place of $\bar{R}$, and Appendix 2, Table 3,

$$0.48\bar{R} = \frac{678.60}{20}\,\frac{1}{F_{RP,0.10,20}} = \frac{678.60}{20}\,\frac{1}{0.4323}$$

$$\bar{R} = \$163.51 \text{ per year before taxes} \quad ans.\ (b)$$

4.10 Equivalence

One of the major purposes of a firm is to show a profit, which it does by committing its funds to ventures which promise to do so. There are always alternatives for

Table 4.10TI Summary of Relationships for Continuous Interest

Item no.	Item	Description	Algebraic relationship	Factor relationship
1	P to S	Moves a fixed sum P to another instant of time n years with calendar	$S = Pe^{in}$	$S = PF_{PS,\bar{i},n}$ (App. 2, Table 1)
2	S to P	Moves a fixed sum S to another instant of time n years against calendar	$P = Se^{-in}$	$P = SF_{SP,\bar{i},n}$ (App. 2, Table 2)
3	R to P	Converts a unaflow R for n years to present worth at start of flow	$P = nR\frac{1 - e^{-in}}{in}$	$P = nRF_{\bar{R}P,\bar{i},n}$ (App. 2, Table 3)
4	R for 1 year to P	Present worth of 1 year of unaflow starting X years hence	$P = R e^{-iX}\frac{1 - e^{i}}{i}$	$P = R F_{SP,\bar{i},X}F_{\bar{R}P,\bar{i},1}$
5	P of flow changing at an exponential rate for n years	Present worth of $R_x = R_0 e^{\pm gx}$ for n years	$P = nR_0\frac{1 - e^{-(i\mp g)n}}{i \mp g}$	$P = nR_0 F_{\bar{R}P,(\bar{i}\mp\bar{g}),n}$
6	P of flow declining in straight line to zero	Flow goes from R_0 at zero time to 0 in n years; total flow Q is $nR_0/2$	$P = Q\left[\frac{2}{in}\left(1 - \frac{1 - e^{-in}}{in}\right)\right]$	$P = QF_{SDP,\bar{i},n}$ (App. 2, Table 4)
7	P to R	Converts a present worth to a unaflow of n years	$R = \frac{P}{n}\frac{1}{(1 - e^{-in})/in}$	$R = \frac{P}{n}\frac{1}{F_{\bar{R}P,\bar{i},n}}$
8	R to S	Converts a unaflow for n years to a future amount, n years hence	$S = nR\,e^{in}\frac{1 - e^{-in}}{in}$	$S = nR\,F_{PS,\bar{i},n}F_{\bar{R}P,\bar{i},n}$
9	S to R	Converts a future sum S, n years from now, to a unaflow; sinking-fund payment	$R = \frac{Se^{-in}}{n}\frac{1}{(1 - e^{-in})/in}$	$R = \frac{S}{n}F_{PS,\bar{i},n}\frac{1}{F_{\bar{R}P,\bar{i},n}}$
10	P to K	Converts a present worth representing n years to a capitalized cost	$K = P\frac{e^{in}}{e^{in} - 1}$	
11	K to R	Converts a capitalized cost to a unaflow	$R = iK$	
12	Before to after tax	Converts before-tax amount to after-tax amount, at tax rate t	After tax $= (1 - t\psi)$ before tax	
13	ψ	Present value of \$1 of depreciation, n' years life for tax purpose	$\psi_{SL} = \frac{1 - e^{-in'}}{in'}$ $\psi_{SD} = \frac{2}{in'}\left(1 - \frac{1 - e^{-in'}}{in'}\right)$ $\psi = 1$ for instantaneous tax benefit $\psi = 0$ for no depreciation, i.e., land	$\psi_{SL} = F_{\bar{R}P,\bar{i},n'}$ (App. 2, Table 3) $\psi_{SD} = F_{SDP,\bar{i},n'}$ (App. 2, Table 4)

making these investments and expenditures. The function of economic analysis is to help in making better choices between alternatives by placing dollar values on quality, quantity, time, and other characteristics of these alternatives. This quantifying needs to be done in a consistent way so that the dollar values are measured by the same yardstick, i.e., expressed on an equivalent basis. Doing so highlights the better alternative, the one with the lowest economic cost or highest economic value.

Discounting and compounding at the same interest rate places the same dollar value on time. However, since the alternatives may have different lives, e.g., low-cost short-life carbon steel vs. high-cost long-life alloy steel, it is also necessary to compare them over the same time interval. Present worth compares equivalent values now; unaflow, the equivalent continuous annual cost, compares the values on a per-year basis; and capitalized cost compares them on a forever basis as a common denominator for all service lives. Comparisons can also be made on the basis of rate of return, discounted cash flow, as developed in the following chapter.

The method to be used for comparing alternatives or ventures and the choice between periodic and continuous interest is left to the analyst. There is no universal or intrinsic answer, and the choice varies with circumstances. Some analysts are more familiar with one approach and therefore prefer it. Some problems are so expressed that one solution is easier or more meaningful by one of the methods. The technique which is felt best by the analyst for getting the solution, however, is not necessarily the one best for presenting the solution to the client. Although an economic specialist may prefer one method or even be equally comfortable with all four, the client is often a manager, who is a generalist by necessity. Since he does not have time to be a specialist in every field, results must be presented in terms familiar to him—simple enough to be grasped on the run. Experience shows that if a solution is presented in unfamiliar or seemingly unrealistic terms, it will not be understood; if not understood, it will not be believed; and if not believed, it will not be accepted.

This chapter has discussed the essence of continuous discounting and compounding, showed how to develop relationships for handling commonly encountered cash flows, including those which may change over time, and illustrated the ease of applying them by simple examples. Most real problems are more complex, not so much in computation, but in defining what the cash flows will prove to be. Often, 95% of the total time in solving a problem is required on such a determination for allocating costs and incomes, determining the applicable tax and other government regulations, and projecting sales, costs, and so on. This circumstance does not mean that economic analysis of cash flows is insignificant, because even with the right cash flows, a wrong analysis or interpretation can lead to the wrong choice of alternatives. Instead, the observation is intended to put the various parts of problem solving into meaningful perspective for understanding and coping better with real situations when they arise.

A summary of the various relationships using continuous interest is given in Table 4.10T1.

4.11 Nomenclature

C_d	Depreciable first cost, $
e	Naperian constant 2.71828 · · ·
$F_{P\bar{R},\bar{i},n}$	Factor to convert P to $\bar{R}$ with continuous compounding; reciprocal of $F_{\bar{R}P,\bar{i},n}$, year^{-1}
$F_{PS,\bar{i},n}$	Factor to convert P to S with continuous compounding, e^{in}, Appendix 2, Table 1, dimensionless
$F_{\bar{R}P,\bar{i},n}$	Factor to convert $\bar{R}$ to P with continuous discounting; Appendix 2, Table 3, years
$F_{SP,\bar{i},n}$	Factor to convert S to P with continuous discounting, e^{-in}, Appendix 2, Table 2, decimal, dimensionless
$F_{SDP,\bar{i},n}$	Factor to convert a unit total flow declining to zero at a constant rate over n years starting with the reference point and with continuous discounting, decimal, dimensionless, approximates ψ_{SD}
g	Constant in exponential-rate flow change, decimal
i	Nominal interest rate, decimal/year
k	Empirical exponent in learning-curve relationship, decimal, dimensionless
K	Capitalized cost, $
M	Profit margin, $
n	Time, years
n'	Time for tax depreciation, years
p	Periods per year
P	Present worth, $
P_n	Present worth for n years duration, $
Q	Total flow, $
r	Nominal rate of return after taxes, decimal/year
$\bar{R}$	Uniform flow, unaflow, $ per year
$\bar{R}_0$	Initial flow rate, $ per year
$\bar{R}_x$	Flow rate at time X, $ per year
S	Future worth, $
SL	Straight-line depreciation
SD	Sum-of-digits depreciation
t	Income tax rate, decimal
T	Time, years
ψ	Factor associated with present worth of tax benefits arising from depreciation, dimensionless
ψ_{SL}	The ψ factor for straight-line depreciation
ψ_{SD}	The ψ factor for sum-of-digits depreciation

4.12 Problems

P1. Develop a relationship for discounting a flow increasing in a straight line from zero at zero time to $\bar{R}_n$ at time n.

P2. Develop a relationship for discounting a flow increasing in a straight line from $\bar{R}_1$ at zero time to $\bar{R}_2$ at time n.

P3. Develop a relationship for discounting a series of periodic cash flows of k payments, Y each, at intervals of n years, the first one beginning n years hence.

P4. A firm has a contributory savings plan whereby each employee can set aside 5% of his gross salary. The firm will match this amount, invest the sums in its capital stock, and reinvest all dividends in capital stock. If an employee's salary is consistently $12,000 per year, how much will he accumulate after 20 years if the company's net earnings average 8% per year and the stock consistently sells at book value?

P5. What is the average rate of growth of the employee's $600 per year portion of the contribution?

P6. Suppose the employee finds an alternative proposition which promises to double his money every 5 years. Will he be better off to participate in the savings plan or forego the company's contribution and invest his contribution in the alternative?

P7. If the parents in Example 4.3E3 continue their monthly savings during the 4 years their child attends college, e.g., for 21 instead of 17 years, how much must their monthly savings be to permit $3,000 per year to be withdrawn uniformly over the 4 years from the seventeenth to the twenty-first birthday?

P8. A new machine costs $8,000 and lasts 10 years, using sum-of-digits depreciation and a 10-year life for tax purposes. If money is worth 10% per year after a 52% tax, how much can be spent to repair an old machine to extend its life 3 years? The repair job can be written off at once for tax purposes. Compare with Prob 3.15P6.

P9. Repeat Example 2.8E6 using continuous discounting. Money is worth 10% per year after a 48% tax rate. Use straight-line depreciation. Machine A will be written off in 8 years for tax purposes, machine B in 10 years. Maintenance costs and savings from quality control are uniform flows in years in which they occur. The salvage value is anticipated and cannot be depreciated for tax purposes.

	A	B
First cost, $	10,000	95,000
Maintenance, $ per year	3,000	1,000
Extra maintenance, year 3, $	4,000	
Extra maintenance, year 4, $	1,500	
Savings from quality control, $ per year		6,000
Salvage value		20,000
Life, years	4	10

P10. A company completed a plant 10 years ago. It was expected to be serviceable for 25 years, but technical advances and accumulated know-how suggest that obsolesence may have progressed faster than expected, so that it may be profitable to displace it now. Assume for simplicity that (1) a new plant would have the same capacity as the old and would produce the same array of products with the same initial revenue for both, so that the advantage of the new is reflected only in its lower operating costs; (2) these savings in operating costs are $180,000 per year; and (3) depreciation on the old plant is $35,000 per year on a straight-line basis, and present salvage value is zero, (4) the tax rate is 50% per year.

If the investment required for the new plant is $1 million and both its useful life and life for tax purposes are 15 years with sum-of-digits depreciation, what is the rate of return to be earned by investment in a new plant?

4.13 References

R1. Hirschmann, W. B.: Profit from the Learning Curve, *Harvard Business Rev.*, January–February, 1964, pp 125–139.

R2. Hirschmann, W. B., and J. R. Brauweiler: Investment Analysis: Coping with Change, *Harvard Business Rev.*, May–June, 1965, pp. 62–72.

R3. Hirschmann, W. B., and J. R. Brauweiler: Continuous Discounting for Realistic Investment Analysis, *Chem. Eng.*, July 19, 1965, pp. 210–214.

R4. Hirschmann, W. B., and J. R. Brauweiler: Realistic Investment Analysis, II, *Chem. Eng.*, August 16, 1965, pp. 132–136.

present worth (P_2) of $3,000 from year 5 to infinity, using Eq. (8.3) and the P/F factor, is

$$P_2 = \frac{3{,}000}{0.05}(P/F, 5\%, 4) = \$49{,}362$$

The two annual costs are converted to a capitalized cost (P_3):

$$P_3 = \frac{A_1 + A_2}{i} = \frac{847 + 5{,}000}{0.05} = \$116{,}940$$

5 The total capitalized cost (P_T) can now be obtained by addition:

$$P_T = P_1 + P_2 + P_3 = \$346{,}997$$

COMMENT In calculating P_2, $n = 4$ was used in the P/F factor because the present worth of the annual $3,000 cost is computed in year 4, since P is always one year ahead of the first A. You should rework the problem using the second method suggested for calculating P_2. ////

Problems P8.17–P8.22

8.4 Capitalized-Cost Comparison of Two Alternatives

When two or more alternatives are compared on the basis of their capitalized cost, the procedure of Example 8.3 is followed. Since the capitalized cost represents the present total cost of financing and maintaining a given alternative forever, the alternatives will automatically be compared for the same number of years (i.e., infinity). The alternative with the smaller capitalized cost will represent the most economical one. As in present worth and all other alternative evaluation methods, it is only the differences in cash flow between the alternatives which must be considered. Therefore, whenever possible, the calculations should be simplified by eliminating the elements of cash flow which are common to both alternatives. Example 8.4 shows the procedure for comparing two alternatives on the basis of their capitalized cost.

Example 8.4 Two sites are currently under consideration for a bridge to cross the Ohio River. The north site would connect a major state highway with an interstate loop around the city and would alleviate much of the local through traffic. The disadvantages of this site are that the bridge would do little to ease local traffic congestion during rush hours, and the bridge would have to stretch from one hill to another to span the widest part of the river, railroad tracks, and local highways below. This bridge would therefore be a suspension bridge. The south site would require a much shorter span allowing for construction of a truss bridge, but would require new road construction.

The suspension bridge would have a first cost of $30 million with annual inspection and maintenance costs of $15,000. In addition, the concrete deck will have be resurfaced every ten years at a cost of $50,000. The truss bridge and approach roads are expected to cost \$12 million and will have annual maintenance costs of \$8,000. The bridge will have to be painted every three years at a cost of \$10,000. In addition, the bridge will have to be sandblasted and painted every ten years at a cost of \$45,000. The cost of purchasing right-of-way is expected to be \$800,000 for the suspension bridge and \$10.3 million for the truss bridge. Compare the alternatives on the basis of their capitalized cost if the interest rate is 6%.

SOLUTION Construct the cash-flow diagrams before you attempt to solve the problem. You should do this *now*.

Capitalized cost of suspension bridge

$$P_1 = \text{present worth of initial cost} = 30.0 + 0.8 = \$30.8 \text{ million}$$

The recurring operating cost is A_1 = \$15,000, while the annual equivalent of the resurface cost is

$$A_2 = 50{,}000(A/F,6\%,10) = \$3{,}794$$

$$P_2 = \text{capitalized cost of recurring costs} = \frac{A_1 + A_2}{i}$$

$$= \frac{15{,}000 + 3{,}794}{0.06}$$

$$= \$313{,}233$$

Finally, the total capitalized cost (P_S) is

$$P_S = P_1 + P_2 = \$31{,}113{,}233 \qquad (\$31.1 \text{ million})$$

Capitalized cost of truss bridge

$$P_1 = 12.0 + 10.3 = \$22.3 \text{ million}$$
$$A_1 = \$8{,}000$$
$$A_2 = \text{annual cost of painting} = 10{,}000(A/F,6\%,3) = \$3{,}141$$
$$A_3 = \text{annual cost of sandblasting} = 45{,}000(A/F,6\%,10) = \$3{,}414$$

$$P_2 = \frac{A_1 + A_2 + A_3}{i} = \$242{,}583$$

The total capitalized cost (P_T) is

$$P_T = P_1 + P_2 = \$22{,}542{,}583 \qquad (\$22.5 \text{ million})$$

Since $P_T < P_S$, the truss bridge should be constructed. ////

Example 6.8
Problems P8.23, P8.24

Example 11.3 Two routes are under consideration for a new interstate highway. The northerly route (N) would be located about five miles from the central business district and would require longer travel distances by local commuter traffic. The southerly route (S) would pass directly through the downtown area and, although its construction cost would be higher, it would reduce the travel time and distance for local commuters. Assume the costs for the two routes are as follows:

	Route N	Route S
Initial cost	\$10,000,000	\$15,000,000
Maintenance cost per year	35,000	55,000
Road-user cost per year	450,000	200,000

If the roads are assumed to last 30 years with no salvage value, which route should be accepted on the basis of a benefit/cost analysis using an interest rate of 5%?

SOLUTION Since most of the costs are already annualized, the EUAC method will be used to obtain the equivalent annual cost. The costs to be used in the B/C ratio are the initial cost and maintenance cost:

$$\text{EUAC}_N = 10{,}000{,}000(A/P, 5\%,30) + 35{,}000 = \$685{,}500$$
$$\text{EUAC}_S = 15{,}000{,}000(A/P, 5\%,30) + 55{,}000 = \$1{,}030{,}750$$

The benefits in this example are represented by the road-user costs, since these are costs "to the public." The benefits, however, are not the road-user costs themselves but the difference in road-user costs if one alternative is selected over the other. In this example, there is a \$450,000 − \$200,000 = \$250,000-per-year benefit if Route S is chosen instead of Route N. Therefore, the benefit (B) of Route S over Route N is \$250,000 per year. On the other hand, the costs (C) associated with these benefits are represented by the difference between the annual costs of Routes N and S. Thus,

$$C = \text{EUAC}_S - \text{EUAC}_N = \$345{,}250 \text{ per year}$$

Note that the route that costs more (Route S) is the one that provides the benefits. Hence, the B/C ratio can now be computed by Eq. (11.1).

$$B/C = \frac{250{,}000}{345{,}250} = 0.724$$

The B/C ratio of less than 1.0 indicates that the extra benefits associated with Route S are less than the extra costs associated with this route. Therefore, Route N would be selected for construction. Note that there is no "do nothing" alternative in this case, since one of the roads *must* be constructed.

COMMENT If there had been disbenefits associated with each route, the difference between the disbenefits would have to be added or subtracted from the net benefits (\$250,000) for Route S, depending on whether the disbenefits for Route S were less than or greater than the disbenefits for Route N. That is, if the disbenefits for Route S were less than those for Route N, the difference between the two would have to be added to the $250,000 benefit for Route S, since the disbenefits involved would also favor Route S. However, if the disbenefits for Route S were greater than those for Route N, their difference should be subtracted from the benefits associated with Route S, since the disbenefits involved would favor Route N instead of Route S. Example 11.6 illustrates the calculations when disbenefits must be considered. ////

Example 11.6
Problems P11.8–P11.12

11.4 Benefit/Cost Analysis for Multiple Alternatives

When only one alternative must be selected from three or more mutually exclusive (stand-alone) alternatives, a multiple alternative evaluation is required. In this case, it is necessary to conduct an analysis on the *incremental* benefits and costs similar to the method used in Chap. 10 for incremental rates of return. The "do nothing" alternative may be one of the considerations.

There are two situations which must be considered with regard to multiple alternative analysis by the benefit/cost method. In the first case, if funds are available so that *more than one* alternative can be chosen from among several, it is necessary only to compare the alternatives against the "do nothing" alternative. The alternatives are referred to as *independent* in this situation. For example, if several flood-control dams could be constructed on a particular river and adequate funding is available for all dams, the B/C ratios should be those associated with a particular dam versus no dam. That is, the result of the calculations could show that three dams along the river would be economically justifiable on the basis of reduced flood damage, recreation, etc., and, therefore, should be constructed.

On the other hand, when only *one* alternative can be selected from among several, it is necessary to compare the alternatives against each other rather than against the "do nothing" alternative. The exact procedure for doing this is discussed in Chap. 17. However, it is important for you to understand at this time the difference between the procedure to be followed when multiple projects are mutually exclusive and when they are not. In the case of mutually exclusive projects, it is necessary to compare them against each other, while in the case of projects that are not mutually exclusive (independent projects), it is necessary only to compare them against the "do nothing" alternative.

Problem P11.13

11.5 Purpose and Formulas of Service-Life Analysis

Basically, service-life analysis is used to determine the number of years an asset must be retained and used to recover its initial cost with a stated return, given the annual cash flow and salvage value. The analysis should be performed using after-tax cash-flow values (CF) (Chaps. 15 and 16), so that the results are more realistic. To find the economic service life of an asset, the following model is utilized.

$$0 = -P + \sum_{j=1}^{n'} (\text{CF})_j (P/F, i\%, j) \qquad (11.2)$$

where $(\text{CF})_j$ = net cash flow at the end of year j ($j = 1, 2, \ldots, n'$). For a given interest rate (i), the value of n' is sought. After n' years (not necessarily an integer), the cash flows will recover the first cost (P) and a return of $i\%$. A common, but incorrect, industrial practice is to determine n' at $i = 0\%$; that is, with no return accounted for; this is illustrated in the Solved Examples section. In this case Eq. (11.2) becomes

$$0 = -P + \sum_{j=1}^{n'} (\text{CF})_j \qquad (11.3)$$

which is used to compute no-interest service life, more commonly called *payback* or *payout period*. If the cash flow (CF) is the same for each year, Eq. (11.3) is usually solved for n' directly.

$$n' = \frac{P}{\text{CF}} \qquad (11.4)$$

For a brief look at payback and some of its fallacies see Solved Examples, *after* you read the next section.

Problem P11.14

11.6 Use of Service Life to Determine Required Life

Equation (11.2) can be used to find the number of years necessary to recover the first cost at a stated rate of return. If the service life (n') is less than the time you would expect to be able to employ or retain the asset, it should be bought. If n' is greater than the expected usable life, the asset should not be bought, since there will not be enough time to recover the investment plus the stated return during the usable life.

Example 11.3 A semiautomatic assembly machine can be purchased for $18,000 with a salvage value of $3,000 and an annual cash flow of $3,000. If a return of 15% is required and the company would never expect such a machine to be used for more than ten years, should it be purchased?

SOLUTION Of course, there are several ways to answer this question—present-worth, EUAG, or rate-of-return analysis. But, let's use the service-life approach. Using Eq. (11.2), we have

$$0 = -18{,}000 + \sum_{j=1}^{n'} (\text{CF})_j (P/F, 15\%, j)$$

We assume the salvage value of $3,000 is correct regardless of how long the asset is retained. We, therefore, can modify the above relation as follows:

$$0 = -18{,}000 + \text{CF}(P/A, 15\%, n) + \text{SV}(P/F, 15\%$$

where $SV(P/F,15\%,n)$ is the present worth of the salvage after n years and the P/A factor has been used where possible. At $n = 15$ years we have

$$P = -18{,}000 + 3{,}000(P/A, 15\%,15) + 3{,}000(P/F, 15\%,15)$$
$$= \$-89.10$$

For $n = 16$, the result is \$+183.30. Interpolation indicates that in $n' = 15.3$ years the first cost plus 15% will be recovered. Since a fair estimate of usability is ten years, the machine should not be purchased.

COMMENT The salvage value and cash flows will be allowed to vary in the material of Chap. 12. ////

Example 11.7
Problems P11.15–P11.20

11.7 Comparison of Two Alternatives Using Service-Life Computation

If capital is tight and the future uncertain (about available money *and* proposed investments), a breakeven (or equivalent-point) service life of two proposals may be computed for use in decision-making. Still, other evaluation methods, such as present-worth, should be pursued, because service-life analysis is considered only a supplementary tool. If a firm is short of capital and requires quick recovery of investment capital, the service-life computations can indicate the speed with which the project will "pay for itself." Therefore, capital recovery being important, service life at a stated rate of return is found by equating alternative present-worth or EUAC values and finding n' by trial and error. ~~Depending on how many years the purchase will reasonably be used, the proposal with the smaller present-worth or EUAC value is selected.~~ The method is the same as that used in rate-of-return breakeven analysis (Sec. 10.5), but with the value of n sought here.

~~Example 11.4~~ A dirt-moving company requires the service of dirt-moving equipment. The service may be acquired by purchasing a mover for \$25,000 having a negligible salvage value, \$5,000 annual operating cost, and a \$12,000 overhaul cost in year 10. Alternatively, the company may lease the mover at a total cost of \$10,000 per year. If all other costs are equal and service is needed for 12 years at a 12% rate of return, use service-life analysis to determine whether the mover should be purchased or leased.

SOLUTION We use the relation $EUAC_{buy} = EUAC_{lease}$ and find the breakeven n value (n').

$$EUAC_{buy} = 25{,}000(A/P, 12\%,n) + 5{,}000 + 12{,}000(P/F, 12\%, 10)(A/P, 12\%,n)$$
$$EUAC_{lease} = \$10{,}000$$

The last term of $EUAC_{buy}$ is used only when $n \geq 10$. Then when $n < 10$, equating EUAC relations gives

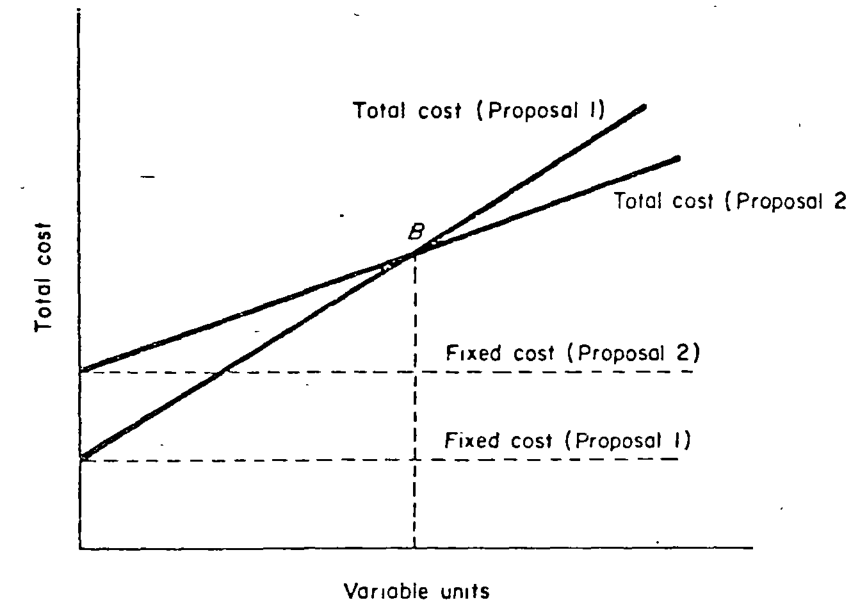


FIG. 12.1 Graphical illustration of breakeven.

operating cost or production cost. Figure 12.1 graphically illustrates the breakeven concept for two proposals (identified as Proposal 1 and Proposal 2). As shown in the figure, the fixed cost (which may be simply the initial investment cost) of Proposal 2 is greater than that of Proposal 1, but Proposal 2 has a lower variable cost (as shown by its smaller slope). The point of intersection (B) of the two lines represents the breakeven point between the two proposals. Thus, if the variable units (such as hours of operation or level of output) are expected to be greater than the breakeven amount, Proposal 2 would be selected, since the total cost of the operation would be lower with this alternative. Conversely, an anticipated level of operation below the breakeven number of variable units would favor Proposal 1.

Instead of plotting the total costs of each alternative and finding the breakeven point graphically, it is generally easier to calculate the breakeven point algebraically. Although the total cost can be expressed as either a present worth or equivalent uniform annual cost, the latter is generally preferable because the variable units are oftentimes expressed on a yearly basis. Additionally, EUAC calculations are simpler when the alternatives under consideration have different lives. In either case, however, the first step in calculating the breakeven point is to ~~*express the total cost of each alternative as a function of the variable that is sought.*~~ Example 12.7 illustrates breakeven calculations.

~~Example 12.7~~ A sheet metal company is considering the purchase of an automatic machine for a certain phase of the finishing process. The machine has an initial cost of \$23,000, a salvage value of \$4,000, and a life of ten years. If the machine is purchased, one operator will be required at a cost of \$12 an hour. The output with this machine would be 8 tons per hour. Annual maintenance and operation cost of the machine is expected to be \$3,500.

Alternatively, the company can purchase a less sophisticated machine for \$8,000, which has no salvage value and a life of five years. However, with this alternative, three laborers will be required at a cost of \$8 an hour and the machine will have an annual maintenance and operation cost of \$1,500. Output is expected to be 6 tons per hour for this machine. All invested capital must return 10%. (*a*) How many tons of sheet metal must be finished per year in order to justify the purchase of the automatic machine? (*b*) If management anticipates a requirement to finish 2,000 tons per year, which machine should be purchased?

SOLUTION

(*a*) The first step is to express each of the variable costs in terms of the units sought, which is tons per year in this case. Thus, for the automatic machine, the annual cost per ton would be

$$\text{Annual cost per ton} = \left(\frac{\$12}{\text{hour}}\right)\left(\frac{1\ \text{hour}}{8\ \text{tons}}\right)\left(\frac{x\ \text{tons}}{\text{year}}\right) = \frac{12}{8}x$$

where x = number of tons per year for break even. Note that the final units are in dollars per year, which is what we want since we are trying to obtain the EUAC. The total EUAC for the automatic machine is

$$\begin{aligned}\text{EUAC}_{\text{auto}} &= 23{,}000(A/P, 10\%, 10) - 4{,}000(A/F, 10\%, 10)\\ &\quad + 3{,}500 + \frac{12}{8}x\\ &= \$6{,}992 + 1.5x\end{aligned}$$

Similarly, the EUAC of the manual machine is

$$\begin{aligned}\text{EUAC}_{\text{manual}} &= 8{,}000(A/P, 10\%, 5) + 1{,}500 + \frac{3(8)}{6}x\\ &= \$3{,}610 + 4x\end{aligned}$$

Equating the two costs and solving for x yields

$$\begin{aligned}\text{EUAC}_{\text{auto}} &= \text{EUAC}_{\text{manual}}\\ 6{,}992 + 1.5x &= 3{,}610 + 4x\\ x &= 1{,}352.8 \text{ tons per year}\end{aligned}$$

Thus, at an output of 1,352.8 tons per year, the EUAC of each method is the same. If the output is expected to be greater than this figure, the automatic machine should be purchased; if the output is to be less, then the less sophisticated machine should be purchased.

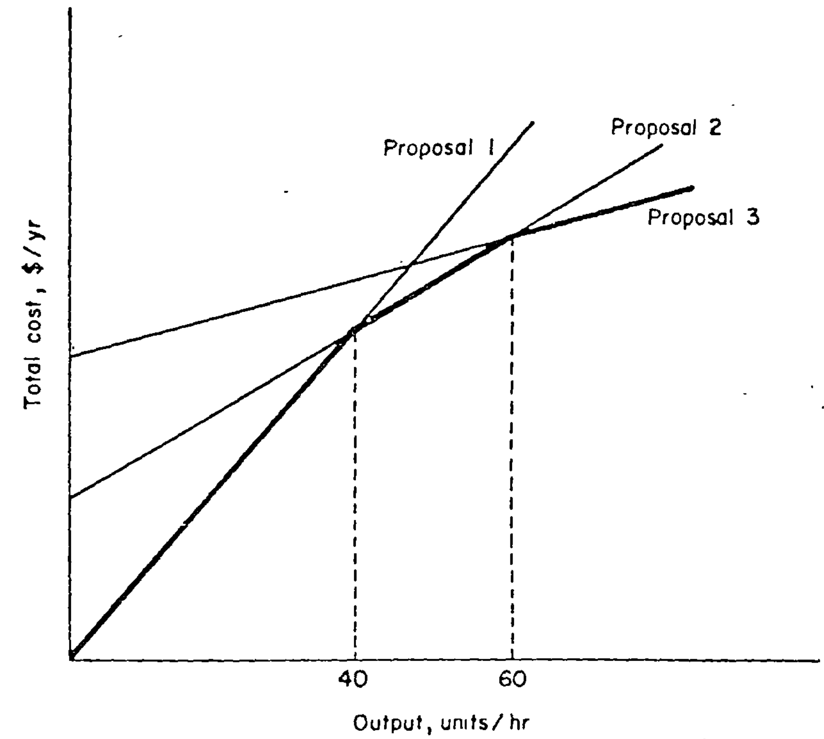


FIG. 12.2 Breakeven points for three proposals.

(*b*) Substituting the expected production level of 2,000 tons per year into the EUAC relations, we have $\text{EUAC}_{\text{auto}} = \$9{,}992$ and $\text{EUAC}_{\text{manual}} = \$11{,}610$. Therefore, purchase the automatic machine.

COMMENT Work the problem on a present-worth basis to satisfy yourself that either method results in the same breakeven point. A question that sometimes arises after the breakeven point is calculated is: How do you know which alternative should be selected when you are either above or below the breakeven point? As shown in Fig. 12.1, the alternative with the smaller slope (i.e., lower variable cost) should be selected when the variable units are above the breakeven point (and vice versa). ////

~~While the preceding example dealt with only two alternatives, the same type~~ of ~~analysis can be made for three or more alternatives. In this case, it becomes necessary to compare the alternatives with each other in order to find their respective breakeven points.~~ The results reveal the ranges through which each alternative would be the most economical one. For example, in Fig. 12.2, if the output is expected to be less than 40 units per hour, Proposal 1 should be selected. Between 40 and 60 units per hour Proposal 2 would be the most economical, and above 60 units per hour Proposal 3 would be favored.

17.2 Selection Using Incremental Rate of Return

You will recall from Secs. 10.5 and 10.6 that the incremental-analysis procedure determines rate of return on the *extra investment* that is required by the plan having the higher investment cost. As discussed there, if the rate of return on the extra investment is greater than the MARR, the plan requiring the extra investment should be selected. This same procedure is followed when analyzing mutually exclusive alternatives, but now it becomes important to determine which alternatives must be compared with each other (and therefore, *which* increments will be involved). In this regard, the most important rule that must be remembered when evaluating alternatives by the incremental-investment rate-of-return method is that *an alternative can never be compared with one for which the incremental investment has not been justified.* The procedure to be used when evaluating multiple, mutually exclusive alternatives can conveniently be summarized as follows:

1 Rank the alternatives in terms of increasing initial investment.
2 Considering the "do nothing" alternative as a defender, compute the overall rate of return for the alternative with the lowest initial investment.
3 If $i <$ MARR, remove the lowest investment alternative from further consideration and compute the overall rate of return for the next higher investment alternative. Repeat this step until $i \geq$ MARR for one of the alternatives. When $i \geq$ MARR, the lowest investment alternative becomes the defender and the next higher investment alternative is the challenger.
4 Determine the incremental costs and incomes between the challenger and the defender.
5 Calculate the rate of return on the incremental investment required in the challenger.
6 If the rate of return calculated (on the increment of investment) in step *5* is greater than the MARR, the challenger becomes the defender and the previous defender is removed from further consideration. Conversely, if the rate of return in step *5* is less than the MARR, the challenger is removed from further consideration and the defender remains as the defender against a new challenger.
7 Repeat steps *4-6* until only one alternative remains.

Note that in the incremental analysis (steps *4-6*), only *two* alternatives are compared at any one time. It is very important, therefore, that the correct alternatives be compared. Unless the procedure is followed as presented above, the wrong alternative can be selected from the incremental analysis. The procedure detailed above is illustrated in Examples 17.1 and 17.2.

Example 17.1 Four different building locations have been suggested, of which only one will be selected. Data for each site are detailed in Table 17.1. Annual CFAT varies due to different tax structures, labor costs, and transportation charges resulting in different annual receipts and disbursements. If the MARR is 10% after taxes, use incremental-rate-of-return analysis to select a building location.

Table 17.1 FOUR ALTERNATE BUILDING LOCATIONS

	Location			
	A	B	C	D
Building cost	\$−200,000	\$−275,000	\$−190,000	\$−350,000
Annual CFAT	+22,000	+35,000	+19,500	+42,000
Life, years	30	30	30	30

SOLUTION The steps outlined above result in the following procedure:

1 Order the alternatives according to increasing initial investment. This is done in the first line of Table 17.2.
2 The next step is to find the lowest investment alternative that has an overall rate of return of at least 10%. Table 17.2 indicates a rate of return of 9.63% for Location C, resulting in its elimination from further consideration. The next alternative, Location A, has an i of 10.49% and replaces "do nothing" as the defender.
3 The incremental investment between alternatives must now be considered. Since all locations have a 30-year life, the relation used to find the incremental i is

$$0 = \text{incremental cost} + \text{incremental CFAT } (P/A,i\%,30) \qquad (17.1)$$

where i is found by trial and error. Note that $(P/A,10\%,30) = 9.4269$; thus any P/A value resulting from Eq. (17.1) greater than 9.4269 indicates the return is less than 10% and, therefore, is unacceptable. Comparing B incrementally to Location A, using Eq. (17.1), results in the equation $0 = -75{,}000 + 13{,}000(P/A,i\%,30)$. A rate of return of 17.28% on the extra investment justifies Location B, thereby eliminating Location A.
4 With B as the defender and D the challenger, the incremental investment yields 8.55%, which is less than 10% and eliminates Location D. Only Locations A and B are justified and B is selected, since it requires the larger investment.

Table 17.2 COMPUTATION OF INCREMENTAL RATE OF RETURN FOR MUTUALLY EXCLUSIVE EQUAL-LIVED PROJECTS

	C	A	B	D
Building cost	\$−190,000	\$−200,000	\$−275,000	\$−350,000
Annual CFAT	19,500	22,000	35,000	42,000
Projects compared	C to none	A to none	B to A	D to B
Incremental cost	\$−190,000	\$−200,000	\$ −75,000	\$ −75,000
Incremental CFAT	19,500	22,000	13,000	7,000
$(P/A,i\%,30)$	9.7436	9.0909	5.7692	10.7143
Incremental i	9.63%	10.49%	17.28%	8.55%
Increment justified?	No	Yes	Yes	No
Project selected	None	A	B	B

COMMENT We should mention here again, just as a word of warning, that an alternative should *always* be compared with an acceptable alternative, noting that the "do nothing" alternative may be the acceptable one. Since C was not justified, Location A was *not* compared to C. Thus, *if* the B-to-A comparison had not indicated that B was incrementally justified, then the comparison D-to-A would have been made, instead of D-to-B.

It is important to understand the use of incremental rate-of-return selection because if it is not properly applied in mutually-exclusive-alternative evaluation, the wrong alternatives may be selected. If the overall rate of return of each alternative is computed, the results are

Location	C	A	B	D
Overall *i*	9.63%	10.49%	12.40%	11.59%

If we now apply *only* the first criterion stated earlier, that is, make the largest investment that has a MARR of 10% or more, we would choose Location D. But, as shown above, this is the wrong selection because the extra investment of $75,000 between Locations B and D will not earn the MARR. In fact, it will earn only 8.55% (Table 17.2). Remember, therefore, that incremental analysis is necessary for selection of one alternative from several when the rate-of-return evaluation method is used.

////

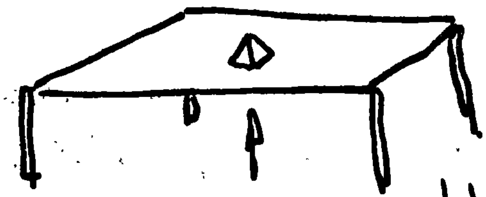
When the alternatives under consideration consist of disbursements only, the "income" is the difference between costs for two alternatives. In this case, there is no need to compare any of the alternatives again the "do nothing" alternative. The lowest-investment-cost alternative is the defender against the next-lowest-investment-cost alternative (challenger). This procedure is illustrated in Example 17.2.

Example 17.2 Four machines can be used for a certain stamping operation. The costs for each machine are shown in Table 17.3. Determine which machine should be selected if the company's MARR is 12%.

Table 17.3 FOUR MUTUALLY EXCLUSIVE ALTERNATIVES

	Machine			
	1	2	3	4
First cost	$-5,000	$-6,500	$-10,000	$-15,000
Annual operating cost	-3,500	-3,200	-3,000	-1,400
Salvage value	+500	+900	+700	+1,000
Life, years	8	8	8	8

EJEMPLO 3.- CONSTRUIMOS UN DADO DE 4 CARAS (TETRAHEDRO) Y EL RESULTADO DE CADA TIRADA SE VE POR ABAJO DE UNA MESA DE VIDRIO



(solamente)

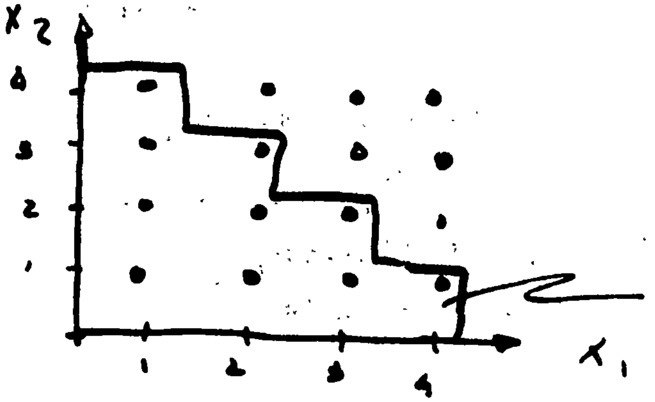
SUPONGAMOS QUE EL DADO ES LANZADO 2 VECES, Y NOS DICEN QUE EL PRODUCTO DEL RESULTADO OBTENIDO ES MENOR QUE SIETE.

a) CUAL ES LA PROBABILIDAD DE QUE ALMENOS SE OBTENGA UN DOS?

b) La PROBABILIDAD QUE EN 2 TIRADAS LA SUMA DEL RESULTADO SE MENOR QUE 7.

c) SI NOS DICEN QUE la SUMA Y EL PRODUCTO SON MENOR QUE 7, Y QUE AL MENOS SE TIENE UN DOS, DETERMINE LA PROBABILIDAD DE CADA POSIBLE SOLUCION DIFERENTE? (DEL OTRO DADO)

SOLUCION.



ESPACIO DE EVENTOS $X_1 \cdot X_2 < 7$

a) LA PROBABILIDAD DE OBTENER UN DOS EN UN DADO ES DE $1/4$ Y LA PROB. DE OBTENER UN DOS EN EL SEGUNDO DADO ES TAMBIÉN DE $1/4$ (EVENTOS INDEPENDIENTES)

10 EVENTOS

$$P(\text{AL MENOS UN } 2) = \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = 0.5$$

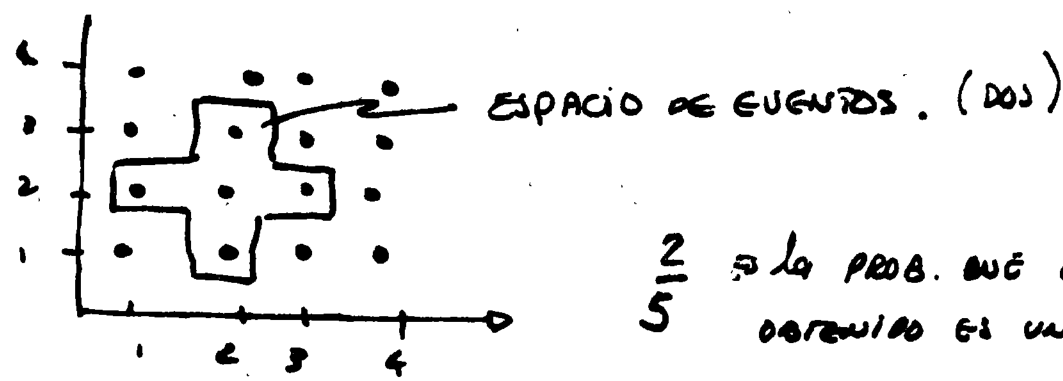
ó

$5 \times 0.1 = 0.5$

DOS

b) EVENTO SEGURO ← POR ENUNCIADO, PROBABILIDAD = 1.

c)



$\frac{2}{5}$ = la PROB. QUE EL VALOR OBTENIDO ES UNO, O TRES

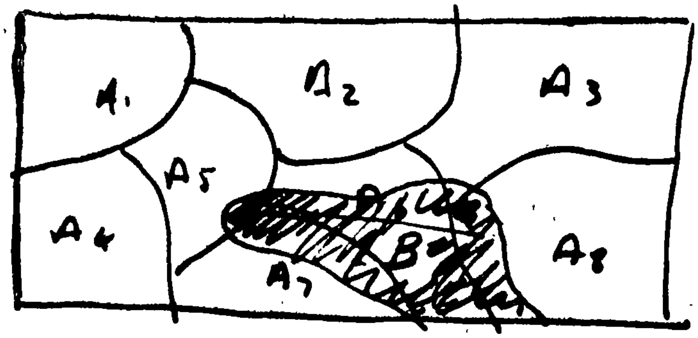
y $\frac{1}{5}$ SEA DOS

TEOREMA DE BAYES.

SIEMPRE Y CUANDO LA PROBABILIDAD CONDICIONADA DE LOS EVENTOS SEA DIFERENTE DE CERO, SE TENDRÁ.

$$P(AB) = P(A)\,P(B/A) = P(B)\,P(A/B)$$

SI APLICAMOS LA RELACIÓN ANTERIOR AL CASO QUE LOS EVENTOS $A_1, A_2, A_3 \ldots A_N$ SEAN MUTUAMENTE EXCLUSIVOS Y EXHAUSTIVAMENTE COLECTIVOS, PARA LO CUAL CONSIDERAREMOS UN ESPACIO UNIVERSAL Y UN SUBESPACIO B.



SUPONGAMOS QUE $P(A_i)$ y $P(B|A_i)$ SON CONOCIDAS PARA TODA $1 \leq i \leq N$ y QUEREMOS DETERMINAR $P(A_i|B)$

PARA EJEMPLIFICAR LO ANTERIOR CONSIDEREMOS QUE A_i REPRESENTA EL EVENTO QUE UNA MANZANA PROVIENE DEL RANCHO i. SEA B EL EVENTO EN EL CUAL LA MANZANA SE VUELVE AZUL, DURANTE EL TRANSPORTE.

$P(B/A_i)$: ES LA PROBABILIDAD QUE LA MANZANA SE TORNO AZUL, DADO QUE PROVIENE DEL RANCHO i.
SIN UNA PREGUNTA MAS INTERESANTE SERA:
DADO QUE LA MANZANA SE VOLVIO AZUL, CUAL ES LA PROBABILIDAD QUE PROVENGA DEL DEL RANCHO i

$P(A_i/B)$ SIEMPRE Y CUANDO $P(B) \neq 0$ y $P(A_i) \neq 0$ PARA TODA i.

SUBSTITUYENDO A_i POR A, PODEMOS ESCRIBIR.

$$P(A_i/B) = \frac{P(A_i)P(B/A_i)}{P(B)}$$

$$P(B) = P(UB) = P[(A_1 + A_2 + \dots A_N)B] =$$

$$= \sum_{i=1}^{n} P(A_iB) = \sum^{n} P(A_i)P(B/A_i)$$

Sustituyendo $P(B)$ en la expresión anterior, tenemos el teorema de Bayes

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

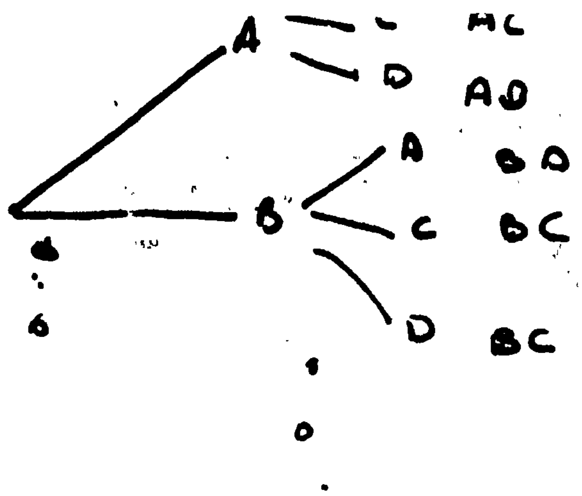
Existe en la literatura controversia respecto la validez del teorema de Bayes. Cuando se utiliza en problemas hay que tener cuidado de no asignar probabilidades 'a priori' aun menos el cual se tiene poca información. — Más adelante vamos a ver ejemplos de aplicación del teorema de Bayes)

Numeración del espacio de eventos permutaciones y combinaciones.

Dados n elementos diferentes entre sí, de cuántas maneras diferentes los podemos acomodar r a la vez. — Esto es supongamos que tenemos los eventos A, B, C, D, si tomamos el ~~[illegible]~~ de 2 en 2, podemos formar los siguientes arreglos.

A B	B A	C A	D A
A C	B C	C B	D B
A D	B D	C D	D C.

Esto es en forma de árbol :



MATEMATICAMENTE LO PODEMOS ESCRIBIR COMO:

$$ {}_nP_K = \binom{n}{k} = \frac{n!}{(n-r)!} $$

COMBINACIONES:

EN EL EJEMPLO ANTERIOR VIMOS LOS POSIBLES ARREGLOS DE CUATRO LETRAS, TOMADAS DE 2 EN DOS, Y AB ERA DIFERENTE DE BA, SIN EMBARGO HAY CASOS EN QUE LOS ~~[illegible]~~ ELEMENTOS DE CADA GRUPO SON IGUALES (i.e. BOLAS BLANCAS Y NEGRAS). — O TAMBIÉN PODEMOS CONSIDERAR AB y BA EN LA MISMA CLASE.

$$ {}_nC_K = \left(\frac{n!}{K!(n-K)!} \right) = \frac{{}_nP_K}{K!} $$

EJEMPLO. DE CUANTAS MANERAS DIFERENTES SE PUEDEN ESCOGER 3 BARAJAS DE UN CONJUNTO DE 8

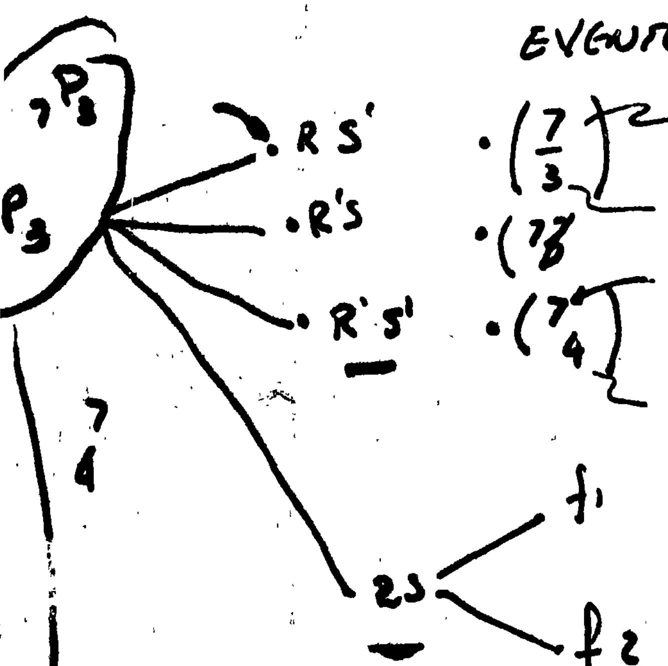
$$ {}_8C_3 = \frac{8!}{3!(5!)} = \frac{8\cdot 7\cdot 6\times 5!}{(3!)\,5!} = 56 $$

SE QUIERE FORMAR UN COMITE DE CUATRO MIEMBROS, EL CUAL DEBE DE SER ESCOGIDO DE UN GRUPO DE 4 HOMBRES R, S, T, U y DE 5 MUJERES V, W, X, Y, Z.

R, S NO PUEDEN FORMAR PARTE DEL COMITE, A MENOS QUE EXISTA UNA MUJER EN EL MISMO

– CUANTOS COMITES DIFERENTES PODEMOS FORMAR.

NOTACION EVENTO X, X ESTA EN EL COMITE.
EVENTO f_n, n MUJERES EN EL COMITE.



R y S ESTAN EN EL COMITE y HAY AL MENOS UNA MUJER y DOS MUJERES. EL CASO DE f_3 ES EL DE LAS 2 PRIMERAS RAMAS DEL ARBOL.

$$2\binom{7}{3} + \binom{7}{4} + \binom{5}{1}\binom{2}{1} + \binom{5}{2} = 125$$

POSIBLES COMITES.

SI NO HUBIESE RESTRICCIONES EL NUMERO DIFERENTES DE COMITES A FORMAR SERIA.

$$\binom{9}{4} = \frac{9!}{4!(5!)} = \frac{9\times8\times7\times6\times5!}{4\times3\times2\times1} = 126$$

¿ENTENDIMOS?

BIEN VAMOS A VER ALGUNOS PROBLEMAS, PARA REAFIRMAR LOS CONCEPTO VISTOS. (NO NECESARIAMENTE EN ORDEN).

SE LANZA UN DADO 2 VECES, CUAL ES LA PROB. DE OBTENER 4, 5, Ó 6 EN LA PRIMERA TIRADA y 1, 2, 3, 4 en la 2ª.

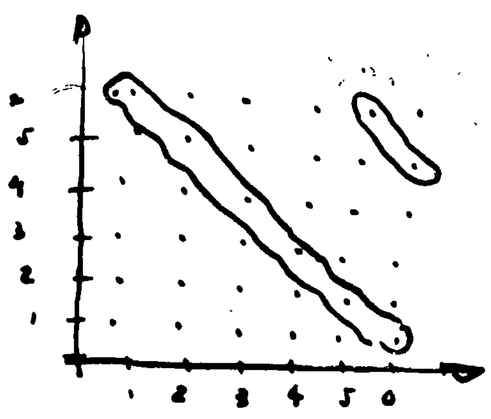
SE A_1 PRIMERA TIRADA
A_2 SEGUNDA TIRADA.

$$P(A_1 A_2) = P(A_1)P(A_2|A_1) = P(A_1)P(A_2)$$

DADO Q SON INDEPENDIENTES

$$= \left(\frac{3}{6}\right)\left(\frac{4}{6}\right) = \frac{1}{3}$$

CUAL ES LA PROB. DE NO OBTENER UN 7 O UN 11 AL TIRAR 2 DADOS.



$$P(A') = 1 - P(A) = 1 - \frac{8}{36} = \frac{7}{9}$$

SI LOS DADOS SE TIRAN 2 VECES (?)

$$\left(\frac{7}{9}\right)\left(\frac{7}{9}\right) = \frac{49}{81}$$

UNA BOLSA CONTIENE 4 bolas blancas y 2 NEGRAS, OTRA BOLSA CONTIENE 3 bolas blancas y 5 NEGRAS. SI SE SACA UNA bola DE CADA BOLSA, CUAL es la PROBABILIDAD QUE:

a) AMBAS SEAN BLANCAS,
b) " " NEGRAS
c) UNA BLANCA Y UNA NEGRA.

B_1 - bola BLANCA BOLSA 1, B_2 bola Blanca bolsa 2

a) $P(B_1 \cap B_2) = P(B_1)P(B_2) = \left(\frac{4}{4+2}\right)\left(\frac{3}{3+5}\right) = \frac{1}{4}$.

b) $P(B_1' \cap B_2') = \left(\frac{2}{4+2}\right)\left(\frac{5}{3+5}\right) = \frac{5}{24}$

c) $1 - P(B_1 \cap B_2) - P(B_1' \cap B_2') =$

$= 1 - \frac{1}{4} - \frac{5}{24} = \frac{13}{24}$

SE DESEAN SENTAR A 5 HOMBRES y 4 DAMAS EN UNA FILA DE sillas, DE MANERA QUE las MUJERES OCUPEN los LUGARES PARES. — DE CUANTAS MANERAS DIFERENTES, SE PUEDEN SENTAR?

$_5P_5 \, _4P_4 = 5! \, 4! = (120)(24) = 2880$.

EN UN GRUPO DE 2000 PERSONAS HAY

612 FUMADORES
670 > DE 25 AÑOS
960 BEBEDORES
86 BEBEDORES QUE FUMAN
290 BEBEDORES > DE 25 AÑOS
158 FUMADORES > DE 25 AÑOS
44 PERSONAS > 25 BEBEDORES Y FUMADORES
250 PERSONAS < 25 AÑOS QUE NI FUMAN NI BEBEN

ES POSIBLE LO ANTERIOR?

CAPITULO 2 VARIABLES ALEATORIAS

UNA VARIABLE ALEATORIA (V.A), ES DEFINIDA COMO UNA FUNCION LA CUAL ASIGNA EL VALOR DE LA V.A. A CADA PUNTO DEL ESPACIO MUESTRA DE UN EXPERIMENTO.

CADA RESULTADO DEL EXPERIMENTO SE DICE QUE GENERA UN VALOR EXPERIMENTAL DE LA V.A.. EL VALOR EXPERIMENTAL DE LA V.A. ES IGUAL AL VALOR ASIGNADO A LA V.A, DEL PUNTO MUESTRA (DEL ESPACIO), QUE CORRESPONDE AL RESULTADO DE UN EXPERIMENTO

EJEMPLO CONSIDERAMOS QUE SE TIRA UNA MONEDA AL AIRE 3 VECES Y CONSIDERAMOS:

$$\text{EVENTO} \begin{Bmatrix} A_n \\ S_n \end{Bmatrix} : \begin{Bmatrix} \text{AGUILA} \\ \text{SOL} \end{Bmatrix} \text{ EN LA TIRADA } n.$$

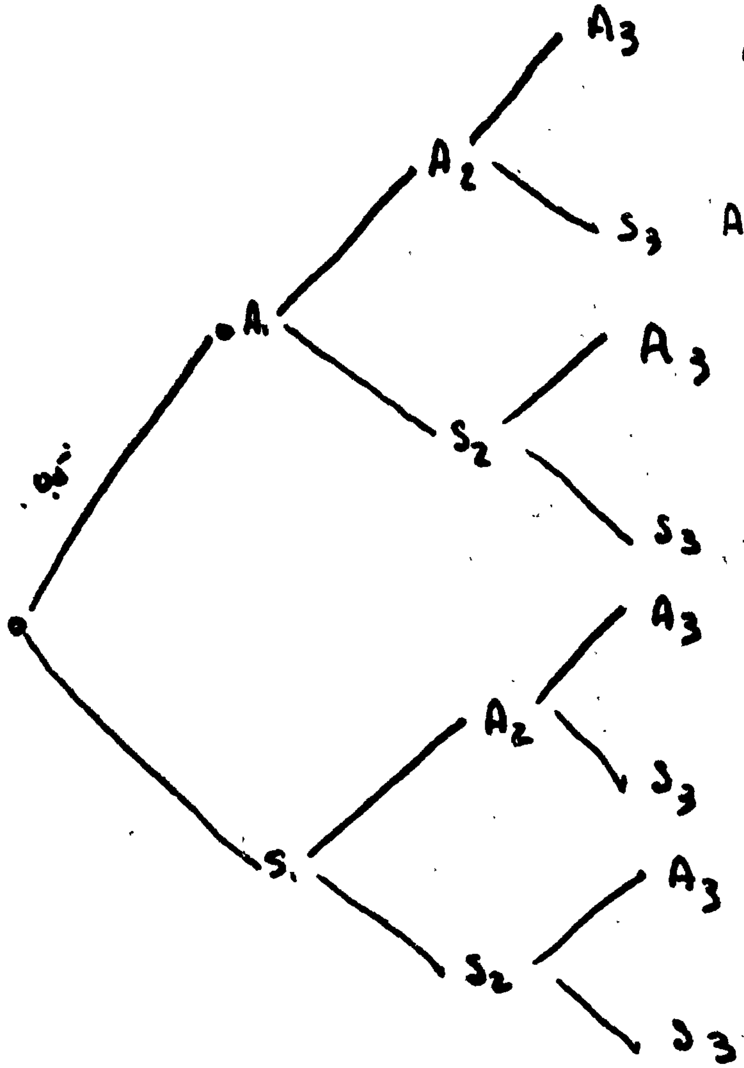
PODEMOS DEFINIR LAS VARIABLES ALEATORIAS DEL ESPACIO MUESTRA DE ESTE EXPERIMENTO, Y CONSIDEREMOS LAS SIGUIENTES:

a = # TOTAL DE AGUILAS EN LAS 3 TIRADAS

r = LONGITUD MAXIMA DE LA CORRIDA DE LAS 3 TIRADAS

(CORRIDA ES UN CONJUNTO SUCESIVO DE TIRADAS LAS CUALES TIENEN EL MISMO RESULTADO)

PREPAREMOS UN ESPACIO MUESTRA ETIQUETADO PARA ESTE EXPERIMENTO Y EL CUAL SERA



PUNTO MUESTRA	P(·)	a	r
$A_1 A_2 A_3$	.125	3	3
$A_1 A_2 S_3$	.125	2	2
$A_1 S_2 A_3$	.125	2	1
$A_1 S_2 S_3$	.125	1	2
$S_1 A_2 A_3$	.125	2	2
$S_1 A_2 S_3$	.125	1	1
$S_1 S_2 A_3$	.125	1	2
$S_1 S_2 S_3$	.125	0	3

SI EL EXPERIMENTO ANTERIOR SE LLEVASE A CABO UNA SOLA VEZ Y EL RESULTADO SERIA EL EVENTO $A_1 S_2 S_3$, DIRIAMOS QUE PARA ESTE CASO EN CUESTION EL VALOR DE las variables aleatorias a y r seria DE 1 y 2 RESPECTIVAMENTE. NOTANDO QUE TUVIMOS QUE CONSTRUIR TODO EL ESPACIO MUESTRA DE EXPERIMENTO, QUE LO DESCRIBE PROBABILISTICAMENTE. SIN EMBARGO NUESTRO INTERES SE RELACIONARA CON LOS VALORES RESULTANTES DE CADA EXPERIMENTO, DE UNA O MAS V.A.

PROBABILIDADES DE FUNCIONES DISCRETAS.

SEA X UNA V.A. DISCRETA Y SUPONGA QUE LOS VALORES QUE PUEDE TENER SON $X_1, X_2, X_3, \ldots$ ARREGLADOS EN MAGNITUD CRECIENTE, SUPONGA QUE A ESTOS VALORES, SE LES ASIGNAN LAS SIG. PROBABILIDADES

$$P(X = x_k) = f(x) \qquad k = 1, 2, 3. \qquad (1)$$

PRESENTEMOS AHORA EL CONCEPTO DE FUNCIÓN DE PROBABILIDAD (f.p) O FUNCION DE DISTRIBUCION DE PROBABILIDAD (f d p) LA CUAL ESTA DADA POR.

$$P(X = x) = f(x)$$

PARA $X = X_k$ SE TIENE LA FORMULA (1), PARA OTROS VALORES DE x $f(x) = 0$. EN GENERAL SI $f(x)$ ES UNA f. d p.

1.- $f(x) \geq 0$

2.- $\sum_x f(x) = 1$

ESTA SUMA DEL PUNTO 2 SE CONSIDERAN TODOS LOS VALORES DE x, EN EL EJEMPLO ANTERIOR SERIA LA PROBABILIDAD DE TODOS LOS CASOS DEL ARBOL
$P(x) = .125 + .125 + \ldots + .125 = 1$

UNA GRAFICA DE $f(x)$ SE LLAMA GRAFICA DE PROBABILIDAD.

EJEMPLO 1 SUPONGA QUE UNA MONEDA SE LANZA AL AIRE 2 VECES Y SU ESPACIO MUESTRA ES {AA, AS, SA, SS}, a) ENCUENTRE LA f.p. DE LA V.A. X.

$P(AA) = \frac{1}{4}$ $P(AS) = \frac{1}{4}$ $P(SA) = \frac{1}{4}$ $P(SS) = \frac{1}{4}$

ENTONCES.

SI $P(X=0) = P(AA) = \frac{1}{4}$

$P(X=1) = P(AS+SA) = P(AS)+P(SA) = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

$P(X=2) = P(SS) = \frac{1}{4}$

EN FORMA TABULAR.

X	0	1	2
f(x)	1/4	1/2	1/4.

b) CONSTRUYA UNA GRAFICA DE PROBABILIDAD.

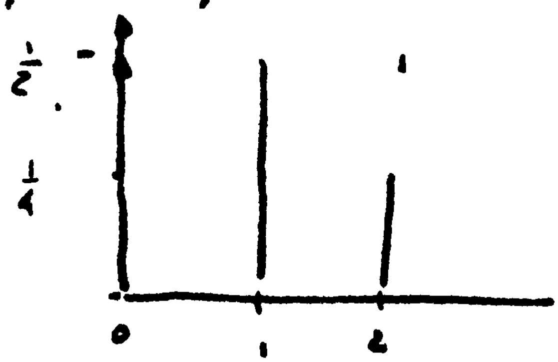
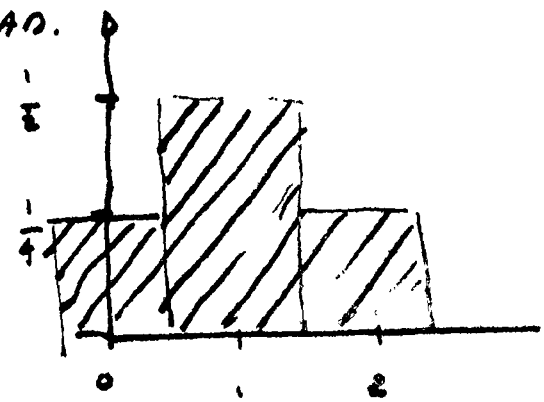


DIAGRAMA DE BARRAS.



HISTOGRAMA.

* VER. Ej HOJA 31 BIS ANTES DEL Ej 2.

EJEMPLO 2.- ENCUENTRE LA DISTRIBUCION DE PROBABILIDAD DE NIÑOS Y NIÑAS EN FAMILIAS CON 3 HIJOS, SUPONIENDO IGUAL PROBABILIDAD A LOS NIÑOS Y LAS NIÑAS. b) CONSTRUYA LA GRAFICA DE PROB.

DETERMINE LA PROBABILIDAD DE OBTENER 3 SEIS EN 5 TIRADAS DE UN DADO.

LAS TIRADAS SERAN REPRESENTADAS POR — — — — —
Y EN CADA — TENDREMOS 6 O 6'

LA PROB. DE OBTENER 6 6 6' 6 6' ES

$$P(6\,6\,6'\,6\,6') = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

SI SUPONEMOS QUE HAY INDEPENDENCIA EN LOS EVENTOS PODEMOS ESCRIBIR [illegible]

$$P = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

PERO TENEMOS ${}_5C_3$ RESULTADOS POSIBLES.

$$P(3 \text{ seis}) = {}_5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{5!}{2!\,3!}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

$$= \frac{125}{3888}$$

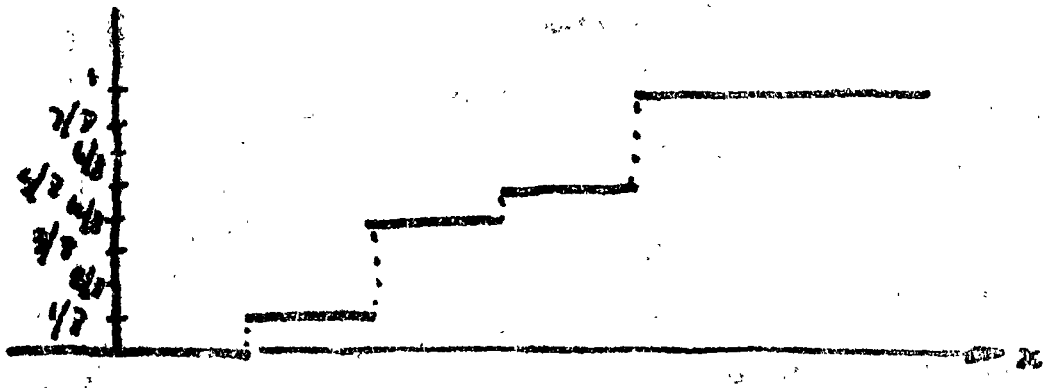
EN GENERAL

$${}_nC_x\, p^x q^{n-x} = \binom{n}{x} p^x q^{n-x}$$

$q = 1-p \qquad x = 3 \qquad p = \frac{1}{6}$

$q = 2 \quad q = 1 - p = \frac{5}{6}$

[illegible]



FUNCIONES CONTINUAS DE DISTRIBUCIÓN DE PROBABILIDAD

SI X ES UNA V.A. CONTINUA, LA PROBABILIDAD DE QUE X TOME UN VALOR PARTICULAR ES CERO, POR LO QUE PODEMOS DEFINIR VALORES ENTRE RANGOS SOLAMENTE.

LA IDEA ES LA MISMA DE LAS FUNCIONES DISCRETAS, EXCEPTO QUE:

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$P(a < X < b) = \int_a^b f(x)\,dx.$$

EJEMPLO: ENCUENTRE UNA CONSTANTE C TAL QUE:

a) $$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{EN OTRO CASO} \end{cases}$$

b) ENCUENTRE $P(1 < X < 2)$

TENEMOS n EVENTOS MUTUAMENTE EXCLUSIVOS, EN DONDE SE TIENEN 2 SOLUCIONES POSIBLES (HOMBRE, MUJER) CON PROBABILIDADES:

$$A = \quad p \quad = \frac{1}{2}$$
$$A' = \quad q = 1 - p \quad = 0.5$$

LA PROBABILIDAD DE OBTENER EXACTAMENTE x A's EN n PRUEBAS ES DE

$$_nC_x \; p^x q^{n-x}$$

SUPONGA QUE A ES UN NIÑO, PARA $n = 3$ TENEMOS.

$$P(\text{EXACTAMENTE } x \text{ NIÑOS}) = P(X=x) = {_3C_x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$$

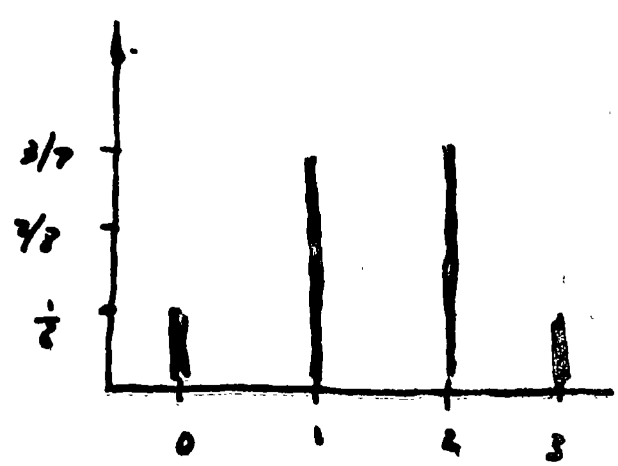
DONDE LA VARIABLE ALEATORIA X REPRESENTA EL # DE NIÑOS EN CADA FAMILIA.

LA FUNCION DE PROBABILIDAD DE X SERA:

$$f(x) = {_3C_x} \left(\frac{1}{2}\right)^3$$

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

Y SU GRAFICA SERA



FUNCION DE DISTRIBUCION PARA VARIABLES ALEATORIAS DISCRETAS.

LA FUNCION DE DISTRIBUCION DE PROBABILIDAD ACUMULATIVA (FDPA) o FUNCION DE DISTRIBUCION (FDP) DE UNA V.A. ESTA DEFINIDA POR

$$f(x) = P(X = x)$$

$$P(X \leq x) = F(x) \qquad P(X \leq x) = F(x)$$

DONDE x ES UN REAL i.e. $-\infty < x < \infty$. LA FDPA SE PUEDE OBTENER DE LA FUNCION DE PROBABILIDAD

$$F(x) = P(X \leq x) = \sum_{u \leq x} f(u)$$

DONDE la SUMATORIA DEBE CONSIDERAR TODO VALOR DE u.

SI X TOMA FINITOS SU FDPA ES ESTA DADA POR.

$$F(x) = \begin{cases} 0 & -\infty < x < x_1 \\ f(x_1) & x_1 \leq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leq x < x_3 \\ \vdots & \\ f(x_1) + \cdots + f(x_n) & x_n \leq x < \infty \end{cases}$$

EJEMPLO – ENCUENTRE LA FDA DE la V.A. X del ejemplo pasado (de las MONEDAS). Y OBTENGA SU GRAFICA.

COMO $f(x)$ CUMPLE CON LA CONDICION 1 SI $C \geq 0$ DEBE SATISFACER LA CONDICION 2.

$$9c \therefore = 1 \qquad \int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 cx^2\,dx = c\int x^2 dx = \left.\frac{cx^3}{3}\right|_0^3$$

$$= 9c$$

y DADO QUE DEBE VALER 1 TENEMOS QUE $C = \frac{1}{9}$

$$P(1 < X < 2) = \int_1^2 \frac{1}{9}x^2\,dx = \left.\frac{x^3}{27}\right|_1^2 =$$

$$= \frac{8}{27} - \frac{1}{27} = \frac{7}{27}.$$

$$P(1 < X < 2) = \frac{7}{27}.$$

EJEMPLO: UNA V.A X TIENE FUNCION DE DENSIDAD

$$f(x) = \frac{c}{x^2+1} \qquad \text{PARA } -\infty < X < \infty$$

a) DETERMINE EL VALOR DE c

b) DETERMINE LA PROB. QUE X^2 ESTE ENTRE $1/3$ y 1

SOLUCION.

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ $\qquad \frac{2\pi}{2} = \pi$

$$\int_{-\infty}^{\infty} \frac{c\,dx}{x^2+1} = c\tan^{-1}x\Big|_{-\infty}^{\infty} = c\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1$$

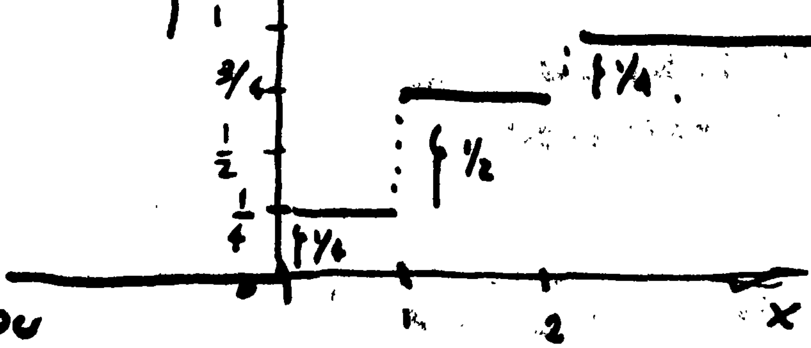
POR LO QUE $c = \frac{1}{\pi}$

b) SI $\frac{1}{3} \leq X^2 \leq 1$ ENTONCES $\frac{\sqrt{3}}{3} \leq X \leq 1$ ó $-1 \leq X \leq -\frac{\sqrt{3}}{3}$

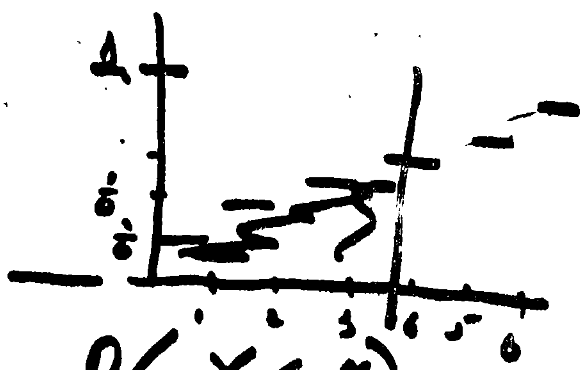
a) LA F.D.A. ES

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & 2 \le x < \infty \end{cases}$$

$$F(x) = \begin{cases} 0 & -\infty \le x < 0 \\ \frac{1}{6} & 0 \le x < 1 \\ \frac{1}{3} & 1 \le x < 2 \\ \frac{1}{2} & 2 \le x < 3 \\ 1 & \end{cases}$$



$P(-\infty \le X \le +\infty) = F(x) = 1$



$P(X \le x)$

NOTESE QUE:

SE TRATA DE UNA FUNCION 'ESCALON' LA CUAL BRINCA EN LOS VALORES 0, 1, 2

- ES MONOTONICAMENTE CRECIENTE

EJEMPLO: ENCUENTRE LA F.D.P.A. DEL EJEMPLO DE LOS NIÑOS (AS) DE FAMILIAS DE 3. Y SU GRAFICA

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/8 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$$

$$\frac{1}{\pi}\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2+1} + \frac{1}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1} = \frac{2}{\pi}\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2+1}$$

$$= \frac{2}{\pi}\left[\tan^{-1}(1) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{6}.$$

DETERMINE LA FUNCION DE DISTRIBUCION DE LA FUNCION DE DENSIDAD DEL PROBLEMA ANTERIOR.

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \frac{1}{\pi}\int_{-\infty}^{x} \frac{du}{u^2+1} = \frac{1}{\pi}\left[\tan^{-1} u \Big|_{-\infty}^{x}\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right] = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x$$

EJEMPLO .- LA FUNC. DE DIST. DE PROB. PARA LA V.A X ES

$$F(x) = \begin{cases} 1 - e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) ENCUENTRE SU FUNCION DE DENSIDAD.

$$f(x) = \frac{d\,F(x)}{dx} = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x < 0 \end{cases}$$

ENCUENTRE LA PROBABILIDAD QUE $X>2$

$$P(X>2) = \int_2^{\infty} 2e^{-2u}\,du = -e^{-2u}\Big|_2^{\infty} = e^{-4}$$

ó $P(X \leq 2) = F(2) = 1 - e^{-4}$

$$P(X>2) = 1 - (1 - e^{-4}) = e^{-4}$$

FUNCIONES DE DISTRIBUCION PARA V.A CONTINUAS.

$$F(x) = P(X \leq x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(u)\,du.$$

EJEMPLO: ENCUENTRE LA FUNC. DE DISTRIBUCION DE:

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & \text{EN OTRO CASO} \end{cases}$$

SI $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$. ENTONCES

SI $0 \leq x < 3$

$$F(x) = \int_0^x f(u)\,du = \int_0^x \frac{1}{9}u^2\,du = \frac{x^3}{27}$$

SI $x \geq 3$

$$F(x) = \int_0^3 f(u) + \int_3^x f(u)\,du = \int_0^3 \frac{1}{9}u^2\,du + \int_3^x 0\,du = 1$$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{27} & 0 \leq x < 3 \\ 1 & x > 3 \end{cases}$$

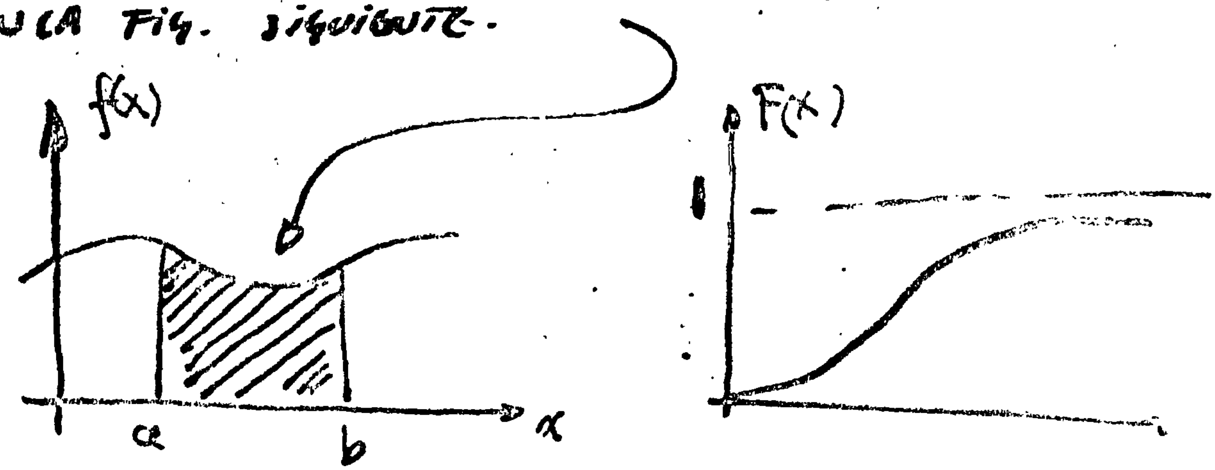
VALE LA PENA EN ESTE PUNTO RECORDAR QUE.

$$\frac{dF(x)}{dx} = f(x)$$

para cualquier punto donde x es continuo.

VEAMOS A CONTINUACION LA INTERPRETACION GEOMETRICA DE LO QUE HEMOS VISTO:

si $f(x)$ ES UNA FUNC. DE DENSIDAD DE UNA V.A X PODEMOS REPRESENTAR GRAFICAMENTE $y = f(x)$ Y LA PROBABILIDAD QUE X ESTE EN EL INTERVALO a, b, ESTO ES $P(a<X<b)$ SE MUESTRA EN LA FIG. SIGUIENTE.



fdp F. ACUMULADA

DISTRIBUCIONES CONJUNTAS.

LA IDEAS ANTERIORES LAS PODEMOS GENERALIZAR PARA EL CASO DE MAS DE UNA V.A. Y VEAMOS EL CASO DE 2 V.A

1.- CASO DISCRETO.

si X y Y SON V.A DISCRETAS, DEFINIMOS SU FUNCION DE PROBABILIDAD CONJUNTA COMO

$$P(X=x, Y=y) = f(x,y)$$

DONDE

1.- $f(x,y) \geq 0$

2.- $\sum_x \sum_y f(x,y) = 1.$

PARA EL CASO CONTINUO — LA FUNCION DE PROBABILIDAD CONJUNTA PARA EL CASO CONTINUO SERA:

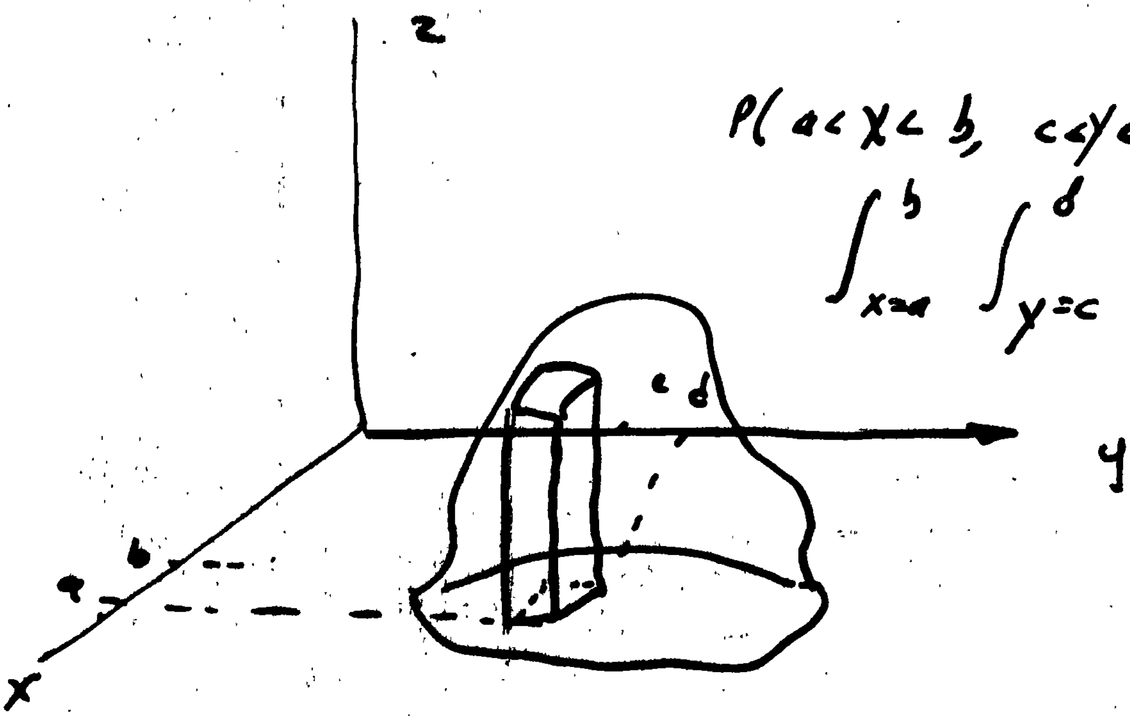
1.- $f(x,y) \geq 0$

2 $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

GRAFICAMENTE SI $z = f(x,y)$ REPRESENTA LA "SUPERFICIE DE PROBABILIDAD

$P(a < X < b,\ c < Y < d) =$

$\int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dx,y)$

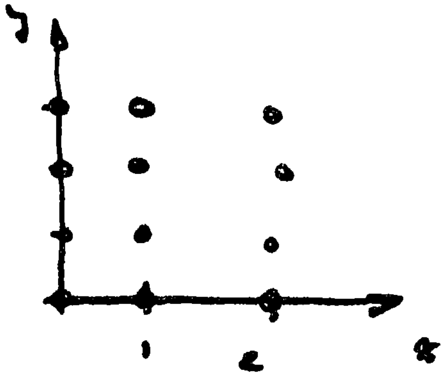


EJEMPLO LA FUNCIÓN CONJUNTA DE PROBABILIDAD DE 2 V.A DISCRETAS X y Y ESTÁ DADA POR $f(x,y) = c(2x+y)$ DONDE x, y PUEDEN TOMAR LOS SIG. VALORES $0 \le x \le 2$, $0 \le y \le 3$ y $f(x,y) = 0$ EN OTRO CASO.

a) ENCUENTRE EL VALOR DE C.

x \ y	0	1	2	3	TOTAL
0	0	c	2c	3c	6c
1	2c	3c	4c	5c	14c
2	4c	5c	6c	7c	22c
TOTAL	6c	9c	12c	15c	42c

$42c \rightarrow c = \frac{1}{42}$



b) DETERMINE $P(X=2, Y=1)$ DE LA TABLA ANTERIOR

$$P(X=2, Y=1) = 5c = \frac{5}{42}$$

c) DETERMINE LA PROBABILIDAD DE $P(X \ge 1, Y \le 2)$
VER ZONA AZUL EN LA TABLA.

$$P(X \ge 1, Y \le 2) = \sum_{x \ge 1} \sum_{y \le 2} f(x,y).$$

$$= (2c + 3c + 4c) + (4c + 5c + 6c)$$

$$= 24c = \frac{24}{42} = \frac{4}{7}$$

LA FUNCION CONJUNTA DE PROBABILIDAD DE DOS V.A. ES.

$$f(x,y) = \begin{cases} cxy & 0<x<4 \quad 1<y<5 \\ 0 & \text{EN OTRO CASO} \end{cases}$$

a) DETERMINE LA CTE C.

$$\int_{x=0}^{4}\int_{y=1}^{5} cxy\,dy\,dx = c\int_0^4 \frac{xy^2}{2}dx\Big|_{y=1}^{5} = c\int_{x=0}^{4}\left(\frac{25}{2}x - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\int_{x=0}^{4} 24x\,dx = c\,(6x^2)\Big|_0^4 = 96c \qquad c = \frac{1}{96}$$

b) $P(1<X<2,\ 2<Y<3)$

$$= \int_{x=1}^{2}\int_{y=2}^{3} \frac{xy}{96}\,dx\,dy = \frac{1}{96}\int_{x=1}^{2} \frac{xy^2}{2}\Big|_{y=2}^{3} dx$$

$$= \frac{1}{96}\int_1^2 \frac{5x}{2}\,dx = \frac{5}{192}\left(\frac{x^2}{2}\right)\Big|_1^2 = \frac{5}{128}$$

DISTRIBUCIONES CONDICIONALES.

SABEMOS QUE SI $P(A)>0$

$$P(B/A) = \frac{P(AB)}{P(A)}$$

SI X y Y SON V.A. DISCRETAS Y TENEMOS LOS EVENTOS $(A: X=x)$, $B(B: Y=y)$, LA FORMULA ANTERIOR SE PUEDE ESCRIBIR COMO:

$$P(Y=y / X=x) = \frac{f(x,y)}{f_1(x)}$$

DONDE $f(x,y) = P(X=x, Y=y)$ ES LA PROBABILIDAD CONJUNTA Y $f_1(x)$ ES LA PROBABILIDAD MARGINAL DE X.

DEFINIMOS

$$f(y/x) = \frac{f(x,y)}{f_1(x)}$$

Y LA LLAMAREMOS LA FUNCION DE DISTRIBUCION CONDICIONAL DE Y DADO X. EN FORMA SIMILAR PODEMOS ESCRIBIR

$$f(x/y) = \frac{f(x,y)}{f_2(y)}.$$

USANDO LA FORMULA ANTERIOR PODEMOS ESCRIBIR, LA PROBABILIDAD DE QUE Y SE ENCUENTRE ENTRE c Y d DADO QUE OCURRIO $x < X < x + dx$. TENEMOS:

$$P(c < Y < d / x < X < x + dx) = \int_c^d f(y/x)\, dy$$

EJEMPLO:

SI X Y Y TIENEN FUNCION DE DENSIDAD DE PROBABILIDAD CONJUNTA QUE ES:

$$f(x,y) = \begin{cases} \frac{3}{4} + xy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{EN OTRO CASO.} \end{cases}$$

DETERMINE $f(y/x)$ PARA $0<x<1$

$$f_1(x) = \int_0^1 \left(\frac{3}{4} + xy\right) dy = \frac{3}{4} + \frac{x}{2}$$

$$f(y/x) = \frac{f(x,y)}{f_1(x)} = \begin{cases} \dfrac{3+4xy}{3+2x} & 0<y<1 \\ 0 & \text{OTROS VALORES } y \end{cases}$$

NOTEMOS QUE PARA OTROS VALORES DE x $f(y/x)$ NO ESTA DEFINIDA.

ULTIMO EJEMPLO DE ESTE CAPITULO.

DOS PERSONAS SE QUEDAN DE VER ENTRE LAS 14:00 hs (2 PM) Y LAS 15:00 (3 PM), Y ACORDARON DE ACUERDO QUE NINGUNO ESPERARA MAS DE 15 min. ¿CUAL ES LA PROBABILIDAD QUE SE ENCUENTREN?

Sol.

SEAN X y Y las V.A. QUE REPRESENTAN EL TIEMPO DE LLEGADA DE CADA PERSONA, MEDIDO EN FRACCIONES DE TIEMPO DESPUES DE LAS 2 PM.

SUPONIENDO QUE INTERVALOS DE TIEMPO IGUAL TIENEN PROBABILIDADES IGUALES DE LLEGADA, LAS FUNCIONES DE DENSIDAD DE LAS V.A X, Y ESTARAN DADAS POR:

$$f_1(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{en otro caso} \end{cases}$$

$$f_2(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & \text{en otro caso.} \end{cases}$$

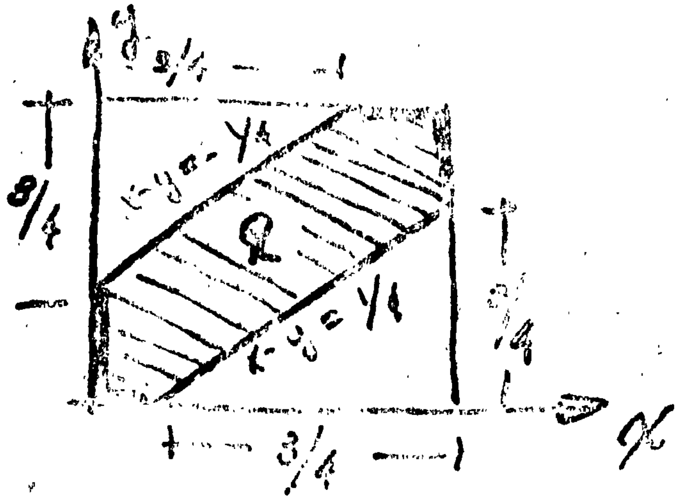
Dado que X e Y son independientes, la func. de densidad conjunta será:

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} 1 & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otro caso.} \end{cases}$$

Dado que el área es igual a $\frac{1}{4}$ de 1. La probabilidad requerida es

$$P(|X-Y| \leq \tfrac{1}{4}) = \int_R \int dx\, dy.$$

Donde R se muestra en la sig. figura.



$$P(|X-Y| \geq \tfrac{1}{4}) = 1 - \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right) = \tfrac{7}{16}$$ dado que

el cuadro de la fig tiene área 1 y las esquinas tienen área $\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)$ c/u, la prob. requerida es de

$7/16$.

EXPECTACION (ESPERANZA) MATEMATICA

ó valor esperado. PARA UNA V.A. X, la cual puede tomar valores $x_1, x_2, \ldots x_n$ LA EXPECTACION MATEMATICA, SE DEFINE COMO.

$$E(x) = x_1 P(X = x_1) + \ldots\ldots x_n P(X = x_n) = \sum_{j=1}^{n} x_j P(X = x_j)$$

$$\text{ó} \quad E(X) = x_1 f(x_1) + \ldots\ldots x_n f(x_n) = \sum_{j=1}^{n} x_j f(x_j) = \sum x f(x)$$

UN CASO ESPECIAL DE LA FORMULA ANTERIOR ES LA

$$E(X) = \frac{x_1 + x_2 + \ldots\ldots x_n}{n}$$

PARA UNA V.A CONTINUA CON f.d.p $f(x)$ SU ESPERANZA MAT. SERA

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

A $E(X)$ COMUNMENTE SE LE LLAMA MEDIA DE X y SE ESCRIBE COMO μ_x ó SIMPLEMENTE μ.

EJEMPLO. SE TIENE UN DADO, CON EL CUAL SE JUEGA DE LA SIG. MANERA, si sale un 2 GANA \$ 20, SI SALE 4 GANA \$ 40, SI SALE 6 PIERDE \$ 30, y NI GANA NI PIERDE SI SALE OTRA CARA (1, 3, 5) DETERMINE LA CANTIDAD DE DINERO QUE ESPERA GANAR.

$$E(X) = 0\left(\frac{1}{6}\right) + 20\left(\frac{1}{6}\right) + 40\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) + (-30)\left(\frac{1}{6}\right) = 5$$

OTRO EJEMPLO; LA FUNCION DE DENSIDAD DE UNA U.A. ESTA DADO POR

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{EN OTRO CASO.} \end{cases}$$

EL VALOR ESPERADO DE X SERA:

$$E(X) = \int_{-\infty}^{\infty} x f(x) = \int_0^2 x\left(\tfrac{1}{2}x\right) dx = \int_0^2 \frac{x^2}{2} dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}$$

VAMOS ALGUNOS TEOREMAS INTERESANTES.

1. SI c = CTE $E(cX) = cE(X)$.
2. SI X y Y SON U.A. INDEPENDIENTES $E(X+Y) = E(X) + E(Y)$
3. SI X y Y SON U.A. INDEPENDIENTES $E(XY) = E(X)E(Y)$

VARIANZA Y DESVIACION STANDARD (ESTANDAR)

DEFINAMOS LA VARIANZA DE UNA U.A X COMO

$$VAR(X) = E\left[(X-\mu)^2\right]$$

QUE ES UN NUMERO ≥0

Y LA DESVIACION ESTANDAR COMO

$$\sigma_X = \sqrt{VAR(X)} = \sqrt{E[X-\mu]^2}$$

LA VARIANCIA SE ESCRIBE TAMBIEN COMO σ^2.

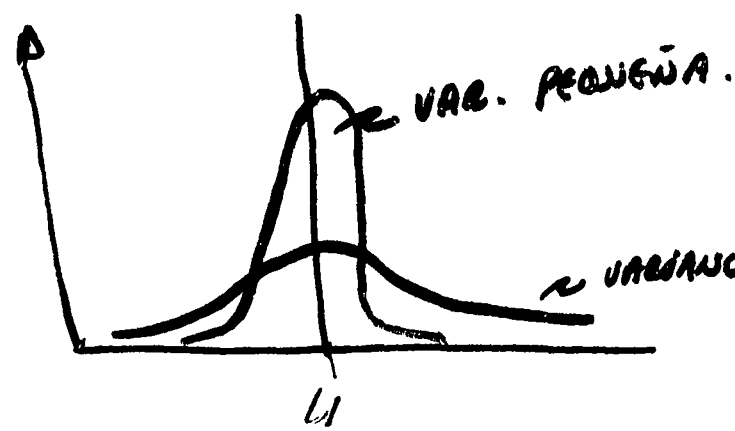
EN EL CASO ESPACIAL EN EL CUAL TODOS LOS EVENTOS TIENEN LA MISMA PROB.

$$\sigma^2 = [(x_1-\mu)^2 + ((x_2-\mu)^2 + \dots (x_n-\mu)^2]/n$$

SI X ES UNA V. A CONTINUA.

$$\sigma_x^2 = E[(x-\mu)^2] = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx.$$

LA VARIACIA Y LA DESV. STANDARD (DS) SON MEDIDAS DE DISPERSION CON RESPECTO A LA MEDIA μ.



EJ. ENCUENTRE LA VARIANZA Y D.S. DEL EJ. ANTERIOR.

$$f(x) = \begin{cases} \frac{1}{2}x & 0 < x < 2 \\ 0 & \text{OTRO CASO} \end{cases}$$

COMO SE VIO $\mu_x = \frac{4}{3}$

$$\sigma^2 = E[(x-\tfrac{4}{3})^2] = \int_{-\infty}^{\infty}(x-\tfrac{4}{3})^2 f(x)\,dx$$

$$= \int_0^2 (x-\tfrac{4}{3})^2(\tfrac{1}{2}x)\,dx = \frac{2}{9} = \text{VARIANZA} = \sigma^2$$

$$\sigma = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3} = \text{D.S.}$$

VEAMOS ALGUNOS TEOREMAS SOBRE LA VARIANZA.

$$\sigma^2 = E[(X-\mu)^2] = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

$[E(X)]^2 = \mu^2$

SI c = CTE.

$$\star \quad Var(cX) = c^2 Var(X).$$

VEAMOS A CONTINUACIÓN ALGUNOS OTROS CONCEPTOS INTERESANTES ANTES DE ENTRAR A RESOLVER ALGUNOS PROBLEMAS.

VARIANZA PARA DISTRIBUCIONES CONJUNTAS. COVARIANZA

SEAN X y Y DOS V.A. CONTINUAS CON FUNC. DE DENSIDAD CONJUNTA DE PROBABILIDAD $f(x,y)$ Y MEDIAS:

$$\mu_x = E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$\mu_y = E(Y) = \int\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy.$$

SUS VARIANZAS.

$$\sigma_x^2 = E[(X-\mu_x)^2] = \int\int_{-\infty}^{\infty} (x-\mu_x)^2 f(x,y)\, dx\, dy.$$

$$\sigma_y^2 = E[(Y-\mu_y)^2] = \int\int (y-\mu)^2 f(x,y)\, dy\, dx$$

Y SU COVARIANZA ESTARA DEFINIDA POR:

$$\sigma_{xy} = Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)]$$

$$\sigma_{xy} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\, dx\, dy.$$

COEFICIENTE DE CORRELACION.

si X y Y son v.a. independientes su

$$COV(X,Y) = \sigma_{xy} = 0$$

si X y Y son totalmente dependientes i.e. X=Y

$$COV(X,Y) = \sigma_{xy} = \sigma_x \sigma_y$$

De aqui podemos medir dependencia existente entre X e Y y esto es.

$$\boxed{\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}} \quad \text{COEF.}$$

y le llamaremos coeficiente de correlacion

$$-1 \leq \rho \leq 1$$

si $\rho = 0$ no estan correlacionadas.

Veamos a continuacion varios ejemplos.

Ej 1.- En una loteria se tienen 200 premios de \$ 5.-, 20 premios de \$ 25.-, 5 premios de . 100.- . Suponga que la loteria vende 10000 boletos ¿Cual seria un precio justo a pagar por boleto?

Solucion sea X la v.a. del dinero ganado.

X (\$)	5	25	100	0
P (X=x)	0.02	0.002	0.0005	0.9775

$$E(X) = 5(0.002) + 25(0.002) + 100(0.0005) + 0(.9775)$$

$E(X) = 0.2$.

