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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

2. FUNDAMENTALS OF RIGID-BODY THREE-DIMENSIONAL KINEMATICS

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2. FUNDAMENTALS OF RIGID-BODY THREE-DIMENSIONAL KINEMATICS.

2.1 INTRODUCTION. The rigid body is defined as a continuum for which, under any physically possible motion, the distance between any pair of its points remains unchanged. The rigid body is a mathematical abstraction which models very accurately the behaviour of a wide variety of natural and man-made mechanical systems under certain conditions. However, as such it does not exist in nature, as neither do the elastic body nor the perfect fluid. The theorems related to rigid body motions are rigorously proved and the foundations for the analysis of the motion of systems of coupled rigid bodies (linkages) are laid down. The main results in this chapter are the theorems of Euler, Chasles, the one on the existence of an instant screw, the Theorem of Aronhold-Kennedy and that of Coriolis.

2.2 MOTION OF A RIGID BODY.

Consider a subset D of the Euclidean three-dimensional physical space occupied by a rigid body, and let $\underset{\sim}{x}$ be the position vector of a point of that body. A rigid body motion is a mapping $\underset{\sim}{M}$ which maps every point $\underset{\sim}{x}$ of D into a unique point $\underset{\sim}{y}$ of a set D', called "the image" of D under $\underset{\sim}{M}$,

$$\underset{\sim}{M} : \underset{\sim}{x} \to \underset{\sim}{y} \tag{2.2.1}$$

such that, for any pair x_1 and x_2, mapped by M into y_1 and y_2, respectively, one has

$$||\underset{\sim}{x}_2 - \underset{\sim}{x}_1|| = ||\underset{\sim}{y}_2 - \underset{\sim}{y}_1|| . \tag{2.2.2}$$

The symbol $||.||$ denotes the Euclidean norm* of the space under consideration.

It is next shown that, under the above difinition, a rigid-body motion preserves the angle between any two lines of a body. Indeed, let $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$

* See Section 1.8

and $\underset{\sim}{x}_3$ be three noncollinear points of a rigid body. Let $\underset{\sim}{M}$ map these points into $\underset{\sim}{y}_1$, $\underset{\sim}{y}_2$ and $\underset{\sim}{y}_3$, respectively. Clearly,

$$||\underset{\sim}{x}_3-\underset{\sim}{x}_2||^2 = (\underset{\sim}{x}_3-\underset{\sim}{x}_2, \underset{\sim}{x}_3-\underset{\sim}{x}_2) = ((\underset{\sim}{x}_3-\underset{\sim}{x}_1) - (\underset{\sim}{x}_2-\underset{\sim}{x}_1), (\underset{\sim}{x}_3-\underset{\sim}{x}_1)-(\underset{\sim}{x}_2-\underset{\sim}{x}_1)) =$$
$$= ||\underset{\sim}{x}_3-\underset{\sim}{x}_1||^2 - 2(\underset{\sim}{x}_3-\underset{\sim}{x}_1, \underset{\sim}{x}_2-\underset{\sim}{x}_1) + ||\underset{\sim}{x}_2-\underset{\sim}{x}_1||^2$$

Similarly,

$$||\underset{\sim}{y}_3-\underset{\sim}{y}_2||^2 = (\underset{\sim}{y}_3-\underset{\sim}{y}_2, \underset{\sim}{y}_3-\underset{\sim}{y}_2) = ||\underset{\sim}{y}_3-\underset{\sim}{y}_1||^2 - 2(\underset{\sim}{y}_3-\underset{\sim}{y}_1, \underset{\sim}{y}_2-\underset{\sim}{y}_1) + ||\underset{\sim}{y}_2-\underset{\sim}{y}_1||^2$$

From the definition of a rigid-body motion, however,

$$||\underset{\sim}{y}_3-\underset{\sim}{y}_2||^2 = ||\underset{\sim}{x}_3-\underset{\sim}{x}_2||^2$$

Thus,

$$||\underset{\sim}{x}_3-\underset{\sim}{x}_1||^2 - 2(\underset{\sim}{x}_3-\underset{\sim}{x}_1, \underset{\sim}{x}_2-\underset{\sim}{x}_1) + ||\underset{\sim}{x}_2-\underset{\sim}{x}_1||^2 = ||\underset{\sim}{y}_3-\underset{\sim}{y}_1||^2$$
$$-2(\underset{\sim}{y}_3-\underset{\sim}{y}_1, \underset{\sim}{y}_2-\underset{\sim}{y}_1) + ||\underset{\sim}{y}_2-\underset{\sim}{y}_1||^2 \qquad (2.2.3)$$

However, again from the rigid-body motion definition,

$$||\underset{\sim}{x}_3-\underset{\sim}{x}_1||^2 = ||\underset{\sim}{y}_3-\underset{\sim}{y}_1||^2 \qquad (2.2.4)$$

and

$$||\underset{\sim}{x}_2-\underset{\sim}{x}_1||^2 = ||\underset{\sim}{y}_3-\underset{\sim}{y}_1||^2 \qquad (2.2.5)$$

Thus clearly, from (2.2.3), (2.2.4) and (2.2.5),

$$(\underset{\sim}{x}_3-\underset{\sim}{x}_1, \underset{\sim}{x}_2-\underset{\sim}{x}_1) = (\underset{\sim}{y}_3-\underset{\sim}{y}_1, \underset{\sim}{y}_2-\underset{\sim}{y}_1) \qquad (2.2.6)$$

which states that the angle (See Section 1.7) between vectors x_3-x_1 and x_2-x_1 remains unchanged.

The foregoing mapping $\underset{\sim}{M}$ is, in general, nonlinear, but there exists a class $\underset{\sim}{Q}$ of mappings $\underset{\sim}{M}$, leaving one point in a body fixed, that are linear. In fact, let O be a point of a rigid body which remains fixed under $\underset{\sim}{Q}$, its position vector being the zero vector $\underset{\sim}{0}$ of the space under study (this can always be rearranged since one has the freedom to place the origin of coordinates in any suitable position). Let $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ be any two points of

this rigid body.

From the previous results,

$$||\underline{x}_i|| = ||\underline{Q}(\underline{x}_i)||, \ i = 1, 2 \tag{2.2.7}$$

$$(\underline{x}_i, \underline{x}_j) = (\underline{Q}(\underline{x}_i), \underline{Q}(\underline{x}_j)), \ i,j=1,2 \tag{2.2.8}$$

Assume for a moment that $\underline{Q}$ is not linear.

Let

$$\underline{e} \equiv \underline{Q}(\underline{x}_1+\underline{x}_2) - \left(\underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2)\right)$$

Then

$$\begin{aligned}||\underline{e}||^2 &= ||\underline{Q}(\underline{x}_1+\underline{x}_2)||^2+||\underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2)||^2-2(\underline{Q}(\underline{x}_1+\underline{x}_2),\underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2))= \\ &= ||\underline{x}_1+\underline{x}_2||^2+||\underline{Q}(\underline{x}_1)||^2+||\underline{Q}(\underline{x}_2)||^2+2(\underline{Q}(\underline{x}_1),\underline{Q}(\underline{x}_2)) \\ &\quad - 2(\underline{Q}(\underline{x}_1+\underline{x}_2),\underline{Q}(\underline{x}_1))-2(\underline{Q}(\underline{x}_1+\underline{x}_2),\underline{Q}(\underline{x}_2))\end{aligned}$$

where the rigidity condition has been applied, i.e. the condition that states that, under a rigid body motion, any two points of the body remain equidistant. Applying this condition again, together with the condition of constancy of the angle between any two lines of the rigid body (eq. (2.2.6)),

$$\begin{aligned}||\underline{e}||^2 &= ||\underline{x}_1||^2+||\underline{x}_2||^2+2(\underline{x}_1,\underline{x}_2)+||\underline{x}_1||^2+||\underline{x}_2||^2+2(\underline{x}_1,\underline{x}_2) \\ &\quad -2(\underline{x}_1+\underline{x}_2,\underline{x}_1)-2(\underline{x}_1+\underline{x}_2,\underline{x}_2)= \\ &= 2||\underline{x}_1||^2+2||\underline{x}_2||^2+4(\underline{x}_1,\underline{x}_2)-\left(2||\underline{x}_1||^2+2||\underline{x}_2||^2+4(\underline{x}_1,\underline{x}_2)\right) \\ &= 0\end{aligned}$$

From the positive-definiteness of the norm, then

$$\underline{e} = \underline{0}$$

thereby showing that

$$\underline{Q}(\underline{x}_1+\underline{x}_2) = \underline{Q}(\underline{x}_1)+\underline{Q}(\underline{x}_2)$$

i.e. $\underset{\sim}{Q}$ is an additive operator*

On the other hand, since $\underset{\sim}{Q}$ preserves the angle between any pair of lines of a rigid body, for any given real number $\alpha>0$, $\underset{\sim}{Q}(\underset{\sim}{x})$ and $\underset{\sim}{Q}(\alpha\underset{\sim}{x})$ are parallel, i.e. linearly dependent (for $\underset{\sim}{x}$ and $\alpha\underset{\sim}{x}$ are parallel as well). Hence,

$$\underset{\sim}{Q}(\alpha\underset{\sim}{x}) = \beta\underset{\sim}{Q}(\underset{\sim}{x}), \quad \beta>0 \tag{2.2.9}$$

Since $\underset{\sim}{Q}$ preserves the Euclidean norm,

$$||\underset{\sim}{Q}(\alpha\underset{\sim}{x})||=||\alpha\underset{\sim}{x}||=|\alpha|.||\underset{\sim}{x}|| \tag{2.2.10}$$

On the other hand, from eq. (2.2.9),

$$||\underset{\sim}{Q}(\alpha\underset{\sim}{x})||=||\beta\underset{\sim}{Q}(\underset{\sim}{x})||=|\beta|.||\underset{\sim}{Q}(\underset{\sim}{x})||=|\beta|.||\underset{\sim}{x}|| \tag{2.2.11}$$

Hence, equating (2.2.10) and (2.2.11), and dropping the absolute-value brackets, for $\alpha,\beta>0$,

$$\alpha = \beta$$

and

$$\underset{\sim}{Q}(\alpha\underset{\sim}{x}) = \alpha\underset{\sim}{Q}(\underset{\sim}{x}) \tag{2.2.12}$$

and hence, $\underset{\sim}{Q}$ is a homogeneous operator. Being homogeneous and additive, $\underset{\sim}{Q}$ is linear. The following has thus been proved.

THEOREM 2.2.1 *If $\underset{\sim}{Q}$ is a rigid body motion that leaves a point fixed, then $\underset{\sim}{Q}$ is a linear transformation.*

From the foregoing discussion, $\underset{\sim}{Q}$ is representable by means of a 3x3 matrix referred to a certain basis (theorem 1.2.1)

If $B=\{e_1,e_2,e_3\}$ is an orthonormal** basis for the 3-dimensional Euclidean

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* This proof is due to Prof. G.S. Sidhu, Institute for Applied Mathematics and Systems Research, U. of Mexico

** $(e_i,e_j) = \delta_{ij}$ (The Kronecker delta)

space, the ith column of the matrix $\underline{Q}$ is formed from the coefficients of $\underline{Q}(\underline{e}_i)$ expressed in terms of B according to Definition 1.2.1. In fact, the resulting matrix is orthogonal. Since $\underline{Q}$ is linear, $\underline{Q}(\underline{x})$ can be expressed simply as $\underline{Q}\underline{x}$. Now if

$$\underline{y} = \underline{Q}\underline{x}$$

then

$$||\underline{y}|| = ||\underline{x}||$$

Hence

$$\underline{y}^T\underline{y} = \underline{x}^T\underline{Q}^T\underline{Q}\underline{x} = \underline{x}^T\underline{x}, \text{ for any } \underline{x}$$

Hence, clearly

$$\underline{Q}^T\underline{Q} = \underline{I}$$

the identity matrix. This result can then be stated as

THEOREM 2.2.2 *A rigid body motion leaving one point fixed is represented with respect to an orthonormal basis by an orthogonal matrix.*

2.3 THE THEOREM OF EULER AND THE REVOLUTE MATRIX.

In the previous sections it was shown that the motion of a rigid body which keeps one of its points fixed can be represented by an orthogonal 3 x 3 matrix. In view of Sect. 1.9 there are two classes of orthogonal matrices, depending on whether their determinant is plus or minus unity. Orthogonal matrices whose determinant is +1 are called proper orthogonal and those whose determinant is -1 are called improper orthogonal.

Proper orthogonal matrices represent rigid body rotations, whereas improper orthogonal matrices represent reflections. Indeed, consider the rotation of axes X_1, Y_1, Z_1 into X_2, Y_2, Z_2 as shown in Fig 2.3.1

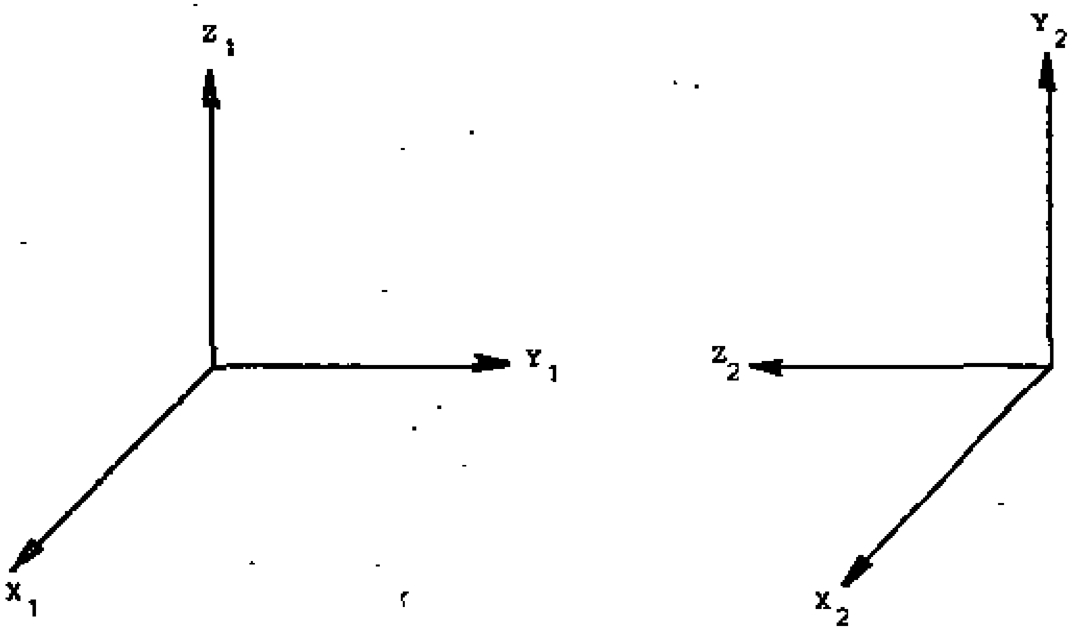

Fig 2.3.1 Rotation of axes

The matrix representation of the above rotation is obtained from the relationship

$$\underline{x}_2=\underline{x}_1, \qquad (2.3.1)$$

$$\underline{y}_2=\underline{z}_1,$$

$$\underline{z}_2=-\underline{y}_1$$

where $\underline{x}_1$, $\underline{x}_2$ represent unit vectors along the $\underline{X}_1$ and $\underline{X}_2$ axes, respectively, etc. From eqs. (2.3.1),

$$(\underline{Q})_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (2.3.2)$$

$(\underline{Q})_1$ means the rotation expressed in terms of the basis $\{\underline{x}_1, \underline{y}_1, \underline{z}_1\}$.

Clearly,

$$\det \underline{Q}=+1$$

and thus it is a proper orthogonal matrix.

On the other hand, consider the reflection of axes $\underline{x}_1, \underline{y}_1, \underline{z}_1$ into

$\{X_2, Y_2, Z_2\}$, as shown in Fig 2.3.2

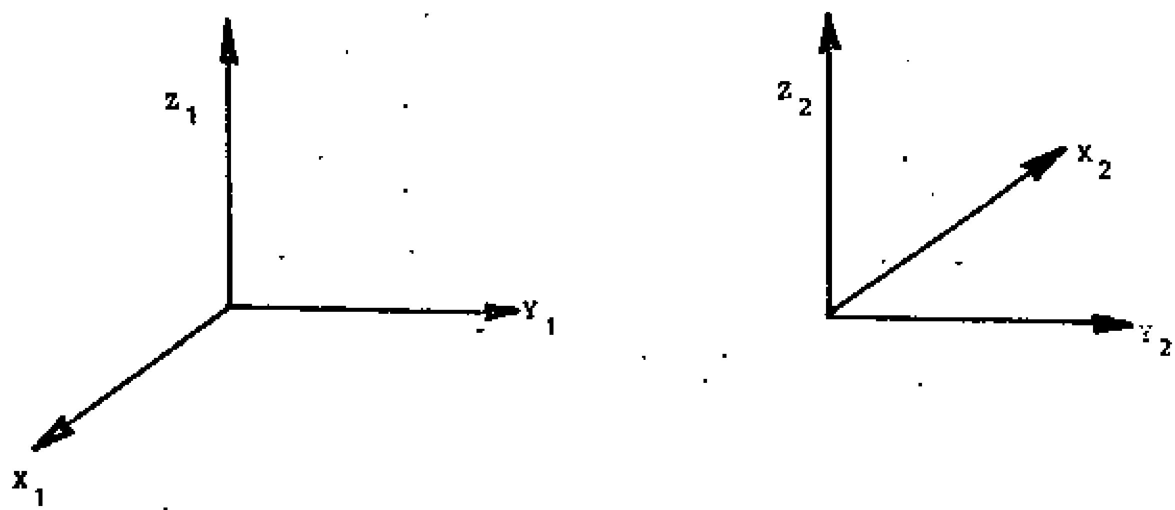

Fig. 2.3.2 Reflection of axes

Now,

$$x_2 = -x_1 \qquad (2.3.3)$$

$$y_2 = y_1$$

$$z_2 = z_1$$

Hence,

$$(\underset{\sim}{Q})_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so,

$$\det \underset{\sim}{Q} = -1$$

i.e., $\underset{\sim}{Q}$ as obtained from (2.3.3) is a reflection. Applications of reflections were studied in Sect. 1.12.

From Corollary 1.9.1 it can be seen that a 3 x 3 proper orthogonal matrix has exactly one eigenvalue equal to +1. Now if $\underset{\sim}{e}$ is the eigenvector of

$\underset{\sim}{Q}$ corresponding to the eigenvalue +1, it follows that

$$\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e}$$

and, furthermore, for any scalar α,

$$\underset{\sim}{Q}\alpha\underset{\sim}{e} = \alpha\underset{\sim}{e}$$

Hence all points of the rigid body located along a line parallel to $\underset{\sim}{e}$ passing through the fixed point O, remain fixed under the rotation $\underset{\sim}{Q}$. Hence, the following result, due to Euler {2.1} :

<u>THEOREM 2.3.1 (Euler)</u>. *If a rigid body undergoes a displacement leaving one of its points fixed, then there exists a line passing through the fixed point, such that all of the points on that line remain fixed during the displacement. This line is called "<u>the axis of rotation</u>" and the angle of rotation is measured on a plane perpendicular to the axis.*

The matrix representing a rotation is sometimes referred to as "<u>the revolute</u>". Clearly, the revolute is completely determined by a scalar parameter, the angle of rotation and a vector, the direction of the axis of rotation*. From the foregoing discussion it is clear that the direction vector of the revolute is obtained as the (unique linearly independent) eigenvector of the revolute associated with its + 1 eigenvalue. The angle of rotation is obtained as follows:

From Euler's Theorem, it is always possible to obtain an (orthonormal) basis B= $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ such that, say $\underset{\sim}{b}_3$, is parallel to the axis of rotation. $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ thus lie in a plane perpendicular to this axis. The rotation would then rotate the vectors through an angle θ. Let $\underset{\sim}{b}_1'$ and $\underset{\sim}{b}_2'$ be the corresponding images of $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ after the rotation under consideration, represented graphically in Fig 2.3.3

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* These parameters are also called "the invariants" of the revolute, for they remain unchanged under different choices of coordinate axes.

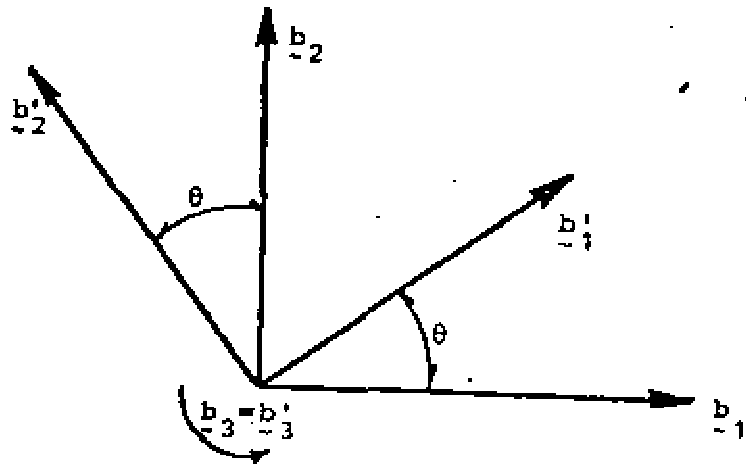

Fig 2.3.3 Rotation through an angle θ about axis b_3.

Then

$$b_1'=\cos\theta b_1+\sin\theta b_2$$

$$b_2'=-\sin\theta b_1+\cos\theta b_2 \qquad (2.3.4)$$

$$b_3'=b_3 .$$

and it follows that

$$(Q)_B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2.3.5)$$

Due to its simple and illuminating form, it seems justified to call matrix (2.3.5) a "canonical form" of the rotation matrix.

Exercise 2.3.1 Devise an algorithm to carry any orthogonal matrix into its canonical form (2.3.5).

Let a revolute matrix Q be given refered to an arbitrary orthonormal basis $A=\{a_1,a_2,a_3\}$, different from B as defined above. Furthermore, let

$$(P)_A = (b_1 \,\vdots\, b_2 \,\vdots\, b_3) \qquad (2.3.6)$$

where

$$\underset{\sim}{b}_j = (b_{1j}, b_{2j}, b_{3j})^T,\ j = 1,2,3$$

b_{ij} being the ith component of $\underset{\sim}{b}_j$ referred to the basis A, i.e.,

$$\underset{\sim}{b}_j = b_{1j}\underset{\sim}{a}_1 + b_{2j}\underset{\sim}{a}_2 + b_{3j}\underset{\sim}{a}_3$$

Since both A and B are orthonormal, $(\underset{\sim}{P})_A$ is an orthogonal matrix. Thus, the canonical form can be obtained from the following similarity transformation

$$(\underset{\sim}{Q})_B = (\underset{\sim}{P}^T)_A (\underset{\sim}{Q})_A (\underset{\sim}{P})_A \qquad (2.3.7)$$

From the cononical form given above, it is apparent that

$$\mathrm{Tr}(\underset{\sim}{Q})_B = 1 + 2\cos\theta$$

from which

$$\theta = \cos^{-1}\{\tfrac{1}{2}(\mathrm{Tr}(\underset{\sim}{Q})_B - 1)\} \qquad (2.3.8)$$

is readily obtained. It should be pointed out that, since the trace is invariant under similarity transformations, i.e. since

$$\mathrm{Tr}(\underset{\sim}{Q})_B = \mathrm{Tr}\ (\underset{\sim}{Q})_A$$

one can compute the rotation angle without transforming the revolute matrix into its canonical form.

Eq. (2.3.8), however, yields the angle of rotation through the $\cos^{-1}$ function, which is even, i.e. $\cos^{-1}(-x) = \cos^{-1}(x)$; hence, the said formula does not provide the sign of the angle. This is next determined by application of Theorem 2.3.2. The proof of this theorem needs some background, which is now laid down.

In what follows, dyadic notation will be used*. Let L be the axis of a

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* For readers unfamiliar with this notation, a short account of algebra of dyadics is provided in Appendix I.

rotation about point O, whose existence is guarenteed by Euler's Theorem. Moreover, let θ be the corresponding angle of rotation, as indicated in Fig 2.3.4, and $\underset{\sim}{e}$ a unit vector parallel to L.

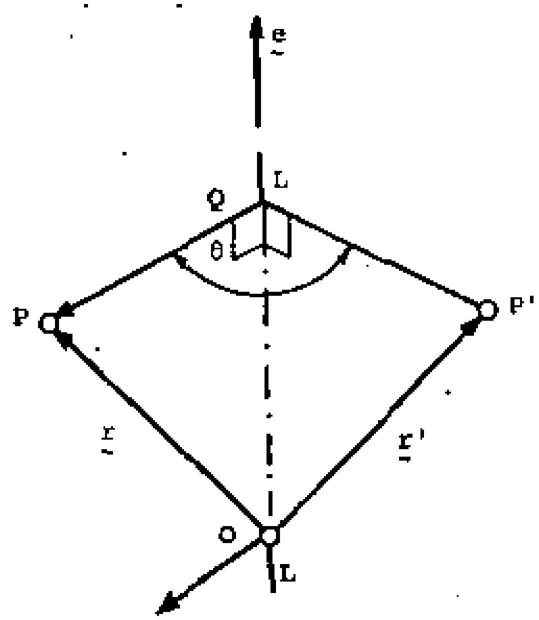

Fig 2.3.4 Rotation about a point.

In Fig 2.3.4 P' is the rotated position of point P. If PQ is perpendicular to L, so is P'Q, because rotations preserve angles of rigid bodies. Thus points P, P', and Q determine a plane perpendicular to L, on which the angle of rotation, θ, is measured. From that figure,

$$\underset{\sim}{r}' = \overrightarrow{OQ} + \overrightarrow{QP}'$$

and

$$\overrightarrow{OQ} = \underset{\sim}{r} - \overrightarrow{QP}$$

Hence

$$\underset{\sim}{r}' = \underset{\sim}{r} - \overrightarrow{QP} + \overrightarrow{QP}' \qquad (2.3.9)$$

Let QP" be a line contained in plane PP'Q, at right angles with line PQ and of length equal to that of QP. Thus, vector $\overrightarrow{QP}'$ can be expressed as a linear combination of vectors $\overrightarrow{QP}$ and $\overrightarrow{QP}"$. But

$$\overrightarrow{QP}" = \underset{\sim}{e} \times \underset{\sim}{r} \qquad (2.3.10)$$

whereas

$$\overrightarrow{QP}^{*} = - \underline{e} \times \overrightarrow{QP}'' = - \underline{e} \times (\underline{e} \times \underline{r}) \qquad (2.3.11)$$

which can readily be proved. Besides, $\overrightarrow{QP}'$ can be expressed as

$$\overrightarrow{QP}' = \overrightarrow{QP} \cos\theta + \overrightarrow{QP}'' \sin\theta$$

which, in view of eqs. (2.3.10) and (2.3.11), yields

$$\overrightarrow{QP}' = - \cos\theta \underline{e} \times (\underline{e} \times \underline{r}) + \sin\theta \underline{e} \times \underline{r} \qquad (2.3.12)$$

Substituting eqs. (2.3.11) and (2.3.12) into eq. (2.3.9) leads to

$$\underline{r}' = \underline{r} + \underline{e} \times (\underline{e} \times \underline{r}) - \cos\theta \underline{e} \times (\underline{e} \times \underline{r}) + \sin\theta \underline{e} \times \underline{r} \qquad (2.3.13)$$

But

$$\underline{e} \times (\underline{e} \times \underline{r}) = (\underline{e} \cdot \underline{r})\underline{e} - (\underline{e} \cdot \underline{e})\underline{r} = (\underline{\underline{e}}\underline{e} - \underline{\underline{1}}) \cdot \underline{r} \qquad (2.3.14)$$

where $\underline{\underline{1}}$ is the identity dyadic, i.e. a dyadic that is isomorphic to the identity matrix. Furthermore

$$\underline{e} \times \underline{r} = \underline{\underline{1}} \cdot \underline{e} \times \underline{r} = \underline{\underline{1}} \times \underline{e} \cdot \underline{r} \qquad (2.3.15)$$

where the dot and the point have been exchanged, what is possible to do by virtue of the algebra of cartesian vectors. Substituting eqs. (2.3.14) and (2.3.15) into eq. (2.3.13) one obtains

$$\begin{aligned}\underline{r}' &= \underline{r} + (1-\cos\theta)(\underline{e}\underline{e} - \underline{\underline{1}}) \cdot \underline{r} + \sin\theta \underline{\underline{1}} \times \underline{e} \cdot \underline{r} = \\ &= \left((1-\cos\theta)\underline{e}\underline{e} + \cos\theta \underline{\underline{1}} + \sin\theta \underline{\underline{1}} \times \underline{e}\right) \cdot \underline{r} = \\ &= \underline{\underline{Q}} \cdot \underline{r} \end{aligned} \qquad (2.3.16)$$

i.e. $\underline{r}'$ has been expressed as a linear transformation of vector $\underline{r}$. The dyadic $\underline{\underline{Q}}$ is, then, isomorphic to the rotation matrix defined in Section 2.2. That is

$$\underline{\underline{Q}} = \underline{e}\underline{e} + (\underline{\underline{1}} - \underline{e}\underline{e})\cos\theta + \sin\theta \underline{\underline{1}} \times \underline{e} \qquad (2.3.17)$$

One can now prove the following

<u>THEOREM 2.3.2</u> *Let a rigid body undergo a pure rotation about a fixed point O and let $\underline{r}$ and $\underline{r}'$ be the initial and the final position vectors of a point of the body (measured from O) not lying on the axis of rotation*

Furthermore let θ and e be the angle of rotation and the unit vector pointing in the direction of the rotation. Then

$$sgn(r \times r' \cdot e) = sgn(\theta)$$

Proof.

Application of eq. (2.3.16) leads to

$$r \times r' = (1-\cos\theta)(e \cdot r) r \times e + \sin\theta\, r \times (e \times r) =$$

$$= (1-\cos\theta)(e \cdot r) r \times e + \sin\theta \left(r^2 e - (r \cdot e) r\right)$$

where

$$r^2 \equiv ||r||^2$$

Thus,

$$r \times r' \cdot e = \sin\theta \left(r^2 - (r \cdot e)^2\right)$$

which can be reduced to

$$r \times r' \cdot e = r^2 \sin\theta \sin^2(r, e)$$

where (r, e) is the angle between vectors r and e. Hence,

$$sgn(r \times r' \cdot e) = sgn(\sin\theta)$$

But

$$sgn(\sin\theta) = sgn(\theta)$$

for sin( ) is an odd function, i.e. sin (-x =-sin(x).

Finally, then

$$sgn(r \times r' \cdot e) = sgn(\theta), \text{q.e.d} \tag{2.3.18}$$

In conclusion, Theorem 2.3.2 allows to distinguish whether a rotation in the specified direction e is either through an angle θ or through an angle -θ.

Exercise 2.3.2 Let p and p' be the initial and the final position vectors of a point P of a rigid body undergoing a screw motion whose rotation matrix is Q. Show that the displacement $p' - Qp$ lies in the null space of $Q - I$.