POR LO QUE DEBERÍA DE SER DE 0.20 ¢, SIN EMBARGO COMO QUIEREN GANAR DINERO, SERÁ UN POCO MAYOR.

Ej LA FUNC. DE DENSIDAD DE UNA V.A X ESTÁ DADA POR

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

ENCUENTRE

a) $E(X)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x(2e^{-2x})\,dx = 2\int_{0^+}^{\infty} x e^{-2x}\,dx$$

$$= 2\left[x\left(\frac{e^{-2x}}{-2}\right) - (-1)\left(\frac{e^{-2x}}{4}\right)\right]_0^{\infty} = \frac{1}{2}.$$

b) $E(X^2)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 2\int_0^{\infty} x^2 e^{-2x}\,dx$$

$$= 2\left[x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}.$$

VEAMOS ALGUNOS PROBLEMAS CON VARIANZA Y D.S.

Ej.- ENCUENTRE LA VARIANZA Y D.S. DE LA SUMA OBTENIDA DE LANZAR 2 DADOS AL AIRE.

SEAN X e Y EL # DE CADA DADO

$$E(X) = E(Y) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + \cdots 6\left(\frac{1}{6}\right) = \frac{7}{2}$$

$$E(X+Y) = E(X) + E(Y) = \frac{7}{2} + \frac{7}{2} = 7$$ QUE ES LA ESPERANZA

$$E(X^2) = E(Y^2) = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + \cdots 6^2\left(\frac{1}{6}\right) = \frac{91}{6}$$

$$VAR(X) = VAR(Y) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

DADO QUE X e Y SON INDEPENDIENTES

$$VAR(X+Y) = Var(X) + Var(Y) = \frac{35}{6}$$

$$\sigma_{X+Y} = \sqrt{Var(X+Y)} = \sqrt{35/6} \qquad \rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

EJEMPLO

DETERMINE LA VARIANZA Y D.S. DE LA V.A

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

COMO YA SE CALCULO CON ANTERIORIDAD

$\mu = E(x) = \frac{1}{2}$ POR LO QUE LA VARIANZA SERA:

$$VAR(x) = E[(x-\mu^2)] = E\left[\left(x - \frac{1}{2}\right)^2\right] = \int_{-\infty}^{\infty} \left(x - \frac{1}{2}\right)^2 f(x)\, dx$$

$$= \int_0^{\infty} (x-\tfrac{1}{2})^2 (2e^{-2x})\,dx = \frac{1}{4}$$

UTILIZANDO OTRO METODO

$$VAR(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\sigma = \sqrt{VAR(X)} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

EJEMPLO DEL PROBLEMA DE LA PAG. 38 ENCUENTRE

a) $E(X) = \sum_x \sum_y x f(x,y) = \sum_x x \left[\sum_y f(x,y)\right]$

$$= 0(6c) + 1(14c) + 2(22c) = 58c = \frac{58}{42} = \frac{29}{21}$$

b) $E(Y) = \sum_x \sum_y y f(x,y) = \sum_y y \left[\sum_x f(x,y)\right]$

$$= 0(6c) + 1(9c) + 2(12c) + 3(15c) = 78c = \frac{78}{42} = \frac{13}{7}$$

c) $E(XY) = \sum_x \sum_y xy f(x,y)$

$$= (0)(0)(0) + (0)(1)(c) + (0)(2)(2c) + (0)(3)(3c) +$$
$$(1)(0)(2c) + (1)(1)(3c) + (1)(2)(3c) + (1)(2)(4c) +$$
$$(1)(3)(5c) + (2)(0)(4c) + (2)(1)(5c) + (2)(2)(6c) +$$
$$(2)(3)(7c) = 102c = \frac{102}{42} = \frac{17}{7}$$

d) $E(X^2) = \sum_x \sum_y x^2 f(x,y) = \sum_x x^2 \left[\sum_y f(x,y)\right]$.

$= (0)^2(6c) + (1)^2(14c) + (2)^2(22c) = 102c = \frac{102}{42} = \frac{17}{7}$.

e) $E(Y^2) = \sum_y y^2 \left[\sum_x f(x,y)\right]$

$= (0)^2(6c) + (1)^2 9c + (2)^2(12c) + (3)^2(15c) = 192c = \frac{192}{42} = \frac{32}{7}$

f) $\sigma_x^2 = VAR(X) = E(X^2) - [E(X)]^2 = \frac{17}{7} - \left(\frac{29}{21}\right)^2 = \frac{230}{441}$

g) $\sigma_y^2 = VAR(Y) = E(Y^2) - [E(Y)]^2 = \frac{17}{7} - \left(\frac{13}{7}\right)^2 = \frac{55}{49}$.

h) $\sigma_{XY} = COV(XY) = E(XY) - E(X)E(Y) = \frac{17}{7} - \left(\frac{29}{21}\right)\left(\frac{13}{7}\right) = -\frac{20}{147}$

i) $\rho = \frac{\sigma_{XY}}{\sigma_{XY}} = \frac{-20/147}{\sqrt{\frac{230}{441}} \times \sqrt{\frac{55}{49}}} = \frac{-20}{\sqrt{230}\sqrt{55}} = -0.2103$

FUNCIONES DE DISTRIBUCION DE PROBABILIDAD.

1.- FUNCION BINOMIAL Ó DISTRIBUCION DE BERNOULLI.

A PRINCIPIO DEL CURSO PAG VIMOS Ó DEDUCIMOS LA FUNCION BINOMIAL.

ESTA FUNCION SE OBTIENE DE LLEVAR REPETIDAMENTE UN EXPERIMENTO (TIRAR UNA MONEDA, UN DADO, SACAR UNA BARAJA ETC)

SEA p LA PROBABILIDAD DE EXITO DE CUALQUIER EVENTO DE BERNOULLI Y $q=1-p$ LA PROBABILIDAD DE FALLA

LA PROBABILIDAD QUE UN EVENTO SUCEDA X VECES EN n INTENTOS ESTA DADO POR:

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Y SUS PROPIEDADES SERAN:

MEDIA $\mu = np$
VARIANZA $\sigma^2 = npq$
D.S. $\sigma = \sqrt{npq}$.

ejemplo CUAL ES LA PROBABILIDAD DE OBTENER 2 ÁGUILAS EN 6 TIRADAS.

$$P(X=2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2} = \frac{6!}{4!\,2!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{15}{64}$$

EN CASO QUE LA FUNCION SEA DISCRETA SE PUEDE ESCRIBIR COMO LA EXPANSION DE BERNOULLI

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots$$
$$= \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} .$$

DISTRIBUCION NORMAL.

ESTA ES UNA DE LAS DISTRIBUCIONES MAS IMPORTANTES, TAMBIEN CONOCIDA COMO DISTRIBUCION GAUSSIANA. Y SU FUNC. DE DE DENSIDAD DE PROBABILIDAD ESTA DADA POR.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < X < \infty$$

DONDE μ Y σ SON SU MEDIA Y SU D.S. RESPECTIVAMENTE.

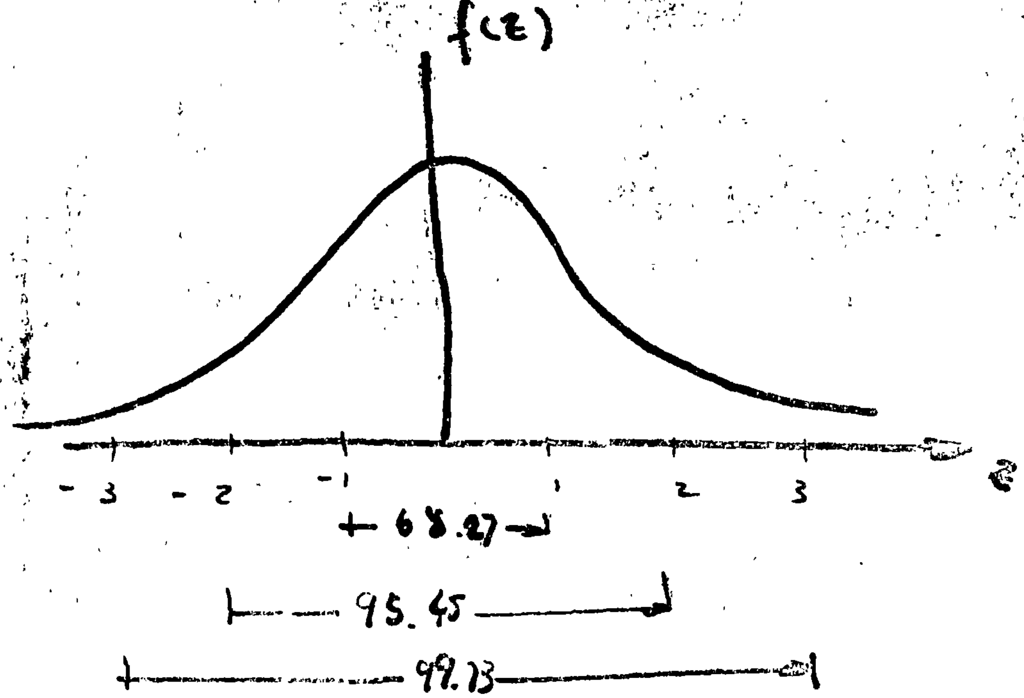
LA FUNCION ACUMULADA NORMAL SERA

$$F(x) = P(X \leq x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(v-\mu)^2/2\sigma^2} dv$$

SI HACEMOS $z = \frac{X-\mu}{\sigma}$ Y $\mu = 0$ Y $\sigma = 1$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

A LA CUAL SE LE LLAMA STANDARD DE LA CURVA NORMAL DE DENSIDAD Ó CURVA NORMALIZADA.



$P(-1 \leq z \leq 1) = 0.6827$

$P(-2 \leq z \leq 2) = .9545$

$P(-3 \leq z \leq 3) = .9973$

DISTRIBUCION DE POISSON.

SEA X UNA V.A. LA CUAL PUEDE TOMAR VALORES 0, 1, 2, ... TAL QUE SU FUNCION DE PROBABILIDAD ESTA DADA POR

$$f(x) = P(X = x) = \frac{\lambda^{x} e^{-\lambda}}{x!}$$

DONDE λ ES UNA CONSTANTE >0, LA CUAL TIENE LAS SIGUIENTES PROPIEDADES:

MEDIA	$\mu = \lambda$
VARIANZA	$\sigma^2 = \lambda$
D.S	$\sigma = \sqrt{\lambda}$

TEOREMA DEL LIMITE CENTRAL.

SEAN $X_1, X_2 \ldots$ V.A. INDEPENDIENTES, LAS CUALES TIENEN LA MISMA DISTRIBUCION, CON MEDIA μ Y VARIANZA σ^2 FINITAS.

SI $S_n = X_1 + X_2 + \ldots + X_n \quad (n = 1, 2 \ldots)$

$$\lim_{n \to \infty} P\left(a \leq \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2} du$$

LO CUAL SIGNIFICA QUE LA V.A. $\dfrac{S_n - n\mu}{\sigma\sqrt{n}}$ ES NORMAL ESTANDARIZADA

FUNCION DE DISTRIBUCION HIPERGEOMETRICA.

SUPONGA QUE UNA CAJA CONTIENE b CANICAS AZULES Y r CANICAS ROJAS. EFECTUEMOS n EXPERIMENTOS, EN LOS CUALES SE SACA UNA CANICA, SE OBSERVA EL COLOR Y SE REGRESA A LA CAJA (MUESTREO CON REEMPLAZO). SI X (ES UNA V.A.) ES EL NUMERO DE CANICAS AZULES OBSERVADO EN n PRUEBAS, LA PROBABILIDAD DE OBTENER EXACTAMENTE x EXITOS ESTA DADO POR:

$$P(X = x) = \binom{n}{x} \frac{b^x r^{n-x}}{(b+r)^n}$$

SI EFECTUAMOS LA PRUEBA SIN REEMPLAZO.

$$P(X = x) = \frac{\binom{b}{x}\binom{r}{n-x}}{\binom{b+r}{n}}$$

Y ES LA DIST. HIPERGEOMETRICA

CUYA MEDIA ES $\mu = \frac{nb}{b+r}$

Y D.S. $\sigma^2 = \frac{nbr(b+r-n)}{(b+r)^2(b+r-1)}$

DISTRIBUCION UNIFORME.

SEA X UNA V.A. UNIFORMEMENTE DISTRIBUIDA EN UN INTERVALO $a \leq x \leq b$ SU f.d.p. ES.

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{EN OTRO CASO.} \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < a \\ \frac{(x-a)}{b-a} & a \leq x < b \\ 1 & x \geq b \end{cases}$$

SU MEDIA SERA $\mu = \frac{1}{2}(a+b)$

$$\sigma^2 = \frac{1}{12}(b-a)^2$$

EXISTEN MUCHAS OTRAS DISTRIBUCIONES.

LA CAUCHY $f(x) = \frac{a}{\pi(x^2+a^2)}$ $a > 0$, $-\infty < x < \infty$

DISTRIBUCION GAMA.

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1}\, e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

DONDE $\Gamma(\alpha)$ ES LA FUNCIÓN GAMA.

DISTRIBUCION χ^2 CHI CUADRADA.

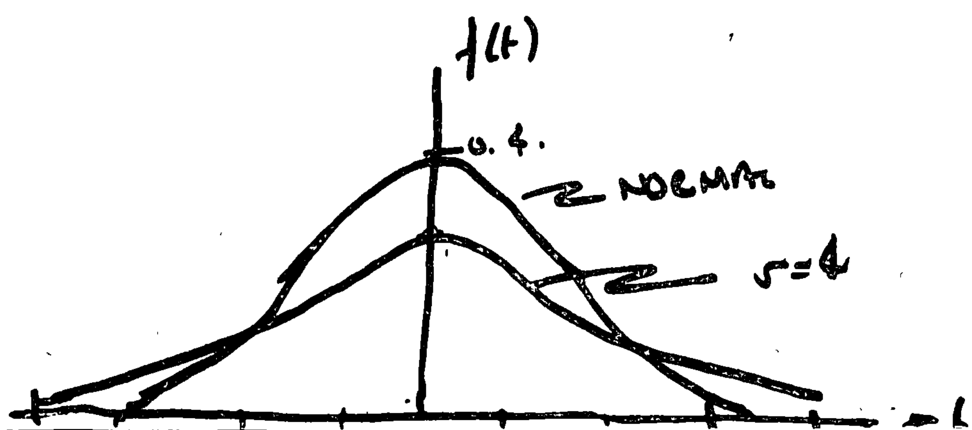
SEA $\chi^2 = X_1^2 + X_2^2 + \dots X_v^2$

$$P(\chi^2 \leq x) = \frac{1}{2^{v/2}\,\Gamma(v/2)} \int_0^x u^{(v/2)-1} e^{-u/2}\, du.$$

DONDE v = GRADOS DE LIBERTAD.

DISTRIBUCIÓN STUDENT (t)

$$f(t) = \frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\,\Gamma\left(\frac{v}{2}\right)} \left(1 + \frac{t^2}{v}\right)^{-\frac{(v+1)}{2}}$$



VAMOS A CONTINUACIÓN UNOS EJEMPLOS.

1) DETERMINE LA PROBABILIDAD DE QUE AL TIRAR UN DADO AL AIRE 5 VECES EL NÚMERO 3.

a) SE TENGA 2 VECES.

QUE RESULTE UNA VEZ ES $p = \frac{1}{6}$ EN UN TIRO
Y QUE NO SALGA $q = 1 - p = \frac{5}{6}$

$$P(x=2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = \frac{625}{3888}$$

b) QUE NO SALGA MÁS DE UNA VEZ. $= P(X \leq 1) = P(X=0) + P(X=1)$

$$= \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = \frac{3125}{3888}.$$

c) QUE SALGA CUANDO MENOS 2 VECES. $P(X \geq 2)$

$P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 + \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 + \binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^1 + \binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0$$

$$= \frac{763}{3888}$$

2) SE TIENEN 2000 FAMILIAS CON 4 HIJOS C/U CUÁNTAS TENDRÁN

a) CUANDO MENOS UN NIÑO

$P(1 \text{ niño}) + P(2 \text{ niños}) + P(3 \text{ niños}) + P(4 \text{ niños})$

$$= \binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + \binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0$$

$$= \frac{15}{16} \qquad 2000\left(\frac{15}{16}\right) = 750$$

3) 20% DE LAS TUERCAS QUE PRODUCE UNA MAQUINA SON DEFECTUOSAS, SE TOMAN ALEATORIAMENTE 4 TUERCAS, CUAL ES LA PROBABILIDAD DE QUE SE TENGA

a) UN PZA. DEFECTUOSA.

$$P(x=1) = \binom{4}{1}(0.2)^1(0.8)^3 = 0.4096$$

b) NINGUNA DEFECTUOSA.

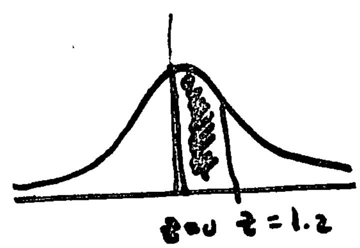
$$P(x=0) = \binom{4}{0}(0.2)^0(0.8)^4 = 0.4096$$

c) MENOS DE 2. $P(x<2) = P(x=0) + P(x=1)$

$$= 0.4096 + 0.4096 = 0.8192$$

DADA UNA CURVA NORMAL ENCUENTRE EL AREA

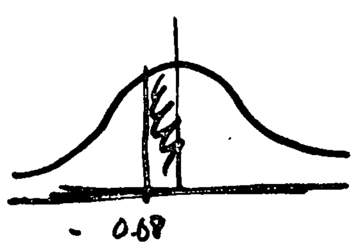
a) ENTRE $z=0$ y $z=1.2$.



DE LAS TABLAS.

$$P(0 \le z \le 2) = \frac{1}{\sqrt{2\pi}}\int_0^{1.2} e^{-u^2/2}\,du = 0.3849.$$

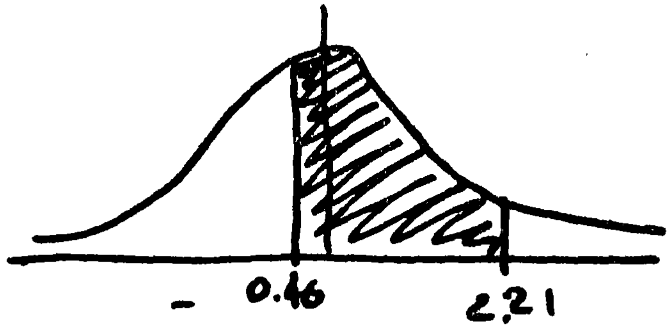
b) ENTRE $z=-0.68$ y $z=0$



DADO QUE LA CURVA ES SIMETRICA DE LA TABLA:

$$P(-0.68 \le z \le 0) = \int_{-0.68}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,du. = 0.2517.$$

c) ENTRE -0.46 y 2.21

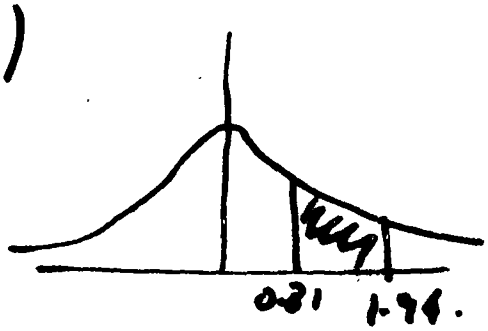


$P(-0.46 \le z \le 2.21)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-0.46}^{2.21} e^{-u^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-0.46}^{0} e^{-u^2/2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{2.21} e^{-u^2/2}$$

$= 0.1772 + 0.4864$

d) $P(0.81 \le z \le 1.94)$

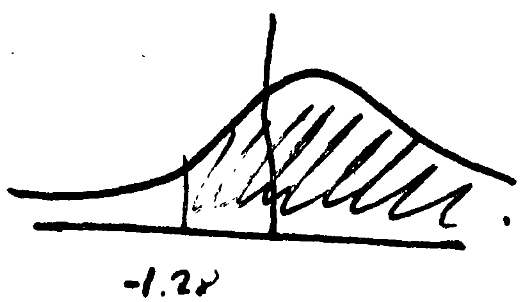


(AREA DE 0 a 1.94) −
(AREA DE 0 a 0.81)

$= 0.4738 - 0.2910 = .1828$

e $P(z \ge -1.28)$ = (AREA -1.28 y 0) + (area a la derecha)

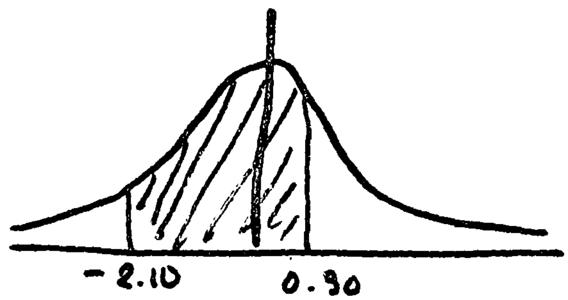
$= 0.3997 + 0.5 = 0.8997$



EL PROMEDIO DE PESO DE 500 ESTUDIANTES ES DE 151 lbs. CON UNA D.S. DE 15 lbs. SUPONIENDO QUE EL PESO ESTA ~~[illegible]~~ NORMALMENTE DISTRIBUIDO. DETERMINE.

a) CUANTO ESTUDIANTES PESAN ENTRE 119.5 y 155.5 lbs.

119.5 EN U. STANDART
$= (119.5 - 151)/15$
$= -2.10$



155.5 EN U. STANDART $= (155.5 - 151)/15 = 0.30$

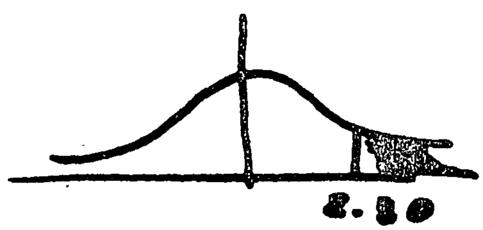
$= 0.4821 + 0.1179 = 0.6000$

POR LO QUE EL NUMERO DE ESTUDIANTES QUE PESAN ENTRE 119.5 y 155.5 ES DE:

$500 (0.600) = 300$

b) EL NUMERO DE ESTUDIANTES QUE PESAN MAS DE 185 lbs. (DIGAMOS 185.5)

185.5 EN U. STANDARD $= (185.5 - 151)/15 = 2.30$



AREA A LA DERECHA 2.30
$=$ -(AREA 0 A 2.30) + (AREA A LA DERECHA DE 0)
$= 0.5 - .4893 = 0.0107$
$500 (0.0107) = 5$

DETERMINE LA PROBABILIDAD DE OBTENER ENTRE 3 y 6 (INCLUSIVE) AGUILAS EN 10 TIRADAS.

a) USANDO DISTRIBUCION BINOMIAL.

$$P(X=3) = \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 = \frac{15}{128}$$

$$P(X=4) = \binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = \frac{105}{512}$$

$$P(X=5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \frac{63}{256}$$

$$P(X=6) \binom{10}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^4 = \frac{105}{512}$$

$$P(3 \leq X \leq 6) = \frac{15}{128} + \frac{105}{512} + \frac{63}{256} + \frac{105}{512}.$$

UNA CAJA CONTIENE 6 CANICAS VERDES y 4 ROJAS. SE TOMA UNA CANICA DE LA CAJA, SE ANOTA EL COLOR, Y NO ES DEVUELTA A LA CAJA. CUAL ES LA PROBABILIDAD DESPUES DE SACAR 5 CANICAS (5 VECES SE EFECTUA EL EXPERIMENTO), SE OBTIENEN 3 CANICAS VERDES

EL # TOTAL DE POSIBILIDADES DE SELECCIONAR 3 VERDES DE 6 VERDES ES $\binom{6}{3}$
EL # TOTAL DE MANERAS ≠ DE SELECCIONAR LAS 2 RESTANTES DE LAS 4 ROJAS ES DE $\binom{4}{2}$, POR LO QUE EL # TOTAL DE MANERAS DE OBTENER 3 VERDES y 2 ROJAS ES DE

$$\binom{6}{3}\binom{4}{2}$$

$6+4=$

DADO QUE EL # DE SELECCIONAR 5 CANICAS DE 10 ES DE

$$\binom{10}{5}$$

$$\frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \frac{10}{21}$$

[illegible] $b=6$, $r=4$, $n=5$, $x=3$

$$\frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = (P\ x=3)$$

centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

DECISIONES DE ACUERDO A LA TEORIA DEL VALOR

ING. RODOLFO FELIX FLORES

OCTUBRE, 1978 .

centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

REPASO DE PROBABILIDAD

M. EN C. MARCIAL PORTILLA ROBERTSON

OCTUBRE, 1978.

CENTRO DE EDUCACION CONTINUA.

DIVISION DE ESTUDIOS SUPERIORES
FAC. DE ING.
U. N. A. M.

REPASO DE PROBABILIDAD.

PROFESOR MARCIAL PORTILLA.

LA INTENCION DE ESTE CURSO, ES LA PRESENTAR LOS CONCEPTOS FUNDAMENTALES DE LA TEORIA DE PROBABILIDADES, PARA QUE MAS ADELANTE, LOS ASISTENTES PUEDAN ENTENDER, FORMULAR Y MANEJAR SITUACIONES PROBABILISTICAS.

CAPITULO 1

TEORIA FUNDAMENTAL DE LAS PROBABILIDADES.
TEOREMA DE BAYES
ESPACIOS DE EVENTOS

CAPITULO 2.

VARIABLES ALEATORIAS
PROBABILIDADES EN ESPACIOS MUESTRA
FUNCIONES DE DISTRIBUCION.

CAPITULO 3.

BERNOULLI
POISSON.
EJEMPLOS. NORMAL, BINOMIAL, LIMITE CENTRAL

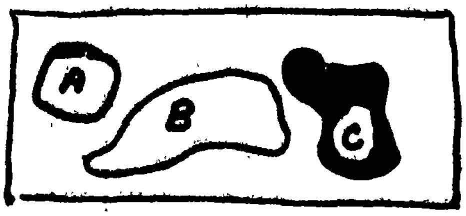
CAPITULO 4.

PROCESOS MARKOVIANOS DISCRETOS
EJEMPLOS.
TEOREMA DEL LIMITE CENTRAL

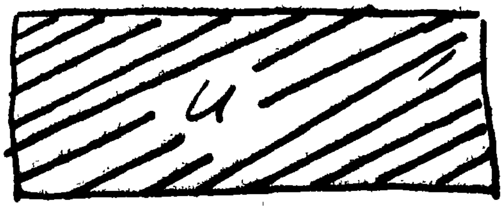
CAPITULO 5.

INTRODUCCION A LA ESTADISTICA

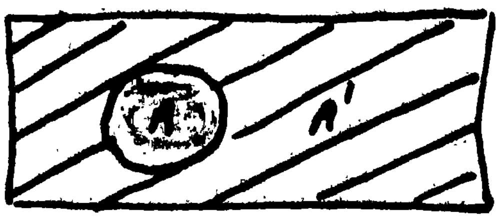
DEFINICIONES



EVENTO (O CONJUNTOS) SON UNA COLECCIÓN DE PUNTOS O AREAS EN UN ESPACIO

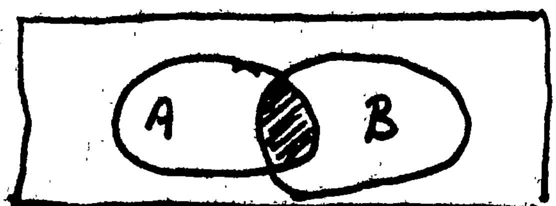


EVENTO O ESPACIO UNIVERSAL U ES EL CONJUNTO DE TODOS LOS PUNTOS EN EL ESPACIO.



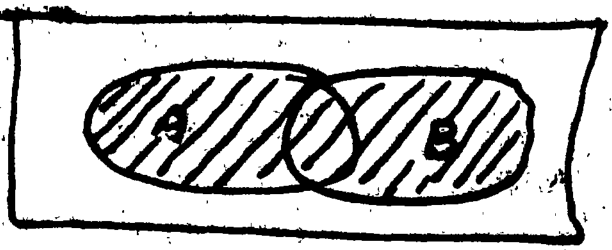
EVENTO A Y SU COMPLEMENTO A'

ESPACIO NULO O VACIO $\emptyset$ (U')



LA INTERSECCION DE DOS CONJUNTOS O EVENTOS $(A \cap B)$ ES LA COLECCIÓN DE PUNTOS QUE ESTAN CONTENIDOS EN A Y EN B.

$A \wedge B$ $A \cap B$ AB



LA UNION DE DOS EVENTOS A Y B $(A \cup B)$ ES LA COLECCIÓN DE PUNTOS QUE ESTA CONTENIDA EN A O EN B O EN AMBOS.

A B

DIAGRAMA DE VENN $A+B$ $A \vee B$

LOS DIAGRAMAS ANTERIORES SE CONOCEN COMO DIAGRAMAS DE VENN. ESCRIBAMOS FORMALMENTE LOS DIAGRAMAS ANTERIORES Y DEFINAMOS UNA ALGEBRA

$A+B = B+A$ LEY CONMUTATIVA

$A+(B+C) = (A+B)+C$ " ASOCIATIVA

$A(B+C) = AB+AC$ " DISTRIBUTIVA

$(A')' = A$

$(AB)' = A'+B'$, TEOREMA DE MORGAN

$AA' = \emptyset$ $\quad (A+B)' = A'B'$

$AU = A$

VEAMOS ALGUNAS OTRAS RELACIONES LAS CUALES SE PUEDEN DEMOSTRAR CON EL ALGEBRA DEFINIDA O CON DIAGRAMAS DE VENN

$A+A = A$

$A+AB = A$

$A+A'B = A+B$

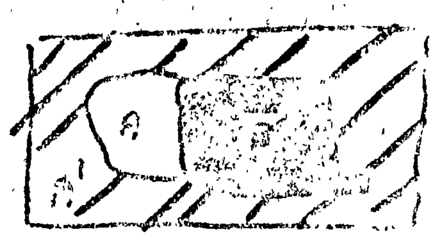
$A+A' = U$

$A+U = U$

$A+BC = (A+B)(A+C)$

$A\emptyset = \emptyset$

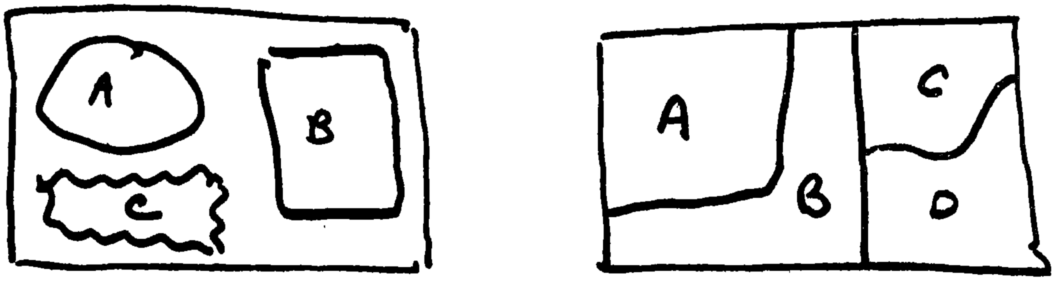
$A(BC) = (AB)C$



CONTINUAREMOS CON ALGUNAS OTRAS DEFINICIONES.

SEAN LOS EVENTOS $A_1, A_2, A_3 \ldots\ldots A_n$, SE DICE QUE SON MUTUAMENTE EXCLUSIVOS SI Y SOLO SI

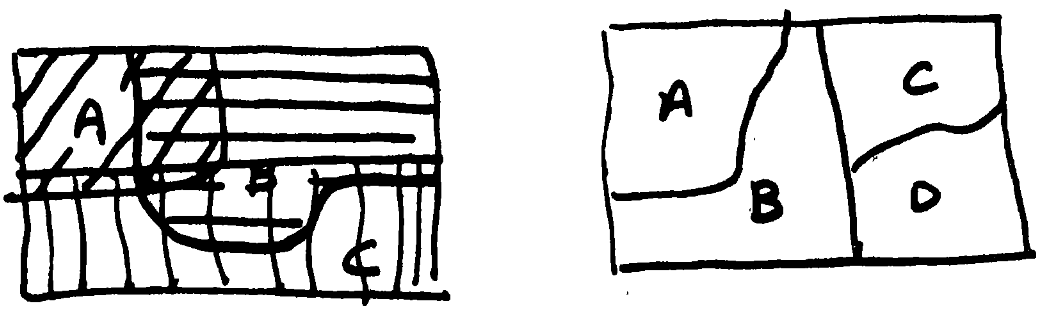
$$A_iA_j = \begin{cases} A_i & \text{para } i=j \\ \emptyset & \text{para } i\neq j \end{cases} \qquad i=1,2,\ldots n \qquad j=1,2,\ldots n.$$



EJEMPLO DE EVENTOS MUTUAMENTE EXCLUSIVOS.

LOS EVENTOS $A_1, A_2, \ldots A_N$ SON COLECTIVAMENTE EXHAUSTIVOS SI Y SOLO SI

$$A_1 + A_2 + A_3 + \cdots A_N = U.$$



SON COLECTIVAMENTE EXHAUSTIVOS.

ESPACIOS MUESTRA Y MODELOS DE EXPERIMENTOS

POR EXPERIMENTO EN ESTA CLASE VAMOS A ENTENDER, CUALQUIER PROCESO O EVENTO OBSERVADO 'NO DETERMINÍSTICO' i.e. EL RESULTADO DE TIRAR UNA MONEDA AL AIRE.

ESPACIO MUESTRA: ES EL RESULTADOS DE LOS POSIBLES EXPERIMENTOS DE UN MODELO (LISTANDO LOS RESULTADOS MUTUAMENTE EXCLUSIVOS Y COLECTIVAMENTE EXHAUSTIVOS).

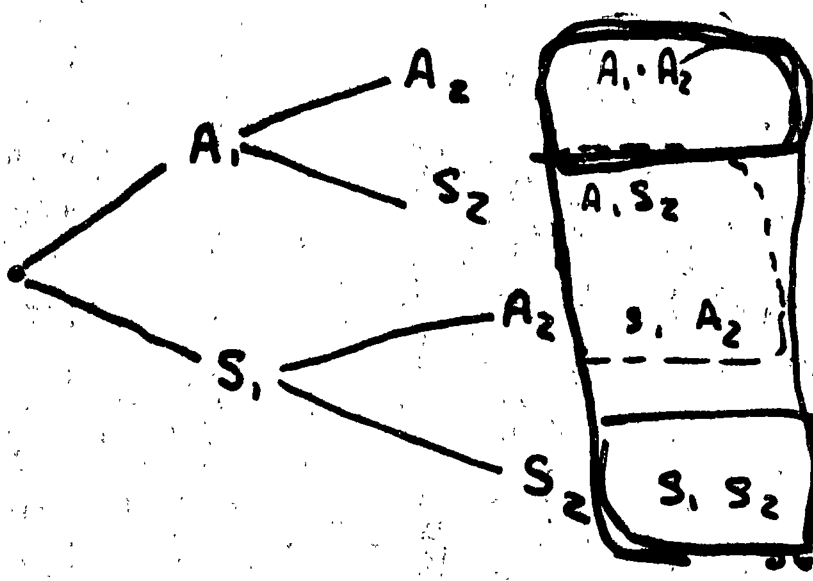
LA LISTA DEL ESPACIO MUESTRA PUEDE HACERSE DE DIFERENTES

FORMAS, SIN EMBARGO LAS MAS COMUNES Y LAS QUE NOS BRINDAN INFORMACION DE UNA MANERA MAS FACIL DE ASIMILAR SON:

(A) CONSIDERE EL EXPERIMENTO DE TIRAR UNA MONEDA AL AIRE, Y SI UTILIZAMOS LA SIG. NOTACION

S_n ~ SOL EN LA ENESIMA TIRADA

A_n - AGUILA - - - -



LA UNION DE ESTOS DOS PUNTOS MUESTRA CORRESPONDE AL EVENTO DE UNA AGUILA EN 2 TIRADAS O QUE EN 2 TIRADAS NO SE REPITA LA MISMA CARA. ESTO ES

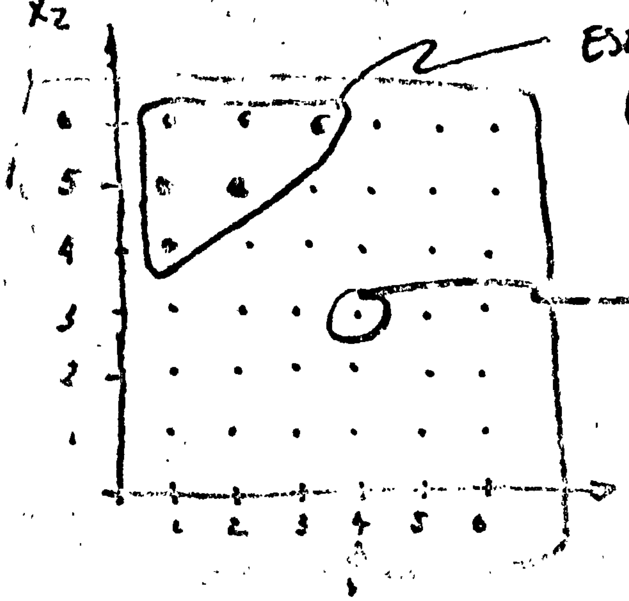
$A + B = B + A$

$(A_1 S_2 + S_2 A_2)$ O

~~$(A_1 A_2 + S_2 S_2)'$~~ ✓

$(A_1 A_2 + S_1 S_2)'$

(B) CONSIDERE EL EXPERIMENTO DE LANZAR UN DADO (DE 6 CARAS MARCADAS 1, 2, 3, 4, 5, 6) 2 VECES. LA PRIMERA TIRADA ES MARCADA CON X_1, Y LA SEGUNDA CON X_2.

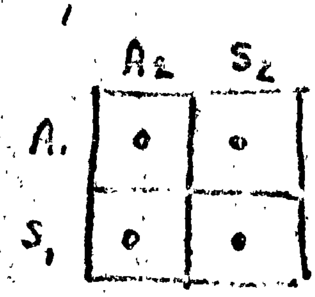


PRESENTEMOS AHORA LA MUESTRA Y ESPACIO DE EVENTOS PARA VARIOS EXPERIMENTOS.

EXPERIMENTO 1.

TIRE UNA MONEDA 2 VECES NOTACIÓN: $\left\{ \begin{matrix} A_n \\ S_n \end{matrix} \right\}$ RESULTADO EN LA TIRADA n.

UTILIZANDO OTRA REPRESENTACIÓN USANDO COMO UN ARREGLO [illegible]



A_1A_2, A_1S_2

EL ESPACIO MUESTRA ES.

$$\left\{ \begin{matrix} A_1A_2, & A_1S_2 \\ S_1A_2, & S_1S_2 \end{matrix} \right.$$

UN EJEMPLO DE EVENTO, PERO NO DE ESPACIO MUESTRA ES

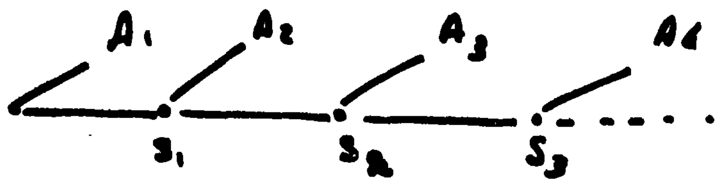
A_1, S_1

OTRO EJEMPLO DE EVENTO, PERO NO EL ESPACIO MUESTRA SERÍA

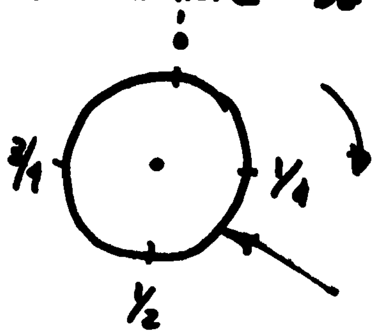
$A_1 S_2$, $(A_1 S_2)'$

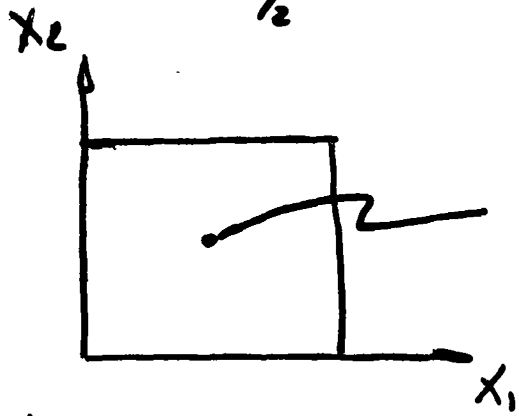
EXPERIMENTO #2 TIRE UNA MONEDA HASTA OBTENER UNA AGUILA.

• A_1 • $S_1 A_2$ • $S_1 S_2 A_3$ • $S_1 S_2 S_3 A_4$ . . . . $S_1 S_2 S_3 .. S_n A_{n+1}$

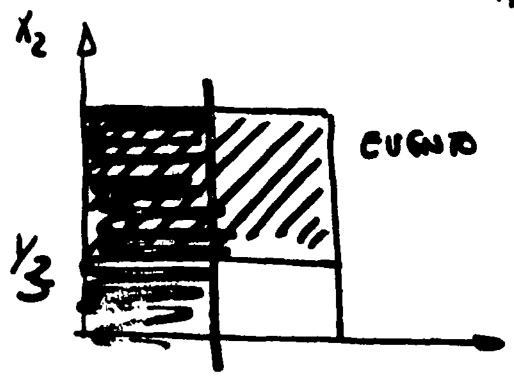


EXPERIMENTO #3 — SE TIENE UN DISCO, CALIBRADO CONTINUAMENTE DE 0 A 1, GIRE 2 VECES, (Y VEA EL RESULTADO)



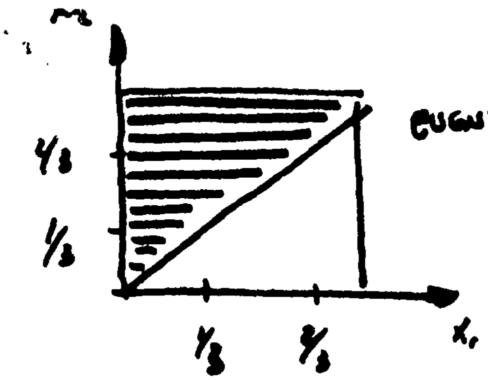


CADA POSIBLE RESULTADO (X_1, X_2) ESTÁ REPRESENTADO POR UN PUNTO EN EL ESPACIO.

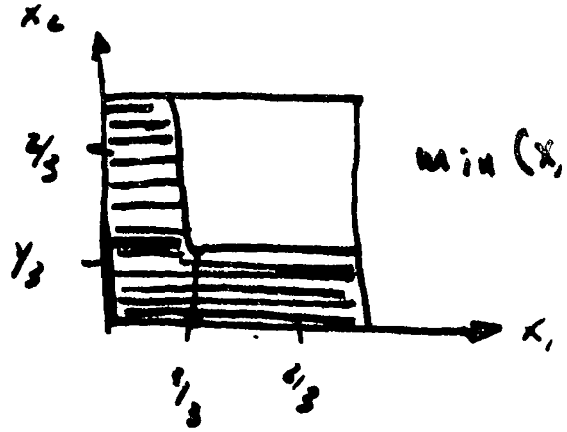


EVENTO $X_2 > \frac{1}{3}$

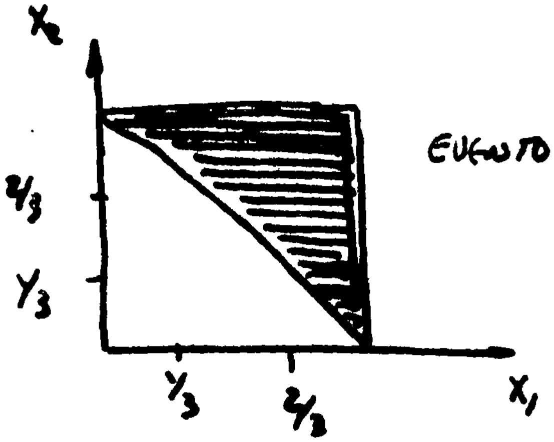
$X_1 < \frac{1}{2}$



EVENTO $X_2 > X_1$



$\min(X_1, X_2) \leq 1/3$



EVENTO $X_1 + X_2 \geq 1$

COMO MEDIR LA PROBABILIDAD (?).

A CADA EXPERIMENTO, NOS GUSTARÍA ASIGNAR PROBABILIDADES A LOS POSIBLES RESULTADOS. LA PROBABILIDAD DE UN EVENTO ES UN NUMERO, QUE REPRESENTA UNA 'POSIBILIDAD', QUE EL RESULTADO (DESEADO) SE OBTENGA EN EL EXPERIMENTO

DESIGNEMOS CON $P(A)$ LA PROBABILIDAD DEL EVENTO A. Y AGREGUEMOS 3 AXIOMAS MAS AL ALGEBRA DE EVENTOS.

1.- PARA CUALQUIER A, $P(A) \geq 0$
2.- $P(U) = 1$ - SEGURO.
3.- SI $AB = \emptyset$ ENTONCES $P(A+B) = P(A) + P(B)$

EXISTEN 4 AXIOMAS MAS EN LA TEORÍA DE PROBABILIDADES, LOS CUALES SE PUEDEN FACILMENTE DEMOSTRAR, Y ESTOS SON:

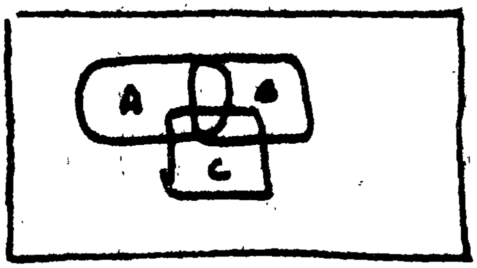
4.- $P(A') = 1 - P(A)$

5.- $P(\emptyset) = 0$ E. IMPOSIBLE.

6.- $P(A+B) = P(A) + P(B) - P(AB)$

7.- $P(A+B+C) = 1 - P(A'B'C')$

EXISTEN VARIAS MANERAS DE ESCRIBIR LA MISMA FORMULA, POR EJEMPLO:



$$P(A+B+C) = P(A) + P(A'B) + P(A'B'C)$$

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

$$P(A+B+C) = P(AB'C') + P(A'BC') + P(A'B'C) + P(AB) + P(AB'C) + P(A'BC)$$

Y HAY MAS.

SIN EMBARGO.

1.- LOS EVENTOS SON OPERADOS Y COMBINADOS DE ACUERDO A LOS 7 AXIOMAS DEL ALGEBRA DE PROBABILIDAD.

2.- LAS PROBABILIDADES DE LOS EVENTOS SON <u>NUMEROS</u> Y PUEDEN SER COMPUTADOS DE ACUERDO A LOS 3 PRIMEROS AXIOMAS

3.- LA ARITMETICA ES OTRO ROLLO.

PROBABILIDAD CONDICIONADA.

LA PROBABILIDAD CONDICIONADA SE DEFINE COMO

$P(A/B)$ (se lee como, la probabilidad de A dado B)

$$P(A/B) = \frac{P(AB)}{P(B)} \quad \text{si} \quad P(B) \neq 0$$

EJEMPLO: SE TIRA UNA MONEDA 2 VECES, Y PEPE QUIEN VIO EL RESULTADO DEL EXPERIMENTO DICE QUE "SALIO SOL EN CUANDO MENOS UNA TIRADA", DADA ESTA INFORMACION PARCIAL, QUEREMOS DETERMINAR LA PROBABILIDAD DE QUE SALIO SOL EN LAS DOS TIRADAS.

LA PROBABILIDAD 'APRIORI' $P(\cdot)$ DE QUE SUCEDA UN EVENTO SERA

	$P(\cdot)$	A	B	AB
S_1, S_2	0.25	✓	✓	✓
S_1, A_2	0.25	✓		
A_1, S_2	0.25	✓		
A_1, A_2	0.25			

$P(A) = 0.75$ $\quad P(B) = 0.25$ $\quad P(AB) = 0.25$

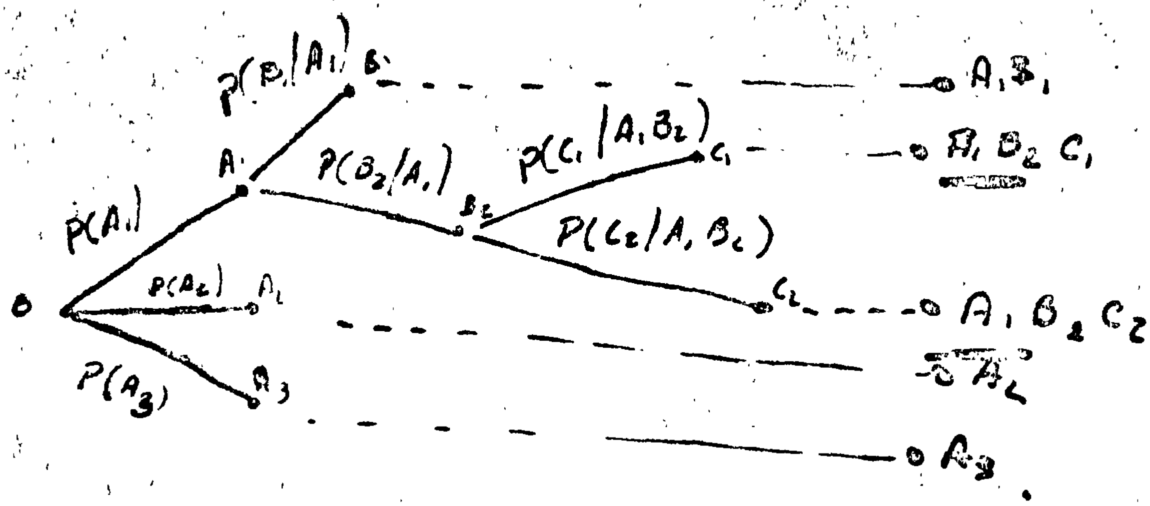
P(AB)

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{0.25}{0.75} = \frac{1}{3}$$

$$P(B/A) = \frac{P(AB)}{P(A)} = \frac{0.25}{0.75} = \frac{1}{3}$$

ARBOLES PROBABILISTICOS PARA EXPERIMENTOS SECUENCIALES.

ESTE TIPO DE ARBOLES SE VIO INFORMALMENTE CUANDO SE VIERON LOS ESPACIOS MUESTRA Y MODELO DE EXPERIMENTOS, SINEMBARGO CONSIDERE EL SIGUIENTE EJEMPLO.



EL ARBOL REPRESENTA UN ESPACIO MUESTRA, EN EL CUAL C_1 O C_2 OCURREN SI Y SOLO SI $A_1 B_2$ OCURRIO CON ANTERIORIDAD.

EL PUNTO (•) DE LA RAMA TERMINAL REPRESENTA LA INTERSECCION DE TODOS LOS EVENTOS QUE OCURRIERON EN EL CAMINO DESDE EL ORIGEN 'O' HASTA EL PUNTO TERMINAL (NODO TERMINAL [illegible])

NOTESE QUE UNICAMENTE LA PRIMER RAMA FUE ETIQUETADA CON SU PROBABILIDAD 'A PRIORI', LAS RAMAS ESTAN ETIQUETADAS CON PROBABILIDADES CONDICIONADAS. LA SUMA DE LAS PROBABILIDADES EN LAS RAMAS SALIENDO DE UN NODO NO TERMINAL, DEBE SUMAR UNO, DE OTRA MANERA LOS NODOS TERMINALES NO PODRIAN SER REPRESENTADOS POR UNA COLECCION [illegible].

EVENTOS INDEPENDIENTES.

DOS EVENTOS SON INDEPENDIENTES, SI Y SOLO SI
$P(A/B) = P(A)$

$$ó \quad P(A) = \frac{P(AB)}{P(B)}$$

HASTA AQUÍ HEMOS TERMINADO CON LOS CONCEPTOS ELEMENTALES DEL CAPITULO 1, VEAMOS A TRAVES DE UNA SERIE DE EJEMPLOS Y PROBLEMAS SI ES QUE ENTENDIMOS Y SABEMOS LO ANTES VISTO.

EJEMPLO 1. SE TIENEN TRES LISTAS DE EVENTOS, LAS CUALES LLAMAREMOS LISTA 1, LISTA 2 Y LISTA 3, TODOS LOS EVENTOS EN LAS LISTAS ESTAN DEFINIDOS EN EL MISMO EXPERIMENTO, Y NINGUN EVENTO TIENE PROBABILIDAD CERO.

LA LISTA UNO CONTIENE LOS EVENTOS $A_1, A_2, \ldots A_K$ LOS LOS CUALES SON MUTUAMENTE EXCLUSIVOS Y COLECTIVAMENTE EXHAUSTIVOS.

LA LISTA 2 CONTIENE LOS EVENTOS $B_1, B_2 \ldots B_K$ MUTUAMENTE EXCLUSIVOS Y COLECTIVAMENTE EXHAUSTIVOS.

LA LISTA 3 CONTIENE LOS EVENTOS $C_1, C_2 \ldots C_K$ Y SON MUTUAMENTE EXCLUSIVOS PERO <u>NO</u> COLECTIVAMENTE EXHAUSTIVOS.

EVALUE:

$$\sum_{j=1}^{5} P(C_i)$$

a) NOS PIDEN EVALUAR LA SUMA DE LAS PROBABILIDADES ASOCIADAS CON LOS EVENTOS. COMO LOS EVENTOS SON MUTUAMENTE EXCLUSIVOS (Y NO COLECTIVAMENTE EXHAUSTIVOS) DEBEMOS DE CALCULAR SU UNION. LA CUAL DEBE DE SER DIFERENTE DE CERO (MAYOR) y no llega a ser EL ESPACIO UNIVERSAL, ESTO ES.

$$0.0 < \sum_{i=1}^{m} P(C_i) < 1.0$$

b) $\sum_{j=1}^{k} P(A_2 / A_j)$.

SABEMOS QUE las A_i SON MUTUAMENTE EXCLUSIVAS, POR LO QUE NOTAMOS QUE $P(A_2 / A_j)$ ES IGUAL A CERO, A MENOS QUE $j=2$. CUANDO $j=2$ TENEMOS QUE $P(A_2/A_2) = 1$ POR LO QUE

$$\sum_{j=1}^{k} P(A_2 / A_j) = 1$$

c) $\sum_{i=1}^{k} \sum_{j=1}^{k} P(A_i A_j)$

LA PROPIEDAD DE QUE LA A_i's SON MUTUAMENTE EXCLUSIVAS REQUIERE QUE $P(A_i A_j) = 0$ A MENOS QUE $j=i$, si $j=i$ TENEMOS $P(A_i A_j) = P(A_j)$

EJEMPLO 2. POR LO QUE SABEMOS, CON PROBABILIDAD 0.8 Alfonso es culpable de un crime y va a ser enjuiciado. Benito y Carlos han sido llamados como testigos, y saben si Alfonso es culpable o inocente.

Benito es amigo de Alfonso, y dira la verdad si Alfonso es inocente, y mentira con probabilidad de 0.2 si Alfonso es culpable. Carlos odia a todo el mundo exepto al juez y dira la verdad si Alfonso es culpable, y mentira con probabilidad de 0.3 si Alfonso es inocente.

a) Determine la probabilidad que los testigos den testimonios contradictorios

b) Cual del los testigos sera mas propenso a mentir en el jusgado

c) Cuál es la probabilidad condicionada que Alfonso es inocente, si Benito y Carlos riedieron testimonios contradictorios

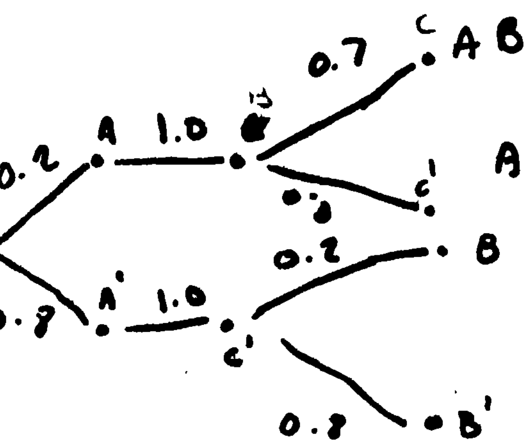
d) ¿Son los eventos 'Benito miente' y 'Carlos miente' independientes?

e) ¿Estan estos eventos (punto d) condicionalmente independientes a un observador quien sabe si Alfonso es culpable?

SOLUCION :

EVENTO A: A ES INOCENTE
EVENTO B: B TESTIFICA QUE A ES INOCENTE
EVENTO C: C TESTIFICA QUE A ES INOCENTE
EVENTO X: LOS TESTIGOS DAN TESTIMONIO CONFLICTIVOS
EVENTO Y: B MIENTE
EVENTO Z: C MIENTA
{ FALSO TESTIMONIO }

$$P(A|X) = \frac{P(AX)}{P(X)} = \frac{0.06}{0.22} = \frac{3}{11}$$



PUNTO MUESTRA	P(·)	X	Y	Z	AX
ABC	0.14				
ABC'	0.06	✓		✓	✓
A'C'B	0.16	✓	✓		
A'C'B'	0.64				
		P(X) = 0.22	P(Y) = 0.16	P(Z) = 0.06	P(AX) = 0.06

PARA DETERMINAR LAS PROBABILIDADES DEL ESPACIO MUESTRA, SIMPLEMENTE SUMEMOS LOS PUNTOS MUESTRA DEL EVENTO EN CUESTION

a) $P(X) = P(BC' + B'C) = P(BC') + P(B'C) = 0.22$

b) $P(Y) = P(AB' + A'B) = 0.0 + 0.16 = 0.16$

$P(Z) = P(AC' + A'C) = P(AC') + P(A'C) = 0.06 + 0.0 = 0.06$

POR LO QUE BENITO (MAS PROBABLEMENTE) RENDIRA FALSO TESTIMONIO

c) $$P(A/X) = \frac{P(AX)}{P(X)} = \frac{0.06}{0.06+0.16} = \frac{3}{11}$$

ES MAS PROBABLE QUE OCURRAN TESTIMONIOS CONTRADICTORIOS SI ALFONSO ES INOCENTE; ASI QUE DADO QUE OCURRIO X, SE DEBERA AUMENTAR LA PROBABILIDAD QUE ALFONSO ES INOCENTE.

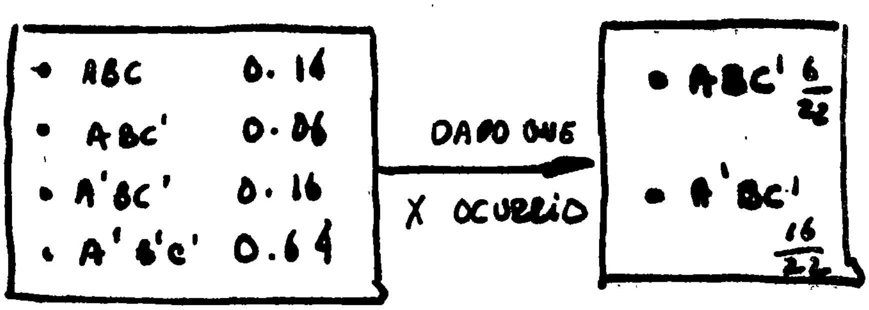
EN ESTE CASO

$$\frac{8}{11} > \frac{1}{5} \qquad P(A|X) = \frac{3}{11} \quad P(BC') = \frac{1}{5}$$

c) SOLUCION ALTERNATIVA.

DADO QUE OCURRIO X, NOS VAMOS AL ESPACIO MUESTRA, Y EXAMINAMOS AQUELLOS PUNTOS CON ATRIBUTO X

LA PROBABILIDAD CONDICIONAL PARA ESOS PUNTOS SE DETERMINA 'ESCALANDO' LAS PROBABILIDADES ORIGINALES APRIORI, POR LA MISMA CONSTANT $[1/P(X)]$ PARA QUE SUMEN UNO



ESPACIO DE PROBABILIDAD CONDICIONADO A LA OCURRENCIA DE X.

ESPACIO ORIGINAL

$$\frac{ABC'}{X} = \frac{0.06}{0.22} = \frac{6}{22}$$

$$\frac{A'BC'}{X} = \frac{0.16}{22} = \frac{16}{22}$$

LO QUE NOS RESTA HACER, ES SUMAR SUMAR LAS PROBABILIDADES CONDICIONALES, DE TODOS LOS PUNTOS EN EL ESPACIO MUESTRA, PARA DETERMINAR LA PROB. CONDICIONAL DEL EVENTO.

$$P(A|X) = P(ABC'|X) = \frac{0.06}{0.22} = \frac{3}{11}.$$

d) PARA PODER DETERMINAR, SI BENITO MIENTE O CARLOS MIENTE SON EVENTOS INDEPENDIENTES EN LA MUESTRA ORIGINAL, DEBEMOS DEMOSTRAR QUE:

$$P(YZ) = P(Y)P(Z)$$

PERO $P(YZ)=0$ SIN EMBARGO $P(Y)>0$ y $P(Z)>0$, POR LO QUE LOS EVENTOS Y, Z NO SON INDEPENDIENTES, Y ESTAS SON MUTUAMENTE EXCLUSIVOS.

e) PARA DETERMINAR SI Y y Z SON CONDICIONALMENTE INDEPENDIENTES DADO A o A', DEBEMOS DEMOSTRAR QUE

$P(YZ|A) \overset{?}{=} P(Y|A)\,P(Z|A)$ y QUE
$P(YZ|A') \overset{?}{=} P(Y|A')\,P(Z|A')$. DEL ESPACIO MUESTRA NOTAMOS QUE EL TERMINO A LA IZQ. DE LA IGUALDAD (?) Y UN TERMINO A LA DERECHA SON CERO, ASÍ QUE LOS EVENTOS Y y Z SON CONDICIONALMENTE INDEPENDIENTES, A LA PERSONA QUE SABE SI ANTONIO ES INOCENTE O CULPABLE.

LO ANTERIOR RESULTA CIERTO, PUES EL TESTIMONIO DE CUANDO LOS TESTIGOS, DEPENDE SOLAMENTE SI ANTONIO ES CULPABLE O INOCENTE.

its importance been more widely appreciated until now? The answer is that many users of probability theory (but certainly not the original developers) considered probabilities to be physical parameters of objects, such as weight, volume, or hardness. For example, there was much mention of "fair" coins and "fair" dice, with the underlying notion that the probability of events associated with these objects could be measured in the real world.

For the past 15 years, however, an important minority of experts on the subject have been advancing the view that probabilities measure a person's state of knowledge about phenomena rather than the phenomena themselves. They would say, for example, that when someone describes a coin as "fair" he really means that on the basis of all evidence presented to him he has no reason for asserting that the coin is more likely to fall heads than tails. This view is modern, but not a product of modern times. It was studied clearly and convincingly 200 years ago but remained buried for a long time.

An example illustrating this view of probability follows: An astronaut is about to be fired into space on a globe-circling mission. As he is strapping himself into his capsule on top of a gleaming rocket, he asks the launch supervisor, "By the way, what's the reliability of this rocket?" The launch supervisor replies "Ninety nine percent—we expect only one rocket in one hundred to fail." The astronaut is reassured but still has some doubts about the success of his mission. He asks, "Are these rockets around the edge of the field the same type as the one I'm sitting on?" The supervisor replies, "They're identical." The astronaut suggests, "Let's shoot up a few just to give me some courage."

The rocket is fitted with a dummy payload, prepared for launching, and fired. It falls in the ocean, a complete failure. The supervisor comments, "Unlucky break, let's try another." Unfortunately, that one also fails by exploding in mid-air. A third is tried with disastrous results as it disintegrates on its pad. By this time, the astronaut has probably handed in his resignation and headed home. Nothing could convince him that the reliability of his rocket is still 99%.

But, in reality, what has changed? His rocket is physically unaffected by the failure of the other rockets. Its guidance system, rocket engine, and life support system are all exactly the same as they were before the other tests. If probability were a state of things, then the reliability of his rocket should still be 0.99. But, of course, it is not. After observing the failure of the first rocket, he might have evaluated the reliability of his rocket at, say, 0.90; after the second failure, at 0.70; and finally after the third failure, at perhaps 0.30. What happened was that his state of knowledge of his own rocket was influenced by what happened to its sister ships, and therefore his estimate of its reliability must decrease. His final view of its reliability is so low that he does not choose to risk his life.

The view of probability as a state of things is just not tenable. Probability should be considered as the reading of a kind of mental thermometer that measures uncertainty rather than temperature. The reading goes up if, as data accumulate, it tends to increase the likelihood of the event under consideration. The reading of 1 corresponds to certainty that the event will occur, the reading of 0 to certainty that it will not occur. The inferential theory of probability is concerned with the question of how the reading ought to fluctuate in the face of new data.

Encoding Experience

Most persons would agree that it would be unwise to make a decision without considering all available knowledge before acting. If someone were offered an opportunity to participate in a game of chance by his best friend, by a tramp, and by a business associate, he would generally have different feelings about the fairness of the game in each case. A major problem is how to encode the knowledge he has in a usable form. This problem is solved by the observation that probability is the appropriate way to measure his uncertainty.

All prior experience must be used in assessing probabilities. The difficulty in encoding prior knowledge as probability is that the prior information available may range in form from a strong belief that results from many years of experience to a vague feeling that arises from a few haphazard observations. Yet there is probably not a person who had no information about an event that was important to him. People who start out saying that they have no idea about what is going to happen can always, when pressed, provide probability assignments that show considerable information about the event in question. The problem of those who would aid decision-makers is to make the process of assigning probabilities as simple, efficient, and accurate as possible.

The Practical Encoding of Knowledge

In the probabilistic phase of decision analysis, we face the problem of encoding the uncertainty in each of the aleatory variables. In organizational decision-making, prior probability distributions (or priors) should be assigned by the people within the organization who are most knowledgeable about each state variable. Thus, the priors on engineering variables will typically be assigned by the engineering department; on marketing variables, by the marketing department; and so on. However, since each case is an attempt to encode a probability distribution that reflects a state of mind and since most individuals have real difficulty in thinking about uncertainty, the method of extracting the priors is extremely important. As people participate in the prior-gathering process, their attitudes are indicated successively by: "This is ridiculous." "It can't be done." "I have told you what you want to know, but it doesn't mean anything." "Yes, it seems to reflect the way I feel." And "Why doesn't everybody do this?" In gathering the information, the analyst must be careful to overcome the defenses the individual develops as a result of being aske[d] for estimates that are often a combination of targets, wishful thinking, and expectations. The biggest difficulty is in conveying to the man that the analyst is interested in his state of knowledge and not in measuring him or setting a goal for him.

If the subject has some experience with probability, he often attempts to make all his priors look like normal distributions, a characteristic known as "bell-shaped" thinking. Although normal distributions are appropriate priors in some circumstances, they should not become foregone conclusions.

Experience has shown certain procedures to be effective in this almost psychoanalytic process of prior measurement. One procedure is to make the measurement in a private interview to eliminate group pressure and to overcome the vague notions that most people exhibit about probabilistic matters. Unless the subjects are already experienced in decision analysis, the distribution of forms on which they are supposed to draw their priors ha[ve] proved worse than useless.

The interview begins with such questions as "What are the chances that x will exceed ten?" This approach is taken because people seem much more comfortable in assigning probabilities to events than they are in sketching a probability density function. The interviewer also skips around, asking the probability that x will be "greater than 50," "less than ten," "greater than 30," often asking the same question again later in the interview. The replies are recorded out of the view of the subject so as to frustrate any attempt at forced consistency on his part. As the interview proceeds, the subject often considers the questions with greater and greater care, so that his answers toward the end of the interview may represent his feelings much better than did his initial answers.

The interviewer can change the form of the questions by asking the subject to divide the possible values of an aleatory variable into n intervals of equal probability. The answers t[o] ll these questions enable the analyst to draw the excess probability distribution for the aleatory variable, a form of representation that seems easy to convey to people without formal probabilistic training.

The result of the interview must be a prior that the subject is willing to live with, regardless of whether it will describe a lottery on who buys coffee or on the disposal of his life savings. The analyst can test the prior by comparing it with known probabilistic mechanisms. For example, if the subject says that some aleatory variable x is equally likely to be less or greater than a, then he should be indifferent about whether he is paid $100 if x exceeds a or if he can call the toss of a coin. If he is not indifferent, then he must change a until he is. The end result of such questions is to produce a prior that the subject is not tempted to change in any way. Although the prior-gathering process is not cheap, the analyst need perform it only on the aleatory variables.

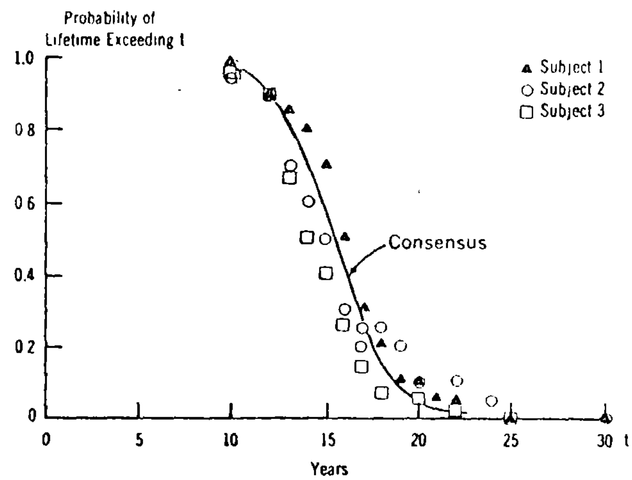
In cases where the interview procedure is t appropriate, the analyst can often obtain .. satisfactory prior by drawing one himself and then letting the subject change it until the subject is satisfied. This technique may also be useful as an educational device in preparation for the interview.

If two or more aleatory variables are dependent, then the procedure requires priors that reflect the dependencies. The technique of prior gathe.ing is generally the same but somewhat more involved. Since the treating of joint variables is a source of expense, the analyst should formulate the problem so as to avoid them whenever possible.

An Actual Probability Assessment

Figure 8 illustrates prior-gathering. The decision in a major problem was thought to depend primarily on the average lifetime of a new material. Since the material had never been made and test results would not be available until three years after the decision was required, it was necessary to encode how r ch knowledge the company now had con-

Fig. 8—Priors on Material Lifetime



cerning the life of the material. This knowledge resided in three professional metallurgists who were experts in that field of technology. These men were interviewed separately according to the principles described. They produced the points labeled "Subjects 1, 2, and 3" in the figure. These results have several interesting features. For example, for $t = 17$, Subject 2 assigned probabilities of 0.2 and 0.25 at various points in the interview. On the whole, however, the subjects were remarkably consistent in their assignments. Subject 3 was more pessimistic about the lifetime than was Subject 1.

Upon conclusion of the interviews, the three subjects were brought together, shown the results, and a vigorous discussion took place. Subjects 1 and 3 each brought forth information of which the other two members of the group were unaware. As the result of this information exchange, the three subjects drew the consensus curve—each said that this curve represented the state of information about the material's life at the end of the meeting. Later, their supervisor said he understood their position on the new material for the first time.

It has been suggested that the proper way to reconcile divergent priors is to assign weights to each, multiply, and add, but this experiment is convincing evidence that any such mechanistic procedure misses the point. Divergent priors are an excellent indicator of divergent states of information. The experience just described not only produced the company's present encoding of uncertainty about the material's lifetime, but at the same time encouraged and effected the exchange of information within the group.

Encoding New Information

Following the encoding of the original information about an aleatory variable by means of a prior probability distribution, or about an event by the assignment of a probability, the question naturally arises as to how these probability assignments should be changed in the light of new information. The answer to this question was provided by Bayes in 1763; it is most easily introduced by considering the case of an event. Suppose that we have assigned some probability $p(A)$ to an event A's occurring and that another event B is statistically related to A. We describe this relationship by a conditional probability of B given A, $p(B|A)$, the probability of B if A occurs; assign this probability also. Now we are told that B has, in fact, occurred. How does this change the probability that A has occurred; in other words, what is the probability of A given $B, p(A|B)$?

Bayes showed that to be logical in this situation, the probability of A given $B, p(A|B)$, must be proportional to the probability of A, $p(A)$, and the probability of B given A, $p(B|A)$. This relationship is expressed as $p(A|B)$ is proportional to $p(A)$ times $p(B|A)$.

The important thing to remember is that any posterior (after new information) probability assignment to an event is proportional to the product of the prior probability assignment and the probability of the new information given that the event in question occurred. The same idea carries over in the much more complicated situations encountered in practice.

Thus, Bayes' interpretation shows how new information must be logically combined with original feelings. Subjective probability assignments are required both in describing the prior information and also in specifying how the new information is related to it. In fact, as already mentioned, Bayes' interpretation is the only method of data processing that ensures that the final state of information will be the same regardless of the order of data presentation.

Encoding Values and Preferences

The other subjective issue that arises in decision analysis is the encoding of values and preferences. It seems just as difficult to obtain an accurate measurement of desires as of information.

The value issue penetrates the core of the decision problem. Whether personal or organizational, the decision will ultimately depend on how values are assigned. If each alternative could produce only a single outcome, it would only be necessary to rank the outcomes in value and then choose the alternative whose outcome was highest in value. However, typically each alternative can produce many possible outcomes, outcomes that are distributed in time and also subject to uncertainty. Consequently, most real decision problems require numerical measures of value and of time and risk preference.

Measuring Value

The application of logic to any decision problem requires as one of its fundamental steps the construction of a value function, a scale of values that specifies the preference of the decision-maker for one outcome compared with another. We can think of the problem as analogous to the one we face if we have someone buy a car for us: We must tell our agent what features of the car are important to us and to what extent. How do we value performance relative to comfort, appearance relative to economy of operation, or other ratings?

To construct a value function in the car urchase problem, we can tell our agent the dollar value we assign to each component of a car's value. We might say, for example, that given our usage characteristics, a car that runs 18 miles to a gallon of gas is worth \$40 a year more to us than a car that runs only 15 miles and that foam rubber seats are worth \$50 more to us than ordinary seats. When we had similarly specified the dollar value of all the possible features of a car, including those whose values might not be additive, our agent would be able to go into the marketplace, determine the value and price of every offered car, and return with the most profitable car for us (which might, of course, be no car at all). In following this philosophy, we do not care if, in fact, there are any cars for sale that have all or any part of the features that we have valued. The establishment of the value function depends remotely, if at all, on the spectrum of cars available.

The main role of the value function is to serve as a framework of discussion for preferences. The value function encodes preferences consistently; it does not assign them. Consequently, the decision-maker or decision analyst can insert alternative value specifications to determine sensitivity of decisions to changes in value function. The process of assigning values will naturally be iterative, with components of value being added or eliminated as understanding of the problem grows.

A question that arises is, "Who should set the values?" In a corporate problem, to what extent do the values derive from management, stockholders, employees, customers, and the public? The process of constructing a value function brings into the open questions that have been avoided since the development of the corporate structure.

Establishing Time Preference

The general tendency of people and organizations is to value outcomes received sooner more highly than outcomes received later. In an organization, this phenomenon usually occurs in connection with a time stream of profit. Time streams that show a greater share of their returns in earlier time periods are generally preferred.

A number of concepts have arisen to cope with time preference in corporations. To illustrate these concepts, let $x(n)$ be the cash flow in year n in the future, positive or negative, where $n = 0$ is the beginning of the present year, $n = 1$ next year, and so on. A positive cash flow indicates that income exceeds expenditures, a negative cash flow implies the reverse. Negative cash flows will usually occur in the early years of the project.

The most elementary approach, the payback period method, rests on the assumption that the cash flow will be negative in early periods and will then become and remain positive for the balance of the project. The payback period is the number of the period in which cumulative cash flow becomes positive.

The payback period came into common use when projects were typically investments in capital equipment, investments characterized by a high initial outlay gradually returned in the course of time. However, only a few modern investments have such a simple structure. The project may contain several interspersed periods of investment and return. There would seem to be little justification for use of the payback period in modern corporate decision-making.

The idea of internal rate of return was introduced as a more sophisticated time preference measure. The internal rate of return is derived from the present value of the project, defined by

$$PV(i) = x(0) + x(1)\left(\frac{1}{1+i}\right) + x(2)\left(\frac{1}{1+i}\right)^2 + \cdots$$

where i is interpreted as an annual interest rate for funds connected with the project. The internal rate of return is the value of i that makes the present value equal to zero; in other words, the solution of the equation

$$PV(i) = 0.$$

A justification offered for the use of internal rate of return is that application of the method to an investment that pays a fixed interest rate, like a bond or a bank deposit, produces an internal rate of return equal to the actual interest rate. Although this property is satisfying, it turns out to be insufficient justification for the method. One defect, for example, is that more than one interest rate may satisfy the equation; that is, it is possible for an investment to have two internal rates of return, such as 8% and 10%. In fact, it can have as many as the number of cash flows in the project minus one. A further criticism of the method is that it purports to provide a measure of the desirability of an investment that is independent of other opportunities and of the financial environment of the firm. Although meticulous use of internal rate of return methods can lead to appropriate time preference orderings, computing the present value of projects establishes the same ordering directly, without the disadvantages of internal rate of return. Furthermore, present value provides a measure of an investment such that the bigger the number, the better the investment. The question that arises is what interest rate i to use in the computation.

Much misunderstanding exists about the implications of choosing an interest rate. Some firms use interest rates like 20% or 25% in the belief that this will maintain profitability. Yet at the same time they find that they are actually investing most of their available capital in bank accounts. The overall earnings on capital investment will therefore be rather low. The general question of selecting i is too complicated to treat here, but the fundamental consideration is the relationship of the firm to its financial environment.

There is a cogent logical argument for the use of present value. If a decision-maker believes certain axioms regarding time streams —axioms that capture such human characteristics as greediness and impatience—then the time preference of the decision-maker for cash streams that are certain must be characterized by the present value corresponding to some interest rate. Furthermore, if a bank is willing to receive and disburse money at some interest rate, then, for consistency, the decision-maker must use this bank interest rate as his own interest rate in the calculation. Present value is therefore a well-founded criterion for time preference.

In this discussion of time preference, there has been no uncertainty in the value of cash streams. Undoubtedly, it was the existence of uncertainty that made payback periods and artificially high interest rate criteria seem more logical than they in fact are. Such procedures confuse the issues of time and risk preference by attempting to describe risk preference as a requirement for even greater rapidity of return. Decision analysis requires a clear distinction between the time and risk preference aspects of decision-making.

Establishing Risk Preference

The phenomenon of risk preference was discussed in connection with the proposition of tossing a coin, double or nothing, for next year's salary: most people will not play. However, suppose they were offered some fraction of next year's salary as an inducement to play. If this fraction is zero, there is no inducement and they will refuse. If the fraction is one they have nothing to lose by playing and they have a .5 probability of ending up with three times next year's salary; clearly, only those with strange motivations would refuse. In experiments on groups of professional men, the fraction required to induce them to play varies from about 60% to 99%, depending on their financial obligations. Obviously, the foot-loose bachelor has a different attitude than does the married man with serious illness in the family.

The characteristic measured in this experiment is risk aversion. Few persons are indifferent to risk—i.e., willing to engage in a fair gamble. Fewer still prefer risk—i.e., willing to engage in the kind of gambles that are unfair, such as those offered at professional gambling establishments. When considering sums that are significant with respect to their financial strength, most individuals and corporations are risk-averse.

A risk-averse decision-maker is willing to forego some expected value in order to be protected from the possibilities of poor outcomes. For example, a man buys life, accident, and liability insurance because he is risk-averse. These policies are unfair in the sense that they have a negative expected value computed as the difference between the premium and the expected loss. It is just this negative expected value that becomes the insurance company's profit from operations. Customers are willing to pay for this service because of their extreme aversion to large losses.

A logical way to treat the problem of risk aversion is to begin with the idea of a lottery. A lottery is a technical term that refers to a set of prizes or prospects with probabilities attached. Thus, tossing a coin for next year's y is a lottery and so is buying a life insurance policy. The axioms that the decision-maker must satisfy to use the theory are:

- Given any two prizes in a lottery, he must be able to state which he prefers or whether he is indifferent between them. His preferences must be transitive: if he prefers prize A to B and prize B to C, he must also prefer A to C.
- If he prefers A to B and B to C, he must be indifferent to receiving B for certain or participating in a lottery with A and C as prizes for some probability of winning A.
- If he prefers A to B, then given the choice of two lotteries that both have prizes A and B, he will prefer the one with the higher probability of winning A.
- He treats as equivalent all lotteries with the same probabilities of achieving the same prizes, regardless of whether the prizes are won in one drawing, or as the result of several drawings that take place at the same time.

It is possible to show that an individual who wants to act in accordance with these axioms possesses a utility function that has two important properties. First, he can compute his utility for any lottery by computing the utility of each prize, multiplying by the probability of that prize, and then summing over all prizes. Second, if he prefers one lottery to another, then his utility for it will be higher.

If the prizes in a lottery are all measured in the same commodity, then, as discussed previously, the certain equivalent of the lottery is the amount of the commodity that has the same utility as the lottery. The concepts of utility and certain equivalent play a central role in understanding risk preference.

In the practical question of measuring risk preference, one approach is to present an individual with a lottery and to ask him his certain equivalent. Or, we can provide the certain equivalent and all prizes but one and let him adjust the remaining prize until the certain equivalent is correct in his view. Finally, we can fix the certain equivalent and prizes and let him adjust the probabilities. All these questions permit us to establish the relationships between points on his utility curve and, ultimately, the curve itself. The interviewing in which the curve is measured is similar to that used for generating priors: the same need for education exists. The same types of inconsistency appear.

Although useful utility curves for individuals and organizations can be found in this manner, most decision-makers prefer to have some guidance in the selection of utility curves. The decision analyst can often provide this guidance by asking whether the decision-makers will accept additional axioms. One such axiom is: if all the prizes in the lottery are increased by some amount Δ, then the certain equivalent of the lottery will increase by Δ. The argument for the reasonableness of the axiom is very simple. The additional amount Δ is money in the bank, no matter which prize in the lottery is won. Therefore, the new lottery should be worth more than the original lottery. The counter argument is that having Δ in the bank changes the psychological orientation to the original lottery.

If this Δ axiom is added to the original set, then it is possible to show not just that a utility curve exists but that it must have a special form called the exponential form. A useful property of this exponential form is that it is described by a single number. This means that the analyst can characterize the utility curve of any individual or organization that wants to subscribe to these axioms by a single number—the risk aversion constant.

It is far easier to demonstrate to a decision-maker the consequences of his having different risk aversion coefficients and to measure his coefficient than it is to attempt to find a complete utility curve that is not of the exponential form. Encoding risk aversion in a single number permits measuring the sensitivity to risk aversion, as discussed earlier. In most practical problems, the entire question of risk aversion appears to be adequately treated by using the exponential form with a risk aversion constant appropriate to the decision-maker.

A cautionary note on the problem of practical measurement of risk aversion: experiments have revealed that the certain equivalents offered by subjects in hypothetical situations differ markedly from those offered when the situations are made real. This difficulty shows that the analyst must treat risk preference phenomena with great care.

Joint Time and Risk Preference

In most problems, both time and risk preference measures are necessary to establish the best alternative. Typically each outcome is represented by a time sequence of dependent uncertain values.

The question of how to describe preferences in such problems is fundamentally related to the way in which information on successive outcomes is revealed and to the extent to which it can help in making future decisions.

Two approaches illustrate the nature of tl problem, each of which is appropriate under certain conditions. The first—that used in the original discussion of the probabilistic phase—is to compute the worth lottery implied by the model and then use the current utility function to develop the certain equivalent worth of the lottery. This approach is appropriate when there is no opportunity to utilize the information about outcomes as it is revealed, and thus where the prime interest is in the position occupied after all outcomes have been revealed.

Another approach is to imagine dealing with two agents. The first is a banker who will always pay immediately the amount specified by a particular company's time preference function applied to any time stream of values that is known with certainty. The other is a risk broker who will always pay the company's certain equivalent for any lottery. When faced with an uncertain stream of income, the company alternately deals with the risk broker to exchange lotteries for certain equivalents an with the banker to convert fixed future payments into present payments. The result of this alternating procedure is ultimately a single equivalent sum to represent the entire future process. Although appealing, the method may lead to the conclusion that the decision-maker should be willing to pay for "peace of mind" even when it has no effect on his financial future.

Thus the time-risk preference question ultimately depends on the decision-maker's tastes and options. The decision analyst can provide guidance in selecting from the many available approaches the one whose implications are best suited to the particular situation.

APPLICATIONS

In brief form, two examples illustrate the accomplishments and potential of decision analysis. In each case, the focus is on the key decision to be made and on the problems peculiar to the analysis.

New Product Introduction

A recent decision analysis was concerned with whether to develop and produce a new product. Although the actual problem was from another industry we shall suppose that it was concerned with aircraft. There were two major alternatives: to develop and sell a new aircraft (A_2) or to continue manufacturing and selling the present product (A_1). The decision was to be based on worth computed as the present value of future expected profits at a discount rate of 10% per year over a 22-year period. Initially, the decision was supposed to rest on the lifetime of the material for which the prior probability distribution, or priors, were obtained (Figure 8); however, a complete decision analysis was desired. Since several hundred million dollars in present value of profits were at stake, the decision analysis was well justified.

In the general scheme of the analysis, the first step was to construct a model for the business, as shown in Figure 9, which was primarily a model of the market. The profit associated with each alternative was described in terms of the price of the product, its operating costs, its capital costs, the behavior of competitors, and the natural characteristics of customers. Suspicion grew that this model did not adequately capture the regional nature of demand. Consequently, a new model was constructed that included the market characteristics region by region and customer by customer. Moving to the more detailed basis affected the predictions so much that the additional refinement was clearly justified. However, other attempts at refinement did not affect the results sufficiently to justify a still more refined model.

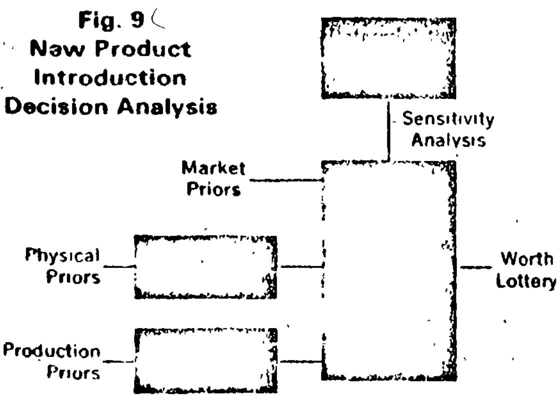


Fig. 9 New Product Introduction Decision Analysis

Next, a sensitivity analysis was performed to determine the aleatory variables. These turned out to be operating cost, capital cost, and a few market parameters. Because of the complexity of the original business model, an approximation was constructed showing how worth depended on these aleatory variables in the area of interest. The coefficients of the approximate business model were established by runs on the complete model.

The market priors were directly assigned with little trouble. However, because the operating and the capital costs were the two most important in the problem, their priors were assigned according to a more detailed procedure. First, the operating cost was related to various physical features of the design by the engineering department; this relationship was called the operating cost function. One of the many input physical variables was the average lifetime of the material whose prior appears in Figure 8. All but two of the 12 physical input variables were independent. The priors on the whole set were gathered and used together with the operating cost function in a Monte Carlo simulation that produced a prior for the operating cost of the product.

The engineering department also developed the capital cost function, which was much simpler in form. The aleatory variables in this case were the production costs for various parts of the product. A simulation produced a prior on capital cost.

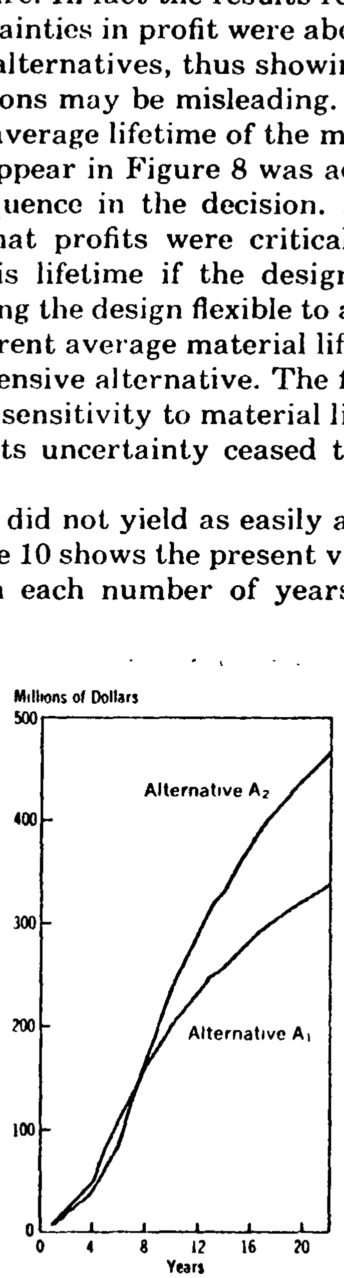
With priors established on all inputs to the approximate business model, numerical analysis determined the worth lottery for each alternative. The worth lotteries for the two alternatives closely resembled those in Figure 4, Part A. The new product alternative A_2 stochastically dominated the alternative A_1 (continuing to manufacture the present product). The result showed two interesting aspects of the problem. First, it had been expected that the worth lottery for the new product alternative would be considerably broader than it was for the old product. The image was that of a profitable and risky new venture compared with a less profitable, but less risky, standard venture. In fact the results revealed that the uncertainties in profit were about the same for both alternatives, thus showing how initial impressions may be misleading.

Second, the average lifetime of the material whose priors appear in Figure 8 was actually of little consequence in the decision. It was true enough that profits were critically dependent on this lifetime if the design were fixed. But leaving the design flexible to accommodate to different average material lifetimes was not an expensive alternative. The flexible design reduced sensitivity to material lifetime so much that its uncertainty ceased to be a major concern.

The problem did not yield as easily as this, however. Figure 10 shows the present value of profits through each number of years t for each alternative. Note that if returns beyond year 7 are ignored, the old product has a higher present value; but in considering returns over the entire 22-year period, the relationship reverses. When managers saw these results they were considerably disturbed. The division in question had been under heavy pressure to show a profit in the near future, and alternative A_2 would not meet that requirement. Thus, the question of time preference that had been quickly passed off as one of present value at 10% per year became the central issue in the decision. The question was whether the division was interested in the quick kill or the long pull.

This problem clearly illustrates the use of decision analysis in clarifying the issues surrounding a decision. A decision that might have been made on the basis of a material lifetime was shown to depend more fundamentally on the question of time preference for profit. The extensive effort devoted to this analysis was considered well spent by the company, which is now interested in instituting decision analysis procedures at several organizational levels.

Fig. 10 Expected Present Value of Profit



Space Program Planning

A more recent application in a quite different area concerned planning a major space program. The problem was to determine the sequence of designs of rockets and payloads that should be used to pursue the goal of exploring Mars. It was considered desirable to place orbiters about Mars as well as to land vehicles on the planet to collect scientific data.

The project manager had to define the design for each mission—that is, the type and number of launch vehicles, orbiters, and landers. The choice of design for the first mission could not logically be made without considering the overall project objectives and the feasible alternatives. Key features of the problem were the time for the development of new orbiting and landing vehicles, cost of each mission, and chances of achieving objectives.

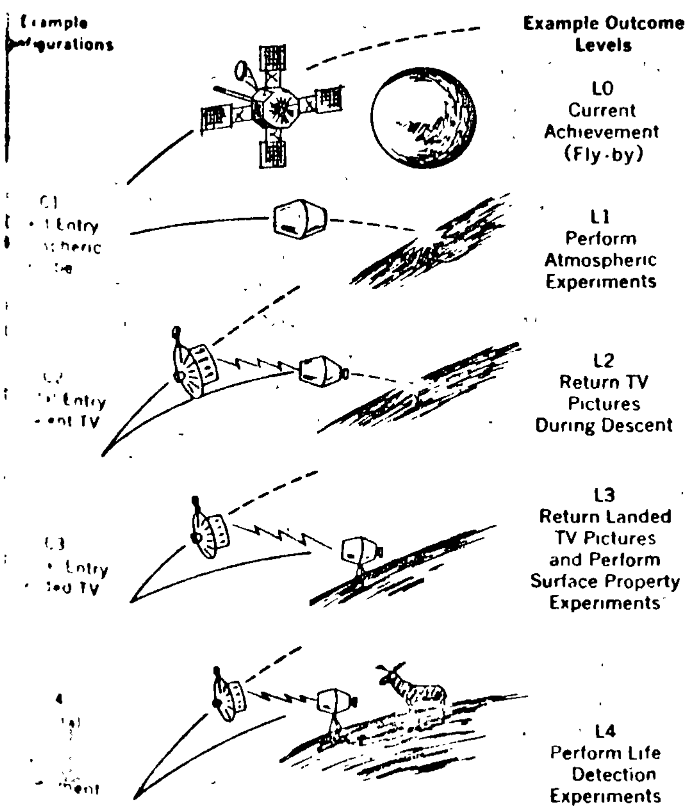
Approach to Solution

To apply decision analysis to the problem posed, a two-phase program was adopted. The first or pilot phase consisted of defining a simplified version of the decision. To the maximum extent possible, however, the essential features of the problem were accurately represented and only the complexity was reduced. This smaller problem allowed easier development of the modeling approach, and exercising of the model provided insight into the level of detail required in structuring the inputs to the decision. The second phase consisted of developing the more realistic and complex model required to decide on an actual mission.

The Pilot Phase

To begin the decision analysis, four possible designs were postulated to represent increasing levels of sophistication. Figure 11 shows these designs and their potential accomplishments. The questions were: what design should be selected for the first opportunity, and what sequence of designs should be planned to follow the first choice? Should the project manager, for example, elect to provide the ultimate level of capability in the initial design in the face of uncertainties in the Martian environment and difficulties in developing complex equipment to survive the prelaunch sterilization environment? Or should he choose a much simpler design that could obtain some information about the Martian environment to be used in developing subsequent, more complex, vehicles.

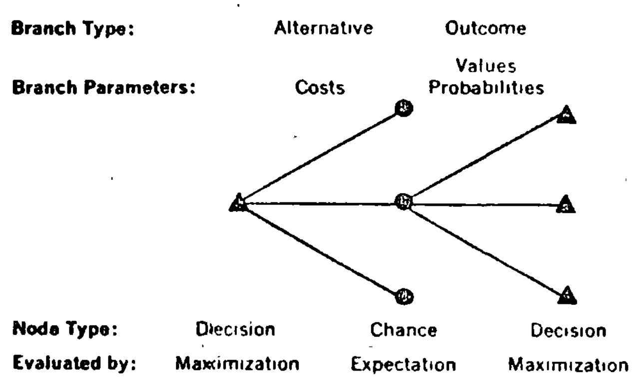
Fig. 11—Configurations and Performance



Decision Trees

The heart of the model used in analyzing the decision was a decision tree that represented the structure of all possible sequences of decisions and outcomes and provided for cost, value, and probability inputs. Such trees contain two types of nodes (decision nodes and chance nodes) and two types of branches (alternative branches and outcome branches), as illustrated in Figure 12. Emanating from each decision node is a set of alternative branches, each branch representing one of the alternatives available for selection at that point of decision. Each chance node is fol-

Fig. 12—Tree Relationships



lowed by a set of outcome branches, one branch for each outcome that may be achieved following that chance node. Probabilities of occurrence and values are assigned to each of these outcomes; costs are assigned to each decision alternative.

Two fundamental operations, expectation and maximization, are used to determine the most economic decision from the tree. At each chance node, the expected profit is computed by summing the probabilities of each outcome, multiplied by the value of that outcome plus expected profit of the node following that outcome. At each decision node, the expected profit of each alternative is calculated as the expected profit of the following node ("successor node") less the cost of the alternative. The optimum decision is found by maximization of these values over the set of possible alternatives, i.e., by selecting the alternative of highest expected profit.

Order of Events

The particular sequence of mission decisions and outcomes was a significant feature of the pilot analysis. As illustrated in Figure 13, the initial event of significance was the selection of the 1973 mission configuration. However, since lead time considerations required that the 1975 configuration decision be made in 1972, the second mission decision had to be made prior to obtaining the first mission results. Similarly, the 1977 decision had to be made before obtaining the results of the 1975 mission, although after the 1973 mission results. In general, then, a mission configuration was made in ignorance of the results of tl previous mission.

Fig. 13 ORDER OF EVENTS

	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977	1978	1979	...
First Flight	S					L	O						
Second Flight					S			L	O				
Third Flight							S			L	O		
Fourth Flight									S			L	O
Fifth Flight											S		...

S = Select L = Launch O = Outcome

Tree Example

A complete decision tree for the pilot pro ect, with the additional assumption that L2 the highest level of success, is presented ; Figure 14. The model that produces the n merical probabilities, values, and costs use in the example will be discussed later. Node at the left side of the tree is the initial decis to select either a $C1$ or a $C2$ for the first laun opportunity. The box designated LO abo this node indicates that the state at this no is the current level of achievement. Suppose $C1$ is selected. The cost of that $C1$ is $850 m lion, indicated by the "−850" that is writte under that branch. As a result of this choic the next node is decision node 2. The box de ignated LO, $C1$ above this node indicates th the state of this node is the current level achievement and a $C1$ is being constructed f the first launch. Now either a $C1$ or $C2$ mu be selected for the second launch. If a $C1$ selected, the cost is $575 million, and the ne node is chance node 7. The two branches fo lowing this node represent the possible ou comes of the first launch. The LO' outcom which would be failure to better LO on th first try, occurs with probability 0.1 wherea the $L1$ outcome occurs with probability 0.9 The value of the LO' outcome is zero, wherea the value of the LO outcome is 1224. Now fo low the case of the $L1$ outcome to decisio node 34. The state $L1$, $C1$ at this node, mean that the highest level of success is $L1$ and tha a $C1$ is being constructed for the next launch Since $L1$ has already been achieved at th point in the tree, a $C2$ is the only design tha may be launched in the third opportunity, at a cost of $740 million. This leads to decision node 35, where the state is $L1$, $C2$.

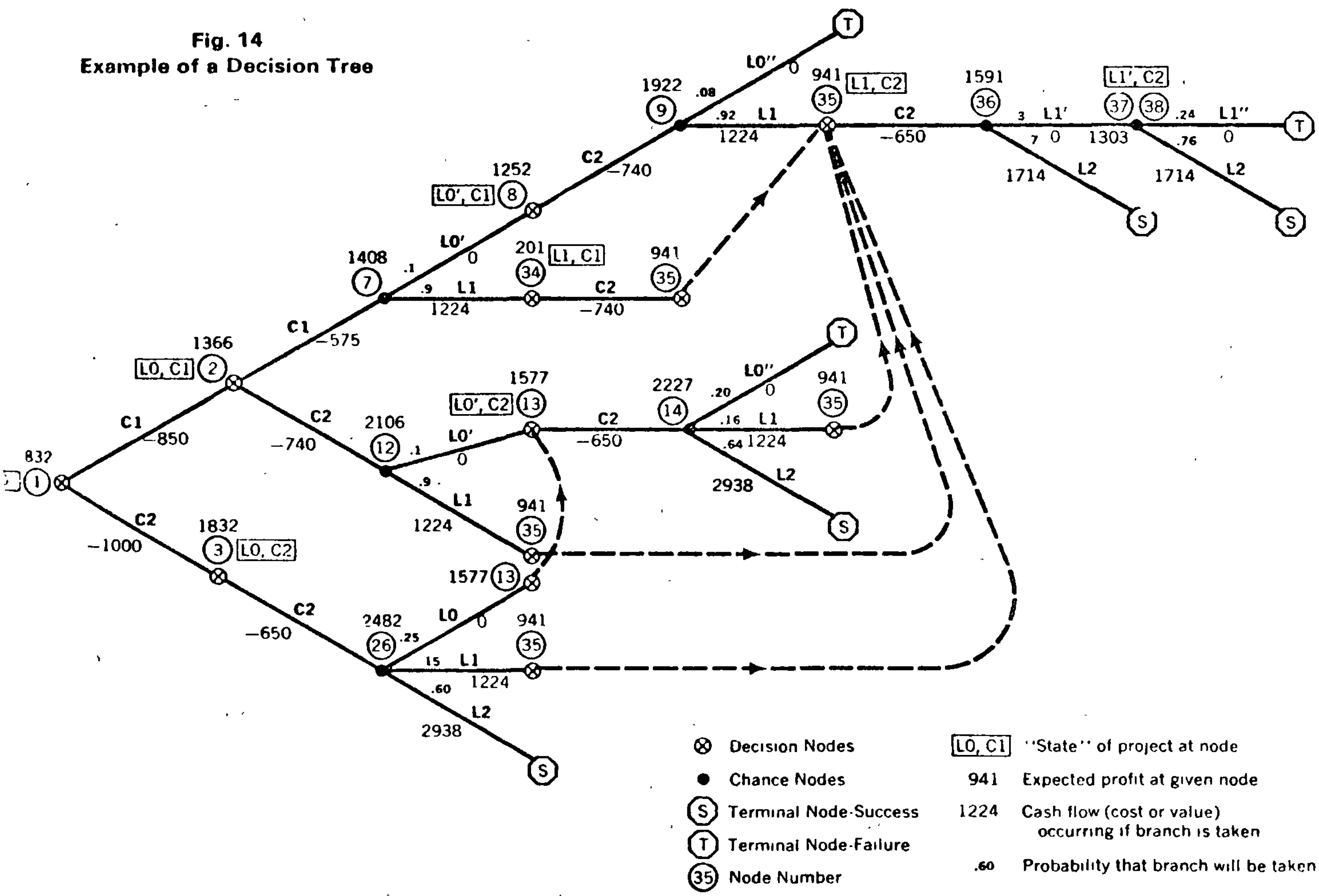
Node 35 in the example tree illustrates coalescence of nodes, a feature vital to maintaining a manageable tree size. Node 35 on the upper path through the tree can be reache from four other paths through the tree as indicated in the exhibit. If the coalescence did not occur, the portion of the tree following node 35 would have to be repeated four additional times. In the full pilot tree, coalescence results in a reduction of the number of branches in the tree by a factor of 30.

Along the path 1-2-7-34-35, at decision node 35, a $C2$ must be selected for the fourth opportunity. At chance node 36, the outcome of the third launch is either an $L1'$ (failure to better $L1$ with one attempt, which leads to node 38), or an $L2$ (which achieves a value of 1714 and successfully completes the program). These outcomes occur with probability 0.3 and 0.7, respectively. If $L1'$ is the outcome, chance node 38 is reached, where the outcome of the fourth launch is represented. The probability of $L1''$ is 0.24, and the probability of $L2$ is 0.76. Note that the probability of 12 has increased over that of node 36 (0.7 to 0.76) because of the experience gained previously.

One can similarly follow and interpret many other paths through the tree. A policy is a complete selection of particular alternatives at all decision nodes. This limits the set of all possible paths to a smaller subset. (It is not possible, for example, to reach node 26 if a $C1$ is chosen at node 1.) The probabilities, values, and cost of these paths then determine the characteristics of the decision policy.

The most economic policy, given the input data specifications, is defined as the policy that maximizes the expected profit of the project, i.e., expected value less expected cost.

Fig. 14
Example of a Decision Tree



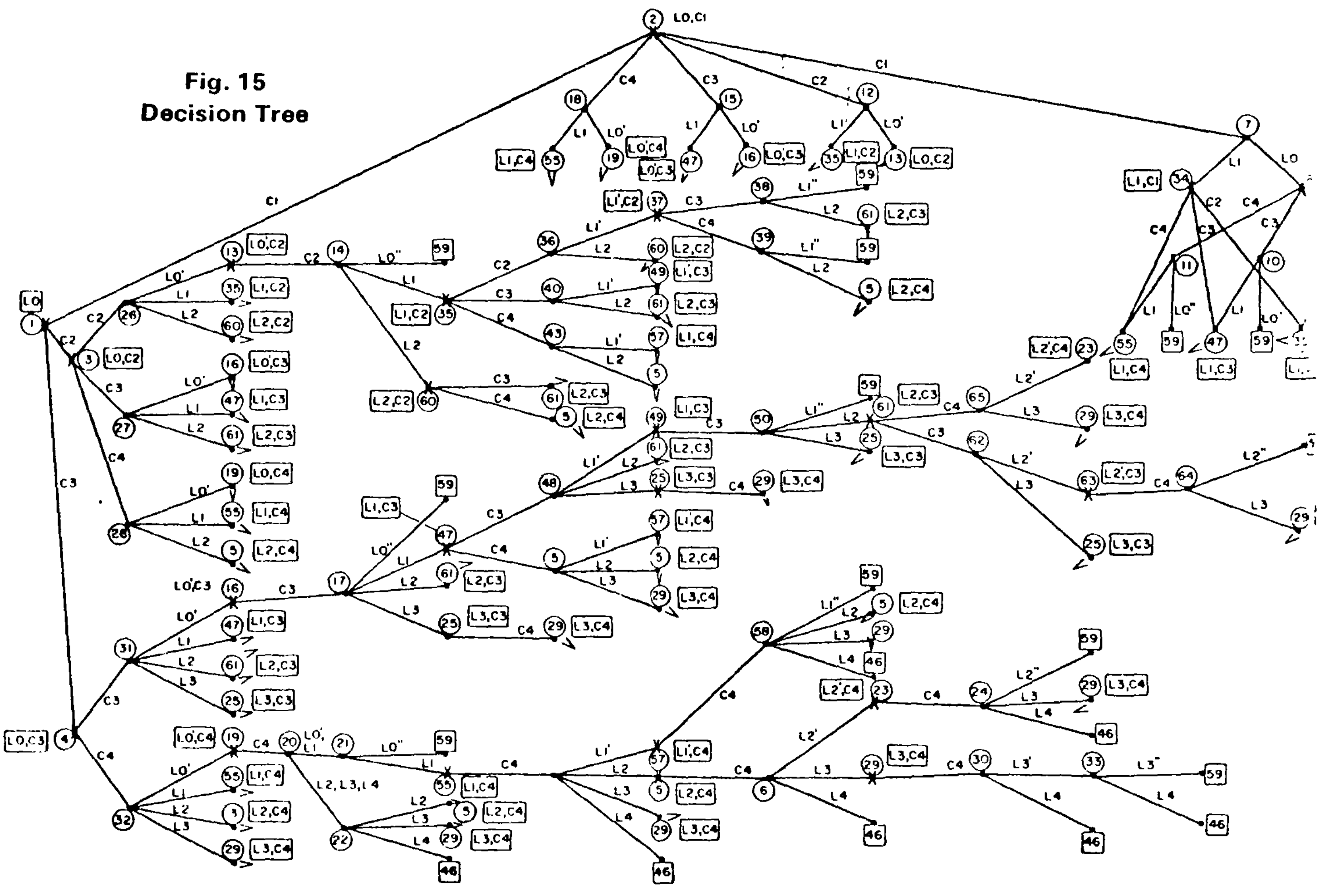
The technique illustrated here eliminates many of the nonoptimum policies from explicit consideration; it is the "roll back" technique that starts from the right side of the tree and progresses left to the beginning of the tree, making all decisions and calculations in *reverse* chronological order. Thus, when each decision is made, only policies that optimize decisions for the following decision nodes are considered.

Consider node 38 in Figure 14. At this chance node the probability of achieving $L1''$, which is worth nothing, is 0.24, and the probability of achieving $L2$, which is worth 1714, is 0.76. Thus, the expected profit of node 38 is: $0.24(0) + 0.76(1714) = 1303$. This number is written near node 38.

The calculations are carried out in this manner backwards through the tree. The first decision node with more than one choice is node 2. If a $C1$ is selected, it costs \$575 million (−575) and leads to node 7 with an expected profit of 1408, which yields $-575 + 1408 = 833$. If a $C2$ is selected, it costs \$740 million (−740) and leads to node 12 with an expected profit of 2106, which yields $-740 + 2106 = 1366$. Since 1366 is greater than 833, the most economic decision is to select a $C2$ at node 2, which results in an expected profit of 1366.

Finally, the first decision is a choice between a $C1$ with an expected profit of 516 or a $C2$ with an expected profit of 832. Maximum expected profit is achieved by the choice of a $C2$ resulting in an expected profit of 832. This

Fig. 15
Decision Tree



Note: Nodes [46] and [59] are the terminal nodes reached when two successive failures force termination of the project prior to achieving L4. Node [46] corresponds to L4 and is reached by achieving a totally successful project. Node [59] is reached when two successive failures force termination of the project prior to achieving L4.
↙ indicates direction of coalesced node bearing same number

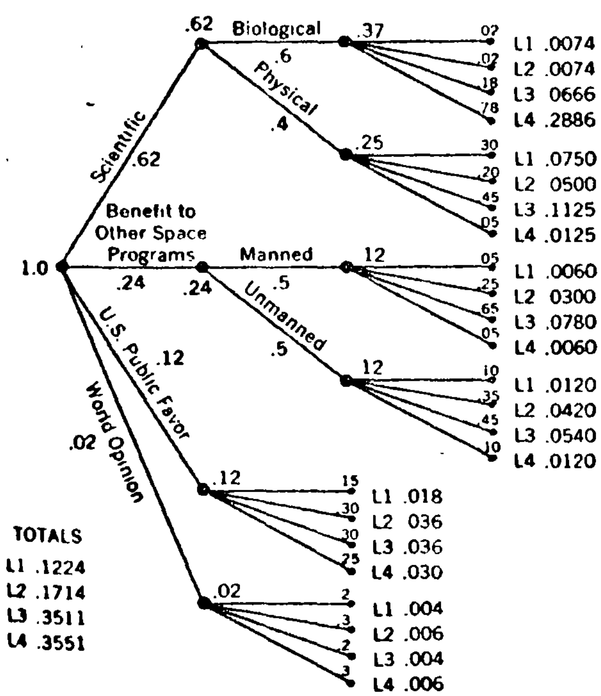
the expected profit of the entire project at the time the first decision is made.

Figure 15 illustrates the complexity of the completed decision tree for the pilot phase of the analysis.

Value Assignment

A particularly important part of this study was the specification of the value to be attached to the outcomes of the program. Since the decision-makers were reluctant to state values in dollar terms, a tree of point values was employed. The value tree is simply a convenient way of showing how the total value of the project is to be broken down into its component outcomes. Figure 16 shows a value tree for the pilot analysis. The points assigned to each tip of the tree are the fraction of total program value assigned to this accomplishment; the values accumulate as the program progresses. A total dollar value assigned to a perfect program therefore determines the dollar values used in the decision tree.

Fig. 16—The Value Tree



To derive a value measure, a value tree is constructed by considering first the major components of value and then the subcategories of each type, which are identified in more and more detail until no further distinction is necessary. Then each tip of the tree (constructed as above) is subdivided into four categories, each corresponding to the contribution of one of the four levels of achievement within the value subcategory represented by that tip.

The number 1.0 attached to the node at the extreme left of the value tree for the pilot analysis represents the total value of all the objectives of the pilot project (thus, the value of achieving *L*1, *L*2, *L*3, and *L*4). The four branches emanating from this node represent the four major categories of value recognized by the pilot model. The figure 0.62 attached to the upper branch represents the fraction of total value assigned to science. Two branches emanate from the science node, and 60% of the science value falls into the category of biological science. The 0.37 attached to the biological science node represents the fraction of *total* value attached to biological science, and is obtained by taking 60% of 0.62 (the fraction of total value attached to all science). Finally, the bottom branch following the biological science node indicates that 78% of the biological science value is achieved by jumping from *L*3 to *L*4.

The final step in value modeling is to obtain the fraction of total value to be attached to achieving each of the four levels. If all the contributions to achieving *L*1 (e.g., contributions to world opinion, U.S. public favor, physical science) are added, the result is the fraction of value that should be attached to achieving *L*1. The same process is followed for reaching *L*2 from *L*1, *L*3 from *L*2, and *L*4 from *L*3. The results of such a calculation are presented in the lower left corner of Figure 16.

Summary

On the basis of the promising results of working with the pilot model, a more complete model was developed to encompass nearly all of the factors involved in selecting the actual mission. It provided a more precise structure for assigning initial values, probabilities, and costs, and for updating probabilities and costs based on results achieved. The following tabulation shows a summary comparison of the complexity of the pilot model with the more complete model.

DECISION TREE COMPARISON TABLE

Pilot	Feature	Full Scale
4	Mission Designs	14
5	Outcomes	56
56	Decision Tree Nodes	3153
1592	Paths Through Tree	354,671,693

Clearly, the full-scale decision tree could not be represented graphically. The tree was constructed and evaluated by computer program specially developed for this application.

A model such as the one described here can be a valuable tool throughout the life of a project. As the project progresses, the knowledge of costs, probabilities, and values will improve as a result of development programs and flights. Improved knowledge can be used in the decision process each time a design must be selected for the next opportunity.

An important additional benefit of this analysis is that it provides a language for communicating the structure of the space project and the data factors relevant to the project decisions. It provides a valuable mechanism for discourse and interchange of information, as well as a means of delegating the responsibility for determining these factors.

FUTURE TRENDS

Decision analysis should show major growth, both in its scope of applications and in its effect on organizational procedures. This section presents various speculations about the future.

Applications

Market Strategy Planning

The importance of decision-making in a competitive environment has stimulated the use of decision analysis in both strategic and tactical marketing planning. The strategic problems are typically more significant because they affect the operations of the enterprise over many years. Strategic analysis entails building models of the company and of its competitors and customers, analyzing their interactions, and selecting strategies that will fare well in the face of competitive activities. Since most of this work is of a highly confidential nature, little has appeared in the public literature; nevertheless, there is reason to believe that many large U.S. corporations are performing work of this kind, however rudimentary it may be. The competitive analyses of a few quite sophisticated companies might rival those conducted in military circles.

Resource Exploration and Development

Resource exploration by mineral industries is a most natural application for decision analysis. Here the uncertainty is high, costs are great, and the potential benefits extremely handsome. At all levels of exploration—from conducting aerial surveys, through obtaining options on drill-test locations, to bidding and site development—decision analysis can make an important contribution. Organizations approaching these problems on a logical, quantitative basis should attain a major competitive advantage.

Capital Budgeting

In a sense, all strategic decision problems of a corporation are capital budgeting problems for its ultimate success depends upon how it allocates its resources. Decision analysis should play an increasingly important role in the selection of projects and in objective comparisons among them. Problems in spending for research and development programs, investment in new facilities, and acquisitions of other businesses will all receive the logical scrutiny of decision analysis. The methodology for treating these problems already exists; it now remains for it to be appreciated and implemented.

Portfolio Management

The quantitative treatment of portfolio management has already begun but it will receive even more formal treatment in the hands of decision analysts. The desires of the investing individual or organization will be measured quantitatively rather than qualitatively. Information on each alternative investment will be encoded numerically so that the effect of adding each to the portfolio can be determined immediately in terms of the expressed desires. The human will perform the tasks for which he is uniquely qualified: providing information and desires. The formal system will complement these by applying rapid logic.

Social Planning

On the frontiers of decision analysis are the problems of social planning. Difficult as it may be to specify the values and the criteria of the business organization, this problem is minor compared with those encountered in the public arena. Yet if decision-making in the public sector is to be logical, there is no alternative.

The problems to which a contribution can be made even at the current stage of development are virtually endless: in decisions associated with park systems, farm subsidies, transportation facilities, educational policy, taxation, defense, medical care, and foreign aid, the question of values is central in every case.

The time may come when every major public decision is accompanied by a decision analysis on public record, where the executive branch makes the decision using values specified by the people through the legislative branch. The breakdown of a public decision problem into its elements can only serve to focus appropriate concern on the issues that are crucial. For the first time, the public interest could be placed "on file" and proposals measured against it. A democracy governed in this fashion is probably not near at hand, but the idea is most intriguing.

Procedures

The effect of decision analysis on organizational procedures should be as impressive as its new applications. Some of the changes will be obvious, others quite subtle.

Application Procedures

Standardization by type of application will produce special forms of analyses for various types of decisions—for example, marketing strategy, new product introduction, research expenditures. This standardization will mean special computer programs, terminology, and specialization of concepts for each application. It will also mean that the important classes of decisions will receive much more effective attention than they do now.

Analytical Procedures

Certain techniques, such as deterministic, stochastic, and economic sensitivity analyses that may be performed with the same logic regardless of the application will be carried out by general computer programs. In fact, the process of development is well under way at the present time. Soon the logical structure of any decision analysis might be assembled from standard components.

Probabilistic Reporting

The introduction of decision analysis should have a major impact on the way organizational reporting is performed externally and internally. Externally, the organization will be able to illustrate its performance not just historically by means of balance sheets and operating statements, but also projectively by showing management's probability distributions on future value. Since these projections would be the result of a decision analysis, each component could be reviewed by interested parties and modified by them for their own purposes. However, management would have a profitable new tool to justify investments whose payoffs lie far in the future.

Organizational management will acquire new and more effective information systems as a result of decision analysis. Internal reporting will emphasize the encoding of knowledge in quantitative form. Instead of sales forecasts for next year, there will be probability distributions of sales. Thus, the state of information about future events will be clearly distinguished from performance goals.

Delegation by Value Function

An important logical consequence of decision analysis is that delegation of a decision requires only transmission of the delegator's present state of information and desires. Since both of these quantities can be made explicit through decision analysis, there should be an increase in the extent and success of delegation. In the external relationships of the firm, the delegation will no doubt appear as an increased emphasis on incentive contracts, where the incentives reflect the value function of the organization to the contractor. This trend is already evident in defense contracting.

Internally, the use of the value function for delegation should facilitate better coordination of the units of the organization. If explicit and consistent values are placed on the outcomes of production, sales, and engineering departments, then the firm can be sure that decisions in each unit are being made consistently with the best overall interests of the firm. The goal is to surround each component of the organization with a value structure on its outputs that encourages it to make decisions as would the chief decision-maker of the organization if he were closely acquainted with the operations of the component.

Organizational Changes and Management Development

The introduction of decision analysis will cause changes in organizational behavior and structure. A change should take place in the language of management, for the concepts discussed in this report are so relevant to the decision-making process that, once experienced in using them, it is difficult to think in any other terms. The explicit recognition of uncertainty and value questions in management discussions will in itself do much to improve the decision-making process.

Special corporate staffs concerned with the performance of decision analysis are already beginning to appear. These people would be specially trained in decision analysis, probability, economics, modeling, and computer implementation. They would be responsible for ensuring that the highest professional standards of logic and ethics are observed in any decision analysis.

Special training for decision analysts will be accompanied by special training for managers. They will need to know much more than they do now about logical structure and probability if they are to obtain full advantage from the decision analyst and his tools. No doubt much of this training will occur in special courses devoted to introducing decision analysis to management. These courses will be similar to, but more fundamental than, the courses that accompanied the introduction of computers into the U.S. economy.

Management Reward

Encouraging managers to be consistent with organizational objectives in decision-making requires adjusting the basis for their rewards to that objective. If rewarded only for short run outcomes, they will have no incentive to undertake the long range projects that may be in the best interest of the organization. It follows that any incentive structure for management will have to reward the quality of decisions rather than the quality of outcomes. The new financial statements that show probability distributions on future profit would be the key to the reward structure. After these distributions had been "audited" for realism, the manager would receive a reward based upon them in a predetermined way. Thus, the manager who created many new investment opportunities for a company could be rewarded for his efforts even before any were fully realized.

To make this system feasible requires distinguishing between two kinds of managers: the one who looks to the future and prepares for it; and the one who makes sure that today's operations are effective and profitable. The distinction is that between an admiral and a captain, or between the general staff and the field commanders. Specialization of function in corporate management with significant rewards and prestige attached to both planning and execution could be the most important benefit of decision analysis.

CONCLUSION

Although an organization can achieve ultimate success only by enjoying favorable outcomes, it can control only the quality of its decisions. Decision analysis is the most powerful tool yet discovered for ensuring the quality of the decision-making process: its ultimate limit is the desire of the decision-maker to be rational.

THE USED CAR BUYER

Man is called upon to make decisions about his home, his business, and his pleasure. These decisions vary in importance, but they have one property in common: most people do not have an orderly procedure for thinking about them. Of course, it is not practical to spend much time and effort thinking about the minor decisions in our lives--yet how can we judge what is practical until we develop a logical framework for decision problems? Our present task is the construction of such a decision procedure.

There are three main points we shall attempt to make about the science of decision making.

1. Probabilistic considerations are essential in the decision-making process;
2. The lessons of the past must be included;
3. The implications of the present decision for the future must be considered.

Let us discuss each of these points. The importance of probability is revealed when we realize that decisions in situations where there is no random element can usually be made with little difficulty. It is only when we are uncertain about which of a number of possible outcomes will occur that we find ourselves with a real decision problem. Consequently, much of our discussion of decision-making will be concerned with the question of how best to incorporate probabilistic notions in our decision procedure.

The question of using previous information in making decisions seems to incite some statisticians to riot, but most of the rest of us think it would be unwise to make a decision without using all our knowledge. If we were offered an opportunity to participate in a game of chance by our best friend, a tramp, and a business associate, we would generally have different feelings about the fairness of the game in each case. Although we might agree on the necessity of considering prior information, it is not clear just how we shall accomplish this objective. The problem is intensified because the prior information available to us may range in form from a strong belief that results from many years of experience to a vague feeling that arises from a few haphazard observations. The decision formalism to be described will allow us to include prior information of any form.

The influence of present decisions upon the future is a point often disregarded by decision-makers. Unfortunately, a decision that seems appropriate in the short run may, in fact, place the decision-maker in a very unfavorable position with respect to the future. For example, a naive taxi driver might be persuaded to take a customer on a long trip to the suburbs by the prospect of the higher fare for such a trip. He might not realize, however, that he will have to return in all likeliho without a paying passenger, and that when all alternatives are consider it could be more profitable for him to refuse the long trip in favor of number of shorter trips that could be made within the city during the s time period. The solution of such problems requires slightly more sophi ticated reasoning than the first two points we have discussed, but it is just as amenable to an analytic approach.

Let us now begin our analysis of decision problems with an example that is so commonplace that there will be every possibility of understan ing the environment of the problem, and yet is sufficiently detailed tha it is not obvious at first glance just how the decision should be approached. A fellow named Joe, of our acquaintance, is in the market for a new car. He has decided to buy a three-year-old Spartan Six sedan, and has surveyed the used-car dealers for such a car. After searching for a while, he has found a car like the one he wants on one dealer's lot. Th going rate for a three-year-old Spartan is $1100, but the price asked by the dealer is only $1000. Consequently, Joe figures that he will make $100 profit by buying this particular car.

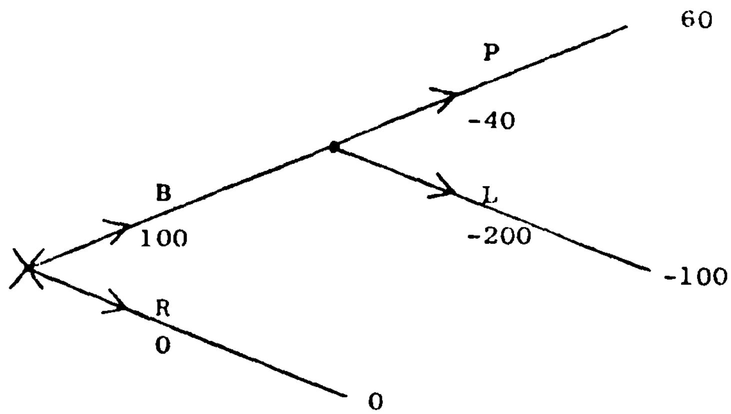
Unfortunately, just as Joe is about to close the deal, he overhears the salesman who has been serving him talking with another salesman. Hi salesman says, "This used-car business is a tough racket. I have a customer interested in the Spartan on our lot, but the practices of our business prevent me from warning him that he may get stuck if he buys it." The other salesman asks, "What do you mean?" Joe's salesman replies, "I worked at a Spartan dealership when that car first came on the market. Spartan made 20% of its cars in a new plant where they were still havin production line troubles; those cars were lemons. The other 80% of total production were pretty good cars." The other salesman asks, "What is th difference between a 'lemon' and a 'peach'?" "Well," says Joe's salesm "every car has 10 major mechanical systems--steering, brakes, transmissi differential, fuel, electric, etc. The peaches all had a serious defect in only one of these 10 systems, but the lemons had serious defects in of the 10 systems." The other salesman replies, "Well, don't feel so b maybe some cars didn't have any defects, or maybe the defects in this c have already been fixed."

"No, that's just it," says Joe's salesman. "Every car produced had either 1 or 6 defects in the ratio I mentioned; and I happen to know, because the previous owner was a friend of mine, that this particular car has never been repaired." "If it is bothering you so much, why don't y tell the guy it's a lemon and forget about it?" says the other salesman. "Ah," answers Joe's man, "that's the trouble. I personally don't know whether or not it is a lemon, and I'm certainly not going to take the chance of losing a sale by worrying a customer unnecessarily." To which the other salesman replies, "It's time for coffee."

We can now imagine the state of our friend Joe. What seemed like a real bargain has turned into a potential nightmare; he can no longer make the $100 profit he had hoped for. Joe's first reaction is to turn and flee, but he has the icy nerves of a decision-maker and so soon regains his composure. Joe realizes that he would be foolish to forego the chance to buy the car he thought he wanted, at this price, without good reason. He decides to call an acquaintance who is a mechanic and get his estimate of what the possible repairs might cost. The mechanic reports that it costs about $40 to repair a single serious defect in one of a car's major systems, but that if 6 defects were to be repaired, the price for all 6 would be only $200.

Now Joe considers the possibilities open to him. He can either buy the car or refuse it. If he decides to buy the car, then his outcome is uncertain. If the car turns out to be a peach, then only one defect will develop and Joe will have made a profit of $60: $100 from buying the car at a low price, less $40 for repairing the one defect. However, if the car should be a lemon, then Joe will lose $100 because it will cost him $200 to repair the 6 defects to be found in a lemon. If, on the other hand, he refuses to buy, then he gains and loses nothing.

We can represent the decision structure of Joe's problem by drawing a decision tree like that shown in Figure 1. The direction of the arrows refers to the time flow of the decision process. In this figure, each directed line segment represents some event in the decision problem. We have used B to indicate the event of Joe's buying the car, and R to indicate his refusing it. P is the event of the car's ultimately turning out to be a peach, while L is the event of the car's being a lemon. The tree as drawn in Figure 1 shows that the car may turn out to be a peach or a lemon regardless of Joe's action. Note that different symbols are used for the node joining the B-R branches and the nodes joining the P-L branches. The X is used to indicate points in the decision tree where the decision-maker must decide on some act; the • is used for nodes where the branch to be taken is subject to chance rather than decision. We shall call these two types of nodes "decision" nodes and "chance" nodes, respectively. In this example, Joe's only decision is whether to buy or refuse to buy; consequently, only the node joining the B-R branches requires an X. The ultimate outcome as to whether the car is a peach or a lemon is governed only by chance and so the P-L branches are joined by a •.



B: Joe buys the car
R: Joe refuses to buy
P: Car is a peach
L: Car is a lemon

Joe's Original Decision Tree

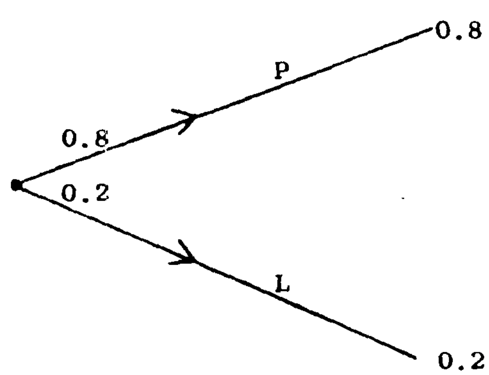
FIGURE 1

Generally, traversing each branch on the decision tree will bring some reward, positive or negative, to the decision-maker. We shall cho as a convention to write this reward under each branch. In Figure 1 we have written 100 under the branch labeled B to represent the immediate profit to Joe in buying the car; 0 is written under R branch, because Joe will neither gain nor lose by refusing to buy. The numbers under the P and L branches refer to the cost of repairing a peach and a lemon, respectively. If the decision-maker follows a tree from its unique star- ing node to all of its tips, then he will experience some pattern of ga and losses according to the branches he actually traverses. The net pr of all such traversals is written at each tip of the tree. Each tip ma designated by the sequence of branches that lead to it. Thus in this c the tip BP is given the value \$60 as the net profit in buying the car an then finding that it is a peach. The tip BL corresponds to a loss of \$1 from buying a lemon, while the tip R is evaluated at zero because the car is refused. These three tips of the tree represent the three possible comes of this decision problem. The outcome BP is favorable to Joe, the outcome BL is unfavorable, and the outcome R is indifferent.

Naturally, Joe would like the outcome to be BP with a profit of \$60, but after hearing the salesmen's conversation he realizes that the likeli- hood of this outcome will be controlled by Nature rather than by himself We can think of Nature as playing a game with Joe, as follows. When she placed the car on the used-car lot, she made it a lemon with probability 0.2 and a peach with probability 0.8. She performed her selection by tossing a coin with probability of "heads" equal to 0.8 and made the car a lemon if the coin came up "tails." Thus the nodes that were chance nodes in Joe's decision tree we can imagine to have been performed by an opponent called Nature who is not malevolent and who selects actions using chance mechanisms.

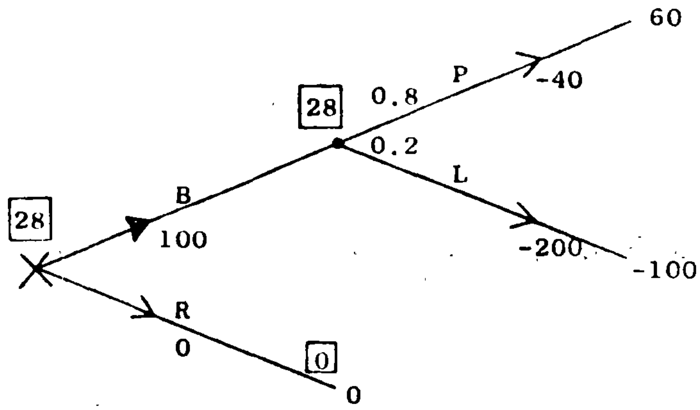
We can draw a tree to show Nature's options, as is done in Figure 2. In Nature's tree, all nodes are chance nodes. We shall write above the beginning of each branch the probability that Nature will follow that branch. In the present example, we know that the probability of a peach is 0.8, the probability of a lemon is 0.2. We also write at each tip of Nature's tree the probability that Nature will produce an outcome corresponding to that tip. In general, these probabilities are calculated by multiplying together the probabilities on all the branches that lead from the initial node on Nature's tree to each tip. In this simple case, all we must do is write 0.8 and 0.2 at the end of both the P and L branches.

The importance of Nature's tree, as we shall see, is that it provides all the probabilistic information that is necessary for the decision tree. To illustrate this point, we recall that we have yet to write probabilities on each chance node of the decision tree. The results of the calculations in Nature's tree allow us to draw Figure 1 in the form of Figure 3. The various features of Figure 3 will be explained gradually. At the moment, our example has such a simple form that it is not at all clear why it is necessary to consider a separate tree for Nature. As our example becomes more complex, the need for Nature's tree will be evident. The numbers in the square boxes at each node in Figure 3 represent the



Nature's Tree

FIGURE 2



Joe's Decision Tree with Probabilities from Nature's Tree

FIGURE 3

net profit to Joe from future activities if he should arrive at such a node. Thus, if Joe is at node B (we label nodes by the branches that must be traversed to reach them), then he expects to earn $60 with the probability 0.8, and lose $100 with probability 0.2. His expected earnings are 0.8(60) + 0.2(-100) = $28. Of course, if Joe decides not to b the car, then he will earn nothing, and so 0 appears in the square box appended to node R.

As a result of evaluating each possible action that Joe might take in terms of its expected value equivalent, we are in a position to help Joe with his decision. If Joe buys the car, then he expects to earn $28. if he refuses to buy, he will earn nothing. If Joe is an expected-value decision-maker, he should decide to buy the car. His recommended action is shown by drawing a solid arrowhead on the B branch leading from the decision node. We then write his expected profit from taking that acti $28, in the square box over the decision node.

As a result of this analysis of the problem, Joe feels a little bet than he did before. He has forsaken all hope of a $100 profit and is c ing around to the idea that it might be wise to settle for an expected profit of $28. However, while he is becoming reconciled to the forces fate, a stranger approaches him and says, "I couldn't help overhearing talking to yourself about your problems. Perhaps I can help you. You I worked in the factory where the substandard Spartans, or lemons as called them, were made. I can tell you whether the car sitting on thi is a lemon simply by looking at the serial number." Joe can hardly bel his ears. At last, a possibility of finding out whether the car is a before buying it.

Joe looks at the man, decides he has an honest appearance, and says, "You are just the kind of help I need. Let's go over to the car and take a look at it. I am eager to find out whether or not it is a good deal." The stranger smiles and replies, "I am sure you are, but you can hardly expect me to go to all the trouble of examining the car and getting myself dirty without some financial consideration." At first Joe is angry about the stranger's mercenary attitude, but then he remembers he is not in a position to throw away potentially useful information if it can be obtained at a reasonable price. He asks for and is granted a few moments to think over the stranger's offer.

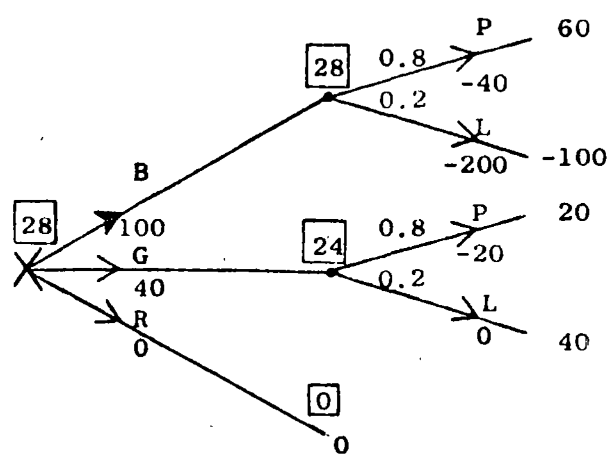
The problem is this; how much is Joe willing to pay the stranger for his information? He reasons as follows. On the basis of the stranger's appearance and manner, Joe decides that he can be trusted in his claim of being able to distinguish peaches from lemons. If the stranger reports that the car is a peach, then Joe will buy it and make an expected profit of $60. If the stranger says it is a lemon, then Joe will refuse to buy it and make nothing. The probability that the stranger will find a peach is 0.8; the probability of finding a lemon is 0.2. Consequently, Joe's expected profit after receiving the information is 0.8(60) + 0.2(0) = $48. Therefore, is the information worth $48? No, because even without it Joe expects to make $28, according to our original analysis. Hence, the net value of the stranger's information to Joe is $20. That is, Joe as an expected-value decision-maker should be willing to pay any amount up to $20 for the stranger's advice.

This figure of $20 seems high to Joe, so he decides to check it in the following way. Joe thinks, without this new information I would buy the car and make an expected profit of $28. If I buy the information, then with probability 0.8 the stranger will report that the car is a peach and his information will be worthless because I am going to buy the car anyway. On the other hand, with probability 0.2 the stranger will find that the car is a lemon, and in this case the information is worth $100 since that is the amount that I would lose if I bought the car and it turned out to be a lemon. Consequently, the expected value of the information to me is 0.8(0) + 0.2(100) = $20, the same as before. Now Joe is convinced that he should pay as much as $20.

We shall call this quantity the expected value of perfect information, or the EVPI. It represents the maximum price that should be paid for any experimental results in a statistical decision situation. This follows since no partial knowledge could ever be worth more than a report of the actual outcome of nature's process. We shall have much more to say of this quantity in our later discussion.

Joe now decides to offer the stranger $15 in hopes of getting the information at a bargain price. However, when he confronts the stranger with this offer, the stranger replies that he couldn't consider the job for less than $25 and suggests that Joe think it over for a while. Joe is upset by this turn of events, but quickly regains his composure. He thinks to himself that the real reason for his difficulties is that he doesn't have a wide enough range of alternatives from which to select an appropriate action. Suddenly he has a brainstorm--maybe he can get the dealer to give him the guarantee on the car! He inquires of the deal whether a guarantee is available. The dealer says, "Yes, there is a guarantee plan; it costs $60 and covers 50% of repair cost." Joe thinks fast and replies, "You certainly don't have much confidence in your cars. If I bought a car and it turned out to be a lemon, I could go broke even on my 50%." The dealer says, "All right. Just for you I will include an anti-lemon feature in the guarantee. If total repairs on the car cost you $100 or more, I will make no charge for <u>any</u> of the repairs. How's that for meeting a customer half-way?" Joe says that's fine and now he would like to think it over again.

At this point Joe realizes that he has a new decision tree. It is shown in Figure 4. This tree differs from the preceding one because there are now three possible actions at the decision node. The new alternative is to buy the car with the guarantee; that is, to hedge against the possibility of getting a lemon by spending $60. This alternative is given the symbol G. We see that although the car might still turn out to be a lemon if this alternative is followed, the costs associated with the two outco



Joe's Decision Tree Including Guarantee Possibility

FIGURE 4

are strikingly different from what they are in the case where the car is bought without such a guarantee.

Let us examine Figure 4 in some detail. The figures written below each branch are again the expected profit from traversing that branch. The numbers on the tips are the total expected profit of the chain of branches leading to that tip. Now, as before, we shall choose to calculate the expected value of each node by using the number on the tips rather than on the branches. However, this choice is arbitrary and will be reversed when a reversal is convenient.

The expected value of the nodes B and R are calculated as before. The value of $40 written under the G branch refers to the fact that our initial profit from buying the car with the guarantee is only $40 because the guarantee itself costs $60. The value of -$20 over the P branch following the G action arises because even a peach will require one repair at at cost of $40, but half of this $40 will be paid by the guarantee. The 0 under the corresponding L branch is a result of the anti-lemon feature of the guarantee. Since the cost of repairs on a lemon will exceed $100, there will be no charge for repairs. Thus the net profit of buying the car with a guarantee and having it turn out to be a peach is $20, while the profit if it turns out to be a lemon is $40. Since Nature's tree of Figure 2 still applies to this case, the probabilities of these two events have values of 0.8 and 0.2, respectively. Hence, the expected earnings from buying the car with the guarantee is 0.8(20) + 0.2(40) = $24. Since this is less than the $28 profit to be expected if the car is bought without the guarantee, the guarantee does not look like a good idea. The choice should once more be to buy the car without any protection, as is indicated by the heavy arrowhead on the B branch.

At this point our knowledgeable stranger returns and once more offers his advice--for a price. Has the advent of the guarantee changed what Joe should pay? Let's find out. If the information is bought, the stranger will find that the car is a peach with probability 0.8. If a peach is reported, then Joe will buy it without a guarantee and make an expected profit of $60. With probability 0.2 the stranger will discover a lemon. In this case, however, Joe is best advised not to refuse the car and make nothing as he did before, but rather to buy it with the guarantee. As the number on the tip of the branch GL in Figure 4 indicates, by taking this action he will earn an expected profit of $40. Thus, the amount that Joe expects to earn by buying the car is 0.8(60) + 0.2(40) = $56. Since Joe expects to earn $28 anyway by buying the car without this information, the value of the additional information to him is $28.

It may at first seem strange that the expected value of perfect information, or EVPI, should increase simply because an alternative has been added to those already available to Joe. However, such an increase has taken place as a result of the fact that Joe is in a better position to make use of information that the car is a lemon than he was previously. We can verify the figure of $28 using the same method employed before. If the stranger reports a peach, then Joe's decision to buy the car will be unchanged; but if a lemon is reported, then Joe will buy the car with, rather than without, the guarantee and so will turn a loss of $100 into a profit of $40. Consequently, his expected profit will increase by $140 with probability 0.2. Thus, the information is worth 0.2(140) = $28 to Joe.

Now, of course, the stranger's asking price of $25 for the perfect information seems quite reasonable. Joe is about to purchase the information when he has another brainstorm. He knows that perfect information is worth $28 to him, and so he reasons that if he can get partial information at a price sufficiently lower than $28 he may be able to increase his profits. He first asks the dealer if he can take the car to his mechanic friend for a checkup. The dealer is willing to allow this, but places a time limit of one hour on the car's absence from the lot. Somewhat elated, Joe calls his friend to ask what kind of tests could be performed in an hour and how much they would cost. The mechanic says that he can only do at the most one or two tests on the car in the time available. He then supplies Joe with the following test alternatives:

1. He can test the steering system alone, at a cost of $9;
2. He can test two systems--the fuel and electrical systems--for a total cost of $13;
3. He can perform a two-test sequence, in which Joe will be able to authorize the second test after the result of the first test is known. Thus, under this alternative, the mechanic will test the transmission, at a cost of $10, report the outcome of the test to Joe, and then proceed to check the differential, at an additional cost of $4, if he is requested to do so.

All the tests will find a defect in each system tested, if a defect exists. The test alternatives are summarized in Table 1.

Including the possibility of no testing, Joe now looks over these test alternatives and decides that it is worthwhile at least to consider testing because the cost of each of these tests is significantly less than the $28 value of perfect information. If all tests had cost over

Table 1

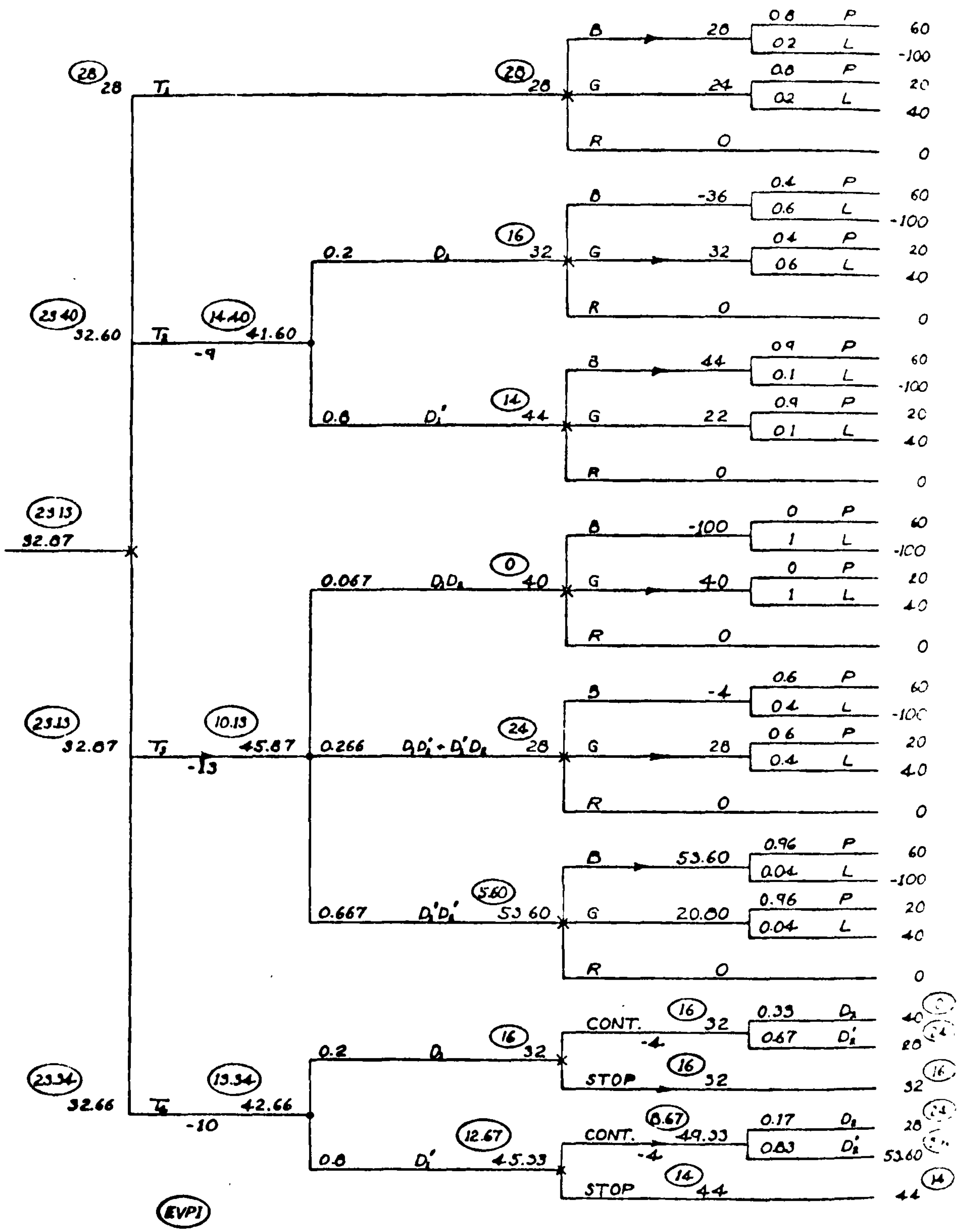
THE TEST ALTERNATIVES

Test	Description	Cost
T_1	Perform no tests	$ 0
T_2	Test steering system	9
T_3	Test fuel and electrical systems (2 systems)	13
T_4	Test transmission with option on testing differential for	10 4

$28, then there would be no point in considering a testing program because each test will generally provide only partial information, and even perfect information is worth a maximum of $28. However, it is still not clear which test, if any, should be performed. Furthermore, Joe would like to know the value of the stranger's information under these new circumstances. These problems will be approached by drawing a new decision tree for Joe and a new tree for Nature. The general structure of the decision tree is shown in Figure 5.

This tree is quite complicated, so we shall explain it in gradual steps. Notice that the first decision to be made is which of the four test options--T_1, T_2, T_3, T_4--to follow. If some tests are made, the mechanic will report the results, and then a decision about buying the car must be made. If the test T_4 is used, of course, then there will also be a step in which the mechanic is advised whether or not to continue the test procedure. Let us now examine the situation resulting from each test in more detail.

If test T_1 is selected, then no physical test is made and Joe is required to make a decision about buying the car immediately. The decision tree from this point on looks just like that of Figure 4. In fact, the numbers that appear in Figure 4 have been reproduced exactly in Figure 5, with the exception that only the numbers on the tips of the branches have been copied because they are sufficient for our purposes. Indeed, a little reflection will reveal that regardless of the test program we follow, we must end up with a decision tree like that of Figure 4. However, although the numbers on the tips of the branches will be the same in all cases, the



Joe's Complete Expected Value Decision Tree

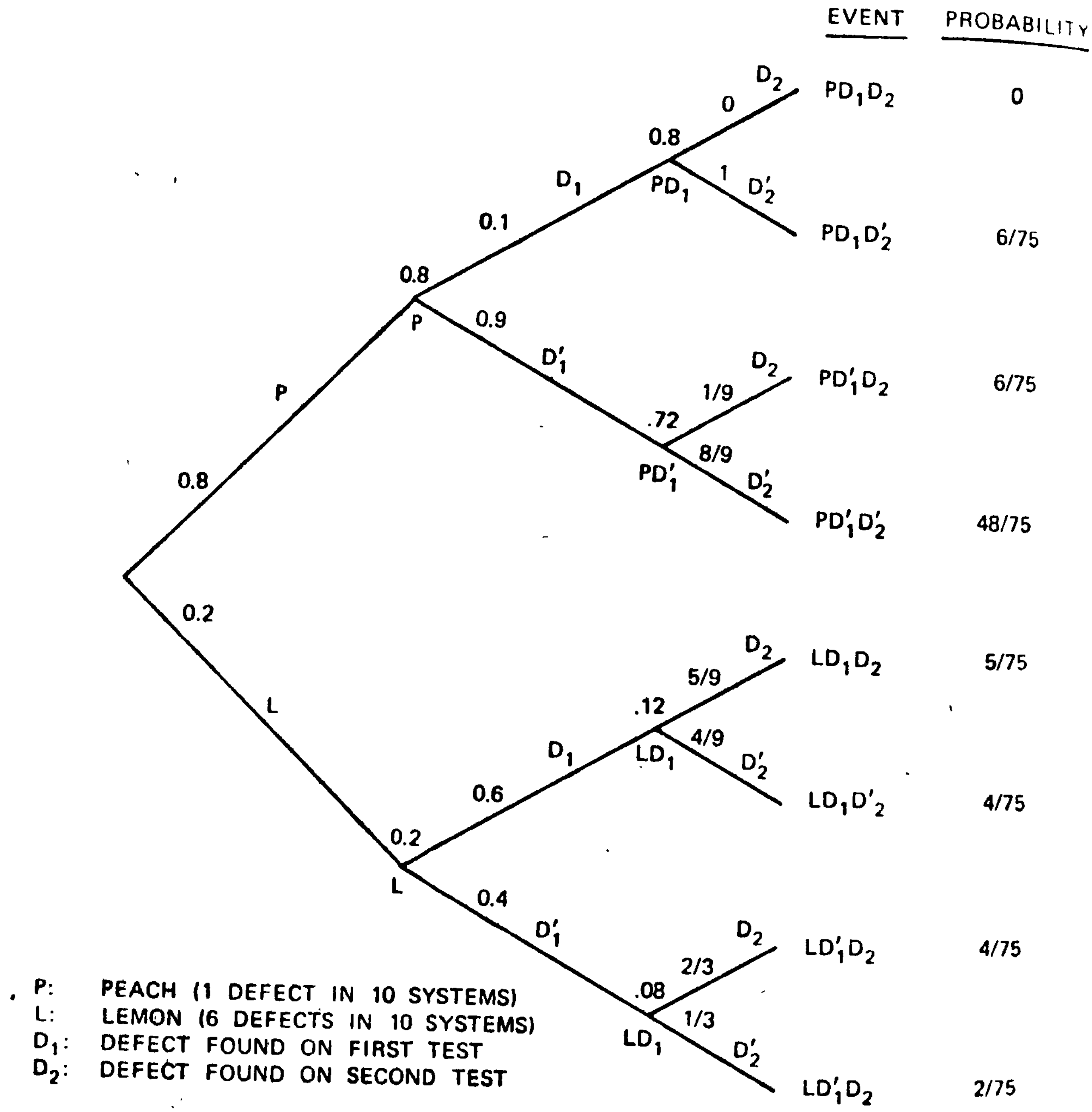
FIGURE 5

probabilities to be written on the branches will differ in each case. The probability of the final outcome of a peach or a lemon will generally depend on the findings of the experimental program until the time the decision on buying the car must be made. For example, if two defects have been found, then the car is a lemon with probability one.

We see that what is now required is a mechanism that will give for each possible result of the experimental program the appropriate probabilities for the ultimate outcome of a peach and a lemon. Nature's tree is just such a mechanism. It is drawn for this problem in Figure 6. In this figure we have used D_1 to represent the event that a defect is discovered in the first test on the car, if such a test is performed, and D_2 is used similarly to indicate the finding of a defect on a second test, if any. The numbers on each branch represent the conditional probabilities of going to each following node, given that the present node has been reached. The numbers on the nodes represent the unconditional probability of occupying that node. The tree can then be explained as follows. Nature first decides whether the car is to be a peach or a lemon with probabilities 0.8 and 0.2, respectively, using some random process like the biased coin-flipping described earlier; thus, $p(P) = 0.8$, $p(L) = 0.2$.

Suppose that the car has turned out to be a peach. Then, using our convention that a node is labeled by the letters on the branches that must be traversed to reach it, we are at node P. Now suppose that one major system of the car is tested. Since the car is a peach, there is probability 1 in 10, or 0.1, that the one defective system will be checked and found defective; thus $p(D_1|P) = 0.1$. If this happens, we proceed to the node PD_1; then, $p(PD_1) = p(P)\ p(D_1|P) = 0.08$. On the other hand, with probability 0.9 no defect is discovered and we reach node PD_1'. Suppose, further, that a second test on another system is now performed. If we are at node PD_1, then the only defective system in the car has already been discovered and there is probability 0 of finding another defect and reaching node PD_1D_2. Under these circumstances, we shall be certain to proceed to node PD_1D_2'. The overall probability of such event as PD_1D_2 is determined by multiplying together the probabilities on all the branches that lead to that tip of the tree. Thus, $p(PD_1D_2) = 0$ and $p(PD_1D_2') = 0.08$.

If the car were a peach, but no defect had been found on the first test, then we would be at node PD_1'. If, now, a second test is performed, it will yield a defect with the probability that the system tested is the one defective system in the remaining nine, or 1/9. Of course, the probability of finding no defect in this situation is then 8/9. The overall probabilities $p(PD_1'D_2) = 0.08$ and $p(PD_1'D_2') = 0.64$ can then be calculated.



Nature's Tree for Complete Decision Problem

FIGURE 6

If Nature selects a lemon initially, then the same sort of reasoning applies. The probability of finding a defect in the first test on a lemon is equal to the chance of testing one of the 6 defective systems out of the 10 systems on the car, or 0.6. If one defect has been found in a lemon, then the probability of finding another is the chance that one of the 5 defective systems among the remaining 9 systems will be inspected, or 5/9. If, on the other hand, no defect is found in the first test on a lemon, then the probability of finding one during the second test is the chance of testing one of the 6 defective systems among the 9 systems remaining, or 2/3. The probabilities of all final outcomes pertaining to the lemon branch of the tree are then computed and written on the tips of the branches.

Figure 6 contains all the information necessary to answer any question about the probabilistic structure of the decision process. We can best see this by returning at this point to our discussion of the test alternatives in Figure 5.

If the alternative T_2, test one system is followed, then the first requirement is that Joe pay $9 for the services of the mechanic. This payment is indicated by the -9 on the T_2 branch. The next event to take place is the report of the mechanic on whether or not he found a defect. His report is a chance event, so indicated by the solid dot that follows branch T_2. The mechanic reports either that he found a defect, D_1, or did not find a defect, D_1'. However, the probability that each of the branches D_1 or D_1' will occur must yet be determined. But $p(D_1) = p(PD_1) + p(LD_1)$ since P and L are mutually exclusive and collectively exhaustive events. By using the results of Nature's tree in Figure 4, we have $p(PD_1) = 0.08$, $p(LD_1) = 0.12$ and so $p(D_1) = 0.2$; of course, $p(D_1') = 0.8$. These two probabilities are recorded on the branches D_1 and D_1' that follow branch T_2 to indicate the nature of the chance point. Once D_1 or D_1' has occurred, Joe faces a decision tree like that of Figure 4, but with different probabilities that will be calculated from Nature's tree in Figure 6. In particular, we require the probabilities $p(P|D_1)$, $p(P|D_1')$ and their complements. These probabilities are easy to obtain because $p(P|D_1) = p(PD_1)/p(D_1)$ by definition, and we have just calculated both probabilities involved in this expression. Thus, $p(P|D_1) = 0.08/0.2 = 0.4$, and $p(L|D_1) = 0.6$. These numbers are entered as the probabilities of a peach and a lemon, respectively, on the branches that follow node T_2D_1 in Figure 5. Similarly, $p(P|D_1') = p(PD_1')/p(D_1') = 0.72/0.80 = 0.9$, again using the results of Figure 6, and $p(L|D_1') = 0.1$. The branches for peach and lemon that follow node T_2D_1' in Figure 5 are labeled with these probabilities.

We have now obtained the complete probabilistic structure of the test T_2. The branches emanating from every chance point have been assigned the appropriate probabilities. It is now possible to determine the expected profit to be obtained by following test T_2. First, we shall compute the decision to be made if a defect is reported. If, in this case, Joe decides to buy the car without a guarantee, he will earn \$60 with probability 0.4 and lose \$100 with probability 0.6. His expected profit is then -\$36. If he hedges by buying with the guarantee, his expected profit is 0.4(20) + 0.6(40) = \$32. If he refuses to buy, he earns nothing. Since \$32 is a better result than no earnings or a loss, Joe should decide to buy the car with a guarantee if he finds himself at this situation. His expected return will be \$32, as indicated in the square boxes following node T_2D_1.

On the other hand, if the mechanic finds no defect in the steering, then Joe will be at node T_2D_1' and will again be faced by a decision. If he buys without a guarantee, his expected profit is 0.9(60) + 0.1(-100) = \$44. If he buys with a guarantee, his expected profit is 0.9(20) + 0.1(40) = \$22. Again, he makes nothing if he refuses to buy. Since \$44 is the maximum return, he should decide to buy the car without the guarantee. The expected earnings of \$44 are written at the end of branch T_2D_1'.

There is but one step remaining in the analysis of test option T_2. If the mechanic reports a defect, Joe expects to earn \$32. If he reports no defect, then Joe expects to earn \$44. These two events happen with probability 0.2 and 0.8, respectively, according to the earlier calculations using Nature's tree. Hence, the expected profit before the results of the test are known, but after the test has been paid for, is 0.2(32) + 0.8(44) = \$41.60. Since Joe must pay the mechanic \$9 to reach this position, his expected earnings from test T_2, including the payment to the mechanic, are \$41.60 - \$9 = 32.60. This number is entered at the left of branch T_2 to indicate the expected profit from following this test program. Since we have already calculated the expected profit of program T_1 to be \$28, it is clear that Joe is better advised to proceed with the test on the steering rather than to make the decision in the absence of this information. By so doing he will increase his expected earnings by \$4.60. Of course, it is still not proved that T_2 is the best test alternative to follow--we have only shown that it is better than T_1. It remains to investigate T_3 and T_4.

Before we do so, however, let us return once more to the concept of the value of perfect information. We have already shown that the partial information supplied by option T_2 is more valuable than its cost. How has this revelation affected our evaluation of the stranger's information? Before the test alternatives were introduced, Joe had calculated that the expected value of perfect information was \$28. As you recall, this figure was determined by calculating first the amount of money Joe could make if the perfect information was available to him ($56) and then subtracting from this quantity the amount he could expect to earn in the absence of this information ($28); thus, EVPI equalled $56 - $28. Now what has changed in these calculations? The $56 profit to be expected by using perfect information has remained unchanged since the introduction of the guarantee plan. However, Joe's expectation without the stranger's information has been increased from $28 to $32.60. Hence, the expected value of perfect information has been lowered to $56 - $32.60 = $23.40.

It is interesting to note how we have vacillated about the value of the stranger's information. Before the advent of the guarantee plan, it was $20 and the stranger's price of $25 seemed too high. Then the guarantee possibility was introduced and the value of perfect information rose to $28. At that point the stranger's $25 price seemed like a bargain. Finally, however, Joe calculated the results to be expected using the test alternative T_2 and saw that the value of perfect information had decreased to $23.40, a figure below the stranger's price. Consequently, Joe is not in a mood to buy at the moment. Although he has not yet evaluated the value of perfect information under test plans T_3 and T_4, at this point he is sure that it cannot possibly be greater than $23.40.

The value of perfect information at each point in the tree will be shown in Figure 5 in the ovals at pertinent nodes. In every case the EVPI is calculated simply by subtracting the expected earnings at each node from the profit to be expected if the perfect information were available. At the two nodes that begin and end branch T_2, the result of the test is not known and so the expected profit using perfect information is still $56. Thus the node to the right of branch T_2 bears the EVPI $14.40 since $56 - $41.60 = $14.40. Perfect information is worth $9 less than it was to the right of branch T_2 because of the payment to the mechanic.

The calculation of the value of perfect information is performed in the same way when the test results are known, but in this case, the expected profit from using the perfect information is different. Consider the situation where a defect has been reported. Joe knows that if the car is a peach he should buy it without the guarantee and make $60, and that if it is a lemon he should buy it with the guarantee and make $40. In the absence of any test result, the stranger would report a peach with probability 0.8 and a lemon with probability 0.2, so that Joe's expected profit would be 0.8(60) + 0.2(40) = $56. However, now that a defect has been reported, the probabilities of a peach and a lemon have changed to 0.4 and 0.6, respectively. Thus, the expected profit using perfect information is now 0.4(60) + 0.6(40) = $48. It is from this quantity that the expected value of state T_2D_1, \$32, must be subtracted in order to obtain the EVPI of \$16 entered in the oval above node T_2D_1.

Similarly, we see that if no defect had been reported, the probabilities of peach and lemon would be 0.9 and 0.1, and the expected profit of using perfect information would be 0.9(60) + 0.1(40) = \$58. When we subtract the \$44 value of node T_2D_1', we obtain the \$14 figure for the EVPI that is pertinent to that node.

There is one other observation we should make. The values of perfect information at nodes T_2D_1 and T_2D_1' are \$16 and \$14. The probabilities of arriving in each of these states is 0.2 and 0.8, respectively. Consequently, the expected value of what the expected value of perfect information will be after the mechanic report is 0.2(16) + 0.8(14) = \$14.40, in agreement with our previous value for this quantity entered in the oval at node T_2. Thus, it is possible to compute the expected value of perfect information at each point in the tree by using only the values of perfect information pertinent to the final decision on buying the car and the probabilistic structure of the tree. We shall have more to say of these quantities later.

Let us now move forward to an analysis of test option T_3. In this case, as you recall, two systems on the car--the fuel and electrical systems--are subjected to test and then the results of both tests are reported to Joe. The possible reports are that 2, 1, or 0 defects were found. These three events are represented by the three branches, D_1D_2, $D_1D_2' + D_1'D_2$, and $D_1'D_2'$ that are drawn to the right of node T_3 in the tree of Figure 5. Note that once more we have written under branch T_3 the amount to be paid to the mechanic for performing the tests. When the mechanic's report is known, Joe must make a decision on buying the car, using a decision tree similar to that shown in Figure 4. The expected earnings at the tips of the tree remain the same, but once more we require a new assignment of the ultimate probabilities of a peach and a lemon as a result of the mechanic's report. These probabilities may be found from Nature's tree in Figure 6. The probabilities necessary are: $p(D_1D_2)$, $p(D_1D_2' + D_1'D_2)$, $p(D_1'D_2')$, $p(P|D_1D_2)$, $p(P|D_1D_2' + D_1'D_2)$ and $p(P|D_1'D_2')$. By using the numbers on the nodes of Nature's tree and the basic relations of probability theory, we obtain the following results:

$$p(D_1D_2) = p(PD_1D_2) + p(LD_1D_2) = 0 + 1/15 = 1/15 = 0.067$$

$$p(D_1D_2' + D_1'D_2) = p(PD_1D_2') + p(LD_1D_2') + p(PD_1'D_2) + p(LD_1'D_2)$$

$$= 6/75 + 4/75 + 6/75 + 4/75 = 4/15 = 0.266$$

$$p(D_1'D_2') = p(PD_1'D_2') + p(LD_1'D_2') = 48/75 + 2/75 = 2/3 = 0.667$$

$$p(P|D_1D_2) = p(PD_1D_2)/p(D_1D_2) = \frac{0}{1/15} = 0$$

$$p(P|D_1D_2' + D_1'D_2) = [p(PD_1D_2') + p(PD_1'D_2)]/p(D_1D_2' + D_1'D_2)$$

$$= [\frac{6/75 + 6/75}{4/15}] = 3/5 = 0.6$$

$$p(P|D_1'D_2') = p(PD_1'D_2)/p(D_1'D_2') = \frac{48/75}{2/3} = 24/25 = 0.96$$

Thus, we see that after Joe has committed himself to the test, there are probabilities of 0.067, 0.266, and 0.667 that the mechanic will report 2, 1, or 0 defects. These numbers are entered in Figure 5 on the three branches leaving the chance node T_3. If two defects are reported, $p(P|D_1D_2)$ shows that Joe will make his decision with the satisfying, but disappointing, knowledge that the car is certain to be a lemon. This information is indicated on the tree by the 0 and 1 entered on the branches P and L that originate in chance nodes $T_3D_1D_2B$, $T_3D_1D_2G$, and $T_3D_1D_2R$. The expected earnings from making each of the decisions B, G, and R are -100, 40, and 0. Consequently, the most profitable act for Joe is to buy the car with the guarantee, even though it is a lemon, and thus earn the $40 profit. This preferred decision is shown by the solid arrowhead on the branch G following node $T_3D_1D_2$; the profit of $40 is recorded in the square box above that node.

The situation when only one defect is reported is very similar. In this case, we observe from $p(P|D_1D_2' + D_1'D_2)$ that the probabilities of a peach and a lemon are 0.6 and 0.4. These probabilities appear on the P and L branches at the ends of the sub-tree that follows node $T_3(D_1D_2' + D_1'D_2)$. The expected earnings of the three acts B, G, and R are 0.6(60) + 0.4(-100) = -$4; 0.6(20) + 0.4(40) = $28; and $0. Once more, the highest expected profit will result if Joe buys the car with the guarantee. Note that he does this even though the car is still more likely to be a peach than a lemon. Again we record the expected profit of $28 in the square boxes over the decision node and indicate the preferred decision with a solid arrowhead.

If no defects are reported, the car is almost certain to be a peach; there is only a 4 percent chance of its being a lemon. When we compute the expected profit of the three decisions following node $T_3 D_1'D_2'$, using the probability 0.96 for a peach and 0.04 for a lemon, we find that buying the car without a guarantee pays $53.60, buying it with a guarantee pays $20.00, and not buying it at all pays nothing. Thus, Joe is best advised to buy the car without the guarantee, as represented by the solid arrowhead on the B branch following node $T_2D_1'D_2'$ and by the $53.60 entered in the square box over that node.

We have now calculated the optimum decision and maximum expected earnings for each possible mechanic's report under test plan T_3. As we know, chance determines the actual reporting, but we also have learned the probabilities of the mechanic's reporting 2, 1, or 0 defects, and have entered them in the decision tree. The expected profit to Joe when he is waiting to learn the test results is thus 0.067(40) + 0.266(28) + 0.667(53.60), or $45.87. Of course, in order to reach a situation with this expected value, Joe had to pay out $13. Hence, his expected earnings from test T_3 are $32.87. Since this number is higher than the expected profit under either the policy of no testing or of testing only one system, the option of testing two systems for $13 is the most favorable yet evaluated. However, its margin over test plan T_2 is only $0.28.

We might, at this point, examine once again the value of the perfect information offered by the stranger. As we found earlier, this quantity can be calculated at each node of the decision tree simply by subtracting from the expected earnings with perfect information the expected earnings at that node as given in the pertinent square boxes. Accordingly, since the expected profit using perfect information is still $56 before the test results are known, the value of perfect information when Joe has decided to use test T_3 is $23.13 (i.e., $56 - $32.87) before he has paid the mechanic, and $10.13 (i.e., $56 - $45.87) after the mechanic has received his $13.

However, after the test results have been reported, the expected profit using perfect information is different from $56. Remember that Joe can make a profit of $60 if he knows the car is a peach, and of $40 if he knows it is a lemon. From our tree we see that the pair [p(P),p(L)] takes on the values (0,1), (0.6,0.4), and (0.96,0.04) according to whether 2, 1, or 0 defects were discovered. Joe's expected profit using perfect information is thus $40, $52, or $59.20, depending on the defect situation

Since we have already calculated the expected values of these states to be \$40, \$28, and \$53.60 without perfect information, the EVPI's for them must be \$0, \$24, and \$5.60, respectively. As before, if we weigh these three numbers with the respective probabilities of 2, 1, or 0 defects being reported, namely, 0.067, 0.266, and 0.667, we obtain the figure of \$10.13, formerly computed as the value of perfect information at node T_3.

An observation of particular importance may be based on these numbers: Although we would expect the amount Joe would be willing to pay the stranger for his perfect information to decrease after he is committed to a test plan, it is not necessary for this situation to obtain for any experimental outcome, but only on the average. Thus, after Joe has decided to follow test plan T_3, he establishes that the value of perfect information to him is only \$23.13. However, if the mechanic should report that he had found exactly one defect in the car, Joe now notices that the value of perfect information has increased to \$24, a net gain of \$0.87. This means that if Joe had decided on T_3, and the stranger's price for his information was \$23.50, Joe would refuse the information and go ahead with the test, but then willingly pay \$24 for the same information if the mechanic reports only one defect.

This result is really not too surprising when we realize that Joe had already considered the change of being placed in a situation where the expected value of perfect information is \$24 when he made his optimum decision at node T_3. When Joe contracted for test plan T_3 he had to consider how every possible outcome of the test--2, 1, or 0 defects--would affect his state of knowledge about the type of car on the lot. If no defects were found, then Joe would be very confident that the car is a peach and would be willing to pay only \$5.60 to remove his remaining uncertainty. If two defects were found, then the car is surely a lemon and the stranger cannot tell Joe anything of value. However, if the mechanic reports one defect, then Joe does not expect to make any more money from this point into the future than he would have made if no tests whatever had been performed; \$28. It is important to note that the value of perfect information is \$24 in this situation rather than the \$28 figure applicable in the absence of tests. This difference is, of course, due to the fact that the probability that the stranger will discover that the car is good has fallen from 0.8 to 0.6. Thus, we see that although the expected value of perfect information cannot increase on an average value basis in such trees, it is possible for it to increase for some of the chance outcomes.

Now let us turn to the evaluation of test plan T_4. Under this option the transmission is tested for \$10; when the outcome of this test is reported, it is possible to have the mechanic test the differential for an additional cost of \$4. Such a test procedure is representative of a large class of experimental plans which we may call sequential test. Such processes are characterized by the option to decide whether or not to continue testing after the results of the initial tests are known.

The decision tree pertinent to T_4 is shown in Figure 5. The development of this tree is once more most easily understood by considering the chronological sequence of the decisions that must be made and their outcomes. The payment of \$10 to initiate this test plan is indicated by a -10 under the branch T_4. The next event that will occur is the report of the mechanic about whether he found a defect in the transmission. Th we establish a chance point that generates branches D_1 and D_1'. Regardless of whether or not a defect has been found, Joe must make a decision on the continuation of the test. His two possible actions, continue on to test the differential, and stop testing, are shown by the two branches named CONTINUE and STOP that leave decision nodes T_4D_1 and T_4D_1'. Both of the CONTINUE branches are labeled -4 to indicate the cost of requesting the testing of the differential.

If Joe decides to stop the testing program after hearing the report on the transmission, he will have to make his final decision on buying the car having only the information that either a defect was or was not found. But these two situations were also encountered under test plan T_2 after the mechanic had made his report. Since Joe finds himself in the same position they must have the same value to him. (Remember that the money paid out for the performance of the test is a fixed cost at this point and so does not affect the <u>future</u> expected earnings.) Consequently, we should enter in the tree at the tips of the T_4D_1 STOP and T_4D_1' STOP branches the same values to be found at nodes T_2D_1 and T_2D_1', respectively. We shall denote these values by $v(T_2D_1)$ and $v(T_2D_1')$; we see that $v(T_2D_1)$ = \$32, $v(T_2D_1')$ = \$44.

The situation if Joe decides to continue testing after hearing the mechanic's report on his first test is analogous but not identical. If the CONTINUE option is followed, the next event to take place is the report by the mechanic on whether he found a defect on his second test. Thus, we create chance points at the T_4D_1 CONTINUE and T_4D_1' CONTINUE nodes and D_2 and D_2' branches emanating from them. However, when we receive the second report from the mechanic, our total information is that in two tests 2, 1, or 0 defects have been found in the car. Thus, we are in the same positions as we were under test option T_3 after the mechanic's report was known. The appropriate value for T_4D_1 CONTINUE D_2 is, therefore, $v(T_3D_1D_2) = 40$; for T_4D_1 CONTINUE D_2' and T_4D_1' CONTINUE D_2 it is $v(T_3D_1D_2') = v(T_3D_1'D_2) = 28$; and for T_4D_1' CONTINUE D_2' it is $v(T_3D_1'D_2') = 53.60$. These numbers have been placed at the pertinent tips of the T_4 test plan tree.

We have now been able to evaluate the terminal points of the T_4 tree by identifying them with nodes that had been considered earlier. It remains to place the relevant probabilities on the chance nodes in this tree so that we can proceed to make a judgment about the utility of this option. Once more we find that Nature's tree of Figure 6 supplies the probabilistic information we require. The probabilities of the branches D_1 and D_1' that leave node T_4 have already been computed in the tree for test plan T_2; they are 0.2 and 0.8. The only remaining probabilities are $p(D_2|D_1)$ and $p(D_2'|D_1)$ to go to the right of node T_4D_1 CONTINUE and the probabilities $p(D_2|D_1')$ and $p(D_2'|D_1')$ to go in the analogous place on the D_1' fork. Our task is again simplified by the fact that the sum of all probabilities emerging from a chance node must be 1. From the definition of conditional probability we can write:

$$p(D_2|D_1) = p(D_1D_2)/p(D_1)$$

and

$$p(D_2|D_1') = p(D_1'D_2)/p(D_1')$$

From Figure 6 we find

$$p(D_2|D_1) = \frac{p(D_1D_2)}{p(D_1)} = \frac{p(PD_1D_2) + p(LD_1D_2)}{p(PD_1D_2) + p(LD_1D_2) + p(PD_1D_2') + p(LD_1D_2')}$$

$$= \frac{1/15}{1/5} = 1/3$$

and

$$p(D_2|D_1') = \frac{p(D_1'D_2)}{p(D_1')} = \frac{p(PD_1'D_2) + p(LD_1'D_2)}{p(PD_1'D_2) + p(LD_1'D_2) + p(PD_1'D_2') + p(LD_1'D_2')}$$

$$= \frac{2/15}{4/5} = 1/6$$

Of course, most of the probabilities in this calculation were computed earlier in the evaluation of test options T_2 and T_3. However, their repetition at this time serves to emphasize the basic role of Nature's tree. Finally we have

$$p(D_2'|D_1) = 1 - p(D_2|D_1) = 2/3$$

and

$$p(D_2'|D_1') = 1 - p(D_2|D_1') = 5/6$$

When the four conditional probabilities we have just found are entered in their appropriate places in the tree for test option T_4, we are ready to proceed with the expected value computation.

At node T_4D_1' CONTINUE there is a 1/3 probability of the value 40 and a 2/3 probability of the value 28. The expected value of this node is thus 1/3(40) + 2/3(28) = \$32, as indicated in the square box. The node T_4D_1 STOP also has a value of \$32; however, in order to reach node T_4D_1 CONTINUE, \$4 must be paid and so when viewed from the left end of the T_4 CONTINUE branch, this action is worth only \$28. Consequently, Joe is be advised to take the stop branch at this juncture and thereby make the value of decision node T_4D_1 equal to \$32. Such a decision has been indicated on the tree.

At node T_4D_1' CONTINUE we see a 1/6 probability of the value 28 and a 5/6 probability of the value 53.60. The expected value of node T_4D_1' CONTINUE is 1/6(28) + 5/6(53.60) = \$49.33. Even after the \$4 expense for continuing the test has been included, this act still has an expected value of \$45.33, an amount slightly in excess of the \$44 value to be expected if branch T_4D_1' STOP is followed. The solid arrowhead and the number in the square box at node T_4D_1' correspond to this decision.

At chance node T_4 there is an 0.2 probability of the mechanic's reporting that he found a defect on the first test and thus causing us to expect a profit of \$32. With probability 0.8 we shall expect earnings of \$45.33 because he has reported no defect. Therefore, the expected value of being at decision node T_4 is 0.2(32) + 0.8(45.33) = \$42.66. Since it is necessary to pay \$10 for the first test, the expected value of test plan T_4 is \$32.66, as shown in the square box to the left of branch T_4.

The expected value of perfect information can be easily calculated for this test plan. All that is necessary is to copy the EVPI numbers corresponding to the value expressions at the tips of the T_3 tree. For example, the EVPI in the oval at node $T_3D_1D_2$ is 0; this figure is placed in the oval at the node T_4D_1 CONTINUE D_2 where $v(T_3D_1D_2)$ has already been copied. When this has been done for all six terminating nodes of the T_4 tree, the EVPI of all other nodes in the tree can be obtained by taking expected values of these quantities at chance nodes and taking the route indicated by the solid arrowhead at decision nodes. The solid arrowhead will always correspond to the act that minimizes the expected value of perfect information. To illustrate, at node T_4D_1 CONTINUE, the value of perfect information will be 0 if a second defect is reported and 24 if not. Weighting with the (1/3, 2/3) probabilities of these events, we obtain \$16 at this node, or \$20 before the \$4 cost of the second test is paid. At node T_4D_1 STOP the expected value of perfect information is

\$16--therefore, the STOP alternative should be selected and the EVPI at node T_4D_1 is \$16. The reader should finish the calculation of the EVPI's in the T_4 tree to satisfy himself that the entries in Figure 5 are correct.

We have now evaluated all four test plans. From Figure 5 we can see that the expected profits from options T_1, T_2, T_3, and T_4 are, respectively, \$28, \$32.60, \$32.87, and \$32.66. Since plan T_3, that of testing two systems, has the highest expected profit, it is the one indicated by a solid arrowhead after the initial decision node. However, the evidence of the tree should be interpreted not to mean that T_3 is the best test plan, but rather that any of the plans T_2, T_3, T_4 will be slightly less than \$5 better than the option of no testing, on the average. The big payoff is not in the selection of a particular test plan, but rather in the decision to do some testing.

Let us review these test plans to show their operational character. If Joe does no testing, he will buy the car without a guarantee. If he follows plan T_2, he will buy the car with the guarantee if a defect is found in the system tested and he will buy it without the guarantee if no defect is discovered. Our evaluation of plan T_3 shows that Joe should buy the car without a guarantee only if no defects are found in the two systems tested, and buy it with the guarantee otherwise.

Finally, if T_4 is chosen, Joe should stop further testing if a defect is discovered on the first test and continue testing otherwise. If a defect is found in the first test on the transmission, then Joe should buy the car with a guarantee, as we see from the decision at node T_4D_1. However, if the transmission is not defective, then depending on whether the further test of the differential does or does not reveal a defect, Joe will either buy the car with or without the guarantee, in that order. This is determined by locating the ultimate outcomes of the T_4D_1' CONTINUE D_2 and T_4D_1' CONTINUE D_2' branches in the T_3 tree. It is interesting to note that the reason the nodes T_4D_1 CONTINUE and T_4D_1 STOP have the same values is that even if the tests were continued at this point, Joe's decision would be to buy the car with a guarantee regardless of how the second test came out. Since the test cannot affect the decision, it is not worthwhile to pay anything for the privilege of making it. The tree implies just this result.

We have now seen that after all the calculations have been performed, the final decision offers no real problem. Since test plan T_3 is most favorable by a small amount, Joe will probably decide to follow it. The expected value of perfect information is \$23.13 when plan T_3 is used; therefore, the stranger's \$25 price for this information once more looks too high. Unless the price is lowered below \$23.13, Joe should proceed with having the fuel and electrical systems tested at a cost of $13. He will buy the car without the guarantee only if no defects are found and with it otherwise. Joe's expected profit from this plan of action is $32.87, an increase of $4.87 over what he expected to make without considering testing. Of course, by this time Joe may have decided that he would rather walk than do all this calculation!

The stranger with the perfect information has witnessed a good deal of vacillation in what Joe is willing to pay him. The EVPI was $20 initially, $28 after the guarantee was introduced, and $23.13 under test plan T_3. From the stranger's point of view, the guarantee was good news, but the test options were bad news. However, even if Joe decides to follow T_3, the stranger can still sell his knowledge to Joe by reducing its price below $23.13. Joe will realize an increase in profit equal to the difference between $23.13 and what he pays the stranger.

Let's suppose, however, that the stranger had stepped away by the time Joe had completed his deliberations and that when he had reappeared, Joe had already paid the mechanic the $13 necessary to carry out test plan T_3. Even at this point, the stranger can make some money if he considers this situation carefully. His immediate problem is that: should he offer his perfect information to Joe at a reduced price before or after Joe has received the test results from the mechanic, and what should his price be? Since Joe already has paid the mechanic, the EVPI to Joe is now $10.13 according to the figure in the rounded box above node T_3; Joe will presumably pay any amount less than $10.13 to get perfect information. Now the probabilities that the mechanic will report 2, 1, or 0 defects are 1/15, 4/15, and 2/3. In fact, it is on the basis of these probabilities and the EVPI of 0, 24, and 5.60 recorded at nodes $T_3D_1D_2$, $T_3(D_1D_2' + D_1'D_2)$ and $T_3D_1'D_2'$ that Joe established the EVPI at node T_3 to be $10.13. However, let us suppose that the stranger had determined the one piece of information that Joe does not have; namely, he has found out whether or the car is a lemon simply by observing the serial number. Using this information, the stranger can calculate new probabilities of the various reports of the mechanic according to whether the car is a peach or a lemon. He will thus obtain an expected value of EVPI after the report is known that will different from Joe's estimate of $10.13. If the stranger's estimate is higher than Joe's, he will do better in his expected value by not offering his perfect information until the outcome of the test is known. On the other hand, if the stranger's estimate is lower than Joe's, he should offer his information immediately.

The calculations involved are quite straightforward. If the stranger determines that the car is a peach, then the three probabilities that should be used to weigh the numbers 0, 24, and 5.60 should be $p(D_1D_2|P)$.

$p(D_1D_2' + D_1'D_2'|P)$, and $p(D_1'D_2'|P)$. If the car is found to be a lemon, then the appropriate probabilities are $p(D_1D_2|L)$, $p(D_1D_2' + D_1'D_2|L)$, and $p(D_1'D_2'|L)$. These probabilities are computed from Nature's tree of Figure 6 as follows:

$$p(D_1D_2|P) = \frac{p(PD_1D_2)}{p(P)} = \frac{0}{0.8} = 0$$

$$p(D_1D_2' + D_1'D_2|P) = \frac{p(PD_1D_2') + p(PD_1'D_2)}{p(P)} = \frac{0.08 + 0.08}{0.08} = 1/5$$

$$p(D_1'D_2'|P) = \frac{p(PD_1'D_2')}{p(P)} = \frac{0.64}{0.8} = 4/5$$

$$p(D_1D_2|L) = \frac{p(LD_1D_2)}{p(L)} = \frac{5/75}{0.2} = 1/3$$

$$p(D_1D_2' + D_1'D_2|L) = \frac{p(LD_1D_2') + p(LD_1'D_2)}{p(L)} = \frac{8/75}{0.2} = 8/15$$

$$p(D_1'D_2'|L) = \frac{p(LD_1'D_2')}{p(L)} = \frac{2/75}{0.2} = 2/15$$

The expected value of the expected value of perfect information that will exist after the results of the test are known is computed for the states of knowledge of Joe, of the stranger when the car is a peach, and of the stranger when the car is a lemon in Table II.

The important thing to note is that the stranger expects the EVPI to be only \$9.28 after the results of the experiment are known if the car is a peach, but \$13.55 if the car is a lemon. In other words, considering that the EVPI of perfect information is \$10.13 in Joe's eyes, the stranger expects the EVPI to be lower than Joe's when the results are reported if the car is a peach and higher if it is a lemon. It is, therefore, prudent for the stranger to sell the perfect information to Joe before the mechanic calls for, say, \$10 if the car is a peach, but to wait until after the mechanic's report before offering it if the car is a lemon. Since $p(D_1D_2) = p(D_1D_2|P) \times p(P) + p(D_1D_2|L) \times p(L)$, etc., and since $p(P) = 0.8$, $p(L) = 0.2$, $10.13 = 0.8(9.28) + 0.2(13.55)$. That is, the expectation of the EVPI from Joe's point of view is the expected value of the EVPI from the stranger's

Table II

The expected value of what the expected value of perfect information will be when the results of test T_3 are known.

	EVPI of Report	Probabilities of the Report as Seen by: Joe	The stranger when the car is a peach	The stranger when the car is a le-
Two defects, D_1D_2	0	$p(D_1D_2) = 1/15$	$p(D_1D_2 \mid P) = 0$	$p(D_1D_2 \mid L) = 1/3$
One defect, $D_1D_2' + D_1'D_2$	24	$p(D_1D_2' + D_1'D_2) = 4/15$	$p(D_1D_2' + D_1'D_2 \mid P) = 1/5$	$p(D_1D_2' + D_1'D_2 \mid L) = 8/15$
No defects, $D_1'D_2'$	5.60	$p(D_1'D_2') = 2/3$	$p(D_1'D_2' \mid P) = 4/15$	$p(D_1'D_2' \mid L) = 2$
EVPIs weighed with probabilities		10.13	9.28	13.55

point of view. This computation provides the essential reconciliation between the viewpoints of the buyer and seller of perfect information.

However, to show that our problem still has hidden facets, we suddenly realize that a competitive-game aspect has appeared. If the stranger offers his information before the test results are known, and if Joe knows that the stranger has reasoned according to the previous paragraph, then Joe is certain that the car is a peach. Similarly, if the offer is made after the test, Joe is certain that the car is a lemon. In either case, Joe will have received perfect information without paying for it. This forces the stranger to randomize his strategy, and so on and on and on. We shall give up trying to help Joe at this point.

Well, at last Joe is driving away in his Spartan, having used test plan T_3 and abided by the results. A most human question is: Did he make a good decision or didn't he? The answer to this question does not depend at all on whether his new car is actually a peach or a lemon. We must make a distinction between a good decision and a good outcome. Joe made a good decision because he based it on logic and his available knowledge. Whether or not the outcome is good depends on the vagaries of chance.

value of clairvoyance on any uncertainty represents an upper bound on what any information-gathering process that offers to shed light on the uncertainty might be worth.

For example, if we find in the medical problem that the value of clairvoyance on whether or not we are going to die from the drug is \$500, then that means that we should not pay more than \$500 for any literature search or anything else that would provide only imperfect information with respect to whether or not we are going to have this problem.

That is a revelation in itself to many people—the fact that one can establish a hard dollars and cents number on the value of information to us in making a decision, and hence can use that number to guide what information-gathering processes we might participate in.

The Medical Problem Evaluated. It is hard to demonstrate very simply how to do such a calculation, but let us try by taking the medical example and putting some numbers in it (see Fig. 6). The patient has the choice of taking the magic medicine or not. If he does not take it, then he is going to get the pain; we will consider that as a reference point of value \$0. If he does take the medicine, let us suppose he has one chance in a thousand of dying and 999 in a thousand of getting the instant cure. We have also put in numbers here saying that the cure is worth \$100 more than the pain. He is a relatively poor person, but he would pay \$100 more for the painless cure than he would for spending a painful day in the hospital. Now for death—what is the value of life to a person? This person has set the value of his life at \$100,000.

Notice that we "set" the value of his life. What is meant by this is that he wants the designers of public highway systems and airplanes to use the number \$100,000 in valuing his life. Why does he not make it a million dollars? If he does, he will have more expensive rides in airplanes, more expensive automobiles, and so forth. He does not get something for nothing. If he makes it too small, he had better be

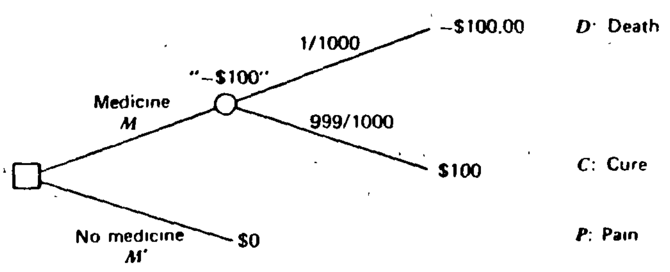


Figure 6 The medical decision.

hospital with pain, then the number p would certainly give us insight into their attitude toward risk and would allow us indeed to start building a description of their risk preference.

The Logical Decision. When this has been done, when we have carried out this procedure and have established preferences, the values placed on outcomes, the attitude toward time, the attitude toward risk (and there is a methodology for doing all of this), when we have established the models necessary for the decision one is making and have assessed probabilities as required on the uncertain variables, then we need nothing but logic to arrive at a decision. And a good decision is now very simply defined as the decision that is logically implied by the choices, information, and preferences that we have expressed. There is no ambiguity from that point on—there is only one logical decision.

This allows us to begin to assign praise or blame to the process of making the decision rather than to the ultimate outcome. We can do an analysis of the decision and make sure it is a high quality decision before we learn whether or not it produced a good outcome. This gives us many opportunities. It gives us the opportunity to revise the analysis—to look for weak spots in it—in other words, to tinker with it in the same way we can tinker with an engineering model of any other process.

The Value of Information

If this were all decision analysis did, it would be impressive enough, but from it we also get other benefits. We obtain sensitivities to the various features of the problem and we learn something that I think is unique to decision analysis called the "value of information." The value of information is what it would be worth to resolve uncertainty once and for all on one or more of the variables of the problem. In other words, suppose we are uncertain about something and do not know what to do. We postulate a person called a "clairvoyant." The clairvoyant is competent and truthful. He will tell us what is going to happen—for a price. The question is what should that price be. What can we afford to pay to eliminate uncertainty for the purpose of making this decision?

Of course we do not have real clairvoyants in the world—at least not very often—but the clairvoyant plays the same role in decision analysis as does the Carnot engine in thermodynamics. It is not the fact that we can or cannot make it, but that it serves as a bench mark for any other practical procedure against which it is compared. So the value of clairvoyance on any uncertainty represents an upper bound on what any information-gathering process that offers to shed light on the uncertainty might be worth.

For example, if we find in the medical problem that the value of clairvoyance on whether or not we are going to die from the drug is $500, then that means that we should not pay more than $500 for any literature search or anything else that would provide only imperfect information with respect to whether or not we are going to have this problem.

That is a revelation in itself to many people—the fact that one can establish a hard dollars and cents number on the value of information to us in making a decision, and hence can use that number to guide what information-gathering processes we might participate in.

The Medical Problem Evaluated. It is hard to demonstrate very simply how to do such a calculation, but let us try by taking the medical example and putting some numbers in it (see Fig. 6). The patient has the choice of taking the magic medicine or not. If he does not take it, then he is going to get the pain; we will consider that as a reference point of value $0. If he does take the medicine, let us suppose he has one chance in a thousand of dying and 999 in a thousand of getting the instant cure. We have also put in numbers here saying that the cure is worth $100 more than the pain. He is a relatively poor person, but he would pay $100 more for the painless cure than he would for spending a painful day in the hospital. Now for death—what is the value of life to a person? This person has set the value of his life at $100,000.

Notice that we "set" the value of his life. What is meant by this is that he wants the designers of public highway systems and airplanes to use the number $100,000 in valuing his life. Why does he not make it a million dollars? If he does, he will have more expensive rides in airplanes, more expensive automobiles, and so forth. He does not get something for nothing. If he makes it too small, he had better be

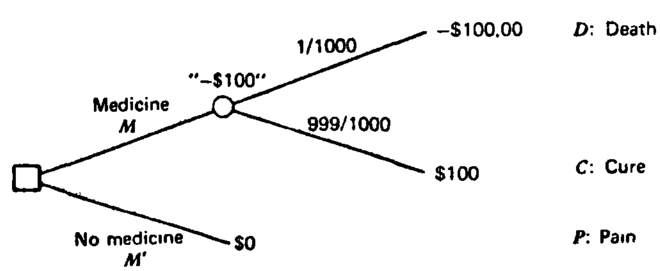


Figure 6 The medical decision.

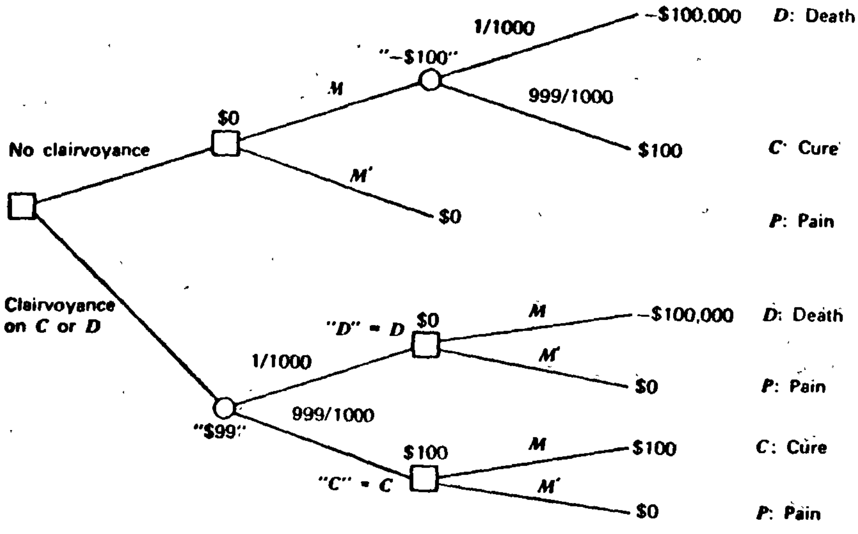


Figure 7 Value of clairvoyance computation.

wearing a helmet every time he enters his car. So it is a decision for him as to what number he wants the decision makers to use in this completely logical world that we are talking about.

The number -$100 in quotes (in Fig. 6) means that our patient has said that one chance in a thousand of losing $100,000 and 999 chances in a thousand of winning $100 has a value to him of −$100. In other words, we have to pay him $100 to get him to take on this uncertain proposition. It is clear that, comparing -$100 to $0, he is better off deciding not to take the medicine. So for him the probabilities, values, and attitude toward risk leading to the −$100 assessment of this whole uncertain proposition, the best decision is to forget about the medicine.

Clairvoyance. Now the clairvoyant arrives. If the individual we are talking about does not patronize the clairvoyant, then he does not take the medicine and makes nothing. If, on the other hand, he does buy the clairvoyance on the question of whether death will occur, what will happen? First, the clairvoyant will tell him whether he is going to die if he takes the medicine (see Fig. 7). We have "*D*" in quotes here, meaning that the clairvoyant says he is going to die, equivalent to his actually dying because the clairvoyant is truly prophetic. "*C*" means the clairvoyant says he is going to be cured. Since the probability the clairvoyant will say he is going to die has to be the same as the probability that he really will die, he has to assign one chance in a thousand to getting that report from the clairvoyant. Now suppose the clairvoyant says he is going to die. Obviously, he ought not to take the medicine in that case, and he will make nothing. If the clairvoyant says he is going to be cured without dying, then he is better off taking the medicine, and he will make $100. Since the payoff from the clairvoyant's saying that he is going to die is $0 and from not going to die is $100, and since there are 999 chances out of a 1000 that the clairvoyant will say he is not going to die, just by looking at that lottery we can see it will be worth almost $100 to him. He has 999 chances out of a 1000 of winning $100, and only one chance in 1000 in winning $0.

Let us suppose he evaluates the whole uncertain proposition at $99. If he does not buy the clairvoyance, he is looking at $0; if he does buy it, he is looking at a proposition that is worth about $99 to him. Thus, the value of the clairvoyance would be $99.

So here is an uncertain proposition with all kinds of big numbers running around in it, yet a very simple calculation based on his attitudes toward risk, life, death, and pain says he should not be willing to pay more than $99 to know for sure whether he would get the unfortunate event of death if he should take the drug.

Similarly, in any other decision problem—and there are some very, very complicated ones, involving many jointly-related variables—we can establish an upper bound on the value of information-gathering on any aspect of that problem. We can subsequently determine the best information-gathering strategy to precede the actual making of the decision.

The Decision Analysis Cycle

Let us begin with a word on methodology and then go on to an example. When doing a decision analysis it helps to organize your thoughts along the following lines. First, constructing a deterministic model of the problem and then measuring the sensitivity to each of the problem variables will reveal which uncertainties are important. Next, assessing probabilities on these uncertainties and establishing risk preference will determine the best decision. Finally, performing a value of clairvoyance analysis allows us to evaluate getting information on each of the uncertainties in the problem. The problem could be very complicated, involving many variables and months of modelling and analysis, but the basic logic is the same. The phases are: deterministic to evaluate sensitivities, probabilistic to find the best decision, and informational to determine in what direction new information would be most valuable. Of course you can repeat the process as many times as is economically valuable.

That is just to give an idea of how one does a professional decision analysis. Let us now turn to a case history to demonstrate the kind of problem that can be attacked in this way. Everything said so far has a naive ring to it. We can talk about betting on next year's salary, but we are really interested in not just the theory of decision analysis, but the practice of it.

CASO MEXICO

A Power System Expansion Decision

Let us take an example from the public area. It concerns the planning of the electrical system of Mexico and is one of the largest decision analyses that has been done. It has been chosen because it comes closest to a problem in systems engineering. The specific question posed was: Should the Mexican electrical system install a nuclear plant and, if so, what should its policy toward nuclear plants in general be? Of course, we can not really answer that question without deciding how they are going to expand, operate, and price their system over time from here on out. So the real question is how to run the electrical system of Mexico for the rest of the century (see Fig. 8).

The Mexican electrical system is nationalized and very large—the size of several United States state-sized electrical systems. Because it is a complete national system, its planners have unique problems and also unique opportunities. The basic idea in working this problem was to look first of all at the various environmental factors that might influence the decision and then to look at the various measures of value that would result from particular methods of operation.