Exercise 2.3.3 Show that the trace of a matrix is invariant under similarity transformations.

Exercise 2.3.4 Show that a revolute matrix $\underset{\sim}{Q}$ has two complex conjugate eigenvalues, λ and $\overline{\lambda}$ ($\overline{\lambda}$ = complex conjugate of λ).

Furthermore, show that

$$\mathrm{Re}\{\lambda\} = \frac{1}{2}(\mathrm{Tr}\underset{\sim}{Q}-1)$$

What is the relationship between the complex eigenvalues of the revolute matrix and its angle of rotation?.

In the foregoing paragraphs the revolute matrix was analyzed, i.e. it was shown how to obtain its invariants when the matrix is known. The inverse problem is discussed next: Given the axis and the angle of rotation, obtain the revolute matrix referred to a specified set of coordinate axes.

It is apparent that the most convenient basis (or coordinate axes) for representing the revolute matrix is the one for which this takes on its canonical form. Let $B = \{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \underset{\sim}{b}_3\}$ be this basis, where $\underset{\sim}{b}_3$ coincides with the given revolute axis, and $\underset{\sim}{b}_1$ and $\underset{\sim}{b}_2$ are any pair of orthonormal vectors lying in the plane perpendicular to $\underset{\sim}{b}_3$.

Hence, $(\underset{\sim}{Q})_B$ appears as in eq. (2.3.5), with given θ. Let $A = \{a_1, a_2, a_3\}$ be an othonormal basis with respect to which $\underset{\sim}{Q}$ is to be represented, and let

$$(\underset{\sim}{P})_A = (\underset{\sim}{b}_1 \;\vdots\; \underset{\sim}{b}_2 \;\vdots\; \underset{\sim}{b}_3)_A$$

be a matrix formed with the vectors of B. Then, it is clear that

$$(\underset{\sim}{Q})_A = (\underset{\sim}{P}^T)_A (\underset{\sim}{Q})_B (\underset{\sim}{P}^T)_A$$

<u>Example 2.3.1</u> Let

$$\underset{\sim}{Q} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

Verify whether it is orthogonal. If it is, does it represent a rotation?.

If so, describe the rotation

<u>Solution:</u>

$$\underset{\sim}{Q}\underset{\sim}{Q}^T = \frac{1}{9}\begin{pmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \underset{\sim}{I}$$

Hence $\underset{\sim}{Q}$ is in fact orthogonal. Next,

$$\det \underset{\sim}{Q} = \frac{2}{3}\begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{vmatrix} + \frac{2}{3}\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{vmatrix} - \frac{1}{3}\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

$$= \frac{2}{3}\left(\frac{4}{9} + \frac{2}{9}\right) + \frac{2}{3}\left(\frac{2}{9} + \frac{4}{9}\right) - \frac{1}{3}\left(\frac{1}{9} - \frac{4}{9}\right) = +1$$

Thus $\underset{\sim}{Q}$ is a proper orthogonal matrix and consequently represents a rotation.

To find the axis of the rotation it is necessary to find a unit vector

$\underset{\sim}{e} = (e_1, e_2, e_3)^T$ such that

$$\underset{\sim}{Q}\underset{\sim}{e} = \underset{\sim}{e},$$

i.e.

$$\frac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Hence

$$-e_1 + e_2 + 2e_3 = 0$$

$$-2e_1 - e_2 + e_3 = 0$$

$$-e_1 - 23_2 - e_3 = 0$$

from which

$$e_1 = e_3$$

$$e_2 = -e_3$$

and so

$$\underset{\sim}{e} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e_3 .$$

Setting $||\underset{\sim}{e}||=1$, it follows that $e_3 = \frac{\sqrt{3}}{3}$, and

$$\underset{\sim}{e} = \frac{\sqrt{3}}{3} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Thus, the axis of rotation is parallel to the vector $\underset{\sim}{e}$ given above.

To find the angle of rotation - is an even simpler matter:

$$\text{Tr } \underset{\sim}{Q} = \frac{1}{3}\,(2+2+2) = 1 + 2\cos\theta$$

Thus $\theta = \cos^{-1}(\frac{1}{2}) = -60°$

where use was made of Theorem 2.3.2 to find the sign of θ.

Example 2.3.2. Determine the revolute matrix representing a rotation of 90° about an axis having three equal direction cosines with respect to the X,Y,Z axes. The matrix should be expressed with respect to these axes.

Solution:

Let $B = \{b_1, b_2, b_3\}$ be an orthonormal basis with respect to which the revolute is represented in its canonical form. Let b_3 be coincident with the axis of rotation. Clearly

$$b_3 = \frac{\sqrt{3}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

It remains only to determine b_1 and b_2. Clearly, these must satisfy

$$b_1 \cdot b_2 = b_1 \cdot b_3 = b_2 \cdot b_3 = 0.$$

Let

$$(b_1) = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}, \quad (b_2) = \begin{pmatrix}\lambda\\\mu\\\nu\end{pmatrix}.$$

Thus, the components of b_1 must satisfy

$$\alpha+\beta+\gamma=0,$$
$$\alpha^2+\beta^2+\gamma^2=1.$$

It is apparent that one component can be freely chosen. Let, for example,

$$\alpha = 0$$

Hence,

$$\beta + \gamma = 0$$
$$\beta^2 + \gamma^2 = 1$$

from which

$$2\beta^2=1. \quad \text{Thus } \beta = \pm\frac{\sqrt{2}}{2}, \gamma = \mp\frac{\sqrt{2}}{2}$$

Thus, choosing the + sign for β,

$$\underset{\sim}{b}_1=\frac{1}{2}\begin{Bmatrix} 0 \\ \sqrt{2} \\ \sqrt{2} \end{Bmatrix}$$

$\underset{\sim}{b}_2$ can be obtained now very easily from the fact that $\underset{\sim}{b}_1, \underset{\sim}{b}_2$ and $\underset{\sim}{b}_3$ constitute an orthonormal right-hand triad, i.e.

$$\underset{\sim}{b}_2=\underset{\sim}{b}_3 \times \underset{\sim}{b}_1=\frac{\sqrt{6}}{6}\left(-2,1,1\right)^T$$

With respect to this basis, then, from eq. (2.3.5) the rotation matrix has the form

$$\left(\underset{\sim}{Q}\right)_B=\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, letting A be the basis defined by the given X, Y and Z axes,

$$\left(\underset{\sim}{P}\right)_A=\begin{bmatrix} 0 & -\sqrt{6}/3 & \sqrt{3}/3 \\ \sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3 \\ -\sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3 \end{bmatrix}$$

and, from eq. (1.5.12), defining the following similarity transformation,

$$\left(\underset{\sim}{Q}\right)_A=\left(\underset{\sim}{P}\right)_A\left(\underset{\sim}{Q}\right)_B\left(\underset{\sim}{P}^T\right)_A$$

With $\left(\underset{\sim}{Q}\right)_B$ in its canonical form, the revolute matrix $\underset{\sim}{Q}$, expressed with respect to the X,Y,Z axes, is found to be

$$\left(\underset{\sim}{Q}\right)_A=\frac{1}{2}\begin{bmatrix} 1 & 1-\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1 & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} & 1 \end{bmatrix}$$

Exercise 2.3.6 If the plane

$$x + y + z + 1 = 0$$

is rotated through 60° about an axis passing through the point (-1, -1, -1)

and with direction cosines $\frac{1}{\sqrt{3}}$, $-\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$, what is the equation of the plane in its new position?.

Exercise 2.3.7. The four vertices of an equilateral tetrahedron are labelled A, B, C, and D. If the tetrahedron is rotated in such a way that A, B, C, and D are mapped into C, B, D, and A, respectively, find the axis and the angle of the rotation.

What are the other rotations similar to the previous one, i.e., which map every vertex of the tetrahedron into another vertex?.

All these rotations, together with the identity rotation (the one leaving the vertices of the tetrahedron unchanged), constitute the symmetry group* of the tetrahedron.

Exercise 2.3.8 Given an axis A whose direction cosines are $(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})$, with respect to a set of coordinate axes XYZ, what is the matrix representation, with respect to these coordinate axes, of a rotation about A through an angle $2\pi/n$?

Exercise 2.3.9 A square matrix $\underset{\sim}{A}$ is said to be idempotent of index k whenever k is the smallest integer for which the k^{th} power of $\underset{\sim}{A}$ becomes the identity matrix. Explain why the matrix obtained in Exercise 2.3.8 should be idempotent of index n.

Exercise 2.3.10 Show that any rotation matrix $\underset{\sim}{Q}$ can be expressed as

$$\underset{\sim}{Q}=e^{\underset{\sim}{A}\theta}$$

where $\underset{\sim}{A}$ is a nilpotent matrix and θ is the rotation angle. What is the relationship between matrix $\underset{\sim}{A}$ and the axis of rotation of $\underset{\sim}{Q}$?

*See Sect. 2.4 for the definition of this term.

Exercise 2.3.11 The equation of a three-axes ellipsoid is given as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

what is its equation after rotating it through an angle θ about an axis of direction numbers (a,b,c)?

2.4 GROUPS OF ROTATIONS.

A group is a set g with a binary operation o such that

i) if a and b $\in g$, then $a o b \in g$

ii) if a,b,c then $a o (b o c) = (a o b) o c$

iii) g contains an element i, called the identity of g under o, such that, for every a $\in g$

$$a o i = i o a = a$$

iv) for every a $\in g$, there exists an element denoted $a^{-1} \in g$, called the inverse of a under o such that

$$a o a^{-1} = a^{-1} o a = i$$

Notice that in the above definition it is not required that the group be commutative, i.e. that $a o b = b o a$ for all a,b$\in g$. Commutative groups are a special class of groups, called abelian groups.

Some examples of groups are:

a) The natural numbers 1,2,..., 12 on the face of a (mechanical, not quartz or similar) clock and the operation $k o m$ corresponding to "shift the clock hand from location k to location k + m", where k and m are natural numbers between 1 and 12. Of course, if k + m>12, the resulting operation is meant to be (k + m)(mod 12).

b) The set of rational numbers with the usual multiplication operation.

c) The set of integers with the usual algebraic addition operation. The set of integers with the multiplication operation do not constitute a group (Why?)

Exercise 2.4.1 Show that the set of all the rotations referred to in Exercise 2.3.5 actually constitute a group.

Exercise 2.4.2 What is the symmetry group* of

i) an icosahedron?

ii) a regular pentagonal prism?

iii) a circular cylinder?

iv) a sphere?

It is clear, from the above discussion, that the set of all orthogonal matrices constitute a group under matrix multiplication. In particular, the set of proper orthogonal matrices constitutes a group under matrix multiplication, but the improper set does not (Why?).

As an application of the group property of rotations or, equivalently, of proper orthogonal matrices, arbitrary rotations can be formed by the composition of successive simple rotations (See Example 2.4.1).

Another application is found in the compositon of rotations using Euler's angles (2.2)

Example 2.4.1 Referring to Fig 2.4.1, find the matrix representation, with respect to the X_1, Y_1, Z_1 axes, of the rotation that carries vertices A and B of the cube into A' and B', respectively, while leaving vertex O fixed. A' and B' lie in the Y_1Z_1 plane and points A', O and D, are collinear, as are B', F and E.

* See Exercise 2.3.5 for a definition of a symmetry group.

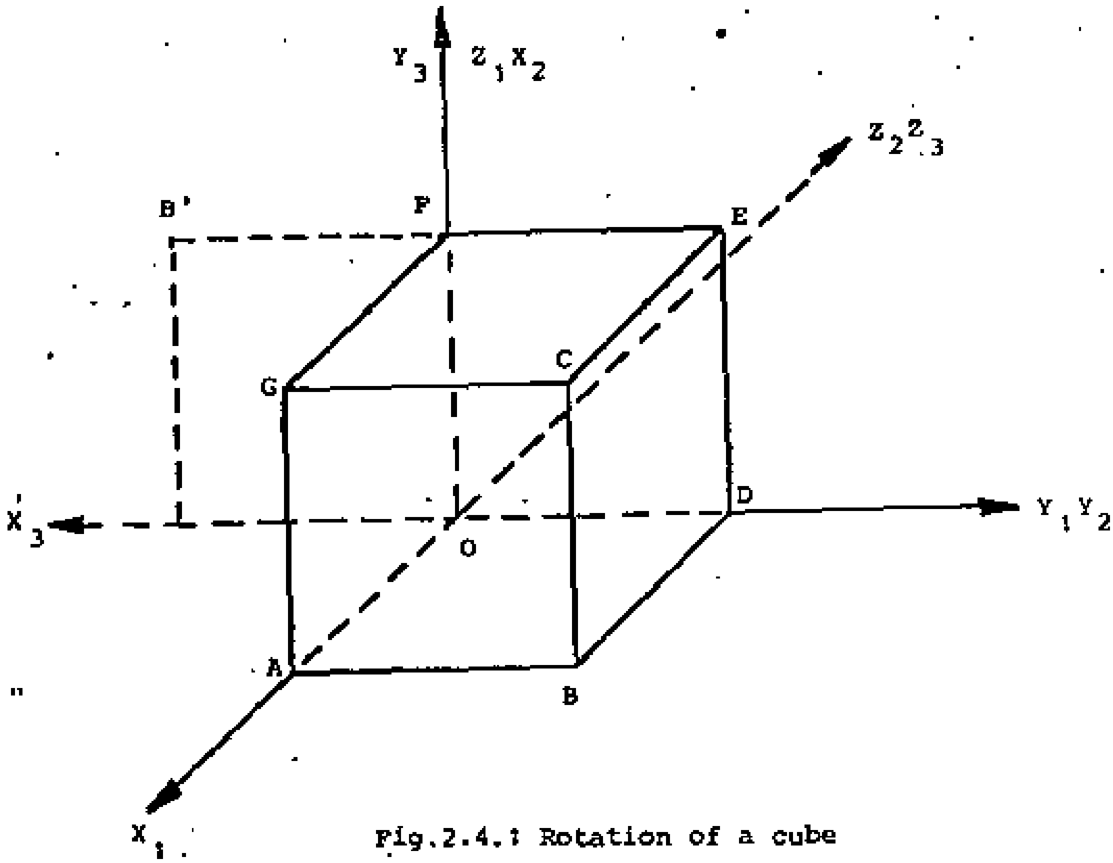

Fig.2.4.1 Rotation of a cube

Solution:

Let $(Q_{12})_1$ be the matrix representing the rotation of axes labelled 1 into those labelled 2(referred to axes 1). Then, letting x_i, y_i and z_i be unit vectors directed along the X_i-, Y_i- and Z_i axes, respectively,

$$Q_{12}x_1 = x_2 = z_1$$

$$Q_{12}y_1 = y_2 = y_1$$

$$Q_{12}z_1 = z_2 = -x_1$$

from which

$$(Q_{12})_1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Next, rotate axes labelled 2 into axes labelled 3. Call this rotation Q_{23}. This rotation would leave axis X_1 fixed whereas it would carry axis Y_1 into Z_1 and axis Z_1 into $-Y_1$. Hence,

$$\underset{\sim}{Q}_{23}\underset{\sim}{x}_1 = \underset{\sim}{x}_1$$

$$\underset{\sim}{Q}_{23}\underset{\sim}{y}_1 = \underset{\sim}{z}_1$$

$$\underset{\sim}{Q}_{23}\underset{\sim}{z}_1 = \underset{\sim}{y}_1$$

and so,

$$\left(Q_{23}\right)_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Let $\underset{\sim}{Q}_{13}$ be the rotation meant to be obtained. Its matrix can be computed then as

$$\left(Q_{13}\right)_1 = \left(Q_{23}\right)_1\left(Q_{12}\right)_1 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which could also have been obtained by noticing that

$$\underset{\sim}{Q}_{13}\underset{\sim}{x}_1 = \underset{\sim}{x}_3 = -\underset{\sim}{y}_1$$

$$\underset{\sim}{Q}_{13}\underset{\sim}{y}_1 = \underset{\sim}{y}_3 = \underset{\sim}{z}_1$$

$$\underset{\sim}{Q}_{13}\underset{\sim}{z}_1 = \underset{\sim}{z}_3 = -\underset{\sim}{x}_1$$

Matrix $\left(\underset{\sim}{Q}_{13}\right)_1$ represents a rotation through an angle $\theta = 120°$ about an axis with direction cosines $-\alpha, \alpha, \alpha$. Although in this example the rotation could be obtained by an alternate method, in many cases, such as the one in Exercise 2.4.3, the use of rotation composition seems to be the simplest method.

<u>Exercise 2.4.3</u> Determine the axis and the angle of the rotation carrying axes X, Y, Z into axes ξ, η, ζ, as shown in Fig. 2.4.2

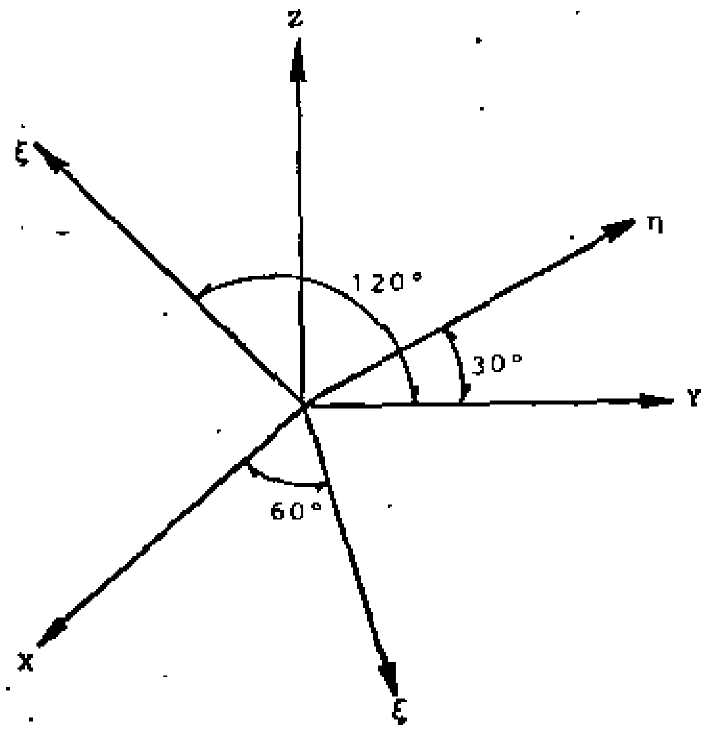

Fig 2.4.2 Rotation of axes

Exercise 2.4.4 The cube appearing in Fig 2.4.1 is rotated 45° about diagonal OC. Find the matrix representation, with respect to X_1, Y_1, Z_1, of this rotation and the distance that vertex B is displaced through.

2.5 RODRIGUES' FORMULA AND CARTESIAN DECOMPOSITION OF THE ROTATION MATRIX.

The image $\underline{r}_2$ of a Cartesian vector $\underline{r}_1$ under a rotation through an angle θ about an axis parallel to the unit vector $\underline{e}$ passing through the origin of coordinates was shown to be (See Section 2.3)

$$\underline{r}_2 = \left((1-\cos\theta)\underline{e}\underline{e}+\cos\theta\underline{1}+\sin\theta\underline{1}\times\underline{e}\right)\cdot\underline{r}_1 \tag{2.5.1}$$

Multiplying both sides of eq. (2.5.1) times $\underline{e}\times$ yields

$$\underline{r}_2-\underline{r}_1=\tan\frac{\theta}{2}\underline{e}\times(\underline{r}_1+\underline{r}_2) \tag{2.5.2}$$

which is called Rodrigues' formula (2.3, 2.4)

Form (2.5.1) of the rotation dyadic is advantageous since it shows explicitly the invariants $\underline{e}$ and θ of the rotation.

Other useful expression of the rotation matrix is now derived. Letting

$$\underline{e} = (u,v,w)^T \qquad (2.5.3)$$

the rotation matrix can be written as (2.5)

$$\underline{Q} = \underline{R} + \underline{T}\cos\theta + \underline{P}\sin\theta \qquad (2.5.4)$$

where

$$\underline{R} = \begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & uw \\ uw & vw & w^2 \end{pmatrix}, \quad \underline{T} = \begin{pmatrix} v^2+w^2 & -uv & -uw \\ -uv & u^2+w^2 & -vw \\ -uw & -vw & u^2+v^2 \end{pmatrix} \qquad (2.5.5a)$$

and

$$\underline{P} = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix} \qquad (2.5.5b)$$

In fact, computing the dyadics involved in expression (2.3.17),

$$\begin{aligned} \underline{e}\underline{e} &= (u\underline{i} + v\underline{j} + w\underline{k})(u\underline{i} + v\underline{j} + w\underline{k}) = \\ &= u^2\underline{i}\underline{i} + uv\underline{i}\underline{j} + uw\underline{i}\underline{k} + \\ &+ uv\underline{j}\underline{i} + v^2\underline{j}\underline{j} + vw\underline{j}\underline{k} + \\ &+ uw\underline{k}\underline{i} + vw\underline{k}\underline{j} + w^2\underline{k}\underline{k} \end{aligned} \qquad (2.5.6)$$

$$\underline{1} = \underline{i}\underline{i} + \underline{j}\underline{j} + \underline{k}\underline{k} \qquad (2.5.7)$$

Hence

$$\begin{aligned} \underline{1} \times \underline{e} &= (\underline{i}\underline{i} + \underline{j}\underline{j} + \underline{k}\underline{k}) \times (u\underline{i}+v\underline{j}+w\underline{k}) = \\ &= u\underline{i}\underline{i}\times\underline{i} + v\underline{i}\underline{i}\times\underline{j} + w\underline{i}\underline{i}\times\underline{k} + \\ &+ u\underline{j}\underline{j}\times\underline{i} + v\underline{j}\underline{j}\times\underline{j} + w\underline{j}\underline{j}\times\underline{k} + \\ &+ u\underline{k}\underline{k}\times\underline{i} + v\underline{k}\underline{k}\times\underline{j} + w\underline{k}\underline{k}\times\underline{k} \end{aligned} \qquad (2.5.8)$$

But

$$\underline{i} \times \underline{i} = \underline{j}\times\underline{j} = \underline{k}\times\underline{k} = \underline{0} \qquad (2.5.9)$$

and

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}\ ,\ \mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j} \qquad (2.5.10a)$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \qquad (2.5.10b)$$

Thus

$$\begin{aligned} \mathbf{1} \times \mathbf{e} = & -w\mathbf{ij} + v\mathbf{ik} \\ & + w\mathbf{ji} - u\mathbf{jk} \\ & - v\mathbf{ki} + u\mathbf{kj} \end{aligned} \qquad (2.5.11)$$

Dyadics (2.5.6) and (2.5.7) can be written in matrix form as

$$(\mathbf{ee}) = \begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & uw \\ uw & vw & w^2 \end{pmatrix},\ (\mathbf{1}\times\mathbf{e}) = \begin{pmatrix} 0 & -w & v \\ w & 0 & -u \\ -v & u & 0 \end{pmatrix} \qquad (2.5.12)$$

and

$$(\mathbf{1} - \mathbf{ee}) = \begin{pmatrix} v^2+w^2 & -uv & -uv \\ -uv & u^2+w^2 & -vw \\ -uw & -uw & u^2+v^2 \end{pmatrix}, \qquad (2.5.13)$$

Substitution of matrices (2.5.12) and (2.5.13) into eq. (2.3.17) leads directly to eq. (2.5.4). This expression of matrix $\mathbf{Q}$ is very useful because it allows one to determine the sign of θ without requiring to compute the image $\mathbf{r}'$ of a vector $\mathbf{r}$ under $\mathbf{Q}$.

Indeed, from eqs. (2.5.5a) and (2.5.5b), it is clear that matrices $\mathbf{R}$ and $\mathbf{I}$ are symmetric, whereas $\mathbf{P}$ is skew symmetric. Hence, and from Theorem 1.7.1, $\mathbf{P}\sin\theta$ can be obtained as

$$\mathbf{P}\sin\theta = \frac{1}{2}(\mathbf{Q} - \mathbf{Q}^T) \qquad (2.5.14)$$

i.e. eq. (2.5.4) can be regarded as the cartesian decomposition (see Section 1.7) of matrix $\mathbf{Q}$. Now, calling e_i the ith component of vector $\mathbf{e}$, as given by eq. (2.5.3) and taking definition (2.5.5b) and eq. (2.5.14) into

account, one obtains

$$-e_1\sin\theta = \frac{1}{2}(q_{23}-q_{32}) \tag{2.5.15a}$$

$$e_2\sin\theta = \frac{1}{2}(q_{13}-q_{31}) \tag{2.5.15b}$$

$$-e_3\sin\theta = \frac{1}{2}(q_{12}-q_{21}) \tag{2.5.15c}$$

Introducing the alternating tensor ε_{ijk} defined as

$$\varepsilon_{ijk} = \begin{cases} +1, \text{if } i,j \text{ and } k \text{ are in cyclic order} \\ -1, \text{ if } i,j \text{ and } k \text{ are in acyclic order} \\ \ \ 0, \text{ if at least one index is repeated} \end{cases}$$

eqs. (2.5.15) can be written as

$$e_i\sin\theta = -\frac{1}{2}\varepsilon_{ijk}(q_{jk}-q_{kj})$$

from which, if e_i does not vanish,

$$\operatorname{sgn}\theta = \varepsilon_{ijk}\operatorname{sgn}\left(\frac{q_{kj}-q_{jk}}{e_i}\right)$$

follows directly.

<u>Exercise 2.5.1</u> Given matrices $\mathbf{T}$ and $\mathbf{P}$, as defined in eqs. (2.5.5a) and (2.5.5b), prove that $\mathbf{T} = -\mathbf{P}^2$ and device an algorithm to compute $\mathbf{P}$ given $\mathbf{T}$.

<u>Exercise 2.5.2</u> Use eq. (2.5.16) to determine the sign of θ for the rotation matrix of Example 2.3.1 and verify the result thus obtained with the one obtained previously.

2.6 GENERAL MOTION OF A RIGID BODY AND CHASLES' THEOREM

In the previous sections only the motion of a rigid body about a fixed point was discussed. There are rigid body motions, however, with no fixed point. Such motions are studied in this section.

Consider a motion under which one point is displaced from A to A' and another one is displaced from R to R', as shown in Fig 2.6.1
This motion can take place in any of three different ways, namely i) any pair of points A,R of the body undergo a displacement to A', R', respectively in such a way that line A' R' is parallel to line AR; this motion is referred to as pure translation; ii) a line of the body remains fixed, in which case, according to Euler's Theorem (Theorem 2.3.1), the motion is referred to as pure rotation;

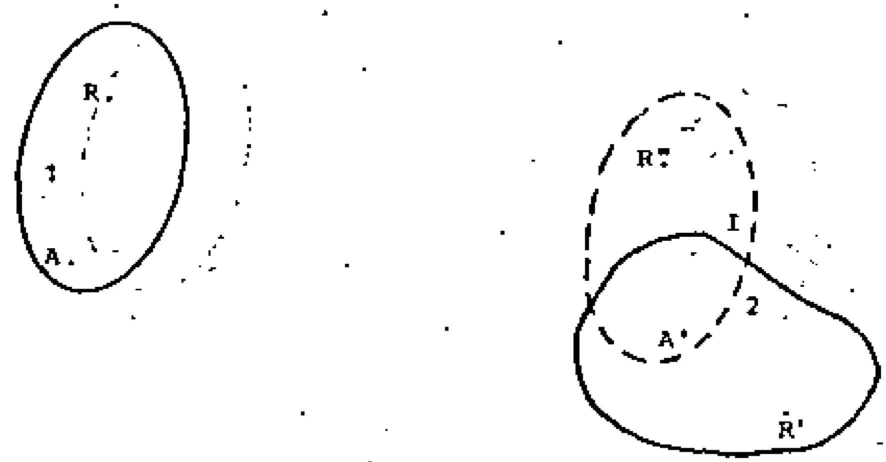

Fig 2.6.1 General motion of a rigid body

iii) no point of the body remains fixed under the motion, in which case it is referred to as general motion.

The motion from configuration 1 to configuration 2 can be regarded as the composition of two motions: first the rigid body is displaced from 1 to I without any rotation. Hence, the lines connecting any pair of points in 1 are parallel to those connecting them in the intermediate configuration I. Since this is a rigid body motion, the length of each segment remains unchanged. Thus, letting $\underline{a}, \underline{a}'$, $\underline{r}, \underline{r}'$ and $\underline{r}''$ be the position vectors of points A,A', R,R' and R", respectively,

$$\underline{r}''-\underline{a}'=\underline{r}-\underline{a} \qquad (2.6.1)$$

Next, to take the body into its final configuration, 2, a rigid body rotation $\underline{Q}$, about point A', must be performed.

Thus,

$$\underline{r}'-\underline{a}'=\underline{Q}(\underline{r}''-\underline{a}') \qquad (2.6.2)$$

Substitution of (2.6.1) into (2.6.2) and rearrangement of the terms yield

$$\underline{r}'=\underline{a}'+\underline{Q}(\underline{r}-\underline{a}) \qquad (2.6.3)$$

which is an expression for the final position R' of any point R of the rigid body in terms of: i) its initial position, $\underline{r}$, ii) the initial and the final position of any other point A, and iii) the rotation $\underline{Q}$ accompanying the motion. The above expression could have also been obtained considering first a rigid body rotation about point A from 1 to an intermediate configuration I' in which all lines connecting any pair of points are parallel to the corresponding lines in 2 and, since the motion is rigid, the segments thus defined are of equal lengths; then, perform a pure translation from I' to 2. Summarizing: The general motion of a rigid body is completely defined by the initial and final positions of any one of its points and the rotation involved.

<u>Exercise 2.6.1</u> Obtain eq. (2.6.3) by performing first a rotation and then a translation.

The main result in this section is <u>Chasles' Theorem</u>, which states that, given <u>any</u> rigid body displacement, it can always be obtained as the rotation about a line of the body, known as "the screw axis", followed by a translation parallel* to the rotation axis. Moreover, the displacements of all points of the body along the screw axis are of minimum magnitude. The displacement vector of a point is defined as the vector between the final and the initial positions of the point, e.g. the displacement of point R in the previous discussion is

$$\underset{\sim}{u}=\underset{\sim}{r}'-\underset{\sim}{r}=\underset{\sim}{a}'+\underset{\sim}{Q}(\underset{\sim}{r}-\underset{\sim}{a})-\underset{\sim}{r}=$$

$$=\underset{\sim}{a}'-\underset{\sim}{Q}\underset{\sim}{a}+(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{r} \qquad (2.6.3a)$$

From eq. (2.6.3) notice that $\underset{\sim}{u}$ is a linear function of one single variable, $\underset{\sim}{r}$. Hence, the norm of $\underset{\sim}{u}$ is a linear function of $\underset{\sim}{r}$ only. The square of this norm is quadratic in $\underset{\sim}{r}$ and is given as

$$\phi(\underset{\sim}{r})=\underset{\sim}{u}^T\underset{\sim}{u}=\underset{\sim}{r}^T(\underset{\sim}{Q}-\underset{\sim}{I})^T(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{r}+2(\underset{\sim}{a}'-\underset{\sim}{Q}\underset{\sim}{a})^T(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{r}+(\underset{\sim}{a}'-\underset{\sim}{Q}\underset{\sim}{a})^T(\underset{\sim}{a}'-\underset{\sim}{Q}\underset{\sim}{a}) \qquad (2.6.4)$$

The theorem is now proved via the minimization of $\phi(\underset{\sim}{r})$. This function has one extremum at the point $\underset{\sim}{r}_0$ where $\phi'(\underset{\sim}{r}_0)=\underset{\sim}{0}$. The derivative $\phi'(\underset{\sim}{r})$ is next computed, and zeroed at $\underset{\sim}{r}_0$.