The Inputs

First, let us discuss the inputs. There are four input models: financial, energy, technology, and market. The financial models are concerned with the financial environment of the Mexican electrical system both in the world and the Mexican financial market. The inputs that these models provide are the amounts of money and the rates at which money can be borrowed from that source over time, with uncertainty if necessary. An input to this model is something called x which is picked up from the lower right. It is the book profit of the system. There is a feedback between the profitability of the system over time and the amount that it can borrow to support future

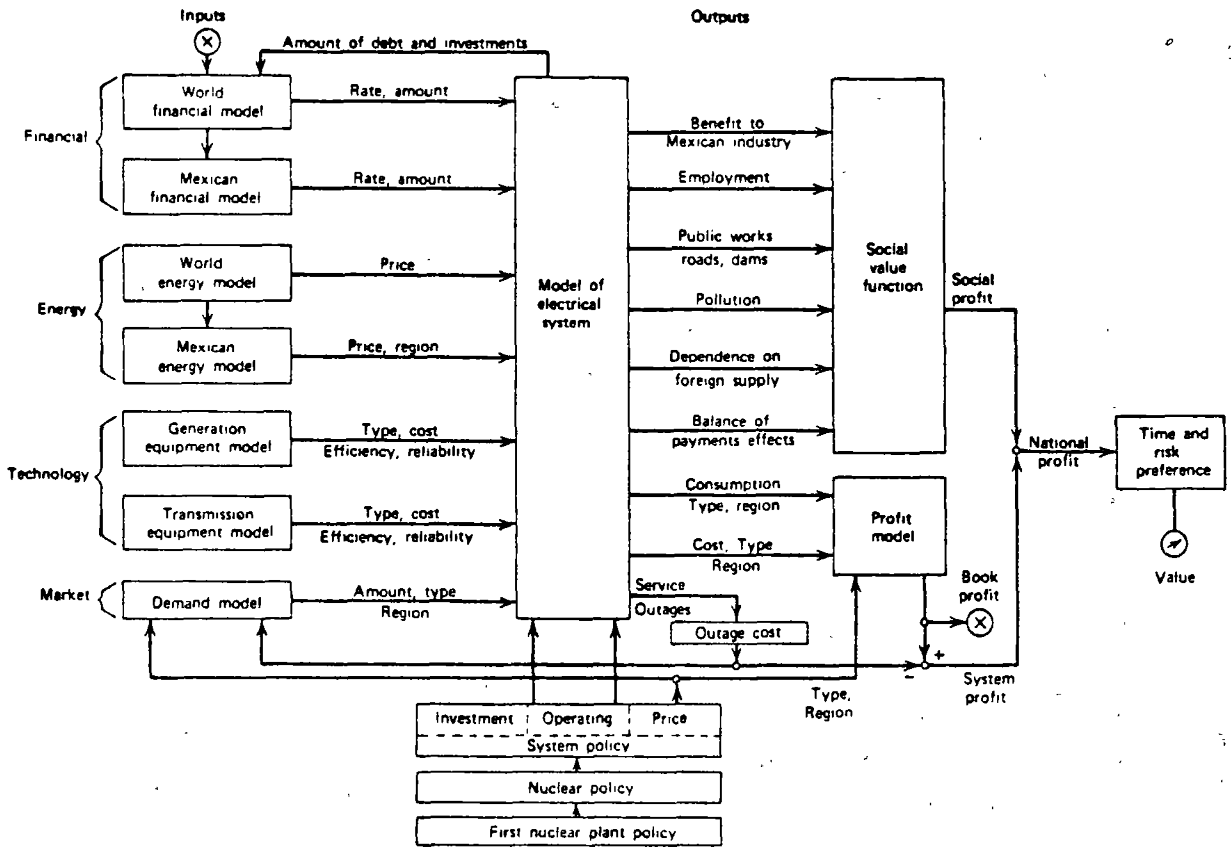


Figure 8 A decision analysis model of the Mexican electrical system.

expansion. The current amounts of debt and investment are also fed back.

The second type of input is energy costs, both in the world market and in the Mexican market. The interesting thing about Mexico is that it has just about every type of energy available: coal, oil, uranium, and thermal fields, and, of course, there are world markets in uranium and oil, at least, whose price movements over time would influence the economics of the Mexican system.

Next comes technology. This model describes generation and transmission equipment according to type, cost, efficiency, reliability. It includes such features as the advent of better reactors in the future and the possibilities of new and improved transmission systems which might make some of their remote hydro locations more desirable.

The last input model is the demand or market model, indicating by type and region the amount of electricity that would be consumed, given a pricing policy and given a quality of service. So these are the inputs to the model of the Mexican electrical system, which can then be run.

The Outputs

We will not go into the details of the rather sophisticated model which was prepared to describe operation and expansion of the Mexican electrical system. Of more interest in this discussion is the kind of outputs that were produced. There were the very logical ones of the consumption of electricity and the cost of producing the electricity by region to give a profit for the electrical system. This profit was what might be called the operating profit or book profit of the system, and is what the investor would see if he looked at the books of the Mexican electrical system. One modification to that profit which was considered was an economic penalty for system outages. A measure of the service provided by the system is added to the book profit to give something called system profit—which the investor does not see, but which the designer of the system does see. This penalty makes him unwilling to make a system that has outages for hours at a time, even though it might be more profitable if he looked only at the book profit.

The Social Value Function. But what is unusual about the outputs here is that many of them do not appear on the balance sheet of the corporation at all, but are what we might call social outputs; they enter into something called the social value function.

The decision maker in this case was the head of the Mexican electrical system. He felt many pressures on his position—not just the regular financial pressures of operating an electrical system, but social pressures coming about from the fact that his is a nationalized industry. For example, one of the things that was of concern to him was the benefit to Mexican industry. What would be the Mexican manufactured component of any system that might be installed? Another one was employment. How many Mexicans would be employed at what level if they went one route as opposed to another? Now we can see that the way we design the system is going to have major impacts on these kinds of outputs. If we have a nuclear system, then we might provide training for a few high-level technicians, but most of the components would be manufactured abroad; we do not have the army of Mexican laborers that we would if we built a hydro system in a remote location.

Another side effect is the public works that are produced by the generation choice. For example, with hydro you have roads and dams—that is access, flood control, and so on, that we would not have if we installed a large nuclear plant in the central valley of Mexico. Balance of payments is still another consideration. Mexico at that time had not devalued its currency; the currency was artificially pegged with respect to the free world rate. The question is, if we are going to have an import quota system to try to maintain this kind of disparity in the price of money, should we include that mechanism within the model or should we say other parts of the government are going to be responsible for making such adjustments. That is what the balance of payment effect is all about.

There are two outputs left that illustrate two different points. One is called dependence on foreign supply. At the time that this study started, there was a worry in the minds of the Mexicans that a nation supplying nuclear equipment might become hostile for some political reason and cut off the supply of repair parts, fuel, or maintenance facilities, much as the United States did with respect to Cuba. If that happened, of course Mexico would be in trouble. The question was, would this have a major effect on the decision, or would it not. They could buy insurance against it by stock-piling uranium until such time as they were able to establish alternate sources of supply. But it was a real worry, because they wanted to make sure they would be protected against any politically generated stoppage of equipment or supplies. By the end of the study, this whole area was of much less importance.

The other output was pollution. Originally the decision makers were not too interested in pollution. They said they could not afford to worry about it. And yet, if you have visited Mexico City, you know that atmospheric polution is very high. By the time this study was over, about one year later, they were very glad that they had provided a place in the model for pollution because they were now getting the same kind of citizen complaint that we get in the United States. Some of the things they were planning, like giant coal plants in the middle of Mexico City, were not acceptable any more.

The social outputs from the operating model entered the social value function to produce what we call "social profit." It represents social effects that do not appear on the balance sheet of the electrical system, per se. Social profit is combined with the system profit to produce national profit. Time and risk preference are expressed on national profit to give an evaluation of the system as a whole.

The problem that remained was to find a way to expand the Mexican electrical system that would produce the highest overall evaluation. Various optimization procedures were used to suggest installations of different types (gas turbines, nuclear, conventional, and hydro plants) to achieve this objective over the rest of the century.

The Nature of Policy

Let us briefly examine the question of what a policy for expansion of such a system means. A common policy in the past had been to establish a so-called plant list, which was a list of when each type of plant would be installed—in 1979 we are going to have an X-type plant in location Y. That is a little bit like asking a new father, "When is your son going to wear size-ten pants?" He could look at projected growth charts and say, "Well, I think it will be when he is nine years old." Another way to answer the question is to say, "Well, I will buy him size-ten pants when his measurements get into such and such a region." This is what we might call a closed-loop policy because we cannot say in advance when we are going to do it, but we have built a rule that will tell us the right time to do it.

So when we ask how is the system going to be expanded from here on out, no one can tell us: They can show us expected times for different things to happen, but indeed, only the program can determine what the effect on expansion of the future evolution of the system's environment will be. It has what we might call a self-healing property. If we foul it up by forcing it to put in a giant plant that it cannot immediately assimilate, then it is self-healing in the sense that it will delay and adjust the sizes and types of future plants until it gets back on the optimum track again. As a matter of fact, it is so much self-healing that it is hard to foul it up very much no matter what we do, because in the course of time it is a growing system that finds a way to get around any of our idiocies. In actuality, when they compared what this optimization system was doing with the designs produced by their conventional techniques using the same information, this system yielded superior results in every case.

The size of the Mexican study is interesting. It took approximately eight man-years, and was completed in one calendar year by a staff of decision analysts from the Stanford Research Institute Decision Analysis Group, plus four representatives of the Mexican Electricity Commission who were very competent in nuclear engineering and power system design. The programs and analyses are now being used in Mexico for continued planning of system expansion.

Other Applications

Other applications include industrial projects—should companies merge, should they bring out a new product, or should they bring a mine into production? All of these things are what we might call fairly conventional decision analyses by the criteria that we in the profession use.

Some interesting decision analyses have been done in the medical area, such as one recently performed on the treatment of pleural effusion, that is, water in the cavity between the lung and the chest wall. This was a one-year study done by a graduate student who, as far as the doctor (who was the lung expert) is concerned, completely encoded everything the doctor knew about pleural effusion. Later the doctor was asked if he developed this symptom would he prefer to be treated by this large decision model or by one of his colleagues. He said, without hesitation, he would rather use the model.

Another study that has just recently been completed is whether to seed a hurricane threatening the coast of the United States. It was based on a large experiment a few years ago on hurricane "Debbie" which indicated, but certainly not conclusively, that seeding a hurricane with silver iodide crystals would cause the wind to diminish about 15 percent. This in turn would lead to something like a 50 percent decrease in damage. The question now is—if you are the decision maker in the White House and here comes a big one, hurricane "Zazie," headed right for Miami—what do you do? Should you send the planes out to seed it knowing that, even so, there is a chance that it might get worse just because of natural causes and wipe out two cities instead of one? Or should you sit on your hands and possibly watch people get killed and property destroyed when they might have been saved? There is a tough problem. It has severe social impacts and is definitely a decision under uncertainty. Study of this problem was presented very recently to the President's Scientific Advisory Committee. They have formed a subpanel to see whether the conclusions should be put into effect.

Conclusion

We have tried to characterize what is a new profession—a profession that brings to the making of decisions the same kind of engineering concern and competence applied to other engineering questions. It seems fair to say that the profession has now come of age. We are able to work on virtually any decision where there is a decision maker who is worried about making that decision, regardless of the context in which it may arise. The only proviso is that the resources that he is allocating must be real world resources. We are not competent to allocate prayer because we can not get our hands on it—or love, which is infinite. But when it comes down to allocating money, or time, or anything else that a person or organization might have to allocate, this logic has a lot to be said for it. And indeed, as we have seen, the key is the idea of separating the good decision from the good outcome. Once we have done that then we have the same ability to analyze, to measure, to compare that gives strength to any other engineering discipline.

Question Period

QUESTION. Is the professional decision maker the man who is right out in the forefront making the decisions in his own name, or will there be a professional decision analyst who is like the ghost writer standing behind the man, the president, the corporate executive?

ANSWER. That is a good question. In the legal profession there is a maxim that the lawyer who defends himself in court has a fool for a client. And I think the same is true of decision analysis. I know that I would never want to be my own decision analyst because I am not detached. I want the answer to come out certain ways, subconsciously. For example, if I want to make a case for why I should buy a new stereo system, I will work like a dog to make sure that I have lots of variables in the analysis indicating that I am

BRUNSWICK CORPORATION*

El "Snurfer"

A mediados de Abril de 1967, Gerry O'keefe, Vicepresidente de mercadotecnia de la compañía Brunswick se encontró con que tenia que decidir cuantos Snurfers se deberían fabricar para la estación de invierno 1967-1968.

El Snurfer era un nuevo producto; introducido por primera vez al mercado por la compañía durante Julio y Agosto de 1966, pero debido a la dificultad para predecir los requerimientos de ventas, la compañía fabricó más Snurfers de los que en realidad se vendieron. O'keefe, no quería volver a enfrentarse a la misma situación en el siguiente periodo, por lo que se encontraba tratando de manejarla.

El Snurfer

El Snurfer no era otra cosa que una patineta (skate board) para ser utilizada en la nieve. Consistía en una tabla de madera moldeada de 1.20m. de largo y 17 cms. de ancho, sobre la cual el patinador esquiaba o mas bien patinaba sobre la nieve. La compañía en un folleto de publicidad lo anunció como sigue:

> Disfrute de la gran emocion que le proporcionará el nuevo gran deporte de patinar sobre la nieve. Los niños, jovenes y adultos podrán combinar toda clase de habilidades para sortear las dificultades de esquiar sobre el nuevo Snurfer Brunswick. Es facil de aprender a manejar y muy divertido. El Snurfer es simplemente el nuevo deporte de moda.

El Snurfer se fabricó de 2 formas, la regular y la super. El modelo regular consistía de madera laminada pintada de amarillo con rayas negras, y huellas de metal para colocar los pies. El super era similar pero tenía incorporada una quilla de metal para mejor maneobrabilidad; en lugar de estar pintado ela barnizado para presentar la apariencia natural de la madera y las huellas para los pies eran más lujosas. Ademas su venta incluía cera para el Snurfer, la cual al colocarse en el parte baja de este, permitía aumentar la velocidad.

El desarrollo del Snurfer, Enero 1966 - Marzo 1967

La idea del Snurfer se originó en Muskegon, Michigan en Enero de 1966, cuando un plomero decidió convertir un esquí para agua en patineta para la nieve, con la finalidad de que sus niños jugaran. La idea le gustó, y experimento varias formas y tamaños y bautizó al artefacto con el nombre de Snurfer.

Durante el mes de Febrero, un empleado de Brunswick Corporation, se percató del juguete y pensó que este sería de interés para la compañía. En Abril de 1966, la compañía negoció la compra de los derechos del diseño y nombre proporcionados por el plomero. El contrato involucraba una suma inicial y un porcentaje sobre las ventas totales del producto. El porcentaje no era valido a menos que se produjeran un mínimo de ventas predeterminado.

Posterior a la forma del contrato, los ingenieros de la compañía iniciaron el estudio para la optimización del modelo. Algunas pruebas se llavaron a cabo en los últimos terrenos nevados, pues el Invierno ya había terminado. Al final de Abril, el proyecto estaba terminado y listo para ser entregado al personal de producción.

Mientras los ingenieros estaban ocupados en el diseño, Noel Biery Jefe de producción y O'keefe estaban tratando de determinar el tamaño del mercado potencial y los canales de distribución. Debido a que el producto aparentemente resultaba más atractivo a los niños se decidió canalizarlo mediante jugueterias. Sin embargo, al encontrarse con un desarrollo lento de la canalización se realizó una demostración en la exposición del juguete en Nueva York. Solo se tenian modelos prototipo en aquella exposición (Marzo 66) y se encontró que aún asi, los resultados eran alentadores. Durante la exposición, el modelo que entonces era único y que despues se convirtió en el regular, fue vendido con un precio de fábrica de $3.60 y se sugería un precio al público equivalente a $5.95 (dólares).

Debido a que a mediados de Abril, los prototipos y sus especificaciones se encontraban muy adelatados, los representantes de la compañía fueron entonces enviados a investigar el mercado y a presionar para lograr las ventas durante el resto del mes.

Al final de Abril, O'keefe tenía que tomar la decisión de continuar o no con el Snurfer, y en caso afirmativo decidir el número de unidades por fabricar. El Departamento de Producción de la compañía, determinó que para realizar el producto antes de la estación de invierno, era necesario tener los requerimientos de producción al final de Abril, por lo que O'keefe se lanzó a ordenar 60,000 unidades, aun cuando solo se tenia la promesa de compra de 3000 de ellas. 50,000 iban a ser de tipo regular el resto super.

Considerando lo anterior, una maquinaria con capacidad para producir 150,000 unidades, se ordenó, con un costo de $50,000.00 (dolares) y con esto, el departamento de producción programó la iniciación del proceso para principios de Septiembre de 1966.

Sin embargo, en Junio ni una orden más aparte de las 3000 se habia recibido y O'keefe, preocupado, se enfrentaba a la toma de una dificil dirección a seguir. Se investigó la causa del fracaso en las ventas, y mediante la vista a varias tiendas de deportes, se encontró una magnifica reacción; en contraste con las jugueterías donde nos se lograban las ventas. Se decidió, por lo tanto, que utilizar las jugueterías como canales de distribución era un error, consecuentemente se cerraron estos canales y se trató de promever la venta atraves de las tiendas de deportes. Desafortunadamente, para esa fecha, este tipo de tiendas habían practicamente completado su inventario para las ventas de invierno, por lo que, aun cuando la reacción era buena, no existía el deseo de ordenar para la estación en puerta. Asi, la gerencia decidió cortar la producción de 60,000 unidades a solo 50,000 y se cambio la proporción entre regulares y super.

El número total de Snurfers vendidos durante la estación de invierno 1966-67 fue menor a 35,000 unidades de las cuales el 60% fueron super. A mediados de Marzo de 1967 se tenian en inventario unos 17,000 Snurfers de los cuales 12,000 eran regulares y 5000 Super.

Producción para 1967 - Abril 1967

Debido a las dificultades y problemas qu O'keefe y Biery experimentaron en 1966, decidieron que los planes para 1967 deberían estar firmemente basados en la experiencia anterior.

Al revisar la situación, se tuvueron razones para pensar que los problemas habían surgido a raíz de canalizar las ventas por jugueterías. Se observó por otra parte, que se podía desarrollar un fuerte grado de habilidad por parte de los entusiastas del Snurfer y que se podían obtener velocidades superiores a los 50 Km/hr. Este hecho, aunado a la magnifica respuesta, aun cuando tardía, recibida por las tiendas de deportes sugirió

INTRODUCTION

Decision analysis is a term used to describe a body of knowledge and professional practice for the logical illumination of decision problems. It is the latest link in a long chain of quantitative advances in management that have emerged from the operations research/management science heritage. It is the result of combining aspects of systems analysis and statistical decision theory. Systems analysis grew as a branch of engineering whose strength was consideration of the interactions and dynamic behavior of complex situations. Statistical decision theory was concerned with how to be logical in simple uncertain situations. When their concepts are merged, they can reveal how to be logical in complex, dynamic, and uncertain situations; this is the province of decision analysis.

Thus, decision analysis focuses logical power to reduce confusing and worrisome problems to their elemental form. It does this not only by capturing structure, but by providing conceptual and practical methods for measuring and using whatever knowledge regarding uncertainty is available, no matter how vague. When all available knowledge has been applied, the problem is reduced to one of preference; thus the best alternative will depend on the desires of the decision-maker. Here again, decision analysis provides conceptual and practical methods for measuring preferences. The problem may require expressing the relative desirability of various outcomes, the effect on desirability of changes in timing, and the tolerance for uncertainty in receiving outcomes. In particular, the impact of uncertainty upon the decision can be measured and interpreted — not left to intuition.

BACKGROUND

History of Quantitative Decision-Making

Operations Research

Operations research was the first organized activity in the scientific analysis of decision-making. It originated in the application of scientific methods to the study of air defense during the Battle of Britain. The development of operations research continued in the U.S. in the Navy's study of antisubmarine and fleet protection problems. After World War II, many of the scientists experienced in operations research decided to apply their new tools to the problems of management.

However, an examination of the transition of operations research from military to civilian problems shows that the limitations inherent in the military applications carried over to the civilian work. Many of the operations researchers trained in the military environment had become used to working only on operationally repetitive problems. In these constantly recurring problems, the impact of the formal analysis became evident to even the most skeptical observers. Some of the researchers, however, concluded that only this type of problem was susceptible to scientific analysis—that is they limited operations research to the study of repetitive processes.

Since repetitive decisions are also important to the civilian world, operations research made substantial headway in its new environment. Yet, the insistence on repetition confined the efforts of operations researchers within the province of lower and middle management, such as inventory control, production scheduling, and tactical marketing. Seldom did the analysts study decision problems relevant to the top executive.

Management Science

In the mid-1950s, operations research spawned an offshoot—management science. This discipline developed in response to a deep concern that the special problems of management were not receiving sufficient attention in operations research circles. This new field grew to emphasize science more than management, however. Management scientists have been accused of having more interest in those problems that are subject to elegant mathematical treatment than in those of the top executive, which are generally less easily quantified.

Although many students of business have considered the problems of top management, they have not generally had the scientific and mathematical training necessary to give substance to their ideas and to allow their application in new situations. When the top manager sought help on a problem, he often had to choose between a mathematician who was more concerned with the idiosyncrasies of the situation than with its essence and an experienced "expert" who might be tempted to apply an old solution to a radically new problem. Thus, the early promise of scientific aids for the executive was slow in materializing.

Decision Analysis

In the last few years, a new discipline, called "decision analysis," has developed from these predecessors. It seeks to apply logical, mathematical, and scientific procedures to the decision problems of top management that are characterized by the following:

- *Uniqueness*. Each is one of a kind, perhaps similar to—but never identical with—previous situations.
- *Importance*. A significant portion of the organization's resources is in question.
- *Uncertainty*. Many of the key factors that must be taken into account are imperfectly known.
- *Long run implications*. The enterprise will be forced to live with the results of the situation for many years, perhaps even beyond the lifetimes of all individuals involved.
- *Complex preferences*. The task of incorporating the decision-maker's preferences about time and risk assumes great importance.

Decision analysis provides a logical framework for balancing all these considerations. It permits mathematical modeling of the decision, computational implementation of the model, and quantitative evaluation of the various courses of action. This report describes and delineates the potential of decision analysis as an aid to top management.

The Timeliness of Decision Analysis

An appropriate question is why decision analysis has only recently emerged as a discipline capable of treating the complexities of significant decision problems. The answer is found in the combination of three factors: historical circumstance, development of complementary capabilities, and the need for increased formalism.

The Computer Revolution

Despite the elaborateness of its logical foundations, decision analysis would be merely an intellectual curiosity rather than a powerful tool if the means were not available to build models and to manipulate them economically. The rapid development of the electronic computer in the past two decades has made feasible what would have been impossible only a quarter of a century ago. The availability of electronic computation is an essential condition for the growth of the decision analysis field.

The Tyranny of the Computer

A powerful tool is always subject to misuse. The widespread use of computers has led some managers to feel that they are losing rather than gaining control over the operations of their organizations. These feelings can lead to a defensive attitude toward the sugtion that computers should be included in the decision-making process.

Decision analysis can play a major role in providing the focus that management requires to control application of computers to management activities. When examined through decision analysis, the problem is not one of management information systems, but one of providing management with structured decision alternatives in which management experience, judgment, and preference have already been incorporated. Since properly applied decision analysis produces insight as well as answers, it places control in, rather than out of, the hands of the decision-maker.

The Need for Formalism

A final force in the current development of decision analysis is the trend toward professional management in present organizations. The one-man show is giving way to committees and boards, and the individual entrepreneur is becoming relatively less important. A concomitant of this change is the need for new professional managers to present evidence of more carefully reasoned and documented decisions. Even the good intuitive decision-maker will have to convince others of the logic of his decisions.

However, the need for more formalism may also be imposed from outside the organization. The nature of competition will mean that when one company in an industry capitalizes on the efficacy of decision analysis, the others will be under pressure to become more orderly in their own decision-making. To an increasing extent, good outcomes resulting from intuitive decisions will be regarded in the same light as winnings at the races—that is, as the result of luck rather than of prudent managerial practice.

The Essence of Decision Analysis

Definition of Decision

In describing decision analysis, the first step is to define a decision. In this report, a decision is considered an irrevocable allocation of resources, in the sense that it would take additional resources, perhaps prohibitive in amount, to change the allocation. Some decisions are inherently irrevocable, such as whether or not to amputate a pianist's hand; others are essentially irrevocable, such as the decision by a major company to enter a new field of endeavor.

Clearly, no one can make a decision unless he has resources to allocate. For example, a manufacturer may be concerned about whether his competition will cut prices, but unless he can change something about the way he does business, he has no decisions to make. Concern without the ability to make decisions is simply "worry." It is not unusual in practice to encounter decision problems that are really worries. Exposing a decision problem as a worry may be very helpful if it allows the resources of the decision-maker to be devoted more profitably to other concerns.

Another common phenomenon is the study, which is an investigation that does not focus on a decision. Until a decision must be made, how can the economic balance of the study be determined? For example, suppose someone requested a study of the automobile in his particular community. The person conducting the study might survey cars' weight, horsepower, displacement, braking ability, seating capacity, make, type, color, age, origin, and on and on. However, if a decision were required concerning the size of stalls in a parking facility, or the length of a highway acceleration lane, the pertinent characteristics would become clear. Further, decision analysis could even determine how extensive a survey, if any, would be economic. Thus, concentrating on a decision to be made provides a direct focus to the analysis that is achievable in no other way. Studies, like worries, are not our concern; decisions are.

The next step is to define a decision-maker: an individual who has the power to commit the resources of the organization. In some cases, the decision-maker may be an organizational entity, such as an executive committee. It is important, however, to distinguish advisory individuals or bodies from those with the power to commit the organization. Study upon study may be performed within an organization advocating or decrying a certain course of action, but until resources are committed, no decision has been made. The first step in any decision analysis is the identification of the responsible party.

The Distinction Between a Good Decision and a Good Outcome

Before there can be a formal discussion of decision analysis, the distinction between a good decision and a good outcome must be understood. A good decision is one based on the information, values, and preferences of a decision-maker. A good outcome is one that is favorably regarded by a decision-maker. It is possible to have good decisions produce either good or bad outcomes. Most persons follow logical decision procedures because they believe that these procedures, speaking loosely, produce the best chance of obtaining good outcomes.

To illustrate this point, suppose that we had agreed to serve as decision analysis consultants to a person who said that he would engage only in gambles that were weighted in his favor. Then this person informed us that he had purchased a ticket in a lottery. There were 100 tickets in the lottery, the prize was $100, and he paid $10 for the ticket. We demonstrate to him that with 1 chance in 100 of winning the $100, his expected income from the ticket is only 1/100 of $100 or $1, so that having paid $10 for the ticket, his expected loss on the entire prospect is $9. Consequently, in view of this person's expressed desire to avoid unfavorable gambles, we say that he has made a bad decision.

However, the next day he receives a check for $100 as a consequence of having won the lottery; everyone agrees that this is a good outcome for him. Yet we must report that his decision was bad in spite of the good outcome, or, perhaps better, that his outcome was good in spite of the bad decision. This would be proper situation to be described as "lucky."

Suppose, however, that the person had paid only 10 cents for his ticket. In this case, his expected income is still $1, but because he spent only 10 cents for the ticket, his net expected earnings are 90 cents. Consequently, we would compliment him on his good decision. Yet if no winnings check appears on the next day, the client has now experienced a bad outcome from his good decision.

The distinction between good outcomes and good decisions is especially important in maintaining a detached, professional attitude toward decision problems. Recriminations based on hindsight in the form of "Why didn't it work?" are pointless unless they reveal that available information was not used, that logic was faulty, or that the preferences of the decision-maker were not properly encoded. The proper framework for discussing the quality of decisions and outcomes is a major aid in using hindsight effectively.

Decision Analysis as a Language and a Philosophy

The decision analysis formalism serves both as a language for describing decision problems and as a philosophical guide to their solution. The existence of the language permits precision in specifying the many factors that influence a decision.

The most important feature of the language is its ability to represent the uncertainty that inevitably permeates a decision problem. The language of probability theory is used with only minor changes in terminology that reflect a subjective interpretation of probabilistic measurement. We regard probability as a state of mind rather than of things. The operational justification for this interpretation can be as simple as noting the changing odds on a sporting contest posted by gamblers as information about the event changes. As new information arrives, a new probability assignment is made. Decision analysis uses the same subjective view of probability. By so doing, statements regarding uncertainty can be much more precise. Rather than saying, "There is some chance that a bad result is likely," or an equivalent ambiguous statement, we shall be able to speak directly of the probability of a bad result. There is no need for vagueness in the language that describes uncertainty. Putting what is not known on the record is the first step to new knowledge.

Decision analysis can also make a major contribution to the understanding of decision problems by providing a language and philosophy for treating values and preferences. "Values" mean the desirability of each outoutcome; "preferences" refer to the attitudes of the decision-maker toward postponement or uncertainty in the outcomes he receives. Placing values and preferences in unambiguous terms is as unusual in current decision-making as is the use of direct probability assignments. Yet both must be done if the procedure is to be used to full advantage.

Later sections of this report describe the theory and practice of assigning probabilities, values, and preferences, but the impact of thinking in such terms can be indicated here. A most important consequence of formal thought is the spontaneous resolution of individual differences that often occurs when the protagonists can deal in unambiguous terms. Two people who differ over the best alternative may find their disagreements in the areas of probability assignment, value, or preference. Thus, two men who are equally willing to take a risk may disagree because they assign different probabilities to various outcomes; or two men who assign the same probability to the outcomes may differ in their aversion to risk. It is unlikely that the nature of the disagreement will emerge without the formal language. More likely, epithets such as "foolhardy" or "rock-bound conservative," will prevent any communication at all.

The decision analyst must play a detached role in illuminating the decision problem if he is to resolve differences. He must be impartial, never committing himself to any alternative, but rather showing how new information or changes in preference affect the desirability of available alternatives. The effectiveness of the decision analyst depends as much on his emotional detachment as on his knowledge of formal tools.

Decision analysis is a normative, rather than a descriptive, approach to decision problems. The decision analyst is not particularly interested in describing how decision-makers currently make decisions; rather he is trying to show how a person subscribing to certain logical rules would make these decisions in order to maximize attainment of his objectives. The decision procedures are derived from logic and from the desires of the decision-maker and are in this sense prescriptive.

Decision analysis is more than a language and a philosophy, but the experience of its users justifies it on this basis alone. By focusing on central issues, the approach often illuminates the best course of action in a way that makes discord evaporate.

Decision Analysis as a Logical and Quantitative Procedure

Decision analysis provides not only the philosophical foundations, but also a logical and quantitative procedure for decision-making. Since decision analysis encodes information, values, and preferences numerically, it permits quantitative evaluation of the various courses of action. Further, it documents the state of information at any stage of the problem and determines whether the gathering of further information is economically justifiable. The actual implementation of decision analysis models is typically a computer program that enables the many facets of the problem to be examined together. Most of this report will describe how the philosophy of decision analysis carries over into practice.

Delegation of Responsibility

Decision analysis provides both philosophical and operational guidelines for delegating responsibility in an organization. If we want someone to make a good decision, we must provide that individual not only with the information but also with the values and preferences that are relevant to the decision. The key principle is that the delegator must supply a subordinate decision-maker with whatever information, values, and preferences required for him to reach the same decision that the delegating individual would have reached in the same situation. While few organizations currently use decision analysis principles in handling the problem of delegation, these principles are available when needed. It is rare that an organization performs a decision analysis on one of its major decisions without simultaneously obtaining new insight into its organizational structure.

THE DECISION ANALYSIS CYCLE

Decision analysis as a procedure for analyzing a decision is described below. This procedure is not an inviolable method of attacking the problem, but is a means of ensuring that essential steps have been consciously considered.

The figure describes decision analysis in the broadest terms. The procedure is iterative and comprises three phases. The first is a deterministic phase, in which the variables affecting the decision are defined and related, values are assigned, and the importance of the variables is measured without any consideration of uncertainty.

The second, or probabilistic, phase introduces probability assignments on the important variables and derives associated probability assignments on values. This phase also introduces the assignment of risk preference, which provides the best solution in the face of uncertainty.

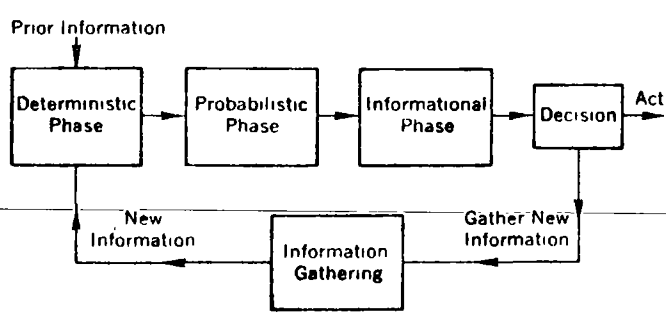
The third, or informational, phase reviews the results of the last two phases to determine the economic value of eliminating uncertainty in each of the important variables in the problem. In some ways, this is the most important phase because it shows just what it could cost in dollars and cents not to have perfect information. A comparison of the value of information with its cost determines whether additional information should be collected.

If there are profitable further sources of information, then the decision should be to gather the information rather than to make the primary decision at this time. Thereupon will follow the design and execution of the information-gathering program, whether it be a market survey, a laboratory test, or military field trials.

The information that results from this program may change the model and the probability assignments on important variables. Therefore, the original three phases must be performed once more. However, the additional work required to incorporate the modifications should be slight and the evaluation rapid. At the decision point, it may again be profitable to gather new information and repeat the cycle or it may be more advisable to act. Eventually, the value of new analysis and information-gathering will be less than its cost, and the decision to act will then be made.

This procedure will apply to a variety of decision situations: in the commercial area, to the introduction of a new product or the change in design of an old one; in the military area, to the acquisition of a new weapon or the best defense against that of a potential enemy; in the medical area, to the selection of a med-

Fig. 1—The Decision Analysis Cycle



... or surgical procedure for a patient; in the social area, to the regulation and operation of public utilities; and finally, in the personal area to selection of a new car, home or career. In short, the procedure can be applied to any decision susceptible to logical analysis.

The Deterministic Phase

Descriptions of the various phases of the procedure follow beginning with the deterministic phase. The deterministic phase is essentially a systems analysis of the problem. Within this phase, efforts devoted to modeling are distinguished from efforts devoted to analysis. The elements of the phase appear in Figure 2.

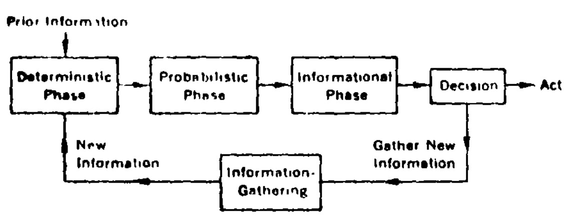


Fig. 2
The Deterministic Phase

MODELING:
- Bound Decision
- Identify Alternatives
- Establish Outcomes
- Select System Variables
- Create Structural Model
- Create Value Model
- Create Time Preference Model

ANALYSIS.
- Measure Sensitivity
  - to Decision Variables
  - to State Variables

Modeling

Modeling is the process of representing the various relationships of the problem in formal, mathematical terms. The first step in modeling is to bound the decision, to specify precisely just what decision must be made. This requires listing in detail the perceived alternatives. Identification of the alternatives will separate an actual decision problem from a worry.

The next step—finding new alternatives—is the most creative part of decision analysis. New alternatives can spring from radically new concepts; more often they may be careful combinations of existing alternatives. Discovering a new alternative can never make the problem less attractive to the decision-maker; it can only enhance it or leave it unchanged. Often the difficulty of a decision problem disappears when a new alternative is generated.

The next step is to specify the various outcomes that the set of alternatives could produce. These outcomes are the subsequent events that will determine the ultimate desirability of the whole issue. In a new product introduction, for example, the outcomes might be specified by sales levels and costs of production or even more simply by yearly profits. Thus, there is a certain amount of arbitrariness in what to call an outcome. For decision analysis, however, an outcome is whatever the decision-maker would like to know in retrospect to determine how the problem came out. In a military problem, the outcome could be a complicated list of casualties, destruction, and armament expenditures; in a medical problem, it could be as simple as whether or not the patient dies.

Now comes the challenging process of selecting the system variables for the analysis, which are all those variables on which the outcomes depend. We can identify the system variables by imagining that we have a crystal ball that will answer any numerical questions relative to the decision problem, except, of course, which alternative to select. We could ask it questions about the outcome variables directly, thereby making them the only system variables in the problem. But typically outcome variables are difficult to think about in advance in the real world, and so we might choose to relate the outcome variables to others that are easier to comprehend. For example, we might like to know the sales level of a new product. Or in lieu of this, we might attempt to relate the sales to our own price and quality and the competitors' price and quality, factors that we might regard as more accessible. These factors would then become system variables in the analysis.

The selection of system variables is therefore a process of successive refinement, wherein the generation of new system variables is curtailed by considering the importance of the problem and the contributions of the variables. Clearly, allocation of the national budget can economically justify the use of many more system variables than can the selection of a new car.

Once we have decided on the system variables to use in the problem, each one must be distinguished either as a variable under the decision-maker's control or as a variable determined by the environment of the problem. System variables that are under the decision-maker's control are called decision variables. The selection of an alternative in a decision problem is really the specification of the setting of the decision variables. For example, in the new product introduction problem, the product price and the size of production facilities would both be decision variables.

System variables in the problem that are determined by the environment are known as state variables. Although state variables may have a drastic effect on the outcomes, they are autonomous, beyond the control of the decision-maker. For example, in the new product introduction, the cost of a crucial raw material or the competitor's advertising level might be state variables.

We shall want to examine the effect of fluctuations in all system variables, whether decision variables or state variables. To aid in this task, the decision-maker or his surrogate must specify for each system variable a nominal value and a range of values that the variable may take on. In the case of a decision variable, the nominal value and range are determined by the decision-maker's preconceptions regarding the interesting alternatives. In the case of state variables, the nominal value and range reflect the uncertainty assigned to the variables. For convenience, we can often think of the nominal value of a state variable as its expected value in the mathematical sense and of the range as the 10th percentile and 90th percentile points of its probability distribution.

Selecting system variables and setting nominal values and ranges require extensive consultation between the decision-maker and the decision analyst. At this stage, it is better to err by including a variable that will later prove to be unimportant than it is to eliminate a variable prematurely.

The next step is to specify the relationships among the system variables. This is the heart of the modeling process—i.e., creating a structural model that captures the essential interdependencies of the problem. This model should be expressed in the language of logic—mathematics—typically by a set of equations relating the system variables. In most decisions of professional interest, these equations will form the basis for a computer program to represent the model. The program provides rapid evaluation of model characteristics at modest cost.

Constructing a model of this type requires a certain sophistication in the process of orderly description and a facility for careful simplification. The procedure is elementary, but not trivial; straightforward, but not pedestrian.

Now the decision-maker must assign values to outcomes. Just as there was difficulty in defining an outcome, so there may be some question about the distinction between an outcome and its value. For example, in a business problem, the decision-maker may think of his future profit as both the outcome and the value associated with it. However, maintaining the generality of the formulation requires creating a distinction between the two.

To illustrate the necessity for this, consider a medical question involving the amputation of an arm. The outcomes of interest might be complete recovery, partial recovery, or death, each with or without the operation. These outcomes would describe the results but would not reveal their value. For example, if the patient were a lawyer, he might consider death by far the most serious outcome and be willing to undergo the amputation if it sufficiently reduced the probability of death. These feelings might be based on the observation that an arm is not essential to his career. To a concert pianist, however, amputation might be worse than death itself, since life without being able to play might be unbearable. Consequently, he would be rational in refusing the amputation even if this choice made his death more likely.

Although in some cases the decision can be reached as a result of ordering outcomes in terms of desirability, most problems of practical interest require a numerical (cardinal) ranking system. Therefore, assigning a value means assigning a numerical value to an outcome. Though there may be many elements of value in the outcome, the final value assignment is a single number associated with that outcome.

In commercial situations, the value assigned to an outcome will typically be some form of profit. In social and military problems, however, the value assignment is more difficult because it requires measuring the value of a human life, or a cultured life, or a healthy life in dollars and cents terms. Though these questions of evaluation may be difficult, logic demands that they be approached directly in monetary terms if monetary resources are to be allocated.

The final step in creating the deterministic model is to specify the time preference of the decision-maker. Time preference is the term used to describe the human phenomenon of impatience. Everyone wants good things to happen to him sooner rather than later. This impatience is reflected in a willingness to consume less now rather than postpone the consumption. The payment of interest on savings accounts and the collection of interest on loans are mere reflections of this phenomenon. Consequently, representing the desires of a decision-maker requires a realistic mechanism for describing his time preference, a mechanism that reduces any time stream of value to a single number called worth.

For a corporate financial decision, worth will often be simply the discounted difference between future income and expenditures using an interest rate that depends upon the relationship of the corporation to its financial environment. In the military or medical fields, worth may be more difficult to establish.

The modeling part of the deterministic phase thus progresses from the original statement of the decision problem to a formal description suitable for detailed examination by logical and computational analysis. The decision-maker's value assignments and his time preference permit rating any outcome that appears as a time stream first as a set of values in time and then as an equivalent worth.

Analysis

Analysis based on the deterministic phase centers on observing how changes in the variables affect worth. Experimentation of this type is known as sensitivity analysis; it is highly effective in refining the formulation of the problem.

The first sensitivity analysis we perform is associated with the decision variables. First, fixing all other state variables in the problem at their nominal values, we then allow one of the decision variables to traverse its assigned range and observe how worth changes. Of course, these observations are usually carried out by computer program. If we find that a particular decision variable has a major effect, then we know that we were correct in including it in the original formulation. But if a decision variable has little or no effect, we are justified in considering its removal as a decision variable. If reflection reveals that the latter is the case, we would say that we have eliminated an impotent decision variable. For example, the time of introduction of a new product might seem to be a decision variable of major importance, but because of the combined effects of competitive reaction and the gaining of production experience, it might turn out to have very little effect. The timing of entry would then be an impotent variable.

Next, we perform sensitivity analyses on the state variables, which are uncertain and over which the decision-maker has no control. With all other system variables at their nominal values, we observe the change in worth while sweeping one state variable over its range. If a state variable has a major effect, then the uncertainty in the variable deserves special attention. Such variables are called aleatory variables to emphasize their uncertainty.

If, however, varying a state variable over its range produces only a minor change in worth, then that variable might well be fixed at its nominal value. In this case, we say that the state variable has become a fixated variable. A state variable may become fixated either because it has an important influence on the worth per unit of its range, but an extremely small range, or because it has little influence on the worth per unit of its range, even though it has a broad range.

There is no reason to conclude that a fixated variable is unimportant in an absolute sense. For example, the corporate tax rate may be a fixated variable in a problem because no change in it is anticipated within the time period under consideration. Yet it is possible that an unforeseen large change in this rate could change a favorable venture into an unfavorable one.

Although sensitivity analysis has been described as if it concerns only changes in one variable at a time, some of the most interesting sensitivity results are often observed when there are simultaneous changes in state variables. Since the possibilities of changing state variables jointly grows rapidly with the number of state variables, an important matter of judgment for the decision analyst is to determine the amount of simultaneous sensitivity analysis that is economic.

The Probabilistic Phase

The net result of the deterministic sensitivity analysis on the autonomous state variables is to divide them into aleatory and fixated classes. The probabilistic phase determines the uncertainty in value and worth due to the aleatory variables. The phase will be divided into steps of modeling and analysis; Figure 3 illustrates its internal structure.

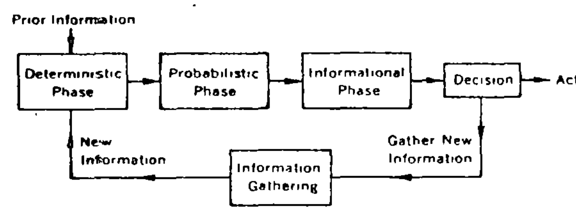


Fig. 3
The Probabilistic Phase

MODELING
- Encode Uncertainty on Aleatory Variables
- Encode Risk Preference

ANALYSIS
- Develop Worth Lotteries and Certainty Equivalent
- Measure Stochastic Sensitivity
- Measure Risk Sensitivity

Modeling Probability Distributions

The first modeling step in the probabilistic phase is the assignment of probability distributions to the aleatory variables. Either the decision-maker or someone he designates must assign the probability that each aleatory variable will exceed any given value. If any set of aleatory variables is dependent, in the sense that knowledge of one would provide information about the others, then the probability assignments on any one variable must be conditional on the values of the others. Gathering these assignments amounts to asking such questions as, "What are the odds that sales will exceed 10 million units in the first year?" (See section entitled "Encoding Knowledge and Preferences.") Strange as such questions may be in the current business world, they could be the standard executive language of tomorrow.

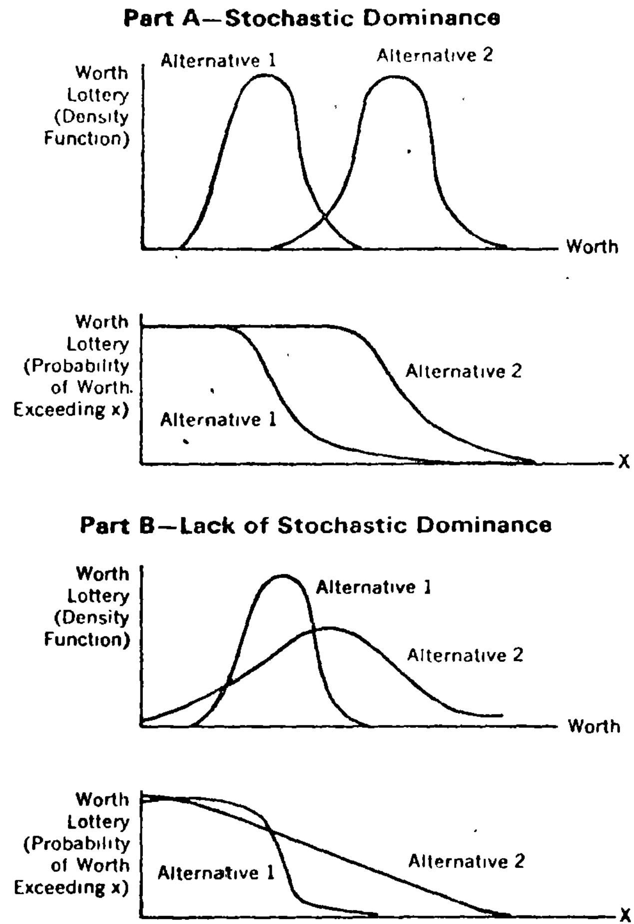
Analysis

With knowledge from the deterministic phase of how the worth depends on the state variables and assigned probability distributions on the aleatory variables, it is a straightforward calculation to determine the probability distribution of worth for any setting of the decision variables; this probability distribution is the "worth lottery." The worth lottery describes the uncertainty in worth that results from the probability assignments to the aleatory variables for any given alternative (setting of decision variables.) Of course, the values of the fixated variables are never changed.

To select a course of action, the analyst could generate a worth lottery for each alternative and then select the one that is more desirable. But how would he know which worth lottery is most desirable to the decision-maker?

One important principle that allows judging one worth lottery as being better than another is that of stochastic dominance, which is illustrated in Figure 4. Part A of this figure shows the worth lottery for two alternatives in both probability densities and excess probability distribution forms. The excess probability distribution, or excess distribution, is the probability that the variable will exceed any given value plotted as a function of that value. Its height at any point is the area under the probability density function to the right of that point. Comparison of the excess distributions for the two alternatives reveals that, for any value of X, there is a higher probability that alternative 2 will produce a worth in excess of that X than will alternative 1. Consequently, a decision-maker preferring more worth to less would prefer alternative 2. If alternative A has an excess distribution that is at least as great as that of alternative B at any point and greater than B at at least one point, alternative A stochastically dominates alternative B. If stochastic dominance exists between two competing alternatives, there is no need to inquire into the risk preference of the decision-maker, who rationally must rule out the stochastically dominated alternatives.

Fig. 4



Part B of Figure 4 illustrates a case in which stochastic dominance does not exist. The excess distributions on worth for the two alternatives cross. If the decision-maker wants to maximize his chance of receiving at least a small amount of worth, he would prefer alternative 1; if he wants to maximize his chance of receiving at least a large amount of worth, he would prefer alternative 2. In situations like this, where stochastic dominance does not apply, the risk preference of the decision-maker must be encoded formally, as shown below.

Just because alternative A stochastically dominates alternative B does not mean that the decision-maker will necessarily achieve a higher worth by following alternative A. For example, if alternative A produces worths of five to 15 with equal probability and alternative B produces worths of zero and ten with equal probability, then A stochastically dominates B. Yet it is possible that A will produce a worth of five while B will produce a worth of ten. However, not knowing how the lottery will turn out, the rational man would prefer alternative A.

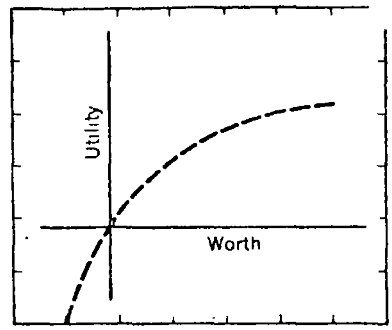
Modeling Risk Preference

If stochastic dominance has not determined the best alternative, the analyst must turn to the question of risk preference. To demonstrate that most individuals are averse to risk, it is only necessary to note that few, if any, are willing to toss a coin, double or nothing, for a year's salary. Organizations typically act in the same way. A realistic analysis of decisions requires capturing this aversion to risk in the formal model.

Fortunately, if the decision-maker agrees to a set of axioms about risk taking (to be described in the following section), his risk preference can be represented by a utility curve like that shown in Figure 5. This curve assigns a utility to any value of worth. As a consequence of the risk preference axioms, the decision-maker's rating of any worth lottery can be computed by multiplying the utility of any possible worth in the lottery by the probability of that worth and then summing over all possible worths. This rating is called the expected utility of the worth lottery.

If one worth lottery has a higher expected

Fig. 5
A Typical Utility Curve



utility than another, then it must be preferred by the decision-maker if he is to remain consistent with the axioms. The analyst is not telling the decision-maker which worth lottery he should prefer but only pointing out to him a way to be consistent with a very reasonable set of properties he would like his preferences to enjoy.

Thus, the utility curve provides a practical method of incorporating risk preference into the model. When faced with a choice between two alternatives whose worth lotteries do not exhibit stochastic dominance, the analyst computes the expected utility of each and chooses the one with the higher expected utility.

Although the expected utility rating does serve to make the choice between alternatives, its numerical value has no particular intuitive meaning. Therefore, after computing the expected utility of a worth lottery, the analyst often returns to the utility curve to see what worth corresponds to this expected utility; we call this quantity the certain equivalent worth of the worth lottery. The name arises as follows: if another worth lottery produced the certain equivalent worth with probability one, then it and the original lottery would have the same expected utilities and hence would be equally preferred by the decision-maker. Consequently, the certain equivalent worth of any worth lottery is the amount of worth received for certain, so that the decision-maker would be indifferent between receiving this worth and participating in the lottery. Since almost all utility curves show that utility increases as worth increases, worth lotteries can be ranked in terms of their certain equivalent worths. The best alternative is the one whose worth lottery has the highest certain equivalent worth.

Analysis

In returning to the analysis of the probabilistic phase, the first step is to compute the certain equivalent worth of each of the alternatives. Since the best decision would be the alternative with the highest certain equivalent worth, the decision probably could be considered solved at this point. The careful analyst, however, will examine the properties of the model to establish its validity and so would not stop here. The introduction of risk preference is another point at which to check the sensitivity of the problem. For example, by setting all decision variables but one to their nominal values and then sweeping this one decision variable through its range, the analyst may find that although this variation changes the worth lottery it does not significantly change the certain equivalent worth. This result would indicate that the decision variable could be fixed at its nominal value.

Aleatory variables receive the same sensitivity analysis by setting one of them equal to a trial value within the range and then allowing the others to have the appropriate conditional joint probability distribution. When the decision variables are given their nominal values, the program will produce a worth lottery and hence a certain equivalent worth for the trial value. Sweeping the trial value from one end of its range to the other shows how much certain equivalent worth is changed. If the change is small, there is evidence that the particular aleatory variable may be changed to a fixated variable. We call this procedure measurement of the stochastic sensitivity of a variable. It is possible that an aleatory variable showing a large deterministic sensitivity could reveal only a small stochastic sensitivity and vice versa. Consequently, any decisions to remove variables from aleatory status on the basis of deterministic sensitivity might well be reviewed at this time by measurement of stochastic sensitivity.

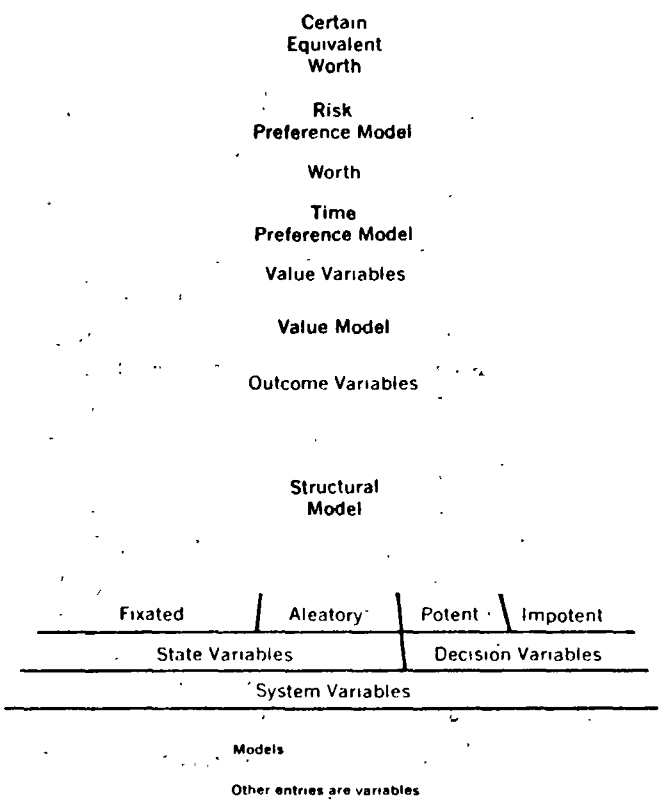
As in the case of deterministic sensitivity, we can measure the stochastic sensitivity of many variables, simultaneously. Once more, the decision analyst must judge how far it is profitable to proceed. Measurement of stochastic sensitivity is a powerful tool for locating the important variables of the problem.

There is one other form of sensitivity analysis available at this point: risk sensitivity. In some cases, it is possible to characterize the utility curve by a single number—the risk aversion constant (just when this is possible will be discussed later). However, when the risk aversion constant is applicable we can interpret it as a direct measure of a decision-maker's willingness to accept a risk. An individual with a small risk aversion constant is quite willing to engage in a fair gamble; he has a tolerant attitude toward risk. As his risk aversion constant increases, he becomes more and more unwilling to participate. If two men share responsibility for a decision problem, the less risk tolerant will assign a lower certain equivalent worth for any given worth lottery than will the other. Perhaps, however, when the certain equivalent worths are computed for all alternatives for both men, the ranking of certain equivalent worths might be the same for both, or at least the same alternative would appear at the top of both lists. Then there would hardly be any point in their arguing over the desirable extent of risk aversion and a possible source of controversy would have been eliminated.

The measurement of risk sensitivity determines how the certain equivalent worths of the most favorable alternatives depend on the risk aversion constant. The issue of risk aversion can often be quickly resolved.

The problem structure, the set of alternatives generated, the probability assignment to aleatory variables, the value assessments, the statement of time preference, and the specification of risk preference combine to indicate

Fig. 6—The Decision Analysis Hierarchy



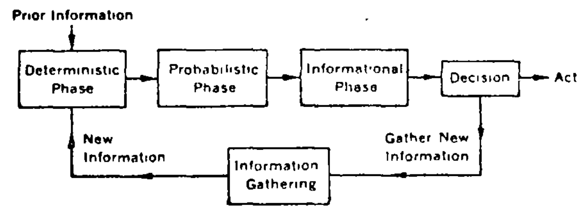


Fig. 7 The Informational Phase

ANALYSIS:

- Measure Economic Sensitivity (Determine Value of Eliminating Uncertainty in Aleatory Variables)

MODELING:

- Explore Feasibility of Information Gathering

the best alternative in the problem. The overall procedure is illustrated by the decision analysis pyramid in Figure 6. However, it still may be best to obtain more information rather than to act. This determination is made in the third phase, as described below.

The Informational Phase

The informational phase is devoted to finding out whether it is worthwhile to engage in a possibly expensive information-gathering activity before making a decision. It is, in the broadest sense, an experimental design procedure from which one very possible result is the decision to perform no experiment at all. Figure 7 shows the steps in the phase.

Analysis

The fundamental idea in the informational phase is that of placing a monetary value on additional information. A key concept in approaching this value is that of clairvoyance. Suppose someone exists who knows in advance just what value a particular aleatory variable would assume in the decision problem—a clairvoyant. How much should the decision-maker be willing to pay him for his services?

To answer this question, recall that the discussion of stochastic sensitivity described how to compute the certain equivalent worth given that an aleatory variable took on a value s. In that procedure, the decision variables were set equal to their best values from the probabilistic phase. Suppose now that we engage the clairvoyant at a cost k, and then he tells us that the aleatory variable will take on the value s. First, we would set the decision variables to take best advantage of this information. However, since the other aleatory variables are still uncertain, they would be described by the appropriate distributions en the available information. The computer program would then determine the expected utility of the entire decision problem including the payment to the clairvoyant, all conditional on his reporting s.

Before engaging the clairvoyant, however, the probability to be assigned to his reporting s as the value of the particular aleatory variable is described by the probability distribution showing the current state of knowledge on this variable. Consequently, we obtain the expected utility of purchasing his information on the variable at a cost k by multiplying the expected utility of the information given that he reports s and costs k, by the current probability that he will report s and then summing over all values of s. The analyst uses the current probability in this calculation because if the clairvoyant is reliable, the chance of his reporting that the variable falls in any range is just the chance that it will fall in that range.

Knowing the expected utility of purchasing the information from the clairvoyant at a cost k, we can gradually increase k from zero until the expected utility of purchasing the information is just equal to the expected utility of proceeding with the decision without clairvoyant information. The value of k that establishes this equivalence is the value of clairvoyance on the aleatory variable.

The value of clairvoyance on an aleatory variable represents an upper bound on the payment for any experimental program designed to provide information on this variable, for no such program could be worth more than clairvoyance. The actual existence of a clairvoyant is not material to this discussion; he is merely a construct to guide our thinking.

We call the process of measuring the value of clairvoyance the measurement of economic sensitivity. If any aleatory variable exhibits high economic sensitivity, it is a prime candidate for an information-gathering program. It is possible, however, for a variable to have a high stochastic sensitivity and a low economic sensitivity because the available alternatives cannot take advantage of the information received about the variable. To determine the importance of joint information, the analyst can measure the value of clairvoyance on more than one variable at a time.

The actual information-gathering programs available will seldom provide perfect information, so they will be less valuable than clairvoyance. Extension of the discussion of clairvoyance shows how their value can be measured. Whereas the clairvoyant reported a particular value s for an aleatory variable, a typical experimental program will provide only a new probability distribution for the aleatory variable. The analyst would then determine the best decision, given this new information, and compute the expected utility of the decision problem. He would next multiply the expected utility by the probability that the exerimental program would come out in this way and then sum over all possible outcomes of the experimental program. The result would be the expected utility of the experimental program at a given cost. The cost that would make the expected utility just equal to the expected utility of the problem without the experimental program would be the value of the experimental program. If the value is positive, it represents the maximum that one should pay for the program. If the value is negative, it means that the experimental program is expected to be unprofitable. Consequently, even though it would provide useful information, it would not be conducted.

Modeling

At this stage, the decision-maker and the analyst must identify the relevant information-gathering alternatives, from surveys to laboratory programs, and find which, if any, are expected to make a profitable contribution to the decision problem. In considering alternatives, they must take into account any deleterious effect of delay in making the primary decision. When the preferred information-gathering program is performed, it will lead, at least, to new probability assignments on the aleatory variables; it might also result in changing the basic structure of the model. When all changes that have been implied by the outcome of the experimental program are incorporated into the model, the deterministic and probabilistic phases are repeated to check sensitivities. Finally, the informational phase determines whether further information-gathering is profitable. At some point, further information will cost more than it is worth, and the alternative that currently has the highest certainty equivalent will be selected for implementation.

The iterative decision analysis described above is not intended to fit any particular situation exactly but, rather, all situations conceptually. A discussion follows on two procedures required to carry out the analysis: encoding knowledge and preferences.

ENCODING KNOWLEDGE AND PREFERENCES

Encoding Knowledge as Probability Distributions

Perhaps the single most unusual aspect of decision analysis is its treatment of uncertainty. Since uncertainty is the central problem in decision-making, it is essential to understand the conceptual and logical foundations of the approach to this issue.

The Importance of Uncertainty

The importance of uncertainty is revealed by the realization that decisions in situations where there is no random element can usually be made with little difficulty. Only when uncertainty exists about which outcome will occur is there a real decision problem.

For example, suppose that we are planning to take a trip tomorrow and that bad weather is forecast. We have the choice of flying or of taking a train. If a clairvoyant told us the consequences of each of these acts, then our decision would be very simple. Thus, if he said that the train would depart at 9:13 A.M. and arrive at 5:43 P.M. and if he described in detail the nature of the train accommodations, the dining car, and the people whom we would meet as traveling companions, then we would have a very clear idea of what taking the train implied. If he further specified that the plane would leave 2 hours late and arrive 2½ hours late, stated that the flight would be especially bumpy during a certain portion of the trip, and described the meals that would be served and the acquaintances we would meet, then the flying alternative would be described as well.

Most of us would have little trouble in making a decision about our means of travel when we considered these carefully specified outcomes in terms of our tastes and desires. The decision problem is difficult because of the uncertainty of departure and arrival times and, in the case of the plane, even whether the trip would be possible at all. The factors of personal convenience and pleasure will be more or less important depending upon the urgency of the trip and, consequently, so will the uncertainties in these factors. Thus we cannot make a meaningful study of decision-making unless we understand how to deal with uncertainty. Of course, in the problems that are of major practical interest to the decision analyst, the treatment of uncertainty is even more pressing.

It is possible to show that the only consistent theory of uncertainty is the theory of probability invented 300 years ago and studied seriously by mathematicians the world over. This theory of probability is the only one that has the following important property: the likelihood of any event's following the presentation of a sequence of points of data does not depend upon the order in which those data are presented. So fundamental is this property that many would use it as a defining basis for the theory.

The Subjective Interpretation of Probability

A reasonable question is: If probability is so essential to decision-making, why hasn't que mediante una distribución cuidadosa acompañada de una buena promoción, se tendrían ventas potanciales excedentes a las proyectadas en 1966. Aunque Biery y O'keefe estaban completamente convencidos del exito futuro del Snurfer, se encontraban ante la incertidumbre de la demanda total del producto para el año en cuestión, asi como la parte correspondiente por destinar a supers. Estaban seguros, eso si, que para maximizar las ganancias del producto, sería necesario estimar el tamaño de la producción de manera cuidadosa y sistemática. Como era de esperarse, la orden de producción tenía que enviarse al Departamento de Producción al final de Abril de 1967.

El primer paso para determinar tal cantidad, fue revisar las estimaciones recientes del costo de los dos modelos. El jefe de producción informó que la maquinaria existente cuyo costo era de $50,000 se encontraba en buenas condiciones y sería capaz de producir 150,000 unidades de cualquier tipo y en cualquier combinación. Para producir entre 150,000 y 200,000 unidades se requería una inversion extra de $15,000. Incrementar la producción sobre las 200,000 unidades requeriría otros $55,000 pero permitiría a la fábrica producir hasta 500,000 unidades al año. Biery decidió que los costos de inversión en maquinaria deberían amortizarse en el año de su adquisición.

Posterior a una consulta con los agentes de ventas, se consideró vender los Snurfers en 1967 a un precio promedio de fabrica (al aumentar o disminuir la cantidad el precio varia) de $4.30 el regular y $5.50 el super. Los costos directos para la compañía fueron $2.50 y $3.20 respectivamente. Por otra parte, los costos indirectos se calcularon para ambos modelos como un 9% sobre la ganancia, los cuales incluían gastos por administración, renta de inmuebles etc., mientras que un 3% adicional se dedicó a gastos por publicidad. El costo por almacenaje del inventario, se cargó a 2% al mes sobre los costos directos y se estimó que todo inventario en exceso tendría que almacenarse por lo regular un promedio de 6 meses.

Con los costos involucrados definidos, Biery se puso a analizar la demanda. Aunque no estaba seguro de qué cigra seleccionaría, estaba consciente de la improbabilidad de introducción de competencia en el mercado. Aun más, se dió cuenta que el Snurfer era un articulo de novedad y que de seguro seguiría la tendencia característica de ese tipo de articulos, como las patinetas y el hula-hula con ventas muy altas por un par de años y disminuyendo rapidamente hasta desaparecer. Por esto, Biery solo se concentró en la venta del producto para la temporada 1967-1968.

Para determinar la demanda, Biery se reunió con O'keefe y juntos analizaron las posibilidades de los Snurfers. Finalmente concluyeron que la demanda media sería de 150,000 unidades. Un hecho era seguro, que no estaría por debajo de las 50,000 ni en exceso de las 300,000, tambien consideraron que había una oportunidad en 4 de que la demanda sería de al menos 190,000 unidades y que existían 3 oportunidades en 4 de que al menos fuera de 125,000 unidades.

Para poder decidir la cantidad de unidades a ordenar, tenían por otro lado que estimar la demanda para los regulares y para los super. Esto, era obviamente necesario pues, se debian adquirir diferentes materias primas y por otra parte no se deseaba que se tuvuiera un resultado final de regulares inventariados con demanda insatisfecha de super; o viceversa. Ambos coincidieron en que la demanda entre modelos podía considerarse independiente de la demanda total; bajo el razonamiento de que el consumidor seleccionaría entre un modelo u otro, exclusivamente de acuerdo a las diferencias entre estos, y que la decisión sobre cual modelo comprar no se encontraba influenciada por el numero total de Snurfers vendidos.

Biery y O'keefe estimaron que el super Snurfer probablemente sería demandado en un 40% del total pero que se podría llegar hasta un 60%. De cualquier forma, la demanda no caería por debajo del 30% en ninguna circunstancia. Por otra parte, consideraron que existía un 75% de oportunidades de que la demanda fuera de un 45% y un 25% de oportunidades de que los supers formaran un 36% de la demanda total.

Resumen de Costos por Unidad

MODELO	PRECIO	Costo VARIABLE	COSTO INDIRECTO Y PUBLICIDAD	TOTAL	Almacen INVENTARIO
Regular	$4.30	2.50	0.22	2.72	0.30
Super	$5.50	3.20	0.28	3.48	0.38

Cantidades a Producir

Para determinar las cantidades de producción a considerar, Biery decidió estudiar las sugerencias presentadas por diferentes personas involucradas con el proyecto. El siguiente resumen muestra las recomendaciones de oficios me y memoranda enviados.

El personal de ventas argumenta que las ganancias totales en ambos modelos, hacen que el costo por almacenamiento de las unidades no vendidas sea practicamente despreciable, por lo que pretenden que una cantidad total de 225,000 Snurfers sea ordenada. De estos 130,000 deberán ser reguIares y 95,000 super.

Por otra parte el jefe de producción sugiere 150,000 unidades, 70,000 super y 80,000 regulares, argumentando que la cantidad no se requerirá ninguna inversión adicional, y que aumentar la proporción de lpss super a 47% en lugar de 40% era conveniente, yq ya que estas tenian un mayor margen de ganancia y que la proporción era más acorde con la experiencia de las ventas anteriores.

Biery consideró que ambos argumentos eran meritorios, pero estaba un poco esceptico al respecto por lo que proponía que una producción de 200,000 unidades repartidas en 85,000 super y 115,000 regulares disminuiría el costo deve ventas perdidas sin incurrir en una mayor inversión en maquinaria. Para asegurarse de lograr la decision correcta, se propuso realizar un análisis de las 3 alternativas para determinar cual era la mejor. Para ello, se hizo asistir de un cuerpo de asesores expertos en analisis de decisiones, en particular investigadores de la "Business School" de la Universidad de Harvard.

Preguntas Guía

1) Que hubiera hecho usted, en caso de haber sido llamado por Mr. Biery?

2) Hubiera usted analizado todas las alternativas?

3) Cuanto es lo que usted le recomendaría a Mr. Biery, pagar por obtener información extra?

4) Sería posible obtener la solución óptima/

(Asuma en su análisis, que los productos no vendidos en la edición 1967-1968 se venderán durante 1968-1969)

centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

EVALUACION DE PROYECTOS Y TOMA DE DECISIONES

TEORIA DE JUEGOS

ING. CARLOS VALENCIA

OCTUBRE, 1978.

Chapter 11

Games and competitive situations

Competitive situations occur when individuals or institutions are at cross-purposes. They are situations in which there is at least some element of conflict. Two firms battling for market share are clearly at cross-purposes. So are gasoline stations waging a price war, a new car dealer and a customer haggling over price and options, a number of companies bidding for a NASA contract, the writer of an insurance contract and the insured, and an IBM representative and a customer working out the details of a multimillion-dollar computer installation.

In most of these examples the individuals or institutions are not entirely at cross-purposes, however. The car dealer and customer, for example, while at cross-purposes regarding price, share the objective of closing a mutually advantageous deal. Most competitive situations, in fact, contain elements of both mutual interest and cross-purpose. This mixture is part of what makes their analysis so challenging.

Competitive situations pervade every sphere of human activity—military strategy, diplomacy, government, politics, business, sports, and private lives are obvious examples. Because they are so pervasive, competitive situations have been studied from many viewpoints, including economics, politics, history, sociology, psychology, military strategy, and the mathematical theory of games. Depending on the administrative situation under consideration, any one or a combination of these viewpoints may aid decision making.

In deciding on which viewpoints to bring to bear on a particular situation—or, indeed, which aspects of a particular viewpoint are relevant—it helps to be aware of the dimensions on which competitive situations differ. For example, two situations may differ in the degree to

which the parties have mutual interests. To facilitate thinking about competitive problems in general, it is well to be aware of the elements shared by all such situations.

This chapter addresses these issues. The first section explores the common elements of competitive situations and the significant dimensions on which they differ by describing in detail the battle among airlines for transcontinental passengers. The next section provides a framework for analyzing some competitive situations by describing how to analyze two-person zero-sum games. In the last section this framework is extended to nonzero-sum games in order to address issues of cooperation and communications.

AN EXAMPLE: THE BATTLE FOR TRANSCONTINENTAL AIR PASSENGERS[1]

Consider the airlines' battle for transcontinental passengers that has been going on since the early 1960s. The long hauls are the routes of greatest profitability, and the New York–California runs have been termed the "essence of the essence." Thus, competition has been fierce among the three largest airlines, American, TWA, and United, which collectively control about 90 percent of the market. In the days of piston aircraft, just before the battle started, TWA was the dominant transcontinental carrier. However, American, then in second place, was more aggressive than the others in introducing jet aircraft. As a result, it surpassed TWA in the early sixties, achieving 38 percent of the market by 1962. For a while TWA, awaiting delivery of its Convair jets, and United, awaiting DC–8s, emphasized services to counter American's jets. This competitive weapon continued to be used after TWA and United became competitive with aircraft. Since the industry is regulated, price competition was largely ruled out.

About 1963, TWA introduced in-flight motion pictures, and it was some two years before American and United followed suit. Later, TWA was to offer a choice of two movies. Shortly thereafter, United introduced stereo entertainment, and the others soon followed. During the mid sixties, United tried single-class service, and after several disastrous years reverted to the traditional coach and first-class service. In a series of moves and countermoves, the three competitors offered increasingly elaborate meal service, including choice of entrees or steak cooked in flight. By 1967, TWA was touting choice of seven entrees in its first-class service, all cooked in flight. United increased the number of main course choices from two to four in its coach section. Each carrier, as it introduced a new innovation, featured it in its advertising, which was constantly being used

[1] We are grateful to Laurence Doty of *Aviation Week and Space Technology* for providing much of the factual material for this section.

to differentiate the line's service in the eyes of the public. American was the business traveler's line, United the vacation traveler's.

Once all the carriers had sufficient jets, they began to escalate the frequency of their flights, in part in response to growing passenger demand. More importantly, the escalation stemmed from the widespread belief that the carrier with the greatest number of departures would get a share of market more than proportionate to its share of departures. This was so because many travelers initially contact the carrier offering the most flights to their destinations when they make reservations. Consequently, number of departures became one of the most competitive weapons.

By the late 1960s, just before the introduction of the wide-body jets, however, airline capacity became scarce. Nonetheless, the carriers continued their capacity war on the transcontinental routes, at some sacrifice to their less desirable routes. For instance, when American added another New York–California flight in 1967, TWA felt it had to delay the introduction of its new Cincinnati–Los Angeles nonstop service to match American's flight. By 1967, flights were so frequent from New York to California that *Aviation Week* (August 14, 1967) called them a "shuttle." American had 16 of the 43 flights a day, TWA had 14, and United had 13. Their market shares were ranked in the same order. Timing of schedules was also important; the lines constantly jockeyed with one another for the more favorable departure times.

The capacity battle intensified with the introduction of the 747s about 1970. Formerly capacity constrained, the carriers suddenly found excess capacity, because the introduction coincided with an economic recession. After load factors (percentage of occupied seats) had fallen under 40 percent, American finally tried to break the cycle by unilaterally cutting back on capacity. It hoped the others would follow. However, United stood pat and TWA increased capacity. Month by month American watched, waited for the other carriers to revise themselves, and lost market share and large sums of money. Finally, it relented and again entered the fray.

In the meantime, plane configuration became the chief competitive weapon; the "battle of the coach lounges" took place. Spurred by empty seats, in 1972 Continental Airlines removed some seats from planes on its Chicago–Los Angeles run and installed a lounge in the coach sections. The big three quickly followed suit on the transcontinental routes. Soon one carrier featured two lounges. Then came the piano bars; first American and then the others added pianos to their lounges, so passengers could gather, play the piano, sing songs, and imbibe. The battle of the lounges abated in mid-1973.

In 1972 there also was a wave of reoutfitting flight attendants, with first one carrier and then another introducing new uniforms. More moves and countermoves on food took place, with one carrier touting Trader Vic food.

Faced with excess capacity, the carriers then tried fare reductions. TWA filed an application with the Civil Aeronautics Board (CAB) for special book-ahead fares where, if the customer booked 90 days in advance, the fare paid was approximately one half. In defense, American soon filed an application for an identical plan. Not to be outclassed, United filed for similar fares with seven-days-ahead booking, but a minimum seven, maximum nine-day stay. Defensively, the others matched the United plan. The United plan met with the greatest success and finally was adopted by all carriers. The net effect, of course, was to lower the average fare collected by each carrier.

Up to this point in the battle, the carriers had been making capacity decisions independently, without consulting one another. Load factors had dropped below break-even to 36 to 38 percent; competition was so fierce that running planes through maintenance and scheduling crews was a problem. In 1972, the CAB began to encourage negotiations among the carriers to limit capacity. The negotiations started in the summer, under protest from the Justice Department and various consumer groups, and by October 1972, the first capacity reductions led to a 10 percent improvement in load factors. Starting in June 1973 fuel shortages provided further incentives to get together, and, after protracted negotiations, further capacity reductions followed. As capacity was being cut back, TWA was hit by a six-week strike, giving a major assist to the two remaining carriers. In early 1973, the carriers used advertising to vie for market share on the basis of quality of service. This was spurred by American, which was trying to recoup market share lost due to poor service stemming from a pilot slowdown in December 1972 to January 1973. Beginning in February 1973, it started touting the improvement in its service, and the others countered by praising their own.

This particular competitive battle well illustrates the richness of competitive situations: the wide variety of weapons used; the constant moves and countermoves, both offensive and defensive; the importance of timing; the uncertainty about opponents' moves, and whether they will succeed; and the great complexity of the total situation.

Elements shared by all competitive situations

Several of the elements common to all competitive situations which are illustrated by the transcontinental air passenger example are discussed below.

The rules of the game. Perhaps most important of all, there are specific rules that govern the behavior of the competitors. These competitive practices are generally agreed upon, general laws as well as specific industry regulations. For instance, the airline industry is a heavily regulated one; competitors may not change fares without prior approval of the CAB.

Potential payoffs and ultimate outcomes. There is a range of outcomes or *payoffs* that can occur for each competitor—in the case of the airlines, the various market shares, passengers carried, or profits. As a result of the actions of the competitors and possibly of events beyond their control, there is an outcome of the situation—one of the potential payoffs. Each competitor considers some outcomes to be more desirable than others—for instance, more market share is better than less. While this seems obvious, each has relative preferences for the various dimensions of the payoff: market share, immediate profits, long-range profits, cash flow, and so forth.

Outcomes determined by competitor choices and other events. Each competitor has open to it a range of potential strategies it can employ. In the airline example a strategy consists of a stance regarding number of departures, schedules, plane configurations, in-flight services, advertising, and so forth. Each competitor has some control over the situation, but it does not have full control. Some of this control is in the hands of the other competitors. American's success, for example, depends in part on its strategy, but it is heavily influenced by what TWA and United do. Furthermore, some elements may not be in the control of any competitor, such as the strike closing TWA in 1973, the economic downturn of 1970, and the pilot slowdown that hit American in December 1972.

Significant differences among competitive situations

There are also various dimensions on which competitive situations can differ significantly. The way these factors can affect the analysis of the competitive situation are noted in the sections below.

Number of competitors. The number of competitors, or distinct sets of interests, is one of the fundamental ways to categorize competitive situations. It is customary to speak of a conflict situation having two competitors as *two-person* and one with more than two competitors as ***n*-person**, although it may just as well be called a many-person situation. The word *person* is game-theory shorthand for a party at interest in a competitive situation; in short, one of the conflicting "sides." In this sense, a person may be an individual, a group of individuals, a corporation, or a nation.

The two-person conflict situation is the common one in which one person and an adversary have conflicting interests. Certainly the seller of a house you would like to buy does not share your interest in a lower price. Two contractors have clear conflicting interests in bidding for a construction contract.

When there are more than two interested parties, the situation becomes more complex. First, there is simply more to keep track of. Second, and more important, there is the possibility that some of the competitors might

form coalitions to deal more effectively with the others. For instance, the Arab nations banded together to set a common oil policy with the developed nations in 1973, even though the individual nations had somewhat differing interests. Similarly, companies form trade associations to lobby for common interests, workers form unions, and nations sign mutual aid treaties. Sometimes the coalitions are only implicit and tacit, such as banks following common policies in setting their prime rates. Also, workers sometimes band together in informal groups to socially control "rate busters," and card players will gang up on the leader to keep anyone from amassing the number of points necessary to win the game.

When one is faced with coalitions, an important analytical issue is their stability. How likely is it that members of the coalition will break with their original coalitions to join others, form new ones, or strike out on their own? Is it advantageous to encourage or discourage this? Which group is advantageous for you to join?

Another implication of *n*-person situations is simply the need to recognize the number of different interests. For instance, suppose you are negotiating to purchase a small machine shop from its founders and their children. The founders want to retire and divorce themselves financially from the enterprise. The children would like to continue in its management and, if they are successful, share in the rewards. If you fail to recognize these different interests, if you consider "the owners" to be monolithic, you risk missing an appropriately structured deal which will be more in the interests of all parties—including yourself.

Degree of mutual versus opposing interest. There are some situations in which the interests of the competitors are strictly opposed. At the end of a poker game, for example, there is usually just an exchange of assets. Since winnings are balanced by losses, their net is equal to zero. In game theory terms, this type of competitive situation is called a *zero-sum game*.

The zero-sum game may be thought of as one extreme—that of pure conflict. At the other extreme are situations of pure common interest, in which the "competitors" win or lose together, and both prefer the same outcome. For instance, in bridge the two partners do their utmost toward achieving full cooperation. Their fates are inextricably intertwined.

It is difficult to find administrative examples of either pure cooperation or pure conflict, since the vast majority of competitive situations lie between these extremes. In most situations the opponents exhibit varying degrees of common interest and competition. Formally, any game that is not strictly competitive is designated a *nonzero-sum game*.

In a labor negotiation, for instance, labor and management may not agree concerning the division of their joint profit, but both probably want to make the joint profit as large as possible. Thus they have both conflicting and common interests. Similarly, the three airlines competing for transcontinental passengers, while they would prefer gaining market share at the others' expense, would mutually prefer competitive alternatives that profitably stimulate passenger demand, or those that permit handling a given number of passengers at lower cost.

The competitive aspects of most business, political, and military conflicts can only be analyzed in a realistic way if the elements of common interest as well as conflict are taken into consideration.

Communication or agreement about actions. In the airline example, the competing carriers first made independent decisions on departures. The eventual result was that departures escalated and load factors dipped below break-even. When the carriers were permitted to decide jointly on departures, the number of flights was reduced to a profitable level.

This difference in behavior illustrates the significance of perhaps the most important distinction that can be made about competitive situations—whether or not the competitors are allowed to communicate explicitly before making their moves. If so, the situation is said to be *cooperative;* otherwise, it is designated *noncooperative*.

In general, the more the players' interests coincide, the more significant is their ability (or inability) to communicate. Where there is pure common interest, the problem is entirely one of communication. In competitive situations in which the decision makers have some common interests and some conflicting interests, communication, if permitted, plays a complex role in determining the outcome. In two-person, pure-conflict situations, communication cannot benefit either competitor.

Sometimes the competitors must take action in the complete absence of communication, as do participants in a sealed bid auction. Under such noncooperative circumstances, the analysis of a competitor's potential actions should influence the other party's actions. Sometimes competitors can communicate to a limited degree, as with public pronouncements, but must stop short of actual agreement on a mutual course of action. For example, the president of TWA might announce that TWA will match American's departures plane for plane. The purpose of this type of communication—threat, promise, or bluff—is to attempt to influence the opponent's behavior. The effect of these limited communications then enters the competitive analysis.

Finally, there is the cooperative situation where the competitors are in full communication and jointly attempt to reach agreement. Promises, threats, and bluffs continue to play a role in attempting to change each other's preferences and attitudes. However, now the adversaries, through dialogue, also attempt to create new alternatives while trying to reach agreement. This is the bargaining situation.

Before leaving the subject of communication, the role of tacit communication bears mentioning. In most marketplace competition, the law forbids collusion. Nonetheless, although competitors do not communicate directly with one another, "understandings" often develop. Price leadership in the steel industry is a good example. The kinds of understandings that emerge and their stability is an important aspect of such competitive situations. So is the way that competitors "signal" their intent to one another, without explicitly communicating. For example, American was apparently unsuccessful in signaling the other airlines to cut back capacity in 1971.

Repeating the competitive situation. Another important dimension of difference is whether the same participants will be involved in a similar situation in the future. For instance, the buyer and seller of a house most likely will not, whereas a particular union and company will be back at the bargaining table at the completion of a just-negotiated contract. Similarly, the competition between the airlines is an ongoing one.

In one-shot situations, competitors are usually out for all they can get. In an ongoing situation, they often behave much differently. All they can get is tempered by what the impact will be on what they might get in the future. If management negotiates too stringent a contract this time, the union may be more militant the next time.

Amount of information each competitor has. Information is one of the most important commodities in a competitive situation. If this were not the case, we would not see the tremendous secrecy with which Detroit's automakers treat their new designs. We would not see a petrochemical manufacturer photographing a competitor's outdoor chemical facilities from the air, so that its chemical engineers could infer the production process from the configuration of the facility and thus estimate the competitor's costs. We would not see frogmen from one oil company checking on the offshore drilling rigs of another.

Indeed, some feel that much can be gained by analyzing a competitive situation, particularly a bargaining one, in terms of exchange of information. What would you like to know about your competitor? What would you like your competitor to believe about you?

There is a host of things about which you might have relatively abundant or limited information. For instance, you may know specifically who your competitor is, or you may not. If you are building contractor submitting a bid to the city of Hartford for the construction of its proposed civic center, you may not know who your competitors are. In order to make a decision about how much to bid, you may have to hypothesize about the typical competitors facing you.

More frequently you know who your competitors are, but there may still be substantial information gaps. You may not know what competitive options your competitors are considering, much less which ones they will choose. Nor will you have a clear understanding of their objectives, or of their views—sanguine or pessimistic—of future conditions in the markets for which you are competing. You may not have information about the innermost workings of your competitor's organization, such as costs or resource allocation. (In the airline industry cost information is publicly reported, for instance, but in most industries it is not.)

Sometimes there is uncertainty about the value of the item for which you are competing. In competing for oil rights leases, for example, bidders usually do not know for certain the value of the reserves on the property. To make matters worse, some competitors may have a better idea than others about the value of the item. For instance, the seller of a company often has important information unavailable to the buyer.

Sometimes, unfortunately, you fail to have complete information about yourself and your organization. What are your objectives? Do you have the resources necessary for the competitive battle that might ensue if a particular course of action is chosen? Apparently GE and RCA did not when they announced plans to become greater factors in the computer industry and then withdrew.[2]

From the discussion and examples cited above, it is evident that decision making in competitive situations is a tricky, delicate, difficult business. In the following sections some formal structure is presented to assist in analyzing competitive situations.

TWO-PERSON ZERO-SUM GAMES

To introduce some of the key elements in the analysis of competitive situations and to put these elements as starkly as possible, we have chosen the simplest of competitive situations. This is the *two-person zero-sum game*, so named because two parties compete for the same resource: what one gains, the other loses.

Although this kind of situation is somewhat rare, many of the basic analytic ideas carry over to the more realistic nonzero-sum context. Furthermore, many people treat competitive situations that are not zero-sum as though they were. It pays to know a little about the zero-sum setting to understand what is wrong with their thinking.

We will use as examples pseudo-administrative problems in contexts with which you are familiar and will place you directly in the position of the decision maker. We use the word "pseudo" advisedly, because we have had to distort real administrative facts somewhat in order to achieve simple, zero-sum settings. First, we look at a situation in which two competitors vie for market share through television advertising. We analyze this situation only in part and then digress to consider three simpler situations which illustrate various solution techniques. We will complete the analysis of the marketing example after discussing these three situations.

[2] William Fruhan, "Pyrrhic Victories for Market Share," *Harvard Business Review*, September–October, 1972.

A marketing example: General Edison versus Westvania

General Edison, the largest manufacturer of electric light bulbs for home use, has as its sole competitor the Westvania Corporation. Consumers purchase their slightly differentiated products infrequently, and both brands are available widely. About three quarters of the purchases are made by consumers who are extremely loyal to one brand or the other; the other customers are not at all brand loyal. The brand these consumers select is exclusively influenced by the advertising to which they have been exposed just prior to each purchase.

The two companies vie for these uncommitted customers (whom we call *the market*) solely through television spot commercials, with advertising commitments made monthly. The Federal Trade Commission watches competition carefully and sees to it that the networks keep the advertising plans of the competitors confidential. It is a long-standing industry tradition that GE buys three spots a day on each network and Westvania purchases two a day.

The television advertising day is divided into three segments—morning, afternoon, and evening. Twenty percent of the bulbs are purchased on the basis of viewing morning advertising, 30 percent on the basis of afternoon viewing, and 50 percent on the basis of evening viewing. Whichever firm buys the most spots during a segment captures the *entire* market resulting from that period. If GE and Westvania buy the same number of spots during any one period, each gets half the purchasing audience; this is the case even if neither buys spots. Since use of bulbs is unaffected by advertising, neither company's advertising affects the size of the market—only market share is related to advertising efforts.

Suppose that you are the advertising director of General Edison and you must decide on its advertising plan for the coming month. Given the situation and industry traditions, you are in a zero-sum situation. Your interests are strictly opposed to Westvania's: what you gain in market share Westvania loses, and vice versa. What will your advertising schedule be? And how much of the coming month's market will you expect to capture as a result?

Let us speculate on how you might think about these questions. You might consider putting all your advertising in the evening. That way you are assured of at least half the market—how much more you get depends on when Westvania uses its two spots. If, for example, Westvania uses one in the morning and one in the afternoon, you will get exactly 50 percent of the market, since you have the majority of evening spots and they have the majority of morning and afternoon spots. Or, if you are lucky, Westvania will put both of its spots in the afternoon. In this case you could get all of the evening plus half of the morning, for a total of 60 percent of the market.

Your thoughts about using all your spots in the evening might tempt you to conclude that Westvania would never use its two spots in the evening. So you might decide to put two in the evening and one in the morning. That way, if Westvania puts its two spots in the afternoon, you will win both the morning and evening purchasers, for a total of 70 percent of the market. However, if Westvania splits its spots between morning and afternoon, you will get 60 percent of the market. Figuring that Westvania might do this, you then think of putting two spots in the afternoon and one in the evening: that way you get 80 percent of the market. But if Westvania knew you were thinking seriously of doing that, it might go with two in the evening—to get 60 percent of the market, leaving you with a mere 40 percent. On the other hand, you could counter their move by going back to your original idea—three spots in the evening—and thereby capture a whopping 75 percent of the market. And so it goes.

A pattern emerges. How well you do with your advertising schedule depends on what your opponent does. You must, therefore, take into account possible competitive moves in deciding on your strategy. And your competitor will take your moves into account. There is a possibility for an endless choir of "I think that they think that I think that they think . . ." Your destinies are inexorably intertwined. How can we make progress in analyzing this problem?

Toward resolving the dilemma

There are three major steps in analyzing a game: (1) understanding the options open to you and your opponent, (2) understanding the well-being of you and your opponent in every combination of strategies, and (3) analyzing and choosing a strategy.

The first thing you need to do is get a clear picture of the choices open to you and to your opponent. It turns out that there are ten distinct options open to you and six open to your opponent in this example. Your options are for two evening spots and one afternoon spot, two evening spots and one morning spot, and so forth. Since there are 16 options for you and your opponent, you need a shorthand to list them succinctly: Let E stand for an evening spot, A for afternoon, and M for morning. Now if you want to represent two evening spots and one afternoon spot, you can simply write EEA. Using this shorthand, your options and your opponent's can be listed, as in Table 11-1. Each option is called a *strategy*.

Table 11-1
List of strategies open to you and your opponent

General Edison's strategies		*Westvania's strategies*
EEE	EMM	EE
EEA	AAA	EA
EEM	AAM	EM
EAA	AMM	AA
EAM	MMM	AM
		MM

In any competitive situation, you need to be aware of all the strategies open to your opponent, or your opponent could possibly slip one past you. You need to also understand your options or you might miss out on a good one, simply because you did not consider it. (Later on we will see that we really have not listed all the options open to you and your opponent in this particular situation, and failure to consider the omitted strategies can result in leaving money on the table.)

The second thing you need to do is to consider how well off you and your opponent would be for any combination of your respective strategies. For example, EEE against AA yield 40 percent market share to your opponent and 60 percent to you. There are lots of ways to indicate how well off each of you would be—tables, graphs, formulas, and words can all be used. The best way depends upon the particular competitive situation. In this case a table seems most useful. Across the top you can list your competitor's strategies, and along the side you can list yours. At each intersection you can list your market share and your competitor's, that is, the two *payoffs*.

Actually you do not have to list both your own and your competitor's payoffs, since this is a zero-sum game. If you list yours, then your competitor's payoffs will be known automatically—if yours is 60 percent, then theirs must be 40 percent. Such a table, called a *payoff table*, is presented for your problem as Table 11-2. For instance, the entry in row EEE and column EM says General Edison gets 65 percent of the market (and Westvania gets 35 percent) if General Edison follows strategy EEE and Westvania chooses EM.

Table 11-2
Market share captured by General Edison

		Westvania's strategies					
		EE	*EA*	*EM*	*AA*	*AM*	*MM*
General Edison's strategies	*EEE*	75	60	65	60	50	65
	EEA	65	75	80	60	65	80
	EEM	60	70	75	70	60	65
	EAA	40	65	55	75	80	80
	EAM	50	60	65	70	75	80
	EMM	35	45	60	70	70	75
	AAA	40	40	30	65	55	55
	AAM	50	50	40	60	65	55
	AMM	50	35	50	45	60	65
	MMM	35	20	35	45	[illegible]	60

Now that you have a reasonably succinct statement of your problem, you can begin to analyze it in order to choose a strategy. Before doing so, we will consider a series of three simpler competitive situations that illustrate analytical approaches, then return to use them on this problem.

Situation 1: Board of directors meeting, Jimenez versus Smith

The Giant Corporation will soon have a vacancy on its board of directors, and the current 12 directors will meet early next month to choose the company's candidate for the position. Pedro Jimenez and Hamilton Smith are the only two being considered. Whoever wins the greater number of votes captures the nomination, and such a nomination is usually tantamount to election. At a subsequent meeting, the board will make a final decision about whether Giant will follow a slow, moderate, or rapid five-year growth plan.

It is thought that a nominee's position on which growth plan Giant should follow will be the biggest factor in determining the number of votes received, since the two are about equally qualified for the position. Jimenez, however, has a slight edge over Smith since he is a Mexican-American, and the board is eager to have an additional minority-group member.

You are Jimenez. After informal conversations with individual directors, you put together Table 11–3, which shows the number of votes you expect to receive if you and Smith take the positions shown. You are the Row player. If, for example, you favor moderate growth, R_M, and the Column player, Smith, favors rapid growth, C_R, then you expect to receive eight votes. Smith, of course, wins the remaining four votes, since this is a zero-sum situation.

Table 11–3
Number of votes won by Jimenez*

Row's (Jimenez's) choices	Column's (Smith's) choices: C_S	C_M	C_R
R_S	7	9	10
R_M	5	7	8
R_R	4	5	7

* Seven votes are necessary for a majority.

You have no a priori reasons to favor one growth plan over another. Your only concern is to win the nomination. Which plan should you favor?

Analysis by dominance. Observe that you win the most votes if you favor the slow-growth plan, R_S—regardless of Column's choice. Strategy

R_S is said to be your *dominant* strategy. Your choice is easy . . . you would be foolish to choose a strategy other than R_S.

What will Column do? For Smith, the strategies C_M and C_R are both dominated by C_S (Remember—he likes the smaller entries.) If Smith's only concern is to maximize his own votes, and, if he perceives the situation as you do in Table 11–3, he will be acting in his best interest if he also chooses to favor the slow-growth plan. Thus, if each player chooses his dominant strategy, the final outcome is seven votes for you and five for Smith.

In this brief analysis we have assumed that each player as his sole objective wishes to maximize the number of votes received. In other words, we have assumed that both Row and Column are so-called *rational* players—that each will endeavor to choose a strategy that will maximize his own ends.

We also have assumed that each player perceived the situation in the same way. That is, we assumed Row and Column both constructed the same payoff table. Of course, there are situations in which this fundamental assumption does not hold. For example, if Smith is already a member of several other boards and feels he is too busy to hold an additional directorship, he may decide to help Jimenez win by as large a vote as possible. In such a situation, Jimenez and Smith do not have the same payoff tables. Jimenez will use Table 11–3, but Smith will construct a payoff table whose entries reflect a different utility of winning each number of votes. In other words, the entries in Smith's table must be weighted to show that he prefers to win as few votes as possible.

In our analysis of zero-sum games we will always make these two fundamental assumptions: (1) perfect rationality—each player is rational, seeking only to maximize his own gain, and (2) perfect information—both players have the same payoff table, and they both know it.

Analysis by iterated dominance. As the date of the meeting approaches, the business outlook for the next few years is growing increasingly rosy due to unexpected events at home and abroad, and you, as Jimenez, feel that fewer of the directors will favor slow growth over the next five years. Accordingly, you decide to change your payoff table to that shown in Table 11–4.

Table 11–4
Revised number of votes won by Jimenez

Row's choices	Column's choices: C_S	C_M	C_R
R_S	9	5	7
R_M	8	7	8
R_R	10	6	5

This time you as Row do not have a clearly dominant strategy. Neither does Column, but Column has a strategy he does *not* want to choose, his *dominated* strategy, C_S. Since every vote not cast for Row is cast for Column, Column always does better by choosing either C_M or C_R, depending on which strategy Row chose.

Assuming Column is a rational player, it follows that he will not choose C_S, so the first column can be eliminated from Table 11–4. The *reduced game* is shown in Table 11–5. In this reduced game you will do best to follow your dominant strategy, R_M, since you always do best by favoring moderate growth.

Table 11–5
Reduced payoff table

Row's choices		Column's choices C_M	C_R
	R_S	5	7
	R_M	7	8
	R_R	6	5

Now, what do you think Column's position will be? Well, if he refers to Table 11–5 he will observe that he does not have a dominant strategy. However, if he notices that you do have one, R_M, and if he assumes you will follow it, then essentially Column is confronted with the reduced game shown in Table 11–6.

Table 11–6
Payoff table reduced again

Row's choices		Column's choices C_M	C_R
	R_M	7	8

Now, *of course*, Column wins more votes if he favors moderate growth over rapid growth. However, you win the nomination since, in the reduced game shown in Table 11–6, you capture a majority of the votes regardless of Column's choice.

Reduction of a game by dominance is a useful first step in analyzing a game. Sometimes the reduced game can be reduced again, as in this situation. Sometimes the second reduction can be reduced still further, and so forth. This is called *analysis by iterated dominance*. In some cases, its application leads to the choice of a best strategy for each player. But usually you are not so lucky, as illustrated by the next example.

Situation 2: Selection of an advertising package, General Truck versus National Motors

General Truck and National Motors comprise a duopoly in the sale of replacement parts for diesel engines. Because of a persistent sluggishness in the U.S. economy during the past few years, replacement part sales have remained fairly constant. While neither General Truck nor National Motors has ever captured more than 65 percent of total industry sales, year-to-year fluctuations in market share have often been dramatic.

Replacement part sales have shown little sensitivity to either price increases or technological innovation. The diesel engine manufacturers supply their customers with a complete maintenance schedule specifying how often each part should be replaced. Moreover, union contracts and ICC regulations require that parts be replaced according to the maintenance schedule. Therefore, replacement part sales are a forced rather than a discretionary purchase.

Over the last ten years technological innovation has usually represented only minor changes in design or materials. These changes have not been the focus of any attempt to create product differentiation.

The principal vehicle for selling replacement parts for diesel engines is advertising. Both General Truck and National Motors advertise extensively in *Modern Diesel Design*, the largest publication devoted entirely to reporting trends and developments in diesel engines and diesel parts. Every November the editors of *Modern Diesel Design* meet with the marketing directors of General Truck and National Motors to agree on an advertising package for the next issue. Because of spacing requirements (neither General Truck nor National Motors wants their advertisements to appear within seven pages of their competitor's advertisements) and *Modern Diesel Design's* policy of limiting advertising copy to 25 pages, the editors usually submit three different package proposals to each company. Each package specifies the size, position, and page of the various advertisements in the issue.

Both General Truck and National Motors may choose any one of the three packages which the editors of *Modern Diesel Design* have proposed. While one package may offer the advantage of more advertising space at the very beginning or end of the issue, another package may propose several positions close to an editorial discussing the diesel parts replacement market.

Each firm is aware of the three packages which the editors of *Modern Diesel Design* have proposed to its competitor. Suppose that you are the marketing director of General Truck. You have just reviewed your three packages and the three packages which the editors have offered National Motors. Which package would you choose?

Several factors will influence your choice. First of all, it will depend upon your appraisal of the attractiveness of each package *Modern Diesel*

Design has offe   you. Second your choice will depend upon your perception of how well each package will fare in light of the three options open to National Motors. Finally your choice will depend upon your estimation of which package National Motors will choose.

The choices open to each firm and the market share gain or loss which you, as the marketing director of General Truck, have assigned to each pair of choices are shown in Table 11–7. For example, if General Truck chooses package G_1 and National Motors chooses package N_3, then General Truck will gain 8 percent of the total replacement market and National Motors will lose 8 percent of the market. As we noted earlier, replacement part sales have remained relatively constant during the past few years. Thus any gain in sales for either company must come at the expense of its competitor. In other words, we have the basis of a zero-sum game—what General Truck gains (loses), National Motors will lose (gain).

Table 11–7
Payoff to General Truck

		National Motors' choices		
		N_1	N_2	N_3
	G_1	−1	−4	8
General Truck's choices	G_2	0	3	6
	G_3	−3	5	−7

As you survey this matrix, or payoff table, you must consider two features of the game while choosing a strategy. First, you must assume that the competition is rational; that is, no matter which package General Truck chooses, National Motors will behave in a manner which will maximize their gain and minimize their risk of loss of market share. This idea of rational behavior also points up another important feature of zero-sum games. Since this is a zero-sum game, whatever General Truck gains National Motors must lose. There is no room for bargaining in any zero-sum game, since neither player has anything to offer his opponent. (We eliminate magnanimous gestures of generosity as irrational.)

Surveying the payoff table, you first examine the matrix to determine if dominant strategies exist for either General Truck or National Motors. Since you do not find any dominant strategies, the worst possible outcomes which can result for each of General Truck's three choices are listed. In the payoff table these are the minimum values in each row—the *security level* for each strategy. Of these three minimum values, you prefer the value 0 (the minimum of G_2), which is greater than −4 (the minimum of G_1) or −7 (the minimum of G_3). This value 0, is the maximum of the minimum values, or the *maximin value*. In Table 11–8 we have reconstructed the original payoff table and have listed the three security levels

Table 11–8
General Truck's security levels

		National Motors' choices			*General Truck's security levels*
		N_1	N_2	N_3	
	G_1	−1	−4	8	−4
General Truck's choices	G_2	0	3	6	0*
	G_3	−3	5	−7	−7

* General Truck's maximin value.

for General Truck in the margin. We have also placed an asterisk next to the maximin value.

You know that if strategy G_2 is chosen, then the worst outcome that can occur is neither a gain nor a loss in market share. Although strategy G_1 might result in the largest gain in market share (8 percent), strategy G_1 might lead to a loss of 4 percent.

To determine the outcome of the game, you must look at the game from the point of view of the competition at National Motors. The marketing director of National Motors will follow a plan of attack similar to the one you followed for General Truck. Since National Motors has no dominant strategies, the worst possible outcomes that can occur for each strategy are listed. However, instead of finding the minimum values for each row, the marketing director of National Motors will list the maximum value of each column. Remembering that the payoffs in the table represent the market share gain (or loss) to General Truck, the marketing director of National Motors must find the maximum value of each column. These maximum values are the maximum loss which could occur from each strategy. Table 11–9 shows that strategy N_1 could result in a maximum loss of 0, strategy N_2 in a maximum loss of 5, and strategy N_3 in a maximum loss of 8.

Of these three maximum values (security levels of National Motors), the marketing director of National Motors prefers the value 0 (the maximum of N_1), which is less than 5 (the maximum of N_2) or 8 (the maximum of N_3). This value, 0, is the minimum of the maximum values, or simply, the *minimax value*. (We have indicated with an asterisk the minimax value in Table 11–9.)

If you, as the marketing director of General Truck, play the maximin strategy (G_2), and the marketing director of National Motors plays the minimax strategy (N_1), neither player will gain nor lose any market share—the market share positions will remain the same.

This game illustrates a special case of two-person zero-sum games where the optimal strategy for each player is to select a single option: the single option is called a *pure strategy*. Row chooses Row's maximin strategy and Column chooses Column's minimax strategy. It turns out that the

Table 11-9
National Motors' security levels

		National Motors' choices N_1	N_2	N_3	General Truck's security levels
General Truck's choices	G_1	−1	−4	8	−4
	G_2	0	3	6	0
	G_3	−3	5	−7	−7
National Motors' security levels		0*	5	8	

* National Motors' minimax value.

maximin value equals the minimax value; this is often called a *saddle point*. In addition, neither player has any incentive to alter its strategy as long as the other player chooses its maximin or minimax strategy. For example, if General Truck plays a maximin strategy, National Motors can only lose by playing a nonminimax strategy: N_2 would lead to a loss of 3 percent of the market, and N_3 would lead to a loss of 6 percent of the market. Similarly, if National Motors plays a minimax strategy, General Truck can only lose by playing a nonmaximin strategy: G_1 would lead to a loss of 1 percent of the market, and G_3 would lead to a loss of 3 percent of the market.

After each player has chosen the appropriate maximin or minimax strategy, both have arrived at a stable outcome. Neither player can gain from unilaterally changing strategy: the players have reached an *equilibrium point*, and the game is over. This situation, unfortunately, does not always happen this simply, as we shall see in the following example.

Situation 3: The fighter aircraft proposal, Excalibur Aviation versus Western Aircraft

For the past three decades Excalibur Aviation and Western Aircraft have dominated the military defense market for bombers, fighters, and attack aircraft. Each firm has worked closely with the Navy, Army, and Air Force in research and development projects geared to maintain U.S. air superiority.

To reduce the risk of dependence on any one company, the military allocates its aircraft demand between the two companies. It does, however, choose a primary supplier and a secondary supplier. In the past, a company's selection as primary supplier has implied a 60 to 65 percent share of the military's demands, with the balance of the total requirement accruing to the secondary supplier. Recently, the military announced that it will not be bound to any fixed allotment of aircraft between the primary and secondary suppliers.

Although Congress annually debates and approves the U.S. level of military defense spending, the effects of detente have substantially limited the number (though not the capability) of aircraft of both the United States and the Soviet Union. Since the Kiev Agreements several years ago, the aircraft arsenals of both nations have remained constant.

Over the years the aircraft division of the Navy, Naval Air Systems Command, has invited both Excalibur and Western to propose the specifications of a fighter aircraft superior to the most modern Soviet design. While the capabilities of the Soviet MIG–28 (the vanguard of their sea-to-air defense system) are well documented, the Navy's request for proposal has failed to define its basic measure of superiority.

The effectiveness of a fighter aircraft is dependent upon maximum speed and range, weapons load, and avionics gear. However, there are tradeoffs in the design of such aircraft. Because no single model can incorporate every advanced technological feature, and because no manufacturer can judge the extent of these tradeoffs until after the prototype stage, it is not unusual for a manufacturer to independently design two different aircraft.

In response to the Navy's newest request for proposal, Excalibur and Western have each developed two different aircraft. While neither company knows which aircraft its competitor will propose to the Navy, both Excalibur and Western know the general characteristics of their competitor's designs.

Suppose that you are the director of military sales for Excalibur Aviation. Your company has just completed testing the new E–11 and E–12 fighters which were developed for the Navy. Which aircraft will you propose to Naval Air Systems Command?

Even though you believe that Excalibur's fighters are superior to either of Western's two new aircraft, your choice of the E–11 or the E–12 will rest on three criteria: (1) the capabilities of each aircraft, (2) an estimation of the performance of each aircraft in comparison with Western's two aircraft, and (3) an estimation of which aircraft Western will propose.

The choices open to each company (either the E–11 or E–12 for Excalibur and either the W–7 or the W–17 for Western) and the gain in Naval fighter aircraft market share which you have assigned to each pair of choices are shown in Table 11–10. For example, if Excalibur chooses

Table 11–10
Market share gain for Excalibur Aviation

		Western's choices W–7	W–17
Excalibur's choices	E–11	9	2
	E–12	3	7

strategy E-11 ai   estern chooses strategy W-17, Excalibur will gain a 2 percent share of the fighter market, and Western will lose a 2 percent share of the market.

Since the number of military fighters has remained constant since the Kiev Agreements, any gain (loss) in one company's market share must represent an equal loss (gain) to the other company. In other words, we have the basis of a zero-sum game.

As director of military sales for Excalibur, your first step in solving this game is to examine the payoff table for dominant strategies. However, since no dominant strategies exist, you then proceed to identify the security levels for each company's strategies and the respective maximin and minimax values. Table 11-11 lists the security levels for both companies and indicates Excalibur's pure maximin value and Western's pure minimax value.

Table 11-11
Security levels for both companies

		Western's choices W-7	Western's choices W-17	Excalibur's security levels
Excalibur's choices	E-11	9	2	2
	E-12	3	7	3*
Western's security levels		9	7†	

* Excalibur's pure maximin value
† Western's pure minimax value

If Excalibur plays its pure maximin strategy, E-12, and Western plays its pure minimax strategy, W-17, Excalibur will gain 7 percent of the market and Western will lose 7 percent of the market. However, the pure maximin value is not equal to the pure minimax value. If Western knew for sure that Excalibur would follow its pure maximin strategy, a rational decision would dictate that Western abandon its pure minimax strategy, W-17, and follow strategy W-7. In this way Western could reduce its loss from 7 percent of the market to only 3 percent of the market. But, on the other hand, if Excalibur knew for sure that Western would not follow its pure minimax strategy, W-17, but rather would follow strategy W-7, an equally rational decision would dictate that Excalibur follow strategy E-11, with a resulting gain of market share from 3 to 9 percent.

We can extend this type of analysis indefinitely for either company by adopting the train of reasoning "If I knew that they knew that I knew that they knew . . ." However, before you rush to conclude that, as director of military sales at Excalibur, you have no concrete rationale for choosing either strategy, let us reexamine what we know about the characteristics of the game. We know that no dominant strategies exist and that no pure

strat   will yield an equilibrium solution with maximin equal to minimax. We have also demonstrated that either company can gain from knowing which strategy its competitor will follow. Therefore, under no circumstances will either company have any incentive to reveal its strategy.

The mixed strategy. Because no pure equilibrium strategy exists, any of the four payoffs is possible. Moreover, in the absence of an equilibrium point, it might be reasonable for Excalibur to consider a different type of strategy in order to keep its opponent from guessing what it plans to do. For example, consider the following plan: You flip a coin. If heads appear, E-11, is chosen; if tails, E-12. This strategy's *expected payoffs* can be incorporated into the payoff table shown in Table 11-12. Notice that by playing this *mixed* strategy[3], Excalibur has at least an *expected* 4½ percent larger market, a higher value than either *pure strategy*.

Table 11-12
Payoff table with a mixed strategy

		Western's choices W-7	Western's choices W-17	Excalibur's security levels
Excalibur's choices	E-11	9	2	2
	E-12	3	7	3
	Mixed (½ E-11, ½ E-12)	6	4½	4½*
Western's security levels		9	7†	

* Excalibur's maximin value.
† Western's minimax value.

There is nothing sacred, of course, about a 50/50 mixed strategy. Your task is to find that combination of pure strategies which will maximize Excalibur's long-run expected payoff. You must also recognize that Western is searching for some combination of its pure strategies which will minimize its long-run loss of market share. It turns out that the optimal strategy[4] for Excalibur is to choose E-11 and E-12 in the ratio of 4 to 7, while the optimal strategy for Western is to play W-7 and W-17 in the ratio of 5 to 6. These strategies have equal expected payoffs to each company, as shown in Table 11-13. In addition, they are characterized by

[3] Recall that a pure strategy is one that dictates the selection of a single option. We can define a mixed strategy as one that directs the player to choose from two or more options according to some probabilistic rule which details with what probability each option is to be selected.

[4] For this simple 2 × 2 payoff table, there is an easy graphical method to compute this mix. In general, for any two-person zero-sum game, mixed strategies can be determined by formulating and solving an appropriate linear program. It is not necessary to acquaint you with the computational details of finding mixed strategies, but we want to point out that there are situations in which it is worthwhile to consider mixed strategies.

Table 11-1[illegible]
Payoff table with optimal mixed strategies

	Western's choices: W-7	W-17	Mixed ($\frac{5}{11}$ W-7, $\frac{6}{11}$ W-17)	Excalibur's security levels
Excalibur's choices: E-11	9	2	5.18	2
E-12	3	7	5.18	3
Mixed ($\frac{4}{11}$ E-11, $\frac{7}{11}$ E-12)	5.18	5.18	5.18	5.18*
Western's security levels	9	7	5.18†	

* Excalibur's maximin value.
† Western's minimax value.

the same property found with pure strategies in the previous example: Either company will do worse if it unilaterally moves away from its optimal mixed strategy. Thus there is no incentive to do so.

Although you now know the appropriate combination mix for each player, you are still faced with the problem of choosing one strategy or the other for Excalibur. Remembering that either company can gain by knowing which strategy its competitor will follow, you must guard against any internal or external influences, preferences, or pressures which would give Western any indication of your choice of aircraft. The only fail-safe method of maintaining the secrecy of your choice and of reflecting the appropriate combination of strategies in your decision-making process is to allow some random device to make your decision for you. For example, you could place four red balls (corresponding to strategy E–11) and seven white balls (corresponding to strategy E–12) in an urn and draw one of the balls. The color of the ball will indicate which strategy you will follow. Although it may seem irresponsible to relinquish control of your decision to a random device, such a device is only a method of ensuring that your choice is made purely and exclusively on the basis of the optimal combination of strategies.

Solution to the marketing example

Earlier in this section, we began to analyze a marketing problem involving two light bulb manufacturers, General Edison and Westvania, in which you were the advertising director for General Edison. Now that we have examined the concepts of iterative dominance, minimax, equilibrium points, and pure and mixed strategies, we are equipped to finish the analysis.

The payoff table for this problem, Table 11–2, shows that some strategies are clearly bad and can be rejected immediately. Considering Table 11–2, General Edison would never want to play strategy AAA, since it is dominated by EEM: GE does better to play EEM instead of AAA,

regardless of what Westvania does. EEM dominates AMM and MMM as well as AAA; and EAM dominates AAM. Thus GE's last five strategies can be effectively deleted. Once this is done, MM can be deleted for Westvania. Using iterated dominance, the table has been reduced to one in which each player has five strategies, as shown in Table 11–14.

Table 11–14
The reduced marketing game

General Edison's strategies	Westvania's strategies: EE	EA	EM	AA	AM
EEE	75	60	65	60	50
EEA	65	75	80	60	65
EEM	60	70	75	70	60
EAA	40	65	55	75	80
EAM	50	60	65	70	75

Next you would analyze the reduced table by finding security levels, as indicated in Table 11–15. You, as advertising director for GE, can expect at least a 60 percent market share if you follow EEA or EEM; Westvania can expect at least 25 percent (that is, no more than 75 percent for GE) if it adopts EE, EA, or AA.

If you adopt a pure strategy and Westvania figures it out, then the best you can expect to do is to win 60 percent of the market. By appropriately choosing sometimes one strategy, sometimes another, however, you can expect to do better than this. The same is true for Westvania. The largest share it can expect to win for any choice of pure strategy is 25 percent, but by following a mixed strategy, it can gain an expected share greater than 25 percent.

Since no optimal pure strategy exists, each company can expect on average to do better than its security level if it follows a suitable mixed strategy. There is an equilibrium outcome (in between the 60/40 and 75/25 split) which is found by adopting appropriate mixed strategies. If you were to do the calculations necessary to determine the optimal mixed strategy, you would find that GE should follow each of the strategies EEE, EEA, and EAM one third of the time, and Westvania should adopt EE 6 out of 15 times, AA 5 out of 15, and AM, 4 out of 15. Then the

Table 11–15
The reduced marketing game with security levels

General Edison's strategies (rows); *Westvania's strategies* (columns)

	EE	EA	EM	AA	AM	*Row minima*
EEE	75	60	65	60	50	50
EEA	65	75	80	60	65	60*
EEM	60	70	75	70	60	60*
EAA	40	65	55	75	80	40
EAM	50	60	65	70	75	50
Column maxima	75*	75*	80	75*	80	

* Pure strategies having positive weights in optimal mixed strategies

expected payoff at GE will be 63⅓ percent of the market; Westvania's payoff will be 36⅔ percent.

Notice that our analysis of this and the previous example has assumed that both opponents are rational. We have said nothing about how to exploit irrational play on the part of one's opponent. It is, however, true that if you follow your optimal strategy, then having an irrational opponent (i.e., one who does not play his "best" strategy) will only increase your expected payoff over what it would have been with a rational opponent.

Summary

In this section we have taken a look at how to analyze a competitive situation, using the well-known two-person zero-sum game as a setting. In the process we have introduced the important concepts of pure strategy, payoff table, dominance, security level, equilibrium, and mixed strategy. In spite of the fact that few administrative settings are zero-sum situations, these concepts serve well in more realistic nonzero-sum situations.

In addition, we have begun to suggest a procedure for analyzing competitive situations. It consists of these steps:

1. Understand the strategies open to you and your opponent.

2. Understand how well off each of you will be for all combinations of strategies by displaying this information in a useful way.
3. Analyze the display to arrive at a preferred course of action, taking into account your opponent's likely strategy.

The particular ways of handling the last step in this procedure are quite mechanical in the case of zero-sum games. In the next section we will show that the analysis of nonzero-sum situations is not so easy.

NONZERO-SUM GAMES

In the preceding section we discussed situations in which persons or organizations were entirely at cross-purposes. In most competitive situations, however, there are elements of mutual interest as well as cross-purpose. The potential value of entering into prechoice communication and making binding agreements is a major difference between zero-sum and nonzero-sum situations. While this section will primarily consider noncooperative situations, in which the competitors may not communicate before making their moves, it will also consider cooperative situations in which the opponents are allowed to make joint decisions and the impact of communication. Threatening, promising, bluffing, bargaining, colluding, and preempting all may play a role, depending on the exact nature of the situation.

We will introduce these concepts in the context of five prototypical competitive situations. Two predominant ones are known as the Prisoner's Dilemma and the Battle of the Sexes. The others can be called no-conflict situations, threat-vulnerable situations, and force-vulnerable situations.

Games with little or no conflict

In *nonzero-sum games,* the payoff tables contain two entries in each cell: the first is the payoff to the Row player and the second is the payoff to the Column player. Game A shown in Table 11–16 is rather easy to

Table 11–16
Matrix games with Pareto-optimal outcomes

Game A (*Row* / *Column*)

	C_1	C_2
R_1	12, 8	7, 5
R_2	10, 2	4, 0

Game B (*Row* / *Column*)

	C_1	C_2
R_1	12, 8	13, 5
R_2	10, 9	4, 0

analyze. This game represents a *no-conflict situation* because both players do as well as possible when each maximizes his or her own return. Mutual interest is overwhelming. Notice, in this game, that by a slight extension of the concept of dominance, the outcome (12, 8) can be said to *dominate* all other outcomes. It is better, for *both* players, than any other outcome.

Now consider game B in Table 11–16. In this game, there is an element of conflict, but it is very weak. Both players still have dominant strategies, and the equilibrium outcome (12, 8) remains, in some sense, the "natural" outcome. Note, however, that Row prefers (13, 5) to the equilibrium outcome, and Column prefers (10, 9). All three of these outcomes have an important property known as *Pareto optimality*. An outcome is said to be Pareto optimal whenever it is *not* dominated by any other outcome. None of the three Pareto-optimal outcomes in game B are dominated by any other outcome. In game A, note that (12, 8) is the *only* Pareto-optimal outcome.

Threat and forcing potentials

Two similar types of competitive situations are referred to as *threat vulnerability* and *force vulnerability*. Suppose a buyer and a seller repeatedly negotiate a contract in the following way. The seller sends a written notice to the buyer indicating the selling price per unit. In reply, the buyer indicates the quantity that will be purchased at the established price. If the buyer is a retailer who must then resell the goods, a hypothetical payoff table might be Table 11–17. If each player chooses his or her

Table 11–17
Threat vulnerability game, profit to buyer and seller
($ thousands)

		Seller's price choices	
		P_1 = *High*	P_2 = *Low*
Buyer's quantity choices	Q_1 = *High*	2, 4	4, 3
	Q_2 = *Low*	1, 2	3, 1

dominant strategy, the payoff will be (2, 4). Neither player is motivated to make a unilateral shift from this outcome. However, the buyer is not satisfied with the outcome (2, 4) since (4, 3) is more attractive. If the game is to be repeated several times, the buyer would like the seller to choose P_2 to give buyer a chance at winning 4.

If the players are allowed to communicate, the buyer can threaten the seller into lowering the price (that is, threaten the seller into giving the buyer a chance at 4) by saying he or she will only buy the smaller quantity, Q_2, if the seller does not choose P_2. The buyer's threat is effective as long as it is not carried out. Once (Q_2, P_1) occurs, however, it is not in the seller's interest to shift to P_2. Nevertheless, the seller is better off to give in rather than to suffer the consequences of the buyer's shift.

Consider now a situation known as a force vulnerability game. As shown in Table 11–18, only Row has a dominant strategy. If Row uses this dominant strategy, the natural outcome is (0, 2). Row is less satisfied with this result than is Column. If the game is played repeatedly, Row can communicate his or her dissatisfaction and try to *force* Column into changing his or her strategy. Since Column prefers (R_2, C_2) to (R_2, C_1), Row can force Column to switch to C_2 by switching from R_1 to R_2.

Table 11–18
Force vulnerability game
($ thousands)

		Column	
		C_1	C_2
Row	R_1	0, 2	2, −1
	R_2	−1, 0	1, 1

As you can see, threats and force are in many ways similar. The key difference is how Row tries to influence Column's behavior. In the force situation, Row tries to get what he wants by playing the strategy that leads to his best outcome, thereby *forcing* Column to do what Row wants. In the threat situation, Row must *threaten* to make a move which will punish Column if Column does not comply. The Row strategy used for threatening is not the one that leads to Row's best outcome, so Row hopes that it will not have to be used.

Opportunities for using threats and force in real-world situations are widespread. Generally, however, they are *not* obvious. The existence of threat and force potentials can be extremely subtle, and often they are noticed by only one of the players. These simple examples should increase both your awareness of such situations and your understanding of their structure. A worthwhile exercise is to try to conceive of real-world situations where threat and forcing potentials exist.

The Prisoner's Dilemma

A class of situations which are not strictly competitive is popularly known as the Prisoner's Dilemma. These are situations in whic' e best outcome for all concerned results when each competitor refrains from trying to maximize his own payoff. A classic example is the airline battle for a share of passengers on a particular route. As discussed in the first section of this chapter, many believe that the carrier with the largest share of departures gets a share of market disproportionately larger than its percentage of departures. For instance, 60 percent of the departures might yield 70 percent of the market. Consequently, airlines have often used the number of departures as a major competitive tool, especially on the long-haul routes such as New York–California. If one of the carriers—say American Airlines—unilaterally increases capacity, hoping to increase its market share, the other carriers, United and TWA, must decide whether or not to follow suit. The nature of the dilemma is this: If they match the increase, all will be worse off, since little new demand will be stimulated, and the airlines will end up flying more empty seats. If the competitors do not match the increase, however, they will be worse off compared to the carrier that increases capacity.

Situations of this type occur so often that they have been studied in detail; in fact, a whole book has been written on the subject.[3] They all share a common structure, that of the so-called Prisoner's Dilemma.

We will first analyze the Prisoner's Dilemma in the context from which it derives its name. Two suspects, Sam and Harry, are taken into custody and separated. The district attorney is certain they are guilty of a particular crime, and the suspects know they are guilty, but the district attorney does not have adequate evidence to convict them.

Each prisoner has two alternatives, to confess to the crime or not to confess. If neither confesses, then the DA will book them on some very minor trumped-up charge, such as illegal possession of a weapon, and they will both receive minor sentences. If they both confess, he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence, whereas the latter will get the book thrown at him. In terms of years in a penitentiary, the situation may be described by the payoffs in Table 11–19. The problem for each prisoner is to decide whether or not to confess. Since they are in separate cells, they cannot communicate before deciding.

Let us look at the problem from Sam's viewpoint. If he could be sure Harry will not confess, perhaps he should not do so either. But on second thought, if Harry does not confess, why shouldn't he, Sam, confess and spend only a half year in prison?

In fact, no matter *what* Harry does, Sam is better off if he confesses. In other words, Sam's confess strategy dominates his do not confess strategy. The only difficulty is, Harry may reason the same way. Thus, if each chooses his dominant strategy, both prisoners end up with eight years.

[3] Anatol Rapoport and A. M. Chammah, *Prisoner's Dilemma: A Study of Conflict and Cooperation* (Ann Arbor: University of Michigan Press, 1965).

Table 11–19
Payoff table for prisoner's dilemma (years in penitentiary)

Sam's choices		*Harry's choices* H_1 *Confess*	*Harry's choices* H_2 *Do not confess*
	S_1 *Confess*	8, 8	½, 10
	S_2 *Do not confess*	10, ½	1, 1

This result is not the best possible, since both prisoners would be better off if neither confesses. In other words, the strategy pair (S_2, H_2) is Pareto optimal. So are the pairs (S_1, H_2) and (S_2, H_1). The final outcome, the only non-Pareto-optimal pair, is inferior for both players.

If the prisoners were allowed to communicate, they might agree to choose the Pareto-optimal pair (S_2, H_2). Notice, however, that this is not an equilibrium pair. Sam and Harry can each do better by making a unilateral change of choice, so there would be good reason for each of them to defect on their bargain. It is to everyone's advantage if no one cheats, and it is to every prisoner's advantage to cheat unilaterally—a very unstable situation. Prechoice communication cannot help in solving the dilemma unless there is some legal or moral force to bind the prisoners to their agreement.

It might be that in this situation the prisoners will choose (S_2, H_2) even if they are not allowed to communicate. Instead of each prisoner asking "When am I best off?" and assuming his opponent will do the same, each prisoner asks, "When are we *both* best off?" If the prisoners held social values that prompted each to ask this question, they then might choose (S_2, H_2).[4] In this case there is an implicit change in the entries in the payoff table, since they must now reflect both length of term and feelings about the common good.

We can apply the insight gained from studying the one-time dilemma facing the two prisoners to a Prisoner's Dilemma situation which is being repeated in time—a battle over advertising radial tires. The heavily watched Monday night NFL football games on ABC–TV represent prime advertising time for this product. For several years, Goodyear was the only tire advertiser during this time. Then Sears also began advertising during the games and continued to share the time with Goodyear for two

[4] Anatol Rapoport, *Fights, Games and Debates* (Ann Arbor: University of Michigan Press, 1960), p. 177.

seasons, after which the latter withdrew, leaving Sears as the only tire advertiser.

The situation facing Sears and Goodyear is shown in Table 11–20. Each manufacturer must decide each year whether (yes = Y) or not (no = N) to advertise radials during the game. Some purely hypothetical payoffs (in millions of dollars of annual contribution, taking advertising into account) are shown. Notice that if both companies advertise, each loses contribution. Apparently the message gets washed out if there is more than one advertiser in a short time period. In this case, neither Sears nor Goodyear sticks in the consumer's mind.

Table 11–20
Change in annual contribution due to TV advertising during NFL game ($ millions)

Goodyear's choices	Sear's choices S_N	Sear's choices S_Y
G_N	0, 0	−2, 3
G_Y	3, −2	−1, −1

Why is this a Prisoner's Dilemma? Observe that if each manufacturer chooses his dominant strategy (G_Y, S_Y), both end up worse off than if they had made the opposite choices. The pair (G_N, S_N) is not an equilibrium pair, however. So it is to both companies' advantage if neither advertises, but it is to each company's advantage to unilaterally decide to advertise.

For several years Goodyear was the only radial tire advertiser during the game. One year Sears also advertised. This amounted to a choice of (G_Y, S_Y) for a payoff of (−1, −1). The next year, Goodyear decided to stay put and continue to advertise, and so did Sears. Once again, the companies lost contribution when each tried to maximize its own return. Finally, in the third year Goodyear withdrew, the choice being (G_N, S_Y). The game is not yet over, however, since Goodyear may decide to advertise radials during a subsequent year. In the meantime, Sears has no incentive to change its strategy.

Since we do not have access to what the companies were actually thinking, we can only conjecture about what they were trying to do. Did Sears choose to advertise during the game because it knew Goodyear would be forced to withdraw after deadlocking at (G_Y, S_Y) for a few years? Or were they just lucky? Did Sears really understand the situation, or did they think that since it was profitable for Goodyear to advertise during the game it would be profitable for them to do so too?

This example serves to point out that you can go astray by assuming

your opponent knows as much about the game as you do. On the other hand, you can also go astray by not ascribing this much understanding to an opponent who understands the game as well as you do—or maybe even better.

In the Prisoner's Dilemma type of competitive situation, the strategy pair that leaves both players best off is not an equilibrium pair. Thus it is to each individual's advantage to defect; but if both defect, both are worse off. If the situation is to be repeated, the competitors may wish to reach an agreement—an explicit or implicit one, depending on the rules of the game—about their respective moves. But with strong incentives to defect, such agreements may turn out to be tenuous.

The Battle of the Sexes situation: American Chemical versus Boston Pharmaceutical

Another important type of competitive situation is illustrated by the following scenario: A husband and wife have two choices for an evening's entertainment, to go to a prize fight or to a ballet. The man prefers the fight and the woman the ballet; however, to both it is more important that they go out together than that they enjoy their preferred entertainment. Any competitive situation which has a payoff table with the same properties as the one for this situation is popularly known as a Battle of the Sexes situation. The following hypothetical new product introduction is an example.

American Chemical Company must decide whether or not to introduce its newest product, so far designated only as Compound K. The company believes that its major competitor, Boston Pharmaceutical, has a very similar product ready to market. Each company has two choices—to introduce the product or not to introduce it. American's new product manager has calculated the expected payoff (present value) to each company for each alternative. The results are shown in Table 11–21. High fixed costs account for the negative entries in the lower right-hand corner. The subscript N indicates the company has decided not to introduce the product; a Y indicates that the company has decided to introduce the product.

Table 11–21
Payoff table for introduction of compound K (millions of dollars of contribution)

American	Boston B_N	Boston B_Y
A_N	0, 0	0, 2
A_Y	2, 0	−3, −3

Notice that both pairs of choices, American introduces and Boston does not (A_Y, B_N), and Boston introduces and American does not, (A_N, B_Y), are equilibrium pairs since for each pair American's choice is best against Boston's, and vice versa. Neither (A_Y, B_Y) nor (A_N, B_N) are equilibrium pairs. While there are two equilibrium pairs, each does not yield the same return to the players. This is in contrast to those strictly competitive situations for which there is more than one equilibrium pair, and every equilibrium pair gives identical returns to each party.

Obviously, American and Boston must make their decisions regarding the introduction of Compound K without conferring. For the sake of discussion, first let us suppose this is the only time the two companies expect to be opposing each other in this type of situation. What should American do? What will Boston do?

Both companies must realize the market is only big enough for one. If they both introduce Compound K, each will lose several million dollars. If Boston, therefore, chooses B_N to prevent a large loss, it is best for American to choose A_Y. But what if Boston expects American to give in? Then both may lose with (A_Y, B_Y) since the unhappy state of affairs is that whatever rationalization Boston has for choosing either B_Y or B_N, there is a similar rationalization for American.

Even though they may not confer, however, there are ways in which the two companies can communicate to influence the outcome. American, for example, may announce that it will introduce Compound K this coming fall. If Boston *believes* this announcement and interprets it to mean American has definitely committed itself to introducing the product, then, acting in its own best interest, it will probably not choose B_Y. Disclosing its plan ahead of time may allow American to preempt the market.

If American intends to beat Boston to the market by announcing that its product will be forthcoming, it must make its announcement credible. For instance, if American lets it be known that it has committed several million dollars to the building of facilities to produce the new product, it will be clear to Boston that its competitor has taken an irrevocable step. In that case, Boston will probably leave the market to American.

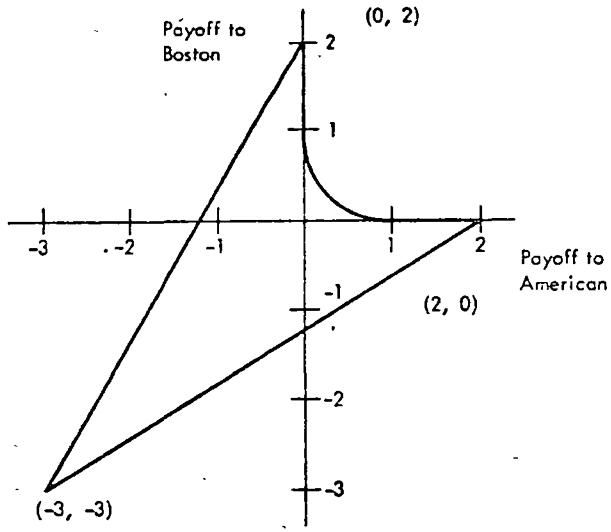
In a one-time Battle of the Sexes situation, it is all-important to preempt the other party. If, on the other hand, the companies expect to be in this type of situation repeatedly with various new products, as may well be the case, they would do well to cooperate to try to obtain either (A_Y, B_N) or (A_N, B_Y), since they are both best off with one or the other of these two Pareto-optimal pairs. *How* they would go about this is the question.

Since the two companies are prohibited from bargaining, they may not arrange to take turns (American totally capturing this market, Boston another, for example) nor may American pay Boston to stay out of the market (for example, make Boston a \$1 million side payment in exchange for sticking to (A_Y, B_N). Even in the absence of prechoice agreements, however, the companies may effectively settle on a pattern of alternation between (A_Y, B_N) and (A_N, B_Y). If they are constantly introducing new prod-

ucts of similar profitabilities, and if each introduction involves high start-up costs, then after one company has spent several million dollars, say, it may not have the resources to introduce another new product immediately. In the meantime, Boston can seize the opportunity to do so. Once American has committed itself to producing Compound K, for example, Boston can introduce another product without immediate direct competition.

The companies might also consider using mixed strategies. Of course, each company has a host of combinations it can consider. A good way to see the complexities of this situation is to make a geometric plot of the possible payoffs, as in Figure 11–1. Along the horizontal axis American's

FIGURE 11–1
Battle of the Sexes graph for American and Boston new product introduction



payoffs are plotted, and along the vertical axis, Boston's payoffs. Only certain combinations are possible: these are shown in the shaded region. To each point in the shaded region there is at least one corresponding pair of strategies having this point as payoffs. Conversely, to each pair of mixed strategies there corresponds a payoff which is one of the points in the shaded region.

If American follows the mixed strategy $.8A_N$, $.2A_Y$ then their expected payoff is (.4, 0) if Boston chooses not to introduce its new product and (−.6, 1.0) if Boston does introduce its product. If Boston has any idea American will follow this strategy, Boston would prefer B_Y. But in that case American does best to choose A_N.

If Boston follows the mixed strategy $.8B_N$, $.2B_Y$, then their expected returns are (0, .4) if American chooses A_N and (1.0, −.6) if it chooses A_Y. Thus, if American expects Boston to follow this mixed strategy, American should choose A_Y. But in that case, it is best for Boston to choose B_X. So if each company expects the other to follow its mixed strategy, the return is (−3, −3), which is all the more reason to play the mixed strategy, which is all the more reason to defect, and so forth.

The difficulty is that this pair of mixed strategies is not in equilibrium. However for any Battle of the Sexes situation, there is a pair of mixed strategies that is in equilibrium. These mixed strategies have the desirable property that they prevent the other player from trying to get more by preempting.

In the situation facing American Chemical and Boston Pharmaceutical, the equilibrium pair of mixed strategies is $^3/_5 A_X$, $^2/_5 A_Y$ and $^3/_5 B_X$, $^2/_5 B_Y$. If American follows this mixed strategy and Boston chooses B_X, the returns are ($^4/_5$, 0); if Boston chooses B_Y, the returns are ($-^6/_5$, 0). Each of Boston's pure strategies is equally good against American's mixed strategy, in the sense that each has an expected return of 0. Thus, there is no incentive to preempt by Boston.

Similarly, an equilibrium mixed strategy removes American's incentive to preempt. As long as one of the companies chooses this mixed strategy, the payoff to the other company does not depend on the strategy chosen.

Regardless of the specific application, any competitive situation which can be described by a payoff table similar to Table 11–21 can be thought about in the same way as this example of new product introduction.[7] In a one-time situation, preemption is all-important. In repeated choice problems, each might also consider choosing this equilibrium pair of mixed strategies. Or, better yet, they might alternate between the equilibrium pairs through either a tacit or an explicit understanding.

Summary

This section introduced the analysis of situations that are not strictly competitive, and the approach differed significantly from that of the strictly competitive situations of zero-sum games. Whereas in the zero-sum case it is never advantageous to disclose one's strategy, in a Battle of the Sexes situation it is all-important to preempt the other party. The zero-sum case is also characterized by the fact that all equilibrium pairs yield the same return to an individual player, whereas in a Battle of the Sexes situation the return to a particular competitor depends on which equilibrium pair is selected. Furthermore, the ability to communicate and collude becomes important in some situations. In others, whether a situation will be repeated many times or is a one-shot decision makes a critical difference. The concept of Pareto optimality is useful in analyzing such situations. A Pareto-optimal outcome is such that neither player can do better, except at his or her opponent's expense. All the common ground has been squeezed out.

When a not strictly competitive, noncooperative situation is repeated many times, certain aspects change. For example, even though formal preplay communication is not allowed, the competitors may develop some form of temporal collusion. Decision makers in a Battle of the Sexes situation may settle into a pattern of choosing between the equilibrium pairs. In the Prisoner's Dilemma, the players may, after a few repetitions, repeatedly choose the Pareto-optimal strategy pair. This situation is unstable, however, since it is to each individual's advantage to cheat unilaterally.

When all is said and done, it is difficult to choose a course of action in a not strictly competitive situation, since one can never be sure what one's opponent will do. However, analysis can provide a framework for thinking about these situations, help prevent foolish moves, suggest creative moves, and give some insight into each situation.

[7] All new product introductions, however, cannot be analyzed this way; some, for instance, have a Prisoner's Dilemma character.

EXERCISES

11.1. In the next classroom session you will be asked to play the role of player 1 or player 2 in a game about to be described. At this point you don't know which role you will actually play. Think hard about the problem now so that when the time comes you will be prepared to act.

The description of the game is:

1. Player 1 has 6 strategies: α_1, α_2, α_3, α_4, α_5, and α_6.
2. Player 2 has 4 strategies: β_1, β_2, β_3, β_4.
3. Each player must choose exactly one of his strategies without any knowledge of the choice made by his adversary.
4. Depending on the choice of strategies (one for each player) there will be a monetary payment from one player to the other. Table A describes the payments from player 2 to player 1. For example, if player 1 chooses α_2 and player 2 chooses β_3, then player 2 must pay player 1 an amount of $15. As another example, if player 1 chooses α_3 and player 2 chooses β_4, then player 2 gives −$12 to player 1—this, of course, means that player 2 gets a positive $12 from player 1.

Table A
Payoffs from player 2 to player 1

	β_1	β_2	β_3	β_4
α_1	2	−1	0	13
α_2	4	0	15	−3
α_3	9	2	4	−12
α_4	3	3	4	5
α_5	3	2	12	3
α_6	1	0	10	−4

Think about the following questions in turn. Do not read the second question until you have responded to the first question, and so on. Record your initial responses.

Imagine that in the next classroom session you and another classroom participant are chosen to play. A fair coin is tossed and you happen to be chosen for *player 1's role* The game will be played just once.

a. What strategy will you choose? (Be prepared to rationalize or justify your choice when the game is discussed in class.)

b. Before the game is actually executed, suppose that someone asks you, as the designated player 1, to sell your rights to play this game for some amount, say \$X. How large does *X* have to be before you will agree to sell?

c. What do you think player 2 will do? (Use probability assessments to reflect your judgments.)

11.2. The same instructions as in Exercise 11.1 apply, but with the payoffs given in Table B. In this exercise, each player has two strategies.

Table B
Payoffs from player 2 to player 1

	β_1	β_2
α_1	0	5
α_2	10	−2

Imagine that in the next classroom session you and another classroom participant are chosen to play. A fair coin is tossed and you happen to be chosen for *player 1's role*. The game will be played just once.

a. What strategy will you choose? (Be prepared to rationalize or justify your choice when the game is discussed in class.)

b. Before the game is actually executed, suppose that someone asks you, as the designated player 1, to sell your rights to play this game for some amount, say \$X. How large does *X* have to be before you will agree to sell?

c. What do you think player 2 will do? (Use probability assessments to reflect your judgments.)

d. Would it be worth any premium to you to be able to talk to your adversary in order to make a deal with him or her before choosing?

e. If your adversary had to announce his or her choice before you were obliged to announce yours, would you gain an appreciable strategic advantage?

11.3. Select a fellow participant to work with you on this exercise. Designate one of you as the A player, the other as the B player.

a. The A player will select either strategy a_1 or a_2. Simultaneously and without communication the B player will select either b_1 or b_2. The payoff will be shown in the cell of Table C corresponding to the two choices. For example, if A picks a_1 and B picks b_2, then the payoff is shown in the upper-right-hand corner of the table as (−5,10). The first of the two numbers in the cell entry is the gain or loss (−) to the A player, the second the gain or loss (−) to the B player. In this example, A would lose \$5 and B would gain \$10.

In this part assume you will play the game only once and that goal is *to do as well as you can for yourself*. You are neither altruistic nor vindictive with respect to your competitor.

Table C

Player A \ *Player B*	b_1	b_2
a_1	(5,5)	(−5,10)
a_2	(10,−5)	(−2,−2)

b. Now play the same game assuming you will play this game an *indefinite* number of times with your competitor. The rest of the rules are the same as (*a*). Remember your goal is to get the most in the long run for yourself. Your objective is neither to do your competitor in or to help him, *nor* is it to do well *relative* to him.

c. Now play the same game assuming you will play the game with your competitor exactly 20 times. Remember, try to maximize *your* take.

11.4. Compare how you behaved in part *c* of Exercise 11.3 with how you would behave in each of the following games under the same rules as Exercise 11.3 (20 plays, no communication).

a.

Player A \ *Player B*	b_1	b_2
a_1	(5,5)	(−50,50)
a_2	(50,−50)	(−3,−3)

b.

Player A \ *Player B*	b_1	b_2
a_1	(5,5)	(−4,6)
a_2	(6,−4)	(−3,−3)

11.5. Play the following game once with no communication:

Player A \ Player B	b_1	b_2
a_1	(1,2)	(3,1)
a_2	(0,−200)	(2,−300)

11.6. Play the game described in Exercise 11.5, but make your decision at a bargaining table with binding contracts. What happened?

Case 11–1A

Fouraker Mining & Metals Corp.

Fouraker Mining & Metals, operator of a medium-scale molybdenum mine in the western United States, was a technically sophisticated but marginally profitable producer of molybdenum ore concentrate. In its efforts to improve operating profits, Fouraker had just entered into an exclusive supply arrangement with Siegel & Company, Inc., to purchase a biochemically produced material called "Flozyme," which greatly increased molybdenum mineral recovery from each ton of ore mined. The arrangement required Fouraker to purchase the additive weekly, in small lots, at a price set each week by Siegel. Walter Lightdale Fouraker's purchasing manager, was in the process of establishing a purchasing strategy under the new arrangement.

Fouraker Mining

Fouraker Mining was started in 1952 by a consulting geologist, Mr. L. Fouraker, and a research metallurgist, Dr. Henry Holmes. It was founded to develop a large, low-grade deposit of molybdenum ore on which Mr. Fouraker had long held mining claims, using chemical processes pioneered by Dr. Holmes.

For the next six years, they struggled to finance both the rapidly expanding pilot plant operation and the exploration and development of the ore body. By 1958 the process had been adequately tested and the mining operation had expanded to the point at which the company was almost breaking even.

In order to develop the ore body fully and to expand the plant to its most efficient size, Mr. Fouraker obtained a commitment for $16 million from a large international mining company in return for a 45 percent interest in the firm. Holmes and Fouraker personally held 10 percent each of the equity, with the remaining interest widely dispersed among individuals who had helped finance Fouraker Mining's first ten years. The balance of the $60 million capitalization was provided by banks and various equipment suppliers. In subsequent years there was a marked increase in scale of operations but the company was only barely profitable ($1.2 million before-tax profit on $37.7 million in sales in 1973.).

Siegel & Company, Inc.

Siegel & Company was a small West Coast producer of proprietary carbohydrate derivatives used in the manufacture of certain prepared foods and drugs. It had been founded shortly after World War II by a young biochemist, Sydney Siegel, in order to commercialize a number of promising new biologically active substances on which he had obtained patents.

Despite the smallness and informality of its operation, Siegel & Company had become extremely profitable in recent years as significant markets began to develop for its highly specialized, costly products. Produced in carefully scheduled and interdependent batches, most of its products were sold exclusively to single users, a consequence of Siegel's past joint-venture method of funding the majority of its research programs. Typically, the joint-venture agreements allowed Siegel to retain patents and the rights to manufacture any resulting products, while the sponsor held exclusive rights to use and/or distribute the products.

Flozyme

Although the basic extractive process for molybdenum was well known and in widespread use, Fouraker Mining had succeeded in greatly enhancing its efficiency by special techniques, including the use of special additives to increase yields. Its continuing research had revealed that use of Flozyme made for a substantial improvement in its recovery of molybdenum minerals with Flozyme was introduced into the process at rates equivalent to a few hundred pounds per week.

An extremely light and chemically unstable powder, Flozyme was a by-product of a complex, biological-organic chemical process. Dr. Holmes had learned of Flozyme's surface-activating behavior from a brief description of the process in a professional journal. On the strength of a few laboratory-scale tests, Holmes recommended that Fouraker Mining undertake large-scale process testing of Flozyme so that its precise effect on molybdenum recovery and the economics associated with its use could be accurately established.

In return for exclusive rights to buy Siegel's entire output of Flozyme—should it prove successful in this application—Fouraker Mining agreed to fund a research program in which both the production and application of the Flozyme by-product were to be investigated. After several years of sporadic activity on this program, Flozyme's effectiveness was proven to Fouraker's satisfaction.

Because Flozyme was a by-product, the yield of the main product was greatly affected by the amount of Flozyme desired. Volume production of Flozyme was possible only through extensive and costly recycling of the main product. Fouraker had learned that Flozyme production added con-

siderably to total process costs—the cost increment becoming larger as Flozyme output increased. Prior to the sale of Flozyme, any by-product material had been disposed of as a waste product.

The purchasing arrangement

The absence of a market for Flozyme outside Fouraker Mining meant that its price would have to be established by negotiation. After lengthy discussions with Fouraker management, Siegel had decided that the simplest approach for the present was to quote weekly an appropriate unit price for the reagent and to let Fouraker place its order based on that price. Both Fouraker and Siegel hoped that prices and quantities would eventually stabilize at levels acceptable to both firms. With this procedure decided upon, Mr. Fouraker assigned the purchase responsibility to his purchasing manager, Mr. Lightdale. Mr. Lightdale was to base his weekly decisions solely on the profit contribution information (Exhibit 1) developed by Fouraker's production superintendent, the chief process engineer, and himself during the final Flozyme tests.

An equivalent tabulation of incremental profits for Siegel & Company, shown in Exhibit 2, was also available from a project report prepared by Siegel chemists at the test program's conclusion. The report was regarded as very reliable and would undoubtedly be used by Siegel in its pricing decision. The same report contained the information shown in Exhibit 1.

Because of Flozyme's great effect on the main process, Sydney Siegel had decided to handle its sale personally in order to keep close watch on the joint process and its combined economics. Weekly, Mr. Siegel would telex a price, in dollars per pound, to Fouraker and Fouraker would, in turn, transmit an order quantity, in 20-pound drums, for delivery two weeks hence. Fouraker had been advised that batches of up to 18 drums per week could be produced and that each batch had an active life of ten days at most. This meant that shipments would have to be used within ten days of manufacture or discarded.

Exhibit 1
Weekly contribution to Fouraker Mining profits resulting from use of Flozyme reagent (in dollars)

Price ($)		Quantity (drums) used per week																		
Per pound	Per drum	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	20	0	570	900	1,210	1,500	1,770	2,020	2,250	2,460	2,650	2,820	2,970	3,100	3,210	3,300	3,370	3,420	3,450	3,460
2	40	0	550	860	1,150	1,420	1,670	1,900	2,110	2,300	2,470	2,620	2,750	2,860	2,950	3,020	3,070	3,100	3,110	3,390
3	60	0	530	820	1,090	1,340	1,570	1,780	1,970	2,140	2,290	2,420	2,530	2,620	2,690	2,740	2,770	2,780	2,770	2,740
4	80	0	510	780	1,030	1,260	1,470	1,660	1,830	1,980	2,110	2,220	2,310	2,380	2,430	2,460	2,470	2,460	2,430	2,380
5	100	0	490	740	970	1,180	1,370	1,540	1,690	1,820	1,930	2,020	2,090	2,140	2,170	2,180	2,170	2,140	2,090	2,020
6	120	0	470	700	910	1,100	1,270	1,420	1,550	1,660	1,750	1,820	1,870	1,900	1,910	1,900	1,870	1,820	1,750	1,660
7	140	0	450	660	850	1,020	1,170	1,300	1,410	1,500	1,570	1,620	1,650	1,660	1,650	1,620	1,570	1,500	1,410	1,300
8	160	0	430	620	790	940	1,070	1,180	1,270	1,340	1,390	1,420	1,430	1,420	1,390	1,340	1,270	1,180	1,070	940
9	180	0	410	580	730	860	970	1,060	1,130	1,180	1,210	1,220	1,210	1,180	1,130	1,060	970	860	730	580
10	200	0	390	540	670	780	870	940	990	1,020	1,030	1,020	990	940	870	780	670	540	390	220
11	220	0	370	500	610	700	770	820	850	820	850	820	770	700	610	500	370	220	50	−140
12	240	0	350	460	550	620	670	700	710	700	670	620	550	460	350	220	70	−100	−290	−500
13	260	0	330	420	490	540	570	580	570	540	490	420	330	220	90	−60	−230	−420	−630	−860
14	280	0	310	380	430	460	470	460	430	380	310	220	110	−20	−170	−340	−530	−740	−970	−1,220
15	300	0	290	340	370	380	370	340	290	220	130	20	−110	−260	−430	−620	−830	−1,060	−1,310	−1,580
16	320	0	270	300	310	300	270	220	150	60	−50	−180	−330	−500	−690	−900	−1,130	−1,380	−1,650	−1,940

Exhibit 2
Weekly contribution to Siegel & Company profits from sales of Flozyme reagent (in dollars)

Price ($)		Quantity (drums) sold per week																		
Per pound	Per drum	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	20	0	450	660	850	1,020	1,170	1,300	1,410	1,500	1,570	1,620	1,650	1,660	1,650	1,620	1,570	1,500	1,410	1,300
2	40	0	470	700	910	1,100	1,270	1,420	1,550	1,660	1,750	1,820	1,870	1,900	1,910	1,900	1,870	1,820	1,750	1,660
3	60	0	490	740	970	1,180	1,370	1,540	1,690	1,820	1,930	2,020	2,090	2,140	2,170	2,180	2,170	2,140	2,090	2,020
4	80	0	510	780	1,030	1,260	1,470	1,660	1,830	1,980	2,110	2,220	2,310	2,380	2,430	2,460	2,470	2,460	2,430	2,380
5	100	0	530	820	1,090	1,340	1,570	1,760	1,970	2,140	2,290	2,420	2,530	2,620	2,690	2,740	2,770	2,780	2,770	2,740
6	120	0	550	860	1,150	1,420	1,670	1,900	2,110	2,300	2,470	2,620	2,750	2,860	2,950	3,020	3,070	3,100	3,110	3,100
7	140	0	570	900	1,210	1,500	1,770	2,020	2,250	2,460	2,650	2,820	2,970	3,100	3,210	3,300	3,370	3,420	3,450	3,460
8	160	0	590	940	1,270	1,580	1,870	2,140	2,390	2,620	2,830	3,020	3,190	3,340	3,470	3,580	3,670	3,740	3,790	3,820
9	180	0	610	980	1,330	1,660	1,970	2,260	2,530	2,780	3,010	3,220	3,410	3,580	3,730	3,860	3,970	4,060	4,130	4,180
10	200	0	630	1,020	1,390	1,740	2,070	2,380	2,670	2,940	3,190	3,420	3,630	3,820	3,990	4,140	4,270	4,380	4,470	4,540
11	220	0	650	1,060	1,450	1,820	2,170	2,500	2,810	3,100	3,370	3,620	3,850	4,060	4,250	4,420	4,570	4,700	4,810	4,900
12	240	0	670	1,100	1,510	1,900	2,270	2,620	2,950	3,260	3,550	3,820	4,070	4,300	4,510	4,700	4,870	5,020	5,150	5,260
13	260	0	690	1,140	1,570	1,980	2,370	2,740	3,090	3,420	3,730	4,020	4,290	4,540	4,770	4,980	5,170	5,340	5,490	5,620
14	280	0	710	1,180	1,630	2,060	2,470	2,860	3,230	3,580	3,910	4,220	4,510	4,780	5,030	5,260	5,470	5,660	5,830	5,980
15	300	0	730	1,220	1,690	2,140	2,570	2,980	3,370	3,740	4,090	4,420	4,730	5,020	5,290	5,540	5,770	5,980	6,170	6,310
16	320	0	750	1,260	1,750	2,220	2,670	3,100	3,510	3,900	4,270	4,620	4,950	5,260	5,550	5,820	6,070	6,300	6,510	6,700

Case 11–3

Maxco Inc. and the Gambit Company

PART I

Maxco, Inc., and the Gambit Company were fully integrated, major oil companies, each with annual sales of over $1 billion and exploration and development budgets of over $100 million. Both firms were preparing sealed bids for an oil rights lease on block A–512 off the Louisiana Gulf Coast. Although the deadline for the submission of bids was only three weeks away, neither firm was very close to a final determination of its bid. Indeed, management at Maxco had yet to decide whether to bid at all, let alone how much to bid. Although Gambit was virtually certain to submit a bid, the level of Gambit's bid was far from settled. This uncharacteristic hesitancy in the preparation of both firms' bids was a direct result of certain peculiarities in the situation surrounding the bidding for block A–512.

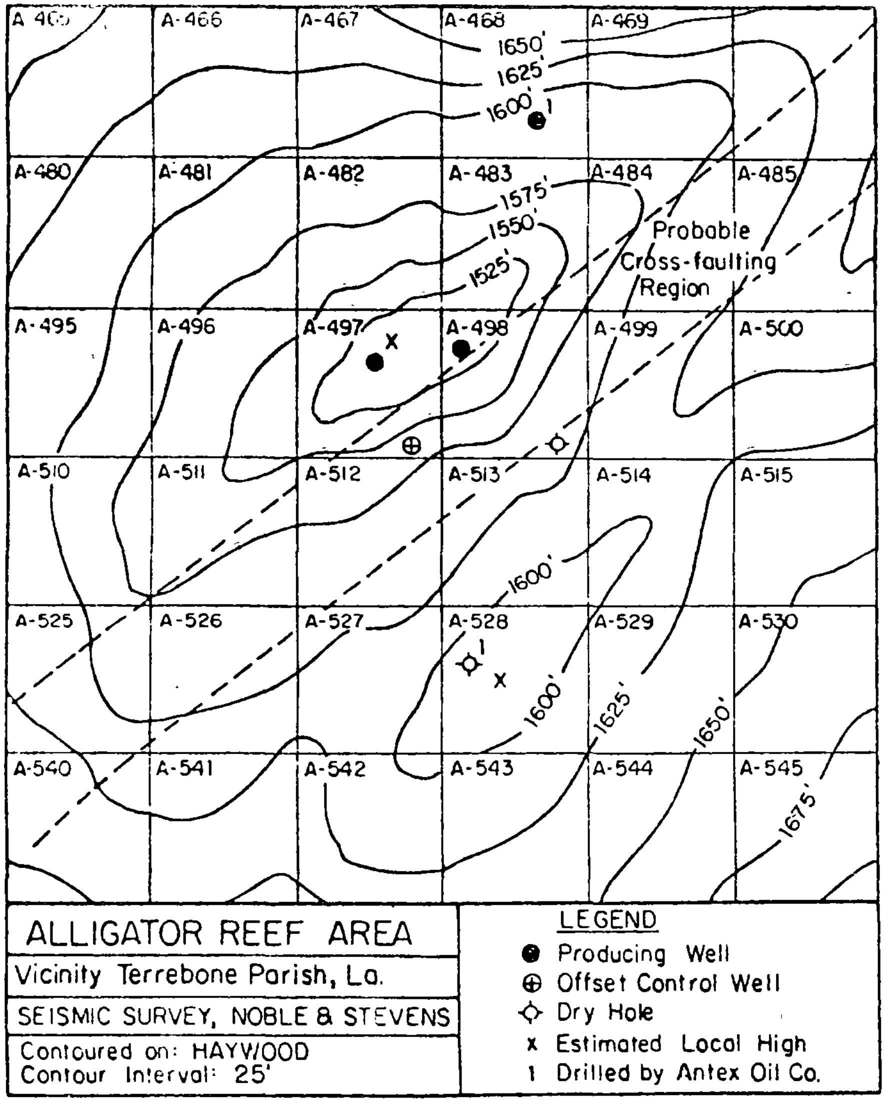
Block A–512 lay in the Alligator Reef area immediately to the south of a known oil-producing region (see Exhibit 1). Just to the north were blocks A–497 and A–498, both of which were already under lease to the Gambit Company. On its leasehold Gambit had two completed wells which had been in production for some time. In addition Gambit had an offset control well in progress near the boundary between its leasehold and block A–512. When this well was completed, Gambit would have access to direct information concerning the value of any oil reserves lying beneath block A–512. Maxco's nearest leasehold, on the other hand, was some seven miles to the southeast. Any bid submitted by Maxco, therefore, would necessarily be based solely on indirect information.

The role of information in bidding oil rights leases

In any bidding situation, information concerning either the object of the bidding or the notions of competing bidders is highly prized. This is ever more the case in bidding for the rights to oil reserves lying, perhaps, thousands of feet below the surface. There are, of course, various kinds of information available to bidders for oil rights. To summarize these various types of information briefly, two categories—direct and indirect—may be established.

Information obtained by drilling on a parcel of land is called direct information. Obviously this is the most precise information obtainable concerning the subsurface structure. From core samples taken up during the drilling operation, and from careful laboratory analysis of these

Exhibit 1
Subsurface m[illegible] the Alligator Reef area



samples, considerable information may be accumulated not only about the presence or absence of oil, but also about the type, thickness, composition, and physical properties of each of the various geologic strata encountered. Such information then provides the driller with a relatively accurate estimate of the oil reserves lying beneath the parcel. Direct information concerning adjacent parcels may be obtained by drilling offset control wells. These wells are offset from the principal producing areas

are located near the boundaries of the leased parcel. Such well[illegible] [illegible] provide a particular lessee with precise and valuable inform[illegible]on about adjacent parcels.

Indirect information is obtained from sources other than drilling and may be roughly divided into two kinds: scouting and nonscouting. Scouting information is gained by observing the operations of other drillers. By counting the sections of drill pipe—each of known length—introduced into a hole, an observer may infer the depth of the hole. By observing the quantity of cement—required by law—used to plug the various porous strata that are encountered, the thicknesses of these strata may be determined. Normally, however, this type of scouting information will not yield nearly the precision available to the driller. It can help in the determination of whether or not oil reserves exist at a particular location, but it is much less useful in determining the size of the reserves.

More definite scouting information may sometimes be obtained by more clandestine means. Eavesdropping on informal conversations in public places, subtle forms of bribery and interrogation, even forcible entry onto a competitor's drilling site may provide much more detailed—and more valuable—information. An extreme anecdote tells of two men who were caught while inspecting a competitor's drilling log—the source document of a driller's direct information. The men were reportedly held at gunpoint for several days in anticipation of the approaching deadline for the submission of bids. Managing to escape the day before the deadline, the two men were able to report back what they had seen in the log. As a result, the operator whose log had been compromised was forced to raise his bid by $7 million.

Less melodramatic, but highly significant, sources of indirect information are available through means other than scouting. Nonscouting information is obtained, first, from published sources, such as government geologic and geophysical surveys, and from reports of previous explorations. Second, nonscouting information may be obtained from local seismic surveys conducted either by in-house personnel or by private contractors. A third source of nonscouting information is found in the trading of dry hole information. The tradition among drilling operators is to reveal their dry hole experiences. The feeling seems to be that there is far more to be gained from the reciprocal exchange of dry hole information than could be gained from watching a competitor pour a considerable investment into a site that is known to be barren. Finally, nonscouting information may also be obtained from independent prospectors, promoters, and traders who may have become familiar with certain tracts in the past and are willing to trade this information, again on a reciprocal basis.

As might be suspected in an environment where information has such a high—and immediate—value, internal security presents a clear and ever-present problem. Bank-type vaults, armed guards, and electrified fences are commonplace. On occasion, entire drilling rigs have been encased in canvas to thwart the efforts of prying eyes. Substantial slowdowns in operations, however, under almost unbearable working conditions have also resulted. Furthermore, a blanket of security must also be placed over the derivation and submission of bids. Information on the level of a particular bid can be even more valuable than information on the value of reserves. When bids were being prepared for the tracts surrounding Prudhoe Bay on Alaska's North Slope, one company packed its entire bidding organization onto a railroad train and ran it back and forth over the same stretch of track until bids had been prepared and submitted and the bidding deadline had passed.

Finally, with information such a prime concern, circulation of false information is often attempted. If operators are successful in leaking false negative information about a particular parcel, they may be able to later "steal" the parcel with a relatively low bid. On the other hand, to divert attention from a particular parcel, operators may feign interest in another one by seeming to conduct tests there.

Maxco's bidding problem

Mr. E. P. Buchanan, Vice President for Exploration and Development, had primary responsibility for preparing Maxco's bid. Mr. Buchanan's information of block A–512 was, as indicated previously, indirect in nature. Although some scouting information on Gambit's offset control well was available to him, the primary basis of his information was a private seismic survey, together with published government geologic maps and reports. Maxco had acquired the survey data, in a jointly financed effort with Gambit, through the use of a private contractor. The contractor, Noble and Stevens, had prepared a detailed survey of the entire Alligator Reef area several years previously when blocks A–497 and A–498 were up for bid. Under the joint financing arrangement, identical copies of the completed report had then been submitted to both Maxco and Gambit. Such an arrangement, while unusual, was not without precedent in known oil-producing areas. Exhibit 1 represents an updated version of a subsurface map included in Noble and Stevens' report.

Based on all of the information available to him, Mr. Buchanan's judgment concerning the monetary value of the oil reserves under block A–512 was essentially captured by the probability mass function given in Exhibit 2. Furthermore, Mr. Buchanan held that Maxco's bid should be based solely on this monetary value of the oil reserves. Since it was known that no nearby blocks were to be put up for bid for at least ten years, Mr. Buchanan did not ascribe any informational value to owning a lease on block A–512.

Mr. Buc[illegible]an also felt—for the present at least—that Gambit's uncertainty was [illegible]ally identical to his own. He was sure, however, that

Gambit's well would be completed by the deadline for the submission of bids. At that time Gambit would know the value of the reserves up to, perhaps, ±5 percent or ±10 percent.

For the past several years, Mr. Buchanan had refused to bid on any parcels of land where he felt he was at a distinct disadvantage to a competing bidder. If a competitor had superior (direct) information about a parcel while Maxco had only indirect information, then Mr. Buchanan preferred not to bid at all.

Less than five months ago, however, in an area not far from Alligator Reef, Mr. Buchanan had *lost* a bid on a block adjacent to a Maxco leasehold. Maxco had gone to the expense of drilling an offset control well on its own block and had found a reasonably large oil reserve. Maxco had then lost the bid, however, to a competitor who was operating solely on the basis of indirect information. In addition, the competitor's winning bid had still been low enough to provide for a substantial profit on the venture.

Thus Mr. Buchanan was considering a change in his policy. While he very much doubted that anyone else would enter the bidding for block A–512, he was beginning to feel that he himself should do so. If he did decide to bid, he then wondered what sort of bid might be reasonable.

PART II

Gambit's bidding problem

Mr. Buchanan's counterpart in the Gambit Company was a Mr. K. R. Mason; primary responsibility for preparing Gambit's bid thus rested with him.

Until Gambit's well on the Alligator Reef leasehold was completed, Mr. Mason's information concerning block A–512 would be indirect in nature. The primary basis of that information was still the private seismic survey, for which Gambit had contracted jointly with Maxco, together with published government geologic maps and reports.

Although Mr. Mason also had detailed production logs on the two producing wells on Gambit's leasehold, he felt that this information was not relevant to the problem of assessing the potential value of block A–512. There was almost certainly some cross-faulting in the Alligator Reef area (see Exhibit 1). Since this cross-faulting would probably terminate the producing area, the principal uncertainty surrounding the value of block A–512 was the precise location of the northernmost cross fault. Thus, Mr. Mason's judgment was also essentially captured by the probability mass function given in Exhibit 2. Although Mr. Mason's judgment certainly did not coincide precisely with Mr. Buchanan's, the facts available to the two men and the economics in the two companies wer[illegible]gely similar. Neither man's estimate of the situation, therefore, differed significantly from Exhibit 2.

Exhibit 2
Probability distribution of monetary values

*Monetary value of oil reserves ($ millions)**	*Probability*
$ 1.7	.03
2.7	.06
3.7	.10
4.7	.17
5.7	.28
6.7	.18
7.7	.08
8.7	.04
9.7	.02
10.7	.01
11.7	.01
12.7	.01
13.7	.01
	1.00

Mean value = $5.83.
* Net present value at 10 percent

This would, of course, change dramatically when Gambit's offset control well was completed. At that time Mr. Mason would be able to reevaluate the property with a much higher degree of precision.

Normally Mr. Mason would then be in a position to submit a bid relatively close to the true value of the block while still allowing a generous margin for profit. Other bidders, not knowing the true value of the block, would be unable to adopt such a strategy. If they bid at all, they would have to either bid relatively low or risk the possibility of "buying in high" to a disastrously unprofitable situation.

Over the past year, however, several operators in the Louisiana Gulf Coast had narrowly lost out when bidding for blocks on which they had direct information. Granted that in no case were extremely large reserves lost, nevertheless operators bidding with nothing but indirect information had been able to "steal away" substantial reserves from operators who were basing their bids on direct information.

With a view toward reassessing his approach to this kind of situation, Mr. Mason thought that it might be useful to prepare a whole schedule of bids. For each possible "true value" of the reserves, Mr. Mason felt that he should be able to establish an appropriate bid—given that value of the reserves. Thus, Mr. Mason felt that he ought to be able to complete a bid schedule similar to that given in Exhibit 3. He was wondering, however, what a reasonable schedule of bids might be like.

Exhibit 3
Gambit's bid schedule

If the true value of the reserves is:	*Then Gambit's bid should be:*
$ 1.7 million	$______ million
2.7 million	______ million
3.7 million	______ million
4.7 million	______ million
5.7 million	______ million
6.7 million	______ million
7.7 million	______ million
8.7 million	______ million
9.7 million	______ million
10.7 million	______ million
11.7 million	______ million
12.7 million	______ million
13.7 million	______ million

Case 11–4A

American Grocery Products (A)

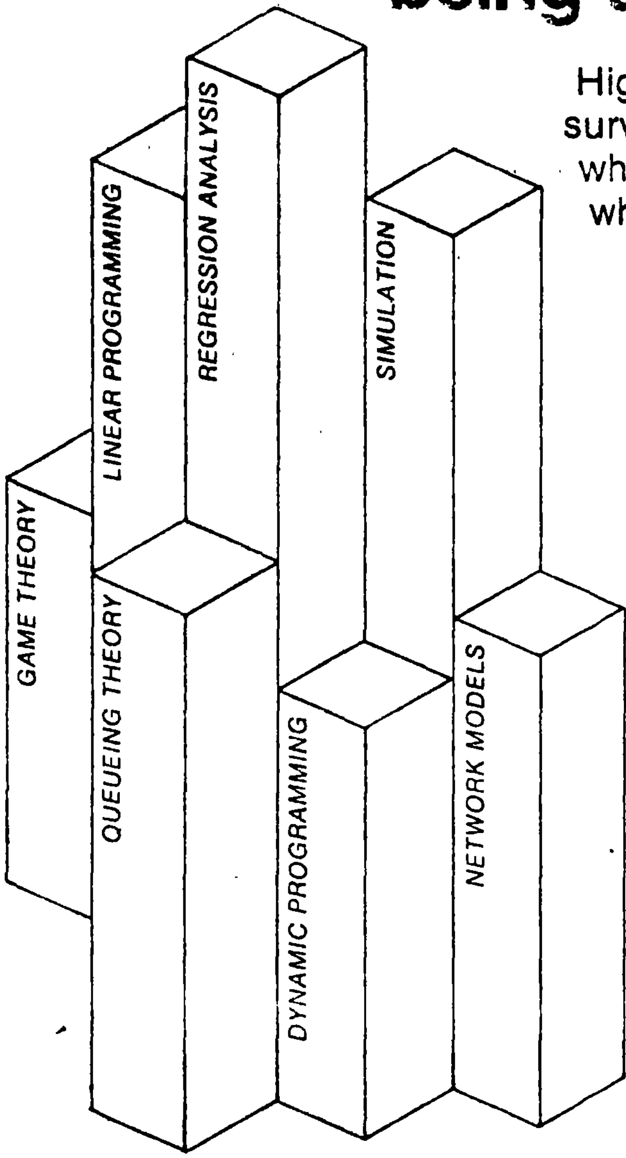
In late January 1970, Mr. John Roberts, Product Manager for American Grocery Products' new hot instant breakfast (code name Product B–14), was formulating a market strategy for introducing his new product. The product had been successfully designed, and limited consumption tests had been conducted to provide some data for predicting its market potential. The New Products Committee, which Mr. Roberts reported to, was then at the stage at which it had to decide whether to test-market or to actually launch the new product.

In 1970, American Grocery Products was one of the largest integrated manufacturers of packaged food products in the United States. The company marketed a wide and diversified line of food and allied products under many major brand names.

Product B–14

B–14 was conceived as a product fulfilling a specific need in the marketplace. In essence, B–14 was almost identical to the cold instant breakfast products that were promoted as meal replacements, but its important contribution was the fact that it was designed to be used with hot or warm water, thus giving a *hot* meal replacement. The product development had been based on the concept that, while many consumers accepted the idea

Are OR techn'qucs being used?



Highlights from recent OR utilization survey of 176 Fortune 500 firms show which methods are used a lot — and which are not. Study focuses on the production industries. Regression analysis, linear programming, and simulation are the most popular.

The 500 largest U.S. industrial firms (Fortune's 1975 listing) were selected for study on the assumption that they were most likely to represent "state-of-the-art" utilization of operations research (OR) techniques. This project was one part of a larger study covering a number of aspects of the OR function. The firms were randomly divided into two equal groups. A separate survey instrument was developed for each group to examine different aspects of the OR function. One instrument examined, among other things, the use of various OR techniques, while the second examined specific areas of OR application, as well as other topics. A total of 176 firms responded to the survey.

In Table I, the comparative use of the seven OR techniques is shown. Respondents were asked to indicate the frequency of use for each tech-

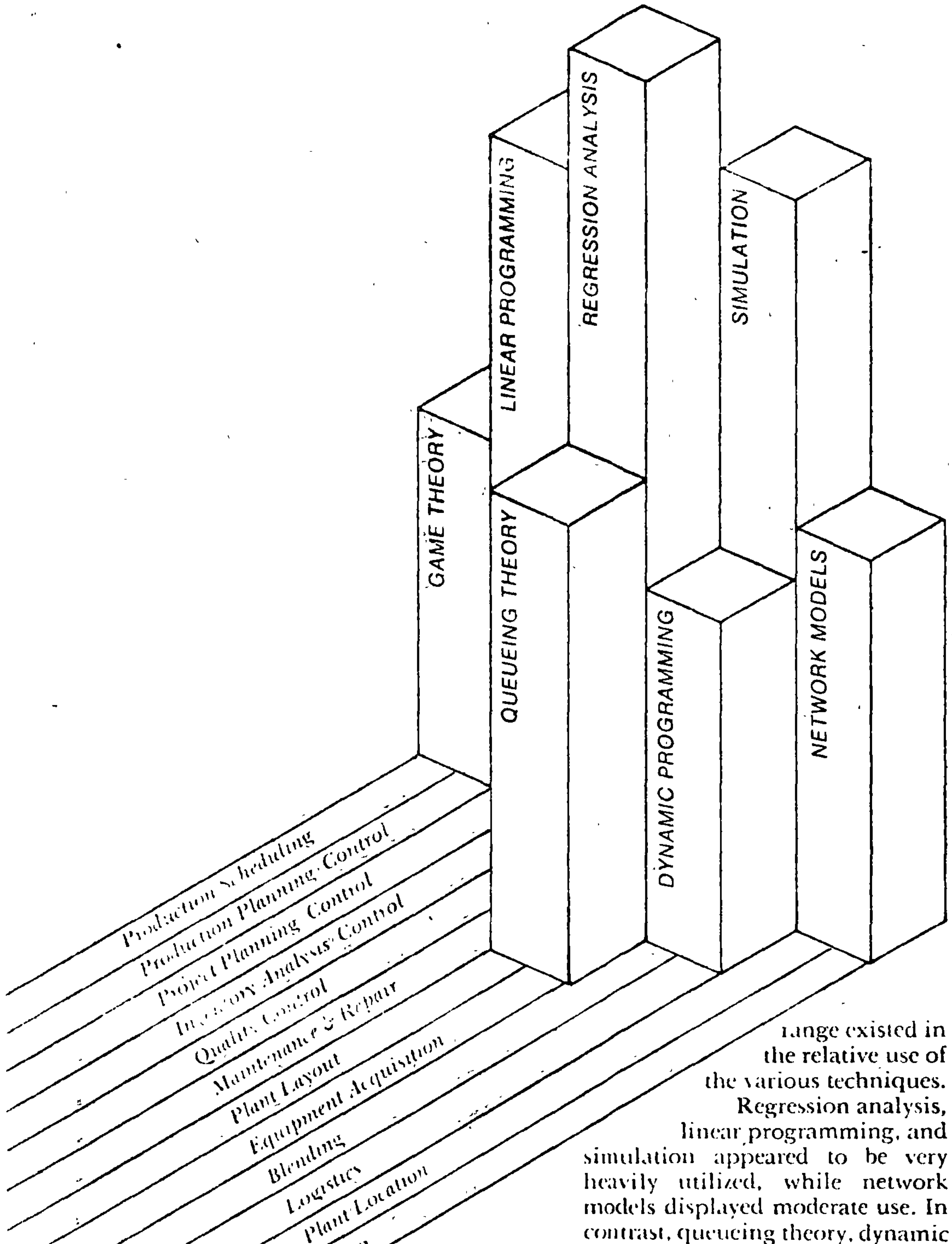
Relative use of operations research techniques

(degree of use in %)

Techniques	Number of respondents	Never 1	2	3	4	Very Frequently 5	Mean
Regression Analysis	74	9.5	2.7	17.6	21.6	48.6	3.97
Linear Programming	78	15.4	14.1	21.8	16.7	32.0	3.36
Simulation (In Production)	70	11.4	15.7	25.7	24.3	22.9	3.31
Network Models	69	39.1	29.0	15.9	10.1	5.8	2.11
Queueing Theory	71	36.6	39.4	16.9	5.6	1.4	1.96
Dynamic Programming	69	53.6	36.2	7.2	0.0	2.9	1.62
Game Theory	67	59.7	25.4	8.9	6.0	0.0	1.61

Table I. Percentages shown here are based on the number of responses to each technique.

Dr. William N. Ledbetter
Associate Professor
Dr. James F. Cox
Assistant Professor
School of Business
Auburn University
Auburn, AL



nique on a five-point scale ranging from "never" to "very frequently." Sample size varied slightly as many of the respondents only checked some of the techniques and left others blank. The blanks could perhaps have been interpreted as meaning "never," but the more conservative approach of only analyzing actual responses was taken. The figures in the table cells are the percentage of the total responses for that particular technique. A mean score was also computed for each technique to facilitate comparisons.

The data indicated that a wide range existed in the relative use of the various techniques. Regression analysis, linear programming, and simulation appeared to be very heavily utilized, while network models displayed moderate use. In contrast, queueing theory, dynamic programming, and game theory showed very low utilization

Applications in production

Table II presents the relative frequency of application of each of the seven techniques in 11 areas of the production function. In addition, respondents were allowed to specify other techniques and other application areas. The figures for these responses are also included in Table II. The figures in each cell of the table represent the number and the percentage (in parentheses) of respondents who indicated they used a technique in a given area. These figures were based on the 73 respondents who completed the production applications portion of the questionnaire.

Linear programming shows a relatively high utilization over a rather wide range of applications More than 40% of the respondents indicated they used linear programming in analyzing problems in the areas of blending (43.8%), plant location (43.8%), and production scheduling (41.1%). Also showing a fairly high utilization was the broad area of logistics (37%), as well as production planning and control (26%). Such a diversity of applications is obviously a major reason for the popularity of linear programming.

Simulation showed by far the greatest breadth of coverage, with only one application area having less than 8% utilization. Six application areas had a 24% or higher utilization rate: inventory analysis and control (37%), production scheduling (35.6%), logistics (32.9%), plant location (31.5%), plant layout (26%), and production planning and control (24.7%).

Queueing theory showed a fairly wide range of application areas but a rather low percentage of utilization. The most frequently mentioned application areas were production scheduling (12.3%) and plant layout (6.8%). It is quite probable that many queueing-type applications are handled with a simulation approach and were thus indicated under the simulation response choice.

As was shown in Table I, regression analysis received the highest rating on overall degree of use within the firm. However, within the production function it was not as extensively used as either simulation, linear programming, or network models. Nevertheless, it showed a modest amount of application within most of the 11 areas with the heaviest concentration in quality control (20.5%) and inventory analysis and control (16.4%).

Under the "other" techniques column, a fairly large number of different techniques were mentioned, including many specialized techniques for particular areas. Although the respondents were allowed the option of listing application areas other than the eleven provided, very few were named. This suggests the list of eleven provides fairly complete coverage of applications within the production function.

The data from Table II provides an interesting insight into the utilization of OR techniques within the production function. They follow rather closely the findings reported in Table I on relative degree of utilization of the techniques within the firm.

How well are they utilized?

As shown in Table III, the number of respondents to a given area ranged from a high of 56 for production scheduling to a low of 16 for maintenance and repair. Of the 56 respondents to the production scheduling area, slightly over half indicated they used linear programming in dealing with problems of this type, and almost half indicated they used simulation. The next most frequently mentioned was the "other" category of techniques which included heuristic programming and material requirements planning. IE

Application of Operations Research Techniques in Production

Application Areas	Linear Programming	Dynamic Programming	Network Models	Simulation	Queueing Theory	Game Theory	Regression Analysis	Other
Production Scheduling	30(41.1)	7(9.6)	6(8.2)	26(35.6)	9(12.3)	0(0.0)	5(6.8)	10(13.7)
Production Planning/Control	19(26.0)	3(4.1)	7(9.6)	18(24.7)	4(5.5)	0(0.0)	3(4.1)	3(4.1)
Project Planning/Control	10(13.7)	1(1.4)	28(38.4)	9(12.3)	2(2.7)	0(0.0)	0(0.0)	3(4.1)
Inventory Analysis/Control	11(15.1)	3(4.1)	3(4.1)	27(37.0)	4(5.5)	1(1.4)	12(16.4)	7(9.6)
Quality Control	2(2.7)	0(0.0)	1(1.4)	2(2.7)	0(0.0)	0(0.0)	15(20.5)	9(12.3)
Maintenance & Repair	0(0.0)	1(1.4)	3(4.1)	8(11.0)	3(4.1)	1(1.4)	4(5.5)	3(4.1)
Plant Layout	13(17.8)	0(0.0)	5(6.8)	19(26.0)	5(6.8)	1(1.4)	2(2.7)	3(4.1)
Equipment Acquisition/Replacement	4(5.5)	0(0.0)	1(1.4)	11(15.1)	1(1.4)	0(0.0)	0(0.0)	7(9.6)
Blending	32(43.8)	0(0.0)	1(1.4)	6(8.2)	1(1.4)	0(0.0)	3(4.1)	1(1.4)
Logistics	27(37.0)	1(1.4)	8(11.0)	24(32.9)	3(4.1)	2(2.7)	6(8.2)	2(2.7)
Plant Location	32(43.8)	2(2.7)	8(11.0)	23(31.5)	1(1.4)	0(0.0)	5(6.8)	4(5.5)
Other	7(9.6)	1(1.4)	2(2.7)	7(9.6)	1(1.4)	1(1.4)	3(4.1)	4(5.5)

Table II. Numbers in parentheses are percentages based on 73 responses. The percentages do not total 100% because many respondents indicated they used more than one technique in a given application area.

Operations Research Technique Utilization in Each Application Area

Application Area	Number of respondents	Linear Programming	Dynamic Programming	Network Models	Simulation	Queueing Theory	Game Theory	Regression Analysis	Other
Production Scheduling	56	53.6	12.5	10.7	46.4	16.1	0.0	8.9	17.9
Production Planning Control	37	51.4	8.1	18.9	48.6	10.8	0.0	8.1	8.1
Project Planning Control	40	25.0	2.5	70.0	22.5	5.0	0.0	0.0	7.5
Inventory Analysis Control	47	23.4	6.4	6.4	57.4	8.5	2.1	25.5	14.9
Quality Control	27	7.4	0.0	3.7	7.4	0.0	0.0	55.6	33.3
Maintenance & Repair	16	0.0	6.3	18.8	50.0	18.8	6.3	25.0	25.0
Plant Layout	32	40.6	0.0	15.6	59.4	15.6	3.1	6.3	9.4
Equipment Acquisition Replacement	22	18.2	0.0	4.5	50.0	4.5	0.0	0.0	31.8
Blending	34	94.1	0.0	2.9	17.6	2.9	0.0	8.8	2.9
Logistics	41	65.9	2.4	19.5	58.5	7.3	4.9	14.6	4.9
Plant Location	43	74.4	4.7	18.6	53.5	2.3	0.0	11.6	9.3

Table III. Figures here are percentages based on number of responses shown at left. Again, percentages do not sum to 100% because of multiple use of the techniques.

REFERENCES FOR THE THEORY OF GAMES

[1] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, 1944, Princeton University Press, 3rd Edition, 1953.

Historically interesting as the first book on the subject, and a very comprehensive one.

[2] J. D. Williams, The Compleat Strategyst, 1954, McGraw-Hill, 2nd Edition, [illegible].

An elementary and entertaining book, to be read for enjoyment.

[3] J. C. C. McKinsey, Introduction to the Theory of Games, 1952, McGraw-Hill.

A well-written text, especially useful as supplementary reading on games in extensive form.

[4] Samuel Karlin, Mathematical Methods and Theory in Games Programming and Economics, Vols. I and II, 1959, Addison-Wesley Publishing Co.

A moderately advanced text, complete and clear and a good survey of the early literature. Vol. II is on infinite games.

[5] G. Owen, Game Theory, 1968, W. B. Saunders Company.

A moderately advanced text, containing a treatment of n-person games.

[6] Melvin Dresher, Theory and Applications of Games of Strategy, 1961, Prentice-Hall, Inc.

Not a text but clear and precise mathematically.

[7] R. D. Luce and H. Raiffa, Games and Decisions, 1957, John Wiley and Sons.

A lucid text with a comprehensive treatment.

[8] D. Blackwell and M. A. Girshick, Theory of Games and Statistical Decisions, 1954, John Wiley and Sons.

The first two chapters are on two-person zero-sum games.

[9] T. Parthasarathy and T. E. S. Raghavan, Some Topics in Two-Person Games, 1971, American Elsevier Publishing Co.

An advanced exposition surveying more recent topics.

[10] A. Rapoport, Two-Person Game Theory, 1966, N-Person Game Theory, 1970, Ann Arbor Science Paperbacks, University of Michigan Press.

Wordy but well-written.

[11] S. Vajda, The Theory of Games and Linear Programming, 1956. An Introduction to Linear Programming and the Theory of Games, 1960 Science Paperbacks, 1966, Methuen and Co.

Elementary and restricted to matrix games.

GAME THEORY AND LINEAR PROGRAMMING

Part I. Game Theory

1. Introduction. The construction of a mathematical model of a field of endeavor in order to understand more fully the phenomena of that field and to make better judgments of what course of action to choose is a fairly recent development in the history of science that now affects such diverse disciplines as chemistry, engineering, medicine, economics, and business management. Evolving from and contributing to this development are several new mathematical fields such as linear programming, statistical decision theory, control theory, and so on, that are concerned with the same underlying objective - to analyze mathematically various possible courses of action in order to determine which course is best according to some criterion. The theory of games is one of these fields. It is characterized by its involvement of two or more participants with conflicting interests who simultaneously choose courses of action to promote their own interests. It has application to the analysis of mathematical models of competitive economic problems, of military strife, of conflicting political interests and so on. In such situations, what is good for one player may be bad for another. Generally, there is no "best" course of action for a player since outcomes depend upon the actions chosen by the other players. Worse yet, players may choose courses of actions that are bad for all of them, as when war breaks out. Accordingly, some game theorists suggest calling the theory of games by another title - the theory of conflict resolution.

The individual most closely associated with the creation of the theory of games is John von Neumann, one of the greatest mathematicians of this century. Although others preceeded him in formulating a theory of games - notably Emile Borel - it was von Neumann who published in 1928 the paper that laid the foundation for the theory of games as it is known today. Von Neumann's work culminated in a fundamental book on the subject written in collaboration with Oskar Morgenstern entitled Theory of Games and Economic Behavior, [1].

Von Neumann's theory is most complete for the class of games called two-person zero-sum games, i.e. games with only two players in which one player wins what the other player loses. For the time being, we restrict attention to such games.

The normal form of a two-person zero-sum game is given by a triplet (X,Y,L), where

(1) X is a set, the set of strategies of player I

(2) Y is a set, the set of strategies of player II

(3) L is a real-valued function defined on $X \times Y$. (Thus, $L(x,y)$ is a real number for every $x \in X$ and every $y \in Y$.)

The interpretation is as follows. Simultaneously, player I chooses $x \in X$ and player II chooses $y \in Y$, each unaware of the choice of the other. Then their choices are made known and I wins the amount $L(x,y)$ from II. Depending on the monetary unit involved, $L(x,y)$ will be cents, dollars, rubles, beads, etc. If L is negative, I pays the absolute value of this amount to II. Thus, $L(x,y)$ represents the winnings of I and the losses of II.

This is a very simple definition of a game; yet it is broad enough to encompass games such as tic-tac-toe and chess. This is done by being sufficiently broadminded about the definition of a strategy. A strategy for a game of chess, for example, is a complete description of how to play the game, of what move to make in every possible situation that could occur. It is rather time-consuming to write down even one strategy, good or bad, for the game of chess. However, several different programs for instructing a machine to play chess have been written. Each program constitutes one strategy. The set of all such strategies for player I is denoted by X. Naturally, in the game of chess it is physically impossible to describe all possible strategies since there are too many; in fact, there are more strategies than there are atoms in the known universe. On the other hand, the number of games of tic-tac-toe is rather small, so that it is possible to study all strategies and find an optimal strategy for each player. Later, when we study the extensive form of a game, we will see that many other types of games may be modeled and described in normal form.

To illustrate the notions involved in games, let us consider the simplest non-trivial case when X and Y each consist of two elements. As an example, take the game that I call the game of Odd-or-Even.

Players I and II simultaneously call out one of the numbers one or two. Player I wins if the sum of the numbers is odd and Player II wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. Hence, $X = \{1,2\}$, $Y = \{1,2\}$, and L is given in the following table.

		II (even) y	
		1	2
I (odd) x	1	-2	+3
	2	+3	-4

$L(x,y)$ = I's winnings = II's losses

It turns out that one of the players has a distinct advantage in this game. Can you tell which one it is?

Let us analyze this game from player I's point of view. Suppose he calls one $3/5^{ths}$ of the time and two $2/5^{ths}$ of the time, at random. In this case,

1. If II calls one, I loses 2 dollars $3/5^{ths}$ of the time and wins 3 dollars $2/5^{ths}$ of the time; on the average, he wins $-2(3/5) + 3(2/5) = 0$ (he breaks even).

2. If II calls two, I wins 3 dollars $3/5^{ths}$ of the time and loses 4 dollars $2/5^{ths}$ of the time; on the average, he wins $3(3/5) - 4(2/5) = 1/5$.

That is, if I mixes his choices in the given way, the game is even every time II calls one, but I wins 20¢ on the average every time II calls two. By employing this simple strategy, I is assured of at least breaking even on the average. Can player I fix it so that he wins a positive amount no matter what II calls?

Let p denote the proportion of times that player I calls one. Let us try to choose p so that player I wins the same amount on the average whether II calls one or two. Then since I's average winnings when II calls one is $-2p + 3(1-p)$, and his average winnings when II calls two is $3p-4(1-p)$, player I should choose p so that

$$-2p+3(1-p) = 3p-4(1-p)$$

$$3-5p = 7p-4$$

$$12p = 7$$

$$p = 7/12.$$

Hence, I calls one with probability 7/12, and two with probability 5/12.

On the average, I wins $-2(7/12)+3(5/12) = 1/12$, or $8\frac{1}{3}$ ¢, every time he plays the game no matter what II does.

The game is clearly in I's favor. Can he do better than $8\frac{1}{3}$ ¢ per game on the average? The answer is: Not unless II helps him. In fact, II could use the same procedure:

call one with probability 7/12

call two with probability 5/12.

If I calls one, II's average loss is $-2(7/12)+3(5/12) = 1/12$. If I calls two, II's average loss is $3(7/12)-4(5/12) = 1/12$.

Hence, I has a procedure that guarantees him at least 1/12 on the average, and II has a procedure that keeps his average loss to at most 1/12. 1/12 is called the <u>value</u> of the game, and the procedure each uses to insure this return is called an <u>optimal strategy</u> or a <u>minimax strategy</u>.

If instead of playing the game, the players agree to call in an arbitrator to settle this conflict, it seems reasonable that the arbitrator should require II to pay $8\frac{1}{3}$¢ to I. For I could argue that he should receive at least $8\frac{1}{3}$¢ since his optimal strategy guarantees him that much on the average no matter what II does. On the other hand II could argue that he should not have to pay more than $8\frac{1}{3}$¢ since he has a strategy that keeps his average loss to at most that amount no matter what I does.

It is useful to make a distinction between a pure strategy and a mixed strategy. We refer to elements of X or Y as <u>pure strategies</u>. The more complex entity that chooses among the pure strategies at random in various proportions is called a <u>mixed strategy</u>. Thus, I's optimal strategy in the game of Odd-or-Even is a mixed strategy; it mixes the pure strategies one and two with probabilities 7/12 and 5/12 respectively. Of course every pure strategy can be considered as the mixed strategy that chooses that pure strategy 100% of the time.

The use of a mixed strategy is just as valuable when a game is to be played only once as it is when the game is played many times. A mixed strategy may be implemented with the aid of a suitable outside random mechanisms, such as tossing a coin, rolling dice, drawing a number out of a hat and so on. The second hand of a watch provides a simple method of randomization provided it is not used too frequently. For example, Player I of Odd-or-Even wants an outside random event with probability 7/12 to implement his optimal strategy. Since 7/12 = 35/60, he could take a quick glance at his watch; if the second hand were between 0 and 35, he would call one, while if it were between 35 and 60, he would call two.

A two-person zero-sum game (X,Y,L) is said to be <u>a finite game</u> if both strategy sets X and Y are finite sets. The fundamental theorem of game theory due to von Neumann states that the situation encountered in the game of Odd-or-Even holds for all finite two-person zero-sum games - namely, for every such game,

there is a number v, called the value of the game,

(1) there is a mixed strategy for I such that I's average gain is at least v no matter what II does, and

(2) there is a mixed strategy for II such that II's average loss is at most v no matter what I does.

This is one form of <u>the minimax theorem</u> to be proved later. If v is zero we say the game is fair. If v is positive, we say the game favors player I, while if v is negative, we say the game favors player II.

<u>Exercise 1.</u> Consider the game of Odd-or-Even with the sole change that [illegible] [illegible] the winner the product, rather than the sum, of the numbers [illegible] Find the table for the payoff function L, and analyze the game [illegible] the value [illegible] strategies of the players.

2. Matrix Games. Consider a two-person zero-sum game in normal form, (X,Y,L), with X and Y finite sets. Such games are sometimes called matrix games because the payoff function L can be represented by a matrix. If $X = \{x_1,\ldots,x_m\}$ and $Y = \{y_1,\ldots,y_n\}$, then by the game matrix or payoff matrix we mean the matrix

$$\underset{\sim}{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{where} \quad a_{ij} = L(x_i,y_j) ,$$

In this form, player I chooses a row, player II chooses a column, and II pays I the entry in the chosen row and column.

Saddle points. Occasionally the game is easily solved. If some entry a_{ij} of the matrix $\underset{\sim}{A}$ has the property that

(1) a_{ij} is the minimum of the i^{th} row, and

(2) a_{ij} is the maximum of the j^{th} column,

then we say a_{ij} is a saddle point. Player I can then win at least a_{ij} by choosing row i. Player II can keep his loss to at most a_{ij} by choosing column j. Hence a_{ij} is the value of the game.

Example. $\begin{pmatrix} 5 & 0 & -4 \\ 2 & 1 & 4 \\ -1 & 0 & 5 \end{pmatrix}$ The central entry is a saddle point, since it is a minimum of its row and maximum of its column.

Thus I chooses 2, and II chooses 2. For large $m \times n$ matrices it is tedious to check each value for the saddle point property. It is easier to compute the minimum of each row and the maximum of each column.

					row min
	3	2	1	0	0
	0	1	2	0	0
	1	0	2	1	0
	3	1	2	2	1
col max	3	2	2	2	

Since no row minimum is equal to a column maximum there is no saddle point. If the 2 in position a_{12} were changed to a 1, a_{12} would be a saddle point.

Mixed strategies. Elements of X and Y are called pure strategies. A mixed strategy for a player is a probability distribution over his pure strategies. A mixed strategy for I will be represented by the m-tuple $(p_1,\dots,p_m)$ where $p_i \geq 0$ for all i, and $\Sigma_1^m p_i = 1$. The interpretation is that if I uses $(p_1,\dots,p_m)$, he chooses x_1 with probability p_1, x_2 with probability p_2, ..., and x_m with probability p_m. In a similar vein, II's mixed strategies are represented by the n-tuples $(q_1,\dots,q_n)$ where $q_j \geq 0$ for all j, and $\Sigma_1^n q_j = 1$.

The pure strategy of choosing row i, say, is represented by the m-tuple $(p_1,\dots,p_m)$ with $p_i = 1$, and the rest of the $p_j = 0$.

Solution of 2×2 matrix games. Consider the 2×2 game matrix $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$. To solve this game (i.e. to find the value and at least one optimal strategy for each player) we proceed as follows.

First, test for a saddle point.

Second, if there is no saddle point, solve by the method of section 1.

To prove the method of section 1 works whenever there is no saddle point, we use the following observation.

Assume there is no saddle point. If $a \geq b$, then $b < c$, otherwise b is a saddle point. Since $b < c$, we must have $c > d$, otherwise c is a saddle point. Continuing thus, we see that $d < a$ and $a > b$. In other words, if $a \leq b$, then $a > b < c > d < a$. By symmetry, if $a \leq b$, then $a < b > c < d > a$. This shows that

if there is no saddle point, then

either $a > b$, $b < c$, $c > d$ and $d < a$,

or $a < b$, $b > c$, $c < d$ and $d > a$.

In equations (1), (2) and (3) below, we develop formulas for the optimal strategies and value of the general 2×2 game. If I chooses the first row with probability p (i.e. uses the mixed strategy $(p, 1-p)$), we equate his average return when II uses columns 1 and 2.

$$ap + d(1 - p) = bp + c(1 - p) .$$

Solving for p, we find

(1) $$p = \frac{c - d}{(a - b) + (c - d)}$$

Since there is no saddle point, $(a - b)$ and $(c - d)$ are either both positive or both negative; hence, $0 < p < 1$. Player I's average return using this strategy is

$$v = ap + d(1 - p) = \frac{ac - bd}{a - b + c - d} .$$

If II chooses the first column with probability q (i.e. uses the strategy $(q, 1 - q)$), we equate his average losses when I uses rows 1 and 2.

$$aq + b(1 - q) = dq + c(1 - q)$$

Hence,

(2) $$q = \frac{c - b}{a - b + c - d} = \frac{c - b}{(a-d)+(c-b)} =$$

Again, since there is no saddle point, $0 < q < 1$. Player II's average loss using this strategy is

(3) $$aq + b(1 - q) = \frac{ac - bd}{a - b + c - d} = v ,$$

the same value achievable by I. This shows that the game has a value, and that the players have optimal strategies. (something the minimax theorem says holds for all finite games).

Example 1. $\underset{\sim}{A} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$

$$p = \frac{-4 - 3}{-2 - 3 - 4 - 3} = \frac{7}{12}$$

q = same

$$v = \frac{8 - 9}{-2 - 3 - 4 - 3} = \frac{1}{12}$$

2. $\underset{\sim}{A} = \begin{pmatrix} 0 & -10 \\ 1 & 2 \end{pmatrix}$

$$p = \frac{2 - 1}{10 + 2 - 1} = \frac{1}{11}$$

$$q = \frac{2 + 10}{10 + 2 - 1} = \frac{12}{11}$$

But q must be between zero and one. What happened? The trouble is we " forgot to test this matrix for a saddle point, so of course it has one". (J. D. Williams the Compleat Strategyst Revised Edition, 1966, McGraw-Hill, page 56.) The lower left corner is a saddle point. So $v = 1$, $p = 0$ and $q = 1$.

Exercises 1. Solve the game with matrix $\begin{pmatrix} -1 & -3 \\ -2 & 2 \end{pmatrix}$.

2. Solve the game with matrix $\begin{pmatrix} 1 & 2 \\ t & 0 \end{pmatrix}$ for each fixed t, and draw the graph of the value of the game as a function of t.

3. Show that if a game matrix has two saddle points, then they have equal values.

Dominance. Sometimes, large matrix games may be reduced in size (hopefully to the 2×2 case) by deleting rows and columns that are obviously bad for the player who chooses them.

Definition. The i^{th} row dominates the k^{th} row if $a_{ij} \geq a_{kj}$ for all j. The j^{th} column dominates the k^{th} column if $a_{ij} \leq a_{ik}$ for all i.

If the i^{th} row dominates the k^{th} row, the k^{th} row may be deleted from the matrix. Player I can do at least as well choosing row i instead of row k. A similar argument shows that if the k^{th} column is dominated it may be removed.

Example. Consider the matrix,

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

The last column is dominated by the middle column. Deleting the last column we obtain:

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{pmatrix}$$

Now the top row is dominated by the bottom row. (Note this is not the case in the original matrix). Deleting the top row we obtain

$$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

This 2×2 matrix does not have a saddle point, so $p = \frac{3}{4}$, $q = \frac{1}{4}$ $v = \frac{7}{4}$. I's optimal strategy in the original game is $(0, 3/4, 1/4)$; II's is $(\frac{1}{4}, \frac{3}{4}, 0)$.