Applying the "chain rule" to ϕ,

$$\phi'(\underset{\sim}{r}) = \left(\frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{r}}\right)^T \frac{\partial \phi}{\partial \underset{\sim}{u}}$$

* The direction of a pure translation of a rigid body is understood here as the direction of the displacement vectors of the points of the body.

where, from eq. (2.6.3a),

$$\frac{\partial u}{\partial r} = Q - I \qquad (2.6.5a)$$

and

$$\frac{\partial \phi}{\partial u} = 2u \qquad (2.6.5b)$$

Thus, letting $u_0 = u(r_0)$, the zeroing of the gradient of ϕ at $r = r_0$ leads to

$$(Q-I)^T u_0 = 0 \qquad (2.6.6)$$

or

$$Q^T u_0 = u_0 \qquad (2.6.6a)$$

Now, if both sides of eq. (2.6.6) are multiplied by Q, one obtains

$$Q u_0 = u_0 \qquad (2.6.6b)$$

thereby concluding that the minimum-norm displacement u_0 lies in the real spectral space of Q, i.e., it is parallel to the axis of rotation of Q.

What is now left to complete the proof of Chasles' Theorem is to determine the set of points of the rigid body having a displacement vector parallel to the rotation axis. This is done next.

Substituting u, evaluated at r_0, as given by eq. (2.6.3a) into eq. (2.6.6), and rearranging terms leads to

$$(Q-I)^T (Q-I) r_0 = (Q-I)^T (Qa-a') \qquad (2.6.6c)$$

from which r_0 cannot be solved for, since $(Q-I)$, and hence $(Q-I)^T(Q-I)$ is singular. In fact, it can be readily proved that this matrix is of rank 2.

Exercise 2.6.1 Prove that $(Q-I)^T(Q-I)$ is of rank 2, except for $Q=I$.

Although r_0 cannot be solved for from the latter equation, interesting results can be derived from it. Indeed, given a point R_0, with position vector r_0, of minimum-magnitude displacement u_0, define a new point S_0,

with position vector $\mathbf{s}_0$ given as

$$\mathbf{s}_0 = \mathbf{r}_0 + \alpha\mathbf{e}$$

where $\mathbf{e}$ is the unit vector parallel to the axis of rotation of $\mathbf{Q}$. Multiplying $\mathbf{s}_0$, as given before, times $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$ gives

$$(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{s}_0=(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})(\mathbf{r}_0+\alpha\mathbf{e})=(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{r}_0+ (\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{e}$$

but $\mathbf{e}$, being parallel to the rotation axis of $\mathbf{Q}$, is in the null space of $\mathbf{Q}-\mathbf{I}$, hence, in the null space of $(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})$. Therefore, the second term in the right-hand side of the latter equation vanishes, the latter equation thus reducing to

$$(\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}-\mathbf{I})\mathbf{s}_0 = (\mathbf{Q}-\mathbf{I})^T(\mathbf{Q}\mathbf{a}-\mathbf{a}')$$

i.e. $\mathbf{s}_0$ also satisfies eq. (2.6.6c). In conclusion, <u>all points $\mathbf{s}_0$ of minimum-magnitude displacement, $\mathbf{u}_0$, lie on a line parallel to the axis of rotation of $\mathbf{Q}$.</u>

<u>Exercise 2.6.2</u> Show that the $\mathbf{r}_0$ satisfying eq. (2.6.6) actually yields a minimum.

From Exercise 2.6.1, if $\mathbf{Q}\neq\mathbf{I}$, the rank of $\mathbf{Q}-\mathbf{I}$ is exactly 2. Therefore, two of the three scalar equations of (2.6.5) are linearly independent. These two equations can be expressed in matrix form as

$$\mathbf{A}\mathbf{r}_0 = \mathbf{c} \tag{2.6.7}$$

where $\mathbf{A}$ is a 2 x 3-rank-two matrix and vectors $\mathbf{r}_0$ and $\mathbf{c}$ are 3-and 2-dimensional, respectively. Now, since the rank of $\mathbf{A}$ is 2, $\mathbf{A}\mathbf{A}^T$, being 2x2, is nonsingular and hence, the minimum norm solution to eq. (2.6.7) is (See Section 1.11)

$$\mathbf{r}_0=\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{c} \tag{2.6.8}$$

The geometric interpretation of the previous result is that $\mathbf{r}_0$, as given by (2.6.8), is perpendicular to the sought axis. This axis is "<u>the screw</u>

axis" and is totally determined by the rotation axis, which gives its direction, and the point r_0 whose position vector, r_0, is given by eq. (2.6.8). The name "screw" comes from the fact that the body moves as if it were fastened to the bolt of a screw whose axis were the screw axis. Other facts motivating the name of the screw axis will be shown later. Another method of finiding a point on the screw axis is via Rodrigues' formula as it appears in (2.6). This procedure can be developed as follows: As was pointed out from eq. (2.6.6), the minimum-norm displacement is parallel to the axis of rotation. Hence, the displacement of R_0 must satisy

$$u_0=r_0'-r_0=\alpha e, \tag{2.6.9}$$

where α is a scalar. Substituting the initial and the final position vectors of R in Rodrigues' formula, eq. (2.5.3),

$$r_0'-r_0=\tan\frac{\theta}{2}e\times(r_0'+r_0) \tag{2.6.10}$$

which, together with eq. (2.5.3) for vectors a and a', denoting the initial and the final positions of point A, yields

$$a'-r_0'-(a-r_0)=\tan\frac{\theta}{2}e\times\left((a'-r_0')+(a-r_0)\right) \tag{2.6.11}$$

From eq. (2.6.9),

$$e\times(r_0'-r_0)=0 \tag{2.6.12}$$

Hence, eq. (2.6.11) becomes

$$a'-a=\alpha e=\tan\frac{\theta}{2}e\times(a'+a)-2\tan\frac{\theta}{2}e\times r_0 \tag{2.6.13}$$

Multiplying both sides of eq. (2.6.13) times $\cot\frac{\theta}{2}e\times$,

$$\cot\frac{\theta}{2}e\times(a'-a)=e\times\left(e\times(a'+a)\right)-2e\times(e\times r_0)=$$

$$=e\times\left(e\times(a'+a)\right)-2(e\cdot r_0)e+2(e\cdot e)r_0 \tag{2.6.14}$$

To determine r_0 from eq. (2.6.14), it is necessary to impose one extra

condition on it, which is done next. Let it be the particular point on the screw axis which is closest to the origin; hence,

$$\mathbf{r}_0 \cdot \mathbf{e} = 0$$

and so, substituting this vector into eq. (2.6.14) and solving for $\mathbf{r}_0$ in the same equation, leads to

$$\mathbf{r}_0 = \frac{1}{2}\cot\frac{\theta}{2}\mathbf{e}\times(\mathbf{a}'-\mathbf{a}) - \frac{1}{2}\mathbf{e}\times\left(\mathbf{e}\times(\mathbf{a}'+\mathbf{a})\right) \qquad (2.6.15)$$

which is an alternate expression for $\mathbf{r}_0$. The foregoing result is summarized next.

THEOREM 2.6.1 (CHASLES). *The most general displacement of a rigid body is equivalent to a translation together with a rotation about an axis parallel to the translation.*

Alernatively, Chasles'Theorem can be stated as follows:

"Given an arbitrary displacement of a rigid body, there exists a set of points of the body, constituting a line, such that all points on that line undergo a displacement parallel to the line, which is of minimum Euclidean norm"

A property of the screw axis is established in the next theorem.

THEOREM 2.6.2. *The displacement vectors of all the points of a rigid body undergoing an arbitrary motion have the same projection along the screw axis.*

Proof:

Let P be an arbitrary point of a rigid body and S a point on the screw axis; let P' and S' represent the corresponding points after the displacement. From eq. (2.6.3), the displacement of P, $\mathbf{u}_p$, is given in terms of the position vectors of P, S and S', by

$$\mathbf{u}_p = \mathbf{s}' - \mathbf{Q}\mathbf{s} + (\mathbf{Q}-\mathbf{I})\mathbf{p} \qquad (2.6.16)$$

The projection of $\underline{u}_P$ onto the screw axis is computed now by obtaining the scalar product of $\underline{u}_P$ times $\underline{u}_S$. From eq. (2.6.16) this becomes

$$\underline{u}_P^T\underline{u}_S=(\underline{s}'-\underline{Q}\underline{s})^T\underline{u}_S+\underline{p}^T(\underline{Q}-\underline{I})^T\underline{u}_S \qquad (2.6.17)$$

where the second term on the right hand side vanishes because, as already shown, $\underline{u}_S$ is an eigenvector of $\underline{Q}$ and $\underline{Q}^T$. Thus, eq. (2.6.17) becomes

$$\underline{u}_P\underline{u}_S=(\underline{s}')^T\underline{u}_S-\underline{s}^T\underline{u}_S=(\underline{s}'-\underline{s})^T\underline{u}_S=\underline{u}_S^T\underline{u}_S$$

From the above expression it follows that the projection of $\underline{u}_P$ onto the screw axis has lenght $||\underline{u}_S||$, q.e.d

Using the same notation as above, the final position vector of a point of a rigid body undergoing an arbitrary motion and its displacement can be expressed as

$$\underline{p}'=\underline{p}+\underline{u}_P \qquad (2.6.18)$$

$$\underline{u}_P=\underline{u}_S+(\underline{Q}-\underline{I})(\underline{p}-\underline{s}) \qquad (2.6.19)$$

Exercise 2.6.2 Derive eq. (2.6.19)

Hence it is clear that the displacement of any point of the rigid body is known if the following quantities are given:

i) The magnitude of the screw displacement, $||\underline{u}_S||$

ii) One point of the screw axis, R_0, whose position vector is $\underline{r}_0$

iii) The axis of rotation, $\underline{e}$

iV) The angle of rotation, θ

Given the above data, vector $\underline{u}_S$ is obtained as

$$\underline{u}_S=||\underline{u}_S||\underline{e} \qquad (2.6.20)$$

and matrix $\underline{Q}$ is given by eqs. (2.5.1) or (2.5.4). Point R_0 and vector $\underline{e}$ completely determine the screw axis, henceforth called L. From Theorem 2.6.2 it is clear that a rigid body undergoing an arbitrary motion, moves

as if it were welded to the bolt of a screw whose axis where L and whose pitch were given by

$$\lambda = \frac{2\pi||\underset{\sim}{u}_S||}{\theta} \tag{2.6.21}$$

For this reason, the pair $(L, \underset{\sim}{Q})$, which completely determines a rigid body motion, is called a "screw", and rigid-body motions are thus referred to as "screw motions". It was shown in section 2.3 how to obtain the matrix $\underset{\sim}{Q}$, given a rigid body motion with a fixed point. Vectors $\underset{\sim}{r}_0$ and $\underset{\sim}{u}_S$, which define L, are obtained from eqs. (2.6.15) or, alternatively, from eq. (2.6.7) and eq. (2.6.20).

The following interesting result is derived immediately from Theorem 2.6.2.

<u>Corollary 2.6.1</u> A rigid body motion is a rotation about a fixed point if and only if the displacement of one point of the body is perpendicular to the screw axis of the motion.

Another useful result is the following

<u>Corollary 2.6.2</u> The difference vector of the displacements of any two of the points of a rigid body undergoing an arbitrary motion is perpendicular to the screw axis.

<u>Exercise 2.6.3</u> Prove corollaries 2.6.1 and 2.6.2

Clearly, the motion of any one plane of a rigid body completely determines the motion of the body. Futhermore, three noncollinear points determine a plane; thus it follows that the motion of any three noncollinear points of a rigid body determine the motion of the body. In other words, knowing the initial and the final positions of three noncollinear points of a rigid body, one can determine the axis, the displacement and the rotation of the corresponding screw. In the following, formulae are derived to compute the screw parameters of a rigid body motion in terms of the motion of

three noncollinear points of the body. It will be shown that these formulae require that the displacements of the involved points be noncoplanar. Now, if three points of a rigid body are collinear, their displacements under any motion are coplanar. The converse, however, is not true, for three noncollinear points of a rigid body could have, under special circumstances, coplanar displacements, as is proved in Theorem 2.6.4. To prove this, a previous result is derived in the following

<u>THEOREM 2.6.3</u> *If the displacements of three noncollinear points of a rigid body are identical, the body undergoes a pure translation.*

<u>Proof:</u>

Let A, B, C be three noncollinear points of a rigid body, and $\underline{a}, \underline{b}, \underline{c}$ their respective position vectors. Using eq. (2.6.19), the displacements of these points can be written as

$$\underline{u}_A = \underline{u}_S + (\underline{Q} - \underline{I})(\underline{a} - \underline{s})$$

$$\underline{u}_B = \underline{u}_S + (\underline{Q} - \underline{I})(\underline{b} - \underline{s})$$

$$\underline{u}_C = \underline{u}_S + (\underline{Q} - \underline{I})(\underline{c} - \underline{s})$$

where $\underline{s}$ is the position vector of a point S on the screw axis.

Subtracting the third equation from the first and the second, and recalling that the three displacements are identical, one obtaines

$$(\underline{Q} - \underline{I})(\underline{a} - \underline{c}) = \underline{0}$$

and

$$(\underline{Q} - \underline{I})(\underline{b} - \underline{c}) = \underline{0}$$

Hence both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ $\in N(\underline{Q}-\underline{I})$*, i.e. $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ lie in the same one-dimensional space spanned by the real eigenvector of $\underline{Q}$. This cannot be so

* $N(\underline{T})$ and $R(\underline{T})$ denote de null space and the range of $\underline{T}$, as defined in Section 1.3

because A,B,C were assumed to be noncollinear. Thus, the only possibility for the two latter equations to hold is that $\underset{\sim}{Q}=\underset{\sim}{I}$, i.e. the motion contains no rotation and hence is a pure translation. q.e.d.

<u>THEOREM 2.6.4</u> *The non-identical* displacements of three points of a rigid body are coplanar if and only if one of the following three conditions is met:*

i) The motion is a pure rotation

ii) The motion is general, but the points are collinear

iii) The motion is general and the points are not collinear, but lie in a plane parallel to the screw axis.

<u>Proof:</u>

("if" part)

i) If the motion is a pure rotation and the origin of coordinates is located along the axis of rotation, the displacement $\underset{\sim}{u}$ of any point of position vector $\underset{\sim}{r}$ is then

$$\underset{\sim}{u}=(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{r}$$

i.e. $\underset{\sim}{u}\in R(\underset{\sim}{Q}-\underset{\sim}{I})$. Since $N(\underset{\sim}{Q}-\underset{\sim}{I})$ is of dimension 1, namely the axis of rotation, then from eq. (1.3.1), $R(\underset{\sim}{Q}-\underset{\sim}{I})$ is of dimension 2, namely a plane passing through the origin, normal to the axis of rotation. Thus, all displacements are coplanar, thereby proving this part.

ii) Let A,B and C be the given three collinear points of the rigid body undergoing a general motion. Let $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ be their respective position vectors. Hence, vectors $\underset{\sim}{c}-\underset{\sim}{a}$ and $\underset{\sim}{b}-\underset{\sim}{a}$ are linearly dependent and they are related by

---

*If the displacements of the three noncollinear points were identical, the motion would be a pure translation, according to Theorem 2.6.3.

$$\underline{c}-\underline{a}=\alpha(\underline{b}-\underline{a}) \qquad \text{(ii.1)}$$

From eq. (2.6.3),

$$\underline{u}_C=\underline{a}'+\underline{Q}(\underline{c}-\underline{a})-\underline{c}$$

$$=\underline{a}'-\underline{a}+\underline{c}+(\underline{Q}-\underline{I})(\underline{c}-\underline{a})-\underline{c}$$

$$=\underline{u}_A+\alpha(\underline{Q}-\underline{I})(\underline{b}-\underline{a}) \qquad \text{(ii.2)}$$

But, also from eq. (2.6.3),

$$(\underline{Q}-\underline{I})(\underline{b}-\underline{a})=\underline{u}_B-\underline{u}_A \qquad \text{(ii.3)}$$

Hence, eq. (ii.2) can be written as

$$\underline{u}_C=(1-\alpha)\underline{u}_A+\alpha\underline{u}_B$$

thus making evident that the three involved displacements are coplanar.

iii) Using eq. (2.6.19), the displacements of points A, B and C are

$$\underline{u}_A=\underline{u}_S+(\underline{Q}-\underline{I})(\underline{a}-\underline{s})$$

$$\underline{u}_B=\underline{u}_S+(\underline{Q}-\underline{I})(\underline{b}-\underline{s})$$

$$\underline{u}_C=\underline{u}_S+(\underline{Q}-\underline{I})(\underline{c}-\underline{s})$$

Since A, B and C lie in a plane parallel to the screw axis, vectors $\underline{b}-\underline{a}$, $\underline{c}-\underline{a}$ and $\underline{u}_S$ are coplanar and hence they can be related as

$$\underline{c}-\underline{a}=\alpha(\underline{b}-\underline{a})+\beta\underline{u}_S$$

or

$$\underline{c}=(1-\alpha)\underline{a}+\alpha\underline{b}+\beta\underline{u}_S$$

Substituting the latter expression into $\underline{u}_C$, after cancellations and rearrangement of terms,

$$\underline{u}_C=\underline{u}_A-\alpha(\underline{Q}-\underline{I})(\underline{a}-\underline{b})$$

But, from the above expressions for $\underline{u}_A$ and $\underline{u}_B$,

$$\underline{u}_A-\underline{u}_B=(\underline{Q}-\underline{I})(\underline{a}-\underline{b})$$

and so, from the latter expressions for $\underset{\sim}{u}_C$,

$$\underset{\sim}{u}_C=(1-\alpha)\underset{\sim}{u}_A+\alpha\underset{\sim}{u}_B$$

thereby showing the linear dependence, i.e. the coplanarity of the three displacements involved.

("only if" part)

If $\underset{\sim}{u}_A$, $\underset{\sim}{u}_B$ and $\underset{\sim}{u}_C$ are coplanar, then

$$\det(\underset{\sim}{u}_A,\underset{\sim}{u}_B,\underset{\sim}{u}_C)=0$$

Introducing eq. (2.6.3), $\underset{\sim}{u}_B$ and $\underset{\sim}{u}_C$ can be written as

$$\underset{\sim}{u}_B=\underset{\sim}{u}_A+(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}-\underset{\sim}{a})$$

$$\underset{\sim}{u}_C=\underset{\sim}{u}_A+(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}-\underset{\sim}{a})$$

Hence, the coplanarity condition can be written, after proper simplifications, as

$$\det\left(\underset{\sim}{u}_A,(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}-\underset{\sim}{a}),(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}-\underset{\sim}{a})\right)=0$$

or, in Gibbs notation:

$$(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}-\underset{\sim}{a})\cdot\underset{\sim}{u}_A=0$$

From (2.6,p.5), the first product can be expressed as

$$(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{Q}-\underset{\sim}{I})(\underset{\sim}{c}-\underset{\sim}{a})=\alpha\underset{\sim}{e}$$

where

$$\alpha=2(1-\cos\theta)\underset{\sim}{e}\times(\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{c}-\underset{\sim}{a})$$

θ and $\underset{\sim}{e}$ being the angle of rotation and the unit vector parallel to the axis of this rotation.

Exercise 2.6.4 Derive the above expression for α

The double product thus can vanish if any one of the following conditions is met:

i) $\underset{\sim}{e}\cdot\underset{\sim}{u}_A=0$

which, from Corollary 2.6.1, states that the body undergoes a pure rotation

ii) $\alpha=0$

which is satisfied under one of the two following conditions:

ii.1) $1-\cos\theta=0$

which implies $\theta=0$; i.e. the motion reduces to a pure translation. This case, however, has been discarded in the present analysis, for the displacements have been assumed to be non-identical (See Theorem 2.6.3)

ii.2) $\underline{e}\times(\underline{b}-\underline{a})\cdot(\underline{c}-\underline{a})=0$

which in turn is satisfied under one of the following two conditions:

ii.2.a) $\underline{e}$, $\underline{b}-\underline{a}$ and $\underline{c}-\underline{a}$ are coplanar, i.e. points A, and C lie on a plane parallel to the rotation axis. (Picture it!)

ii.2.b) $(\underline{b}-\underline{a})\times(\underline{c}-\underline{a})=0$

which implies that A, B and C are collinear, thus completing the proof.

COROLLARY 2.6.3 *Assume a rigid body undergoes an arbitrary motion and choose any three noncollinear points of the body, A, B and C. Letting $\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$ be the three resulting displacements, then the two difference vectors $\underline{u}_A-\underline{u}_C$ and $\underline{u}_B-\underline{u}_C$ are parallel if and only if the points lie in a plane parallel to the screw axis if the motion is general. If the motion is a pure rotation, these points lie in a plane parallel to the axis of rotation.*

Exercise 2.6.5 Prove Corollary 2.6.3

Further consequences of Theorem 2.6.4 are next stated.

Corollary 2.6.4 *The displacements of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation*

Exercise 2.6.6 Prove Corollary 2.6.4. Hint; Use eq. (2.6.19) and the fact that the two terms of its right-hand side are linearly independent; in fact, orthogonal.

Corollary 2.6.5 *If two, and only two, displacements of three noncollinear points of a rigid body are parallel, then either i) the parallel displacements are identical and belong to points lying on a line parallel to the screw axis, or ii) the parallel displacements are different from each other, in which case the motion is a pure rotation whose axis is parallel to the line connecting the two points of parallel displacements.*

Corollary 2.6.6 *If one, and only one of three points of a rigid body has a zero displacement and other two points, noncollinear with the former, have parallel but different displacements, then the body undergoes a pure rotation, whose axis is determined by the intersection of the plane containing the three given points with a second plane defined by the displaced positions of the points.*

Exercise 2.6.7 Prove Corollaries 2.6.5 and 2.6.6.

The formulae that allow the computation of the screw parameters are next derived. It will be assumed that the displacements of three noncollinear points are known and the two cases that could arise are dealt with. These cases are: i) the resulting displacements are noncoplanar, ii) these displacements are coplanar and the motion is either pure rotation or general but the three points lie in a plane parallel to the screw axis.

First Case. The displacements are noncoplanar

Let A,B,C and A',B',C' be the initial and the displaced positions of three noncollinear points. Denoting by $\underline{a},\underline{b},\underline{c}$, $\underline{a}',\underline{b}'$ and $\underline{c}'$ the corresponding position vectors, the displacement vectors are, then

$$u_A = a' - a \qquad (2.6.22a)$$

$$u_B = b' - b \qquad (2.6.22b)$$

$$u_C = c' - c \qquad (2.6.22c)$$

Now, the direction vector, e, of the screw axis can be obtained as follows: Theorem 2.6.2 suggests one way to obtain e, namely, determine the unique vector along which u_A, u_B and u_C all have the same projections. If the tails of all these vectors are attached to one point, say O, then it becomes evident that the vector e is perpendicular to the plane determined by the tips of the three vectors u_A, u_B, u_C. Thus, e is parallel to the cross product of vectors $u_A - u_C$ and $u_B - u_C$. Normalizing e to have unit lenght, one immediately has

$$e = \frac{(u_A - u_C) \times (u_B - u_C)}{||(u_A - u_C) \times (u_B - u_C)||} \qquad (2.6.23)$$

To determine the magnitudes of the screw displacement, $||u_S||$, all that is needed is to project any one of u_A, u_B or u_C onto e. Hence,

$$||u_S|| = |u_A \cdot e| \qquad (2.6.24)$$

where the absolute value has been taken because eq. (2.6.23) determines the vector e up to a change-of-sign (the order of the vectors in the cross product could have been changed). Vector u_S can now be computed as

$$u_S = ||u_S|| e \ \mathrm{sgn}(u_A \cdot e) \qquad (2.6.25)$$

where sgn is the signum function, i.e., sgn(x) is -1 if $x<0$; it is + 1 if $x>0$ and it is irrelevant if $x=0$. The latter ideterminacy of sgn(0) causes no difficulty for, if $u_A \cdot e=0$, $||u_S||=0$, and, from corollary 2.6.1,

the motion is one of rotation about one fixed point.

Now, notice that, if vector $\underset{\sim}{u}_S$ is subtracted from the vectors defined in (2.6.23), the new vectors $\underset{\sim}{u}'_A$, $\underset{\sim}{u}'_B$ and $\underset{\sim}{u}'_C$ lie in the same plane (Why?). To completely determine the screw, the rotation angle θ and the location of the screw axis are next determined. Since $\underset{\sim}{u}'_A$, $\underset{\sim}{u}'_B$ and $\underset{\sim}{u}'_C$ are coplanar, they can be regarded as the displacements of three points of a rigid body undergoing pure rotation. Let A", B" and C" be the final positions of points A, B and C undergoing displacements $\underset{\sim}{u}'_A$, $\underset{\sim}{u}'_B$ and $\underset{\sim}{u}'_C$. Since these vectors are coplanar, the mediator planes π_1 and π_2 of segments AA" and BB" can be projected as the dotted lines appearing in Fig 2.6.2. Then the screw axis L is defined by the intersection I of π_1 and π_2. This intersection is next determined.

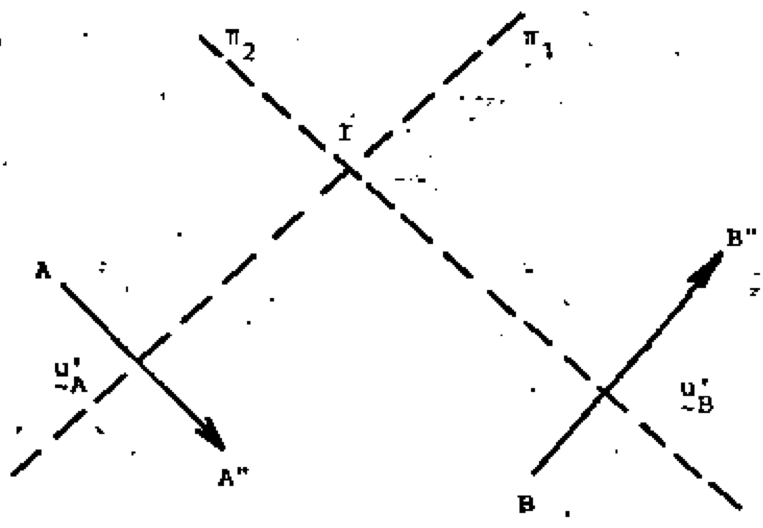

Fig 2.6.2. Determination of the screw axis.

Let M and N be the middle points of segments AA" and BB", their positon vectors being denoted by $\underset{\sim}{m}$ and $\underset{\sim}{n}$, respectively.

Clearly,

$$\underline{m}=\underline{a}+\frac{1}{2}\underline{u}'_A \tag{2.6.26a}$$

$$\underline{n}=\underline{b}+\frac{1}{2}\underline{u}'_B \tag{2.6.26b}$$

The equations of planes π_1 and π_2 are thus

$$(\underline{r}-\underline{m})\cdot\underline{u}'_A=0 \tag{2.6.27a}$$

$$(\underline{r}-\underline{n})\cdot\underline{u}'_B=0 \tag{2.6.27b}$$

respectively.

The set of points $\underline{r}$, satisfying both eqs. (2.6.27), yield line L, the screw axis. The angle of rotation, θ, is then simply, angle AIA" (or equivalently, angle BIB" or angle CIC"). This angle is readily obtained once the line L is known, for it can then be computed from the dot product of vectors $\overrightarrow{IA}$ and $\overrightarrow{IA"}$, both lying in a plane normal to L.

Example 2.6.1 - Determine the screw of the displacement of the cube of Fig 2.6.3 as it is moved from configuration 1 to configuration 2.

Assume the sides of the cube have lenght h.

Solution I:

The problem is first solved via a minimization procedure.