A row (column) may also be removed if it is dominated by a probability combination of other rows (columns). If for some $0 < p < 1$, $pa_{i_1 j} + (1 - p)a_{i_2 j} \geq a_{kj}$ for all j, then the k^{th} row is dominated by the mixed strategy that chooses row i_1 with probability p and row i_2 with probability $1 - p$. Player I can do at least as well using this mixed strategy instead of choosing row k. (In addition, any mixed strategy choosing row k with probability p_k may be replaced by the one in which k's probability is split between i_1 and i_2: i_1's probability is increased by $p \cdot p_k$ and i_2's probability is increased by $(1 - p)p_k$.) A similar argument may be used for columns.

Consider the matrix $\underset{\sim}{A}$. The middle column is dominated by the outside columns taken with probability 1/2 each. With the central column deleted, the middle row is dominated by the combination of the top row with probability 1/3 and the bottom row with probability 2/3. The reduced matrix is easily solved. The value is $v = 54/12 = 9/2$. Of course, mixtures of more than two rows (columns) may be used to dominate and remove other rows (columns). For example, the mixture of columns one two and three with probabilities 1/3 each in matrix $\underset{\sim}{B}$ dominates the last column, and so the last column may be removed.

$$\underset{\sim}{A} = \begin{pmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 6 \\ 9 & 3 \end{pmatrix}$$

$$\underset{\sim}{B} = \begin{pmatrix} 1 & 3 & 5 & 3 \\ 4 & 0 & 2 & 2 \\ 3 & 7 & 3 & 5 \end{pmatrix}$$

Not all games may be reduced by dominance. In fact, even if the matrix has a saddle point, there may not be any dominated rows or columns. The 3×3 Example of a game with a saddle point (pg. 9) demonstrates this.

<u>Exercise 4.</u> Reduce by dominance to 2×2 games and solve.

(a) $$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$$

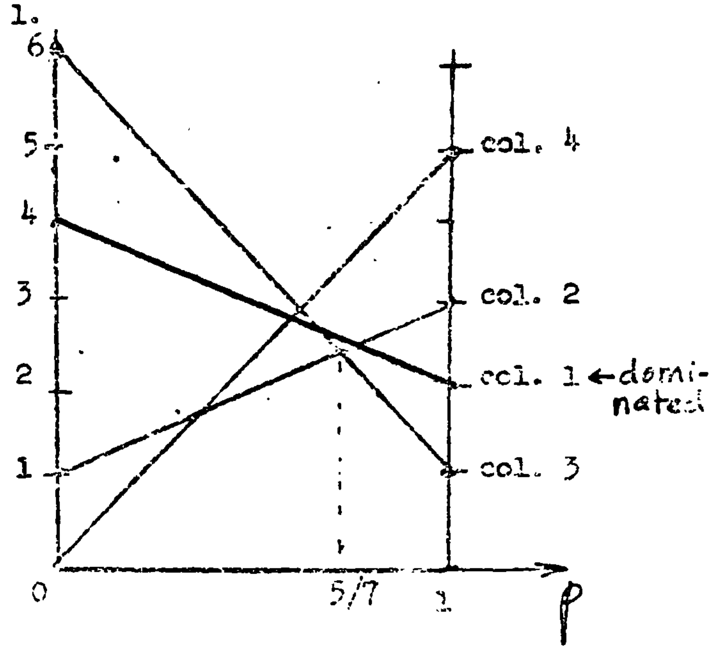
(b) $$\begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}.$$

<u>Solving $2 \times n$ and $m \times 2$ games.</u> Games with matrices of size $2 \times n$ or $m \times 2$ may be solved with the aid of a graphical interpretation. Take the following example.

$$\begin{matrix} p \\ 1-p \end{matrix} \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

2×n
lower envelope
maximin

Suppose Player I chooses the first row with probability p and the second row with probability $1 - p$. If II chooses Column 1, I's average payoff is $2p + 4(1 - p)$. Similarly, choices of Columns 2, 3 and 4 result in average payoffs of $3p + (1 - p)$, $p + 6(1 - p)$, and $5p$ respectively. We graph these four linear functions of p for $0 \le p \le 1$.



For a fixed value of p, Player I can be sure that his average winnings is at least the minimum of these four functions evaluated at p. This is known as the <u>lower envelope</u> of these functions. Since I wants to maximize his guaranteed average winnings, he wants to find p that maximizes this lower envelope. According to the drawing, this should occur at the intersection of the lines for Columns 2 and 3. This essentially, involves solving the game in which II is restricted to Columns

2 and 3. The value of the game $\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}$ is $v = \frac{17}{7}$, I's optimal strategy is $\left(\frac{5}{7},\frac{2}{7}\right)$, and II's optimal strategy is $\left(\frac{5}{7}, \frac{2}{7}\right)$. Subject to the accuracy of the drawing, we conclude therefore that in the original game I's optimal strategy is $\left(\frac{5}{7},\frac{2}{7}\right)$, II's is $\left(0,\frac{5}{7},\frac{2}{7},0\right)$ and the value is $\frac{17}{7}$. The accuracy may be checked: <u>Given any guess at a solution to a game, there is a sure-fire test to see if the guess is correct</u>, as follows. If I uses the strategy $\left(\frac{5}{7},\frac{2}{7}\right)$, his average payoff if II uses Columns 1, 2, 3 and 4, is $\frac{18}{7}$, $\frac{17}{7}$, $\frac{17}{7}$ and $\frac{25}{7}$ respectively. Thus his average payoff is at least $\frac{17}{7}$ no matter what II does. Similarly, if II uses $\left(0,\frac{5}{7},\frac{2}{7},0\right)$, his average loss is (at most) $\frac{17}{7}$. Thus, $\frac{17}{7}$ is the value, and these strategies are optimal.

We note that the line for Column 1 plays no role in the lower envelope (that is, the lower envelope would be unchanged if the line for Column 1 were removed from the graph). This is a test for domination. Column 1 is, in fact, dominated by Columns 2 and 3 taken with probability 1/2 each. The line for Column 4 does appear in the lower envelope, and hence Column 4 can not be dominated.

As an example of a 2 × 2 (m×2) game, consider the following matrix. If q is the probability that II chooses Column 1, then II's average loss for I's three possible choices of rows is given in the accompanying graph. Here, player II looks at the largest of his

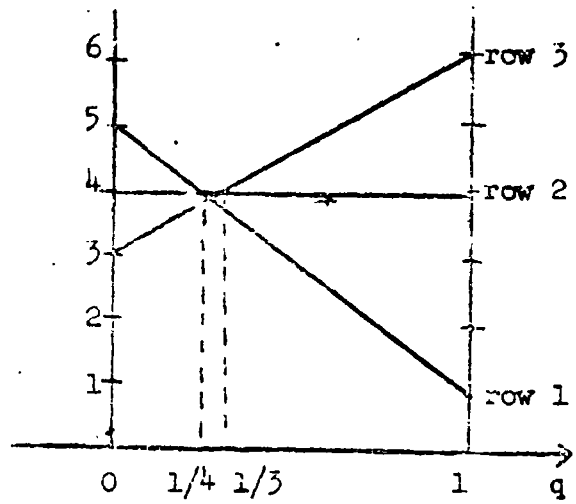
$$\begin{matrix} q & 1-q \\ \end{matrix}$$
$$\begin{pmatrix} 1 & 5 \\ 4 & 4 \\ 6 & 3 \end{pmatrix}$$

m × 2

upper envelope

minimax

average losses for a given q. This is the upper envelope of the function. II wants to find q that minimizes this upper envelope. From the graph, we see that any value of q between 1/4 and 1/3 inclusive achieves this minimum. The value of the game is 4, and I has an optimal pure strategy: row 2.



Exercise 5. Solve $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$

Exercise 6. Reduce to 3×2 by dominance and solve.

$$\begin{pmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{pmatrix}$$

In general, the sure-fire test may be stated thus. For a given game, conjectured optimal strategies $(p_1,\cdots,p_m)$ and $(q_1,\cdots,q_n)$ are indeed optimal if the minimum of I's average payoffs using $(p_1,\cdots,p_m)$ is equal to the maximum of II's average payoffs using $(q_1,\cdots,q_n)$.

Exercise 7. Show that for the game with matrix → the mixed strategies $p = \left(\frac{6}{37}, \frac{20}{37}, 0, \frac{11}{37}\right)$ and $q = \left(\frac{14}{37}, \frac{4}{37}, 0, \frac{19}{37}, 0\right)$ are optimal for I and II respectively. What is the value?

$$\begin{pmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{pmatrix}$$

Exercise 8. Given that $p = \left(\frac{52}{143}, \frac{50}{143}, \frac{41}{143}\right)$ is optimal for I in the game with matrix → what is the value?

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$

3. Upper and lower values. Consider an arbitrary finite two-person zero-sum game (X,Y,L) with $m \times n$ game matrix $\underset{\sim}{A}$. The sets of mixed strategies of players I and II will be denoted respectively by X^* and Y^*,

$$X^* = \{\underset{\sim}{p} = (p_1,\ldots,p_m) : p_i \geq 0,\ i = 1,\ldots,m \text{ and } \Sigma_1^m p_i = 1\}$$

$$Y^* = \{\underset{\sim}{q} = (q_1,\ldots,q_n) : q_j \geq 0,\ j = 1,\ldots,n \text{ and } \Sigma_1^n q_j = 1\} .$$

It is useful to think of the elements of X^* and Y^* as row vectors. The m-dimensional unit vector $\underset{\sim}{e}_k$ with a one for the k^{th} component and zeros elsewhere may be identified with the pure strategy of choosing row k. Thus, we may consider X to be a subset of X^*. Similarly, Y may be considered to be a subset of Y^*.

If player II uses $\underset{\sim}{q} \in Y^*$ and player I chooses row i, the average payoff to I is

$$(1) \qquad \sum_{j=1}^{n} a_{ij}q_j .$$

If player I chooses the row at random according to $\underset{\sim}{p} \in X^*$, the average payoff to I becomes

$$(2) \qquad \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}q_j\Big)p_i = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j = \underset{\sim}{p}\,\underset{\sim}{A}\,\underset{\sim}{q}^T$$

where $\underset{\sim}{q}^T$ represents the transpose of $\underset{\sim}{q}$, and is thus a column vector. We could if we like consider the game (X,Y,L) in which the players are allowed to use mixed strategies as a new game (X^*,Y^*,L^*), where $L^*(\underset{\sim}{p},\underset{\sim}{q}) = \underset{\sim}{p}\,\underset{\sim}{A}\,\underset{\sim}{q}^T$, though this game would no longer be a finite game.

Suppose that I were able to guess correctly that II has decided to use $\underset{\sim}{q} \in Y^*$. Then he would choose that row i that maximizes (1); or, equivalently, he would choose that $\underset{\sim}{p} \in X^*$ that maximizes (2). His average payoff would be

$$\max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \max_{\underset{\sim}{p} \in X^*} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T . \tag{3}$$

To see that these quantities are equal, note that the left side is the maximum of $\underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T$ over $\underset{\sim}{p} \in X$, and so, since $X \subset X^*$, must be less than or equal to the right side. The reverse inequality follows since (2) is an average of the quantities in (1) and so must be less than or equal to the largest of the values in (1).

Any $\underset{\sim}{p} \in X^*$ that achieves the maximum in (3) is called a Bayes strategy against $\underset{\sim}{q}$. In particular, any row i that achieves the maximum of (3) is a (pure) Bayes strategy against $\underset{\sim}{q}$. There always exist pure Bayes strategies against q for every $q \in Y^*$ in finite games.

This notion represents a practical way of playing a game: Make a guess at the probabilities that you think your opponent will play his various pure strategies, and choose a Bayes strategy against this. Of course this may be a dangerous procedure. Your opponent may be better at this type of guessing than you. (See Exercise 1.)

Let us now analyze the general matrix game from player II's viewpoint. If he uses $\underset{\sim}{q} \in Y^*$, he is sure of not losing more than (3) on the average. He would lose this amount on the average if, for example, I were to use a Bayes strategy against $\underset{\sim}{q}$. Therefore, the best that II can be sure of doing by the use of any $\underset{\sim}{q} \in Y^*$ is to keep his average loss to the minimum of (3) over $\underset{\sim}{q} \in Y^*$, denoted by $\overline{V}$ and called the upper value of the game.

(4) $$\overline{V} = \min_{\underset{\sim}{q} \in Y^*} \max_{1 \le i \le m} \sum_{j=1}^{n} a_{ij} q_j = \min_{\underset{\sim}{q} \in Y^*} \max_{\underset{\sim}{p} \in X^*} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T .$$

One may think of $\overline{V}$ as the smallest average loss that player II can assure for himself no matter what I does. Any $\underset{\sim}{q} \in Y^*$ that achieves the minimum in (4) is called a minimax strategy for II. It minimizes his maximum loss. There always exists a minimax strategy in finite games: the quantity (3), being the maximum of m linear functions of $\underset{\sim}{q}$, is a continuous function of $\underset{\sim}{q}$ and since Y^* is a closed bounded set, this function assumes its minimum over Y^* at some point of Y^*.

A similar analysis may be carried through from player I's viewpoint. If I uses $\underset{\sim}{p} \in X^*$, he is assured of winning on the average at least

(5) $$\min_{1 \le j \le n} \sum_{i=1}^{m} p_i a_{ij} = \min_{\underset{\sim}{q} \in Y^*} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T .$$

The column j or the mixed strategy $\underset{\sim}{q}$ that achieves the minimum in (5) is called a Bayes strategy for II against $\underset{\sim}{p}$. Therefore, the best that I can be sure of winning on the average by the use of any $\underset{\sim}{p} \in X^*$ is

(6) $$\underline{V} = \max_{\underset{\sim}{p} \in X^*} \min_{1 \le j \le n} \sum_{i=1} p_i a_{ij} = \max_{\underset{\sim}{p} \in X^*} \min_{\underset{\sim}{q} \in Y^*} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T .$$

The quantity $\underline{V}$ is called the lower value of the game. It is the maximum amount that I can guarantee himself no matter what II does. Any $\underset{\sim}{p} \in X^*$ that achieves the maximum in (6) is called a minimax strategy for I. Perhaps maximin strategy would be more appropriate terminology in view of (6), but from symmetry (either player may consider himself player II for purposes of analysis) the same word to describe the same idea may be preferrable and it is certainly the customary terminology. As in the analysis for player II, we see that player I always has a minimax strategy. This observation is worth stating as a lemma.

Lemma 1. In a matrix game, both players have minimax strategies.

It is easy to argue that the lower value is less than or equal to the upper value. For if $\overline{V} < \underline{V}$ and if I can assure himself of winning at least $\underline{V}$, player II cannot assure himself of losing at most $\overline{V}$. It is worth stating this fact as a lemma too.

Lemma 2. $\underline{V} \leq \overline{V}$.

This lemma also follows from the general mathematical principle that $\max_x \min_y f(x,y) \leq \min_y \max_x f(x,y)$. To see this principle, note that $\min_{y'} f(x,y') \leq \max_{x'} f(x',y)$ for every x and y, so that taking $\max_x$ on the left does not change the inequality, nor does taking $\min_y$ on the right, which gives the result.

When $\underline{V} = \overline{V}$ a very nice situation exists. Player II can assure himself of not losing more on the average than player I can assure himself of winning.

Definition. If $\underline{V} = \overline{V}$, we say the value of the game exists and is equal to the common value of $\underline{V}$ and $\overline{V}$, denoted simply by V. If the value of the game exists, we refer to minimax strategies as optimal strategies.

All the simple games studied in Section 2 were seen to have values. In the next section we prove the minimax theorem, that all matrix games have values.

Another simple observation is useful in this regard. This concerns the invariance of the minimax strategies under the operations of adding a constant to each entry of the game matrix, and of multiplying each entry of the game matrix by a positive constant. The game having matrix $\underset{\sim}{A} = (a_{ij})$ and the game having matrix $\underset{\sim}{A}' = (a'_{ij})$ with $a'_{ij} = a_{ij} + b$, where b is an arbitrary real number, are very closely related. In fact, the game with matrix $\underset{\sim}{A}'$ is equivalent to the game in which II pays I the amount b (or I pays II the amount $-b$), and then I and II play the game with matrix $\underset{\sim}{A}$. Clearly any strategies used in the game with matrix $\underset{\sim}{A}'$ give I b plus the payoff using the same strategies in the game with matrix $\underset{\sim}{A}$. Thus, any minimax strategy for either player in one game is also minimax in the other, and the upper (lower) value of the game with matrix $\underset{\sim}{A}'$ is b plus the upper (lower) value of the game with matrix $\underset{\sim}{A}$.

Similarly, the game having matrix $\underset{\sim}{A}'' = (a''_{ij})$ with $a''_{ij} = ca_{ij}$, where c is a positive constant, may be considered as the game with matrix $\underset{\sim}{A}$ with a change of scale (a change of monetary unit if you prefer). Again, minimax strategies do not change, and the upper (lower) value of $\underset{\sim}{A}''$ is c times the upper (lower) value of $\underset{\sim}{A}$. We combine these observations as follows.

Lemma 3. If $\underset{\sim}{A} = (a_{ij})$ and $\underset{\sim}{A}' = (a'_{ij})$ are matrices with $a'_{ij} = ca_{ij} + b$, where $c > 0$, then the game with matrix $\underset{\sim}{A}$ has the same minimax strategies for I and II as the game with matrix $\underset{\sim}{A}'$. If $\underline{V}$ and $\overline{V}$ are the lower and upper values of the game with matrix $\underset{\sim}{A}$, then game with matrix $\underset{\sim}{A}'$ has lower value $\underline{V}' = c\underline{V} + b$, and upper value $\overline{V}' = c\overline{V} + b$.

In particular, the value V of the game with matrix $\underset{\sim}{A}$ exists if and only if the value V' of the game with matrix $\underset{\sim}{A}'$ exists, and then $V' = cV + b$.

Exercises

1. Consider the game with matrix $\begin{pmatrix} 0 & 7 & 2 & 4 \\ 1 & 4 & 8 & 2 \\ 9 & 3 & -1 & 6 \end{pmatrix}$. Past experience in playing the game with player II enables player I to arrive at a set of probabilities reflecting his belief of the column that II will choose. I thinks that with probabilities 1/5, 1/5, 1/5, and 2/5, II will choose columns 1, 2, 3, and 4 respectively.

(a) Find for I a Bayes strategy against (1/5, 1/5, 1/5, 2/5).

(b) Suppose II guesses correctly that I is going to use a Bayes strategy against (1/5, 1/5, 1/5, 2/5). Instruct II on the strategy he should use - that is, find II's Bayes strategy against I's Bayes strategy against (1/5, 1/5, 1/5, 2/5).

2. The game with matrix $\underset{\sim}{A}$ has value zero, and (6/11, 3/11, 2/11) is optimal for I.

$$\underset{\sim}{A} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ -3 & 3 & 0 \end{pmatrix}$$

(a) Find the value of the game with matrix $\underset{\sim}{A}'$ and an optimal strategy for I.

$$\underset{\sim}{A}' = \begin{pmatrix} 5 & 3 & 7 \\ 9 & 5 & 1 \\ -1 & 11 & 5 \end{pmatrix}$$

(b) Find an optimal strategy for II in both games.

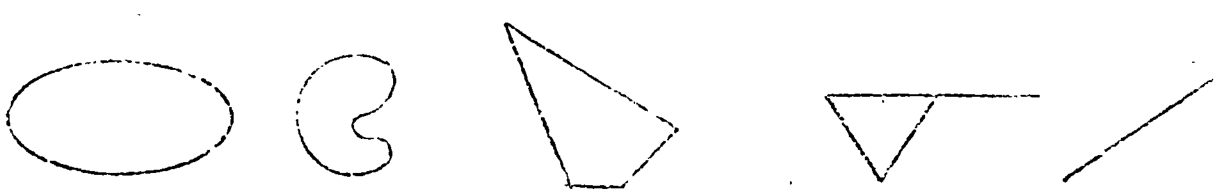
3. Solve the game with matrix $\begin{pmatrix} 0 & -1 & 1 \\ a & 0 & -a \\ -b & b & 0 \end{pmatrix}$ where $a > 0$ and $b > 0$.

4. The minimax theorem. We set out now to prove the minimax theorem. Our starting point is the celebrated separating hyperplane theorem which we state without proof. Later, we will give a constructive method of finding the value and optimal strategies of a matrix game via the simplex method. When we show that this method works for all matrix games, we will have given a second and complete proof of the minimax theorem. However, the following geometric proof has more intuitive appeal.

Let $\mathbb{R}$ denote the real line, and $\mathbb{R}^k$ denote k-dimensional real vector space (the set of all k-tuples of real numbers). The separating hyperplane theorem states that any two disjoint convex sets can be separated by a hyperplane. We first define the notion of a convex set.

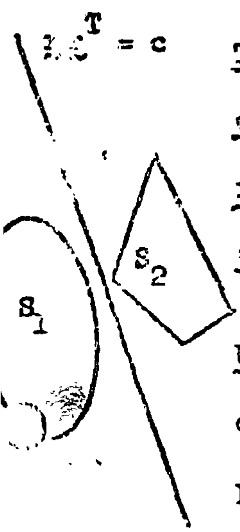
Definition. A set $S \subset \mathbb{R}^k$ is said to be convex if for every $\underset{\sim}{x} \in S$ and every $\underset{\sim}{y} \in S$ and every α, $0 \leq \alpha \leq 1$, the point $\alpha\underset{\sim}{x} + (1-\alpha)\underset{\sim}{y} \in S$.

Note: For fixed $\underset{\sim}{x}$ and $\underset{\sim}{y}$, the set of points $\{\underset{\sim}{z} = \alpha\underset{\sim}{x} + (1-\alpha)\underset{\sim}{y}: 0 \leq \alpha \leq 1\}$ is the line segment joining the points $\underset{\sim}{x}$ and $\underset{\sim}{y}$. Therefore, a set in $\mathbb{R}^k$ is convex if it contains the line segment joining any two of its points.



CONVEX NOT CONVEX CONVEX NOT CONVEX CONVEX

Two sets are said to be disjoint if they have no points in common. The set of points z in $\mathbb{R}^k$ satisfying an equation of the form $z r^T = c$ for some $r \in \mathbb{R}^k$, $r \neq 0$ (the vector of all zeros), and some real number c, is called a hyperplane. The vector r is the vector normal or perpendicular to this hyperplane.



Theorem 1. The separating hyperplane theorem. Let S_1 and S_2 be disjoint convex subsets of $\mathbb{R}^k$. Then there exists a vector $r \in \mathbb{R}^k$, $r \neq 0$, and a real number c such that $s_1 r^T \leq c \leq s_2 r^T$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Remark: The hyperplane $z r^T = c$ divides $\mathbb{R}^k$ into two regions, one of which contains S_1 and the other S_2. The hyperplane may contain points of S_1 and points of S_2. See Karlin [4] Vol I, Appendix B.1, or Blackwell and Girshick [8] for a proof.

Now consider a finite game (X,Y,L) with $m \times n$ game matrix A. Let $a_1, a_2, \ldots, a_m$ represent the rows of A. For $i = 1, \ldots, m$, a_i is an n-dimensional row vector representing the payoff vector to I if he chooses row i. That is, the j^{th} component of a_i is the amount II pays I if I chooses i and II chooses j. If I uses $p \in X^*$, the vector of average payoffs to I is

$$(1) \qquad \sum_{i=1}^{m} p_i a_i = p A .$$

The j^{th} component of pA is the amount II pays I on the average if I uses $p \in X^*$ and II chooses j. Let

(2) $$S = \{z = pA : p \in X^*\}$$

or equivalently,

(3) $$S = \{z = \sum_1^m p_i a_i : p_i \geq 0,\ i = 1,\ldots,m,\ \sum_1^m p_i = 1\} .$$

The set $S \subset \mathbb{R}^n$ is the set of average payoff vectors available to I through the use of mixed strategies. The original game (X,Y,L) with I being allowed the use of mixed strategies is equivalent to the following game: I chooses a point $z \in S$ and simultaneously II chooses a coordinate $j \in \{1,2,\ldots,n\}$. Then II pays I the j^{th} coordinate of z.

The first indication that the separating hyperplane theorem might have some use in analysis of games is that S is convex.

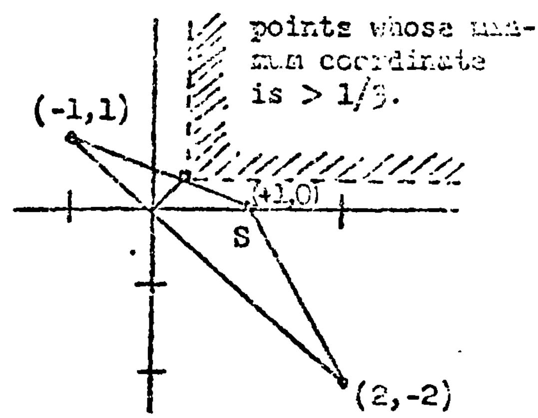
Lemma 1. S is convex.

Proof. Let $z \in S$ and $z' \in S$ and $0 \leq \alpha \leq 1$. We are to show that $\alpha z + (1-\alpha)z' \in S$. Since $z \in S$ and $z' \in S$ there is a $p \in X^*$ and a $p' \in X^*$ such that $z = pA$ and $z' = p'A$. But $\alpha z + (1-\alpha)z' = \alpha pA + (1-\alpha)p'A = (\alpha p + (1-\alpha)p')A$. It is sufficient to show that $\alpha p + (1-\alpha)p' \in X^*$. The i^{th} coordinate is $\alpha p_i + (1-\alpha)p_i'$ which is non-negative, and the sum of the coordinates is $\sum_1^m(\alpha p_i + (1-\alpha)p_i') = \alpha \sum_1^m p_i + (1-\alpha)\sum_1^m p_i' = \alpha + (1-\alpha) = 1$. Therefore, $\alpha p + (1-\alpha)p' \in X^*$ completing the proof.

The set S as defined by (3) is known as the convex hull of the vectors $a_1,\ldots,a_m$. As an example, consider the game with matrix $\underset{\sim}{A}$. $\underset{\sim}{A} = \begin{pmatrix} 2 & -2 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$

[illegible] (the triangle [illegible]

plus the boundary) formed by joining the three points $\underset{\sim}{a}_1 = (2,-2)$, $\underset{\sim}{a}_2 = (-1,1)$ and $\underset{\sim}{a}_3 = (1,0)$. I chooses a point $\underset{\sim}{z} \in S$, and II chooses a coordinate $j \in \{1,2\}$, and II pays I z_j. To attain the lower value of the game I would choose $\underset{\sim}{z}$ so that $\min_j z_j$ is



a maximum; that is, he would choose that $\underset{\sim}{z} \in S$ whose minimum coordinate is a maximum. The minimum coordinate of (2,-2) is -2. The point (1,0) is better; its minimum coordinate is 0. Traveling up the line from (1,0) to (-1,1) the minimum coordinate increases until we reach (1/3,1/3); then it starts to decrease. We cannot obtain a point of S with minimum coordinate larger than 1/3, since the set of points whose minimum coordinate is greater than 1/3 is disjoint from S. The lower value is thus 1/3; it is attained if I chooses $(1/3,1/3) \in S$. This point is a probability mixture of the pure strategies for I of choosing row two and row three, since (1/3,1/3) is on the line segment joining (-1,1) and (1,0). To find the correct mixture we solve

$p(-1,1) + (1-p)(1,0) = (1/3,1/3)$ for p: $p = 1/3$. Therefore the mixed strategy (0,1/3,2/3) is minimax for I.

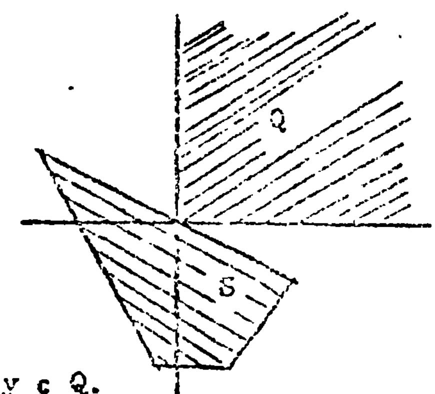
<u>Theorem 2.</u> <u>The minimax theorem.</u> <u>Every finite two-person zero-sum game has a value.</u>

Proof. We are to show $\underline{V} = \overline{V}$. It is always true that $\underline{V} \leq \overline{V}$ (Lemma 2 of Section 3). We are to show $\overline{V} \leq \underline{V}$. We may assume without loss of generality that $\underline{V} = 0$, since a game with arbitrary $\underline{V}$ could be reduced to this case by subtracting $\underline{V}$ from every element of $\underset{\sim}{A}$ (Lemma 3 of Section 3).

Therefore, we assume $\underline{V} = 0$ and attempt to show $\overline{V} \leq 0$. That is, we attempt to show that there is a mixed strategy $\underset{\sim}{q}$ for II such that $\underset{\sim}{p}\underset{\sim}{A}\underset{\sim}{q}^T \leq 0$ for all $\underset{\sim}{p} \in X^*$, or equivalently $\underset{\sim}{z}\underset{\sim}{q}^T \leq 0$ for all $\underset{\sim}{z} \in S$. But

$$(4) \quad \underline{V} = \max_{p \in X^*} \min_{1 \leq j \leq n} \sum_{i=1}^{m} p_i a_{ij} = \max_{\underset{\sim}{z} \in S} \min_{1 \leq j \leq n} z_j = 0$$

means that for every $\underset{\sim}{z} \in S$ there is a $j \in \{1,\ldots,n\}$ such that $z_j \leq 0$. In other words, no point of S has all coordinates positive; that is, S and the positive quantant $Q = \{\underset{\sim}{y} \in \mathbb{R}^n : y_j > 0,\ j = 1,\ldots,n\}$ are disjoint. Since both are convex, the separating hyperplane theorem states that there is a vector $\underset{\sim}{r} \neq \underset{\sim}{0}$ such that



$$(5) \qquad \underset{\sim}{z}\underset{\sim}{r}^T \leq \underset{\sim}{y}\underset{\sim}{r}^T \quad \text{for all } \underset{\sim}{z} \in S \text{ and all } \underset{\sim}{y} \in Q.$$

Letting $\underset{\sim}{y} \in Q$ tend to $\underset{\sim}{0}$, we see that

$$(6) \qquad \underset{\sim}{z}\underset{\sim}{r}^T \leq 0 \quad \text{for all } \underset{\sim}{z} \in S.$$

Note that no component of $\underset{\sim}{r}$ can be negative. For if $r_k < 0$, then taking $\underset{\sim}{y} \in Q$ with the k^{th} component, y_k, tending to infinity, and the rest of the components fixed would entail $\underset{\sim}{y}\,\underset{\sim}{r}^T$ tending to $-\infty$, thus contradicting (5). Not all components of $\underset{\sim}{r}$ are zero (since $\underset{\sim}{r} \neq \underset{\sim}{0}$), so that $\sum_1^n r_j > 0$. Therefore, if we let $\underset{\sim}{q} = \underset{\sim}{r}/\sum_1^n r_j$, we have $\underset{\sim}{q} \in Y^*$ and from (6) we have $\underset{\sim}{z}\,\underset{\sim}{q}^T \leq 0$ for all $\underset{\sim}{z} \in S$, as was to be shown. This completes the proof.

Remark 1. In this proof, the minimax strategy $\underset{\sim}{q}$ for II is seen to be the normal to the separating hyperplane, divided by a constant to make the sum of its components equal to one. In the example preceding the minimax theorem, the separating hyperplane must be the line containing the line segment from (1,0) to (-1,1). This has slope -1/2. The slope of the line perpendicular to this is therefore 2. One normal to the separating hyperplane is therefore $\underset{\sim}{r} = (1,2)$; the sum of the components is 3 so that (1/3,2/3) is minimax for II. One may easily use the methods of Section 2 to check these calculations.

Remark 2. The given proof of the minimax theorem applies as well to the class of semi-infinite games (S,Y,L) in which S is a convex subset of $\mathbb{R}^n$, Y is $\{1,2,\ldots,n\}$ and $L(\underset{\sim}{z},j) = z_j$. The proof shows that the game has a value and player II has a minimax strategy. If S is closed and bounded as well as convex, then player I has a (pure) minimax strategy $\underset{\sim}{z} \in S$.

Exercises.

1. Consider the game with matrix $\underset{\sim}{A}$.

$$\underset{\sim}{A} = \begin{pmatrix} 2 & 4 \\ 6 & -2 \\ 5 & \cdot 1 \\ 4 & 3 \\ 7 & 0 \end{pmatrix}$$

(a) Draw a rough plot of the set S of equation (3).

(b) Note that row 2 is dominated by row 5. How can you tell this fact from the plot?

(c) Note that row 3 is dominated by a mixture of rows 4 and 5. How can you tell this fact from the plot?

(d) Find player I's minimax $\underset{\sim}{z} \in S$. To what mixed strategy over the rows of $\underset{\sim}{A}$ does it correspond?

(e) Find player II's minimax strategy.

2. Let $S = \{\underset{\sim}{z} = (z_1,z_2):(z_1 - 1)^2 + z_2^2 \leq 25\}$ (the circular disc of radius 5 centered at (1,0).) Consider the game $(S,\{1,2\},L)$ where $L(\underset{\sim}{z},j) = z_j$, $j = 1,2$. Find the value, an optimal pure strategy for I and an optimal strategy for II.

5. The principle of equilibrium. For a matrix game with $m \times n$ matrix $\underset{\sim}{A}$ and value V, an optimal strategy $\underset{\sim}{p} = (p_1,\ldots,p_m)$ for I is characterized by the property that

$$(1) \qquad \sum_{i=1}^{m} p_i a_{ij} \geq V \quad \text{for all } j = 1,\ldots,n.$$

Similarly, a strategy $\underset{\sim}{q} = (q_1,\ldots,q_n)$ is optimal for II if and only if

$$(2) \qquad \sum_{j=1}^{n} a_{ij} q_j \leq V \quad \text{for all } i = 1,\ldots,m.$$

When both players use their optimal strategies the average payoff, $\sum\sum p_i a_{ij} q_j$, is exactly V. This may be seen from the inequalities

$$V = \sum_{j=1}^{n} V q_j \leq \sum_{j=1}^{n} (\sum_{i=1}^{m} p_i a_{ij}) q_j = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

$$(3) \qquad = \sum_{i=1}^{m} p_i (\sum_{j=1}^{n} a_{ij} q_j) \leq \sum_{i=1}^{m} p_i V = V.$$

Since this begins and ends with V we must have equality throughout.

The following simple theorem - the principle of equilibrium - gives conditions for equality to be achieved in (1) for certain values of j, and in (2) for certain values of i.

Theorem. Consider a game with $m \times n$ matrix $\underset{\sim}{A}$ and value V. Let $\underset{\sim}{p} = (p_1,\ldots,p_m)$ be any optimal strategy for I and $\underset{\sim}{q} = (q_1,\ldots,q_n)$ be any optimal strategy for II. Then

$$(4) \qquad \sum_{j=1}^{n} a_{ij} q_j = V \quad \text{for all } i \text{ for which } p_i > 0 \text{ and}$$

(5) $\sum_{i=1}^{m} p_i a_{ij} = V$ for all j for which $q_j > 0$.

Proof. Suppose there is a k such that $p_k > 0$ and $\sum_{j=1}^{n} a_{kj} q_j \neq V$. Then from (2) $\sum_{j=1}^{n} a_{kj} q_j < V$. But then from (3) with equality throughout

$$V = \sum_{i=1}^{m} p_i \left(\sum_{j=1}^{n} a_{ij} q_j \right) < \sum_{i=1}^{m} p_i V = V .$$

The inequality is strict since it is strict for the k^{th} term of the sum. This contradiction proves the first conclusion. The second conclusion follows analogously.

Another way of stating the first conclusion of this theorem is: if there exists an optimal strategy for I giving positive probability to row i, then every optimal strategy of II gives I the value of the game if he uses row i.

Although this theorem will not give us a method of solving an arbitrary game, it is quite useful in certain classes of games for helping direct us toward the solution. The procedure this theorem suggests is to try to find a solution to the set of equations (4) formed by those i for which you think it likely that $p_i > 0$. Or try to solve the set of equations (5) formed by those j for which you think it likely that $q_j > 0$.

As an example of this consider the game of odd-or-even in which both players simultaneously call out one of the numbers zero, one, or two. The matrix is

$$\text{Odd} \quad \overset{\text{Even}}{\begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix}}$$

Again it is difficult to guess who has the advantage. If we play the game a few times we might become convinced that Even's optimal strategy gives positive weight (probability) to each of the columns. *If* so, Odd's optimal stragegy $\underset{\sim}{p}$ must satisfy

$$(6)\qquad \begin{aligned} p_2 - 2p_3 &= V \\ p_1 - 2p_2 + 3p_3 &= V \\ -2p_1 + 3p_2 - 4p_3 &= V, \end{aligned}$$

three equations in four unknowns. But don't forget that $\underset{\sim}{p} \in X^*$ implies

$$(7)\qquad p_1 + p_2 + p_3 = 1 .$$

This gives four equations in four unknowns. Hopefully, we can solve this system of equations. First we work with (6); add the first equation to the second,

$$(8)\qquad p_1 - p_2 + p_3 = 2V .$$

Then the second equation to the third.

$$(9)\qquad -p_1 + p_2 - p_3 = 2V.$$

Taken together (8) and (9) imply that $V = 0$. But before we can say anything, we must complete the solution for the p_i to see if $p_i \geq 0$. Adding (7) to (9), we find $2p_2 = 1$, so that $p_2 = 1/2$. The first equation of (6) implies $p_3 = 1/4$ and (7) implies $p_1 = 1/4$. Therefore

$$(10)\qquad \underset{\sim}{p} = (1/4, 1/2, 1/4)$$

is a strategy for I that keeps his average gain to zero no matter what II does. Hence the value of the game is at least zero, and of course, $V = 0$ if our guess that II's optimal strategy gives positive weight to all columns is correct. To complete the solution, we note that if the optimal $\underset{\sim}{p}$ for I gives positive weight to all rows, then II's optimal strategy $\underset{\sim}{q}$ must satisfy the same set of equations (6) and (7) with p replaced by q (because the game matrix here is symmetric). Therefore,

$$\underset{\sim}{q} = (1/4, 1/2, 1/4) \tag{11}$$

is a strategy for II that keeps his average loss to zero no matter what I does. The value of the game is zero and (10) and (11) are optimal for I and II respectively.

<u>Nonsingular game matrices.</u> Let us extend the method used to solve this example. Let the game matrix $\underset{\sim}{A}$ be $m \times m$. <u>Suppose that $\underset{\sim}{A}$ is non-singular and that</u> II <u>has an optimal strategy giving positive weight to each of his columns.</u> Then every optimal strategy $\underset{\sim}{p}$ for I satisfies

$$\sum_{i=1}^{m} p_i a_{ij} = V \qquad \text{for } j = 1, \ldots, m . \tag{12}$$

This may be represented in vector notation as

$$\underset{\sim}{p}\underset{\sim}{A} = V\underset{\sim}{1} \tag{13}$$

where $\underset{\sim}{1} = (1, 1, \ldots, 1)$ represents the m-dimensional row vector of all 1's. We note that V cannot be zero since (13) would imply that $\underset{\sim}{A}$ was singular. Since $\underset{\sim}{A}$ is non-singular, $\underset{\sim}{A}^{-1}$ exists. Multiplying both sides of (13) on the right by $\underset{\sim}{A}^{-1}$ yields

$$\underset{\sim}{p} = V \, \underset{\sim}{1} \, \underset{\sim}{A}^{-1}. \tag{14}$$

If the value of V were known, we would have found the unique optimal strategy for I. To find V, we may use the equation $\sum_1^m p_i = 1$, or in vector notation $\underset{\sim}{p} \, \underset{\sim}{1}^T = 1$. Multiplying both sides of (14) on the right by $\underset{\sim}{1}^T$ yields

$$1 = V \, \underset{\sim}{1} \, \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T \quad \text{or} \quad V = 1/\underset{\sim}{1} \, \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T. \tag{15}$$

The left equation shows that $\underset{\sim}{1} \, \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T$ cannot be zero. The unique optimal strategy for I is therefore

$$\underset{\sim}{p} = \underset{\sim}{1} \, \underset{\sim}{A}^{-1}/\underset{\sim}{1} \, \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T. \tag{16}$$

If now $p_i > 0$ for all i, we can find the optimal strategy for II by the same method. The result would be

$$\underset{\sim}{q}^T = \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T/\underset{\sim}{1} \, \underset{\sim}{A}^{-1} \, \underset{\sim}{1}^T. \tag{17}$$

One might expect this to work for all games with square matrices for which $\underset{\sim}{A}^{-1}$ exists, because the $\underset{\sim}{p}$ of (16) gives $\sum p_i a_{ij} = V$ for all j, and the $\underset{\sim}{q}$ of (17) gives $\sum a_{ij} q_j = V$ for all i where V satisfies (15). The trouble is that either $\underset{\sim}{p}$ or $\underset{\sim}{q}$ or both may contain negative components. If all components of $\underset{\sim}{p}$ and $\underset{\sim}{q}$ of (16) and (17) are non-negative, then $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are optimal and V of (15) is the value of the game.

If the value of a game is zero, this method cannot work directly since (13) implies that A is singular. However, the addition of a positive constant to all entires of the matrix to make the value positive, may change the game matrix into being nonsingular. The previous example is a case in point. The matrix is singular so it would seem that the above method would not work. Yet if 1, say, were added to each entry of the matrix to obtain the matrix A, then A is nonsingular and we may apply the above method. Let us carry through the computations. By some method or another A^{-1} is obtained. Then $1 A^{-1} 1^T$, the sum of the elements of A^{-1}, is found to be 1. Therefore, we

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & -3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{16}\begin{pmatrix} 13 & -2 & -7 \\ -2 & 4 & 6 \\ -7 & 6 & 5 \end{pmatrix}$$

may compute $p = 1 A^{-1} = (1/4, 1/2, 1/4)$, and $q^T = A^{-1} 1^T = (1/4, 1/2, 1/4)^T$. Since both are nonnegative, both are optimal and 1 is the value of the game with matrix A.

It turns out that an arbitrary $m \times n$ matrix game whose value is not zero may be solved by choosing some suitable square submatrix A of order $k \times k$, and applying the above methods and formulae (15), (16), and (17). This is known as the Shapley-Snow Theorem. See Karlin [4] Vol. I, Section 2.4 for a discussion and proof. We will eventually see how to solve games by the simplex method of linear programming, which is an efficient method not only for solving equations of the form (13), but also for finding which square submatrix to choose.

<u>Diagonal games</u>. We apply these ideas to the class of diagonal games - games whose game matrix A is square and diagonal,

(18)
$$A = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & d_m \end{pmatrix}$$

Suppose all diagonal terms are positive, $d_i > 0$. The set of equations (18) becomes

(19) $$p_i d_i = V \qquad i = 1,\ldots,m$$

whose solution is simply

(20) $$p_i = V/d_i \qquad i = 1,\ldots,m .$$

To find V, we sum both sides over i to find

(21) $$1 = V \sum_{i=1}^{m} 1/d_i \quad \text{or} \quad V = \left(\sum_{i=1}^{m} 1/d_i\right)^{-1} .$$

Similarly, the equations for player II yield

(22) $$q_i = V/d_i \qquad i = 1,\ldots,m.$$

Since V is positive from (21), $p_i > 0$ and $q_i > 0$ for all i, so that (20) and (22) give optimal strategies for I and II respectively, and (21) gives the value of the game.

As an example, consider the game with matrix $\underset{\sim}{C}$. From (20) and (22) the optimal strategy is proportional to the reciprocals of the diagonal elements,

$$\underset{\sim}{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

1, 1/2, 1/3, 1/4. The sum of these reciprocals is 1 + 1/2 + 1/3 + 1/4 = 25/12. Therefore, the value is V = 12/25, and the optimal strategies are

$$\underset{\sim}{p} = \underset{\sim}{q} = (12/25,\ 6/25,\ 4/25\ \ 3/25).$$

Triangular games. A class of games for which the equations (12) are easy to solve are the games with triangular matrices - matrices with zeros above or below the main diagonal. Unlike for diagonal games, the method does not always work to solve triangular games because the resulting $\underset{\sim}{p}$ or $\underset{\sim}{q}$ may have negative components. Nevertheless, it works often enough to merit special mention. Consider the game with triangular matrix $\underset{\sim}{T}$. The equations (12) become

$$\underset{\sim}{T} = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} p_1 \qquad\qquad\qquad\qquad &= V \\ -2p_1 + p_2 \qquad\qquad\quad &= V \\ 3p_1 - 2p_2 + p_3 \qquad &= V \\ -4p_1 + 3p_2 - 2p_3 + p_4 &= V \end{aligned}$$

These equations may be solved one at a time from the top down to give

$$p_1 = V \qquad p_2 = 3V \qquad p_3 = 4V \qquad p_4 = 4V .$$

Since $\sum p_i = 1$, we find $V = 1/12$ and $p = (1/12, 1/4, 1/3, 1/3)$. The equations for the q's are

$$\left.\begin{aligned} q_1 - 2q_2 + 3q_3 - 4q_4 &= V \\ q_2 - 2q_3 + 3q_4 &= V \\ q_3 - 2q_4 &= V \\ q_4 &= V \end{aligned}\right\} \qquad \left\{\begin{aligned} q_1 &= 4V \\ q_2 &= 4V \\ q_3 &= 3V \\ q_4 &= V \end{aligned}\right.$$

Since the p's and q's are non-negative, $V = 1/12$ is the value $\underset{\sim}{p} = (1/12, 1/4, 1/3, 1/3)$ is optimal for I, and $\underset{\sim}{q} = (1/3, 1/3, 1/3, 1/12)$ is optimal for II.

Exercises.

1. Consider the game with matrix $\begin{pmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}$.

   (a) Note that this game has a saddle point.

   (b) Show that the inverse of the matrix exists.

   (c) Show that II has an optimal strategy giving positive weight to each of his columns.

   (d) Why then, don't equations (17) give an optimal strategy for II?

2. Consider the diagonal matrix game with matrix (13).

   (a) Suppose one of the diagonal terms is zero. What is the value of the game?

   (b) Suppose one of the diagonal terms is positive and another is negative. What is the value of the game?

   (c) Suppose all diagonal terms are negative. What is the value of the game?

3. Player II chooses a number $j \in \{1,2,3,4\}$, and player I tries to guess what number II has chosen. If he guesses correctly and the number was j, he wins 2^j dollars from II. Otherwise there is no payoff. Set up the matrix of this game and solve.

4. II chooses a number $j \in \{1,2,3,4\}$. I tries to guess what it is. If he guesses correctly, he wins 1 from II. If he overestimates he wins 1/2 from II. If he underestimates, there is no payoff. Set up the matrix of this game and solve.

5. I chooses a number $i \in \{1,2,\ldots,n\}$. II tries to guess what it is. If he guesses correctly, there is no payoff. If he guesses too low, he loses 1 to I. If he guesses too high, he loses 1/2 to I. Solve.

6. Solve the games with the following matrices.

(a) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3/2 & 1 & 1 \\ 1 & 1 & 4/3 & 1 \\ 1 & 1 & 1 & 5/4 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix}$.

6. Symmetric games. A game is symmetric if the rules do not distinguish between the players. For symmetric games, both players should have the same options (the game matrix should be square), and the payoff if I uses i and II uses j should be the negative of the payoff if I uses j and II uses i. (The game matrix should be skew-symmetric $\underset{\sim}{A}^T = -\underset{\sim}{A}$, or $a_{ij} = -a_{ji}$.)

Definition. A finite game is said to be symmetric if its game matrix is square and skew-symmetric.

Strictly speaking, we should say the game is symmetric if after some rearrangement of the rows and columns the game matrix is skew-symmetric.

The game of paper-scissors-rock is an example. Players I and II simultaneously display one of the three objects: paper, scissors, or rock. If they both choose the same object to display, there is no payoff. If they choose different objects, then scissors win over paper (scissors cut paper), rock wins over scissors (rock breaks scissors), and paper wins over rock (paper covers rock). If the payoff upon winning or losing is one unit, then the matrix of the game is as follows.

II

			paper	scissors	rock
(1)	I	paper	0	-1	1
		scissors	1	0	-1
		rock	-1	1	0

This matrix is skew-symmetric so the game is symmetric. The diagonal elements of the matrix are zero. This is true of any skew-symmetric matrix, since $a_{ii} = -a_{ii}$ implies $a_{ii} = 0$.

A contrasting example is the game of matching pennies. The two players simultaneously choose to show a penny with either the heads or the tails side facing up. One of the players, say player I, wins if the choices match. The other player, player II, wins if the choices differ. Although there is a great deal of symmetry in this game, we do not call it a symmetric game. Its matrix is

$$\text{(2)} \qquad \text{I} \quad \begin{array}{c} \\ \text{heads} \\ \text{tails} \end{array} \begin{array}{c} \text{II} \\ \begin{array}{cc} \text{heads} & \text{tails} \end{array} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

Example of Latin Square Game. (see page 45)

This matrix is not skew-symmetric.

We expect a symmetric game to be fair $(V = 0)$. That is indeed the case.

Theorem. A finite symmetric game has value zero. Any strategy optimal for one player is optimal for the other also.

Proof. Let p be any strategy for I. If II uses the same strategy the average payoff is zero:

$$(3) \qquad \underset{\sim}{p}\, \underset{\sim}{A}\, \underset{\sim}{p}^T = -\underset{\sim}{p}\, \underset{\sim}{A}^T \underset{\sim}{p}^T = -(\underset{\sim}{p}\, \underset{\sim}{A}^T \underset{\sim}{p}^T)^T = -\underset{\sim}{p}\, \underset{\sim}{A}\, \underset{\sim}{p}^T = 0.$$

It follows that $\min_{q} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T \le 0$ for all $\underset{\sim}{p}$ so that $\underline{v} \le 0$. Similarly, $\max_{p} \underset{\sim}{p} \underset{\sim}{A} \underset{\sim}{q}^T \ge 0$ for all $\underset{\sim}{q}$ so that $\overline{v} \ge 0$. Since the value of the game exists, it must be equal to zero. Now suppose $\underset{\sim}{p}$ is optimal for I. Then $\sum_{i=1}^{m} p_i a_{ij} \ge 0$ for all j. Hence $\sum_{j=1}^{m} a_{ij} p_j = -\sum_{j=1}^{m} p_j a_{ji} \le 0$ for all i, so that $\underset{\sim}{p}$ is also optimal for II. By symmetry, if $\underset{\sim}{q}$ is optimal for II, it is optimal for I also, and the proof is complete.

Example. Two players simultaneously choose an integer between 1 and n inclusive, $(n \ge 3)$. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1. The player that chooses a number two or more larger than his opponent loses 2. The payoff matrix is

$$(4) \qquad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \end{array} \\ \left(\begin{array}{cccccc} 0 & -1 & 2 & 2 & 2 & \cdots \\ 1 & 0 & -1 & 2 & 2 & \cdots \\ -2 & 1 & 0 & -1 & 2 & \cdots \\ -2 & -2 & 1 & 0 & -1 & \cdots \\ -2 & -2 & -2 & 1 & 0 & \cdots \\ \vdots & & & & & \ddots \end{array} \right) \end{array}$$

The game is symmetric so the value is zero and the players have identical optimal strategies. We see that 1 dominates 4,5,6,..., so we may restrict attention to the upper left 3×3 submatrix. We suspect that there is an optimal strategy for I with $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$. If so, it would follow from the principle of equilibrium (since $q_1 = p_1 > 0$, $q_2 = p_2 > 0$, $q_3 = p_3 > 0$ is optimal for II) that

$$(5)\qquad \begin{aligned} p_2 - 2p_3 &= 0 \\ -p_1 + p_3 &= 0 \\ 2p_1 - p_2 &= 0 \end{aligned}$$

We find $p_2 = 2p_3$ and $p_1 = p_3$ from the first two equations, and the third equation is redundant. Since $p_1 + p_2 + p_3 = 1$, we have $4p_3 = 1$; so $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 1/4$. Since p_1, p_2 and p_3 are positive, this gives the solution: $\underset{\sim}{p} = \underset{\sim}{q} = (1/4, 1/2, 1/4, 0, 0,\ldots)$ is optimal for both players.

<u>Latin square games</u>. A Latin square is an n by n array of n different letters such that each letter occurs once and only once in each row and each column. The 5×5 array at the right is an example. If in a Latin square each letter is assigned a numerical value, the resulting matrix is the matrix of a Latin square game. Such games have simple solutions. The value is the average of the numbers in a row, and the strategy that chooses each pure strategy with equal probability $1/n$ is optimal for both players. The reason is not very deep. The conditions for optimality are satisfied.

$$\begin{pmatrix} a & b & c & d & e \\ b & e & a & c & d \\ c & a & d & e & b \\ d & c & e & b & a \\ e & d & b & a & c \end{pmatrix}$$

$$a = 1 \quad b = 2$$
$$c = d = 3 \quad e = 6$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 6 \\ 2 & 6 & 1 & 3 & 3 \\ 3 & 1 & 3 & 6 & 2 \\ 3 & 3 & 6 & 2 & 1 \\ 6 & 3 & 2 & 1 & 3 \end{pmatrix}$$

In the example above, the value is $(1 + 2 + 3 + 3 + 6)/5 = 3$, and $(1/5,1/5,1/5,1/5,1/5)$ is optimal for both players. The game of matching pennies is a Latin square game. Its value is zero and $(1/2,1/2)$ is optimal for both players.

<u>Exercises</u>

1. Two players simultaneously choose an integer between 1 and n inclusive, $(n \geq 5)$. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 2. The player that chooses a number two or more larger than that chosen by his opponent loses 1. (a) Set up the game matrix. (b) It turns out that the optimal strategy satisfies $p_1 > 0, \ldots, p_5 > 0, p_6 = 0, \ldots, p_n =$

Solve for the optimal $\underset{\sim}{p}$. (It is not too difficult since you can argue $p_1 = p_5$ and $p_2 = p_4$ by symmetry.) Check that in fact the strategy you find is optimal.

2. Solve.

(a) $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & -2 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 & -1 & 5 \\ 4 & -1 & 5 & 1 \\ -1 & 5 & 1 & 4 \\ 5 & 1 & 4 & -1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$

3. Magic square games. A magic square is an $n \times n$ array of the first n^2 integers with the property that all row and column sums are equal. Show how to solve all games with magic square game matrices. Solve the example.

$$\begin{pmatrix} 13 & 2 & 7 & 12 \\ 11 & 8 & 1 & 14 \\ 4 & 15 & 10 & 5 \\ 6 & 9 & 16 & 3 \end{pmatrix}$$

7. The extensive form of a game. The normal form of a game is a compact way of describing a game and it is very convenient mathematically. However, the flavor of many games is lost in such a simple model. The notions of move and position, of bluffing and signaling and so on are not apparent in the normal form of a game. There is another mathematical model of a game that is built on the basic notions of position and move. This model we call the extensive form of a game. Three new concepts make their appearance in the extensive form of a game: The game tree, chance moves, and information sets.

The game tree. Many games can be modeled as a directed graph in which vertices represent positions and edges represent moves. A directed graph is a set T of points called vertices together with a set E of ordered pairs of points of T called edges. Thus (T,E) is a directed graph if $T = \{a,b,c,d,e\}$ and $E = \{(a,b),(b,c),(a,c),(c,b),(c,d),(e,e)\}$ (Figure 1). A path beginning at a vertex $t_1 \in T$ and ending at a vertex $t_2 \in T$ is a sequence of edges $e_1,\ldots,e_n \in E$ such that t_1 is the first component of e_1, t_2 is the second component of e_n, and for $j = 1,\ldots,n-1$, the second component of e_j is the same as the first component of e_{j+1}. An initial vertex is a vertex that does not appear as the second component of any edge. A terminal vertex is a vertex that does not appear as the first component of any edge. In Figure 1, a is the only initial vertex and d the only terminal vertex.

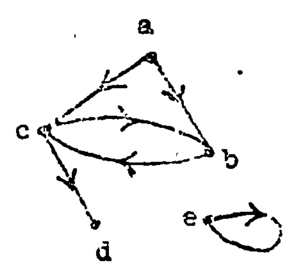


Figure 1.

In game theory we deal with a particular type of directed graph called a tree. A tree is a directed graph in which there is an initial vertex t_0 such that for every vertex $t \in T$, there is a unique path beginning at t_0 and ending at t.

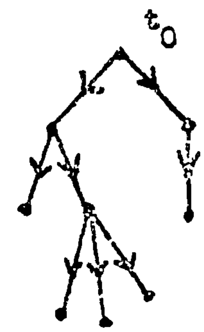


Figure 2.

The existence and uniqueness of the path implies that a tree is connected, has a unique initial vertex, and has no loops

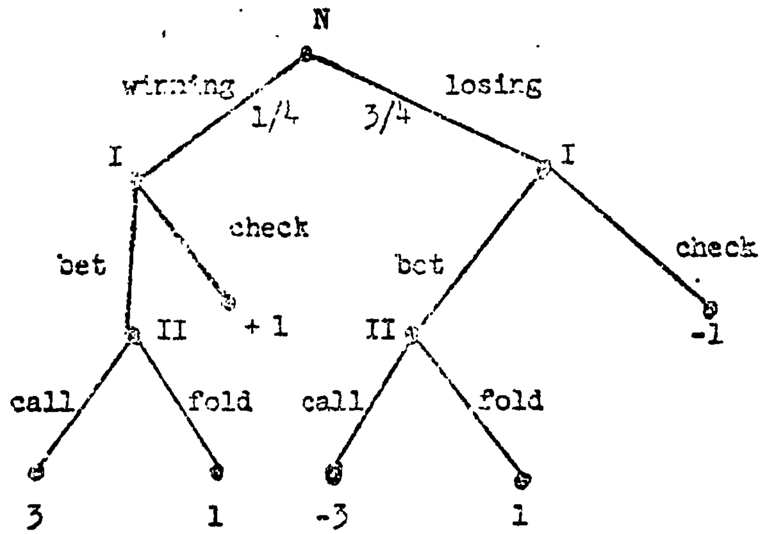
Figure 2 shows a tree. In a game on such a graph, each vertex (position) is assigned to one of the players who is to choose the next edge (move) from that position. Some vertices, however, will be singled out as positions from which a chance move is made.

Chance moves. Many games involve chance moves. The rolling of dice, the dealing of cards, and the spinning of the wheel of fortune are all examples of chance moves occurring in games. Poker and bridge are typical of games in which chance moves play an important role. Even in chess, there is generally a chance move to determine which player gets the white pieces (and therefore the first move). It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move.

Information. Another important aspect we must consider in studying the extensive form of games is the amount of information available to the players about past moves of the game. In poker for example, the first move is the chance move of shuffling and dealing the cards; each player is aware of certain aspects of the outcome of this move (the cards he received) but he is not informed of the complete outcome (the cards received by the other players). This leads to the possibility of "bluffing." The following example involving chance moves and information is a model of a situation that sometimes occurs in the game of stud poker.

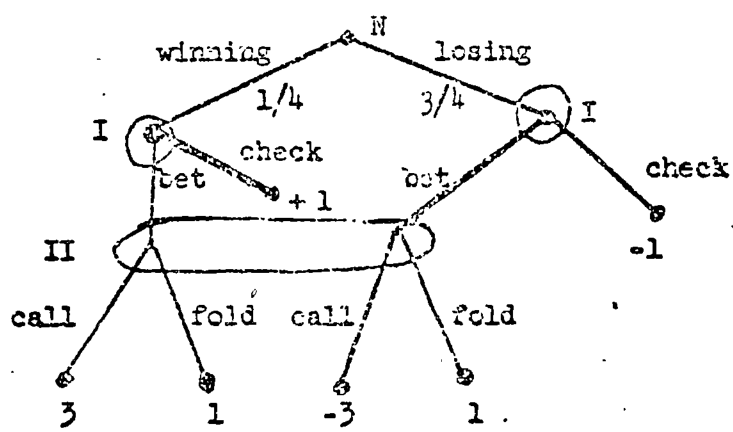
The game of bluffing. Player I is dealt a card from a deck. It is a winning card with probability $1/4$ and a losing card with probability $3/4$. Player I may then check or bet. If he checks, then he wins 1 dollar (the ante) from II if he has a winning card, and he loses 1 dollar to II otherwise. If I bets, player II - not knowing what card player I has - may fold or call. If II folds, he loses 1 dollar to I no matter what card I has. If II calls, I wins 3 dollars (the ante plus the bet) from II if he has a winning card, and loses 3 dollars to II otherwise.

Let us draw the tree for the game of bluffing. There are at most three moves in this game: (1) the chance move that chooses a card for I, (2) I's move in which he checks or bets, and (3) II's move in which he folds or calls. To each vertex of the game tree, we attach a label indicating which player is to move from that position. Chance moves we generally refer to as moves by nature and use the label N. The tree becomes



Each edge is labelled to identify the move. (The arrows are omitted for the sake of clarity. Moves are assumed to proceed down the page.) Also, the moves leading from a vertex at which nature moves are labelled with the probabilities with which they occur. At each terminal vertex, we write the numerical value of I's winnings (II's losses).

From the tree we should be able to reconstruct all the essential rules of the game. That is not the case with the tree given above since we have not indicated that at the time II makes his decision he does not know which card I has received. That is, when it is II's turn to move, he does not know at which of his two possible positions he is. We indicate this on the diagram by encircling the two positions in a closed curve, and we say that these two vertices constitute <u>an information set</u>. The two vertices at which I is to move constitute two separate information sets since he is told the outcome of the chance move. This must also be indicated on the diagram by drawing small circles about these vertices. We may delete one of the labels indicating II's vertices since they belong to the same information set. It is really the information set that must be labeled. The completed game tree becomes



The diagram now contains all the essential rules of the game. It is assumed that both players know the rules of the game. That is, both players are assumed to know the game tree. Games in which one (or both) of the players does not know some of the payoffs, or some of the probabilities of chance moves, or some of the information sets, or even whole branches of the tree, are called games with incomplete information, or pseudogames.

Not every set of vertices can form an information set. In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it. Furthermore, it is important that the edges from each vertex of an information set have the same set of labels. The player moving from such an information set really chooses a label. It is presumed that a player makes just one choice from each information set.

We summarize these ideas formally.

Definition. A finite two-person zero-sum game in extensive form is given by

1) a finite tree with vertices T and edges E,
2) a payoff function that assigns a real number to each terminal vertex,
3) a set T_0 of vertices (representing positions at which chance moves occur) and for each $t \varepsilon T_0$ a probability distributions on the edges leading from t,
4) a partition of the rest of the vertices (not terminal and not in T_0) into two groups of information sets $T_{11}, T_{12}, \ldots, T_{1k_1}$ (for player I) and $T_{21}, T_{22}, \ldots, T_{2k_2}$ (for player II), and
5) for each information set T_{jk}, a set of labels L_{jk}, and for each $t \varepsilon T_{jk}$ a one-to-one mapping of L_{jk} onto the set of edges leading from t.

The information structure in a game in extensive form can be quite complex. It may involve lack of knowledge of the other player's moves or of some of the chance moves. It may indicate a lack of knowledge of how many moves have already been made in the game (as is the case with player II in Figure 3). It may describe situations in which one player has forgotten a move he has made earlier (as is the case with player I in Figure 4).

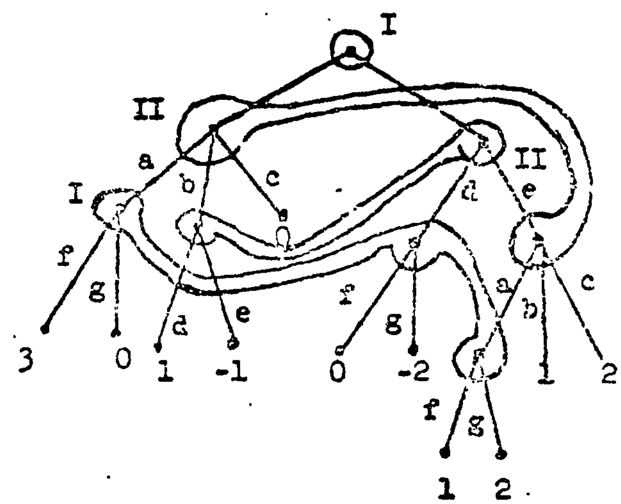


Figure 3.

In fact, one way to try to model the game of bridge as a two-person zero-sum game involves the use of this idea. In bridge, there are four individuals forming two teams or partnerships of two players each. The interests of the members of a partnership are identical, so it makes sense to describe this as a two-person game. But the members of one partnership make bids alternately based on cards that one member knows and the other does not. This may be described as a single player who alternately remembers and forgets the outcomes of some of the previous random moves.

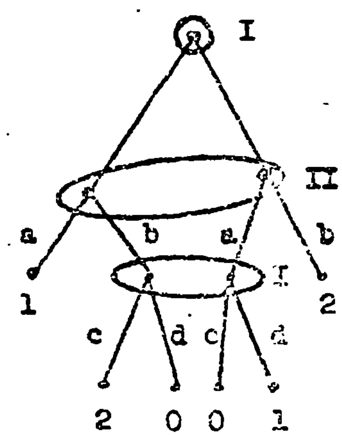


Figure 4.

A kind of degenerate situation exists when an information set contains two vertices which are joined by a path, as is the case with I's information set in Figure 5. We take it as a convention that a player makes one choice from each information set during a game. That choice is used no matter how many times the information set is reached. In Figure 5, if I chooses a there is no problem. If I chooses b, then in the lower of I's two vertices the a is superfluous, and the tree may really be reduced to Figure 6. Instead of using the above convention, we may if we like assume in the definition of a game in extensive form that no information set contains two vertices joined by a path.

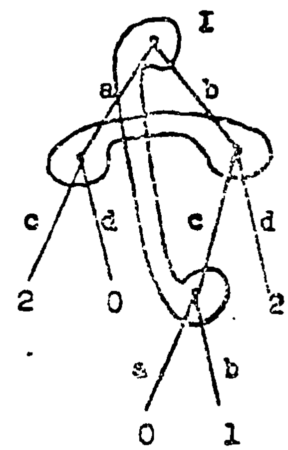


Figure 5.

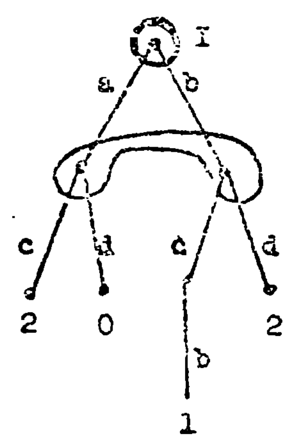


Figure 6.

Exercises.

1. Player II chooses one of two boxes in which to hide an object. Then, player I, not knowing which box contains the object, chooses one of the boxes to search. If the object is in box # 1 and I searches there, then (by a chance move) with probability 1/2 he finds the object, and with probability 1/2 he is given no information. If the object is in box # 2 and I searches there, then with probability 1/3 he finds the object

and with probability 2/3 he is given no information. Also, if he searches the wrong box, he is given no information. I wins one from II if he finds the object; otherwise there is no payoff. Draw the game tree.

2. Draw the game tree for problem 1, if when I is unsuccessful in his attempt to find the object, he is given a second chance to search for the object with the same probabilities of success. (Player II does not get to hide the object again.)

3. A statistical game. Player I has two coins. One is fair (probability 1/2 of heads and 1/2 of tails) and the other is biased with probability 1/3 of heads and 2/3 of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to II. Then II must guess whether I chose the fair or baised coin. If II is correct there is no payoff. If II is incorrect, he loses 1. Draw the game tree.

4. A fair coin (probability 1/2 of heads and 1/2 of tails) is tossed and the outcome is shown to player I. On the basis of the outcome of this toss, I decides whether to bet 1 or 2. Then player II hearing the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Finally, player I (or, more realistically, his partner), remembering the amount bet and II's guess, but not remembering the outcome of the toss, may double or pass. II wins if his guess is correct and loses if his guess is incorrect. The

absolute value of the amount won is [the amount bet (+1 if the coin comes up heads)] (x 2 if I doubled). Draw the game tree.

5. The Kuhn poker model. Two players are each dealt one card at random from a deck of three cards {1,2,3}. (There are six possible equally likely outcomes of this chance move.) Then player I checks or bets. If I bets, II may call or fold. If I checks, II may check or bet. If I checks and II bets, then I may call or fold. If both players check, the player with the higher card wins one. If one player bets and the other folds, the player who bet wins 1. If one player bets and the other calls, the player with the higher card wins 2. Draw the game tree. (H. W. Kuhn, "A simplified two-person poker" Contributions to the Theory of Games, vol. I, pg. 97, 1950, Ed. Kuhn and Tucker, Princeton University Press.)

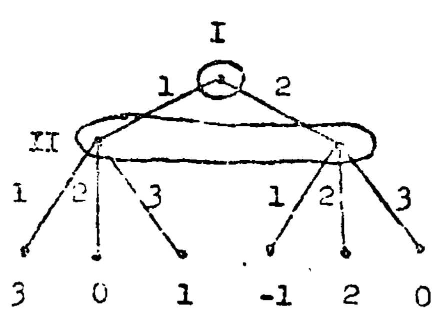
8. Relation between the normal and extensive forms of a game.

The notion of a game in normal form is quite simple. It is described by a triplet (X,Y,L) as in Section 1. The extensive form of a game on the other hand is quite complex. It is described by the game tree with non-terminal vertices labeled as a chance move or as a move of one of the players, with all information sets specified, with probability distributions given for all chance moves, and with a payoff attached to each terminal vertex. It would seem that the theory of games in extensive is much more comprehensive than the theory of games in normal form. This is not the case.

First, let us check that a game in normal form can be put into extensive form. In the normal form of a game, the players are considered to make their choices simultaneously, while in the extensive form of a game simultaneous moves are not allowed. However, simultaneous moves may be made sequentially as follows. We let one player, say player I, move first, and then let player II move without knowing the outcome of I's move. This lack of knowledge may be described by the use of an appropriate information set. The example below illustrates this.

$$\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

matrix game



equivalent extensive form

To go in the reverse direction from the extensive form of a game to the normal form requires consideration of the notion of a pure strategy and a convention regarding random payoffs.

Pure strategies. Given a game in extensive form, we first find X and Y, the sets of pure strategies of the players to be used in the normal form. A pure strategy for player I is a rule that tells him exactly what move to make in each of his information sets. Let $T_{11}, \ldots, T_{1k_1}$ be the information sets for player I and let $L_{11}, \ldots, L_{1k_1}$ be the corresponding sets of labels. A pure strategy for I is a k_1-tuple $\underset{\sim}{x} = (x_1, \ldots, x_{k_1})$ where for each i, x_i is one of the elements of L_{1i}. If there are m_i elements in L_{1i}, the number of such k_1-tuples and hence the number of I's pure strategies is $m_1 \cdot m_2 \cdot \ldots \cdot m_{k_1}$. The set of all such strategies is X. Similarly, if $T_{21}, \ldots, T_{2k_2}$ represent II's information sets and $L_{21}, \ldots, L_{2k_2}$ the corresponding sets of labels, a pure strategy for II is a k_2-tuple $\underset{\sim}{y} = (y_1, \ldots, y_{k_2})$ where $y_j \in L_{2j}$ for each j. Player II has $n_1 \cdot n_2 \cdot \ldots \cdot n_{k_2}$ pure strategies if there are n_j elements in L_{2j}. Y denotes the set of these strategies.

Random payoffs. A referee, given $\underset{\sim}{x} \in X$ and $\underset{\sim}{y} \in Y$, could play the game, playing the appropriate move from $\underset{\sim}{x}$ whenever the game enters one of I's information sets, playing the appropriate move from $\underset{\sim}{y}$ whenever the game enters one of II's information sets, and playing the moves at random with the indicated probabilities at each chance move. The actual outcome of the game for given $\underset{\sim}{x} \in X$ and $\underset{\sim}{y} \in Y$ depends on the chance moves selected, and is therefore a random quantity. Strictly speaking, random payoffs were not provided for in our definition of games in normal form. However, we are quite used to replacing random payoffs by their average values (expected values) when the randomness is due to the use of mixed strategies by the players. We adopt the same convention in dealing with random payoffs when the randomness is due to the chance moves.

<u>Convention. If for fixed pure strategies of the players, $\underset{\sim}{x} \in X$ and $\underset{\sim}{y} \in Y$, the payoff is a random quantity, we replace the payoff by the average value, and denote this average value by</u> $L(\underset{\sim}{x},\underset{\sim}{y})$.

For example, if for given strategies $\underset{\sim}{x} \in X$ and $\underset{\sim}{y} \in Y$, player I wins 3 with probability 1/4, wins 1 with probability 1/4, and loses 1 with probability 1/2, then his average payoff is $\frac{1}{4}(3) + \frac{1}{4}(1) + \frac{1}{2}(-1) = \frac{1}{2}$, so we let $L(\underset{\sim}{x},\underset{\sim}{y}) = 1/2$.

Therefore, given a game in extensive form, we say (X,Y,L) is the equivalent normal form of the game if X and Y are the pure strategy spaces of players I and II respectively, and if $L(\underset{\sim}{x},\underset{\sim}{y})$ is the average payoff for $\underset{\sim}{x} \in X$ and $\underset{\sim}{y} \in Y$.

Let us find the equivalent normal form to the game of bluffing described in the previous section, whose tree is given on page 51. Player I has two information sets; in each set he must make a choice from among two options. He therefore has 2 · 2 = 4 pure strategies. We may denote them by (b,b): bet with a winning card or a losing card.

(b,c): bet with a winning card, check with a losing card.

(c,b): check with a winning card, bet with a losing card.

(c,c): check with a winning card or a losing card.

Therefore, $X = \{(b,b),(b,c),(c,b),(c,c)\}$. We include in X all pure strategies whether good or bad (in particular, (c,b) seems a rather perverse sort of strategy.)

Player II has only one information set. Therefore, $Y = \{c,f\}$ where

c: if I bets, call.

f: if I bets, fold.

Suppose I uses (b,b) and II uses c. Then if I gets a winning card (which happens with probability 1/4), he bets, II calls, and I wins 3 dollars. But if I gets a losing card (which happens with probability 3/4), he bets, II calls, and I loses 3 dollars. I's average or expected winnings is

$$E((b,b),c) = \frac{1}{4}(3) + \frac{3}{4}(-3) = -\frac{3}{2}$$

This gives the upper left entry in the following matrix. The other entries may be computed similarly and are left as exercises.

Let us solve this game by the methods of section 2. We third row is dominated by the first row, and the fourth row is dominated by the second row. In terms of the original form of the game, this says something you may already have suspected: that if

$$\begin{array}{c} \\ (b,b) \\ (b,c) \\ (c,b) \\ (c,c) \end{array} \begin{array}{c} \begin{array}{cc} c & f \end{array} \\ \begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \\ -2 & 1 \\ -1/2 & -1/2 \end{pmatrix} \end{array}$$

I gets a winning card, it cannot be good for him to check - by betting he will win at least as much, and maybe more. With the bottom two rows eliminated the matrix becomes $\begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \end{pmatrix}$ whose solution is easily found. The value is $V = -1/4$. I's optimal strategy is to mix (b,b) and (b,c) with probabilities 1/6 and 5/6 respectively, while II's optimal strategy is to mix c and f with equal probabilities 1/2 each. The strategy (b,b) is player I's bluffing strategy. Its use entails betting with a losing hand. The strategy (b,c) is player I's "honest" strategy; bet with a winning hand and check with a losing hand. I's optimal strategy requires some bluffing and some honesty.

In Exercise 4 of the previous section, there are six information sets for I each with two choices. The number of I's pure strategies is therefore $2^6 = 64$. II has 2 information sets each with two choices. Therefore, II has $2^2 = 4$ pure strategies. The game matrix for the equivalent normal form has dimension 64×4. Dominance can help reduce the dimension to a manageable size.

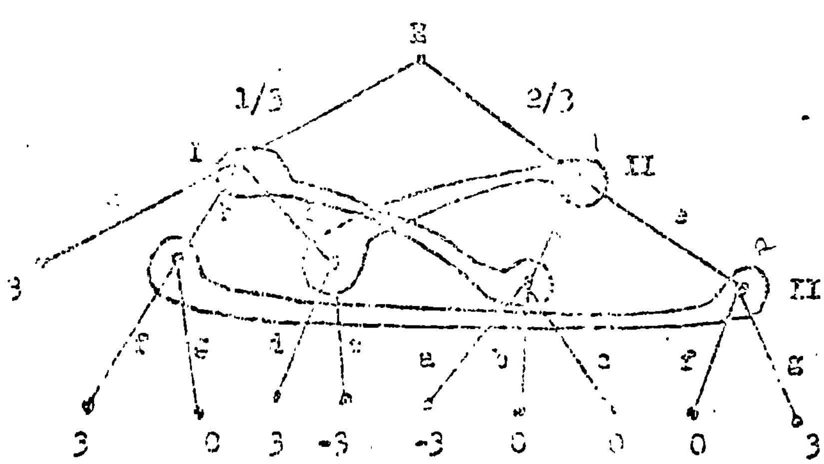
<u>Games of perfect information</u>. If each information set of each player consists of a single vertex, we say the game has <u>perfect information</u>.

In a game of perfect information, the players know all the past moves of the game even the chance ones. Examples include tic-tac-toe, chess, craps, monopoly, etc.

Games of perfect information have a particularly simple mathematical structure. One can show that every game of perfect information has a saddle point, and that the saddle point can be found by removing dominated rows and columns. See McKinsey [3] or Blackwell and Girshick [8] for a proof. This has an interesting implication for the game of chess for example. Since there are no chance moves, every entry of the game matrix for chess must be either +1 (a win for player I), or -1 (a win for player II), or 0 (a draw). A saddle point must be one of these numbers. Thus, either player I can guarantee himself a win, or player II can guarantee himself a win, or both players can assure themselves at least a draw. From the game-theoretic viewpoint, chess is a very simple game. Of course, the real game of chess is so complicated, there is virtually no hope of ever finding an optimal strategy. In fact, it is not yet understood how humans can play the game so well.

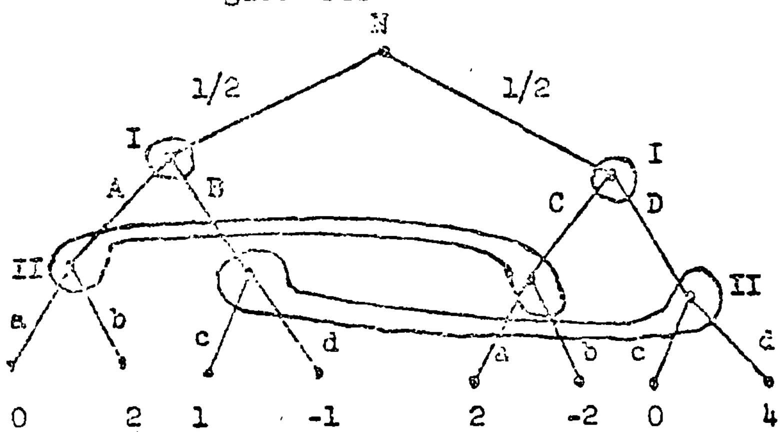
Exercises

1. (a) Find the equivalent normal form of the game with the game tree.



(b) Solve the game.

2. (a). Find the equivalent normal form of the game with the game tree:



(b). Solve the game.

3. Coin A has probability 1/2 or heads and 1/2 of tails. Coin B has probability 1/3 of heads and 2/3 of tails. Player I must predict "heads" or "tails." If he predicts heads coin A is tossed. If he predicts tails, coin B is tossed. Player II is informed as to whether I's prediction was right or wrong (but he is not informed of the prediction or the coin that was used), and then must guess whether coin A or coin B was used. If II guesses correctly he wins 1 dollar from I. If II guesses incorrectly and I's prediction was right, I wins 2 dollars from II. If both are wrong there is no payoff.

(a) Draw the game tree.

(b) Find the equivalent normal form of the game.

(c) Solve.

4. Find the equivalent normal forms of the games of the following exercises of the previous section and solve.

(a) Exercise 1.

(b) Exercise 2.

(c) Exercise 3.

(d) Exercise 4.

EVALUACION DE PROYECTOS Y TOMA DE

DECISIONES

DECISION ANALISIS

NOVIEMBRE, 1978.

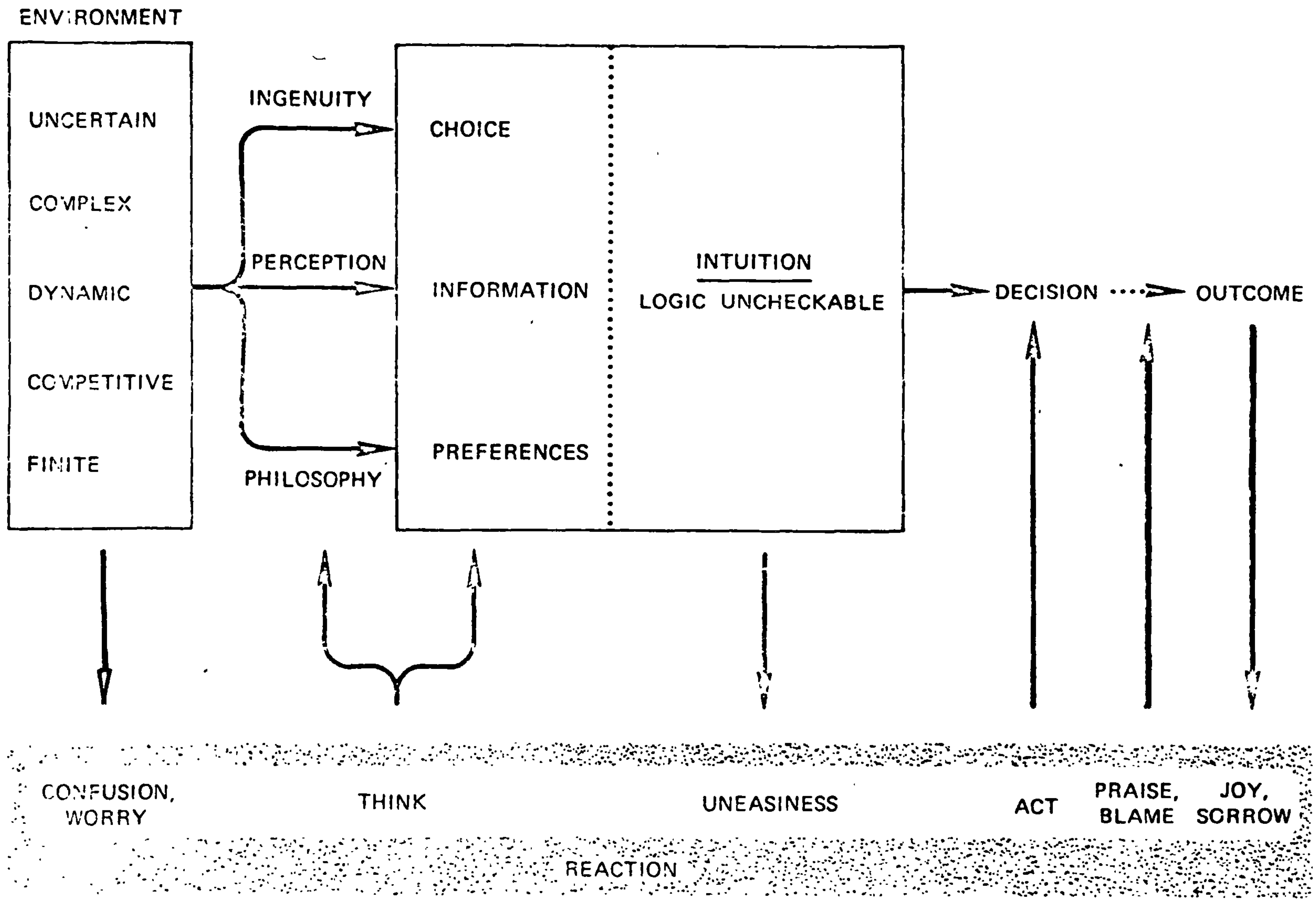
EES 231

DECISION ANALYSIS

Professor Ronald A. Howard

Department of Engineering-Economic Systems
Stanford University

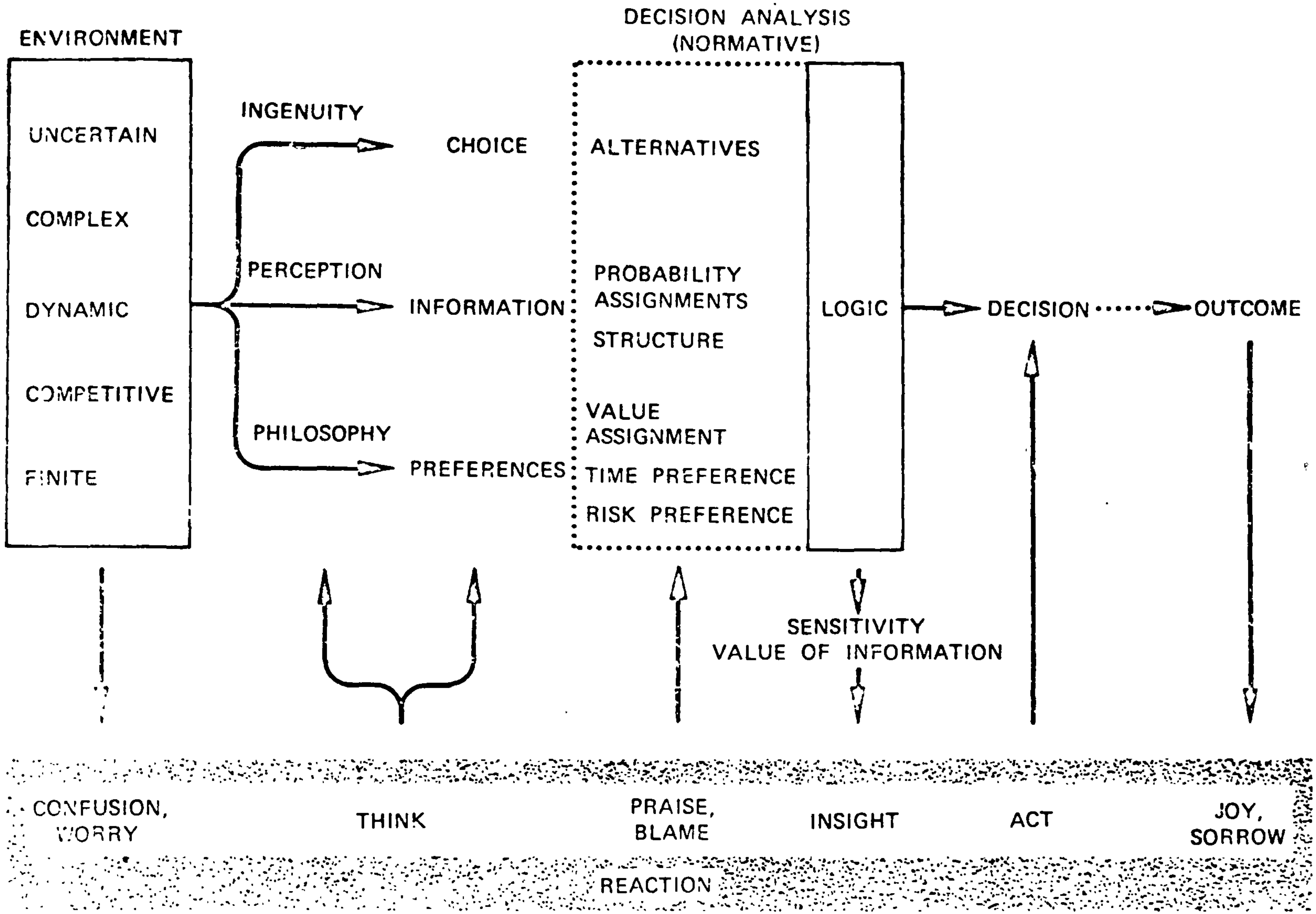
DECISION MAKING (DESCRIPTIVE)



-1-

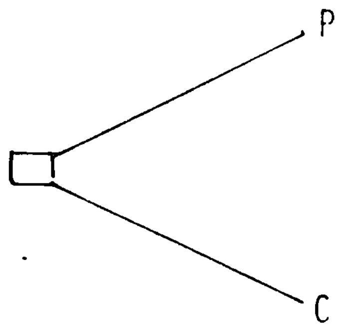
EES 231

DECISION MAKING



PREFERENCE ASSESSMENT

VALUE

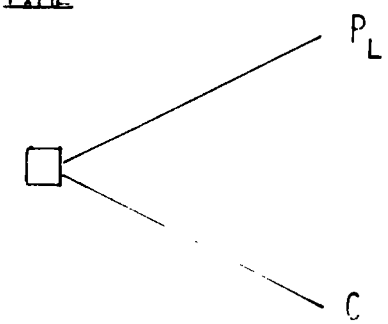


P: ONE DAY IN HOSPITAL WITH SEVERE PAIN, THEN CURE

SEVERE PAIN = PULLING WISDOM TOOTH WITHOUT ANESTHETIC

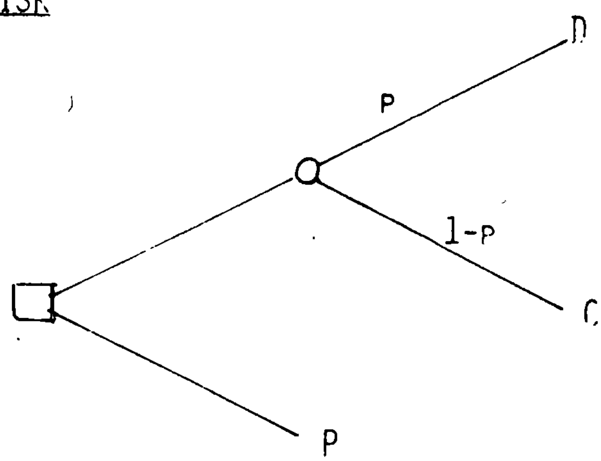
C: INSTANT CURE

TIME



P_L: P ONCE A YEAR FOR LIFE

RISK



D: DEATH

JOINT TIME-RISK WHEN $P \Rightarrow P_L$

TOWARD A THEORY OF DECISION

THE DEFINITION OF A DECISION

DECISIONS ≠ WORRIES

ELEMENTS OF DECISION MAKING

- UNCERTAINTY
  - THE JOURNEY
  - AXIOMS — PROBABILITY THEORY
  - PROBABILITY = STATE OF MIND, NOT OF THINGS
    - THE ASTRONAUT
  - ENCODING OF EXPERIENCE
- VALUES
  - PREFERENCES FOR OUTCOMES
  - ECONOMIC
  - LIFE AND LIMB
- CRITERIA
  - TIME PREFERENCE
    - GREED - IMPATIENCE
  - RISK TOLERANCE
    - COMPARISON OF LOTTERIES
    - BETTING ON SALARY
- STRUCTURE
  - CAPTURING PROBLEM RELATIONSHIPS
  - THE EMBODIMENT OF LOGIC

CAVEATS

I. GOOD DECISIONS ≠ GOOD OUTCOMES

(LOGICAL) (DESIRABLE)

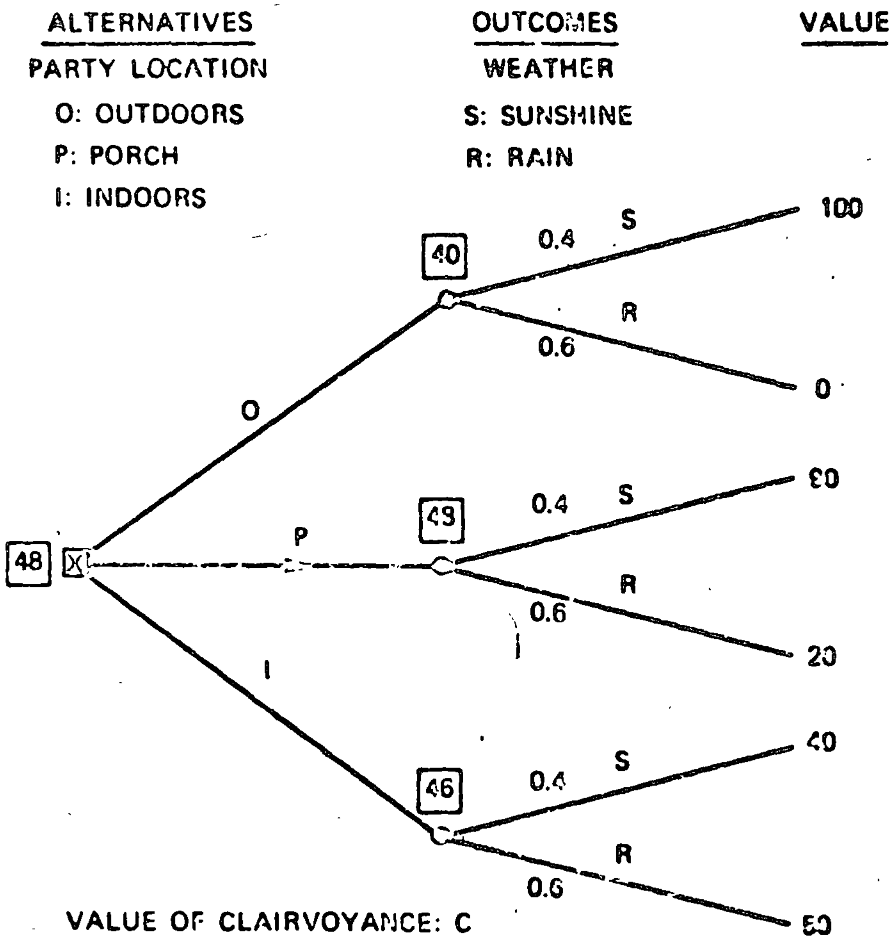
- THE LOTTERY TICKET
- ORGANIZATIONAL REWARD
- THE CAPTAIN AND THE ADMIRAL
- THE C.P.D.A.

II. THEORY IS NORMATIVE, PRESCRIPTIVE
<u>NOT</u> DESCRIPTIVE

INDIVIDUAL FREEDOM AND

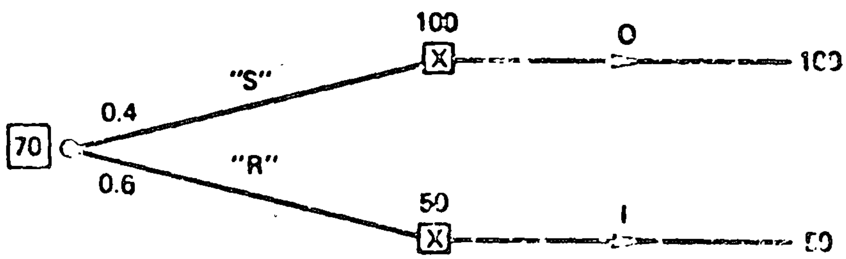
MY GAMBLING BANKER

A PARTY PROBLEM



WIZARD

VALUE OF CLAIRVOYANCE: C

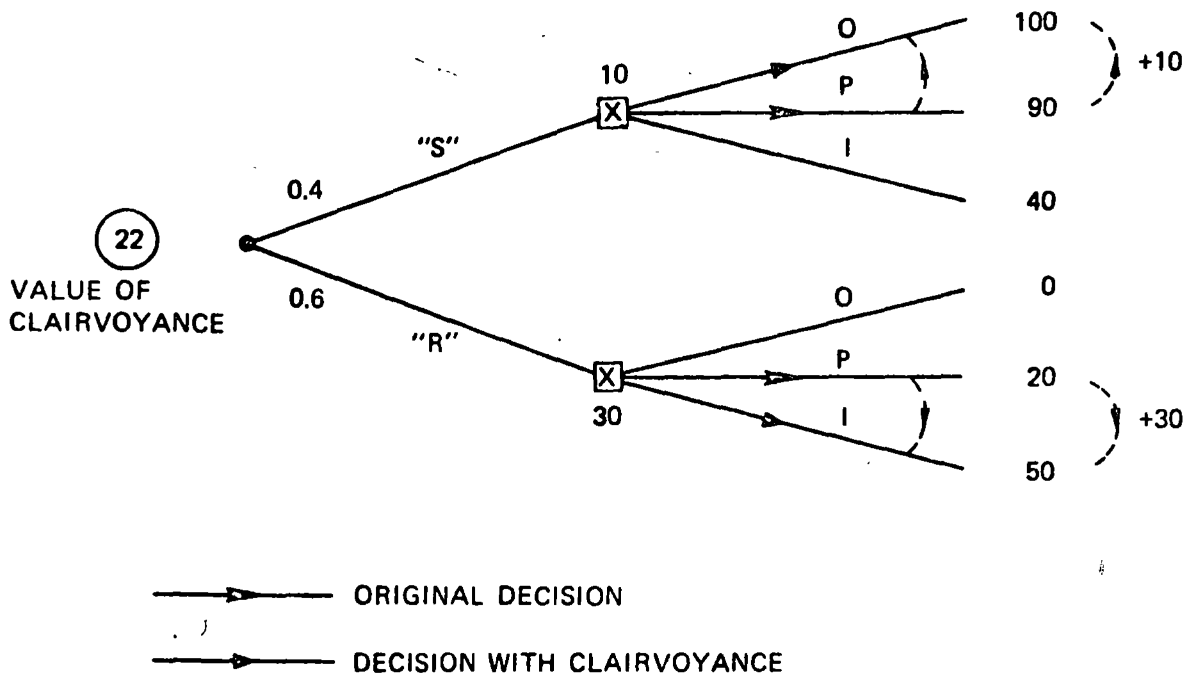


EXPECTED VALUE WITH CLAIRVOYANCE = 70
EXPECTED VALUE WITHOUT CLAIRVOYANCE = 48
EXPECTED VALUE OF CLAIRVOYANCE = 22

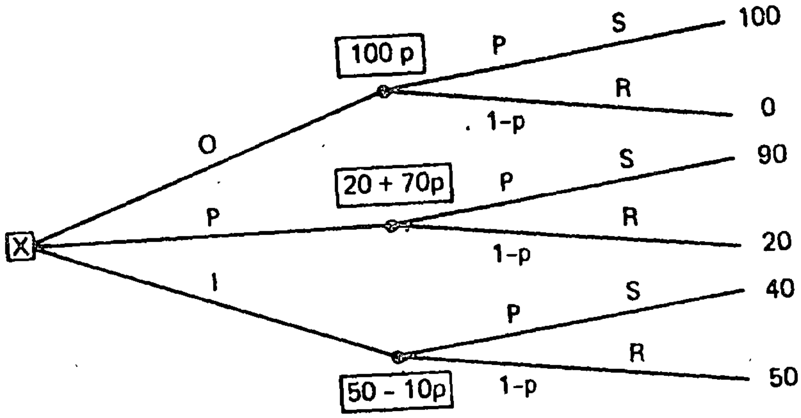
A PARTY PROBLEM

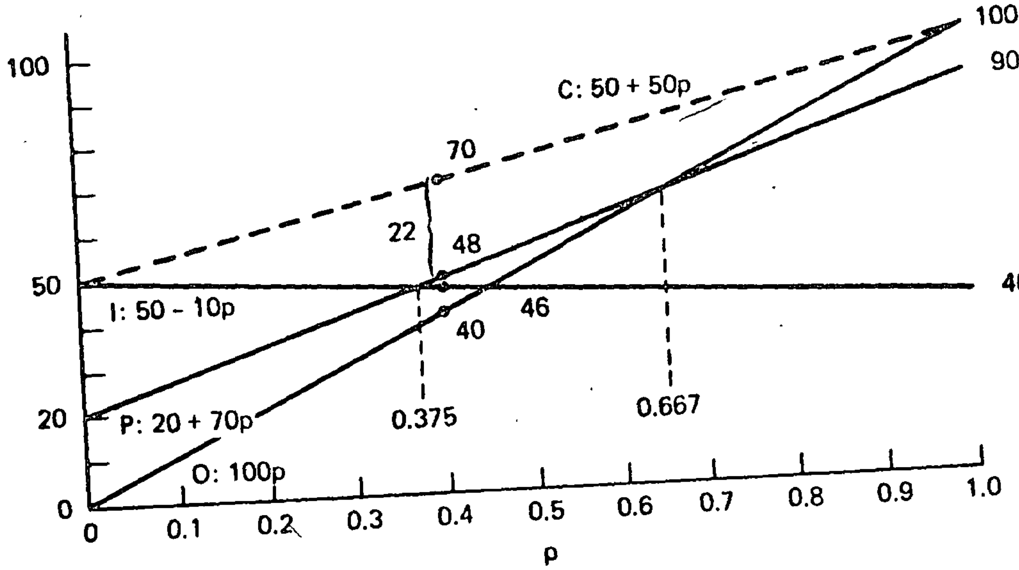
VALUE OF CLAIRVOYANCE – ALTERNATE METHOD

"EFFECT ON DECISION"

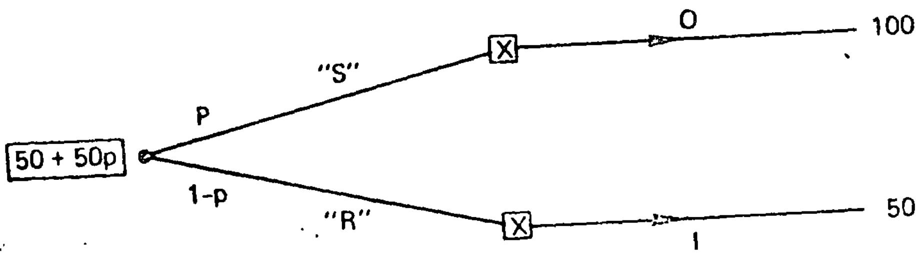


A PARTY PROBLEM

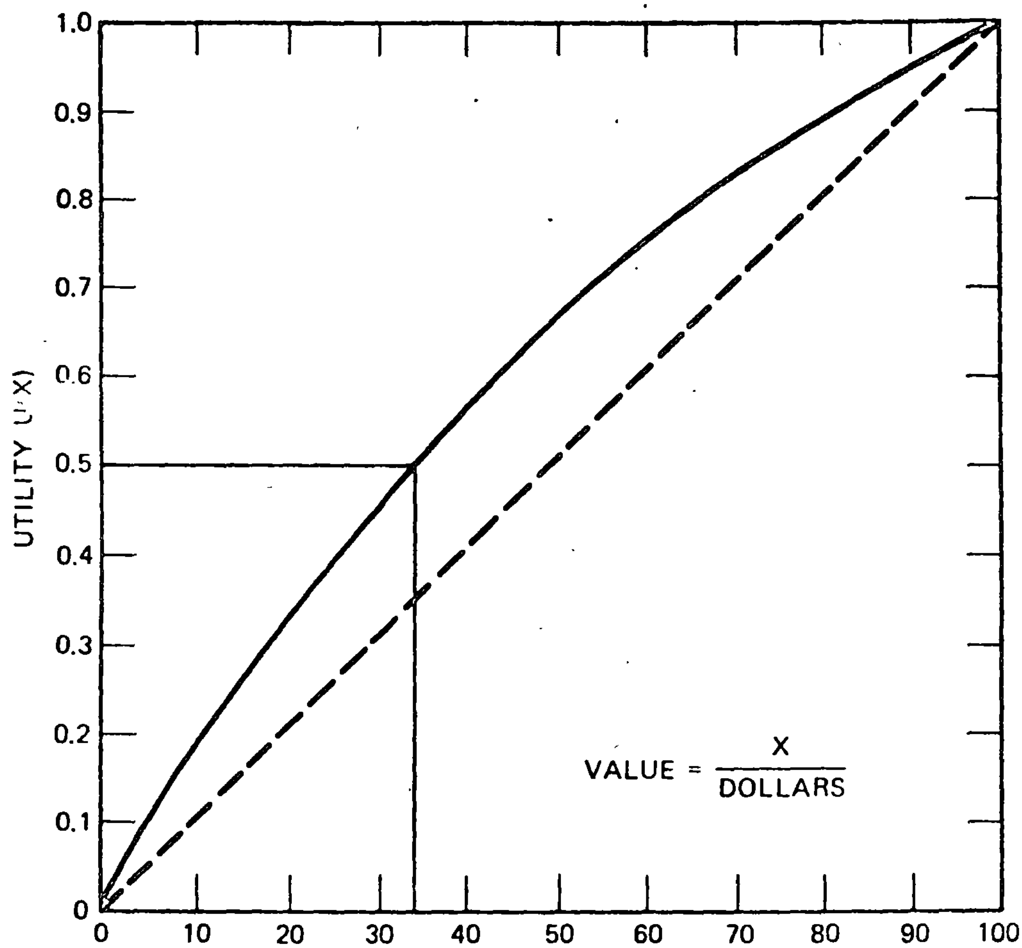




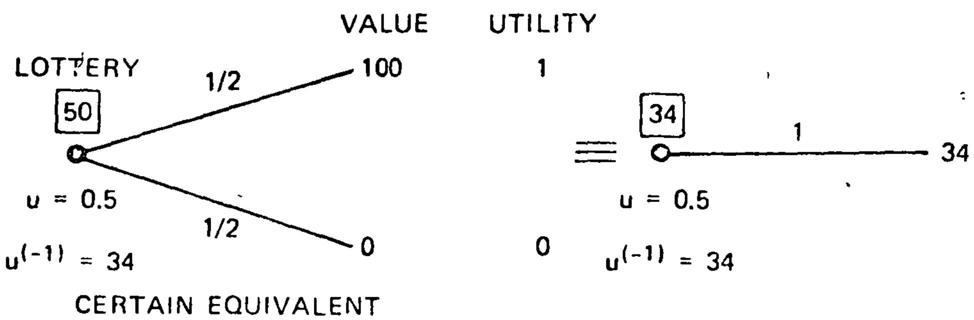
VALUE OF CLAIRVOYANCE



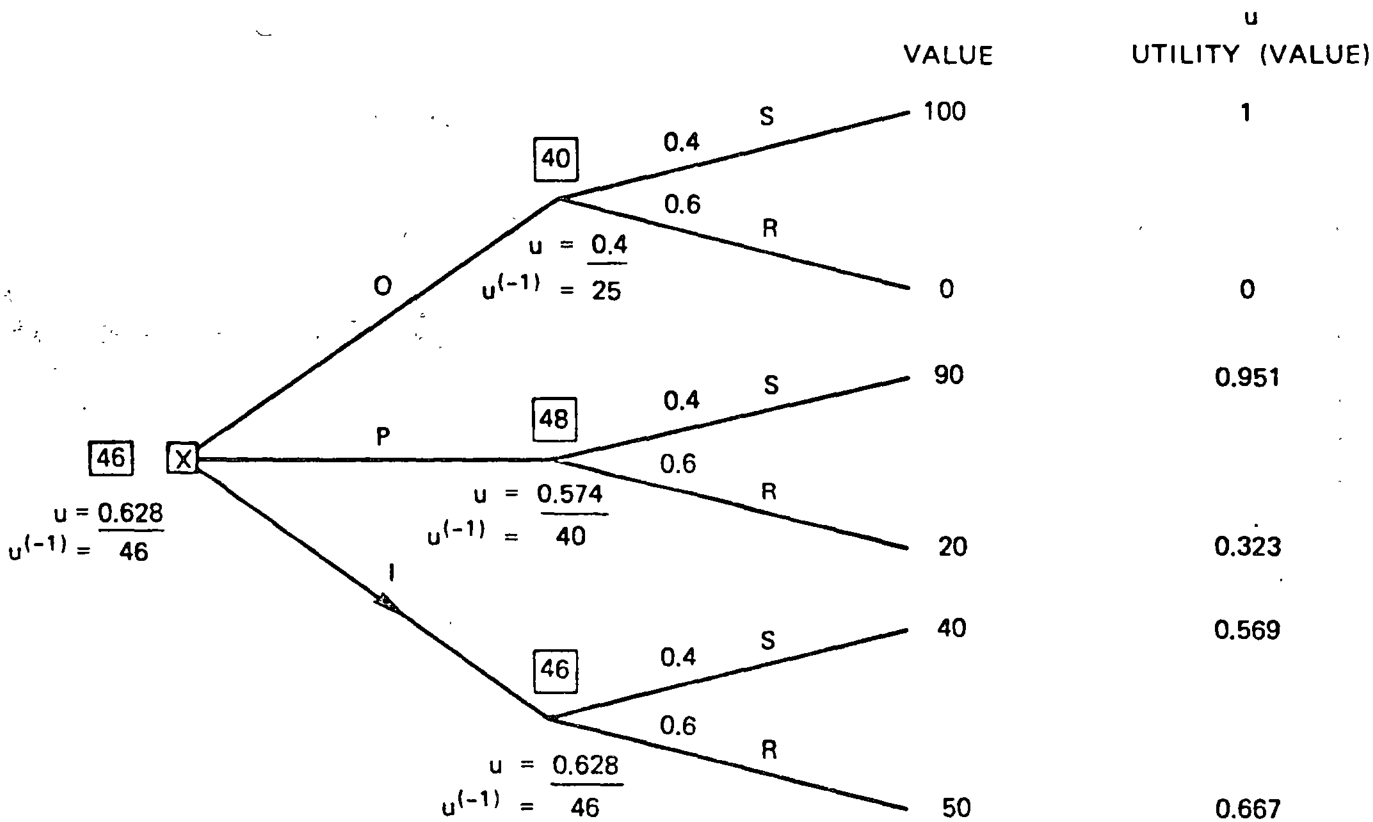
RISK TOLERANCE



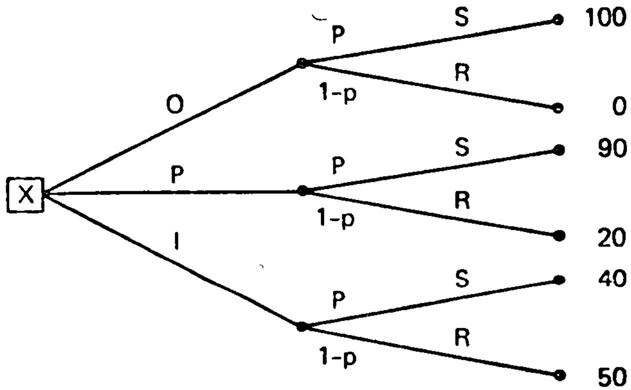
→ THE UTILITY OF A LOTTERY IS ITS EXPECTED UTILITY



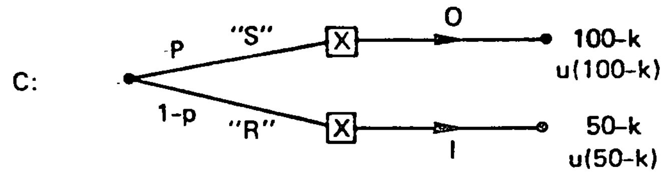
A PARTY PROBLEM



A PARTY PROBLEM



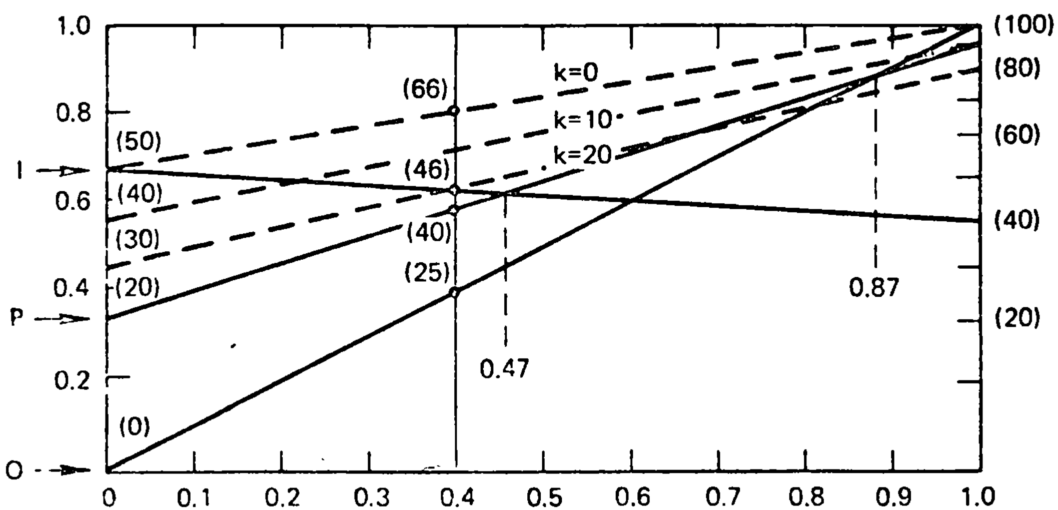
LOTTERY ON CLAIRVOYANCE AT COST k

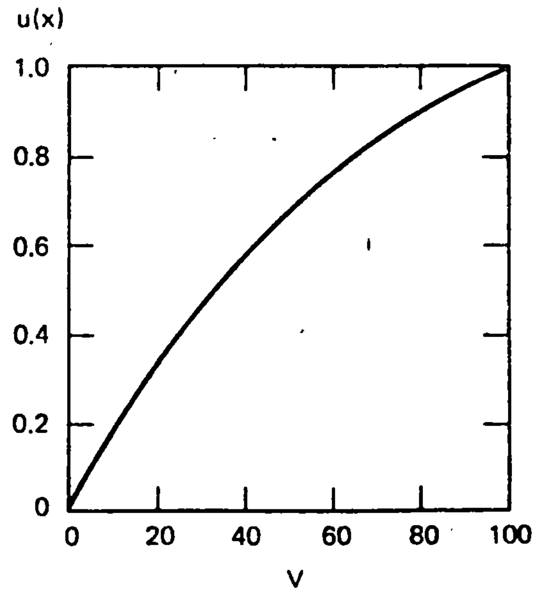


u(c) = pu(100-k) + (1-p)u(50-k)

FIND k SUCH THAT u(c) = u(c′)

k = 20 WHEN p = 0.4





A PARTY PROBLEM

EFFECT OF RISK TOLERANCE ON VALUE OF INFORMATION

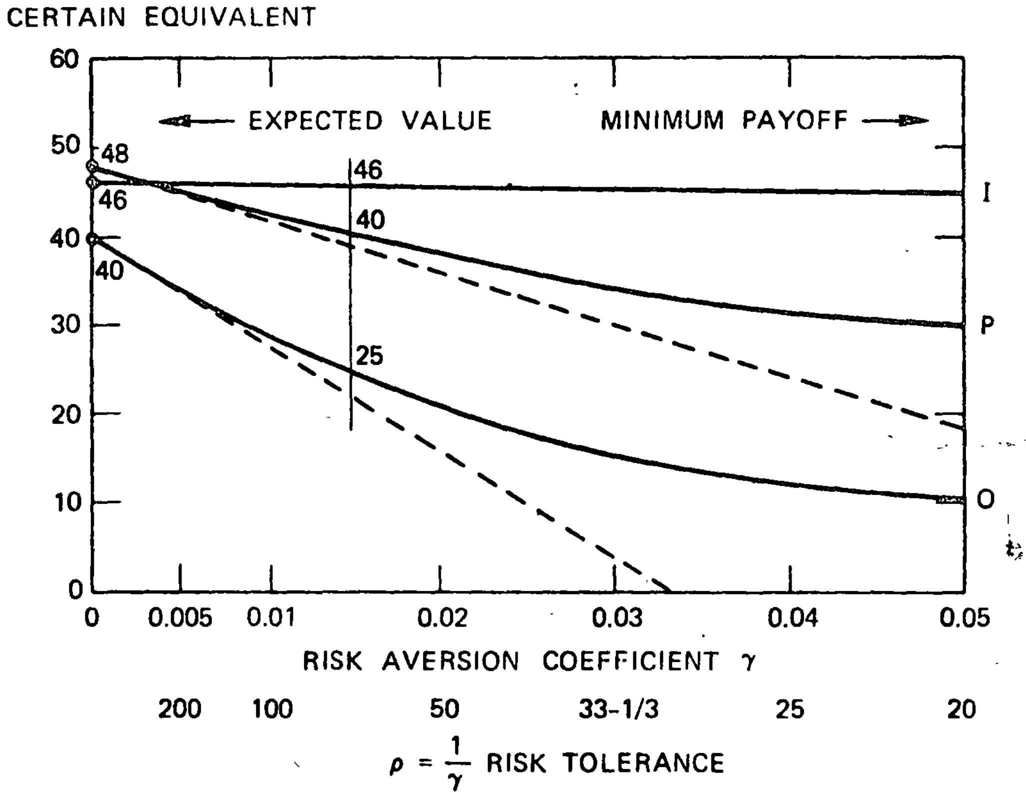
WITH p = 0.4	BEST PRIOR ALTERNATIVE	VALUE OF CLAIRVOYANCE
RISK INDIFFERENCE	P	22
RISK AVERSION	I	20

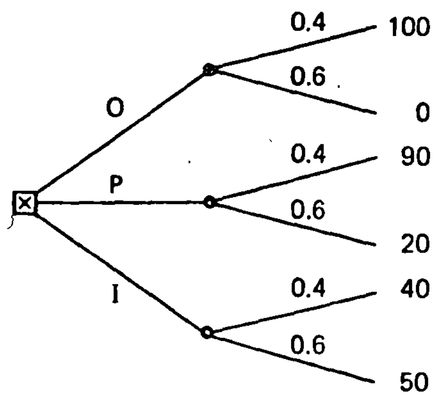
WHY? BECAUSE RISK AVERTER IS USING MORE CONSERVATIVE I ALTERNATIVE IN THE ABSENCE OF CLAIRVOYANCE

HOWEVER,

WITH p = 0.5	BEST PRIOR ALTERNATIVE	VALUE OF CLAIRVOYANCE
RISK INDIFFERENCE	P	20
RISK AVERSION	P	24

RISK SENSITIVITY PROFILE





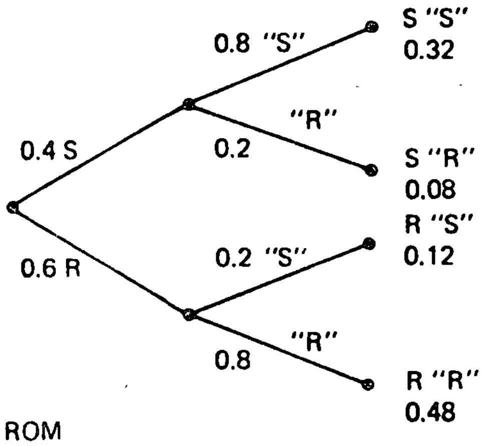
	EXPECTED VALUE	VARIANCE
O	40	2400
P	48	1176
I	46	24

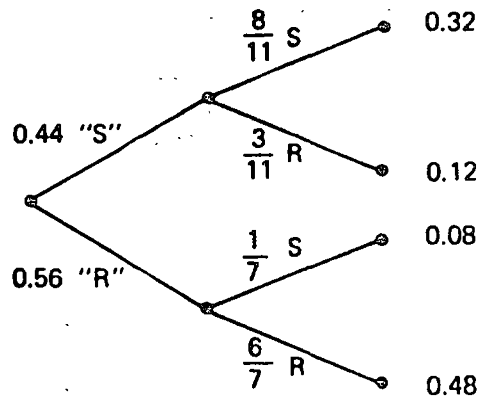
INITIAL SLOPE OF PROFILE IS - 1/2 (VARIANCE)

A PARTY PROBLEM

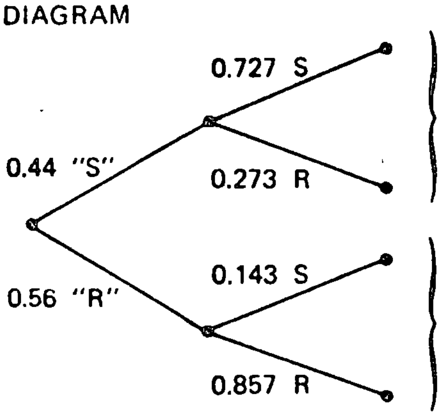
ACME RAIN DETECTOR

- CORRECTLY INDICATES TOMORROW'S WEATHER WITH PROBABILITY 0.8
- "S" INDICATES SUNSHINE
- "R" INDICATES RAIN





FROM
EXPECTED
VALUE
DIAGRAM



OUTDOOR, EXPECTED VALUE = 100 p
= 72.70

INDOOR, EXPECTED VALUE = 50-10 p
= 48.57

EXPECTED PROFIT USING ACME = 0.44(72.70) + 0.56(48.57)

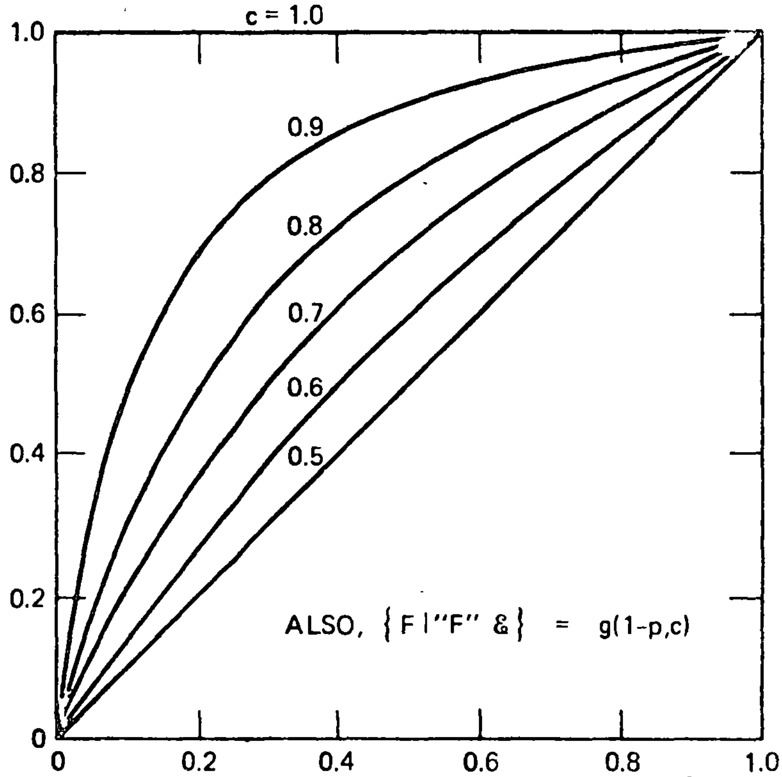
= 59.19

EXPECTED PROFIT WITHOUT ACME = <u>48.00</u>

EXPECTED PROFIT INCREASE 11.19

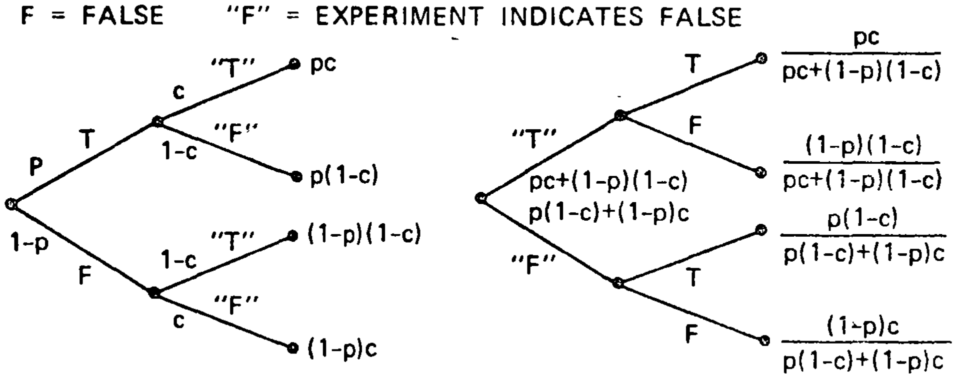
EFFECT OF EXPERIMENTATION

$$\{T \mid \text{"T"}\ \&\} = \frac{pc}{pc+(1-p)(1-c)} = g(p,c) = g(c,p)$$



T = TRUE "T" = EXPERIMENT INDICATES TRUE $p = \{T \mid \&\}$

F = FALSE "F" = EXPERIMENT INDICATES FALSE

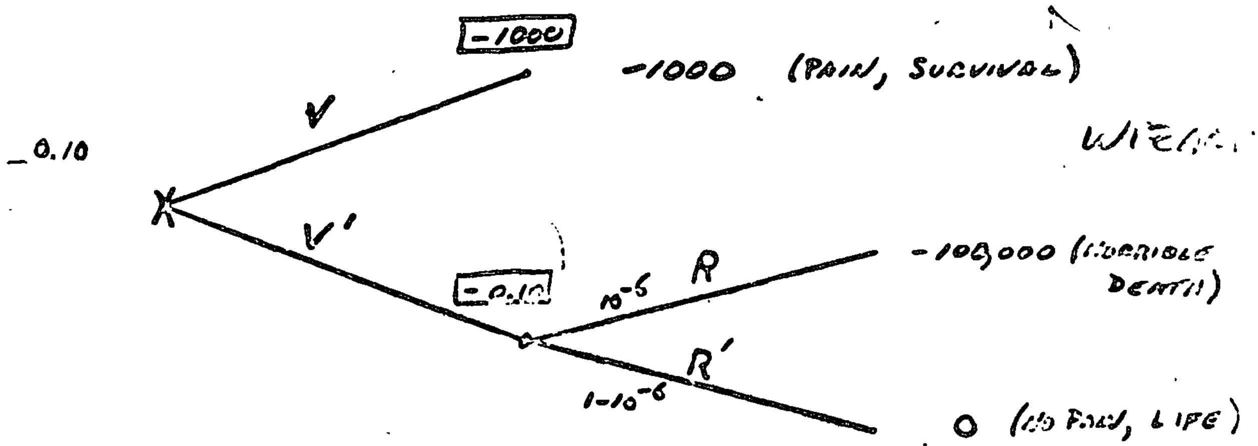


c = PROBABILITY OF CORRECT INDICATION

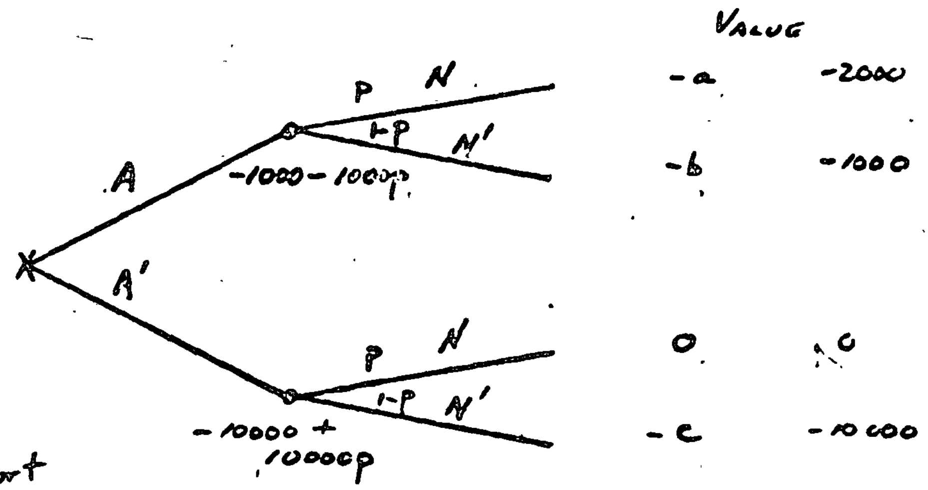
PROBLEM: ADMINISTER RABIES VACCINE?

V: ADMINISTER VACCINE (ASSUME COMPLETELY EFFECTIVE)

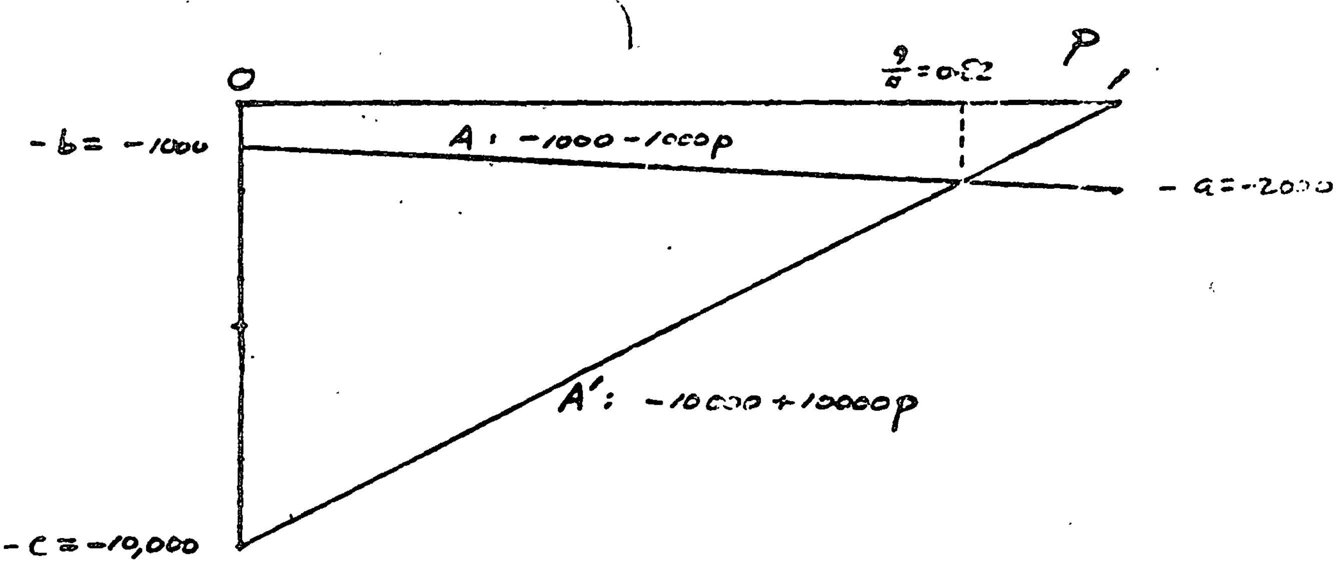
R: DOG WAS RABID ≡ CHILD CONTRACTS RABIES ≡ DEATH



PROBLEM: ABORT POSSIBLY DEFECTIVE FETUS?



A: Abort
N: Normal



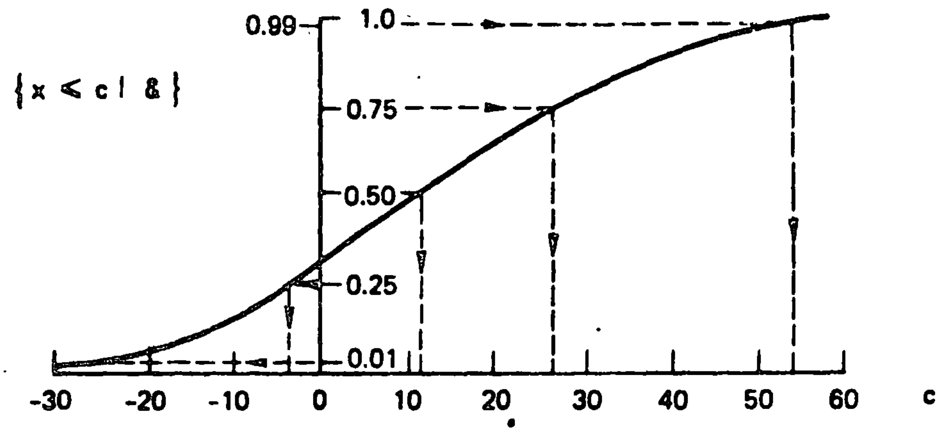
PAY TO THE BEARER THE SUM OF $1000 IF THE WEIGHT OF THE PROJECTOR SATISFIES THE REQUIREMENT:

( )

PAY TO THE BEARER THE SUM OF $1000 IF THE TOSSING OF A COIN SATISFIES THE REQUIREMENT:

( )

PROBABILITY ASSESSMENT



DEFINE FRACTILE ${}^{<}x(f)$

$$\{x \le {}^{<}x(f) \mid \&\} = f$$

f	${}^{<}x(f)$
0.01	-26
0.25	- 4
0.50	11
0.75	27
0.99	54

$$\{{}^{<}x(0.25) < x \le {}^{<}x(0.75) \mid \&\} = \{x \le {}^{<}x(0.75) \mid \&\} - \{x \le {}^{<}x(0.25) \mid \&\}$$

$$= 0.75 - 0.25 = \underline{\underline{0.50}}$$

INTERVAL (${}^{<}x(0.25)$, ${}^{<}x(0.75)$) IS CALLED INTERQUARTILE INTERVAL.

$\{x$ IN INTERQUARTILE INTERVAL $\mid \&\} = 0.50.$

PROBABILITY ASSESSMENT

NOTATION

x: random variable

A: event

$\mathcal{S}$: state of information

$\{x|\mathcal{S}\}$: density function of x given $\mathcal{S}$

$\{A|\mathcal{S}\}$: probability of A given $\mathcal{S}$

$\langle x|\mathcal{S}\rangle$ = expectation of x given $\mathcal{S}$ $= \int_x x\{x|\mathcal{S}\}$

$\langle x^n|\mathcal{S}\rangle$ = n^{th} moment of x given $\mathcal{S}$ $= \int_x x^n\{x|\mathcal{S}\}$

${}^v\langle x|\mathcal{S}\rangle$ = variance of x given $\mathcal{S}$ $= \langle x^2|\mathcal{S}\rangle - \langle x|\mathcal{S}\rangle^2$

$\{x,y|\mathcal{S}\}$ = joint density function of x and y

$\mathcal{E}$: total experience

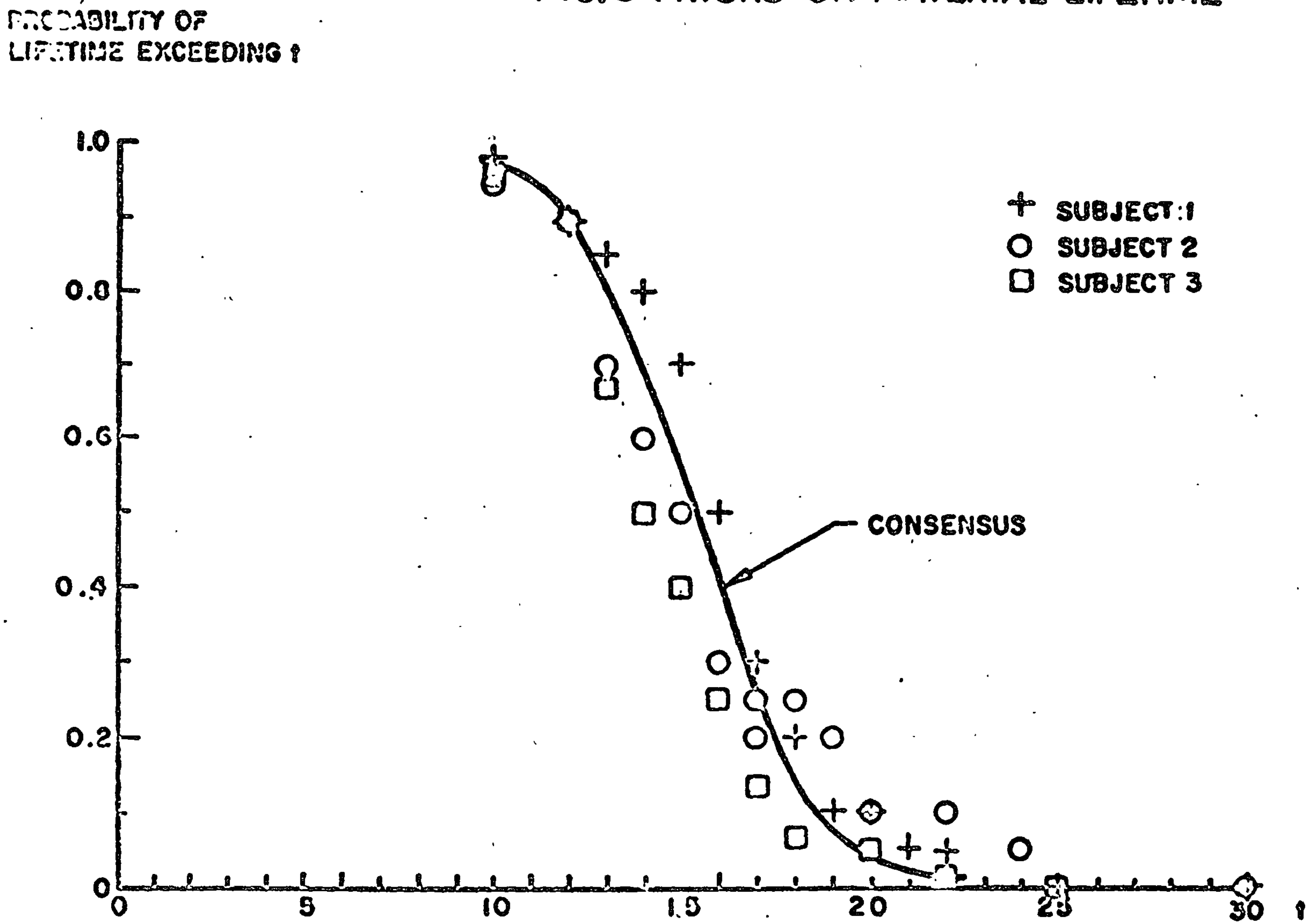
$\{x|\mathcal{E}\}$: "prior" on x

$$\{x|y\,\mathcal{S}\} = \frac{\{x,y|\mathcal{S}\}}{\{y|\mathcal{S}\}}$$ = conditional density function of x given y and $\mathcal{S}$

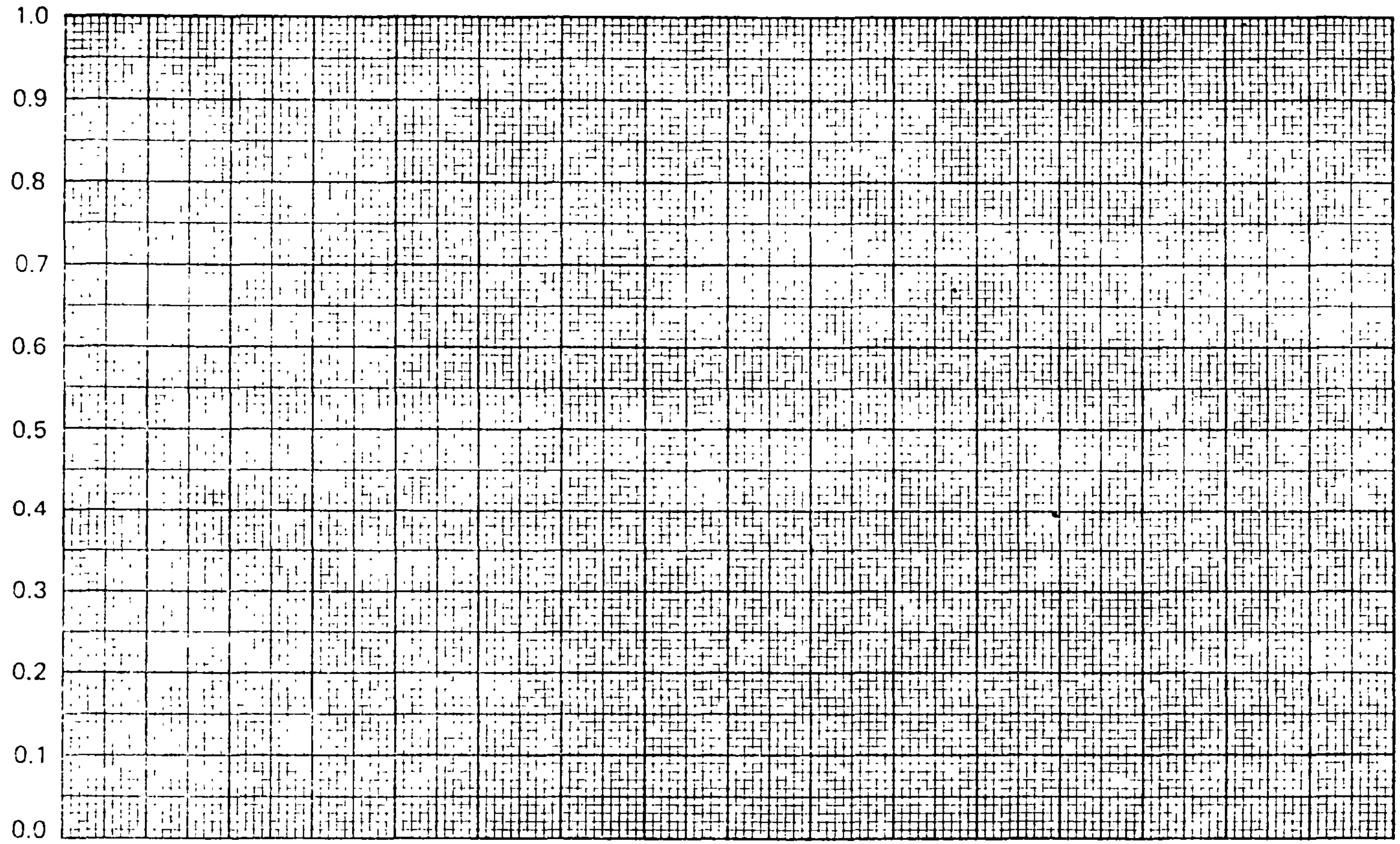
$$\{x|\mathcal{S}\} = \int_y \{x,y|\mathcal{S}\}$$

$$\{x|\mathcal{S}\} = \int_y \{x|y\,\mathcal{S}\}\{y|\mathcal{S}\}$$ <u>expansion</u>

FIG. 3 PRIORS ON MATERIAL LIFETIME



SUBJECT ____________________ DATE ____________________



VARIABLE ______________________________

SRI—PROBABILITY ENCODING FORM—MAY 1969

Probability Assessment

Consider repeated tossing of a fair coin

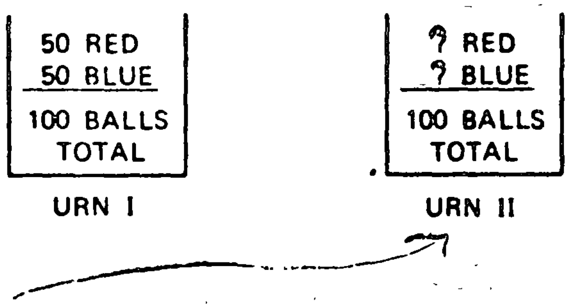
H = Head T = Tail

Let n = number of tosses required to complete first H H H sequence

Ex. H T T H H H T H · · · $\Longrightarrow$ n = 6

$$\{n \leq n(f) \mid \mathcal{E}\} = f$$

f	0.01	0.25	0.50	0.75	0.99
n(f)					



FILLED FROM LARGE SUPPLY OF BOTH RED AND BLUE BALLS BY COLOR BLIND CHILD.

A BALL IS DRAWN — IF YOU GUESS ITS COLOR CORRECTLY YOU WIN $100, OTHERWISE NOTHING.

WOULD YOU PREFER TO PLAY THIS GAME EXACTLY ONCE

A) WITH URN I

OR

B) WITH URN II

	30 RED BALLS
60	? BLUE BALLS ? YELLOW BALLS
	90 BALLS TOTAL

THE COLORBLIND CHILD STRIKES AGAIN.

WE PLAY A GAME WITH THE FOLLOWING PAYOFF CHOICE EXACTLY ONCE.

PAYOFF SCHEME \ BALL DRAWN	RED	BLUE	YELLOW
I	$100	0	0
II	0	$100	0

TWO OTHER PAYOFF SCHEMES MIGHT BE:

PAYOFF SCHEME \ BALL DRAWN	RED	BLUE	YELLOW
III	0	$100	$100
IV	$100	0	$100

NOW WE MAKE THE PAYOFF SCHEME DEPEND ON THE OUTCOME OF THE TOSS OF A FAIR COIN, AND ON WHICH OF TWO OPTIONS WE SELECT.

OPTION \ TOSS OUTCOME	HEADS	TAILS
A	I	III
B	II	IV

IF I > II, III > IV THEN A > B

BUT

OPTION \ BALL	RED	BLUE	YELLOW
A	(100,0)	(0,100)	(0,100)
B	(0,100)	(100,0)	(0,100)

RESULT: A ~ B

[NOTE: (x, y) MEANS LOTTERY: $p\{\text{win } x\} = \frac{1}{2}$, $p\{\text{win } y\} = \frac{1}{2}$]

"A LIE THAT YOU HAVE HEARD A HUNDRED TIMES IS MUCH MORE CREDIBLE THAN A FACT YOU HAVE NEVER HEARD BEFORE."

OLD ADAGE

"OUR PASSIONS, OUR PREJUDICES, AND DOMINATING OPINIONS, BY EXAGGERATING THE PROBABILITIES WHICH ARE FAVORABLE TO THEM AND BY ATTENUATING THE CONTRARY PROBABILITIES, ARE THE ABUNDANT SOURCES OF DANGEROUS ILLUSIONS."

IT IS SEEN IN THIS ESSAY THAT THE THEORY OF PROBABILITIES IS AT BOTTOM ONLY COMMON SENSE REDUCED TO CALCULUS; IT MAKES US APPRECIATE WITH EXACTITUDE THAT WHICH EXACT MINDS FEEL BY A SORT OF INSTINCT WITHOUT BEING ABLE, OFTTIMES, TO GIVE A REASON FOR IT.

LAPLACE
A PHILOSOPHICAL ESSAY ON PROBABILITIES

"A NEW TRUTH DOES NOT TRIUMPH BY CONVINCING ITS OPPONENTS AND MAKING THEM SEE THE LIGHT. BUT RATHER BECAUSE ITS OPPONENTS EVENTUALLY DIE AND A NEW GENERATION GROWS UP THAT IS FAMILIAR WITH IT."

MAX PLANCK

UTILITY

NOTATION

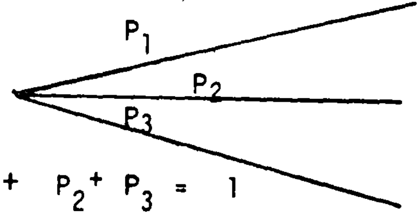
PRIZES A, B

$A \succ B$	I PREFER A TO B
$A \sim B$	I AM INDIFFERENT BETWEEN A AND B
$A \succsim B$	I LIKE A AT LEAST AS MUCH AS B

LOTTERY

A LOTTERY IS A SET OF PRIZES (PROSPECTS) WITH ASSOCIATED PROBABILITIES

L =



A HOPE DIAMOND

B 1 TON CHEESE

C A CASE OF PNEUMONIA

$$P_1 + P_2 + P_3 = 1$$

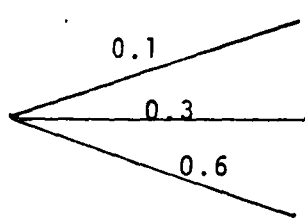
$$L = (P_1, A;\ P_2, B;\ P_3, C)$$

IF THE PRIZES IN A LOTTERY ARE ALL MEASURED IN TERMS OF A SINGLE COMMODITY (LIKE MONEY), THEN WE CAN THINK OF A LOTTERY AS A RANDOM VARIABLE.

LOTTERIES:

DISCRETE

X:

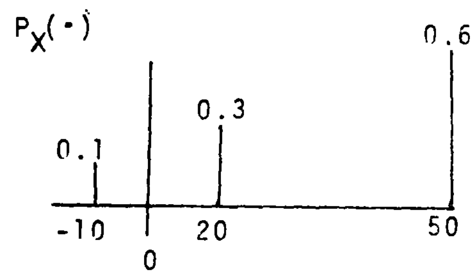


\$

-10

20

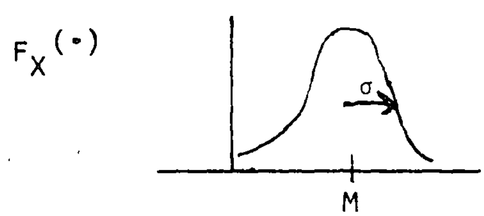
50



MASS FUNCTION

CONTINUOUS $X: \ F_X(X_0) = F_N(X_0 | M, \sigma)$

\$

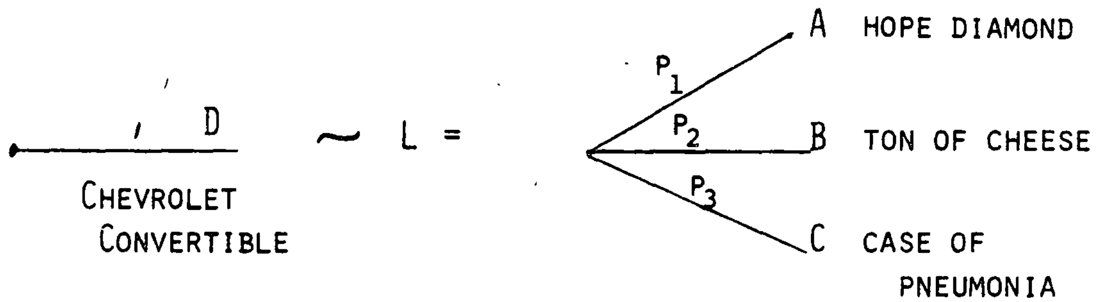


CERTAIN EQUIVALENT

THE CERTAIN EQUIVALENT OF A LOTTERY IS A PRIZE SUCH THAT THE INDIVIDUAL IS INDIFFERENT BETWEEN RECEIVING THE PRIZE AND PARTICIPATING IN THE LOTTERY.

NOTATION: $\tilde{L}$ IS THE CERTAIN EQUIVALENT OF A LOTTERY L

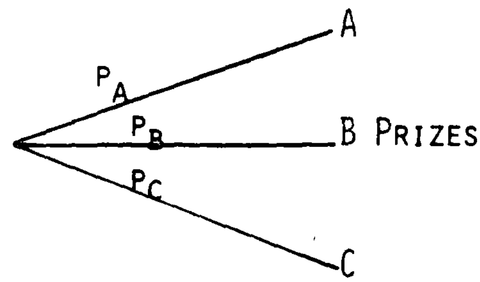
IF



THEN $\tilde{L} = D$

IF THE LOTTERY IS A RANDOM VARIABLE x
THEN $\tilde{x}$ IS ITS CERTAIN EQUIVALENT

UTILITY

LOTTERY: 

AXIOMS

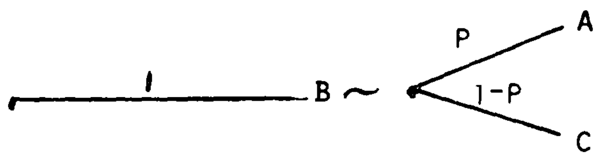
1) <u>ORDERABILITY</u> OF PRIZES

$A > B, \quad A \gtrsim B, \quad A \sim B, \quad A \lesssim B, \quad A < B$

TRANSITIVITY IF $A > B$, $B > C$ THEN $A > C$

2) <u>CONTINUITY</u>

IF $A > B > C$

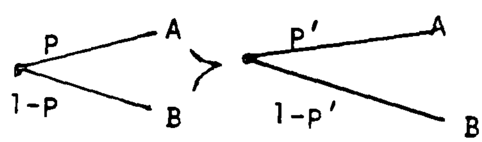
THEN FOR SOME P 

B IS THEN CALLED THE CERTAIN EQUIVALENT OF THE LOTTERY

3) <u>SUBSTITUTABILITY</u>

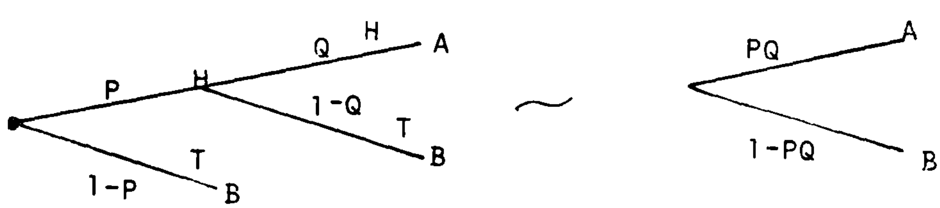
A LOTTERY AND ITS CERTAIN EQUIVALENT ARE INTERCHANGEABLE WITHOUT AFFECTING PREFERENCES

4) <u>MONOTONICITY</u>

IF $A > B$, THEN 

IF AND ONLY IF $P > P'$

5) <u>DECOMPOSABILITY</u>



1 - P + P - PQ

UTILITY

AN INDIVIDUAL WHOSE PREFERENCES SATISFY THE UTILITY AXIOMS MAY ENCODE THESE PREFERENCES IN A UTILITY FUNCTION $u(\cdot)$ DEFINED ON THE PRIZES. THE FUNCTION $u(\cdot)$ HAS TWO IMPORTANT PROPERTIES:

1) THE UTILITY OF ANY LOTTERY IS THE EXPECTED UTILITY OF ITS PRIZES

2) IF LOTTERY L_1 IS PREFERRED TO LOTTERY L_2

$$L_1 \succ L_2$$

THEN $u(L_1) > u(L_2)$

THE UTILITY FUNCTION IS A PREFERENCE THERMOMETER

THE PREFERENCES REPRESENTED BY THE UTILITY FUNCTION ARE UNCHANGED IF THE FUNCTION IS SUBJECTED TO A LINEAR TRANSFORMATION OF THE FORM

$$u'(x) = \alpha + \beta u(x), \quad \beta > 0$$

Utility

Expected Utility Derivation

Consider finite number of prizes (rewards) - $R_1, R_2, \ldots, R_N$

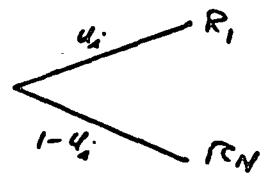
<u>Orderability</u> allows labelling the prizes so that

$$R_1 \succ R_2 \succ \ldots \succ R_N$$

← perhaps some $\succeq$

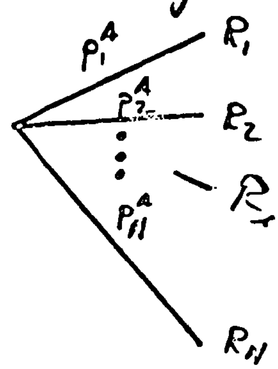
<u>Continuity</u> provides that for some u_i, $0 \le u_i \le 1$

$R_i \sim$



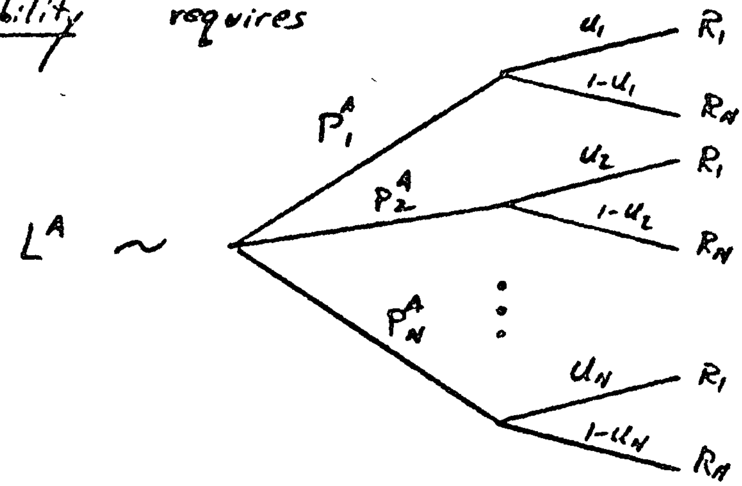
<u>Decomposability</u> implies that all lotteries can be reduced to single stage form

Define Lottery A $\quad L^A$:

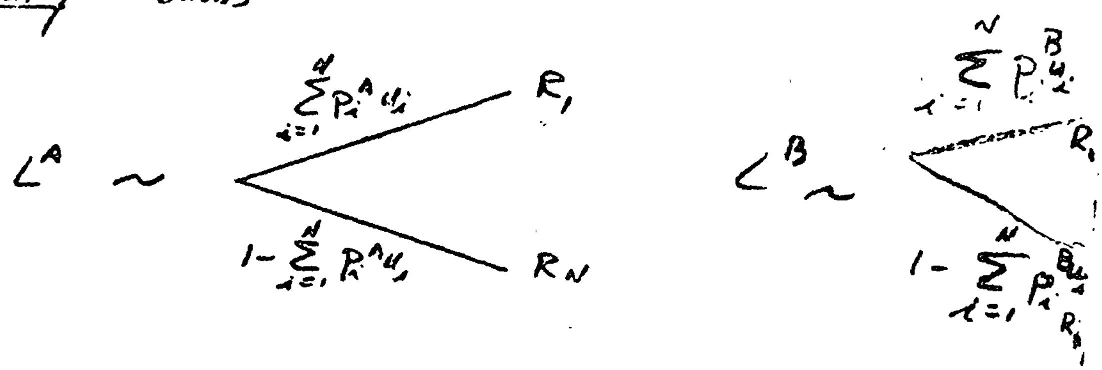


Expected Utility Derivation

Substitutability requires



Decomposability allows

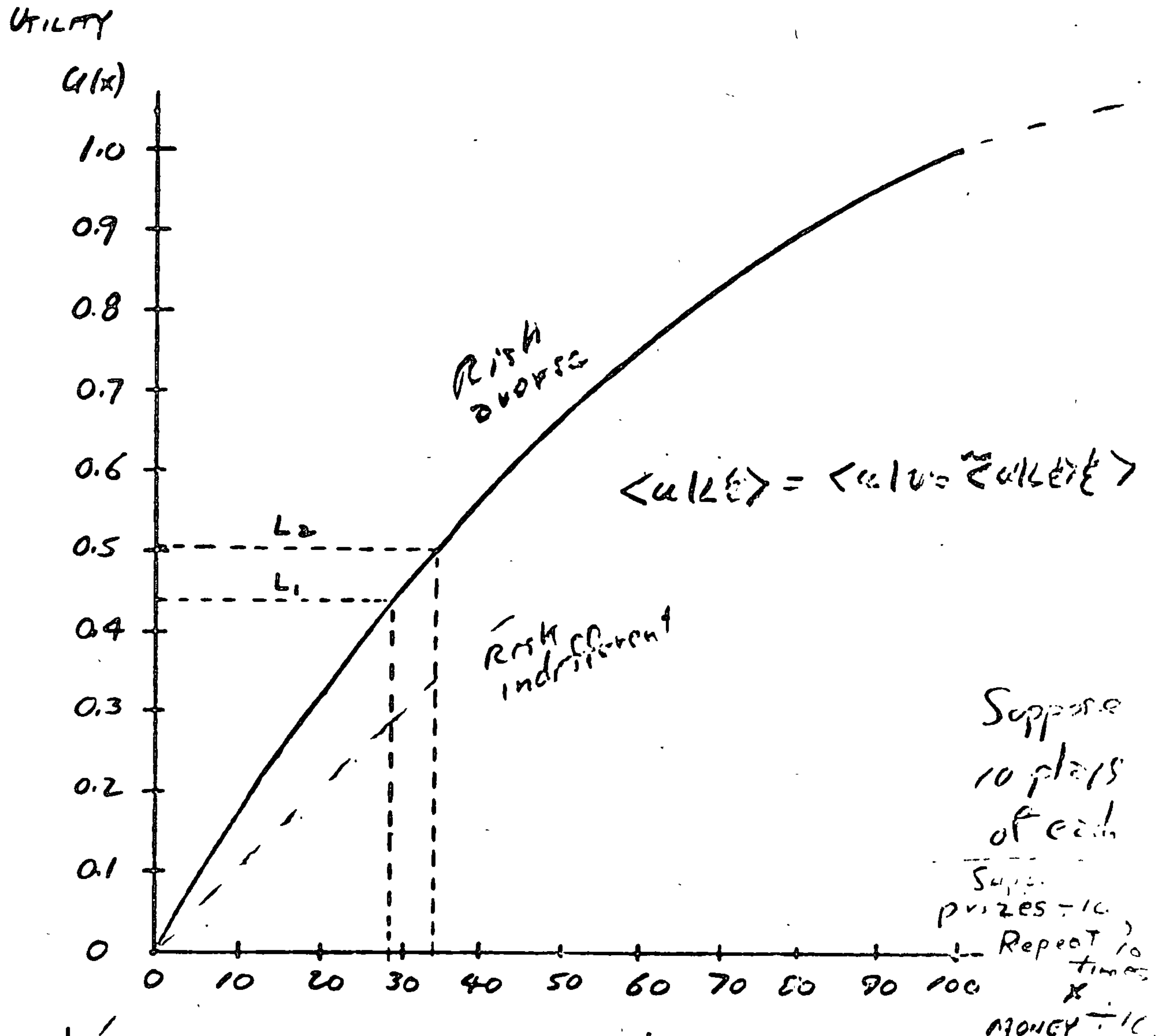


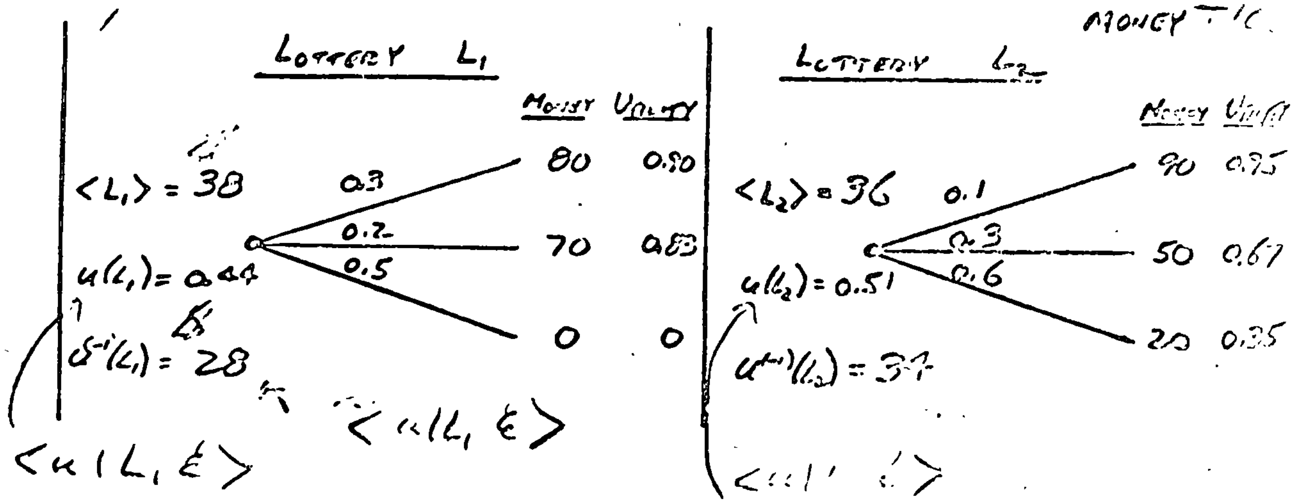
Monotonicity forces

$$L^A > L^B \text{ if and only if } \sum_{i=1}^{N} P_i^A u_i > \sum_{i=1}^{N} P_i^B u_i$$

$$L^A > L^B \iff \langle u \mid L^A, \mathcal{E} \rangle > \langle u \mid L^B, \mathcal{E} \rangle$$

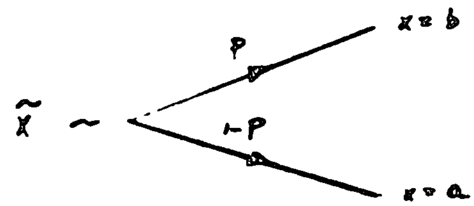
UTILITY


UTILITY
u(x)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 10 20 30 40 50 60 70 80 90 100
MONEY
Risk averse
Risk indifferent
L2
L1
Suppose 10 plays of each
prizes
Repeat 10 times


LOTTERY L1
Money Utility
80 0.90
70 0.83
0 0
0.3
0.2
0.5
⟨L1⟩ = 38
u(L1) = 0.44
⟨u | L1, ε⟩
LOTTERY L2
Money Utility
90 0.95
50 0.67
20 0.35
0.1
0.3
0.6
⟨L2⟩ = 36
u(L2) = 0.51

UTILITY

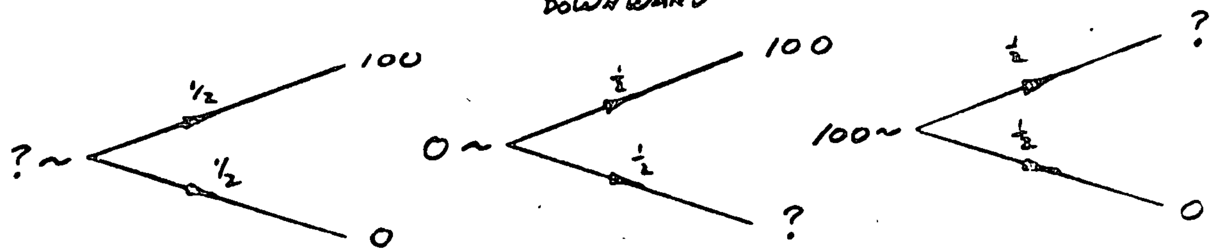
ESTABLISHMENT OF UTILITY CURVES



$u(100) = 1$
$u(0) = 0$

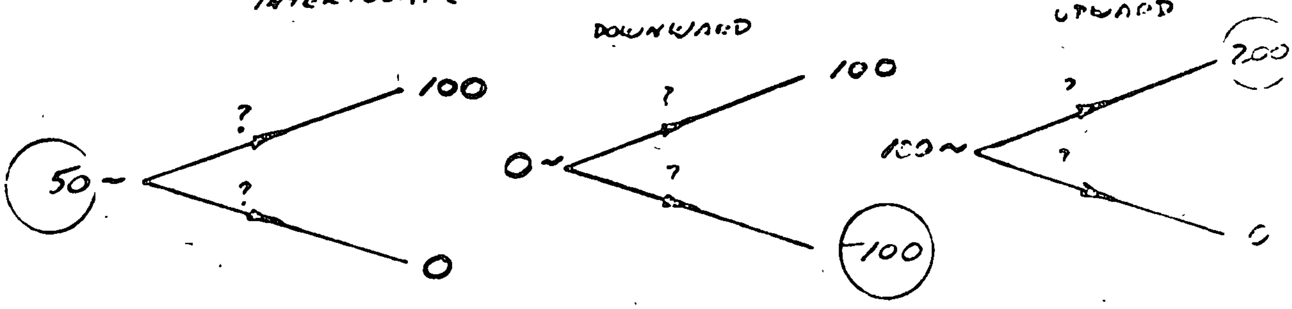
EQUIPROBABLE LOTTERIES : $p = 1/2$

INTERPOLATE — EXTRAPOLATE DOWNWARD — EXTRAPOLATE UPWARD



ASSIGNMENT OF PROBABILITY

INTERPOLATE — EXTRAPOLATE DOWNWARD — EXTRAPOLATE UPWARD



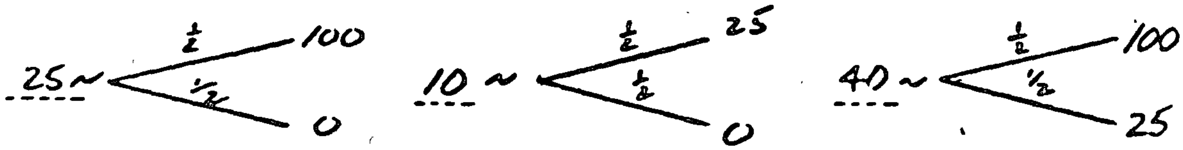
THIS METHOD CAN EVALUATE THE UTILITY OF ANY POINT ◯ IN TERMS OF KNOWN UTILITIES

RISK PREFERENCE
100
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
-110
-120
-130
DOLLARS

Measurement of Risk Tolerance

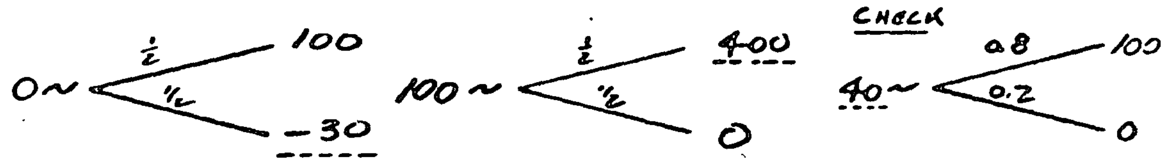
$u(0)=0 \qquad u(100)=1$

---- Response



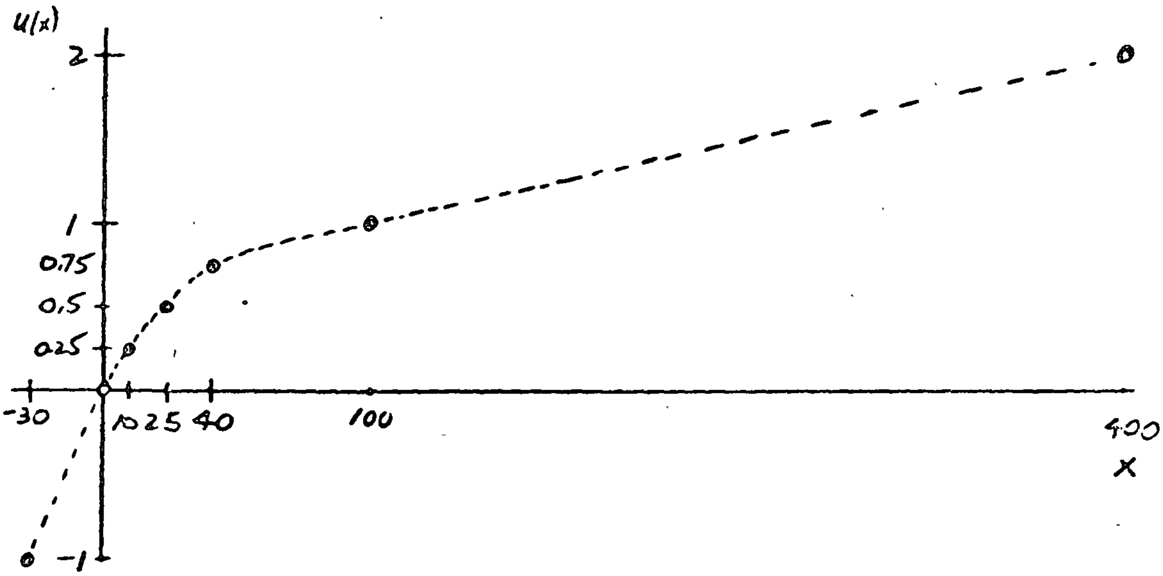
$u(25) = \frac{1}{2}u(100) + \frac{1}{2}u(0)$ $\qquad u(10) = \frac{1}{2}u(25) + \frac{1}{2}u(0)$ $\qquad u(40) = \frac{1}{2}u(100) + \frac{1}{2}u(25)$

$u(25) = 0.5$ $\qquad u(10) = 0.25$ $\qquad u(40) = 0.75$

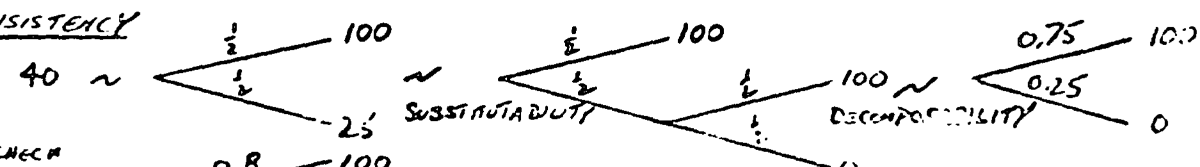


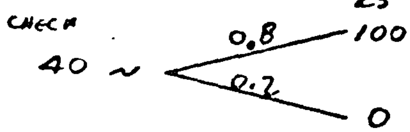
$u(0) = \frac{1}{2}u(100) + \frac{1}{2}u(-30)$ $\qquad u(100) = \frac{1}{2}u(400) + \frac{1}{2}u(0)$ $\qquad u(40) = 0.8u(100) + 0.2u(0)$

$u(-30) = -1$ $\qquad u(400) = 2$ $\qquad = 0.8 \ ?$



Inconsistency





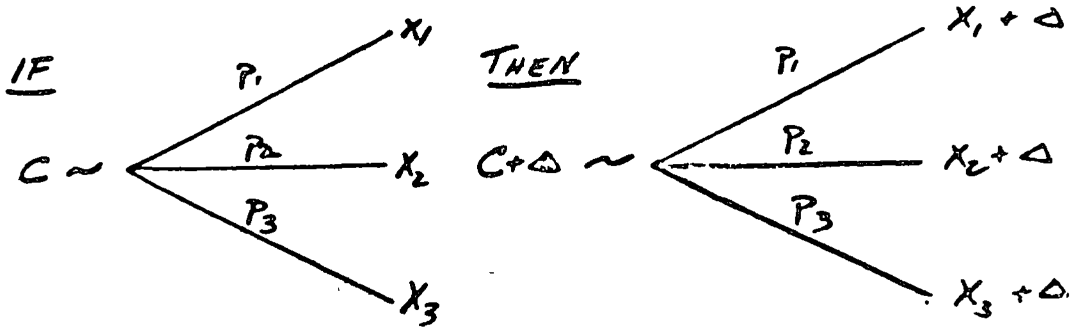
Violates Monotonicity since lottery with higher probability of better prize must be preferred.

UTILITY

CONSIDER A <u>POSSIBLE</u> 6^{th} AXIOM,

THE DELTA PROPERTY:

AN INCREASE OF ALL PRIZES IN A LOTTERY BY AN AMOUNT Δ INCREASES THE CERTAIN EQUIVALENT BY Δ.



CONSEQUENCES

1) THE UTILITY CURVE MUST BE EITHER A STRAIGHT LINE OR AN EXPONENTIAL

$$u(x) = a + bx \quad \text{or} \quad u(x) = a + be^{-\gamma x}$$

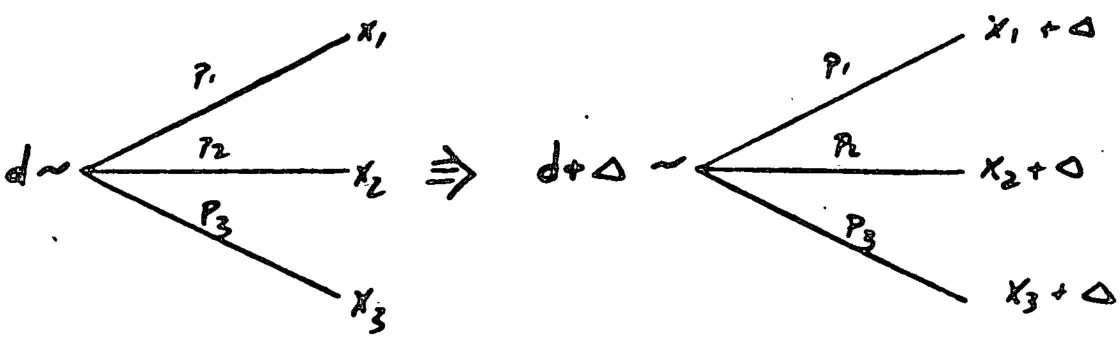
γ IS CALLED THE RISK AVERSION COEFFICIENT

2) THE [illegible] PAYMENT FOR A LOTTERY WILL BE THE SAME AS THE CERTAIN EQUIVALENT.

Utility

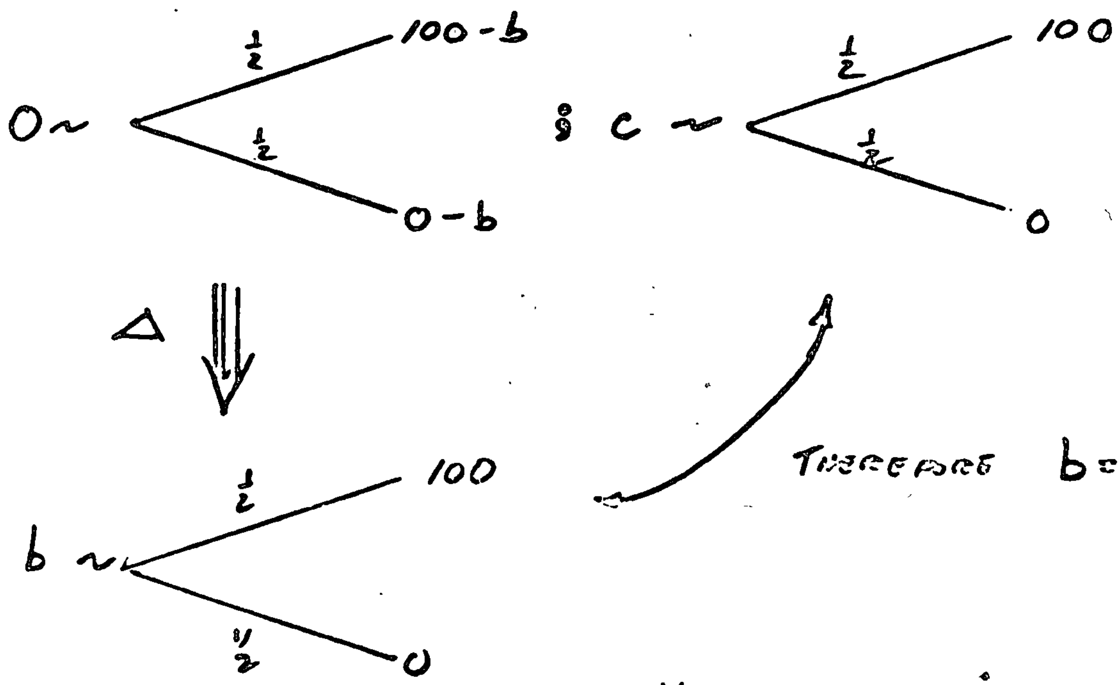
The Delta Property

If an individual satisfies the delta property:



Then his breakeven payment b and certain equivalent c are equal

Consider 0-100 lottery:



Therefore $b = c$

UTILITY

IMPLICATIONS OF DELTA PROPERTY

$L: x$ $\qquad \sim\langle x+\Delta\rangle = \sim\langle x\rangle + \Delta$

$$u^{-1}\left(\langle u(x+\Delta)\rangle\right) = u^{-1}\left(\langle u(x)\rangle\right)+\Delta$$

$$\langle u(x+\Delta)\rangle = u\left[u^{-1}(\langle u(x)\rangle)+\Delta\right]$$

$$= u\left[\sim\langle x\rangle+\Delta\right]$$

$$\langle u(x)\rangle = \int dx_0\, f_x(x_0)\, u(x_0) \qquad \langle u(x+\Delta)\rangle = \int dx_0\, f_x(x_0)\, u(x_0+\Delta)$$

1) $\dfrac{d}{d\Delta}\langle u(x+\Delta)\rangle = \int dx_0\, f_x(x_0)\, u'(x_0+\Delta) = u'(\sim\langle x\rangle+\Delta)$

2) $\dfrac{d^2}{d\Delta^2}\langle u(x+\Delta)\rangle = \int dx_0\, f_x(x_0)\, u''(x_0+\Delta) = u''(\sim\langle x\rangle+\Delta)$

2)÷1) $$\frac{\int dx_0\, f_x(x_0)\, u''(x_0+\Delta)}{\int dx_0\, f_x(x_0)\, u'(x_0+\Delta)} = \frac{u''(\sim\langle x\rangle+\Delta)}{u'(\sim\langle x\rangle+\Delta)}$$

with $\Delta=0$ $$\frac{\int dx_0\, f_x(x_0)\, u''(x_0)}{\int dx_0\, f_x(x_0)\, u'(x_0)} = \frac{u''(\sim\langle x\rangle)}{u'(\sim\langle x\rangle)}$$

Many different $f_x(\cdot)$ will generate same $\sim\langle x\rangle$ and hence same right hand side. For consistency, $\dfrac{u''(\cdot)}{u'(x)}$ must be a constant.

UTILITY

IMPLICATIONS OF DELTA PROPERTY

$$\frac{u''(x)}{u'(x)} = -\gamma$$

Integrate

$$\ln u'(x) = -\gamma x + k_0$$

$$u'(x) = k_1 e^{-\gamma x}$$

$$u(x) = k_2 e^{-\gamma x} + k_3$$

EXPONENTIAL

If $\gamma = 0$

$$u'(x) = k_1$$

$$u(x) = k_1 x + k_4$$

LINEAR

UTILITY

EXPONENTIAL UTILITY

$$u(x) = a + b e^{-\gamma x}$$

A CONVENIENT FORM IS

$$u(x) = \frac{1 - e^{-\gamma x}}{1 - e^{-\gamma}}$$

FOR WHICH $u(0) = 0$, $u(1) = 1$.

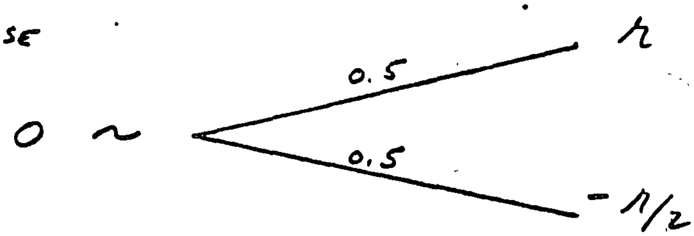
AS $\gamma \to 0$, WE FIND $\lim_{\gamma \to 0} u(x) = \lim_{\gamma \to 0} \frac{1 - e^{-\gamma x}}{1 - e^{-\gamma}} = x$

THEREFORE, WHEN THE RISK AVERSION COEFFICIENT IS ZERO, THE UTILITY CURVE IS A STRAIGHT LINE AND THE INDIVIDUAL IS RISK INDIFFERENT.

WHEN THE RISK AVERSION COEFFICIENT γ IS $\left\{ \begin{matrix} \text{POSITIVE} \\ \text{NEGATIVE} \end{matrix} \right\}$, THE INDIVIDUAL IS RISK-$\left\{ \begin{matrix} \text{AVERTING} \\ \text{PREFERRING} \end{matrix} \right\}$.

EXPONENTIAL: DETERMINATION OF γ and ρ

SUPPOSE



$$u(x) = \frac{1-e^{-\gamma x}}{1-e^{-\gamma}}$$

OR

$$u(x) = -e^{-\gamma x}$$

$$\gamma = \frac{1}{\rho}$$

$$u(0) = 0.5\, u(r) + 0.5\, u(-r/2)$$

USE $\quad u(x) = -e^{-\gamma x}$

$$-1 = -0.5\, e^{-\gamma r} - 0.5\, e^{\gamma r/2}$$

$$2 = e^{-\gamma r} + e^{\gamma r/2}$$

let $\quad \alpha = \gamma r$

$$2 = e^{-\alpha} + e^{\alpha/2}$$

SOLVE: $\quad \alpha = 0.96242365 = \gamma r = r/\rho$

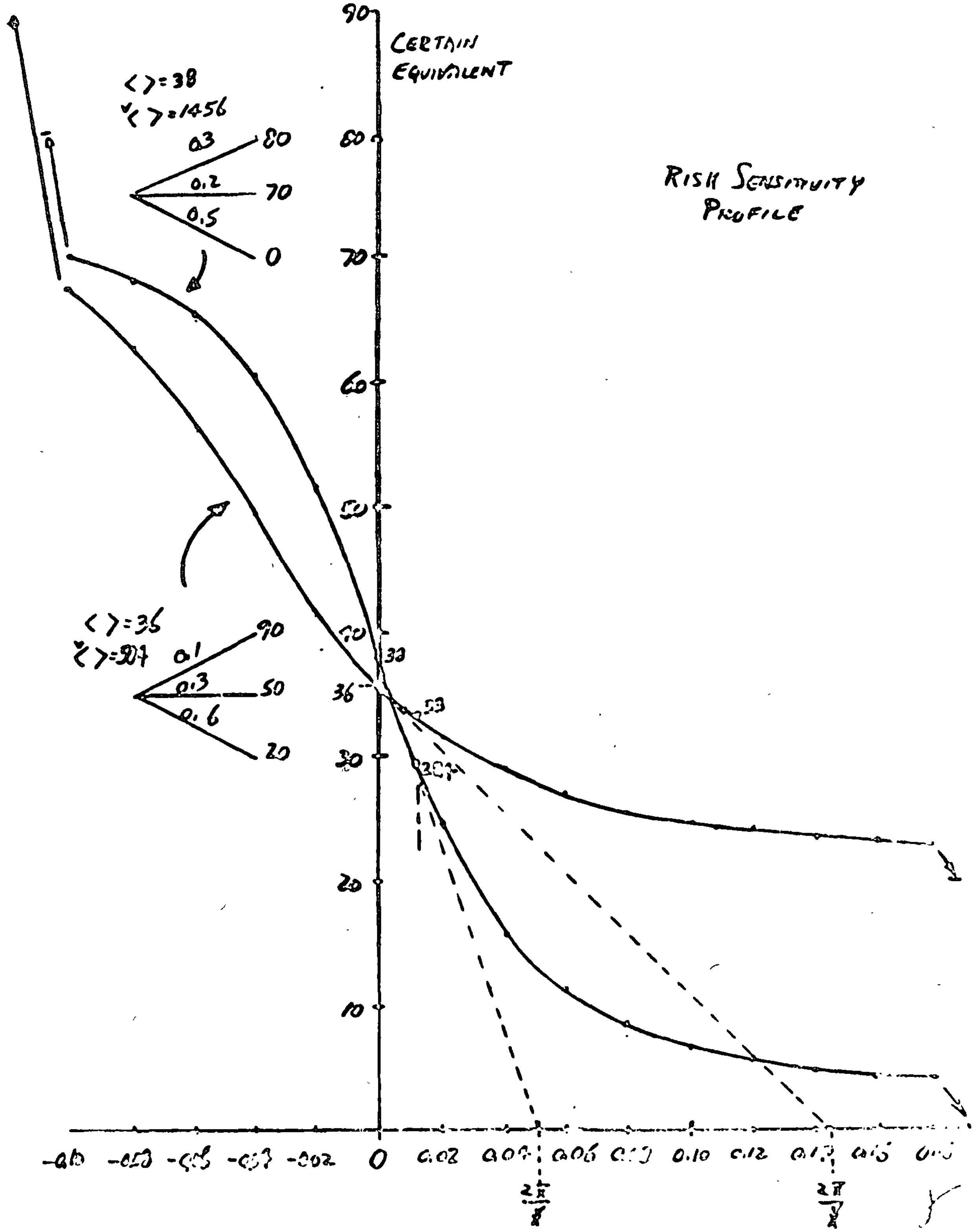
$$\therefore \quad \rho = \frac{r}{\alpha} = 1.03904346\,r$$

APPROXIMATELY,

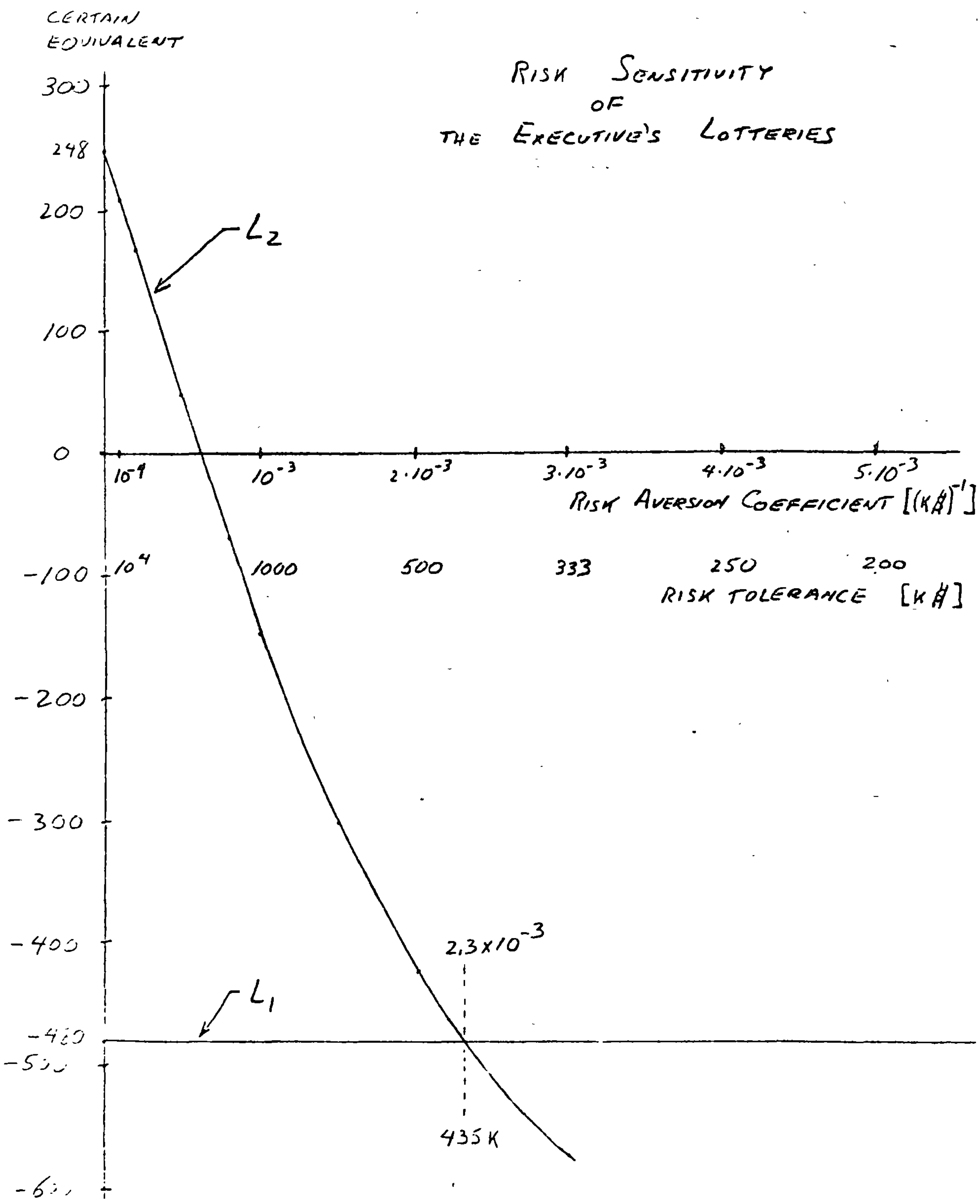
$$\underline{\rho = r}$$

BETTER,

$$\rho = 1.04\, r \quad (\text{ADD } 4\%)$$

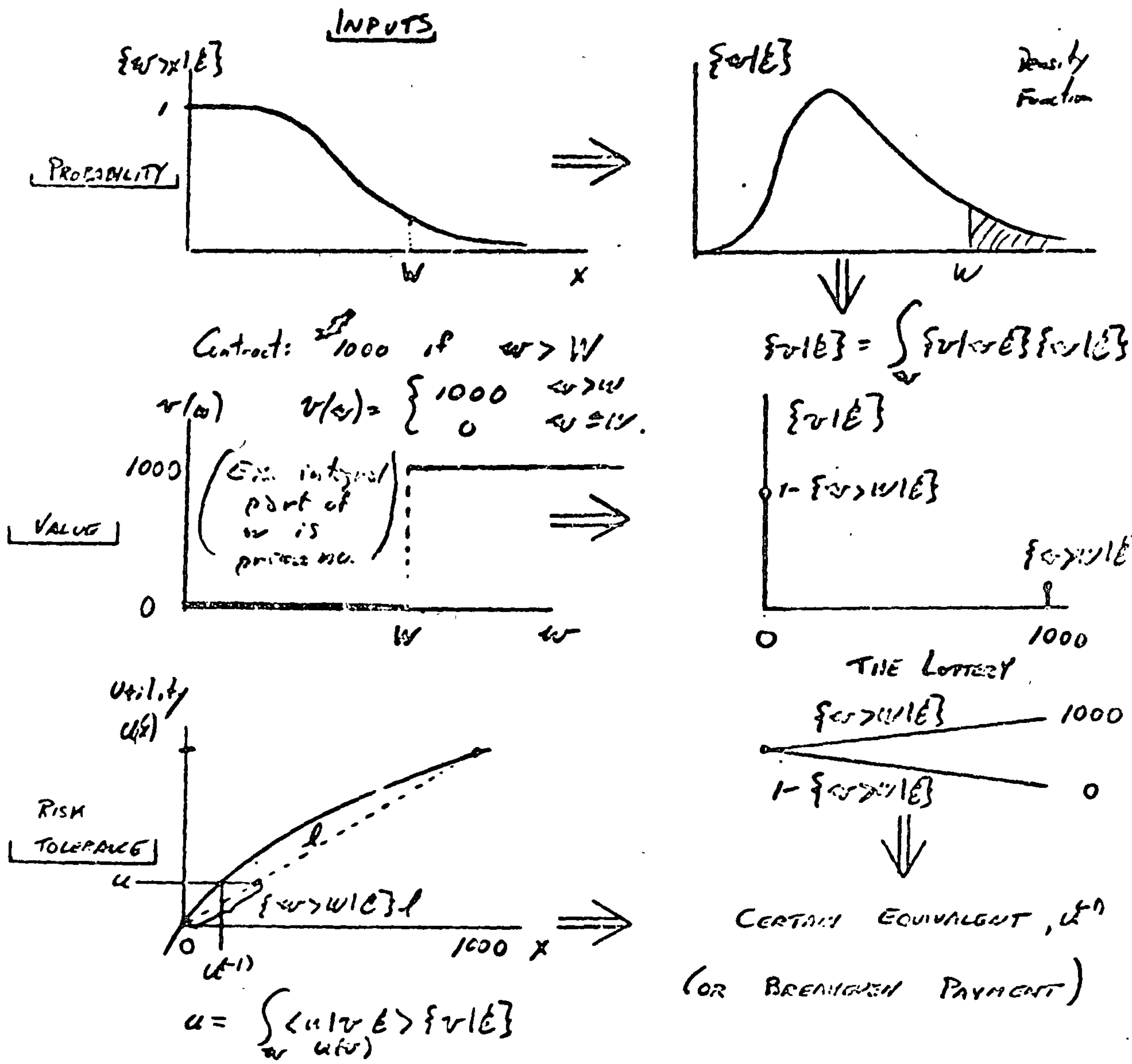


CERTAIN EQUIVALENT
RISK SENSITIVITY PROFILE
⟨ ⟩=38
v⟨ ⟩=1456
0.3
0.2
0.5
80
70
0
⟨ ⟩=36
v⟨ ⟩=504
0.1
0.3
0.6
90
50
20


RISK SENSITIVITY OF THE EXECUTIVE'S LOTTERIES
CERTAIN EQUIVALENT
300
248
200
100
0
-100
-200
-300
-400
-500
L2
L1
10-4
10-3
2·10-3
3·10-3
4·10-3
5·10-3
RISK AVERSION COEFFICIENT [(K$)-1]
10^4
1000
500
333
250
200
RISK TOLERANCE [K$]
2,3x10-3
435K

The Projector Story

w = Weight of Projector (pounds)
$v(w)$ = Contract payoff given w (dollars)
$u(x)$ = Utility of x dollars



RISK TOLERANCE

DO YOU PREFER L_1 OR L_2?

L_1: Certainty of receiving $1 million

L_2: { 10 chances in 100 of receiving $5 million
89 chances in 100 of receiving $1 million
1 chance in 100 of receiving $0 }

DO YOU PREFER L_3 OR L_4?