Step i): Determination of the revolute. For this purpose, assume a rigid-body rotation about point B, as shown in Fig 2.6.4

From Fig 2.6.4 it is clear that the cube undergoes a rotation about line EB, thereby saving the analysis performed in example 2.4.1 to determine the axis of rotation. Thus,

$$\underline{e}=\frac{\overrightarrow{EB}}{||EB||}=\frac{\sqrt{3}}{3}(-\underline{e}_x+\underline{e}_y+\underline{e}_z) \tag{2.6.28}$$

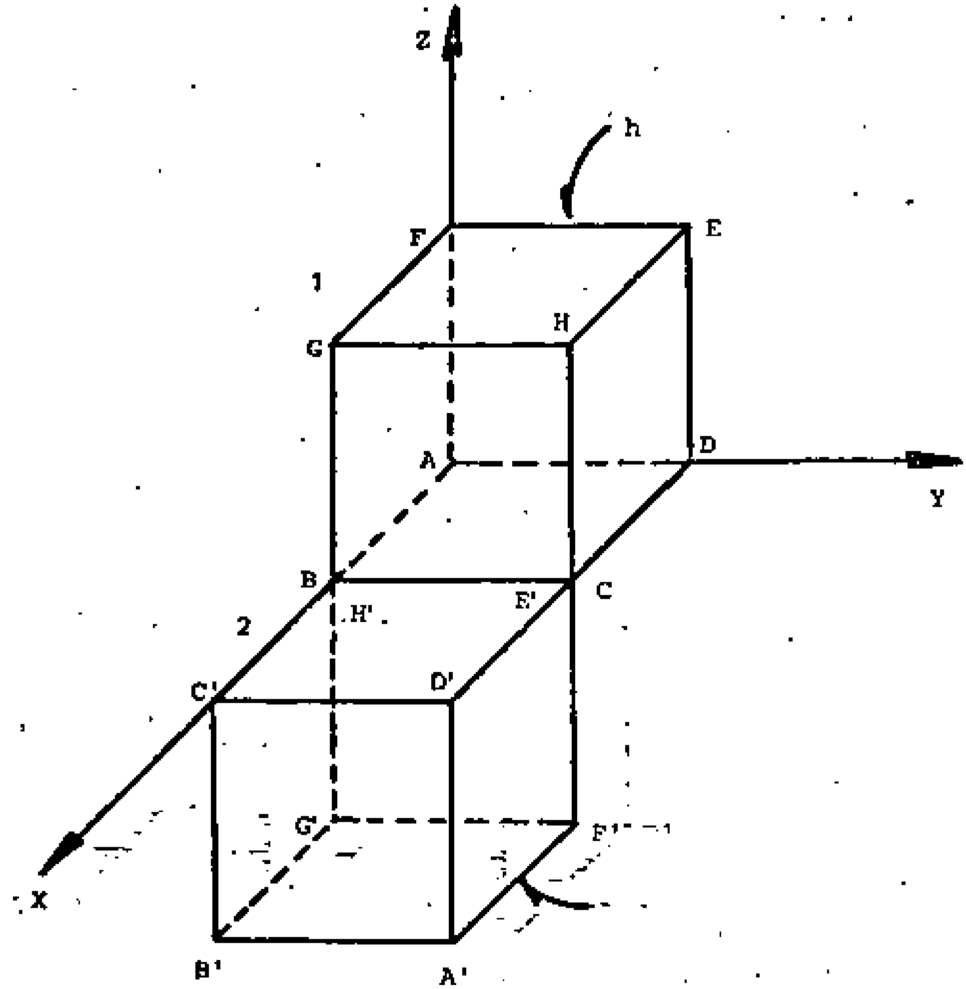

Fig 2.6.3 Motion of a cube

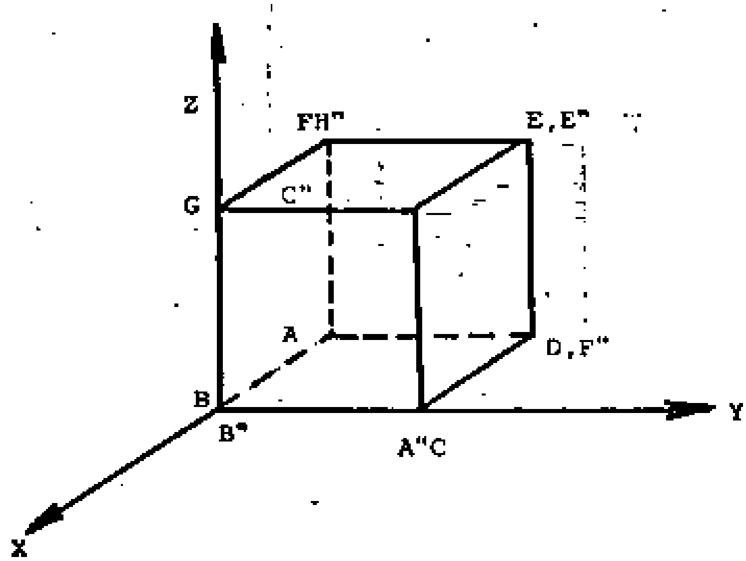

Fig 2.6.4 Rotation of a cube about a fixed point

or, in matrix form,

$$\{e\} = \frac{\sqrt{3}}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

Let M be the intersection of the line BE and a plane perpendicular to it but containing point A (the plane also contains point A", since AA" is perpendicular to BE, as can readly be checked).

Let $\underline{m}$ be the position vector of M. Now $\underline{m}-\underline{a}$ is perpendicular to BE, and M is contained in line BE. BE is specified as the intersection of the planes

$$x+z=0 \tag{2.6.29a}$$

$$y-z=0 \tag{2.6.29b}$$

Since AM is perpendicular to BE, $\underline{m}-\underline{a}$ must be perpendicular to vector $\underline{e}$ of (2.6.28). Hence, the coordinates of M(x,y,z) must satisfy the relation

$$x+h-y-z=0 \tag{2.6.29c}$$

which, together with eqs. (2.6.29), determines M, namely

$$x=-\frac{h}{3}$$
$$y=\frac{h}{3} \tag{2.6.30}$$
$$z=\frac{h}{3}$$

Hence,

$$\underline{m}=\frac{h}{3}(-\underline{e}_x+\underline{e}_y+\underline{e}_z) \tag{2.6.31}$$

It can also be readily checked that $\underline{m}-\underline{a}''$ is perpendicular to $\underline{e}$, as expected, if point A" were to lie in the plane perpendicular to EB. The angle of rotation, θ, can now be computed from the relationship

$$\cos\theta=\frac{(\underline{a}-\underline{m})\cdot(\underline{a}''-\underline{m})}{||\underline{a}-\underline{m}||^2}$$

i.e.

$$\cos\theta=-\frac{1}{2},\ \sin\theta=\frac{\sqrt{3}}{2} \qquad (2.6.32)$$

Knowing the axis of rotation, EB, and the angle fo rotation, θ, the revolute matrix is now readily constructed form eqs. (2.5.4), where

$$u=\frac{\sqrt{3}}{3},\ v=\frac{\sqrt{3}}{3},\ w=\frac{\sqrt{3}}{3} \qquad (2.6.33)$$

and so

$$(\underline{P})=\frac{\sqrt{3}}{3}\begin{pmatrix}0 & 1 & -1\\ -1 & 0 & -1\\ 1 & 1 & 0\end{pmatrix},\quad (\underline{R})=\frac{1}{3}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ -1 & 1 & 1\end{pmatrix}\quad (\underline{T})=\frac{1}{3}\begin{pmatrix}2 & 1 & 1\\ 1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}$$

Therefore,

$$(\underline{Q})=\begin{pmatrix}0 & 0 & -1\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},\quad (\underline{Q}-\underline{I})=\begin{pmatrix}-1 & 0 & -1\\ -1 & -1 & 0\\ 0 & -1 & -1\end{pmatrix}$$

Step ii) : Determination of the screw axis. The minimum-magnitude displacement $\underline{u}_R$ of point R is obtained from eq. (2.6.3a), expressed in terms of the coordinate axes of Fig.2.6.3. Thus,

$$(\underline{u}_R)=\begin{pmatrix}2h-x-z\\ h-x-y\\ -h+y-z\end{pmatrix} \qquad (2.6.36)$$

and

$$\phi(\underline{r})\equiv||u_R||^2=2x^2+2y^2+2z^2+2xz+2xy-2yz-6hx-4hy-2hz+6h^2 \qquad (2.6.37)$$

Hence

$$(\phi'(\underline{r}))=2\begin{pmatrix}2x+y+z-3h\\ x+2y-z-2h\\ x-y+2z-h\end{pmatrix} \qquad (2.6.38)$$

and, equating $\phi'(\underset{\sim}{x})$ to zero, a set of three linearly dependent equations is obtained, from which the following linearly independent set is sorted out.

$$\begin{pmatrix} 2 & 1 & 1 \\ & & \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} h \qquad (2.6.39)$$

This has a minimum-norm solution (according to eq. (2.6.8) given by

$$(\underset{\sim}{r}_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \frac{h}{3} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad (2.6.40)$$

thereby determining the screw axis, which passes through point R_0 (whose position vector is $\underset{\sim}{r}_0$), as given by eq. (2.6.40) and is parallel to vector $\underset{\sim}{e}$, as given by eq. (2.6.28). In order to compute the pitch of the screw, λ, it is necessary to compute $||\underset{\sim}{u}_S||$, which, from Theorem 2.6.2, is given as

$$||\underset{\sim}{u}_S|| = |\underset{\sim}{u}_D \cdot \underset{\sim}{e}| = \frac{2\sqrt{3}}{3}h \qquad (2.6.41a)$$

and

$$\underset{\sim}{u}_S = \frac{2\sqrt{3}}{3}h\underset{\sim}{e} \qquad (2.6.41b)$$

The pitch is, then, from eq. (2.6.21a),

$$\lambda = 2\sqrt{3}\ h \qquad (2.6.42)$$

Solution II

An alternative solution is now given, using eqs. (2.6.23), (2.6.24) and (2.6.27). In order to simplify the computations, choose the displacements of points C, D and G to determine the screw. Thus,

$$\mathbf{u}_C=\mathbf{c}'-\mathbf{c}=h(\mathbf{e}_x-\mathbf{e}_y)$$

$$\mathbf{u}_D=\mathbf{d}'-\mathbf{d}=2h\mathbf{e}_x \qquad (2.6.43)$$

$$\mathbf{u}_G=\mathbf{g}'-\mathbf{g}=-2h\mathbf{e}_z$$

$$(\mathbf{u}_C-\mathbf{u}_G)\times(\mathbf{u}_D-\mathbf{u}_G)=2h^2(-\mathbf{e}_x+\mathbf{e}_y+\mathbf{e}_z)$$

from which

$$||(\mathbf{u}_C-\mathbf{u}_G)\times(\mathbf{u}_D-\mathbf{u}_G)||=2\sqrt{3}\,h^2$$

and so,

$$\mathbf{e}=\frac{\sqrt{3}}{3}(-\mathbf{e}_x+\mathbf{e}_y+\mathbf{e}_z), \text{ or } \mathbf{e}=\frac{\sqrt{3}}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix} \qquad (2.6.44)$$

which is identical with the value previously obtained in (2.6.28).

$||\mathbf{u}_S||$ is obtained from

$$||\mathbf{u}_S||=|\mathbf{u}_D\cdot\mathbf{e}|=\frac{2\sqrt{3}}{3}h$$

and so

$$\mathbf{u}_S=\frac{2}{3}h(\mathbf{e}_x-\mathbf{e}_y-\mathbf{e}_z) \qquad (2.6.45)$$

where the sign of $\mathbf{u}_S$ has been reversed, as compared with that of $\mathbf{e}$, because $\mathbf{u}_D\cdot\mathbf{e}<0$. Next form the vectors,

$$\mathbf{u}'_C=\mathbf{u}_C-\mathbf{u}_S=h(\frac{1}{3}\mathbf{e}_x-\frac{1}{3}\mathbf{e}_y+\frac{2}{3}\mathbf{e}_z) \qquad (2.6.46a)$$

$$\mathbf{u}'_D=\mathbf{u}_D-\mathbf{u}_S=h(\frac{4}{3}\mathbf{e}_x+\frac{2}{3}\mathbf{e}_y+\frac{2}{3}\mathbf{e}_z) \qquad (2.6.46b)$$

$$\mathbf{u}'_G=\mathbf{u}_G-\mathbf{u}_S=h(-\frac{2}{3}\mathbf{e}_x+\frac{2}{3}\mathbf{e}_y-\frac{4}{3}\mathbf{e}_z) \qquad (2.6.46c)$$

which can be readily verified to be coplanar, as expected. Next, the rquations of planes π_1 and π_2 are obtained. Let

$$\mathbf{m}=\mathbf{c}+\frac{1}{2}\mathbf{u}'_C=\frac{h}{6}(7\mathbf{e}_x+5\mathbf{e}_y+2\mathbf{e}_z) \qquad (2.6.47a)$$

$$\underset{\sim}{n}=\underset{\sim}{d}+\frac{1}{2}\underset{\sim}{u}'_D=\frac{h}{3}(2\underset{\sim}{e}_x+4\underset{\sim}{e}_y+\underset{\sim}{e}_z) \qquad (2.6.47b)$$

The equation of plane π_1 is, then,

$$x-y+2z-h=0 \qquad (2.6.48a)$$

and that of plane π_2 is

$$2x+y+z-3h=0 \qquad (2.6.48b)$$

Next, a point I on the axis of rotation, contained in a plane perpendicular to this axis and passing through points C and C", is located. Let $\underset{\sim}{r}_I$ be the position vector of this point. Then, $\underset{\sim}{r}_I$ clearly must satisfy eqs. (2.6.48a and b), for it is a point of the intersection of π_1 and π_2. In addition, $\underset{\sim}{r}_I-\underset{\sim}{c}$ must be perpendicular to that intersection, whose direction cosines are already known from eq. (2.6.44). This latter condition is expressed then as

$$-x+y+z=0 \qquad (2.6.48c)$$

which, together with eqs. (2.6.48 a and b) constitutes a linear algebraic system of 3 equations and 3 unknowns. Its solution is

$$(\underset{\sim}{r}_I)=\frac{h}{3}\begin{pmatrix}3\\2\\1\end{pmatrix} \qquad (2.6.49)$$

which is a solution identical to that obtained in eq. (2.6.40).

The angle of rotation is now obtained from the relationship

$$\cos\theta=\frac{(\underset{\sim}{c}-\underset{\sim}{r}_I)\cdot(\underset{\sim}{c}+\underset{\sim}{u}'_C-\underset{\sim}{r}_I)}{||\underset{\sim}{c}-\underset{\sim}{r}_I||} \qquad (2.6.50)$$

where

$$(\underset{\sim}{c}-\underset{\sim}{r}_I)=\frac{h}{3}\begin{pmatrix}0\\1\\-1\end{pmatrix},\quad \underset{\sim}{c}+\underset{\sim}{u}'_C-\underset{\sim}{r}_I=\frac{h}{3}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

Thus,

$$(\underline{c}-\underline{r}_I)\cdot(\underline{c}+\underline{u}_C'-\underline{r}_I)=-\frac{h^2}{9} \qquad (2.6.51)$$

and

$$||\underline{c}-\underline{r}_I||^2=\frac{2}{9}h^2 \qquad (2.6.52)$$

Substitution of eqs. (2.6.51) and (2.6.52) into eq. (2.6.50) yields, then,

$$\cos\theta=-\frac{1}{2} \text{ or } \theta=-120° \qquad (2.6.53)$$

where the minus sign was found by application of the result of eq. (2.5.16) The screw displacement, $||\underline{u}_S||$ is obtained from eq. (2.6.45) as

$$||\underline{u}_S||=\frac{2\sqrt{3}}{3} \qquad (2.6.54)$$

and the pitch, λ, is obtained from eq. (2.6.21a) as,

$$\lambda=2\sqrt{3}h$$

which results are identical to those of eqs. (2.6.41a) and (2.6.42)

One third method to obtain the point $\underline{r}_0$ on the screw axis closest to the origin is now presented as it appears in (2.2, p.11).

Let $\underline{a}$ and $\underline{a}'$ be the initial and the final position vectors, respectively, of a given point A of a rigid body, which are known. Also let $\underline{r}$ and $\underline{r}'$ be the initial and the final position vectors of another point R, both yet unknown. If point R is to lie on the screw axis, then $\underline{u}_R=\underline{r}'-\underline{r}$ is parallel to the axis of rotation, $\underline{e}$, as was found previously. From Rodrigues' Formula, eq. (2.6.3),

$$\underline{a}'-\underline{a}=\tan\frac{\theta}{2}\underline{e}\times(\underline{a}'+\underline{a}) \qquad (2.6.56)$$

and

$$\underline{r}'-\underline{r}=\tan\frac{\theta}{2}\underline{e}\times(\underline{r}'+\underline{r})$$

Subtracting eq. (2.6.56b) from eq. (2.6.56a) and taking into account that $\underline{r}'-\underline{r}$ is parallel to $\underline{e}$, i.e., writing $\underline{r}'-\underline{r}=\alpha\underline{e}$, α being a scalar,

$$\underset{\sim}{a}'-\underset{\sim}{a}-\alpha\underset{\sim}{e}=\tan\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{a}'+\underset{\sim}{a})-\tan\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{r}'+\underset{\sim}{r}) \quad (2.6.57)$$

Since $\underset{\sim}{r}'-\underset{\sim}{r}=\alpha\underset{\sim}{e}$, it follows that

$$\underset{\sim}{e}\times\underset{\sim}{r}'=\underset{\sim}{e}\times\underset{\sim}{r}$$

Hence, eq. (2.6.57) can be written as

$$\underset{\sim}{a}'-\underset{\sim}{a}-\alpha\underset{\sim}{e}=\tan\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{a}'+\underset{\sim}{a})-2\tan\frac{\theta}{2}\underset{\sim}{e}\times\underset{\sim}{r} \quad (2.6.57a)$$

Multiplying both sides of eq. (2.6.57a) times $\cot\frac{\theta}{2}\underset{\sim}{e}\times$, one obtains

$$\cot\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{a}'-\underset{\sim}{a})=\underset{\sim}{e}\times\left(\underset{\sim}{e}\times(\underset{\sim}{a}'+\underset{\sim}{a})\right)-2\underset{\sim}{e}\times(\underset{\sim}{e}\times\underset{\sim}{r})=\underset{\sim}{e}\times\left(\underset{\sim}{e}\times(\underset{\sim}{a}'+\underset{\sim}{a})\right)-2(\underset{\sim}{e}\cdot\underset{\sim}{r})\underset{\sim}{e}+2\underset{\sim}{r} \quad (2.6.58)$$

If $\underset{\sim}{r}$ is chosen to be the position vector of the point on the screw axis closest to the origin, then

$$\underset{\sim}{e}\cdot\underset{\sim}{r}=0$$

and vector $\underset{\sim}{r}_0$ thus can be obtained from eq. (2.6.58) as

$$\underset{\sim}{r}_0=\frac{1}{2}\cot\frac{\theta}{2}\underset{\sim}{e}\times(\underset{\sim}{a}'-\underset{\sim}{a})-\frac{1}{2}\underset{\sim}{e}\times\left(\underset{\sim}{e}\times(\underset{\sim}{a}'+\underset{\sim}{a})\right) \quad (2.6.59)$$

<u>Second Case.- The displacements are coplanar</u>

This could be due to two possibilities: either the points lie in a plane parallel to the screw axis (or to the axis of rotation if the motion is a pure rotation) or they do not, but the motion is then necessarily a pure rotation.

<u>First possibility. The three points lie in a plane prallel either to the screw axis or to the axis of rotation.</u> From Corollary 2.6.3 the differences of displacement vectors are parallel and hence the cross product appearing in eq. (2.6.23) vanishes thus rendering the computation of $\underset{\sim}{e}$ indeterminate. This vector can be computed, nevertheless, attending the aforementioned Corollary and the fact that it is perpendicular to vector $\underset{\sim}{u}_A-\underset{\sim}{u}_C$, according to Corollary 2.6.2. The condition that $\underset{\sim}{e}$ is contained in the plane of the given points A,B and C is expressed as

$$\underline{e}=\alpha(\underline{a}-\underline{c})+\beta(\underline{b}-\underline{c}) \tag{2.6.60}$$

The perpendicularity condition between $\underline{e}$ and $\underline{u}_A-\underline{u}_C$ is expressed in turn as

$$(\underline{u}_A-\underline{u}_C)^T\underline{e}=0 \tag{2.6.61}$$

Substitution of eq. (2.6.60) into eq. (2.6.61) yields

$$\alpha(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})+\beta(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})=0 \tag{2.6.62}$$

Hence

$$\alpha = -\beta\,\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})} \tag{2.6.63}$$

provided $\underline{u}_A-\underline{u}_C$ is not orthogonal to $\underline{a}-\underline{c}$. If this is so, then from eq. (2.6.62) $\beta=0$ and, since $\underline{e}$ has been defined as of magnitude equal to unity, then

$$\underline{e}=\frac{\underline{a}-\underline{c}}{||\underline{a}-\underline{c}||} \tag{2.6.64}$$

Now, since points A, B and C are not collinear, then it cannot happen that $\underline{u}_A$ and $\underline{u}_C$ be orthogonal to both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$. If, however, $\underline{u}_A-\underline{u}_C$ is orthogonal to $\underline{b}-\underline{c}$, then $\alpha=0$, from eq. (2.6.62), and, in this instance,

$$\underline{e}=\frac{\underline{b}-\underline{c}}{||\underline{b}-\underline{c}||} \tag{2.6.65}$$

If neither α nor β vanish, then from eq. (2.6.63) and the condition imposed on $\underline{e}$ as being of magnitude unity,

$$\frac{1}{\beta^2}=||\underline{a}-\underline{c}||^2\left(\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})}\right)^2-2\,(\underline{a}-\underline{c})^T(\underline{b}-\underline{c})\frac{(\underline{u}_A-\underline{u}_C)^T(\underline{b}-\underline{c})}{(\underline{u}_A-\underline{u}_C)^T(\underline{a}-\underline{c})}$$
$$+||\underline{b}-\underline{c}||^2 \tag{2.6.66}$$

Eq. (2.6.66) yields β. With the value of β known, α is then computed

from eq. (2.6.63). Thus, $\underline{e}$ is finally computed from eq. (2.6.60).

Second possibility. The motion is pure rotation. If the three points are noncollinear and the displacements are nonidentical and parallel but vectors $u_A - u_C$ and $u_B - u_C$ are nonparallel, then, from Theorem 2.6.4 and Corollary 2.6.3, the motion is one of pure rotation. In this case the axis of rotation can be obtained simply from the intersection of the mediator planes of segments AA' and BB'. The perpendiculars to the axis of rotation, traced from A and A' intersect that axis at a common point, I. The angle of rotation is then, simply AIA', thereby completing the motion parameters.

The computation of the screw parameters is realised by SUBROUTINE SCREW, which considers all cases that could arise regarding the relationships amongst all three displacement vectors. These possible cases are shown in the "tree" diagram appearing in Fig. 2.6.5. SCREW uses the following auxiliary subroutines:

1. SUBROUTINE COPL 1 computes the screw parameters when the motion is pure rotation. It distinguishes amongst the different particular cases with the aid of the integer variable INDEX
2. SUBROUTINE COPL 2 computes the screw parameters when the points lie in a plane parallel either to the screw axis or to the axis of rotation. Two different cases could arise, which are distinguished with the aid of the integer variable INDE.
3. SUBROUTINE GENMOT computes the screw parameters when the motion is general and the three given displacements are noncoplanar.

The computation procedure for each case is next described. All over, the vectors referred to are the given displacement vectors, $\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$, of the three given points, A, B and C, whose position vectors in the reference configuration are $\underline{a}_1$, $\underline{b}_1$ and $\underline{c}_1$, whereas those in their displaced configura

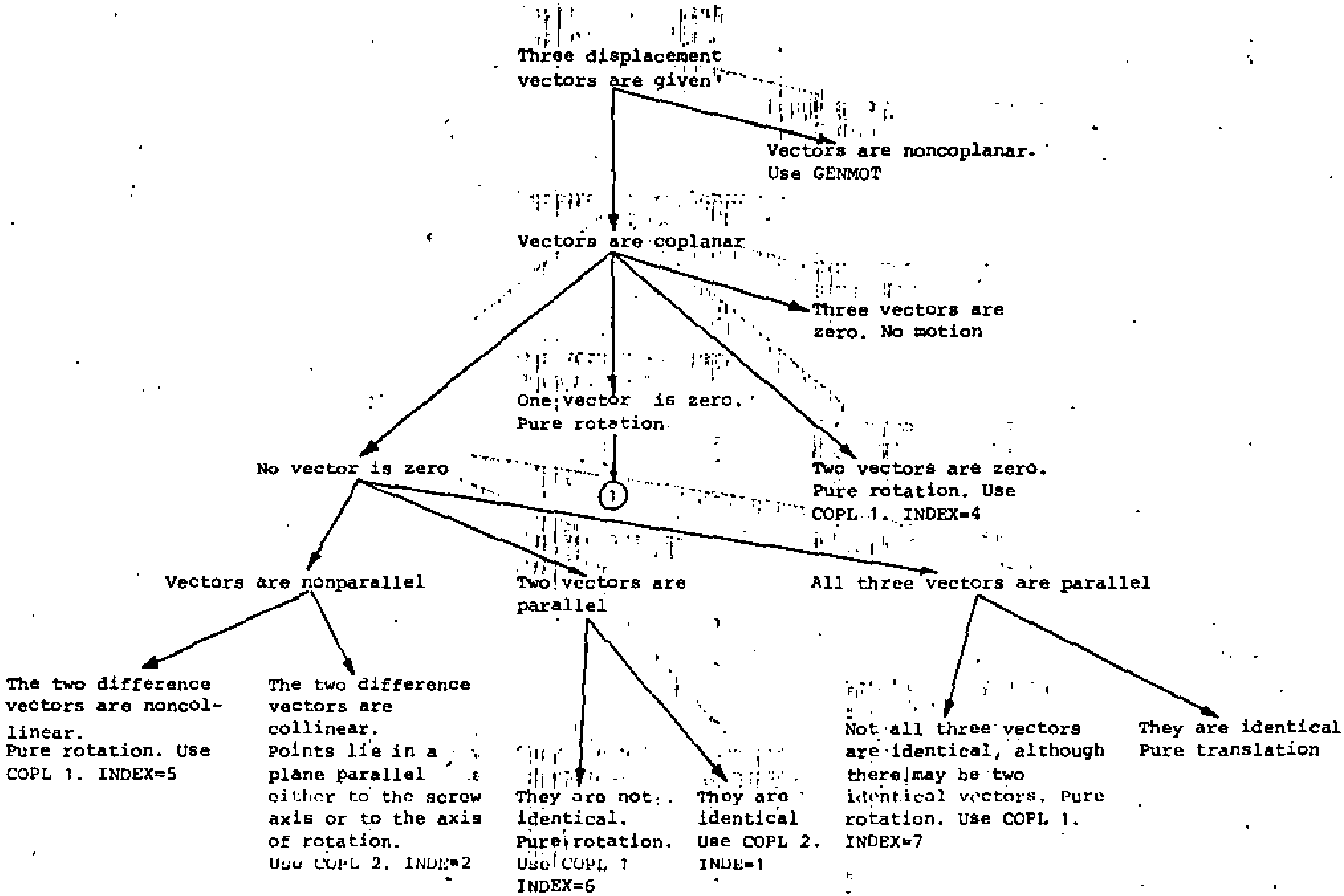

Fig 2.6.5 Tree diagram showing the different possible relationships amongst the displacements of three noncollinear points defining a rigid-body motion.

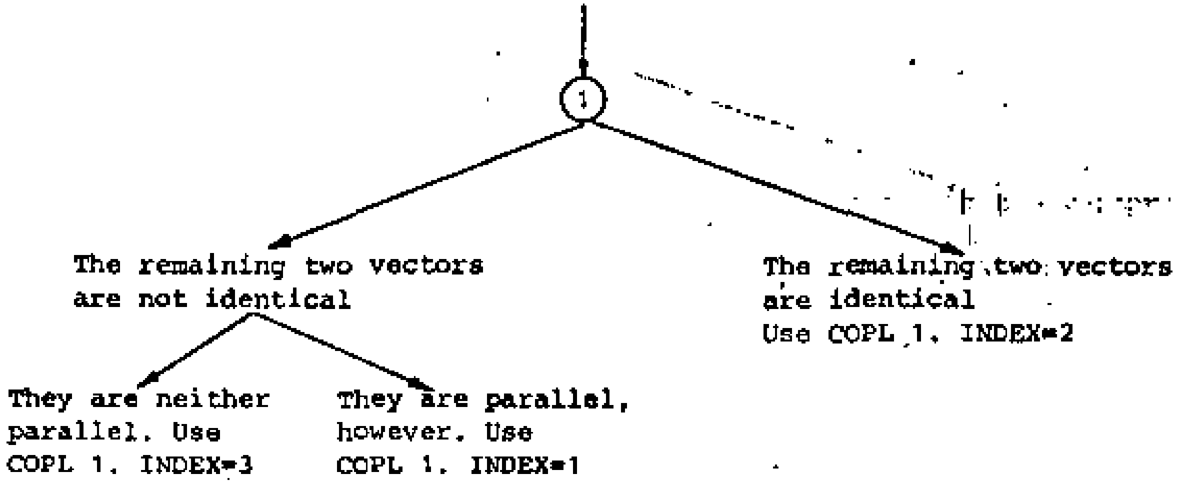

Fig 2.6.5 (continues)

tions are $\underline{a}_2$, $\underline{b}_2$ and $\underline{c}_2$.

INDEX = 1. One vector is zero and the remaining two vectors are not identical; they are parallel, however. It follows from Corollary 2.6.1 that the motion is pure rotation. The location of the axis of rotation follows from the fact that the axis of rotation is contained in the intersection of two planes, Π_1 and Π_2, where Π_1 is the plane of the three given points and Π_2 is defined by the displaced positions of these points. Notice that the point of zero displacement is contained in both Π_1 and Π_2.

Proof

Let C be the point of zero displacement and the origin of coordinates. Furthermore, the displacements $\underline{u}_A$ and $\underline{u}_B$ are given as

$$\underline{u}_A = (\underline{Q}-\underline{I})\underline{a}_1, \; \underline{u}_B = (\underline{Q}-\underline{I})\underline{b}_1 \qquad (2.6.67)$$

Since $\underline{u}_A$ and $\underline{u}_B$ are parallel, there exists a scalar α such that

$$\underline{u}_A + \alpha \underline{u}_B = \underline{0} \qquad (2.6.68)$$

Substituting eqs. (2.6.67) into eq. (2.6.68) yields

$$(\underline{Q}-\underline{I})(\underline{a}_1 + \alpha \underline{b}_1) = \underline{0} \qquad (2.6.69)$$

which means vector $\underline{a}_1 + \alpha \underline{b}_1$ is parallel to the axis of rotation, i.e. the axis of rotation is contained in the plane determined by A,B and C. Moreover, since

$$\underline{a}_2 = \underline{a}_1 + \underline{u}_A, \; \underline{b}_2 = \underline{b}_1 + \underline{u}_B \qquad (2.6.70)$$

The displacements of these points are given by

$$\underline{u}'_A = (\underline{Q}-\underline{I})\underline{a}_2, \; \underline{u}'_B = (\underline{Q}-\underline{I})\underline{b}_2 \qquad (2.6.71)$$

Introducing eqs. (2.6.70) into eqs. (2.6.71) and then simplifying

the resulting expression with the aid of eqs. (2.6.67), one obtains

$$u'_A = Qu_A, \quad u'_B = Qu_B \tag{2.6.72}$$

eqs. (2.6.68) and (2.6.72) lead to

$$u'_A + \alpha u'_B = Q(u_A + \alpha u_B) = 0 \tag{2.6.73}$$

But, introducing eqs. (2.6.71) into eq. (2.6.73),

$$(Q-I)(a_2 + \alpha b_2) = 0 \tag{2.6.74}$$

which implies that vector $a_2 + \alpha b_2$ is parallel to the axis of rotation, i.e. this axis is contained in the plane defined by the points A_2, B_2 and C_2, thereby completing the proof. Other parameters are computed using the general procedure previously outlined.

INDEX = 2 One vector is zero and the remaining two are identical. The motion is pure rotation, due to Corollary 2.6.1, and the axis of rotation is defined by a line passing through the point of zero displacement in the direction of the line connecting the other two points.

Proof

Let C be the point of zero displacement. The displacements of the other two points are

$$u_A = (Q-I)a_1, \quad u_B = (Q-I)b_1 \tag{2.6.75}$$

Since $u_A = u_B$, it follows that

$$u_A - u_B = (Q-I)(a_1 - b_1) = 0 \tag{2.6.76}$$

which implies that vector $a_1 - b_1$ is parallel to the axis of rotation, i.e. the line connecting points A and B is parallel to the axis of rotation, q.e.d.

INDEX = 3. One vector is zero and the remaining two vectors are not parallel. The motion is pure rotation, due to Corollary 2.6.1, and the axis of rotation passes through the point of zero displacement, in the direction of the cross product of the two nonzero displacement vectors, which is a consequence of Theorem 2.6.4 and Corollary 2.6.1.

INDEX = 4. Two vectors are zero. The motion is pure rotation and the axis of rotation is defined by the two points of zero displacement.

INDEX = 5. No vector is zero and all three vectors are nonparallel amongst them but coplanar. Furthermore, the two arising difference vectors are noncollinear. According to Theorem 2.6.4 and Corollary 2.6.3, then, the motion is pure rotation and the screw parameters can be computed using the general procedure.

INDEX = 6. No vector is zero but two vectors are parallel and different. Moreover, the vectors are coplanar. The motion is pure rotation due to Corollary 2.6.5 and the axis of rotation is perpendicular to the plane of the given vectors. Its location can be determined using the general procedure, already outlined for pure rotation.

<u>Proof</u>

Let $\underline{u}_A$ and $\underline{u}_B$ be parallel but different. Then the following relationship holds

$$\underline{u}_B = \alpha \underline{u}_A \tag{2.6.77}$$

Let $\underline{e}$ be the unit vector along the screw axis. Then, from Theorem 2.6.2,

$$\underset{\sim}{u}_B \cdot \underset{\sim}{e} = \underset{\sim}{u}_A \cdot \underset{\sim}{e} \tag{2.6.78}$$

Substituting eq. (2.6.77) into eq. (2.6.78), one obtains

$$(1-\alpha)\underset{\sim}{u}_A \cdot \underset{\sim}{e} = 0 \tag{2.6.79}$$

which vanishes if either $\alpha=1$ or if $\underset{\sim}{u}_A \cdot \underset{\sim}{e}=0$. The first condition is impossible to meet because $\underset{\sim}{u}_A$ and $\underset{\sim}{u}_B$ have been assumed to be different. Hence the only possibility for eq. (2.6.79) to hold is

$$\underset{\sim}{u}_A \cdot \underset{\sim}{e} = 0$$

which indicates that the motion is one of pure rotation, according to Corollary 2.6.1, q.e.d.

INDEX=7.—No vector is zero and all three vectors are parallel to each other. Furthermore, not all three vectors are identical to each other, although there may be a pair of identical vectors. The motion is one of pure rotation and the axis of rotation is determined by the intersection of the plane defined by the given points in their reference configuration with that defined by the points in their final configuration.

Proof

It was shown in the case for which INDEX=6 that the existence of at least two parallel nonidentical vectors guarantees that the motion is one of pure rotation.

It will be shown first that the plane of the three given points contains the axis of rotation. In fact, the corresponding displacements are given by

$$\underset{\sim}{u}_A = (\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{a}_1,\quad \underset{\sim}{u}_B=(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{b}_1,\quad \underset{\sim}{u}_C=(\underset{\sim}{Q}-\underset{\sim}{I})\underset{\sim}{c}_1 \tag{2.6.80}$$

which are all parallel to each other. Thus, the diferences

$$\mathbf{u}_A-\mathbf{u}_C=(\mathbf{Q}-\mathbf{I})(\mathbf{a}_1-\mathbf{c}_1),\quad \mathbf{u}_B-\mathbf{u}_C=(\mathbf{Q}-\mathbf{I})(\mathbf{b}_1-\mathbf{c}_1) \tag{2.6.81}$$

are also parallel to each other. Thus, there exists a scalar such that

$$\mathbf{u}_A-\mathbf{u}_C+\alpha(\mathbf{u}_B-\mathbf{u}_C)=\mathbf{0} \tag{2.6.82}$$

But, substituting eqs. (2.6.81) into eq. (2.6.82),

$$(\mathbf{Q}-\mathbf{I})\left\{\mathbf{a}_1-\mathbf{c}_1+\alpha(\mathbf{b}_1-\mathbf{c}_1)\right\}=\mathbf{0} \tag{2.6.83}$$

which implies that the vector $\mathbf{a}_1-\mathbf{c}_1+\alpha(\mathbf{b}_1-\mathbf{c}_1)$, contained in the plane ABC, is parallel to the axis of rotation. Next, consider the position vectors of the points in their displaced positions

$$\mathbf{a}_2=\mathbf{a}_1+\mathbf{u}_A,\quad \mathbf{b}_2=\mathbf{b}_1+\mathbf{u}_B,\quad \mathbf{c}_2=\mathbf{c}_1+\mathbf{u}_C \tag{2.6.84}$$

The displacements of these points are, after substitutions and cancellations,

$$\mathbf{u}'_A=\mathbf{Q}\mathbf{u}_A,\quad \mathbf{u}'_B=\mathbf{Q}\mathbf{u}_B,\quad \mathbf{u}'_C=\mathbf{Q}\mathbf{u}_C \tag{2.6.85}$$

i.e. $\mathbf{u}'_A$, $\mathbf{u}'_B$ and $\mathbf{u}'_C$ are all parallel to each other. Hence, the differences

$$\mathbf{u}'_A-\mathbf{u}'_C=(\mathbf{Q}-\mathbf{I})(\mathbf{a}_2-\mathbf{c}_2),\quad \mathbf{u}'_B-\mathbf{u}'_C=(\mathbf{Q}-\mathbf{I})(\mathbf{b}_2-\mathbf{c}_2) \tag{2.6.86}$$

are also parallel to each other. Hence, there exists a scalar β such that

$$\mathbf{u}'_A-\mathbf{u}'_C+\beta(\mathbf{u}'_B-\mathbf{u}'_C)=\mathbf{0} \tag{2.6.87}$$

Substitution of eqs. (2.6.86) into eq. (2.6.87) yields then

$$(\mathbf{Q}-\mathbf{I})\left\{\mathbf{a}_2-\mathbf{c}_2+\beta(\mathbf{b}_2-\mathbf{c}_2)\right\}=\mathbf{0} \tag{2.6.88}$$

which means that the vector $\mathbf{a}_2-\mathbf{c}_2+\beta(\mathbf{b}_2-\mathbf{c}_2)$, contained in the plane $A_2B_2C_2$, is parallel to the axis of rotation. Moreover, both planes, ABC and $A_2B_2C_2$, are nonparallel, for the vectors

$\underline{u}_A$, $\underline{u}_B$ and $\underline{u}_C$ have been assumed to be not all three identical to each other. Hence both planes intersect along a line which is the axis of rotation, q.e.d.

So far all cases leading necessarily to a pure rotation motion have been discussed. Next the case in which the given displacement vectors are coplanar but the motion is either a pure rotation or general, is discussed. In this case the arising difference vectors are parallel and hence the given points lie in a plane parallel either to the axis of rotation or to the screw axis. This case is handled by subroutine COPL 2, which identifies each possible different subcase with the aid of the integer variable INDE.

INDE = 1. No vector is zero and two vectors are identical. The motion is either general or a pure rotation, but the screw axis or, correspondingly, the axis of rotation, is parallel to the line defined by the points with identical displacements.

Proof

Let B and C be the two points with identical displacements. These displacements can be expressed using eqs. (2.6.3a) as

$$\underline{u}_B=\underline{u}_A+(\underline{Q}-\underline{I})(\underline{b}_1-\underline{a}_1) \qquad (2.6.95a)$$

$$\underline{u}_C=\underline{u}_A+(\underline{Q}-\underline{I})(\underline{c}_1-\underline{a}_1) \qquad (2.6.95b)$$

Subtracting eq. (2.6.95b) from eq. (2.6.95a) one obtains

$$\underline{u}_B-\underline{u}_C=(\underline{Q}-\underline{I})(\underline{b}_1-\underline{c}_1)=\underline{0} \qquad (2.6.96)$$

which states that the vector connecting points B and C is parallel to the axis of rotation of matrix $\underline{Q}$. Hence, line BC is parallel to the screw axis. This axis is located following the general procedure previously outlined.

INDE = 2. No vector is zero and no two vectors are parallel, but they are coplanar. The motion is either general or a pure rotation and the given points lie in a plane parallel either to the screw axis or to the axis of rotation, according to Theorem 2.6.4 and Corollary 2.6.4. The direction of the screw axis or, correspondingly, of the axis of rotation, is found using eqs. (2.6.60)-(2.6.66). Summarizing, one has the following

THEOREM 2.6.5 *The motion of a rigid body is determined, i.e. its screw parameters can be computed, if, and only if, the positions of three noncollinear points of the body are known in both its reference and its final configurations.*

Subroutines SCREW, COPL 1, COPL 2, and GENMOT, implementing the foregoing computations, use LOCAT 1, LOCAT 2, ANGLE, CYCLIC, EXCHGE, CROSS and SCAL as subsidiary subroutines. Listings of all these subroutines appear in Figs 2.6.6 - 2.6.16

Exercise 2.6.8 In a manufacturing process it is required to position the workpiece of Fig 2.6.17 in configuration 2 starting from configuration 1, by means of an arm fastened to the bolt of a screw. Determine the location of the axis of this screw as well as its pitch. If the operation is to take place in n screw revolutions plus a fraction, what is the value of this fraction?

```
      SUBROUTINE SCREW(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
C
C
C   THIS SUBROUTINE COMPUTES THE SCREW-PARAMETERS OF A RIGID BODY MOTION
C
C
C   INPUT:
C   THE X,Y AND Z-COORDINATES OF THREE NONCOLLINEAR POINTS OF THE
C    RIGID BODY IN BOTH ITS INITIAL (AIN,BIN,CIN 3-DIMENSIONAL VECTORS)
C    AND IN ITS FINAL (AFIN,BFIN,CFIN 3-DIMENSIONAL VECTORS)
C    CONFIGURATIONS.
C
C   OUTPUT:
C    1.) THE DIRECTION E, OF THE SCREW AXIS
C    2.) THE LOCATION RHO OF THE POINT ON THE SCREW AXIS LYING CLOSEST
C        TO THE ORIGIN.
C    3.) THE ANGLE OF ROTATION (SIGN WITH RESPECT TO THE DIRECTION OF E
C        INCLUDED), THETA.
C    4.) THE SCALAR DISPLACEMENT DISPL, ALONG E (SIGN WITH RESPECT TO
C        THE DIRECTION OF E INCLUDED).
C
C   SUBSIDIARY SUBROUTINES:
C
C  COPL1(*****).- CONTAINS THE SAME PARAMETERS AS SCREW PLUS  INDEX AND
C                 IN, WHICH DEFINE EACH PARTICULAR POSSIBLE CASE.
C                 COMPUTES THE SCREW PARAMETERS WHEN THE MOTION IS PURE
C                 ROTATION.
C  COPL2(*****).- COMPUTES THE SCREW PARAMETERS WHEN THE GIVEN POINTS
C                 LIE IN A PLANE PARALLEL TO THE SCREW AXIS. THE MOTION
C                 IS EITHER GENERAL OR PURE ROTATION.
C  GENMOT(****).- COMPUTES THE SCREW PARAMETERS WHEN THE MOTION IS
C                 GENERAL AND THE GIVEN DISPLACEMENTS ARE NONCOPLANAR.
C  CROSS(A,B,C).- COMPUTES THE CROSS PRODUCT OF VECTORS A AND B, IN THIS
C                 ORDER, AND STORES THE PRODUCT IN VECTOR C.
C  SCAL(A,B,C) .- COMPUTES THE SCALAR PRODUCT OF VECTORS A AND B AND
C                 STORES THE PRODUCT IN THE SCALAR S.
C
C
C  3-DIMENSIONAL VECTORS A,B,C ARE AUXILIARY FIELDS.
C
      REAL  AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),UA(3),UB(3)
     -      ,UC(3),A(3),B(3),C(3),E(3),RHO(3)
       LOGICAL LO(3)
       COMMON ZERO
C
C  COLLINEARITY OF GIVEN POINTS IS VERIFIED. WHEN POINTS ARE COLLINEAR,
C   ZERO IS SET EQUAL TO -1. AND SUBROUTINE RETURNS TO MAIN PROGRAM.
      DO 10 I=1,3
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (first part)

```
            LO(I)=.FALSE.
            A(I)=AIN(I)-CIN(I)
   10    B(I)=BIN(I)-CIN(I)
         CALL CROSS(A,B,C)
         CALL SCAL(C,C,S)
         S=SQRT(S)
         IF(S-ZERO) 20,20,30
   20    ZERO=-1.
         WRITE(6,1000)
         RETURN
C
C   DONE
C   COMPABILITY IS VERIFIED. IF THIS IS NOT MET, THEN ZERO IS SET EQUAL
C   TO -2.,-3.,OR -4., DEPENDING UPON WETHER DISTANCE AC,BC, OR AB DOES
C   NOT REMAIN CONSTANT THROUGHOUT THE MOTION.
   30    DO 40 I=1,3
            C(I)=AFIN(I)-CFIN(I)
   40    CONTINUE
         CALL SCAL(A,A,S1)
         S1=SQRT(S1)
         CALL SCAL(C,C,S2)
         S2=SQRT(S2)
         IF(ABS(S1-S2).LE.ZERO) GO TO 50
         ZERO=-2.
         WRITE(6,1010)
         RETURN
   50    DO 60 I=1,3
            C(I)=BFIN(I)-CFIN(I)
   60    CONTINUE
         CALL SCAL(B,B,S1)
         CALL SCAL(C,C,S2)
         S1=SQRT(S1)
         S2=SQRT(S2)
         IF(ABS(S1-S2).LE.ZERO) GO TO 70
         ZERO=-3
         WRITE(6,1020)
         RETURN
   70    DO 80 I=1,3
            A(I)=AIN(I)-BIN(I)
   80    B(I)=AFIN(I)-BFIN(I)
         CALL SCAL(A,A,S1)
         CALL SCAL(B,B,S2)
         S1=SQRT(S1)
         S2=SQRT(S2)
         IF(ABS(S1-S2).LE.ZERO) GO TO 90
         ZERO=-4
         WRITE(6,1030)
         RETURN
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (second part)

```
C
C   DONE
C   DISPLACEMENT VECTORS ARE COMPUTED
   90    DO 100 I=1,3
                 UA(I)=AFIN(I)-AIN(I)
                 UB(I)=BFIN(I)-BIN(I)
  100    UC(I)=CFIN(I)-CIN(I)
C
C   DONE
C   NUMBER OF ZERO-DISPLACEMENTS IS DETERMINED AND STORED IN NUZE.
C   DISPLACEMENT MAGNITUDES ARE TEMPORARILY STORED IN E. IF NUZE.EQ.0
C   THEN NUZE IS SET EQUAL TO 4.
         CALL SCAL(UA,UA,E(1))
         CALL SCAL(UB,UB,E(2))
         CALL SCAL(UC,UC,E(3))
         NUZE=0
         DO 110 I=1,3
                 E(I)=SQRT(E(I))
                 IF(E(I).GT.ZERO) GO TO 110
                 NUZE=NUZE+1
                 LO(I)=.TRUE.
  110    CONTINUE
         IF(NUZE.EQ.0) NUZE=4
C
C   DONE
C   EACH CASE(NUZE=0,1,2,3) IS NOW INVESTIGATED
         GO TO(111,211,311,411),NUZE
  111    DO 120 I=1,3
                 IF(LO(I)) IN=I
  120    CONTINUE
         GO TO(121,131,141),IN
  121    CALL CROSS(UB,UC,C)
         DO 130 I=1,3
                 A(I)=UB(I)-UC(I)
  130    CONTINUE
         GO TO 160
  131    CALL CROSS(UA,UC,C)
         DO 140 I=1,3
                 A(I)=UA(I)-UC(I)
  140    CONTINUE
         GO TO 160
  141    CALL CROSS(UA,UB,C)
         DO 150 I=1,3
                 A(I)=UA(I)-UB(I)
  150    CONTINUE
  160    CALL SCAL(C,C,S1)
         CALL SCAL(A,A,S2)
         S1=SQRT(S1)
```

2.6.6 Listing of SUBROUTINE SCREW (third part)

```
      S2=SQRT(S2)
      INDEX=3
      IF(S1.LE.ZERO) INDEX=1
      IF(S2.LE.ZERO) INDEX=2
      GO TO 230
211   INDEX=4
      DO 221 I=1,3
             IF(LO(I)) GO TO 221
             IN=I
221   CONTINUE
230   CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -           IN)
      RETURN
311   ZERO=-5
      WRITE(6,1040)
      RETURN
C
C  DONE
C  ONE, TWO AND THREE-ZERO-DISPLACEMENT CASES WHERE ALREADY DEALT WITH
C  NO-ZERO DISPLACEMENT CASE IS NEXT INVESTIGATED.
C  PARALLELISM OF DISPLACEMENTS IS FIRST DETERMINED. CROSS PRODUCT
C  MAGNITUDES ARE TEMPORARILY STORED IN E.
411     CALL CROSS(UA,UB,A)
        CALL CROSS(UA,UC,B)
        CALL CROSS(UB,UC,C)
        CALL SCAL(A,A,E(1))
        CALL SCAL(B,B,E(2))
        CALL SCAL(C,C,E(3))
        DO 510 I=1,3
510            LO(I)=.FALSE.
        DO 520 I=1,3
               E(I)=SQRT(E(I))
               IF(E(I).GT.ZERO) GO TO 520
               LO(I)=.TRUE.
520     CONTINUE
        IF(LO(1).OR.LO(2).OR.LO(3)) GO TO 525
C
C  DONE
C  NO TWO DISPLACEMENT VECTORS WERE FOUND TO BE PARALLEL.  COPLANARITY
C  IS NEXT VERIFIED.
        CALL SCAL(UC,A,S)
        IF(ABS(S).LE.ZERO) GO TO 523
        CALL GENMOT(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
        RETURN
C
C  COLLINEARITY OF DIFFERENCE VECTORS IS VERIFIED.
C  DIFFERENCES OF DISPLACEMENT VECTORS ARE TEMPORARILY STORED IN A
C  AND B.  THE CROSS PRODUCT OF THE LATTER IS STORED IN C.
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (fourth part)

```
 523      DO 524 I=1,3
                A(I)=UA(I)-UC(I)
 524      B(I)=UB(I)-UC(I)
          CALL CROSS(A,B,C)
          CALL SCAL(C,C,S)
          S=SQRT(S)
          IF(S.LE.ZERO) GO TO 700
          INDEX=5
          CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
                   IN)
          RETURN
C
C  DONE
C  DETERMINES WHICH VECTORS ARE PARALLEL BY SETTING LO(I) EQUAL TO .TRUE
 525      DO 530 I=1,3
                IF(LO(I)) GO TO 528
                GO TO 530
 528            IN=I
                INDEX=6
 530      CONTINUE
C
C  DONE
C  INVESTIGATES IF ALL THREE VECTORS ARE PARALLEL
          DO 540 I=1,2
                IP1=I+1
                DO 540 J=IP1,3
                      IF(LO(I).AND.LO(J)) INDEX=7
 540      CONTINUE
          JNDEX=INDEX-5
          GO TO(550,600),JNDEX
C
C  DONE
C  DETERMINES IF, FOR TWO PARALLEL VECTORS, THESE ARE IDENTICAL
 550      GO TO(551,561,571),IN
 551      DO 552 I=1,3
                A(I)=UA(I)
                C(I)=UC(I)
                UA(I)=C(I)
 552      UC(I)=A(I)
          GO TO 571
 561      DO 562 I=1,3
                A(I)=UA(I)
                B(I)=UB(I)
               UA(I)=B(I)
 562      UB(I)=A(I)
 571      DO 572 I=1,3
                A(I)=UB(I)-UC(I)
 572      CONTINUE
```

g 2.6.6 Listing of SUBROUTINE SCREW (fifth part)

```
      CALL SCAL(A,A,S)
      S=SQRT(S)
      IF(S.LE.ZERO) GO TO 750
      CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -            IN)
      RETURN
C
C  DONE
C  DETERMINES IF ALL THREE PARALLEL VECTORS ARE IDENTICAL
  600 DO 610 I=1,3
            A(I)=UA(I)-UC(I)
  610 B(I)=UB(I)-UC(I)
      CALL SCAL(A,A,S1)
      CALL SCAL(B,B,S2)
      S1=SQRT(S1)
      S2=SQRT(S2)
      IF(S1.LE.ZERO.AND.S2.LE.ZERO) GO TO 910
      CALL COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDEX,
     -            IN)
      RETURN
C
C  DONE
  700 INDE=2
      GO TO 900
  750 INDE=1
  900 CALL COPL2(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL,INDE,
     -            IN)
      RETURN
C
C  IF MOTION IS PURE TRASLATION, ZERO IS SET EQUAL TO -6
  910 ZERO=-6
      WRITE(6,1050)(UA(I),I=1,3)
      RETURN
 1000 FORMAT(5X,'POINTS ARE COLLINEAR. MOTION IS UNDEFINED'/)
 1010 FORMAT(5X,'MOTION IS NOT RIGID. LENGTH AC DOES NOT REMAIN',
     -       ' CONSTANT.'/)
 1020 FORMAT(5X,'MOTION IS NOT RIGID. LENGTH BC DOES NOT REMAIN',
     -       ' CONSTANT.'/)
 1030 FORMAT(5X,'MOTION IS NOT RIGID. LENGTH AB DOES NOT REMAIN',
     -       ' CONSTANT.'/)
 1040 FORMAT(5X,'NO MOTION. ALL THREE DISPLACEMENTS VECTORS ARE',
     -       ' ZERO.'/)
 1050 FORMAT(5X,'THE MOTION IS PURE TRANSLATION. '//15X,'THE ',
     -       'DISPLACEMENT HAS THE FOLLOWING X-,Y-AND Z COMPONENTS :'
     -       /15X,F12.5,5X,F12.5,5X,F12.5/)
      END
```

Fig 2.6.6 Listing of SUBROUTINE SCREW (sixth and last part)

```
      SUBROUTINE COPL1(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL
     -                ,INDEX,IN)
C
C  THIS SUBROUTINE COMPUTES THE SCREW PARAMETERS E,RHO,THETA AND DISPL
C  WHEN THE RIGID BODY UNDER STUDY UNDERGOES A PURE ROTATION.
C  THE SUBROUTINE PARAMETERS WERE DEFINED IN SUBROUTINE SCREW, EXCEPT
C  FOR INDEX AND IN. THESE ARE DEFINED NEXT.
C  INDEX = 1, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE PARALLEL, BUT NOT IDENTICAL.
C  INDEX = 2, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE IDENTICAL
C  INDEX = 3, IF ONLY ONE DISPLACEMENT IS ZERO AND THE OTHER TWO
C             DISPLACEMENTS ARE NOT IDENTICAL.
C  INDEX = 4, IF EXACTLY TWO DISPLACEMENTS ARE ZERO.
C  INDEX = 5, IF NO DISPLACEMENTS IS ZERO AND ALL DISPLACEMENTS ARE
C             NONPARALLEL, PROVIDED THE TWO DISTINCT DISPLACEMENT
C             DIFFERENCES ARE NONCOLLINEAR.
C  INDEX = 6, IF NO DISPLACEMENT IS ZERO AND EXACTLY TWO VECTORS ARE
C             PARALLEL BUT DIFFERENT.
C  INDEX = 7, IF ALL THREE DISPLACEMENTS ARE PARALLEL BUT NOT ALL THREE
C             ARE IDENTICAL.
C
C  IN, DETECTS WHICH VECTORS ARE PARALLEL OR IDENTICAL, IF AT ALL.
C  SUBSIDARY SUBROUTINES :
C            LOCAT1(*****).- COMPUTES VECTOR RHO, WHEN NO TWO
C                            DISPLACEMENT VECTORS ARE PARALLEL.
C            LOCAT2(*****).- COMPUTES VECTOR E AND RHO WHEN AT LEAST
C                            TWO DISPLACEMENT VECTORS ARE PARALLEL.
C            ANGLE (*****).- COMPUTES THE ANGLE OF ROTATION.
C            CYCLIC(A,B,C).- PERFORMS A CYCLIC CHANGE OF VECTORS A,B
C                            & C, I.E., A IS SET EQUAL TO B, B IS SET
C                            EQUAL TO C,...
C            EXCHGE (A,B) .- EXCHANGES THE LOCATIONS OF FIELDS A AND B
C
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3),
     -     UA(3),UB(3),UC(3),PERP(3),A(3),B(3)
C
C  COMPUTES THE DISPLACEMENTS
C
      DO 10 I=1,3
         UA(I)=AFIN(I)-AIN(I)
         UB(I)=BFIN(I)-BIN(I)
   10 UC(I)=CFIN(I)-CIN(I)
      GO TO(100,100,100,400,500,600,700),INDEX
C
C  DONE
C  IN WAS SET IN SUBROUTINE SCREW EQUAL TO 1,2 OR 3, DEPENDING ON WHICH
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (first part)

```
C     VECTOR IS ZERO, UA, UB, OR UC, RESPECTIVELY
  100     GO TO(120,110,110),IN
  110     CALL CYCLIC(UA,UB,UC)
          CALL CYCLIC(AIN,BIN,CIN)
          IF(IN.EQ.2) GO TO 120
          CALL CYCLIC(UA,UB,UC)
          CALL CYCLIC(AIN,BIN,CIN)
  120     GO TO(130,200,300),INDEX
C
C     COMPUTATION OF RHO AND E WHEN INDEX=1
  130     WRITE(6,1000)
          CALL LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
          GO TO 320
C
C     DONE
C     COMPUTATION OF THE DIRECTION OF THE AXIS OF ROTATION WHEN INDEX=2
  200     WRITE(6,1010)
          DO 210 I=1,3
  210            E(I)=BIN(I)-CIN(I)
          CALL SCAL(E,E,X)
          X=SQRT(X)
          GO TO 305
C
C     DONE
C     COMPUTATION OF THE DIRECTION OF THE AXIS OF ROTATION WHEN INDEX=3
  300     WRITE(6,1020)
          CALL CROSS(UB,UC,E)
          CALL SCAL(E,E,X)
          X=SQRT(X)
  305     DO 307 I=1,3
  307            E(I)=E(I)/X
C
C     DONE
C     COMPUTATION OF THE POINT OF THE AXIS OF ROTATION LYING CLOSEST TO
C     THE ORIGIN.
          CALL SCAL(AIN,E,S)
          DO 310 I=1,3
  310            RHO(I)=AIN(I)-S*E(I)
C
C     DONE
C     COMPUTATION OF THE ANGLE OF ROTATION
  320     CALL ANGLE(BIN,UB,E,RHO,THETA)
          DISPL=0.
          RETURN
C
C     DONE
C     INVESTIGATES THE CASE WHEN TWO DISPLACEMENTS ARE ZERO. IN WAS SET
C     IN SUBROUTINE SCREW EQUAL TO 1,2 OR 3, DEPENDING ON WETHER UA,UB, OR
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (second part)

```
C     UC RESPECTIVELY, IS DIFFERENT FROM ZERO. THEN THE SCREW PARAMETERS
C     ARE COMPUTED.
  400     WRITE(6,1030)
          GO TO(420,410,410),IN
  410     CALL CYCLIC(AIN,BIN,CIN)
          CALL CYCLIC(UA,UB,UC)
          IF(IN.EQ.2) GO TO 420
          CALL CYCLIC(AIN,BIN,CIN)
          CALL CYCLIC(UA,UB,UC)
  420     DO 430 I=1,3
  430         E(I)=CIN(I)-BIN(I)
          CALL SCAL(E,E,X)
          X=SQRT(X)
          DO 440 I=1,3
  440         E(I)=E(I)/X
          CALL SCAL(BIN,E,S)
          DO 450 I=1,3
  450         RHO(I)=BIN(I)-S*E(I)
          CALL ANGLE(AIN,UA,E,RHO,THETA)
          DISPL=0.
          RETURN
C
C     DONE
C     COMPUTES THE SCREW PARAMETERS WHEN NO DISPLACEMENT IS ZERO, AND
C     ALL THREE VECTORS ARE NONPARALLEL. FURTHERMORE, THE TWO DIFFERENCE
C     VECTORS ARE NONCOLLINEAR. HENCE THE MOTION IS A PURE ROTATION.
  500     WRITE(6,1040)
          GO TO 610
C
C     COMPUTES THE SCREW PARAMETERS WHEN NO DISPLACEMENT IS ZERO BUT
C     EXACTLY TWO VECTORS ARE PARALLEL AND DIFFERENT. IN IS SET IN SUB-
C     ROUTINE SCREW EQUAL TO 1,2 OR 3 DEPENDING UPON WETHER UC,UB OR UA IS
C     THE NONPARALLEL VECTOR.
  600     WRITE(6,1050)
          IF(IN.NE.2) GO TO 610
          CALL CYCLIC(AIN,BIN,CIN)
          CALL CYCLIC(UA,UB,UC)
  610     CALL CROSS(UA,UC,E)
          CALL SCAL(E,E,S)
          S=SQRT(S)
          DO 620 I=1,3
              E(I)=E(I)/S
  620     CONTINUE
          CALL LOCAT1(AIN,CIN,UA,UC,RHO)
          CALL ANGLE(AIN,UA,E,RHO,THETA)
          DISPL=0.
          RETURN
C
```

g 2.6.7 Listing of SUBROUTINE COPL1 (third part)

```
C   DONE
C   COMPUTES PARAMETERS FOR NO-ZERO-DISPLACEMENT-CASE WITH ALL THREE
C   VECTORS PARALLEL BUT NO TWO VECTORS  IDENTICAL TO EACH OTHER.
 700      WRITE(6,1060)
          CALL LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
          CALL ANGLE(AIN,UA,E,RHO,THETA)
          DISPL=0.
          RETURN
1000      FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     -    'PARALLEL'/5X,'BUT DISTINTCT. THE MOTION IS PURE ROTATION.',
     -    ' INDEX=1'/)
1010      FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     -    'IDENTICAL.'/5X,'THE MOTION IS PURE ROTATION. INDEX=2'/)
1020      FORMAT(5X,'ONE DISPLACEMENT IS ZERO AND THE OTHER TWO ARE ',
     -    'NEITHER IDENTICAL'/5X,'NOR PARALLEL. THE MOTION IS PURE ',
     -    'ROTATION. INDEX=3'/)
1030      FORMAT(5X,'TWO DISPLACEMENTS ARE ZERO. THE MOTION IS PURE',
     -    ' ROTATION.'/5X,'INDEX=4'/)
1040      FORMAT(5X,'THE DISPLACEMENTS ARE COPLANAR,THE TWO DISPLACEMENT',
     -    ' DIFFERENCES'/5X,'ARE NONCOLLINEAR AND NO DISPLACEMENT IS ',
     -    'ZERO. THE MOTION IS PURE'/5X,'ROTATION. INDEX=5'/)
1050      FORMAT(5X,'TWO DISPLACEMENTS ARE PARALLEL BUT DIFFERENT AND',
     -    ' NO DISPLACEMENT'/5X,'IS ZERO. THE MOTION IS PURE ROTATION.',
     -    ' INDEX=6'/)
1060      FORMAT(5X,'THREE DISPLACEMENTS ARE PARALLEL, BUT NOT ALL THREE',
     -    ' ARE IDENTICAL.'/5X,'THE MOTION IS PURE ROTATION. INDEX=7'/)
          END
```

Fig 2.6.7 Listing of SUBROUTINE COPL1 (fourth and last part)

```
      SUBROUTINE COPL2(AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL
     -                 ,INDE,IN)
C
C     THIS SUBROUTINE COMPUTES THE SCREW PARAMETERS E,RHO,THETA AND DISPL
C     WHEN THE DISPLACEMENTS OF THE THREE GIVEN POINTS ARE COPLANAR, IN
C     WHICH CASE THE POINTS LIE IN A PLANE PARALLEL EITHER TO THE SCREW
C     AXIS OR TO THE AXIS OF ROTATION.
C     THE SUBROUTINE PARAMETERS WERE DEFINED IN SUBROUTINE SCREW EXCEPT
C     FOR INDE AND IN. THESE ARE DEFINED NEXT.
C     INDE = 1, WHEN TWO OF THE SAID DISPLACEMENTS ARE IDENTICAL. IN,
C               DETECT WHICH VECTORS ARE PARALLEL, IF AT ALL.
C     INDE = 2, WHEN ALL THREE DISPLACEMENTS ARE NONPARALLEL BUT THE
C               CORRESPONDING TWO DIFFERENCE DISPLACEMENT VECTORS ARE
C               COLLINEAR.
C     SUBSIDIARY SUBROUTINES WERE ALREADY DESCRIBED IN SUBROUTINE SCREW.
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3),
     -     UA(3),UB(3),UC(3)
      COMMON ZERO
C
C     COMPUTES THE DISPLACEMENTS
C
      DO 10 I=1,3
         UA(I)=AFIN(I)-AIN(I)
         UB(I)=BFIN(I)-BIN(I)
   10 UC(I)=CFIN(I)-CIN(I)
      GO TO(100,200),INDE
C
C     DONE
C     COMPUTES THE SCREW PARAMETERS WHEN INDE =1
C     RELABELS THE POINTS AND THEIR DISPLACEMENTS
  100 WRITE(6,1000)
      GO TO(110,120,130),IN
  110 CALL EXCHGE(AIN,CIN)
      CALL EXCHGE(UA,UC)
      GO TO 130
  120 CALL EXCHGE(AIN,BIN)
      CALL EXCHGE(UA,UB)
  130 DO 140 I=1,3
         E(I)=CIN(I)-BIN(I)
  140 CONTINUE
      CALL SCAL(E,E,S)
      S=SQRT(S)
      DO 150 I=1,3
         E(I)=E(I)/S
  150 CONTINUE
      CALL SCAL(UA,E,DISPL)
C
```

Fig 2.6.8 Listing of SUBROUTINE COPL2 (first part)

```
C     DONE
C     ELIMINATES THE TRANSLATION PART OF THE MOTION
C
      DO 160 I=1,3
             S=DISPL*E(I)
             UA(I)=UA(I)-S
             UB(I)=UB(I)-S
  160 UC(I)=UC(I)-S
      GO TO 310
C
C     DONE
C     COMPUTES THE SCREW PARAMETERS WHEN INDE =2
C     DIFFERENCES ARE TEMPORARILY STORED IN AFIN, BFIN AND CFIN
C
  200 WRITE(6,1010)
      DO 210 I=1,3
             AFIN(I)=UA(I)-UC(I)
             BFIN(I)=BIN(I)-CIN(I)
  210 CFIN(I)=AIN(I)-CIN(I)
      CALL SCAL(AFIN,BFIN,PRO1)
      CALL SCAL(AFIN,CFIN,PRO2)
      CALL SCAL(BFIN,BFIN,BC)
      CALL SCAL(CFIN,CFIN,AC)
      BC=SQRT(BC)
      AC=SQRT(AC)
      IF(ABS(PRO2).GT.ZERO) GO TO 240
      DO 230 I=1,3
             E(I)=CFIN(I)/AC
  230 CONTINUE
      GO TO 290
  240 IF(ABS(PRO1).GT.ZERO) GO TO 260
      DO 250 I=1,3
             E(I)=BFIN(I)/BC
  250 CONTINUE
      GO TO 290
  260 QUOT=PRO1/PRO2
      CALL SCAL(BFIN,CFIN,PRO3)
      BETA=(AC*AC*QUOT-PRO3-PRO3)*QUOT+BC*BC
      BETA=1./BETA
      BETA=SQRT(BETA)
      ALPHA=-BETA*QUOT
      DO 280 I=1,3
             E(I)=ALPHA*CFIN(I)+BETA*BFIN(I)
  280 CONTINUE
C
C     COMPUTES THE SCREW DISPLACEMENT
  290  CALL SCAL(UA,E,DISPL)
C
```

Fig 2.6.8 Listing of SUBROUTINE COPL2 (second part)

```
C     ELIMINATES THE TRANSLATION PART OF THE MOTION
        DO 300 I=1,3
               S=DISPL*E(I)
               UA(I)=UA(I)-S
  300   UB(I)=UB(I)-S
  310   CALL LOCAT1(AIN,BIN,UA,UB,RHO)
        CALL ANGLE(AIN,UA,E,RHO,THETA)
        RETURN
 1000   FORMAT(5X,'TWO DISPLACEMENTS ARE IDENTICAL. THE POINTS ',
     -         'CORRESPONDING'/5X,'TO THESE DISPLACEMENTS LIE ON A LINE ',
     -         'PARALLEL TO THE SCREW'/5X,'AXIS (OR TO THE AXIS OF '
     -         ,'ROTATION, IF THE MOTION IS PURE'/5X,'ROTATION). INDE=1'/)
 1010   FORMAT(5X,'THE TWO DISPLACEMENT DIFFERENCES ARE COLLINEAR. THE ',
     -         'GIVEN POINTS'/5X,'LIE IN A PLANE PARALLEL TO THE SCREW ',
     -         'AXIS (OR TO THE AXIS OF'/5X,'ROTATION, IF THE MOTION IS ',
     -         'PURE ROTATION). INDE=2'/)
C
C     DONE
        END
```

2.6.8 Listing of SUBROUTINE COPL2 (third and last part)

```
      SUBROUTINE GENMOT( AIN,BIN,CIN,AFIN,BFIN,CFIN,E,RHO,THETA,DISPL)
C
C   THIS PROGRAM COMPUTES THE SCREW PARAMETERS WHEN THE MOTION
C   IS GENERAL AND THE RESULTING DISPLACEMENTS ARE NONCOMPLANAR.
C   SUBSIDIARY SUBROUTINES WERE  ALREADY DESCRIBED IN SUBROUTINE
C   SCREW.
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),E(3),RHO(3)
      REAL  UA(3), UB(3),UC(3)
      COMMON  ZERO
C
C  COMPUTES THE DISPLACEMENTS
      DO 10 I=1,3
            UA(I)=AFIN(I)-AIN(I)
            UB(I)=BFIN(I)-BIN(I)
   10 UC(I)=CFIN(I)-CIN(I)
C
C  DONE
C  COMPUTES VECTOR E
C  STORES DIFFERENCE VECTORS TEMPORARILY IN AFIN AND BFIN
      DO 20 I=1,3
            AFIN(I)=UA(I)-UC(I)
            BFIN(I)=UB(I)-UC(I)
   20 CONTINUE
      CALL CROSS(AFIN,BFIN,CFIN)
      CALL SCAL(CFIN,CFIN,S)
      S=SQRT(S)
      DO 30 I=1,3
            E(I)=CFIN(I)/S
   30 CONTINUE
C
C  DONE
C  COMPUTES DISPL
      CALL SCAL(UA,E,DISPL)
C
C  DONE
C  STORES DISPLACEMENT VECTOR (UA*E)*E TEMPORARILY IN RHO AND COMPUTES
C  VECTORS UA', UB' AND UC', AND STORES THEM IN UA, UB AND UC,
C  RESPECTIVELY.
      DO 40 I=1,3
            RHO(I)=DISPL*E(I)
             UA(I)=UA(I)-RHO(I)
             UB(I)=UB(I)-RHO(I)
             UC(I)=UC(I)-RHO(I)
   40 CONTINUE
C
C  DONE
C  DETECTS PARALLELISM AMONGST THE MODIFIED DISPLACEMENT VECTORS AND
```

Fig. 2.6.9 Listing of SUBROUTINE GENMOT (first part).

```
C   COMPUTES THE SCREW PARAMETERS.
       CALL CROSS(UA,UB,AFIN)
       CALL CROSS(UB,UC,BFIN)
       CALL CROSS(UC,UA,CFIN)
       CALL SCAL(AFIN,AFIN,RHO(1))
       CALL SCAL(BFIN,BFIN,RHO(2))
       CALL SCAL(CFIN,CFIN,RHO(3))
       DO 60 I=1,3
             RHO(I)=SQRT(RHO(I))
             IF(RHO(I).GT.ZERO) GO TO 50
             CALL CYCLIC(AIN,BIN,CIN)
             CALL CYCLIC(UA,UB,UC)
 50          I=3
 60    CONTINUE
       CALL LOCAT1(AIN,BIN,UA,UB,RHO)
       CALL ANGLE(AIN,UA,E,RHO,THETA)
C
C   DONE
       WRITE(6,100)
100    FORMAT(5X,'THE MOTION IS GENERAL AND THE GIVEN DISPLACEMENTS ARE'
     -       /5X,'NONCOPLANAR'/)
       RETURN
       END
```

Fig 2.6.9 Listing of SUBROUTINE GENMOT (second and last part)

```
      SUBROUTINE LOCAT1(AIN,BIN,UA,UB,RHO)
C
C   THIS SUBROUTINE COMPUTES VECTOR RHO, I.E., THE POINT ON THE AXIS OF
C   A PURE ROTATION LYING CLOSEST TO THE ORIGIN.
C   PROCEDURE :
C   THE PSEUDO-INVERSE FORMULA (BEN-ISRAEL A. AND GREVILLE T.N.E.,
C   GENERALIZED INVERSES THEORY AND APPLICATIONS, WILEY N. YORK, 1974)
C   IS APPLIED TO FIND THE MINIMUM-NORM SOLUTION TO THE OVERDETERMINED
C   LINEAR 2X3 SYSTEM A*X=B, THESE EQUATIONS BEING THOSE OF TWO NON-
C   PARALLEL-PLANES. THIS FORMULA THUS FINDS THE POINT OF THE LINE
C   DEFINED BY THE INTERSECTION OF TWO NON-PARALLEL PLANES LYING CLOSEST
C   TO THE ORIGIN.
C   THESE PLANES ARE THE MEDIATOR PLANES OF SEGMENTS APIN-AIN AND BPIN-
C   BIN.
C
C
C   COMPUTES THE POSITION VECTORS OF THE MID-POINTS OF THE GIVEN
C   SEGMENTS. EACH IS THEN TEMPORARILY STORED IN RHO AND TEMP, THEN
C   CONSTRUCTS VECTOR B.
C
C
      REAL AIN(3),BIN(3),UA(3),UB(3),RHO(3),TEMP(3)
      DO 10 I=1,3
            RHO(I)=AIN(I)+UA(I)*0.5
            TEMP(I)=BIN(I)+UB(I)*0.5
   10 CONTINUE
C
C  BUILDS MATRIX A*(ATRANSP)
      CALL SCAL(UA,UA,A11)
      CALL SCAL(UB,UB,A22)
      CALL SCAL(UA,UB,A12)
      CALL SCAL(UA,RHO,B1)
      CALL SCAL(UB,TEMP,B2)
      DEN=A11*A22-A12*A12
      IF(ABS(DEN).LE.ZERO) GO TO 30
      X1=(B1*A22-B2*A12)/DEN
      X2=(B2*A11-B1*A12)/DEN
      DO 20 I=1,3
            RHO(I)=UA(I)*X1+UB(I)*X2
   20 CONTINUE
      RETURN
   30 WRITE(6,50)
      DO 40 I=1,3
            RHO(I)=UA(I)+UB(I)
            RHO(I)=RHO(I)*0.5
   40 CONTINUE
   50 FORMAT(//5X,'MATRIX A*(AT) IS SINGULAR'/)
      RETURN
      END
```

Fig 2.6.10 Listing of SUBROUTINE LOCAT1

```
      SUBROUTINE LOCAT2(AIN,BIN,CIN,AFIN,BFIN,CFIN,RHO,E)
C
C   THIS SUBROUTINE COMPUTES VECTORS RHO AND E WHEN ALL THREE RESULTING
C   DISPLACEMENTS ARE PARALLEL BUT NOT TWO VECTORS ARE IDENTICAL TO EACH
C   OTHER.
C   PROCEDURE :
C   EACH PLANE IS DETERMINED BY A TRIAD OF NONCOLLINEAR POINTS (AIN,BIN,
C   CIN, -AFIN,BFIN,CFIN). VECTOR E IS DETERMINED BY THE CROSS PRODUCT
C   OF THE NORMALS TO THE PLANES. RHO IS COMPUTED EXACTLY AS IN LOCAT1.
C
C
      REAL AIN(3),BIN(3),CIN(3),AFIN(3),BFIN(3),CFIN(3),DIF1(3),
     -     DIF2(3),PROD(3),RHO(3),E(3)
      DO 10 I=1,3
            DIF1(I)=BIN(I)-AIN(I)
            DIF2(I)=CIN(I)-AIN(I)
 10   CONTINUE
      CALL CROSS(DIF1,DIF2,PROD)
      DO 20 I=1,3
            DIF1(I)=BFIN(I)-AFIN(I)
            DIF2(I)=CFIN(I)-AFIN(I)
 20   CONTINUE
      CALL CROSS(DIF1,DIF2,RHO)
      CALL CROSS(PROD,RHO,E)
      CALL SCAL(E,E,S)
      S=SQRT(S)
      IF(ABS(S).LE.ZERO) GO TO 40
C
C   BUILDS MATRIX A*(ATRANSP) AND VECTOR B.
      CALL SCAL(PROD,PROD,A11)
      CALL SCAL(RHO,RHO,A22)
      CALL SCAL(PROD,RHO,A12)
      CALL SCAL(PROD,CIN,B1)
      CALL SCAL(RHO,CFIN,B2)
      DEN=A11*A22-A12*A12
      IF(ABS(DEN).LE.ZERO) GO TO 40
      T1=(B1*A22-B2*A12)/DEN
      T2=(B2*A11-B1*A12)/DEN
      DO 30 I=1,3
            E(I)=E(I)/S
            RHO(I)=PROD(I)*T1+RHO(I)*T2
 30   CONTINUE
      RETURN
 40   DO 50 I=1,3
            DIF1(I)=CFIN(I)-CIN(I)
 50   CONTINUE
      CALL CROSS(DIF1,PROD,E)
      CALL SCAL(E,E,S)
      S=SQRT(S)
      DO 60 I=1,3
            E(I)=E(I)/S
 60   CONTINUE
      CALL SCAL(AIN,E,T)
      DO 70 I=1,3
            RHO(I)=AIN(I)-T*E(I)
 70   CONTINUE
      RETURN
      END
```

Fig 2.6.11 Listing of SUBROUTINE LOCAT2

```
      SUBROUTINE ANGLE(AIN,UA,E,RHO,THETA)
C
C   THIS SUBROUTINE COMPUTES THE ANGLE OF ROTATION OF A PURE-ROTATION
C   MOTION.
C   PROCEDURE:
C   USE IS MADE OF RODRIGUES' FORMULA (BISHOP K.E.,'RODRIGUES' FORMULA
C   AND THE SCREW MATRIX', JOURNAL OF ENGINEERING FOR INDUSTRY, TRANS.
C   ASME, SERIES B, VOL. 91, FEB. 1969) :
C                                          R2-R1=TAN(THETA/2)EX(R1+R2)
C   WHERE R1 AND R2 ARE THE INITIAL AND THE FINAL POSITION VECTORS OF
C   ONE POINT OF THE BODY NOT LYING ON THE AXIS OF ROTATION. THE ORIGIN
C   IS ASSUMED TO BE LOCATED AT ONE POINT OF THE AXIS OF ROTATION. THETA
C   AND E ARE THE ANGLE OF ROTATION AND THE UNIT VECTOR PARALLEL TO THE
C   AXIS OF ROTATION, RESPECTIVELY.
C
C
      REAL AIN(3),UA(3),E(3),RHO(3),TEMP(3)
      DO 10 I=1,3
            AIN(I)=AIN(I)-RHO(I)
            UA(I)=AIN(I)+UA(I)
            TEMP(I)=AIN(I)+UA(I)
            UA(I)=UA(I)-AIN(I)
   10 CONTINUE
      CALL CROSS(E,TEMP,AIN)
      QUOT=0
      DO 20 I=1,3
            IF(ABS(AIN(I)).LE.ZERO) GO TO 20
            QUOT=UA(I)/AIN(I)
            THETA=ATAN(QUOT)*2.0
            GO TO 30
   20 CONTINUE
   30 IF(ABS(QUOT).GT.ZERO) RETURN
      THETA=ATAN(1.0)*4.0
      RETURN
      END
```

Fig 2.6.12 Listing of SUBROUTINE ANGLE

```
      SUBROUTINE  CYCLIC (A,B,C)
C
C   THIS SUBROUTINE PERFORMS A CYCLIC RELABELLING OF VECTORS A, B, C,
C   I.E. VECTORS A,B AND C ARE RELABELLED B, C AND A RESPECTIVELY.
C
C
      REAL A(3),B(3),C(3),AUX(3)
      DO 10 I=1,3
            AUX(I)=A(I)
            A(I)=B(I)
            B(I)=C(I)
            C(I)=AUX(I)
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.13 Listing of SUBROUTINE CYCLIC

```
      SUBROUTINE  EXCHGE (A,B)
C
C    THIS SOBROUTINE EXCHANGES THE FIELDS OF A AND B, I.E., IT
C    RETURNS B AS A AND A AS B .
C
C
      REAL A(3),B(3),AUX(3)
      DO 10 I=1,3
            AUX(I)=A(I)
            A(I)=B(I)
            B(I)=AUX(I)
   10 CONTINUE
      RETURN
      END
```

Fig 2.6.14 Listing of SUBROUTINE EXCHGE

```
      SUBROUTINE CROSS  (A,B,C)
C
C  THIS SUBROUTINE PERFORMS THE CROSS PRODUCT A AND B, IN THIS ORDER,
C  AND STORES THIS PRODUCT IN C.
C
C
      REAL A(3),B(3),C(3)
      DO 10 K=1,3
         C(K)=0.
         DO 10 L=1,3
            DO 10 M=1,3
               N=(L-K)*(M-L)*(K-M)
               C(K)=C(K)-N*A(L)*B(M)/2.
   10 CONTINUE
      RETURN
      END
*
```

Fig 2.6.15 Listing of SUBROUTINE CROSS

```
      SUBROUTINE SCAL (A,B,S)
C
C  THIS SUBROUTINE PERFORMS THE SCALAR PRODUCT OF VECTORS A AND B
C  AND STORES THIS PRODUCT IN S.
C
C
      REAL A(3),B(3)
      S=0.
      DO 10 I=1,3
         S=S+A(I)*B(I)
   10 CONTINUE
      RETURN
      END
*
```

Fig 2.6.16 Listing of SUBROUTINE SCAL

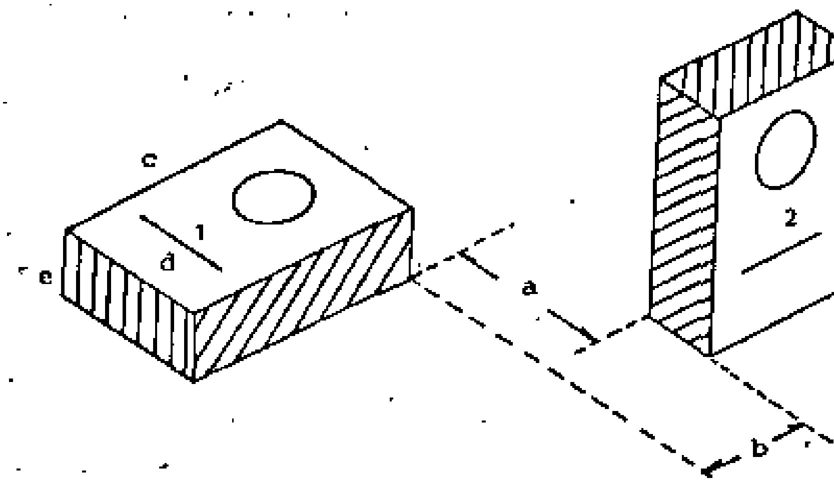

Fig. 2.6.17 Motion of a workpiece

Exercise 2.6.9 (Taken from [2.7]) Let $\underline{a}$, $\underline{r}$, $\underline{a}'$ and $\underline{r}'$ be the position vectors of the initial and the displaced positions of points A and R of a rigid body under the screw motion

$$\underline{r}'=\underline{a}'+\underline{Q}(\underline{r}-\underline{a})$$

$\underline{Q}$ being the rotation of the screw. Show that the set of points of the body that, under the given motion remain equidistant from a fixed point P, lie in a plane.

2.7 VELOCITY OF A POINT OF A RIGID BODY ROTATING ABOUT A FIXED POINT.

In the previous sections the motion of a rigid body when moving between two finitely separated configurations was analyzed. In this section and the following ones, the motion of a rigid body between two infinitesimally separated configurations is analyzed. The variables involved in the body motion are considered to be functions of time and results concerning their time derivatives are obtained.

Let $\underline{y}(t)$ be the image of vector $\underline{x}$ under a pure rotation $\underline{Q}(t)$. Clearly,

$\mathbf{x}$ is an independent variable; however, its image, $\mathbf{y}(t)$, is a function of time. If the origin of coordinates is placed at the fixed point, then

$$\mathbf{y}(t)=\mathbf{Q}(t)\mathbf{x} \tag{2.7.1}$$

Differentiating the above equation with respect to time, one obtains

$$\mathbf{y}(t)=\dot{\mathbf{Q}}(t)\mathbf{x} \tag{2.7.2}$$

which is an expression for the velocity of the point located by vector $\mathbf{x}$ in its initial configuration, at time t. Expression (2.7.2), however, is not practical to compute the velocity of the said point, for it requires knowledge of the point position in its initial configuration. Solving for $\mathbf{x}$ in eq. (2.7.1) and introducing the corresponding value in eq. (2.7.2) yields

$$\mathbf{v}(t)=\dot{\mathbf{y}}(t)=\dot{\mathbf{Q}}(t)\mathbf{Q}^{T}(t)\mathbf{y}(t) \tag{2.7.3}$$

which is an expression for the velocity of a point of a rigid body moving about a fixed point, in terms of the current position vector of the moving point. The matrix poduct, $\dot{\mathbf{Q}}(t)\mathbf{Q}^{T}(t)$, called the angular velocity* of the rigid body, represented by $\mathbf{\Omega}(t)$, is a skew symmetric matrix. Then, the velocity $\mathbf{v}(t)$ can be expressed as

$$\mathbf{v}(t)=\mathbf{\Omega}(t)\mathbf{y}(t) \tag{2.7.4a}$$

where

$$\mathbf{\Omega}(t)=\dot{\mathbf{Q}}(t)\mathbf{Q}^{T}(t)$$

Exercise 2.7.1 Show that, if $\mathbf{Q}(t)$ is orthogonal, then $\dot{\mathbf{Q}}(t)\mathbf{Q}^{T}(t)$ is skew symmetric.

* Truesdell (2.8) prefers to call it "the spin" and so it is found also under this name in the literature

Exercise 2.7.2 Show that the velocity of a point of a rigid body moving about a fixed point is perpendicular to its position vector (directed from the fixed point). Since $\Omega(t)$ is skew symmetric and 3x3 it is totally determined by three independent scalars, thus being isomorphic to a cartesian vector, $\omega(t)$, called also the angular velocity of the rigid body. Using cartesian vector notation, the velocity $v(t)$ then can be expressed as

$$v(t)=\omega(t)\times r(t) \qquad (2.7.5)$$

Exercise 2.7.8 Obtain the components ω_i of vector ω in terms of the components Ω_{ij} of matrix Ω.

Equation 2.7.5 makes the result of Exercise 2.7.2 apparent.

Since $\Omega(t)$ is skew symmetric and 3x3, it has one zero eigenvalue, as is shown below. Furthermore, its other two eigenvalues are complex (and conjugate, of course). Indeed, assume $Q(t)$ is in its canonical form, i.e.

$$\{Q(t)\}=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ \sin\theta & 0 & 1\end{pmatrix} \qquad (2.7.6)$$

$$\{\dot{Q}(t)\}=\begin{pmatrix}-\sin\theta & -\cos\theta & 0\\ \cos\theta & -\sin\theta & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta} \qquad (2.7.7)$$

From the above expressions,

$$\dot{Q}(t)Q^T(t) = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\dot{\theta} \qquad (2.7.8)$$

which makes evident that all vectors of the form $\{e\} = \{0,0,\alpha\}^T$, α being any scalar, correspond to a zero eigenvalue. The other two eigenvalues

are readily found to be

$$\lambda_1=\dot{\theta}i, \lambda_2=-\dot{\theta}i \qquad (2.7.9)$$

where i is the imaginary unity, $\sqrt{-1}$.

The null space of $\underline{\Omega}(t)$ (see Sec. 1.3) is, from the foregoing discussion, of dimension 1, i.e. a line. All the points lying on that line have zero velocity, the line thus being called "the instant axis of rotation" of the rigid body.

A cone rolling without slipping on a plane is a simple example of a rigid body rotating about a fixed point, its apex; its instant axis of rotation is clearly, the element of the cone touching instantaneously the plane. Another example would be a sphere rotating on a plane in such a way that the contact point remains fixed; the instant axis of rotation of the sphere is thus the diameter passing through the point of contact.

Exercise 2.7.4 A cone of revolution rolls on a conic surface, also of revolution, without slipping, in such a way that both appices are coincident. What is the instant axis of rotation of the cones in motion?.

Exercise 2.7.5 Show that the spin matrix Ω can be written as

$$\underline{\Omega}=\underline{A}\dot{\theta}$$

where $\underline{A}$ is a constant matrix and $\dot{\theta}$ is the time derivative of θ , the rotation angle..

2.8 VELOCITY OF A MOVING POINT REFERRED TO A MOVING OBSERVER.

In what follows, an observer will be understood to be a set of coordinate axes provided with a clock (2.8,p.26). Assume a point P_o, located by vector $\underline{x}_o$, is the origin of a coordinate system in motion with respect to another coordinate system, which will be arbitrarily referred to as "fixed".

The latter system constitutes a fixed observer, whereas the first, a moving one.

Let $\underset{\sim}{x}$ be the position vector of a point P, in motion with respect to both observers. Vectors and matrices expressed with respect to the fixed observer will be indexed with letter F, whereas those expressed with respect to the moving one will be indexed with letter M. Let $\underset{\sim}{\xi}$ be the position vector of P in the moving observer and $\underset{\sim}{Q}$ the rotation dyadic from the fixed observer to the moving one.

Hence,

$$(\underset{\sim}{x})_F = (\underset{\sim}{x}_o)_F + (\underset{\sim}{\xi})_F \qquad (2.8.1)$$

where it is understood that all three vectors are functions of time. The velocity of P is obtained differentiating both sides of eq. (2.8.1) with respect to time, i.e.

$$(\underset{\sim}{v})_F = (\underset{\sim}{v}_o)_F + (\dot{\underset{\sim}{\xi}})_F \qquad (2.8.2)$$

where v_o is the velocity of point P_o and, since

$$(\underset{\sim}{\xi})_F = (\underset{\sim}{Q})_F(\underset{\sim}{\xi})_M \qquad (2.8.3)$$

then

$$(\dot{\underset{\sim}{\xi}})_F = (\dot{\underset{\sim}{Q}})_F(\underset{\sim}{\xi})_M + (\underset{\sim}{Q})_F(\dot{\underset{\sim}{\xi}})_M =$$

$$= (\underset{\sim}{\Omega})_F(\underset{\sim}{\xi})_F + (\underset{\sim}{Q})_F(\dot{\underset{\sim}{\xi}})_M \qquad (2.8.4)$$

Thus, eq. (2.8.2) becomes

$$(\underset{\sim}{v})_F = (\underset{\sim}{v}_o)_F + (\underset{\sim}{\Omega})_F(\underset{\sim}{\xi})_F + (\underset{\sim}{Q})_F(\dot{\underset{\sim}{\xi}})_M =$$

$$= (\underset{\sim}{v}_o)_F + (\underset{\sim}{\Omega})_F(\underset{\sim}{Q})_F(\underset{\sim}{\xi})_M + (\underset{\sim}{Q})_F(\dot{\underset{\sim}{\xi}})_M \qquad (2.8.5)$$

where the first two terms of the right hand side represent the velocity of P as if it were one point of the rigid body defined by the moving observer*

* See Section 2.9

and the last term is the velocity of P as measured by the moving observer. Matrix $(\underset{\sim}{Q})_F$ transfers the description of velocity $(\dot{\underset{\sim}{\xi}})_M$ to the fixed observer. Thus, eq. (2.8.5) states that the velocity of point P equals that of point P as if it were fixed to the moving coordinate axes, plus the velocity of point P as if the moving observer were fixed.

Given any two points, P_1 and P_2, moving with velocities $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, respectively, "the relative velocity of P_2 with respect to P_1" is defined as

$$\underset{\sim}{v}_{2/1}=\underset{\sim}{v}_2-\underset{\sim}{v}_1 \quad (2.8.6)$$

Similarly, the relative angular velocity of body 2, moving with angular velocity $\underset{\sim}{\Omega}_2$, with respect to body 1, moving with angular velocity $\underset{\sim}{\Omega}_1$, is defined as

$$\underset{\sim}{\Omega}_{2/1}=\underset{\sim}{\Omega}_2-\underset{\sim}{\Omega}_1 \quad (2.8.7)$$

or, alternatively,

$$\underset{\sim}{\omega}_{2/1}=\underset{\sim}{\omega}_2-\underset{\sim}{\omega}_1 \quad (2.8.7)$$

2.9 GENERAL MOTION OF A RIGID BODY

Let a rigid body B undergo the most general motion, i.e., in general, no point of B remains fixed. Let $\underset{\sim}{v}_P$ be the velocity of one of its points, P, with position vector $\underset{\sim}{x}_P$ and angular velocity $\underset{\sim}{\Omega}$.

Thus, the relative velocity, $\underset{\sim}{v}-\underset{\sim}{v}_P$, of any other point R(located by $\underset{\sim}{x}$) with respect to P, is given by

$$\underset{\sim}{v}-\underset{\sim}{v}_P=\underset{\sim}{\Omega}(\underset{\sim}{x}-\underset{\sim}{x}_P) \quad (2.9.1)$$

for $\underset{\sim}{v}-\underset{\sim}{v}_P$ is the velocity that R would have if P were fixed. From eq. (2.9.1),

$$\underset{\sim}{v}=\underset{\sim}{v}_P+\underset{\sim}{\Omega}(\underset{\sim}{x}-\underset{\sim}{x}_P) \quad (2.9.2)$$

is the velocity of point R, and is given in terms of the velocity of another point, P, the angular velocity $\underset{\sim}{\Omega}$ and the position vector of R with respect to P.

Given two rigid bodies in motion, body 1 rolls without slipping with respect to body 2 if, and only if, there exists a set of points on both 1 and 2 such that the relative velocity of points on that set is zero. Regarding the velocity $\underline{v}$, as given by eq. (2.9.2) as the relative velocity of one point R of body B with respect to body C, the fixed observer, the condition for B to roll without slipping on C is that there exists a set of points, whose position vector is given by $\underline{x}$, for which $\underline{v}=0$. But for this to happen, the condition is

$$\underline{v}_p = -\underset{\sim}{\Omega}(\underline{x}-\underline{x}_p) \qquad (2.9.3)$$

which states that $\underline{v}_p$ is in the range (See section 1.3) of $\underset{\sim}{\Omega}$. However, it was shown in Section 2.8 that the null space of $\underset{\sim}{\Omega}$ if of dimension 1; hence -eq. (1.3.1)- the range of $\underset{\sim}{\Omega}$ is of dimension 2, thereby existing vectors in E^3 not belonging to the range of $\underset{\sim}{\Omega}$. If $\underline{v}_p$ happens to lie outside the range of $\underset{\sim}{\Omega}$, it is impossible to find a vector $\underline{x}$ for which eq. (2.9.3) is satisfied. Those vectors lying outside the range of $\underset{\sim}{\Omega}$ lie necessarily on its null space, i.e., on a line parallel to the eignevector of $\underset{\sim}{\Omega}$ corresponding to its zero eigenvalue or, equivalently, are parallel to the vector $\underline{\omega}$ associated with $\underset{\sim}{\Omega}$. In case $\underline{v}_p$ has a nonzero component along the null space of $\underset{\sim}{\Omega}$, body B is said <u>to slide</u> on body C, which is the case of the worm-gear or of the hypoid gear couplings. In these couplings there are power losses due to the involved sliding and, since the dissipated power is proportional to the sliding velocity, the contact between the two mechanical elements under consideration should take place along points of minimum magnitude of sliding velocity. For hypoid gears this set

of points lie on a line which, paralleling Chasles' Theorem, is called "the instant axis of the screw motion of body B with respect to C" or, for short, "the instant screw axis". Indeed, let the sliding velocity be given by eq. (2.9.2). Finding the points of minimum magnitude of sliding velocity corresponds to finding the vectors x of expression (2.8.2) which minimize the quadratic form $\phi(x)=v^T v$, which has a stationary value (Section 1.10) when $\phi'(x)$ vanishes.

Applying the "chain rule" to $\phi(x)$,

$$\phi'(x)=2\left(\frac{\partial v}{\partial x}\right)^T v \tag{2.9.4a}$$

where, from eq. (2.9.2),

$$\frac{\partial v}{\partial x} = \Omega. \tag{2.9.4b}$$

Thus, points x_0 yielding an extremum of $\phi(x)$ satisfy the equation

$$\Omega^T v = -\Omega v = 0 \tag{2.9.5}$$

Exercise 2.9.1 Show that the gradient of $\phi(x)$ is twice the left hand side of eq. (2.9.5).

Since Ω has one zero eigenvalue (and only one), eq. (2.9.5) states that the minimum-magnitude velocity, given by

$$v_0 = v_P + \Omega(x_0 - x_P) \tag{2.9.6}$$

is in the nullspace of Ω, i.e. is a vector parallel to ω. Notice that eq. (2.9.6) does not yield a unique vector x_0 for, if any vector $\alpha\omega$ (in the null space of Ω) is added to x_0, the velocity v_N of the new point, is given by

$$v_N = v_P + \Omega(x_0 + \alpha\omega - x_P) = v_P + \Omega(x_0 - x_P)$$

Hence, the points of minimum-magnitude velocity lie in a line passing through one point x_0, in the direction of ω, this line being the instant

screw axis of the motion under study. The particular point P_0 on the screw axis, located by $\underset{\sim}{x}_0$, is chosen such that $\underset{\sim}{x}_0$ be normal to the screw axis; thus, $\underset{\sim}{x}_0$ happens to be the minimum-norm vector satisfying (2.9.6). This vector can be found in a similar way as vector $\underset{\sim}{r}_0$ of eq. (2.6.8) was found, namely, choose two linearly independent equations out of eq (2.9.5) and form the system

$$\underset{\sim}{A}\underset{\sim}{x}_0=\underset{\sim}{b} \qquad (2.9.7)$$

where $\underset{\sim}{A}$ is a 2x3 matrix and $\underset{\sim}{b}$ is a two-dimensional vector. Hence, the minimum-norm solution $\underset{\sim}{x}_0$ is given as

$$\underset{\sim}{x}_0=\underset{\sim}{A}^T(\underset{\sim}{A}\underset{\sim}{A}^T)^{-1}\underset{\sim}{b} \qquad (2.9.8)$$

An alternative way of finding $\underset{\sim}{x}_0$ is now presented, expressing eq (2.9.5) in cartesian vector form, namely,

$$-\underset{\sim}{\omega}\times\left(\underset{\sim}{v}_P+\underset{\sim}{\omega}\times(\underset{\sim}{r}_0-\underset{\sim}{r}_P)\right)=\underset{\sim}{0} \qquad (2.9.9)$$

which can be expanded as

$$\underset{\sim}{\omega}\times\underset{\sim}{v}_P+\underset{\sim}{\omega}\times\left(\underset{\sim}{\omega}\times(\underset{\sim}{r}_0-\underset{\sim}{r}_P)\right)=\underset{\sim}{0}$$

or, expanding the double cross product*,

$$\underset{\sim}{\omega}\times\underset{\sim}{v}_P+\left(\underset{\sim}{\omega}\cdot(\underset{\sim}{r}_0-\underset{\sim}{r}_P)\right)\underset{\sim}{\omega}-\underset{\sim}{\omega}^2(\underset{\sim}{r}_0-\underset{\sim}{r}_P)=\underset{\sim}{0}$$

If now $\underset{\sim}{r}_0$ is specified to be normal to $\underset{\sim}{\omega}$, i.e., to be the minimum-norm vector of all those satifying eq. (2.9.9), then it can be obtained from the above equation as

$$\underset{\sim}{r}_0=\underset{\sim}{r}_P+\frac{1}{\underset{\sim}{\omega}^2}\left(\underset{\sim}{\omega}\times\underset{\sim}{v}_P-(\underset{\sim}{\omega}\cdot\underset{\sim}{r}_P)\underset{\sim}{\omega}\right) \qquad (2.9.10)$$

which is an expression similar to that appearing in eq. (2.6.15).

---

* $\underset{\sim}{\omega}^2\equiv\underset{\sim}{\omega}\cdot\underset{\sim}{\omega}$

The foregoing results are summarized in a theorem similar to that of Chasles'.

THEOREM 2.9.1 *Any rigid body motion is equivalent to a screw motion, composed of a velocity $\underline{v}_0$ and a spin about an axis parallel to $\underline{v}_0$. The points whose velocity is $\underline{v}_0$ are located on a line parallel to $\underline{v}_0$, called the instant screw axis and $\underline{v}_0$ is of minimum magnitude. The screw axis passes through point P_0 whose position vector is given either by eq. (2.9.8) or by eq. (2.9.10)*

The counterpart of Theorem 2.6.2 now follows:

THEOREM 2.9.2 *The velocities of all the points of a rigid body undergoing an arbitrary motion have identical projections along the instant screw axis.*

Proof:

The velocity of any point of the rigid body can be written as

$$\underline{v}=\underline{v}_P+\underline{\omega}\times(\underline{r}-\underline{r}_P)$$

Dot multiplying both sides of the above equation times $\underline{\omega}$ (a vector parallel to the screw axis) yields

$$\underline{v}\cdot\underline{\omega}=\underline{v}_P\cdot\underline{\omega}+\underline{\omega}\times(\underline{r}-\underline{r}_P)\cdot\underline{\omega}$$

But the second term of the right hand side clearly vanishes. Hence

$$\underline{v}\cdot\underline{\omega}=\underline{v}_P\cdot\underline{\omega} \qquad \text{q.e.d.}$$

By virtue of the latter result, the projection of the velocities of all the points of a rigid body in motion along the screw axis is given by $||\underline{v}_0||$, which is called "the sliding". The pitch of the instant screw is given by

$$\lambda=\frac{2\pi||\underline{v}_0||}{||\underline{\omega}||} \qquad (2.9.11)$$

which is the counterpart of eq. (2.6.21a)

After Theorem 2.9.2, there follows one method of determining the orientation $\underset{\sim}{e}$ of the screw axis of a rigid body motion when the non-coplanar velocities of three points A, B and C of a rigid body are known. Indeed, paralleling the derivation of eq. (2.7.24) one obtains

$$\underset{\sim}{e} = \frac{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\times(\underset{\sim}{v}_B-\underset{\sim}{v}_C)}{\|(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\times(\underset{\sim}{v}_B-\underset{\sim}{v}_C)\|} \qquad (2.9.12)$$

Exercise 2.9.1 Show that the velocity of all the points of a rigid body, three of whose points, A, B and C, have velocities $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$, respectively, have identical projections along vector $\underset{\sim}{e}$, as given by eq. (2.9.12).

The sliding velocity, $\underset{\sim}{v}_0$, can then be obtained as

$$\underset{\sim}{v}_0=(\underset{\sim}{v}_A\cdot\underset{\sim}{e})\underset{\sim}{e}\ \mathrm{sgn}\ (\underset{\sim}{v}_A\cdot\underset{\sim}{e}) \qquad (2.9.13)$$

where the signum function has been introduced in order to make the directions of both $\underset{\sim}{v}_0$ and $\underset{\sim}{e}$ coincident.

To completely determine the instant screw, only the angular velocity $\underset{\sim}{\omega}$ needs be computed. This is done in what follows, after deriving some results similar to those of Section 2.6

Corollary 2.9.1 *If at least one point of a rigid body has a velocity which is normal to its angular velocity or zero, the body undergoes a pure rotation.*

The foregoing Corollary is a direct consequence of Theorem 2.9.2 and so its proof is left as an exercise for the reader.

Exercise 2.9.2 Prove Corollary 2.9.1

THEOREM 2.9.3 *The difference vector of the velocities of any two points of a rigid body undergoing an arbitrary motion is perpendicular to the instant screw axis.*

Proof

Let $\underline{v}_A$ and $\underline{v}_B$ be the velocities of two points, A and B, of a rigid body. From eq. (2.9.2) these are related by

$$\underline{v}_B=\underline{v}_A+\underline{\Omega}(\underline{b}-\underline{a})$$

Hence, the difference, $\underline{d}$, is given by

$$\underline{d} = \underline{v}_B-\underline{v}_A=\underline{\Omega}(\underline{b}-\underline{a})$$

which makes it clear that $\underline{d}$ lies in the range (See Section 1.3) of $\underline{\Omega}$, thereby being normal to the null space of $\underline{\Omega}$, i.e., normal to the screw axis. Alternately this result can be proved resorting to Gibbs' notation. This way, $\underline{d}$ can be written as

$$\underline{d} = \underline{\omega}\times(\underline{b}-a)$$

and hence

$$\underline{d}\cdot\underline{\omega}= \underline{\omega}\times(\underline{b}-\underline{a})\cdot\underline{\omega}$$

which vanishes because the double product at the right hand side contains two identical vectors, q.e.d.

THEOREM 2.9.4 *If the velocities of three noncollinear points of a rigid body are identical, the body undergoes a pure translation.*

Proof

Let $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$ be the respective velocities of points A,B and C. Referring these to the velocity of an arbitrary point P, one obtains

$$\underline{v}_A=\underline{v}_P+\underline{\Omega}(\underline{a}-\underline{p})$$

$$\underline{v}_B=\underline{v}_P+\underline{\Omega}(\underline{b}-\underline{p})$$

$$\underline{v}_C=\underline{v}_P+\underline{\Omega}(\underline{c}-\underline{p})$$

Subtracting the third equation from the first two yields

$$\underline{v}_A-\underline{v}_C=\underline{\Omega}(\underline{a}-\underline{c})=\underline{0}$$

$$\underline{v}_B-\underline{v}_C=\underline{\Omega}(\underline{b}-\underline{c})=\underline{0}$$

which implies that both $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ lie the null space of $\underline{\Omega}$. This space, however, is of dimension 1, as discussed in Section 2.7. Since points A, B and C are noncollinear, vectors $\underline{a}-\underline{c}$ and $\underline{b}-\underline{c}$ are linearly independent and hence cannot be simultaneously in the null space of $\underline{\Omega}$, unless $\underline{\Omega}=\underline{0}$, the motion thus reducing to a pure translation, q.e.d.

<u>THEOREM 2.9.5</u> *The nonidentical velocities of three points of a rigid body are coplanar if and only if one of the following conditions is met:*

i) The motion is a pure rotation

ii) The motion is general, but the points are collinear

iii) The motion is general and the points are not collinear, but lie in a plane parallel to the screw axis.

<u>Proof</u>

"if" part:

i) If the motion is a pure rotation, the velocity of any point with position vector $\underline{r}$ is given by

$$\underline{v} = \underline{\Omega}\underline{r}$$

which states that $\underline{v}$ lies in the range (See Section 1.) of $\underline{\Omega}$, which is of dimension 2, as was discussed in Section 2.7. This means that all velocity vectors lie in a plane perpendicular to the null space of $\underline{\Omega}$, i.e. perpendicular to the axis of rotation, thereby showing that these velocities are coplaner.

ii) Let A, B and C be three collinear points of the rigid body and $\underline{a}$, $\underline{b}$ and $\underline{c}$ be their respective position vectors. The velocities of these points, referred to an arbitrary point with position vector $\underline{p}$ are

$$\underline{v}_A=\underline{v}_P+\underline{\Omega}(\underline{a}-\underline{p})$$

$$\underline{v}_B=\underline{v}_P+\underline{\Omega}(\underline{b}-\underline{p})$$

$$\underline{v}_C=\underline{v}_P+\underline{\Omega}(\underline{c}-\underline{p})$$

Since the points are collinear, their position vectors are related by

$$c-a=\alpha(b-a)$$

Now, adding Ωa to v_C and subtracting it simultaneously from the same expression, one obtains

$$v_C=v_P+\Omega(a-p)+\Omega(c-a)$$

whose first two terms can be readily identified with v_A. Moreover, substituting $c-a$ in the third term of the latter equation by $\alpha(b-a)$, as given above, leads to

$$v_C=v_A+\alpha\Omega(b-a)$$

But

$$\Omega(b-a)=v_B-v_A$$

Hence, the expression for v_C is transformed into

$$v_C=v_A+\alpha(v_B-v_A)$$

or, equivalently,

$$v_C=(1-\alpha)v_A+\alpha v_B$$

thereby proving the linear dependence, i.e. the coplanarity of v_A, v_B and v_C.

iii) The velocities of the three given points, A, B and C, are referred to that of a point P on the screw axis. These velocities take on the form appearing in ii. Thus, the velocity of P, v_P, is parallel to the screw axis. On the other hand, the fact that A, B and C lie in a plane parallel to the screw axis allows one to establish the following relationship

$$c-a=\alpha(b-a)+\beta v_P$$

or, equivalently,

$$c=(1-\alpha)a+\alpha b+\beta v_P$$

Substituting the latter expression in $\underset{\sim}{v}_C$ as given in ii leads to

$$\underset{\sim}{v}_C = \underset{\sim}{v}_P + \underset{\sim}{\Omega}(\underset{\sim}{c}-\underset{\sim}{p}) =$$
$$= \underset{\sim}{v}_P + \underset{\sim}{\Omega}(\underset{\sim}{a}-\underset{\sim}{p}) - \alpha\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a}) + \beta\underset{\sim}{\Omega}\underset{\sim}{v}_P$$

whose two first terms can be readily identified as $\underset{\sim}{v}_A$, its fourth term vanishing because it lies in the null space of $\underset{\sim}{\Omega}$. Hence

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A - \alpha\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a})$$

But

$$\underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a}) = \underset{\sim}{v}_B - \underset{\sim}{v}_A$$

Thus, the latter expression for $\underset{\sim}{v}_C$ is transformed into

$$\underset{\sim}{v}_C = \underset{\sim}{v}_A - \alpha\underset{\sim}{\Omega}(\underset{\sim}{v}_B - \underset{\sim}{v}_A)$$

which shows the linear dependence of the three given velocity vectors, i.e. its coplanarity.

"only if" part:

Assuming that the velocities $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ of three given points A, B and C are coplanar, the following relationship holds

$$\det(\underset{\sim}{v}_A, \underset{\sim}{v}_B, \underset{\sim}{v}_C) = 0$$

Referring $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ to $\underset{\sim}{v}_A$ one has

$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a})$$
$$\underset{\sim}{v}_C = \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{c}-\underset{\sim}{a})$$

Thus, the above expression for the determinant becomes

$$\det\left(\underset{\sim}{v}_A, \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a}), \underset{\sim}{v}_A + \underset{\sim}{\Omega}(\underset{\sim}{c}-\underset{\sim}{a})\right) = 0$$

Subtracting the first column of this determinant from the remaining ones does not change the value of the determinant. Hence

$$\det(\underset{\sim}{v}_A, \underset{\sim}{\Omega}(\underset{\sim}{b}-\underset{\sim}{a}), \underset{\sim}{\Omega}(\underset{\sim}{c}-\underset{\sim}{a})) = 0$$

which is equivalent to

$$\mathbf{\Omega}(\mathbf{b}-\mathbf{a})\times\mathbf{\Omega}(\mathbf{c}-\mathbf{a})\cdot\mathbf{v}_A=0$$

Introducing Gibbs' notation, and expanding the resulting expression,

$$\mathbf{\Omega}(\mathbf{b}-\mathbf{a})\times\mathbf{\Omega}(\mathbf{c}-\mathbf{a})=\left(\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\right)\times\left(\boldsymbol{\omega}\times(\mathbf{c}-\mathbf{a})\right)=$$
$$=\left(\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})\right)\boldsymbol{\omega}-\left(\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\cdot\boldsymbol{\omega}\right)(\mathbf{c}-\mathbf{a})$$

where the expression in brackets in the second term of the rightmost hand side clearly vanishes. Hence

$$\mathbf{\Omega}(\mathbf{b}-\mathbf{a})\times\mathbf{\Omega}(\mathbf{c}-\mathbf{a})\cdot\mathbf{v}_A=\left(\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})\right)\boldsymbol{\omega}\cdot\mathbf{v}_A$$

which vanishes under one of the following conditions:

i) $\boldsymbol{\omega}\cdot\mathbf{v}_A=0$

which implies, under Corollary 2.9.1, that the motion is a pure rotation

ii) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\mathbf{0}$

which means that points A, B and C are collinear

iii) $\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})=0$

which indicates that vectors $\boldsymbol{\omega}$, $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are coplanar, q.e.d.

A direct consequence of the foregoing result is the following

<u>Corollary 2.9.2</u> *Assume a rigid body under motion and choose any three noncollinear points A, B and C of the body. Letting $\mathbf{v}_A$, $\mathbf{v}_B$ and $\mathbf{v}_C$ be the three involved velocities, then the difference vectors $\mathbf{v}_A-\mathbf{v}_C$ and $\mathbf{v}_B-\mathbf{v}_C$ (and, consequently, $\mathbf{v}_A-\mathbf{v}_B$) are parallel if and only if the points lie in a plane parallel to the screw axis when the motion is general. If the motion reduces to a pure rotation, then the said plane is parallel to the axis of rotation.*

<u>Exercise 2.9.3</u> Prove Corollary 2.9.2

More results connected with the present discussion are the following

Corollary 2.9.3 The velocities of any two points of a rigid body cannot be parallel and different, unless the body undergoes a pure rotation

Corollary 2.9.4 If two, and only two, velocities of three noncollinear points of a rigid body are parallel, then either i) the parallel velocities are identical and belong to points lying on a line parallel to the screw axis, or ii) the parallel velocities are different from each other, in which case the motion is a pure rotation whose axis is parallel to the line connecting the two points of parallel velocities.

Corollary 2.9.5 Given three noncollinear points, A, B and C, of a rigid body in motion, such that $\underset{\sim}{v}_C=\underset{\sim}{0}$ and $\underset{\sim}{v}_A$ and $\underset{\sim}{v}_B$ are parallel but distinct, i.e. $\underset{\sim}{v}_B=\beta\underset{\sim}{v}_A$, then the body undergoes a pure rotation and its axis passes through C and is parallel to vector $\underset{\sim}{b}-\beta\underset{\sim}{a}$, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ being the position vectors of A and B, respectively. If $\underset{\sim}{v}_A=\underset{\sim}{v}_B$, then the axis of rotation is parallel to line AB.

The computation of $\underset{\sim}{\omega}$ given the velocities of three noncollinear points is next discussed. Two cases are considered: i) the arising difference vectors are noncollinear, and ii) these vectors are collinear.

In what follows let A, B and C be the three involved points, $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ being their corresponding velocities. Then,

i) The difference vectors are not collinear.

$$\underset{\sim}{v}_{C/A}=\underset{\sim}{\omega}\times(\underset{\sim}{c}-\underset{\sim}{a}) \tag{2.9.14}$$

Hence

$$\underset{\sim}{v}_{B/A}\times\underset{\sim}{v}_{C/A}=\underset{\sim}{v}_{B/A}\times\left(\underset{\sim}{\omega}\times(\underset{\sim}{c}-\underset{\sim}{a})\right)=$$
$$=\left(\underset{\sim}{v}_{B/A}\cdot(\underset{\sim}{c}-\underset{\sim}{a})\right)\underset{\sim}{\omega}-(\underset{\sim}{v}_{B/A}\cdot\underset{\sim}{\omega})(\underset{\sim}{c}-\underset{\sim}{a}) \tag{2.9.15}$$

But

$$\mathbf{v}_{B/A}\cdot\boldsymbol{\omega} = \left(\boldsymbol{\omega}\times(\mathbf{b}-\mathbf{a})\right)\cdot\boldsymbol{\omega} = 0 \quad (2.9.16)$$

Thus, $\boldsymbol{\omega}$ can be solved for from eq. (2.9.15) as

$$\boldsymbol{\omega} = \frac{\mathbf{v}_{B/A}\times\mathbf{v}_{C/A}}{\mathbf{v}_{B/A}\cdot(\mathbf{c}-\mathbf{a})} \quad (2.9.17)$$

which is the desired expression for $\boldsymbol{\omega}$, irrespective of whether the motion is general or a pure rotation.

ii) The difference vectors are collinear. In this case the points lie in a plane parallel to the instant screw axis. Due to the collinearity of the difference vectors, the cross product of the left-hand side of eq. (2.9.15) vanishes, thus making it impossible to compute $\boldsymbol{\omega}$ using the procedure of case i). Thus, a different approach is introduced. According to Corollary 2.9.2, the given points lie in a plane parallel to the instant screw axis, i.e. to $\boldsymbol{\omega}$. Hence, the following holds.

$$\boldsymbol{\omega} = \alpha(\mathbf{a}-\mathbf{c})+\beta(\mathbf{b}-\mathbf{c}) \quad (2.9.18)$$

According to Theorem 2.9.3,

$$(\mathbf{v}_A-\mathbf{v}_C)\cdot\boldsymbol{\omega} = 0 \quad (2.9.19)$$

or, substituting eq. (2.9.18) into eq. (2.9.19),

$$\alpha(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{a}-\mathbf{c})+\beta(\mathbf{v}_A-\mathbf{v}_C)\cdot(\mathbf{b}-\mathbf{c}) = 0 \quad (2.9.20)$$

Now, several possibilities can arise, namely

ii.1) The relative velocity $\mathbf{v}_A-\mathbf{v}_C$ is perpendicular to $\mathbf{a}-\mathbf{c}$, in which case (2.9.20) holds only if β vanishes. Indeed, $\mathbf{v}_A-\mathbf{v}_C$ cannot be simultaneously perpendicular to both $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{c}$ for these vectors are nonparallel, given the assumed noncollinearity of points A, B and C. Hence

$$\boldsymbol{\omega} = \alpha(\mathbf{a}-\mathbf{c})$$

where α is computed from

$$\underset{\sim}{v}_{B/C} = \underset{\sim}{\omega}\times(\underset{\sim}{b}-\underset{\sim}{c}) \qquad (2.9.21)$$

i.e.

$$\underset{\sim}{v}_B-\underset{\sim}{v}_C = \alpha(\underset{\sim}{a}-\underset{\sim}{c})\times(\underset{\sim}{b}-\underset{\sim}{c}) \qquad (2.9.21a)$$

Dot-multiplying the latter equation times $\underset{\sim}{v}_C$,

$$(\underset{\sim}{v}_B-\underset{\sim}{v}_C)\cdot\underset{\sim}{v}_C=\alpha(\underset{\sim}{a}-\underset{\sim}{c})\times(\underset{\sim}{b}-\underset{\sim}{c})\cdot\underset{\sim}{v}_C$$

from which

$$\alpha = \frac{(\underset{\sim}{v}_B-\underset{\sim}{v}_C)\cdot\underset{\sim}{v}_C}{(\underset{\sim}{a}-\underset{\sim}{c})\times(\underset{\sim}{b}-\underset{\sim}{c})\cdot\underset{\sim}{v}_C}$$

readily follows. In the latter equation it might have happened that the dot product vanishes. In this case, α cannot be solved for due to the arising indeterminacy. This indeterminacy, however, can be resolved by dot multiplying times $\underset{\sim}{v}_A$ or $\underset{\sim}{v}_B$, eq.(2.9.21a), instead.

ii.2) The relative velocity $\underset{\sim}{v}_A-\underset{\sim}{v}_C$ is perpendicular to $\underset{\sim}{b}-\underset{\sim}{c}$, in which case eq. (2.9.20) holds only if α vanishes, resorting to the same argument as in ii.1. Hence

$$\underset{\sim}{\omega} =\beta(\underset{\sim}{b}-\underset{\sim}{c})$$

the constant β being determined as in ii.1.

ii.3) No inner product in (2.9.20) vanishes. Hence β can be solved for as

$$\beta = -\frac{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{a}-\underset{\sim}{c})}{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{b}-\underset{\sim}{c})}\alpha$$

Substituting the latter expression into eq. (2.9.18),

$$\underset{\sim}{\omega} =\left((\underset{\sim}{a}-\underset{\sim}{c})-\frac{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{a}-\underset{\sim}{c})}{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{b}-\underset{\sim}{c})}(\underset{\sim}{b}-\underset{\sim}{c})\right)\alpha \qquad (2.9.22)$$

where α can be computed as before. Indeed, substituting eq. (2.9.22) into eq. (2.9.21), one obtains

$$\underset{\sim}{v}_{B/C} = \alpha\left((\underset{\sim}{a}-\underset{\sim}{c}) - \frac{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{a}-\underset{\sim}{c})}{(\underset{\sim}{v}_A-\underset{\sim}{v}_C)\cdot(\underset{\sim}{b}-\underset{\sim}{c})}(\underset{\sim}{b}-\underset{\sim}{c})\right) \times (\underset{\sim}{b}-\underset{\sim}{c}) \qquad (2.9.23)$$

Hence

$$\underset{\sim}{v}_{B/C} = \alpha(\underset{\sim}{a}-\underset{\sim}{c})\times(\underset{\sim}{b}-\underset{\sim}{c}) \qquad (2.9.24)$$

Dot-multiplying both sides of eq. (2.9.24) times $\underset{\sim}{v}_C$, α can be solved for as

$$\alpha = \frac{\underset{\sim}{v}_{B/C}\cdot\underset{\sim}{v}_C}{(\underset{\sim}{a}-\underset{\sim}{c})\times(\underset{\sim}{b}-\underset{\sim}{c})\cdot\underset{\sim}{v}_C} \qquad (2.9.25)$$

Again, if $\underset{\sim}{v}_{B/C}\cdot\underset{\sim}{v}_C$ happens to vanish then eq.(2.9.25) should be dot-multiplied times either $\underset{\sim}{v}_A$ or $\underset{\sim}{v}_B$ instead

The computation of the instant-screw parameters (the instant-screw axis, the sliding velocity and the spin) is carried on by SUBROUTINE INSCRU, which parallels SUBROUTINE SCREW and thus considers all cases that could arise regarding the relationships amongst all three velocity vectors. These possible cases are shown in the "tree" diagram appearing in Fig 2.9.1. INSCRU uses the following auxiliary subroutines:

1. SUBROUTINE COP1 computes the instant-screw parameters when the motion is pure rotation. It distinguishes amongst the different cases with the aid of the integer variable INDEX
2. SUBROUTINE COP2 computes the instant-screw parameters when the points lie in a plane parallel either to the instant-screw axis or to the instant axis of rotation. Two different cases could arise, which are distinguished with the aid of the integer variable INDE.
3. SUBROUTINE GEMO computes the instant-screw parameters when the motion is general and the three given velocities are noncoplanar.

The computation procedure for each case is next described. All over, the three given points are A, B and C, their respective position vectors being

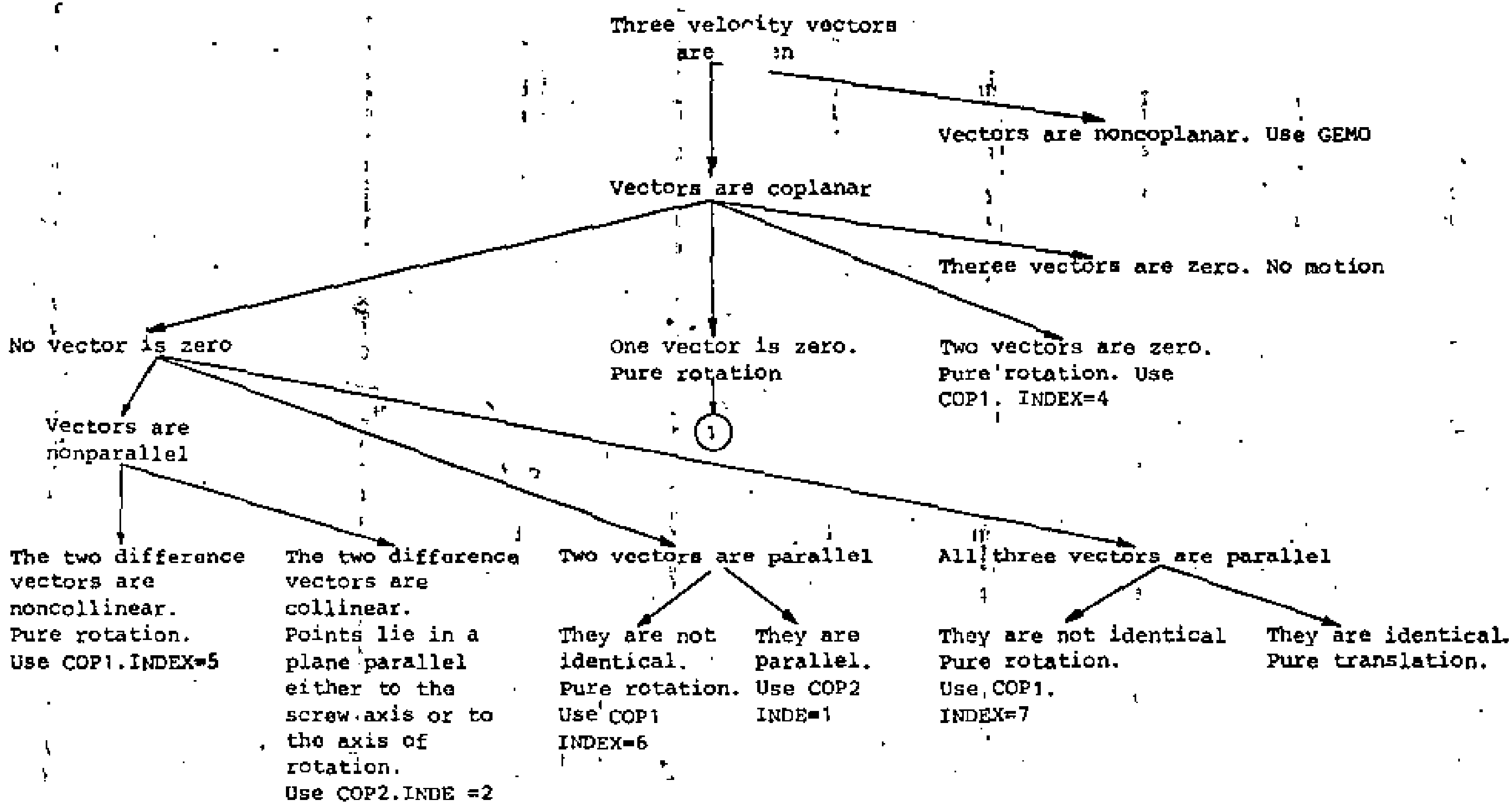

Fig 2.9.1 Tree diagram showing all possible relationships amongst the velocities of three noncollinear points of a rigid body.

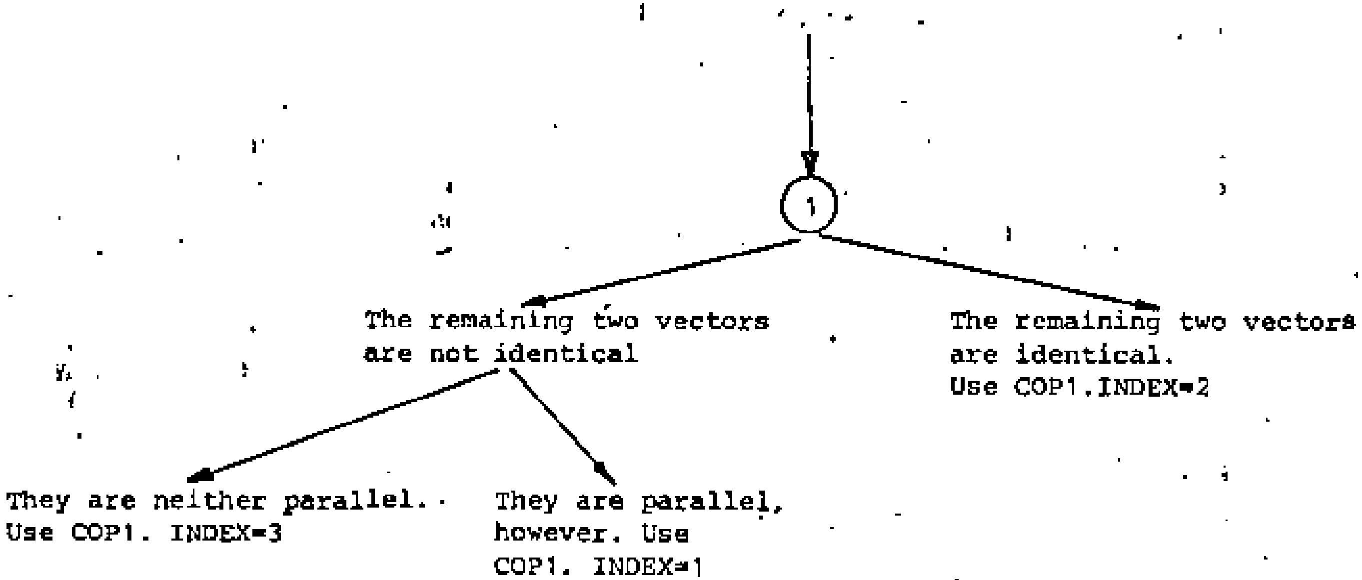

Fig 2.9.1 (continues)

a, b and c. Their velocities are $\underline{v}_A$, $\underline{v}_B$ and $\underline{v}_C$

INDEX = 1. One vector is zero and the remaining two vectors are not identical; they are parallel, however. The motion is pure rotation according to Corollary 2.9.1 and the axis of rotation is located following Corollary 2.9.5

INDEX = 2. One vector is zero and the remaining two vectors are identical. The motion is pure rotation again and the axis of rotation passes through the point of zero velocity in the direction of the line joining the other two points, according to Corollary 2.9.5

INDEX = 3. One vector is zero and the remaining two vectors are not parallel. The motion is pure rotation as before, attending Corollary 2.9.1, and the axis of rotation passes through the point of zero velocity in the direction of the normal to the plane defined by the other two velocities.

INDEX = 4. Two vectors are zero. The motion is pure rotation and the axis of rotation is defined by the two points of zero velocity.

INDEX = 5. No vector is zero and all three vectors are nonparallel amongst them but coplanar. Furthermore, the two arising difference vectors are noncollinear. According to Theorem 2.9.5 and Corollary 2.9.2 then, the motion is pure rotation and the instant-screw parameters can be computed using the general procedure.

INDEX = 6. The vectors are coplanar and no vector is zero but two vectors are parallel and different. According to Corollary 2.9.4 the motion is pure rotation and the axis of rotation is perpendicular to the plane of the velocity vectors. This axis is located

using the general procedure.

INDEX = 7. No vector is zero and all three vectors are parallel to each other. Furthermore, not all three vectors are identical to each other. There may be, nevertheless, a couple of identical vectors. The motion is one of pure rotation and the instant axis of rotation is determined by the intersection of the plane of the given points with a second plane defined next. Let A', B' and C' be points whose position vectors are

$$\underline{a}' = \underline{a}+\underline{v}_A, \quad \underline{b}' = \underline{b}+\underline{v}_B, \quad \underline{c}' = \underline{c}+\underline{v}_C$$

The second plane is that defined by the noncollinear points A', B' and C'

Proof

According to Corollary 2.9.3 the existence of at least two parallel and distinct velocities guarantees that the motion is pure rotation. Thus, there exists a point O in the body whose velocity is zero. Placing the origin of coordinates at O, then,

$$\underline{v}_A = \underline{\omega}\times\underline{a}, \quad \underline{v}_B = \underline{\omega}\times\underline{b}, \quad \underline{v}_C = \underline{\omega}\times\underline{c}$$

From the parallelism condition, one has

$$\underline{v}_B = \beta\underline{v}_A, \quad \underline{v}_C = \gamma\underline{v}_A \,.$$

Hence

$$\underline{\omega}\times(\underline{b}-\beta\underline{a})=0 \quad \text{and} \quad \underline{\omega}\times(\underline{c}-\gamma\underline{a})=0$$

which implies that vectors $\underline{b}-\beta\underline{a}$ and $\underline{c}-\gamma\underline{a}$ are parallel to the axis of rotation. In other words, the planes defined by points A, B, O and A,C,O contain the axis of rotation. Since $\underline{v}_A \neq \underline{0}$, AO cannot be the axis of rotation; hence, points A, B, C and O are coplanar and their plane contains the axis of rotation.

On the other hand, recalling the definition of points A', B' and C', whose position vectors are

$$\underset{\sim}{a}' = \underset{\sim}{a}+\underset{\sim}{v}_A,\ \underset{\sim}{b}' = \underset{\sim}{b}+\underset{\sim}{v}_B,\ \underset{\sim}{c}' = \underset{\sim}{c}+\underset{\sim}{v}_C$$

The velocities of these points are then

$$\underset{\sim}{v}'_A = \underset{\sim}{\omega}\times(\underset{\sim}{a}+\underset{\sim}{v}_A) = \underset{\sim}{v}_A + \underset{\sim}{\omega}\times\underset{\sim}{v}_A$$

$$\underset{\sim}{v}'_B = \underset{\sim}{\omega}\times(\underset{\sim}{b}+\underset{\sim}{v}_B) = \underset{\sim}{v}_B + \underset{\sim}{\omega}\times\underset{\sim}{v}_B$$

$$\underset{\sim}{v}'_C = \underset{\sim}{\omega}\times(\underset{\sim}{c}+\underset{\sim}{v}_C) = \underset{\sim}{v}_C + \underset{\sim}{\omega}\times\underset{\sim}{v}_C$$

i.e. these velocities are parallel and related by

$$\underset{\sim}{v}'_B = \beta\underset{\sim}{v}'_A,\ \underset{\sim}{v}'_C = \gamma\underset{\sim}{v}'_A$$

Hence

$$\underset{\sim}{\omega}\times(\underset{\sim}{b}' - \beta\underset{\sim}{a}') = \underset{\sim}{0},\ \underset{\sim}{\omega}\times(\underset{\sim}{c}' - \gamma\underset{\sim}{a}') = \underset{\sim}{0}$$

which, by arguments similar to those resorted to previously, imply that points A', b', C' and O are coplanar, their plane containing the axis of rotation. Furthermore, since not all three vectors $\underset{\sim}{v}_A$, $\underset{\sim}{v}_B$ and $\underset{\sim}{v}_C$ are identical to each other, planes ABC and A' B' C' are not parallel. Their intersection, clearly, is the axis of rotation, q.e.d.

So far all cases leading necessarily to a pure rotation motion have been discussed. Next the case in which the given velocity vectors are coplanar but the motion is either a pure rotation or general, is discussed. In this case the arising, difference vectors are parallel and hence the given points lie in a plane parallel either to the instant axis of rotation or to the instant screw axis, depending upon whether the motion is a pure rotation or general. This case is handled by Subroutine COP2, which identifies each possible different subcase with the aid of the integer variable INDE.

INDE = 1. No vector is zero and two vectors are identical. The motion is either general or a pure rotation, but the screw axis or, correspondingly, the axis of rotation, is parallel to the line defined by the points with identical velocities.

Proof

Let B and C be the two points with identical velocities. These can be expressed as

$$\underline{v}_B = \underline{v}_A + \underline{\omega}\times(\underline{b}-\underline{a}),\quad \underline{v}_C = \underline{v}_A + \underline{\omega}\times(\underline{c}-\underline{a})$$

Subtracting the latter from the former equation,

$$\underline{v}_B - \underline{v}_A = \underline{\omega}\times(\underline{b}-\underline{c}) = \underline{0}$$

which implies that line BC is parallel to $\underline{\omega}$, i.e. to either the instant axis of rotation or to the instant screw axis.

INDE = 2. No vector is zero and no two vectors are parallel, but they are coplanar. The motion is either general or a pure rotation and the given points lie in a plane parallel either to the instant screw axis or to the instant axis of rotation, according to Theorem 2.9.5 and Corollary 2.9.2. The angular velocity is found by application of eqs. (2.9.18) -(2.9.25)

Thus, one has the following

THEOREM 2.9.6 *The instant motion of a rigid body is determined, i.e. its instant-screw parameters can be computed, if, and only if, the velocities of three noncollinear points of the body are known*

Subroutines INSCRU, COP1, COP2, and GEMO implement the foregoing computations. They use LOCAT1, LOCAT2, SPIN, CYCLIC, EXCHGE, CROSS and SCAL as subsidiary subroutines. Listings of INSCRU, COP1, COP2, GEMO, LOCAT1, LOCAT2 and SPIN appear in Fig 2.9.2-2.9.8.

2.10 THEOREMS RELATED TO THE VELOCITY DISTRIBUTION IN A MOVING RIGID BODY.

Some results concerning the velocity field in a rigid body in motion are now obtained, the main result of this section being the Aronhold-Kennedy Theorem. First a very useful result is proved.

THEOREM 2.10.1 *The velocities of two points of a rigid body have identical components along the line connecting them.*

Proof:

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be the position vectors of two points, A and B, of a rigid body in motion. Thus, for any configuration,

$$(\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{b}-\underset{\sim}{a})=\text{const} \tag{2.10.1}$$

from the rigidity condition. Differentiating both sides of eq. (2.10.1),

$$(\dot{\underset{\sim}{b}}-\dot{\underset{\sim}{a}})\cdot(\underset{\sim}{b}-\underset{\sim}{a})=0$$

or, alternatively,

$$\underset{\sim}{v}_B\cdot(\underset{\sim}{b}-\underset{\sim}{a})=\underset{\sim}{v}_A\cdot(\underset{\sim}{b}-\underset{\sim}{a}),\text{q.e.d.} \tag{2.10.2}$$

This theorem is used to check the compatibility of the given velocities of a rigid body in subroutine INSCRU of Sect. 2.9.

Exercise 2.10.1 The triangular plate of Fig 2.10.1 is constrained to move in such a way that vertex C remains on the Z-axis, while vertex A remains on the x-axis and side AB remains on the X-Y plane. Vertex C has a velocity $\underset{\sim}{v}_C=-5\underset{\sim}{e}_z$ m/sec.

i) Determine the velocity of vertices A and B

ii) Determine the angular velocity of the plate

iii) Locate the instant screw axis of the motion of the plate, and compute the pitch of its screw.

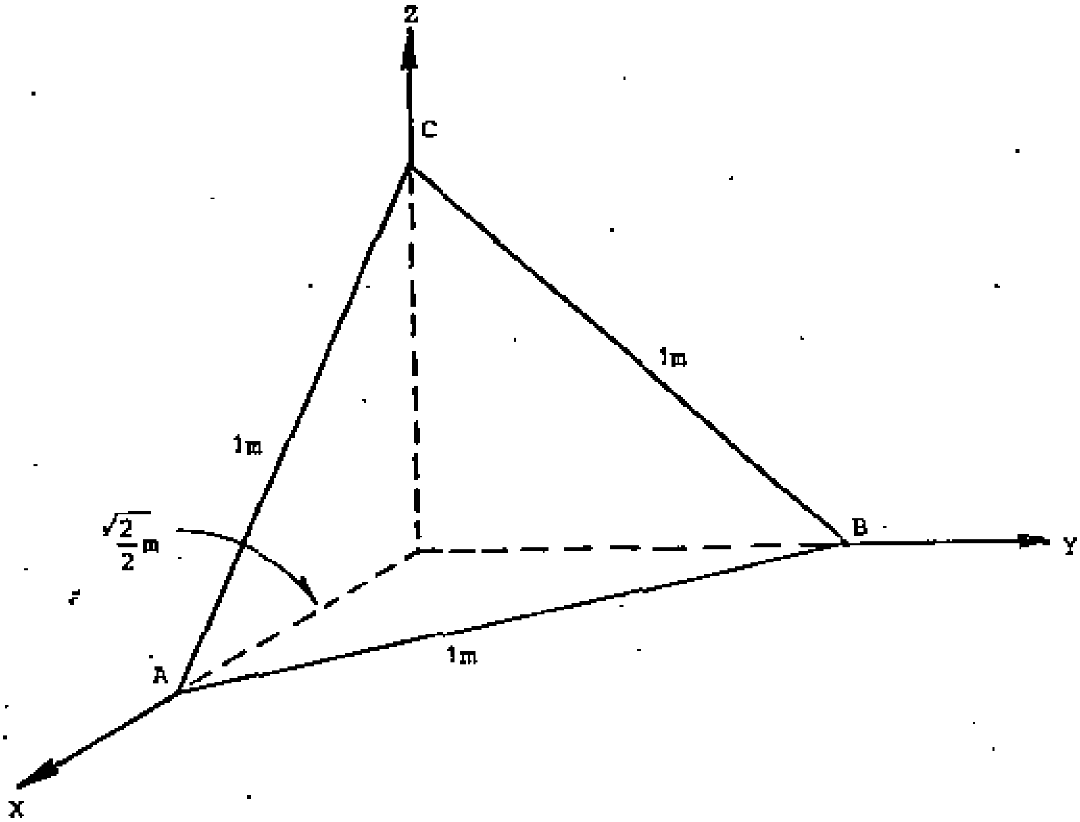

Fig 2.10.1 Triangular plate in constrained motion

THEOREM 2.10.2 (Aronhold-Kennedy). Given three rigid bodies in motion, the resulting three instant screw axes have one common normal intersecting all three axes.

Proof:

Referring to Fig 2.10.2 let S_B and S_C be the instant screw axes of bodies B and C, respectively, with respect to body A; let $\underline{v}_B$ and $\underline{v}_C$ be the relative sliding velocities of the instant screws S_B and S_C, with respect to A. Finally, let $\underline{\Omega}_B$ and $\underline{\Omega}_C$ be the relative angular velocities of bodies B and C, respectively, with respect to body A and $\underline{c}$, the common normal to S_B and S_C, joining both axes. It will be shown that the third instant screw axis, $S_{B/C}$, passes through the common normal B*C*.

Let P be any point of the three-dimensional Euclidean space, with position vector $\underline{r}$. Points P_A, P_B and P_C of bodies A, B and C, coincide at P. Let

$\underset{\sim}{v}_{PB}$, $\underset{\sim}{v}_{PB}$ and $\underset{\sim}{v}_{PC}$ be the velocities of each of these points. Furthermore, let $\underset{\sim}{v}$ be the relative velocities of P_B with respect to P_C and let B* and C* be the points in which the common normal intersects S_B and S_C.

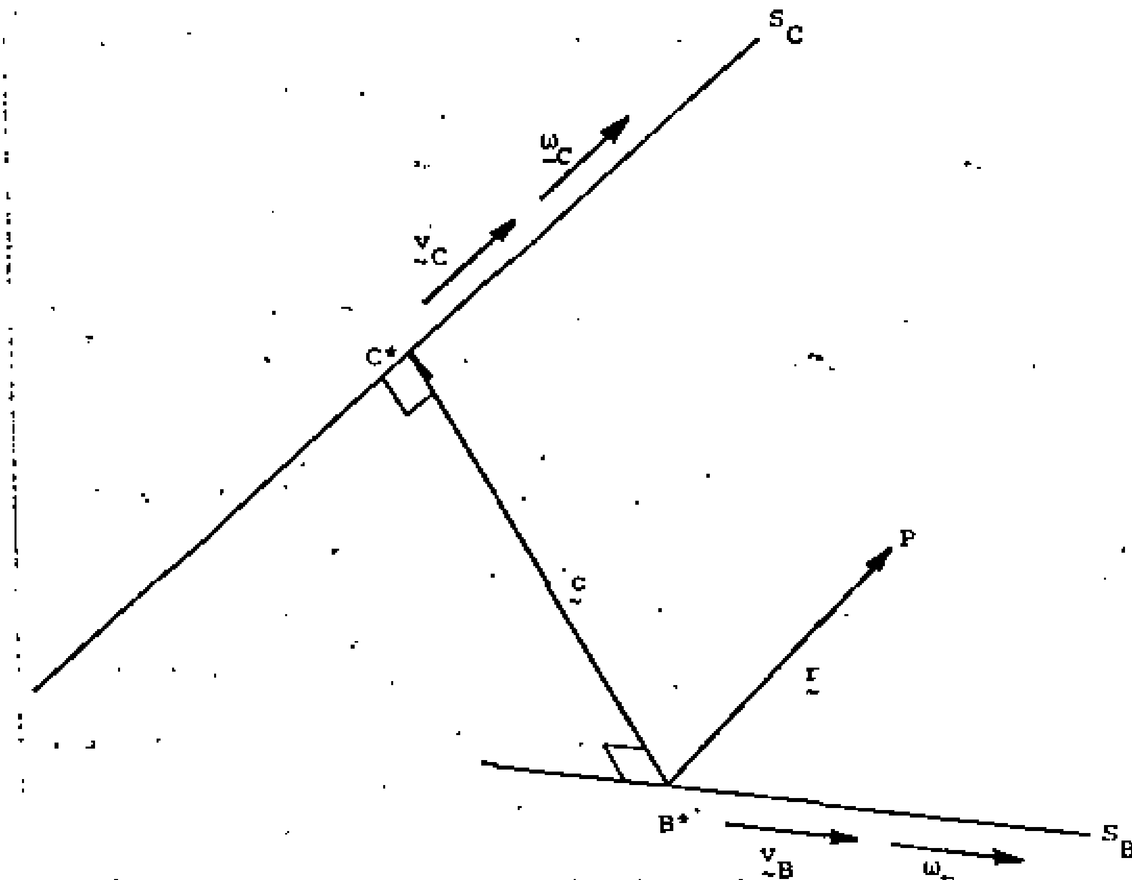

Fig 2.10.2 Instant screws of two bodies in motion with respect to a third one.

Thus,

$$\begin{aligned} \underset{\sim}{v} &= \underset{\sim}{v}_{PB} - \underset{\sim}{v}_{PC} = \\ &= (\underset{\sim}{v}_B + \underset{\sim}{\Omega}_B \underset{\sim}{r}) - \left(\underset{\sim}{v}_C + \underset{\sim}{\Omega}_C (\underset{\sim}{r} - \underset{\sim}{c})\right) = \\ &= \underset{\sim}{v}_{B/C} + \underset{\sim}{\Omega}_{B/C} \underset{\sim}{r} + \underset{\sim}{\Omega}_C \underset{\sim}{c} \end{aligned} \qquad (2.10.3)^*$$

It is next shown that, if P is a point of the relative instant screw axis $S_{B/C}$, then it lies on the line defined by points P* and C*. This is done

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* $\underset{\sim}{v}_{B/C}$ is to be interpreted as the relative velocity of B* with respect to C*.

via the minimization of the quadratic form

$$\phi(\underset{\sim}{r})=\underset{\sim}{v}^T\underset{\sim}{v} \qquad (2.10.4)$$

$\phi(\underset{\sim}{r})$ has an extremum at a point $\underset{\sim}{r}_0$ where its gradient vanishes. The said gradient is, applying the "chain rule" again,

$$\phi'(\underset{\sim}{r})=2\Omega^T_{B/C}\underset{\sim}{v} \qquad (2.10.5a)$$

Thus, at point $\underset{\sim}{r}_0$,

$$\underset{\sim}{\Omega}_{B/C}(\underset{\sim}{v}_{B/C}+\underset{\sim}{\Omega}_C\underset{\sim}{c}+\underset{\sim}{\Omega}_{B/C}\underset{\sim}{r}_0)=\underset{\sim}{0} \qquad (2.10.5b)$$

from which $\underset{\sim}{r}_0$ cannot be solved for, since $\underset{\sim}{\Omega}_{B/C}$ is singular, of rank two. One possible way to find $\underset{\sim}{r}_0$ is imposing on it the minimum-norm condition, as done proviously in similar instances. Another possible way is to write eq. (2.10.5) in Gibbs' notation as

$$\underset{\sim}{\omega}_{B/C}\times(\underset{\sim}{\omega}_{B/C}\times\underset{\sim}{r}_0)+\underset{\sim}{\omega}_{B/C}\times(\underset{\sim}{v}_{B/C}+\underset{\sim}{\omega}_C\times\underset{\sim}{c})=\underset{\sim}{0}$$

Expanding the first term and imposing the condition that $\underset{\sim}{\omega}_{B/C}\cdot\underset{\sim}{r}_0$ be zero, one obtains

$$-\underset{\sim}{\omega}^2_{B/C}\underset{\sim}{r}_0+\underset{\sim}{\omega}_{B/C}\times(\underset{\sim}{v}_{B/C}+\underset{\sim}{\omega}_C\times\underset{\sim}{c})=\underset{\sim}{0}$$

from which,

$$\underset{\sim}{r}_0=-\frac{1}{\underset{\sim}{\omega}^2_{B/C}}(\underset{\sim}{v}_{B/C}+\underset{\sim}{\omega}_C\times\underset{\sim}{c})\times\underset{\sim}{\omega}_{B/C} \qquad (2.10.6)$$

which is parallel to vector $\underset{\sim}{c}$. Since $\underset{\sim}{r}_0$ is parallel to vector $\underset{\sim}{c}$, the common normal to axes S_B and S_C, then $S_{B/C}$ passes through line B'C', q.e.d.

Exercise 2.10.3 Show that $\underset{\sim}{r}_0$, as given by eq. (2.10.6), is parallel to $\underset{\sim}{c}$.

One application of the Aronhold-Kennedy Theorem arises in the pitch surface synthesis of the coupling of two bodies whose relative motion is the composition of sliding and rotation, as is the case in hypoid gears. This is shown in the following example.

Example 2.10.1 Let L_1 and L_2 be the non-intersecting non-parallel axes of two shafts required to be coupled. These axes are shown in Fig 2.10.3.

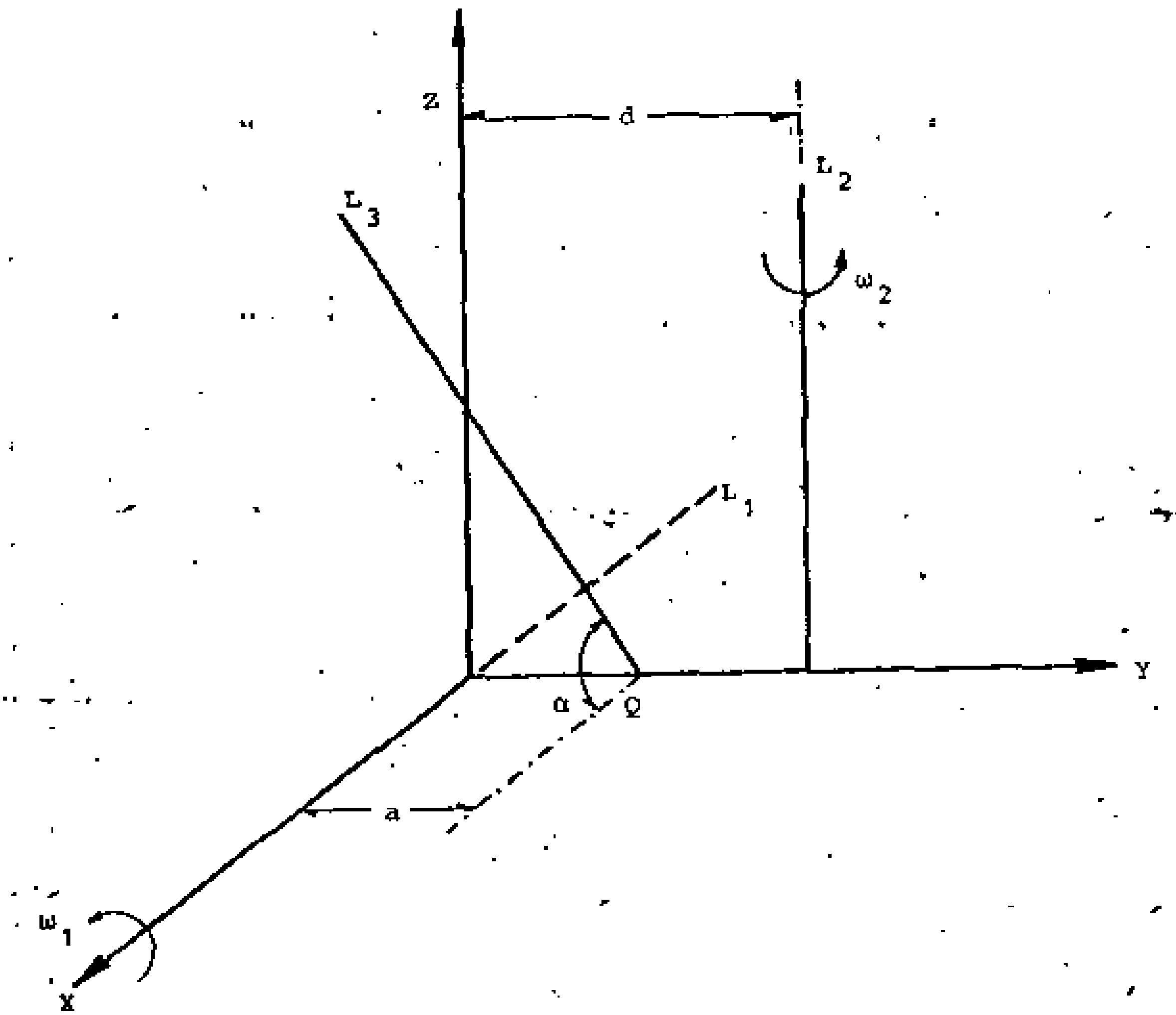

Fig 2.10.3 An application of the Aronhold-Kennedy Theorem

In order to have the most efficient coupling, it is required that this takes place along points of minimum-magnitude relative velocity, i.e., on the instant screw axis of shaft 2 with respect to shaft 1.

From the A-K Theorem, that set of points constitutes line L_3, normal to the Y-axis, a distance a from L_1. Hence, line L_3 is determined by distance a and angle α. Point Q, the intersection of line L_3 and the Y-axis is found from the minimality condition on the relative velocity magnitude. Let $\mathbf{v}_{Q1}$

and v_{Q2} be the velocities of points Q1 and Q2, with respect to the fixed axes X, Y, Z.

Thus,

$$v_{Q1} = a\omega_1 e_z \qquad (2.10.7a)$$

$$v_{Q2} = (d-a)\omega_2 e_x \qquad (2.10.7b)$$

Assuming a required reduction m, i.e.,

$$\omega_2 = m\omega_1 \qquad (2.10.8)$$

eq. (2.10.7b) can be rewritten as

$$v_{Q2} = (d-a) m\omega_1 e_x \qquad (2.10.7c)$$

Next the quadratic form $\phi(a)$, obtained squaring the relative velocity magnitude, is minimized.

$$\phi(a) = (v_{Q2} - v_{Q1}) \cdot (v_{Q2} - v_{Q1}) = = \omega_1^2 \left(m^2 (d-a)^2 + a^2\right) \qquad (2.10.9)$$

$\phi(a)$ has an extremum when $\phi'(a)$ becomes zero, i.e.

$$\phi'(a) = \omega_1^2 \left(-2m^2 (d-a) + 2a\right) = 0 \qquad (2.10.10)$$

from which the minimizing value of a is obtained as

$$a = \frac{m^2 d}{1+m^2} \qquad (2.10.11)$$

Angle α is now obtained from the relationship

$$\cos\alpha = \frac{|\omega_{2/1} \cdot e_x|}{||\omega_{2/1}||}$$

Thus

$$\cos\alpha = \frac{1}{\sqrt{1+m^2}} \qquad (2.10.12)$$

Summarizing, the pitch surface (on 