L_3: { 10 chances in 100 of receiving $5 million
90 chances in 100 of receiving $0 }

L_4: { 11 chances in 100 of receiving $1 million
89 chances in 100 of receiving $0 }

Risk Tolerance

$$L_1 = [1, 1] \qquad L_2 = [0.10, 5;\ 0.89, 1;\ 0.01, 0]$$

(payoffs in $ millions)

$$L_1 \succ L_2 \Rightarrow u(L_1) > u(L_2)$$

$$u(1) > 0.10\,u(5) + 0.89\,u(1) + 0.01\,u(0)$$

$$0.11\,u(1) > 0.10\,u(5) + 0.01\,u(0)$$

$$L_3 = [0.10, 5;\ 0.90, 0] \qquad L_4 = [0.11, 1;\ 0.89, 0]$$

$$L_3 \succ L_4 \Rightarrow u(L_3) > u(L_4)$$

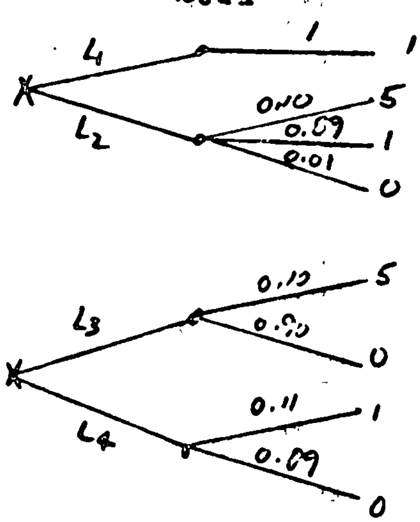
$$0.10\,u(5) + 0.90\,u(0) > 0.11\,u(1) + 0.89\,u(0)$$

$$0.10\,u(5) + 0.01\,u(0) > 0.11\,u(1)$$

<u>A Contradiction</u> "Allais Problem"

Other Representations

Trees



Matrix (Savage)

	Ticket Number 1	2-11	12-100
✓ L_1	1	1	1
L_2	0	5	1
✓ L_4	1	1	0
L_3	0	5	0

(a / b marked between columns 1 and 2-11 for each pair)

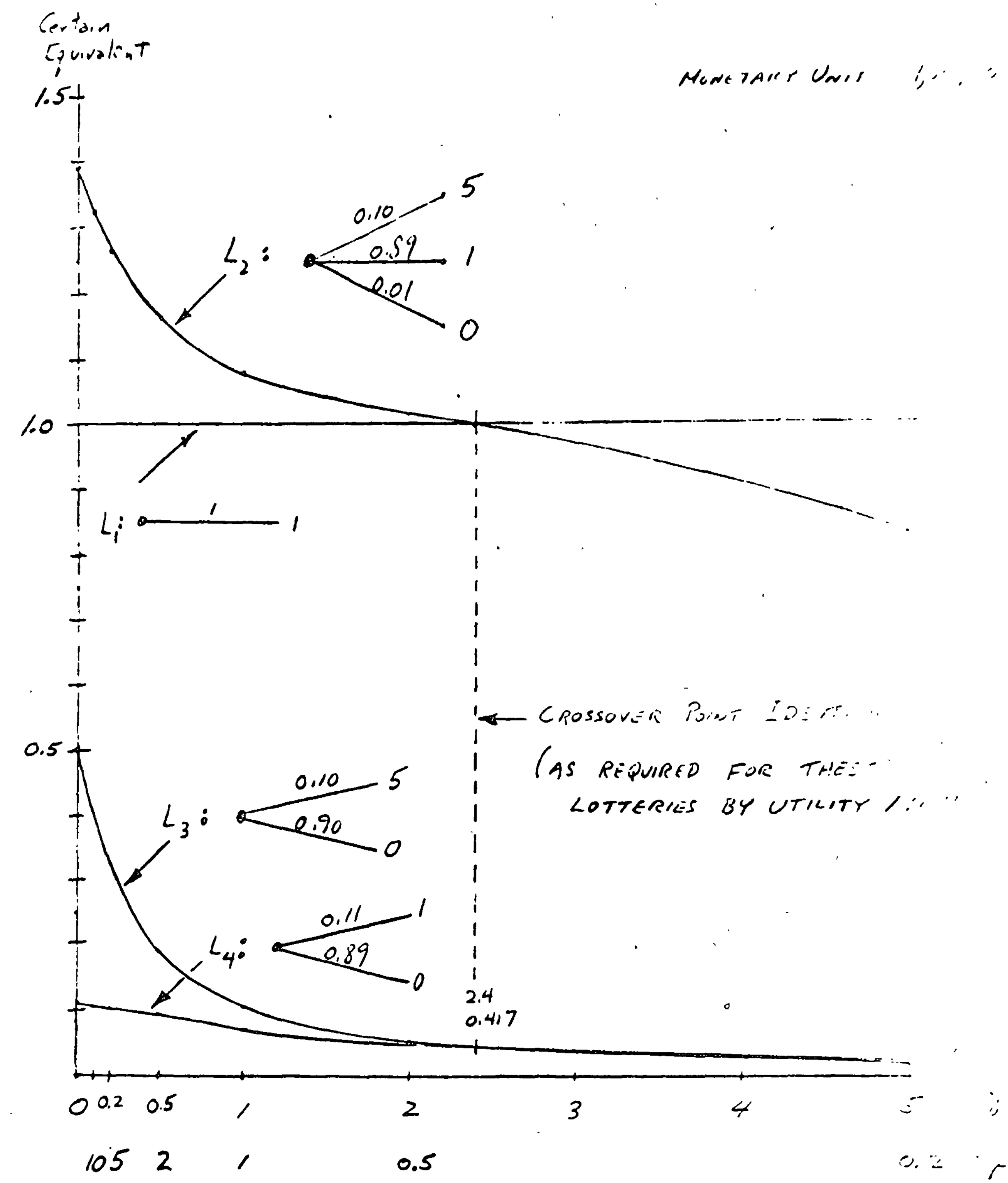
12-100 don't care

1-11

a

"certain lotteries are special"

"tough luck" ticket



[In above terms, ρ = 0.01, γ = 100]

THE CERTAIN EQUIVALENTS ARE

$\tilde{L}_1$ = $1,000,000 $\tilde{L}_2$ = $46,052 $\tilde{L}_3$ = $1054 $\tilde{L}_4$ = [illegible]

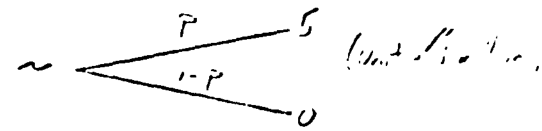
Risk Attitude

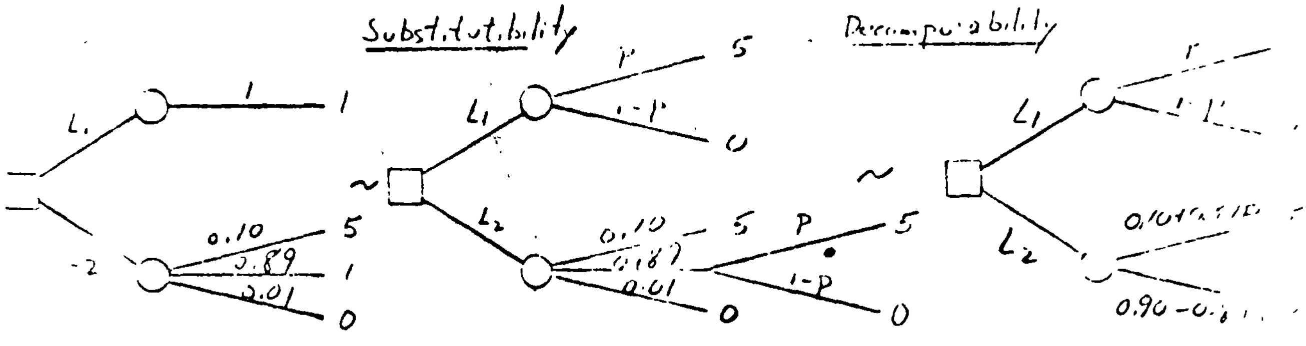
Preferences $L_1 \succ L_2$, $L_3 \succ L_4$ are inconsistent with risk aversion

Proof:

By orderability $5 \succ 1 \succ 0$

By continuity, for some p, $1 \sim$

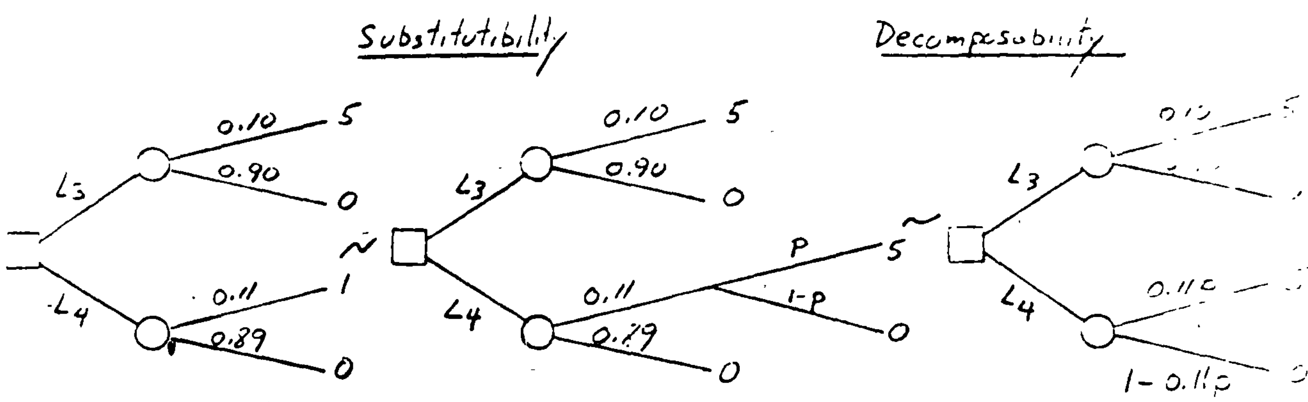




By monotonicity, $L_1 \succ L_2 \Rightarrow \quad p > 0.10 + 0.89p$

$$0.11p > 0.10$$

$$p > 10/11$$



By monotonicity, $L_3 \succ L_4 \Rightarrow \quad 0.10 > 0.11p$

$$p < 10/11 \qquad \text{contradiction}$$

i.e., if $p > 10/11$, monotonicity is violated

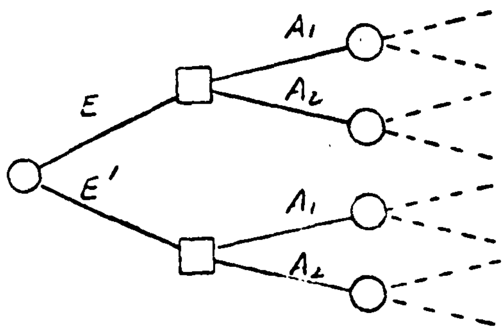
The only consistent preferences: $L_1 \succ L_2$, $L_4 \succ L_3$

$L_2 \succ L_1$, $L_3 \succ L_4$

$L_1 \sim L_2$, $L_3 \sim L_4$

Risk Attitude

Sure-Thing Principle (Savage)



If $[A_1 \succ A_2 \mid E]$ and $[A_1 \succ A_2 \mid E']$

then $A_1 \succ A_2$

Proof from Risk Axioms:

1) $[A_1 \succ A_2 \mid E] \Rightarrow \quad \langle u \mid A_1 E \mathcal{E} \rangle > \langle u \mid A_2 E \mathcal{E} \rangle$

2) $[A_1 \succ A_2 \mid E'] \Rightarrow \quad \langle u \mid A_1 E' \mathcal{E} \rangle > \langle u \mid A_2 E' \mathcal{E} \rangle$

By expansion $\langle u \mid A_i \mathcal{E} \rangle = \langle u \mid A_i E \mathcal{E} \rangle \{E \mid A_i \mathcal{E}\} + \langle u \mid A_i E' \mathcal{E} \rangle \{E' \mid A_i \mathcal{E}\}$

$= \langle u \mid A_i E \mathcal{E} \rangle \{E \mid \mathcal{E}\} + \langle u \mid A_i E' \mathcal{E} \rangle \{E' \mid \mathcal{E}\}$

Multiply 1) by $\{E \mid \mathcal{E}\}$, 2) by $\{E' \mid \mathcal{E}\}$, and add to obtain

$$\langle u \mid A_1 \mathcal{E} \rangle > \langle u \mid A_2 \mathcal{E} \rangle$$

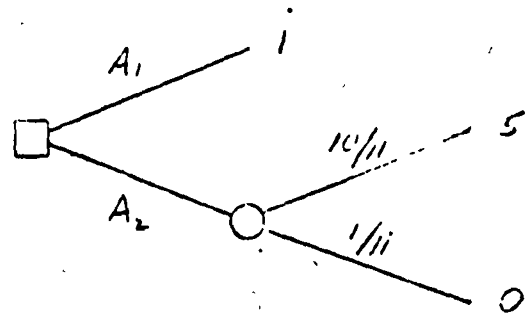
Therefore, $\quad A_1 \succ A_2$

Other implications are obvious, e.g. if $[A_1 \succ A_2 \mid E]$ and $[A_1 \sim A_2 \mid E']$ then $A_1 \succ A_2$

RISK ATTITUDE

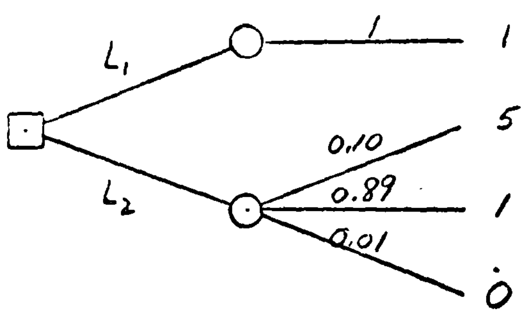
ALLAIS

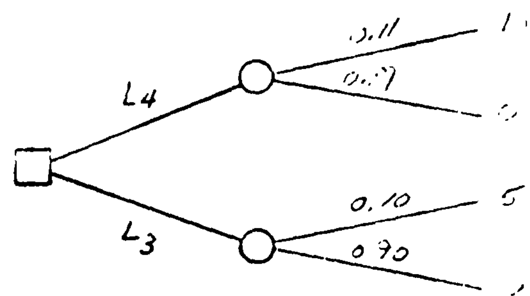
Consider



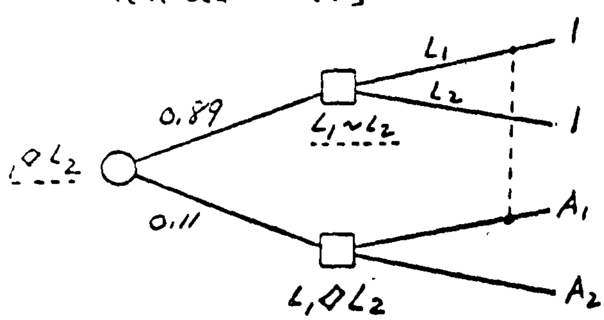
Suppose $A_1 \diamond A_2$ where $\diamond$ is one of $[\succ, \sim \prec]$

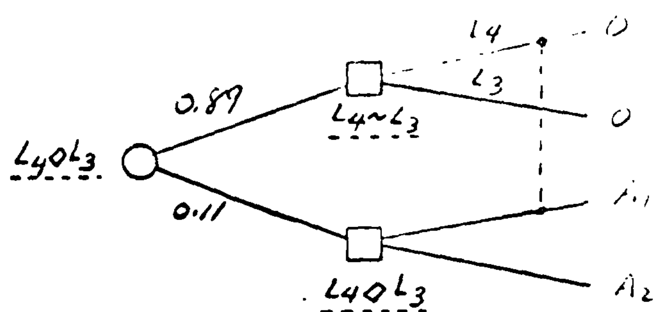
Original Trees for L_1, L_2 for L_3, L_4





Reversed Trees





Therefore, $L_1 \diamond L_2$ IF AND ONLY IF $L_4 \diamond L_3$

MODEL SEQUENCE

PILOT PHASE

PURPOSE: TO UNDERSTAND AND ESTABLISH EFFECTIVE COMMUNICATION REGARDING THE NATURE OF THE DECISIONS AND THE MAJOR ISSUES AND UNCERTAINTIES SURROUNDING THEM.

CONTENT:

- SIMPLIFIED DECISION MODEL
- TENTATIVE PREFERENCE STRUCTURE
- ROUGH CHARACTERIZATION OF UNCERTAINTY

RESULTS:

- PRELIMINARY RECOMMENDATIONS
- GUIDANCE IN CONSTRUCTING FULL-SCALE MODEL

FULL-SCALE MODEL

PURPOSE: TO DETERMINE THE MOST DESIRABLE STRATEGIES GIVEN THE AVAILABLE ALTERNATIVES, INFORMATION AND PREFERENCES.

CONTENT:

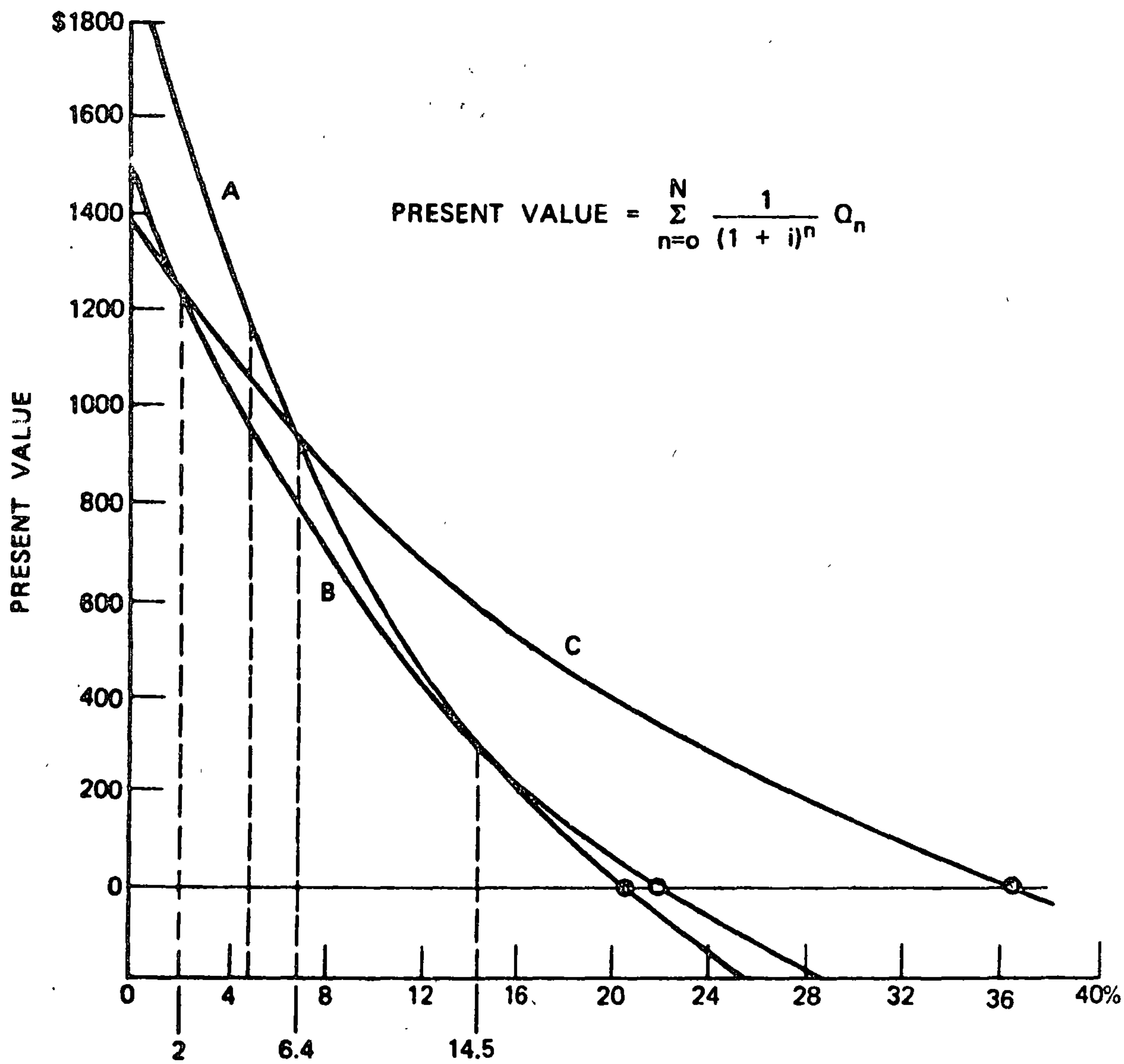
- BALANCED, REALISTIC DECISION MODEL
- CERTIFIED PREFERENCES
- CAREFUL REPRESENTATION OF IMPORTANT UNCERTAINTIES

RESULT:

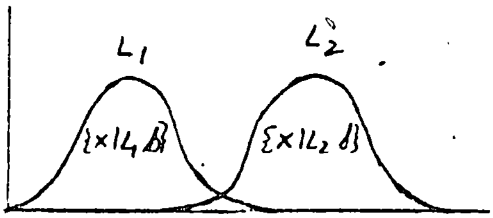
- DECISION RECOMMENDATIONS

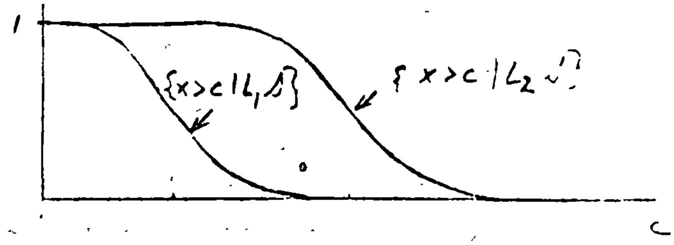
CASH FLOWS FOR INVESTMENTS A, B, AND C

INVESTMENT	Q_0	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}
A	-1000	200	200	200	200	200	200	200	200	200	1200
B	-1000	250	250	250	250	250	250	250	250	250	250
C	-1000	500	500	500	300	200	150	100	50	50	50

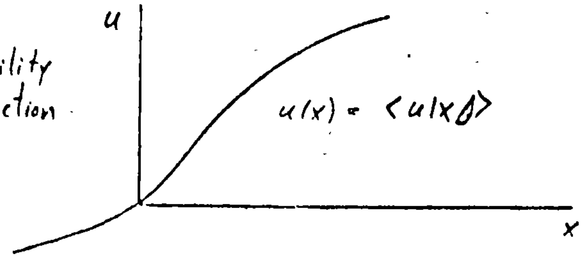


Stochastic Dominance : Proof





Utility Function



$u(x)$ Monotonically non-decreasing:
$u(c_2) \geq u(c_1)$
if $c_2 > c_1$
and $u'(x) \geq 0$ for all x

For any L, $\langle u|L\mathcal{S}\rangle = \int_x \langle u|Lx\mathcal{S}\rangle \{x|L\mathcal{S}\} = \int_x \langle u|x\mathcal{S}\rangle \{x|L\mathcal{S}\}$

To Prove: If $\{x>c|L_2\mathcal{S}\} \geq \{x>c|L_1\mathcal{S}\}$ for all c, Then $\langle u|L_2\mathcal{S}\rangle \geq \langle u|L_1\mathcal{S}\rangle$

Proof: $\langle u|L_2\mathcal{S}\rangle \geq \langle u|L_1\mathcal{S}\rangle$ If $\int_x \langle u|x\mathcal{S}\rangle\{x|L_2\mathcal{S}\} \geq \int_x \langle u|x\mathcal{S}\rangle\{x|L_1\mathcal{S}\}$

or if $d = \int_x \langle u|x\mathcal{S}\rangle\left(\{x|L_2\mathcal{S}\} - \{x|L_1\mathcal{S}\}\right) \geq 0$

Let $(i=1,2)$ $f_i(\cdot) = \{x|L_i\mathcal{S}\}$; ${}^{\leq}P_i(c) = \int_{-\infty}^{x} dx\, f_i(x)$; ${}^{>}P_i(c) = 1 - {}^{\leq}P_i(c)$

Hypothesis implies ${}^{>}P_2(c) \geq {}^{>}P_1(c)$ for all c

$$d = \int_x \langle u|x\mathcal{S}\rangle\left(\{x|L_2\mathcal{S}\} - \{x|L_1\mathcal{S}\}\right) = \int_{-\infty}^{\infty} dx\, u(x)\left[f_2(x) - f_1(x)\right]$$

$$\overset{\text{by parts}}{=} \underbrace{u(x)\left[{}^{\leq}P_2(x) - {}^{\leq}P_1(x)\right]\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} dx\left[{}^{\leq}P_2(x) - \text{[illegible]}\right]$$

$$d = \int_{-\infty}^{\infty} dx\left[{}^{\leq}P_1(x) - {}^{\leq}P_2(x)\right]u'(x) = \int_{-\infty}^{\infty} dx\left[(1-{}^{>}P_1(x)) - (1-{}^{>}P_2(x))\right]u'(x) = \int_{-\infty}^{\infty} dx\left[{}^{>}P_2(x) - {}^{>}P_1(x)\right]u'(x)$$

≥ 0 by hypothesis [illegible]

$\therefore d \geq 0$ Q.E.D.

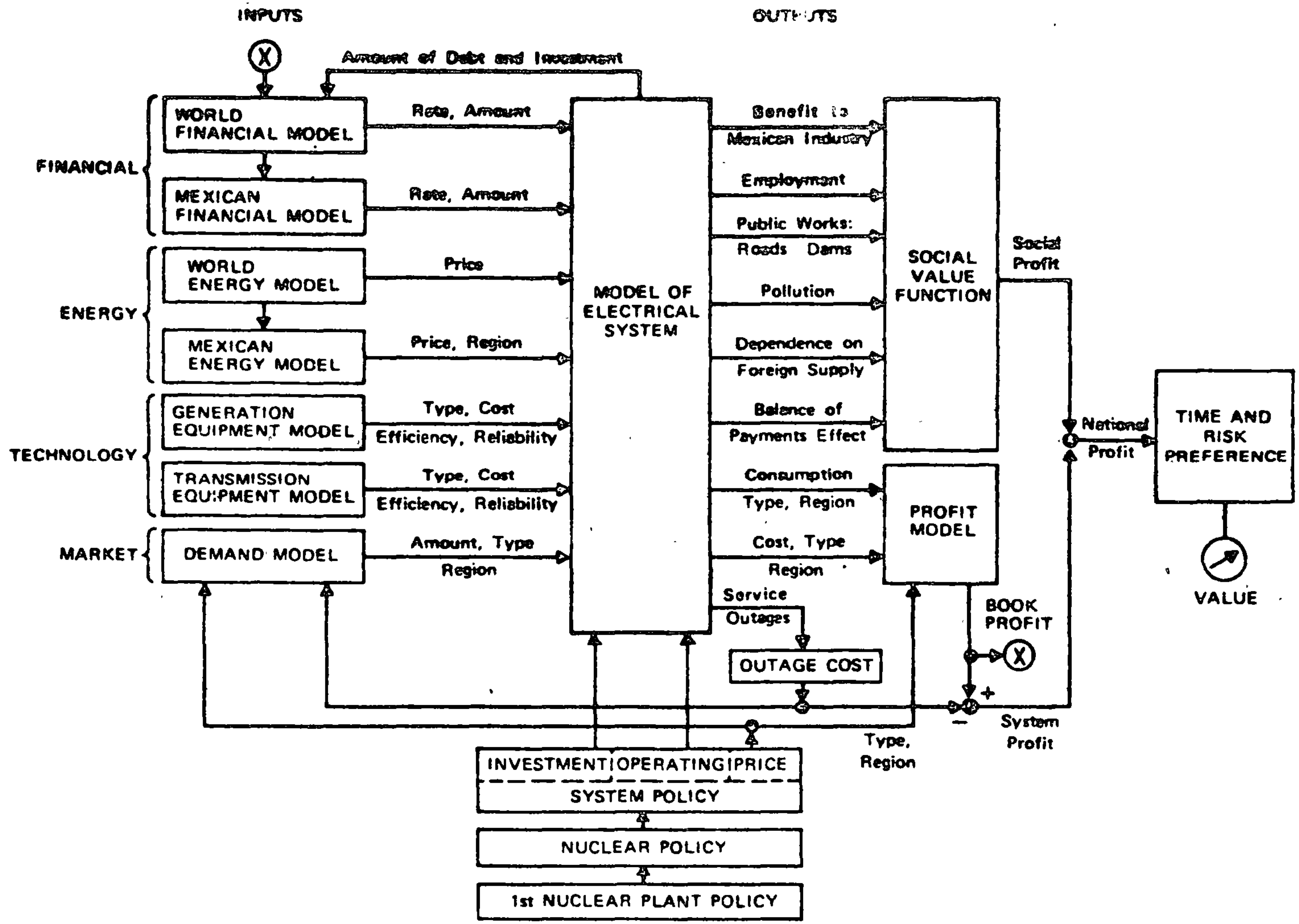


FIG. 1 A DECISION ANALYSIS MODEL OF THE ELECTRICAL SYSTEM

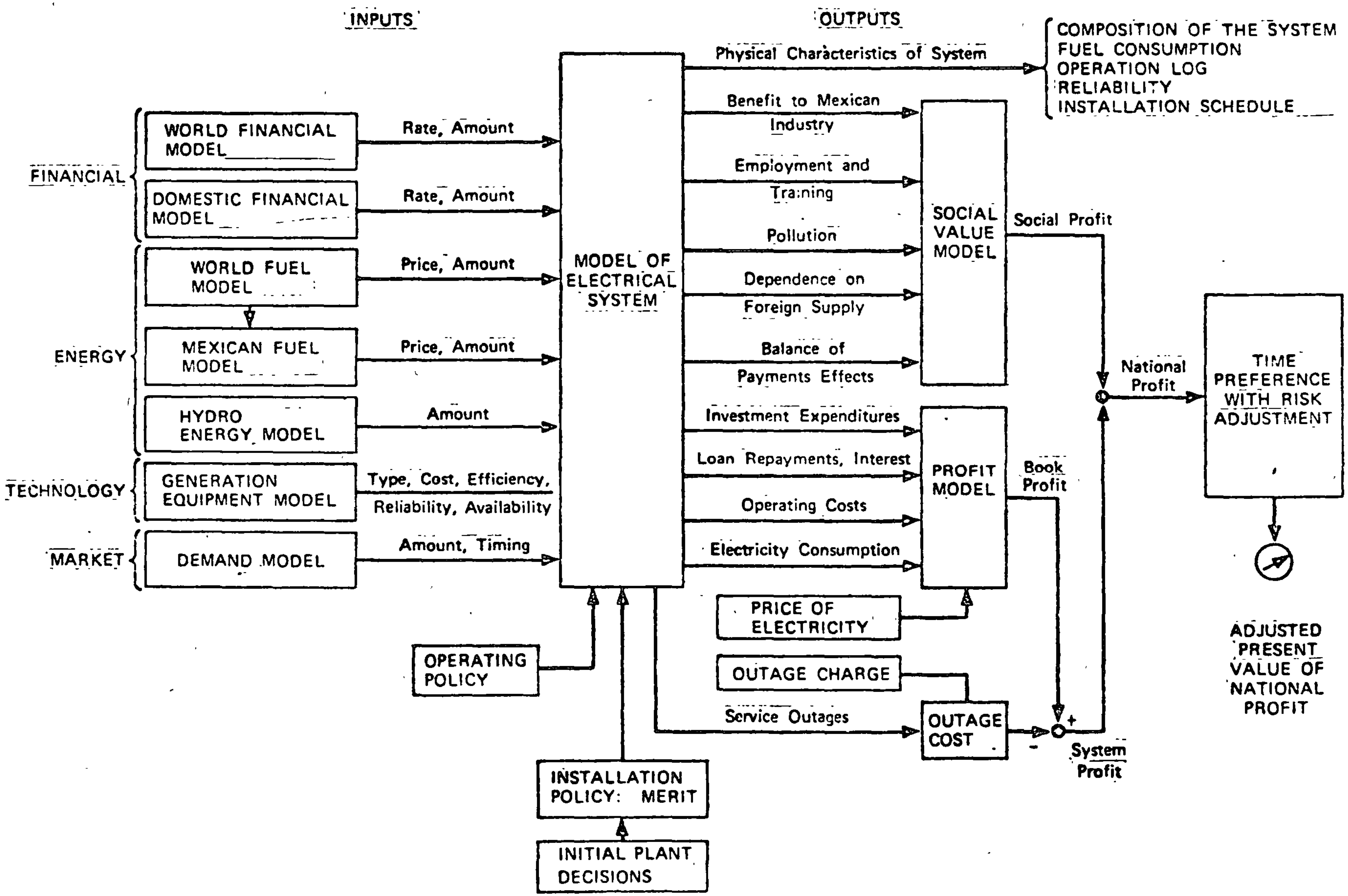
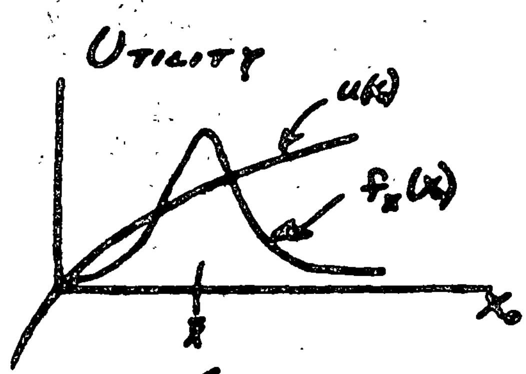


FIG. 2 THE IMPLEMENTED VERSION OF THE MODEL



$$u(\tilde{x}) = \int dx_0 \, u(x_0) f_x(x_0)$$

Expand $u(\cdot)$ about $\bar{x}$

$$u(x_0) \approx u(\bar{x}) + (x_0 - \bar{x}) u'(\bar{x}) + \tfrac{1}{2}(x_0 - \bar{x})^2 u''(\bar{x})$$

$$u(\tilde{x}) = \int dx_0 \, u(x_0) f_x(x_0) \approx u(\bar{x}) + 0 + \tfrac{1}{2} \check{x} \, u''(\bar{x})$$

$$u(\tilde{x}) \approx u(\bar{x}) + \tfrac{1}{2} \check{x} \, u''(\bar{x})$$

From Expansion, $u(\tilde{x}) \approx u(\bar{x}) + (\tilde{x} - \bar{x}) u'(\bar{x})$

∴ Therefore

$$\tilde{x} - \bar{x} \approx \tfrac{1}{2} \check{x} \frac{u''(\bar{x})}{u'(\bar{x})}$$

$$\tilde{x} \approx \bar{x} - \tfrac{1}{2} \check{x} \left(- \frac{u''(\bar{x})}{u'(\bar{x})} \right)$$

Define Risk Aversion Coefficient $r(x) = -\dfrac{u''(x)}{u'(x)}$

Invariant to Linear Transformation

then $\tilde{x} \approx \bar{x} - \tfrac{1}{2} \check{x} \cdot r(\bar{x})$

For constant Risk Aversion, $r(x) = -\dfrac{u''(x)}{u'(x)} = \gamma$

$$-\ln u'(x) = \gamma x + c$$

$$\tilde{x} \approx \bar{x} - \tfrac{1}{2} \gamma \check{x}$$

$$u'(x) = e^{-(\gamma x + c)}$$

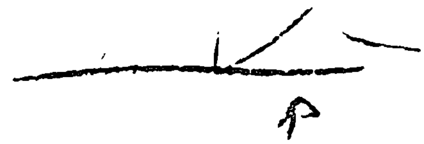
$$u(x) = a + b e^{-\gamma x}$$

EXPONENTIAL

Utility

Exponential

$$u(x) = \frac{1-e^{-\gamma x}}{1-e^{-\gamma}}$$



Consider a lottery v described by $f_v(\cdot)$

$$u_v = \int_{-\infty}^{\infty} dv_0 \, u(v_0) \, f_v(v_0)$$

$$= \int_{-\infty}^{\infty} dv_0 \, \frac{1-e^{-\gamma v_0}}{1-e^{-\gamma}} \, f_v(v_0)$$

$$= \frac{1}{1-e^{-\gamma}} \left[\int_{-\infty}^{\infty} dv_0 \, f_v(v_0) - \int_{-\infty}^{\infty} dv_0 \, e^{-\gamma v_0} f_v(v_0) \right]$$

$$u_v = \frac{1}{1-e^{-\gamma}} \left[1 - \int_{-\infty}^{\infty} dv_0 \, e^{-\gamma v_0} f_v(v_0) \right]$$

But

$$f_v^e(s) = \int_{-\infty}^{\infty} dv_0 \, e^{-s v_0} f_v(v_0) = \overline{e^{-sv}}$$

$$\therefore \quad u_v = \frac{1}{1-e^{-\gamma}} \left[1 - f_v^e(\gamma) \right]$$

* If it exists

Now

$$u(\tilde{v}) = \frac{1-e^{-\gamma \tilde{v}}}{1-e^{-\gamma}} = u_v$$

$$\therefore \quad e^{-\gamma \tilde{v}} = f_v^e(\gamma)$$

$$\boxed{\tilde{v} = -\frac{1}{\gamma} \ln f_v^e(\gamma)}$$

As $\gamma \to 0$

$$\tilde{v} = \lim_{\gamma \to 0} \frac{-\ln f_v^e(\gamma)}{\gamma} = \lim_{\gamma \to 0} - \frac{\frac{f_v^{e\prime}(\gamma)}{f_v^e(\gamma)}}{1}$$

$$= \lim_{\gamma \to 0} -f_v^{e\prime}(\gamma) = \bar{v}$$

UTILITY

EXPONENTIAL

Consider two <u>independent</u> lotteries v_1, v_2

$$\tilde{v}_1 = -\frac{1}{\gamma} \ln f_{v_1}^{e}(\gamma)$$

$$\tilde{v}_2 = -\frac{1}{\gamma} \ln f_{v_2}^{e}(\gamma)$$

Let $v = v_1 + v_2$

$$\tilde{v} = -\frac{1}{\gamma} \ln f_{v}^{e}(\gamma)$$

$$f_v^{e}(s) = f_{v_1}^{e}(s) f_{v_2}^{e}(s)$$

$$\tilde{v} = -\frac{1}{\gamma} \ln \left[f_{v_1}^{e}(\gamma) f_{v_2}^{e}(\gamma) \right]$$

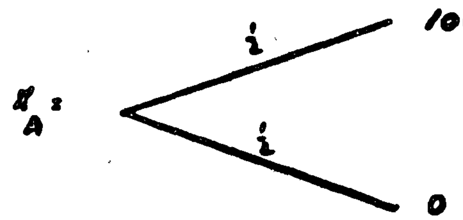
$$= -\frac{1}{\gamma} \ln f_{v_1}^{e}(\gamma) - \frac{1}{\gamma} \ln f_{v_2}^{e}(\gamma)$$

$$\tilde{v} = \tilde{v}_1 + \tilde{v}_2$$

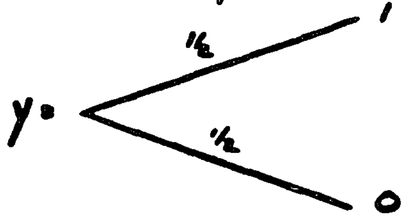
UTILITY

CONSIDER TWO LOTTERIES

A: PLAYING ONE LOTTERY $(1/2, \$0;\ 1/2, \$100)$

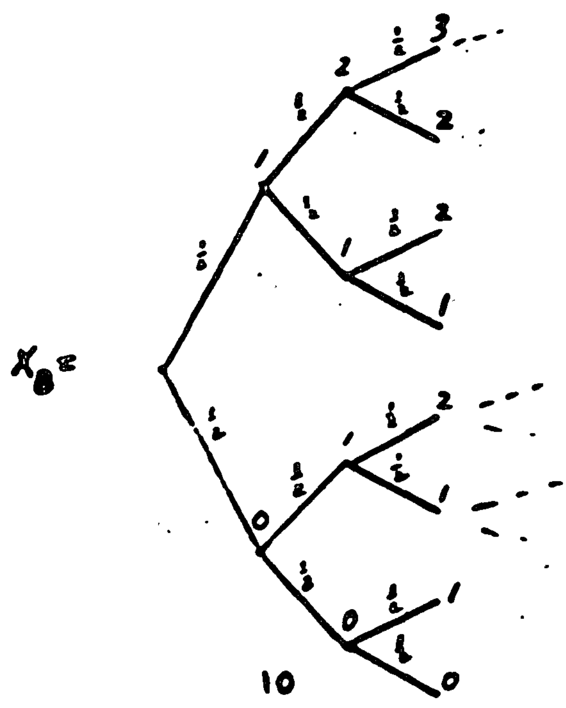


B: PLAYING ONE HUNDRED TIMES THE LOTTERY $y = (1/2, \$0;\ 1/2, \$1)$



$\bar{y} = 0.5$

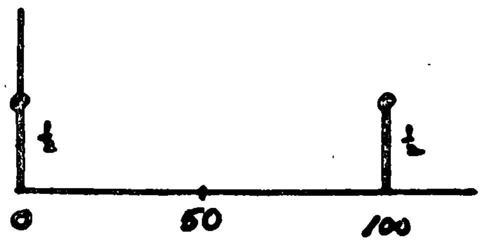
$\check{y} = 0.5$



TOTAL	PROB
100	$(\frac{1}{2})^{100}$
99	$100(\frac{1}{2})^{100}$
98	$\frac{100(99)}{2}(\frac{1}{2})^{100}$
50	$\frac{100!}{50!\,50!}(\frac{1}{2})^{100}$
2	$\frac{100(99)}{2}(\frac{1}{2})^{100}$
1	$100(\frac{1}{2})^{100}$
0	$(\frac{1}{2})^{100}$

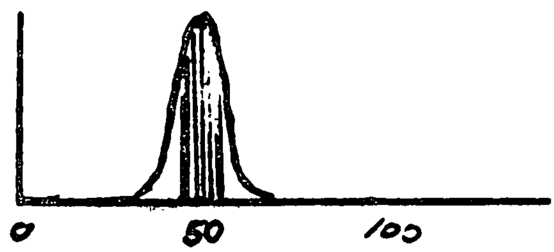
Utility

X_A :



$$\begin{cases} \bar{X}_A = 50 \\ \overset{s}{X}_A = 50 \\ \check{X}_A = 2500 \end{cases}$$

X_B :



$$\begin{cases} \bar{X}_B = 50 \\ \overset{s}{X}_B = 5 \\ \check{X}_B = 25 \end{cases}$$

FOR CONSTANT RISK AVERTER $\tilde{X}_B = 100\,\tilde{Y}$

EXAMPLE: $\gamma = 0.01386$

$$\tilde{X}_A = 34 \approx \bar{X}_A - \tfrac{1}{2}\gamma\check{X}_A = 50 - \tfrac{1}{2}(0.01386)\,2500$$
$$= 50 - 17.33$$
$$= \underline{32.67}$$

$$\tilde{X}_B = 100\,\tilde{Y} \approx 100(\bar{Y} - \tfrac{1}{2}\gamma\check{Y}) = 100(0.5 - \tfrac{1}{2}(0.01386)\,0.25)$$
$$\left[\text{or } = \bar{X}_B - \tfrac{1}{2}\gamma\check{X}_B = 50 - \tfrac{1}{2}(0.01386)\,25\right] = 50 - 0.1733$$
$$= \underline{49.83}$$

Cost of Risk Bearing = Risk Premium $= \bar{X} - \tilde{X} \approx \tfrac{1}{2}\gamma\check{X}$

If n players with same γ share risk equally, $\check{X}$ and hence the risk premium are approximately divided by n.

	A	B
Risk Premium	17.33	0.1733

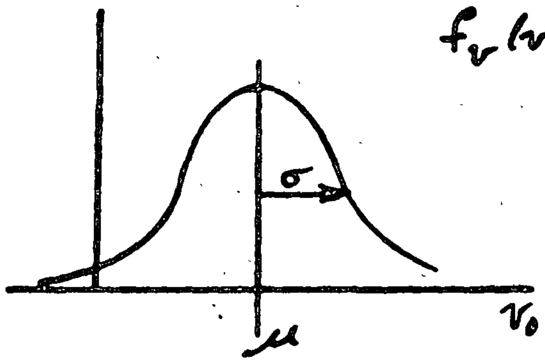
UTILITY

CERTAINTY EQUIVALENT OF NORMAL LOTTERY

UTILITY: $u(x) = \dfrac{1-e^{-\gamma x}}{1-e^{-\gamma}}$ LOTTERY, $v \sim f_v(\cdot)$
EXP. TRANSFORM: $f_v^e(s)$

CERTAINTY EQUIVALENT: $\tilde{v} = -\frac{1}{\gamma} \ln f_v^e(\gamma)$

CONSIDER NORMAL v

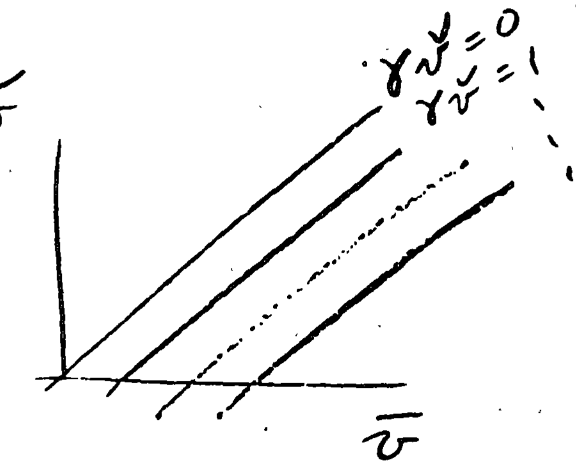


$$f_v(v_0) = f_n(v_0|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(v_0-\mu)^2}{2\sigma^2}} \quad -\infty < v_0 < \infty$$

$\bar{v} = \mu$

$\overline{v^2} = \sigma^2 + \mu^2$ $f_v^e(s) = e^{-\mu s + \frac{\sigma^2 s^2}{2}}$

$\check{v} = \sigma^2$



$$\tilde{v} = -\frac{1}{\gamma} \ln f_v^e(\gamma)$$

$$= -\frac{1}{\gamma} \ln e^{-\mu\gamma + \frac{\sigma^2\gamma^2}{2}}$$

$$= -\frac{1}{\gamma}\left[-\mu\gamma + \frac{\sigma^2\gamma^2}{2}\right]$$

$$= \mu - \frac{1}{2}\sigma^2\gamma$$

$$= \bar{v} - \frac{1}{2}\check{v}\gamma \quad \text{, EXACTLY}$$

$$\frac{\tilde{v}}{\bar{v}} = 1 - \frac{1}{2}\frac{\check{v}}{\bar{v}}\gamma$$

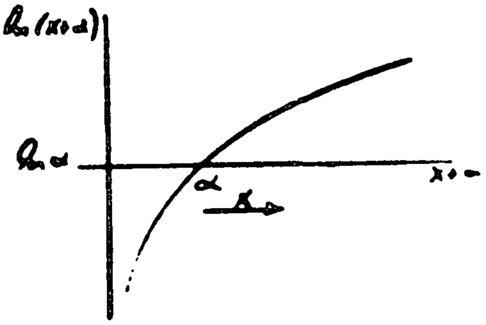
UTILITY

LOGARITHMIC

Let y = total worth ≥ 0 for everyone (e.g.) includes, e.g., right to earn money

$$du = a \cdot \frac{dy}{y}$$ Equity of % raises

$u = a \ln y + b$, or $\ln y$ by linear transformation



Let α = total worth before lottery
x = lottery

Then $u(x) = \ln(x+\alpha)$

RISK AVERSION COEFFICIENT

$$u'(x) = \frac{1}{x+\alpha} \qquad u''(x) = \frac{-1}{(x+\alpha)^2}$$

$$r(x) = \frac{-u''(x)}{u'(x)} = \frac{1}{x+\alpha}$$

RISK AVERSION DECREASES WITH CAPITAL

CERTAINTY EQUIVALENT

$$u(\tilde{x}) = \langle u(x) \rangle$$

$$\ln(\tilde{x}+\alpha) = \overline{\ln(x+\alpha)}$$

$$\tilde{x}+\alpha = e^{\overline{\ln(x+\alpha)}}$$

$$\tilde{x} = e^{\overline{\ln(x+\alpha)}} - \alpha$$

$$\overline{\ln(x+\alpha)} = \int_{-\alpha}^{\infty} dx_0\, f_x(x_0) \ln(x_0+\alpha)$$

UTILITY

LOGARITHMIC

IF x IS discrete , $p_i = p(x = x_i)$

then
$$\overline{\ln(x+\alpha)} = \sum_i p_i \ln(x_i+\alpha)$$
$$= \sum_i \ln(x_i+\alpha)^{p_i}$$

$$e^{\overline{\ln(x+\alpha)}} = e^{\sum_i \ln(x_i+\alpha)^{p_i}}$$
$$= \prod_i e^{\ln(x_i+\alpha)^{p_i}}$$
$$= \prod_i (x_i+\alpha)^{p_i}$$

$$\tilde{x}+\alpha = e^{\overline{\ln(x+\alpha)}} = \prod_i (x_i+\alpha)^{p_i} \quad \Leftarrow$$
$$\tilde{x} = \prod_i (x_i+\alpha)^{p_i} - \alpha$$

$\therefore$ $\tilde{x}+\alpha$ is geometric mean of absolute payments
TO AN INDIVIDUAL WITH A LOGARITHMIC UTILITY CURVE
LOTTERIES ARE EQUAL IF THEIR GEOMETRIC MEANS ARE EQUAL

$\bar{x}+\alpha$ is arithmetic mean of absolute payments

Since the geometric mean is always less than or equal to the arithmetic mean

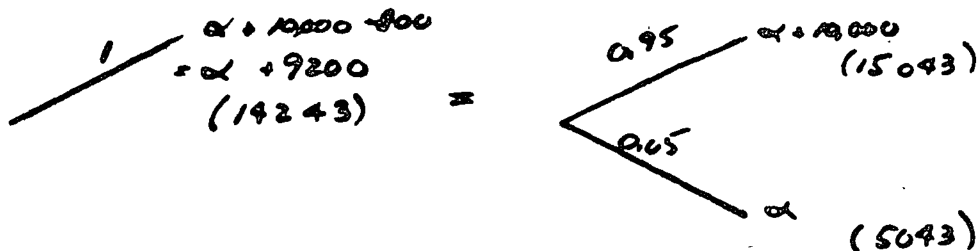
$$\tilde{x}+\alpha \le \bar{x}+\alpha$$
$$\tilde{x} \le \bar{x}$$

LOGARITHMIC

INSURANCE

IF AN INDIVIDUAL HAS A PROPERTY WORTH 10,000 THAT WILL BE LOST WITH PROBABILITY 0.05 AND THE PREMIUM FOR INSURANCE IS 800, HOW MUCH MUST HIS OTHER WORTH, α, BE FOR HIM TO BE SELF-INSURING?

WITH INSURANCE — WITHOUT INSURANCE

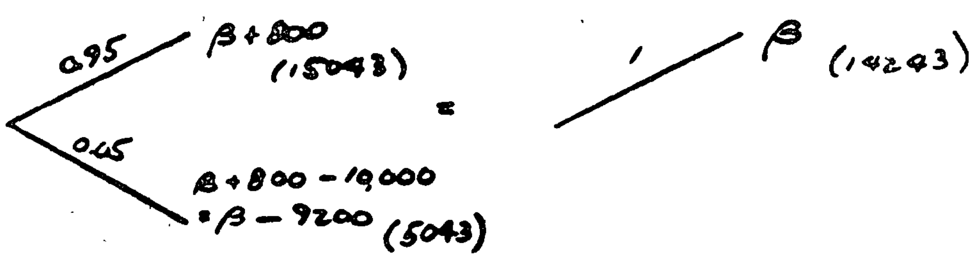


$$\alpha + 9200 = (\alpha + 10{,}000)^{0.95}\,\alpha^{0.05}$$

$$\alpha = 5043$$

WHAT MUST BE THE WORTH β OF THE INSURER?

WITH INSURANCE — WITHOUT INSURANCE



$$(\beta + 800)^{0.95}(\beta - 9200)^{0.05} = \beta$$

OBVIOUSLY, $\beta = \alpha + 9200 = 14243$

INSURER AND INSURED FACE SAME SET OF LOTTERIES

LOGARITHMIC

INSURANCE (CONT.)

IF THE PREMIUM WERE 600, THEN

$\alpha = 20478 \qquad \beta = 29878$

NO ONE WOULD SELL THE INSURANCE FOR LESS THAN 500, SINCE THIS IS THE EXPECTED LOSS

DIVISION OF RISK

AN INDIVIDUAL HAS GOODS WORTH 4000 IN HIS OWN COUNTRY AND 8000 ABROAD. SHIPS WITH PROBABILITY 0.1 OF PERISHING CAN BRING ALL OR ANY PART OF HIS FOREIGN GOODS HOME. WHAT ARE HIS FOREIGN HOLDINGS WORTH?

1) IF HE USES 1 SHIP

0.9 → 12000; 0.1 → 4000 $=$ 1 → $4000 + \tilde{x}$

$$(12000)^{0.9}(4000)^{0.1} = 4000 + \tilde{x}$$

$$\tilde{x} = 6751$$

2) IF HE USES 2 SHIPS

0.81 → 12000; 0.18 → 8000; 0.01 → 4000 $=$ 1 → $4000 + \tilde{x}$

$$(12000)^{0.81}(8000)^{0.18}(4000)^{0.01} = 4000 + \tilde{x}$$

$$\tilde{x} = 7033$$

IN EITHER CASE, $\bar{x} = 7200$

UTILITY

EXPOSITION OF A NEW THEORY ON THE MEASUREMENT OF RISK

by DANIEL BERNOULLI

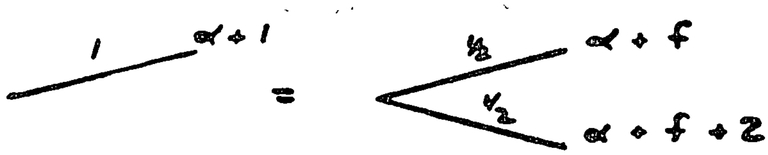
"Specimen Theoriae Novae de Mensura Sortis"
Commentarii Academiae Scientiarum Imperialis Petropolitanae, Tomus V [Papers of the Imperial Academy of Sciences in Petersburg, Vol. V], 1738, pp. 175-192

DANIEL BERNOULLI
FLUID MECHANICS
(1700-1782)

LOGARITHMIC

WHAT FRACTION f OF NEXT YEAR'S SALARY MUST AN INDIVIDUAL RECEIVE TO INDUCE HIM TO PLAY DOUBLE OR NOTHING WITH NEXT YEAR'S SALARY?

WORTH = α, MEASURED IN UNITS OF NEXT YEAR'S SALARY



$$\alpha+1 = (\alpha+f)^{1/2}(\alpha+f+2)^{1/2}$$

$$\alpha^2+2\alpha+1 = (\alpha+f)(\alpha+f+2)$$

$$\alpha^2+2\alpha+1 = \alpha^2+2f\alpha+2\alpha+f^2+2f$$

OR

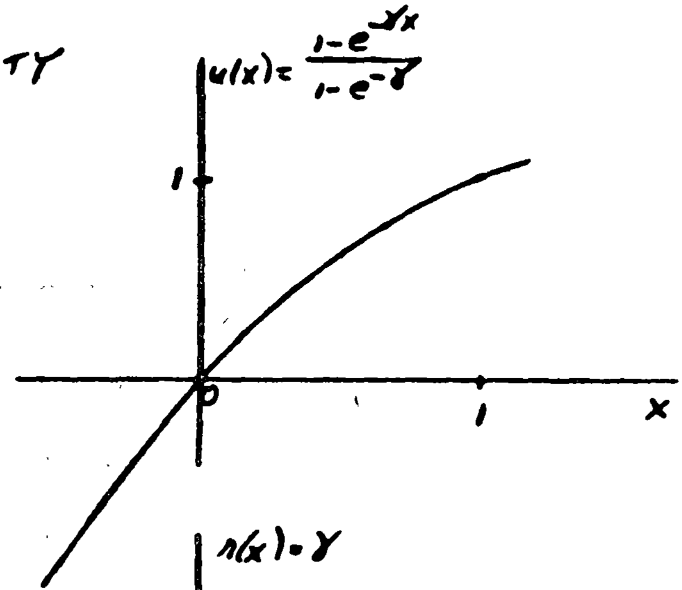
$$f^2+2(\alpha+1)f-1=0$$

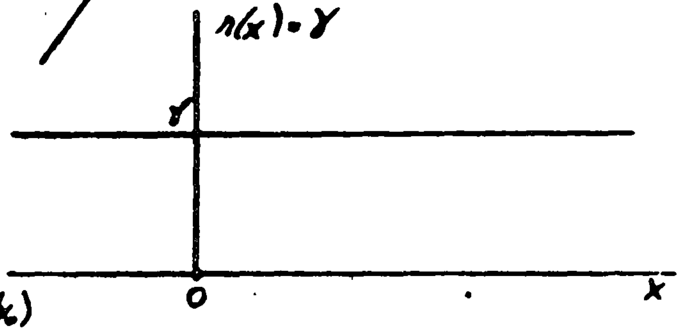
$$f = \tfrac{1}{2}\left[-2(\alpha+1) \pm \sqrt{4(\alpha+1)^2+4}\right]$$

$$= -(\alpha+1)+\sqrt{(\alpha+1)^2+1}$$ choose plus sign

α	f	α	f
−1	1.0	−0.9	0.905
0	$-1+\sqrt{2}=0.414$	−0.8	0.820
1	$-2+\sqrt{5}=0.236$	−0.7	0.744
2	$-3+\sqrt{10}=0.162$	−0.6	0.677
3	$-4+\sqrt{17}=0.123$	−0.5	0.618
4	$-5+\sqrt{26}=0.099$	−0.4	0.566
5	$-6+\sqrt{37}=0.083$	−0.3	0.521
		−0.2	0.481
		−0.1	0.445

UTILITY





EXPONENTIAL

$$u(x) = \frac{1-e^{-\gamma x}}{1-e^{-\gamma}}$$

$$u'(x) = \frac{\gamma e^{-\gamma x}}{1-e^{-\gamma}}$$

$$u''(x) = \frac{-\gamma^2 e^{-\gamma x}}{1-e^{-\gamma}}$$

$$r(x) = -\frac{u''(x)}{u'(x)} = \gamma$$

CERTAIN EQUIVALENT

$$u(\tilde{x}) = \int dx_0\, u(x_0) f_x(x_0) = \int dx_0\, \frac{1-e^{-\gamma x_0}}{1-e^{-\gamma}} f_x(x_0)$$

$$\frac{1-e^{-\gamma \tilde{x}}}{1-e^{-\gamma}} = \frac{1}{1-e^{-\gamma}}\left[\int dx_0\, f_x(x_0) - \int dx_0\, e^{-\gamma x_0} f_x(x_0)\right] = \frac{1}{1-e^{-\gamma}}\left[1 - f_x^e(\gamma)\right]$$

$$e^{-\gamma \tilde{x}} = f_x^e(\gamma)$$

$$\underline{\tilde{x} = -\frac{1}{\gamma} \ln f_x^e(\gamma)}$$

As $\check{x} \to 0$, $f_x(x_0) \to \delta(x_0 - \bar{x})$, $f_x^e(\gamma) \to e^{-\gamma \bar{x}}$, $\tilde{x} \to \bar{x}$

As $\gamma \to 0$, $f_x^e(\gamma) \to 1 + \gamma f_x^{e\prime}(0) + \frac{1}{2}\gamma^2 f_x^{e\prime\prime}(0) = 1 - \gamma \bar{x} + \frac{1}{2}\gamma^2 \overline{x^2}$,

$$\ln f_x^e(\gamma) \to \left(-\gamma \bar{x} + \tfrac{1}{2}\gamma^2 \overline{x^2}\right) - \tfrac{1}{2}\left(-\gamma \bar{x} + \tfrac{1}{2}\gamma^2 \overline{x^2}\right)^2$$

$$\ln f_x^e(\gamma) \to -\gamma \bar{x} + \tfrac{1}{2}\gamma^2\left(\overline{x^2} - \bar{x}^2\right) = -\gamma \bar{x} + \tfrac{1}{2}\gamma^2 \check{x}$$

$$\tilde{x} = -\frac{1}{\gamma} \ln f_x^e(\gamma) \to \bar{x} - \tfrac{1}{2}\gamma \check{x}$$

Utility

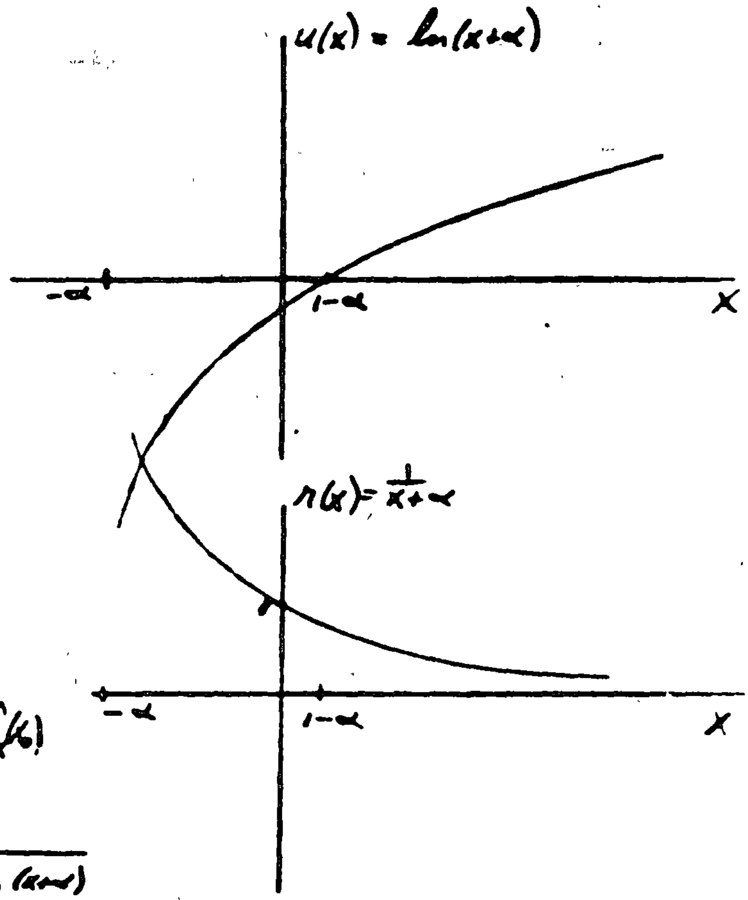
Logarithmic

$$u(x) = \ln(x+\alpha) \qquad -\alpha < x$$

$$u'(x) = \frac{1}{x+\alpha}$$

$$u''(x) = \frac{-1}{(x+\alpha)^2}$$

$$r(x) = \frac{-u''(x)}{u'(x)} = \frac{1}{x+\alpha}$$



Certain Equivalent

$$u(\tilde{x}) = \int dx_0\, u(x_0) f_x(x_0) = \int_{-\alpha}^{\infty} dx_0 \ln(x_0+\alpha) f_x(x_0)$$

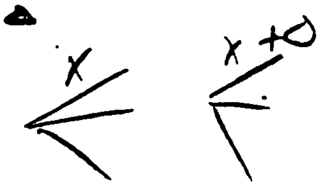
$$\ln(\tilde{x}+\alpha) = \overline{\ln(x+\alpha)}$$

$$\tilde{x}+\alpha = e^{-\overline{\ln(x+\alpha)}}$$

$$\tilde{x} = e^{-\overline{\ln(x+\alpha)}} - \alpha$$

EFFECT OF Δ ON CERTAIN EQUIVALENT

Consider lottery x. Increase payoffs by Δ to form lottery $x(\Delta)$. How does certain equivalent $\tilde{x}(\Delta)$ of $x(\Delta)$ compare with certain equivalent $\tilde{x}$ of x $(= x(0))$.



From approximation,

$$\tilde{x} \approx \bar{x} - \tfrac{1}{2} r(\bar{x}) \check{x}$$

$$\tilde{x}(\Delta) = \bar{x}(\Delta) - \tfrac{1}{2} r(\bar{x}(\Delta)) \check{x}(\Delta)$$

Since $x(\Delta) = x + \Delta$, $\bar{x}(\Delta) = \bar{x} + \Delta$, $\check{x}(\Delta) = \check{x}$

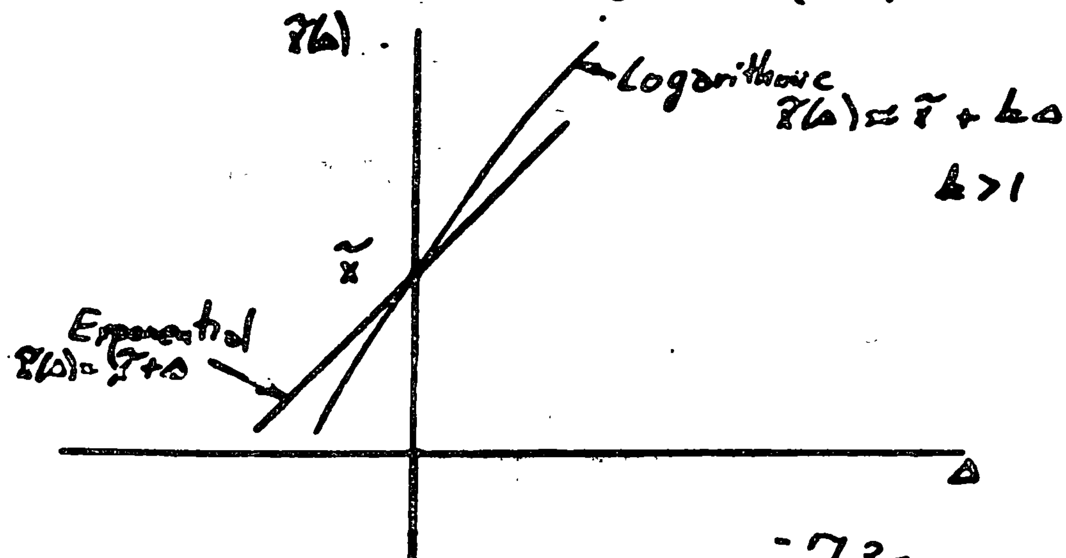
$$\begin{aligned} \tilde{x}(\Delta) &\approx \bar{x} + \Delta - \tfrac{1}{2} r(\bar{x}+\Delta)\check{x} \\ &\approx \bar{x} + \Delta - \tfrac{1}{2}\left[r(\bar{x}) + \Delta r'(\bar{x})\right]\check{x} \\ &\approx \tilde{x} + \Delta \underbrace{\left[1 - \tfrac{1}{2} r'(\bar{x}) \check{x}\right]}_{k} \end{aligned}$$

For exponential $u(x)$, $r'(x) = 0$, $k = 1$, $\tilde{x}(\Delta) = \tilde{x} + \Delta$

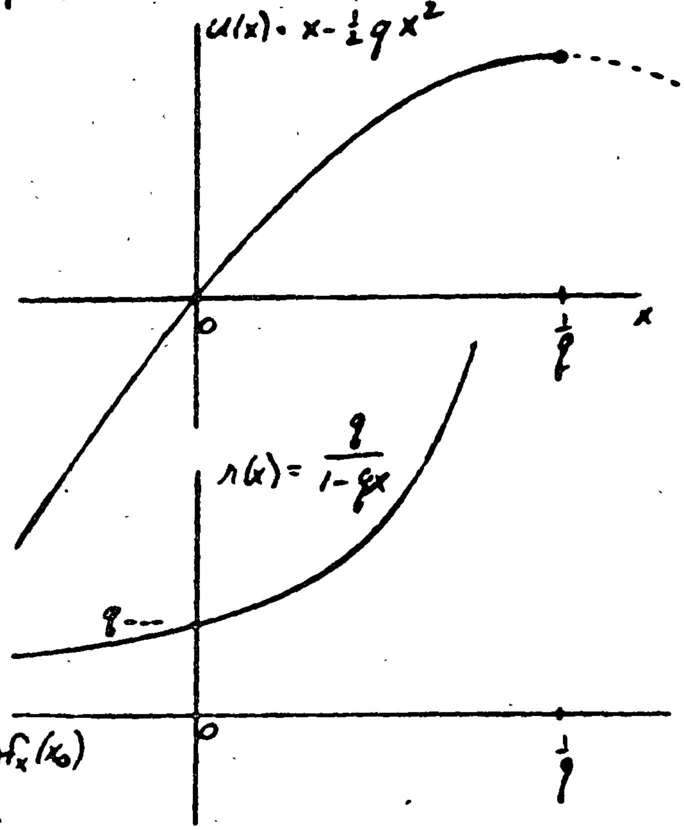
For decreasing risk aversion, $r'(x) < 0$, $k > 1$ $\tilde{x}(\Delta) \approx \tilde{x} + k\Delta$

Examples: logarithmic $r(x) = \frac{1}{x+\alpha}$, $r'(x) = -\frac{1}{(x+\alpha)^2}$

$$k = 1 + \frac{\check{x}}{2(\bar{x}+\alpha)^2} > 1$$



Utility



Quadratic

$$u(x) = x - \tfrac{1}{2} q x^2 \qquad x < \tfrac{1}{q}$$

$$u'(x) = 1 - qx$$

$$u''(x) = -q$$

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{q}{1-qx}$$

Certain Equivalent

$$u(\tilde{x}) = \int dx_0\, u(x_0) f_x(x_0) = \int dx_0 \left(x_0 - \tfrac{1}{2} q x_0^2\right) f_x(x_0)$$

$$\tilde{x} - \tfrac{1}{2} q \tilde{x}^2 = \bar{x} - \tfrac{1}{2} q \overline{x^2}$$

$$-\tfrac{1}{2} q \tilde{x}^2 + \tilde{x} - \bar{x} + \tfrac{1}{2} q \overline{x^2} = 0$$

$$\tilde{x} = -\frac{1}{q}\left[-1 \pm \sqrt{1 - 2q\bar{x} + q^2 \overline{x^2}}\,\right]$$

$$= \frac{1}{q}\left[1 \pm \sqrt{(1-q\bar{x})^2 + q^2(\overline{x^2} - \bar{x}^2)}\,\right]$$

$$\tilde{x} = \frac{1}{q}\left[1 - \sqrt{(1-q\bar{x})^2 + q^2 \check{x}}\,\right] \qquad \text{EXACT}$$

$$\tilde{x} = \frac{1}{q}\left[1 - (1-q\bar{x})\sqrt{1 + \frac{q^2}{(1-q\bar{x})^2}\check{x}}\,\right]$$

As $\check{x} \to 0$,

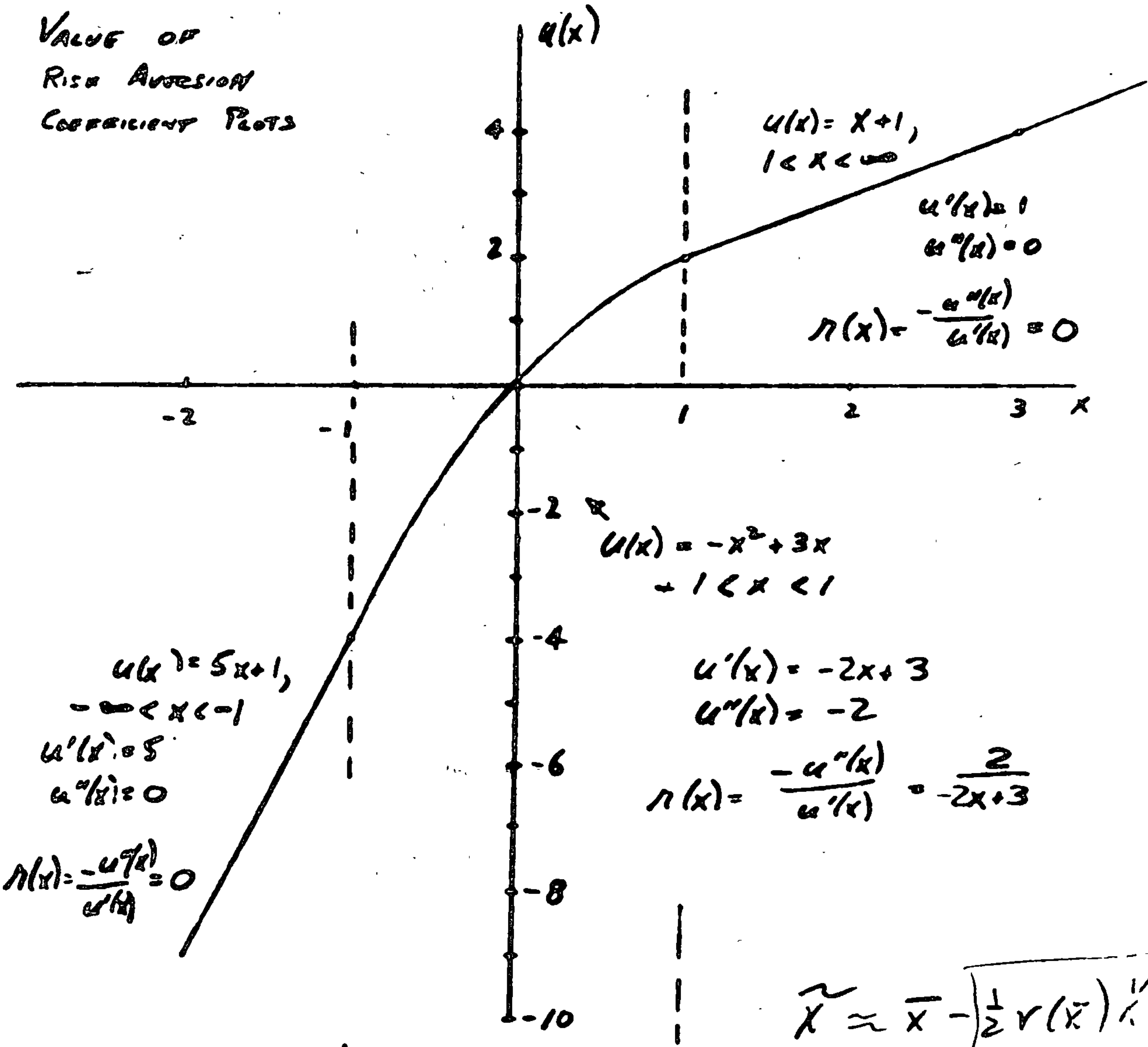
$$\tilde{x} \approx \frac{1}{q}\left[1 - (1-q\bar{x})\left(1 + \frac{1}{2}\frac{q^2}{(1-q\bar{x})^2}\check{x}\right)\right]$$

$$\approx \bar{x} - \frac{1}{2}\frac{q}{1-q\bar{x}}\check{x}$$

$$\approx \bar{x} - \tfrac{1}{2} r(\bar{x})\check{x}$$

Utility

Value of Risk Aversion Coefficient Plots

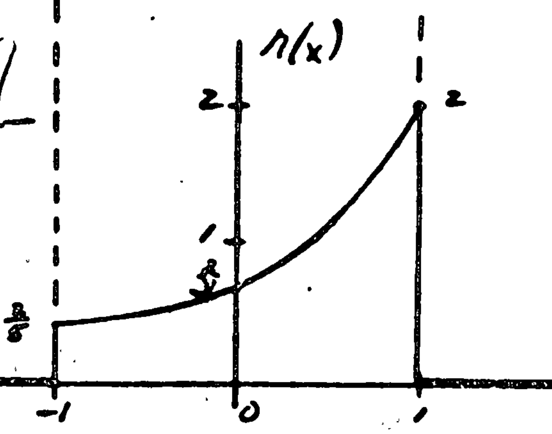


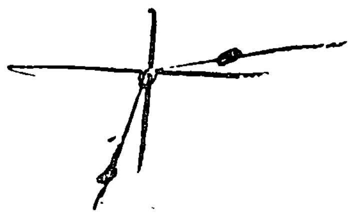
$$\tilde{x} \approx \bar{x} - \frac{1}{2} r(\bar{x})$$

½ 1 + 0

½ 0 + 0

Risk Premium





A Fundamental Theorem

$$V = EV + VE$$

$$\overset{\vee\prime}{x} = \bar{\overset{\vee}{x}}'' + \overset{\vee}{\bar{x}}''$$

"The Prior Variance is the Sum of the Mean of the Posterior Variance and the Variance of the Posterior Mean."

Consider two jointly related variables x and y. Consider the revelation of y to be experimental evidence.

Then $\{x|\mathcal{E}\}$ is prior

$\{y|x\mathcal{E}\}$ is likelihood function

$\{x|y\mathcal{E}\}$ is posterior

$\{y|\mathcal{E}\}$ is preposterior

Bayes' Theorem

$$\{x|y\mathcal{E}\} = \frac{\{y|x\mathcal{E}\}\{x|\mathcal{E}\}}{\{y|\mathcal{E}\}}$$

Expansion

$$\{y|\mathcal{E}\} = \int_x \{y|x\mathcal{E}\}\{x|\mathcal{E}\}$$

Moments	Mean	Variance
Prior	$\langle x|\mathcal{E}\rangle$	$\overset{\vee}{\langle} x|\mathcal{E}\rangle = \langle x^2|\mathcal{E}\rangle - \langle x|\mathcal{E}\rangle^2$
Posterior	$\langle x|y\mathcal{E}\rangle$	$\overset{\vee}{\langle} x|y\mathcal{E}\rangle = \langle x^2|y\mathcal{E}\rangle - \langle x|y\mathcal{E}\rangle^2$

A Fundamental Theorem

$${}^{\vee}\langle x|\varepsilon\rangle = \langle {}^{\vee}\langle x|y\varepsilon\rangle|\varepsilon\rangle + {}^{\vee}\langle\langle x|y\varepsilon\rangle|\varepsilon\rangle$$

$$\langle {}^{\vee}\langle x|y\varepsilon\rangle|\varepsilon\rangle = \langle \langle x^2|y\varepsilon\rangle - \langle x|y\varepsilon\rangle^2|\varepsilon\rangle$$

$$= \langle x^2|\varepsilon\rangle - \langle\langle x|y\varepsilon\rangle^2|\varepsilon\rangle$$

$${}^{\vee}\langle\langle x|y\varepsilon\rangle|\varepsilon\rangle = \langle\langle x|y\varepsilon\rangle^2|\varepsilon\rangle - \langle\langle x|y\varepsilon\rangle|\varepsilon\rangle^2$$

$$= \langle\langle x|y\varepsilon\rangle^2|\varepsilon\rangle - \langle x|\varepsilon\rangle^2$$

Add

$$\langle {}^{\vee}\langle x|y\varepsilon\rangle|\varepsilon\rangle + {}^{\vee}\langle\langle x|y\varepsilon\rangle|\varepsilon\rangle = \langle x^2|\varepsilon\rangle - \langle x|\varepsilon\rangle^2$$

$$= {}^{\vee}\langle x|\varepsilon\rangle$$

Recall

Expansion

$$\{x|\varepsilon\} = \int_y \{x|y\varepsilon\}\,\{y|\varepsilon\}$$

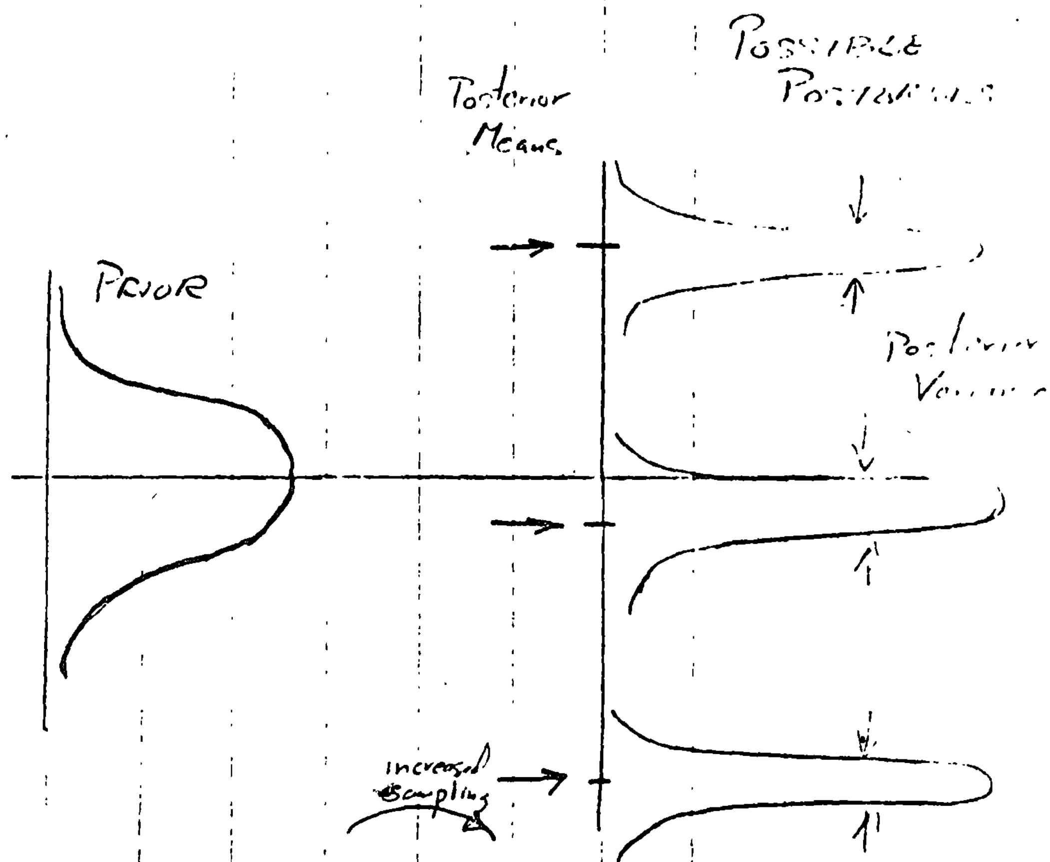
$$\langle x|\varepsilon\rangle = \int_x x\,\{x|\varepsilon\} = \int_x x \int_y \{x|y\varepsilon\}\,\{y|\varepsilon\}$$

$$= \int_y \int_x x\,\{x|y\varepsilon\}\,\{y|\varepsilon\}$$

$$= \int_y \langle x|y\varepsilon\rangle\,\{y|\varepsilon\}$$ Expansion for Moments

$$\langle\langle x|y\varepsilon\rangle|\varepsilon\rangle = \int_y \langle x|y\varepsilon\rangle\,\{y|\varepsilon\} = \langle x|\varepsilon\rangle$$

A Fundamental Theorem



$$\check{X}' = \bar{\check{X}}'' + \underline{\check{X}}''$$

The effect of increased sampling is dispersion of the posterior mean and contraction of the posterior variance.

With no sampling $\check{X}' = \bar{\check{X}}''$, $\underline{\check{X}}'' = 0$

With infinite sampling $\check{X}' = \underline{\check{X}}''$, $\bar{\check{X}}'' = 0$

ASISTENTES AL CURSO EVALUACION DE PROYECTOS Y TOMA DE DECISIONES OCTUBRE DE 1978.

1. FRANCISCO BAHAMONDE TORRES
S.A.R.H.
DIR. GRAL. DE PROTECCION Y ORD. ECO.
REFORMA 107-1°
MEXICO, D.F.
TEL. 566.06.88 EXT.154

EDIF. 1 ENTRADA B DEPTO. 501
UNIDAD LINDAVISTA VALLEJO
MEXICO 14, D.F.
TEL. 587.37.92

2. ARIEL BAUTISTA GARCIA
CENTRO SAHOP
DELEGACION ASENT. HUMANOS
IGLESIAS 202 P.A.
PACHUCA, HGO.
TEL. 2.45.80

TIBO PARAISO 105
REAL DE MINAS
PACHUCA, HGO.
TEL. 2.11.31

3. ENRIQUE BETANCOURT GONZALEZ
COLEGIO DE BACHILLERES
AV. CUAUHTEMOC 1236
MEXICO 13, D.F.
TEL. 559.76.97

AV. SN. JERONIMO 1491
MEXICO 20, D.F.
TEL. 595.09.93

4. RAUL CASTILLO GUERRERO
FIDEICOMISO LAZARO CARDENAS
MIGUEL A. DE QUEVEDO 276
MEXICO 21, D.F.
TEL. 554.11.45

REP. DE ARGENTINA 96-14
MEXICO 1, D.F.
TEL. 554.11.45

5. ROBERTO COLSA DIAZ
CIA. DE LUZ Y FZA. DEL CENTRO, S.A.
MELCHOR OCAMPO 171
MEXICO 17, D.F.
TEL. 566.47.13

J. SANCHEZ AZCONA 1412 No.401
COL. DEL VALLE
MEXICO 12, D.F.
TEL. 566.47.13

6. VICTOR CASTORENA DAVIS
IGUALA 36-2
COL. ROMA SUR
MEXICO 7, D.F.
TEL. 574.44.32

7. JOSE CARMEN CONTRERAS GUTIERREZ
S.A.R.H.
PLAZA DE LA REP. 31
MEXICO 1, D.F.
TEL. 592.18.44

JOSE MA. MORELOS 215
COL. LOMABONITA
NTEZATHUALCOYOTL, EDO. DE MEX.

8. PORFIRIO CORDOVA SANCHEZ
SUBDIRECCION DE INVERSIONES DEL GOBIERNO DEL EDO. DE MEX.
PALACIO DE GOBERNO, TOLUCA
ESTADO DE MEX.
TEL. 406.13 y 40690

9. CARLOS DE ICAZA LOZANO
S.A.R.H.
REFORMA 20-1°
MEXICO , D.F.
TEL. 591.14.61

AV. CORDOBA 511
COL. VALLE DORADO
TLALNEPANTLA, EDO. DE MEX.
TEL. 379.06.82

10. MA. MARGARITA DEL VALLE
DIR. GRAL. DE BIBLIOTECAS
U. N. A. M.
TEL. 548.65.60 EXT.525

AV. COPILCO 300-11-203
COL. COPILCO
MEXICO 20, D.F.
TEL. 548.23.41

11. ANTONOVICTOR ECHEGOYEN GUZMAN
HULES MEXICANOS S.A.
LEIBNITZ 14
MEXICO 5, D.F.

OSA MAYOR 43
COL. PRADO CHURUBUSCO
TEL. 670.54.12

12. RICARDO ESCOBEDO VILLALON
ESTUDIOS Y PROYECTOS INTEGRALES S.C
DIRECTOR
GALILEO 20-401
MEXICO 5, D.F.
TEL. 531.82.42 y 43

MONTE CHIMBORAZO 513-5
LOMAS DE CHAPULTEPEC
MEXICO 10, D.F.
TEL. 540.21.78

13. JORGE FLORES CANCHE
S.A.R.H.
REFOR MA 51-15
MEXICO 1, D.F.
TEL. 535.54.83

MORAS 345-1
MEXICO 12, D.F.
TEL. 575.48.73

14. FRANCO DEL VALLE TEOBALDO N.
S.A.H.P.
XOLA 1755-4°
MEXICO 12, D.F.
TEL. 519.80.48

CALLE 5 # 339
COL. ALDAMA .
MEXICO 15, D.F.
TEL. 355.29.77

15. PAULINO ALBERTO FRIAS RODRIGUEZ
FIDEICOMISO DE PROGRAMACION Y
PRESUPUESTO
MIGUEL A. DE QUEVEDO 276
MEXICO 21, D.F.
TEL. 554.11.45

VALLE DE ATENCO 19
FRACC. VERGEL DE COYOACAN
MEXICO 22, D.F.

16. MIGUEL ANGEL GALAVIZ HIGUERA
S.A.H.O.P.
REFRMA 77-10
MEXICO , D.F.
TEL. 591.18.69

MEDELLIN 155-202
COL. ROMA
MEXICO 7, D.F.
TEL. 564.59.66

17. ALFREDO GAMEZ LOPEZ
UNIVERIDAD VERACRUZANA
FACULTAD DE INGENIERIA
XALAPA, VER.
TEL. 7.32.14

BETANCOURT 67
XALAPA, VER.
TEL. 772.34

18.	MANUEL GARCIA CARMONA SANTIESTEBAN SECRETARIA DE MARINA AZUETA NO. 1 MEXICO 1, D.F. TEL. 521.79.95	CAPRICORNIO 75 COL. PRADO CHURUBUSCO MEXICO 13, D.F. TEL. 581.47.49
19.	ERNESTO GONZALEZ OCAÑA S.A.R.H. P. DE LA REFORMA 51 MEXICO 1, D.F. TEL. 535.54.83	1a. CERRADA AV. 599 No.4 U. ARAGON MEXICO 14, D.F. TEL. 760.06.79
20.	RAUL GONZALEZ VELASCO CIA. DE LUZ Y FUERZA DEL CENTRO MELCHOR OCAMPO NO. 171 MEXICO 17, D.F. TEL. 518.00.80	CONTADORES 56 JARDINES DE CHURUBUSCO MEXICO 8, D.F. TEL. 670.29.73
21.	THELMA GUTIERREZ SANCHEZ P E M E X MARINA NAL. 329 DEPTO. DE SISTEMAS ECONOMISTA MEXICO 17, D.F. TEL. 531.37.95	TEJOMULCO 22 STA. CRUZ A. XOCHIMILCO MEXICO 23, D.F.
22	JORGE HERNANDEZ R. I.S.T.M.E. AV. LEGARIA COL. PENSIL MEXICO, D.F. TEL. 399.69.22 EXT.124	
23.	SAMUEL HUERTA PARRA S.C.T. INSURGENTES SUR 465 MEXICO, D.F.	ANDADOR 28 DIV. DELNTE.21 VILLA COAPA MEXICO 22, D.F. TEL. 594.09.60
24.	PABLO JARDON HERRERA CIA. DE LUZ Y FZA. DEL CENTRO S.A. MELCHOR OCAMPO 171 MEXICO 17, D.F. TEL. 518.00.80 EXT.213	HIDALGO 28 TLACOPAC MEXICO 20, D.F.
25.	JACINTO LIMA M. BANCO INTERAMERICANO DE DESARROLLO REFORMA 379-7° MEXICO 1, D.F. TEL. 33.57.09	SUR 113 # 514 SECTOR POPULAR MEXICO, D.F. TEL. 670.50.93
26.	MATIAS LOPEZ JIMENEZ DIR. GRAL. DE AEROPUERTOS S.A.H.O.P. XOLA Y AVE. UNIVERSIDAD 1755-4°	CALLE SUR 111 A # 328 HEROES DE CHURUBUSCO MEXICO 13, D.F. 519.80.46

27. JORGE ROBERTO LOPEZ RAMIREZ
S.A.R.H.
PLAZA DE LA REP. 31-7°
ME XICO, D.F.
TEL. 535.1325

FCO. PIMENTEL 27 ALTOS 5
COL. SAN RAFAEL
MEXICO 4, D.F.
TEL. 535.36.11

28. JOSE LUIS LUJAN HUERTA
S. C. T.
DIR. GRAL. DE OBRAS MARITIMAS
INSURGENTES SUR 465
MEXICO, D.F.
TEL. 564.76.68

RET 9 FCO. DEL PASO Y T.
EDIF. 19 A-201
MEXICO 9, D.F.
TEL. 552.78.63

29. MARIO GERARDO MACAY LIM
S. A. R. H.
REFORMA 51-15 °
MEXICO 1, D.F.
TEL. 566.97.69

CALLE BARRILITO 279
COL. ESPERANZA
NETZAHUALCOYOTL, EDO. DE MEX.

30. JOSE LUIS MADRID FRANCO
S.A.H.O.P.
XOLA 1755
MEXICO, D.F.
TEL. 530.99.71

31. SALVADOR MANRIQUE LEIJA
DIR. GRAL. DE PLANIFICACION
D. D. F.
SUBJEFE DE OFICINA
PINO SUAREZ NO. 15-1°
MEXICO 1, D.F.

32. CESAR R. MARGAIN Y R.S.
D. D. F.
DIR. GRAL. DE PLANIFICACION
PLAZA DE LA CONSTITUCION
MEXICO 1, D.F.
TEL. 521.30.94

AV. CHAPULTEPEC 456
MEXICO 7, D. F.
TEL. 521.30.94

33. MANUEL MARTINEZ GARRIDO
PROFESOR
U. N. A. M.
TEL. 548.99.42

CERRO DEL PEÑON 117-4
MEXICO 21, D.F.
TEL. 689.13.56

34. IGNACIO MONSALVO ESCAMILLA
S. A. R. H.
REFORMA 20-1º 103
MEXICO 1, D.F.
TEL. 546.64.84

GPE. VICTORIA 286
MEXICO 14, D.F.
TEL. 781.45.69

35. MARRON HERNANDEZ RIGOBERTO
S.A.R.H.
IGNACIO RAMIREZ No. 20
MEXICO, D.F.
TEL: 592-36-84

CABO SAN ROQUE No. 49
COL. GABRIEL HERNANDEZ
MEXICO 14, D.F.
TEL: 577-19-25

36. JESUS LUMBRERAS GUERRERO
COLEGIO DE BACHILLERES
AV. CUAUHTEMOC No. 1236-8o. PISO
STA. CRUZ ATOYAC
MEXICO 12, D.F.
TEL: 559-55-22

ANTILLAS No. 507-1
COL. PORTALES
MEXICO 13, D. F.
TEL: 532-89-97

37. MANUEL ARTEMIO MIRANDA MENDEZ
COCA COLA DE MEXICO
RIO AMAZONAS No. 43-4o. PISO
MEXICO 5, D.F.
TEL: 591-00-66

GUERRERO No. 325
EDIF. IGNACIO RAMIREZ
ENT. "E", DEPTO. 1217
MEXICO 3, D.F.
TEL: 597-19-04

38. MARTHA MARGARITA MONOBE HDEZ.
S.A.R.H.
REFORMA No. 51-15o. PISO
COL. SAN RAFAEL
MEXICO 4, D.F.
TEL: 566-97-92

AV. AMERICAS No. 81-201
COL. MODERNA
MEXICO 13, D.F.
TEL: 566-60-27

39. MARIO MONTERO MANJARREZ
DIREC. GRAL. DE OBRAS PUBLICAS-DDF
TLALOC No. 1, 1er. PISO
MEXICO, D.F.
TEL: 592-31-96

ORIENTE 53 No. 338-4
COL. VILLA DE CORTES
MEXICO 13, D.F.
TEL: 590-37-61

40. GABINO GASPAR MONTERROSA REYES
S.A.R.H.
REFORMA 107-1er. PISO
MEXICO 13, D.F.
TEL: 566-95-58

SARATOGA No. 1017-202
COL. PORTALES
MEXICO 13, D.F.
TEL: 532-46-59

41. IGNACIO NIEVA CALDERON
BCO. INTERAMERICANO DE DESARROLLO
REFORMA No. 379-1er. PISO
COL. CUAUHTEMOC
MEXICO 5, D.F.
TEL: 533-57-09

ING. CARLOS DAZA No. 212-12
COL. GPE. INSURGENTES
MEXICO 14, D.F.
TEL: 759-19-04 533-57-09

42. JUAN MANUEL ORENDAY GONZALEZ
S.A.H.O.P.
XOLA 1755-4o. PISO -DEPTO. TECNICO
MEXICO 12, D.F.
TEL: 519-80-46

DIV. DEL NORTE No. 1251-103
COL. VERTIZ NARVARTE
MEXICO 13, D.F.
TEL: 519-80-46

43. JAVIER PALACIOS MONTOYA
COM. PLAN NAL. HIDRAULICO -SARH
TEPIC No. 40
COL. ROMA
MEXICO 7, D.F.
TEL:

JUAN A. GUTIERREZ No. 67-4
COL. MOCTEZUMA
MEXICO 9, D.F.
TEL: 762-32-03

44. MAURO PEÑA V.
COLEGIO DE BACHILLERES
AV. CUAUHTEMOC 1236
MEXICO 13, D.F.
TEL: 559-66-73

ATLETAS 51-101
COL. COUNTRY CLUB
MEXICO 21, D.F.
TEL: 544-59-45

45. WALTER PINEDA PINEDA
THE COCA COLA EXPORT COORP.
AMAZONAS No. 43-4o. PISO
COL. CUAUHTEMOC
MEXICO 5, D.F.
TEL: 591-00-66

EDIF. PEDRO MORENO ENT. "C"
DEPTO. 309 - TLATELOLCO
MEXICO 3, D.F.
TEL: 583-18-31

46. HECTOR F. QUIROZ VILCHIS
S.A.R.H.
PLAZA DE LA REPUBLICA 31-7o. PISO
MEXICO 1, D.F.
TEL: 535-13-25

BELGICA No. 1003
COL. PORTALES
MEXICO 13, D.F.
TEL: 532-26-00

47. TARSICIO RAMIREZ CARREÑO
S.A.H.O.P.
BLVD. XOLA No. 1755
MEXICO 13, D.F.
TEL: 519-18-02

AV. MORELOS No. 605, EDIF. C-2
COL. BALBUENA
MEXICO 8, D.F.
TEL: 768-73-09

48. LUCIO RIOS AVILES
DIREC. GRAL. AEROPUERTOS-SAHOP
XOLA 1755
COL. NARVARTE
MEXICO 12, D.F.

FCO. MORAZAN 490-4
JAMAICA
MEXICO 8, D.F.
TEL: 530-30-53

49. FRANCISCO RIOS HERNANDEZ
S.A.R.H.
P. DE LA REFORMA 51-15o. PISO
MEXICO 4, D.F.
TEL: 566-97-83

CLAVELINAS No. 181
COL. NUEVA STA. MARIA
MEXICO 16, D.F.
TEL: 556-46-03

50. ROBERTO RIVERA RIOS
DIREC.GRAL. CARRET. FEDERALES
S.A.H.O.P.
XOLA Y UNIVERSIDAD
MEXICO 12, D.F.
TEL: 519-92-21

ANICETO ORTEGA No. 1044-6
COL. DEL VALLE
MEXICO 12, D.F.
559-24-68

51. ROGELIO RIVERO CARRARO
SECRETARIA DE PROGRAM. Y PRES.
IZAZAGA No. 38
MEXICO 1, D.F.
TEL: 521-69-27

FERROCARRIL INDUSTRIAL # 56
COL. LA RAZA
MEXICO 15, D.F.
TEL: 355-17-56

52. JAVIER ROMERO VIGUERAS
S.A.H.O.P.
REFORMA 77-10o.
MEXICO 4, D.F.
TEL: 546-80-17

LUIS QUINTERO No. 43
ATZACOALCO
MEXICO 14, D.F.
TEL:

53. LUCAS RUIZ GONZALEZ
I.S.T.M.E
AV. LEGARIA No.
COL. PENSIL
MEXICO, D.F.
TEL: 399-69-22 - 124

AV. UNIVERSIDAD No. 903-15
COL. DEL VALLE
MEXICO 12, D.F.

54. VICTOR MANUEL RUIZ PADILLA
DESARROLLO DE SERV. CORPORATIVOS, S.C.
CULIACAN No. 108-2o. PISO
COL. HIPODROMO CONDESA
MEXICO 11, D.F.
TEL: 564-85-00

CHOLULTECAS No. 62
MEXICO 15, D.F.
TEL: 583-60-84

55. JOSE LUIS SANCHEZ CONTRERAS
SECRETARIA DE PROGRAM. Y PRESUP.
JOSE MA. IZAZAGA 38-5o. PISO
MEXICO 1, D.F.
TEL: 521-62-41

SAN DIEGO No. 32
COL. CAPISTRANO
A. DE ZARAGOZA, EDO. DE MEX.
TEL: 398-20-07

56. RAFAEL SAINZ ZAMORA
S.A.R.H.
P. DE LA REFORMA No. 20-105
MEXICO 1, D.F.
TEL: 546-46-76

PENNSYLVANIA No. 280-4o.-2B
COL. NAPOLES
MEXICO 18, D.F.

57. LUIS ALFONSO SOSA VILLASEÑOR
PEMEX
MARINA NACIONAL No. 329
MEXICO, D.F.
TEL: 531-37-95

CARR. DESIERTO LEONES 6563
COL. V. ALVARO OBREGON
MEXICO 20, D.F.
TEL: 566-67-77 Clave 5326

58. ADAN SOTO GOMEZ
S.A.H.O.P.
AV. REFORMA 77-7o. PISO
MEXICO, D.F.
TEL: 546-80-17

SUR 143 No. 1903
COL. GABRIEL R. MILLAN
MEXICO 8, D.F.
TEL: 657-04-73

59. ROBERTO TAVERA CARMONA
S.A.H.O.P.
XOLA Y AV. UNIVERSIDAD
MEXICO 12, D.F.
TEL: 530-33-11

ESCULTORES No. 80
CD. SATELITE, EDO. DE MEX.
TEL: 562-82-62

60. VICTOR MANUEL TORRES DELGADO
D.D.F. - DELEG. XOCHIMILCO
PROLONG. GLADIOLAS No. 161
COL. XOCHIMILCO
MEXICO 23, D.F.
TEL: 676-06-68 676-05-76

RAYON NORTE No. 213-409
TOLUCA, MEX.
TEL: 559-91

61. JAVIER VALADEZ BECERRA
ESC. CIENC. QUIM. DE LA U.A.Z.
ANTONIO DOVALI S/N
ZACATECAS, ZAC.
TEL: 2 19-99

DR. IGNACIO HIERRO No. 407
ZACATECAS, ZAC.
TEL: 2 50-15

62. MANUEL VARELA CAMACHO
BUFETE INDUSTRIAL
TOLSTOI No. 22
MEXICO 5, D.F.
TEL: 533-15-00

MANUEL GAMIO No. 557
COL. SINATEL
MEXICO 13, D.F.
TEL: 581-07-80

63. GANDHI ZILLI V.
SECRETARIA DE MARINA
REFORMA No. 40
MEXICO 1, D.F.
TEL: 535-23-51

REFORMA No. 40
MEXICO 1, D.F.
TEL: 535-23-51

64. JESUS ZUÑIGA RUIZ
COM.AGUAS DEL V. DE MEX.S.A.R.H.
BALDERAS 55-4o. PISO
MEXICO 1, D.F.
TEL: 585-50-66

ADOLFO PRIETO No. 125-404-B
COL. DEL VALLE
MEXICO 12, D.F.
TEL: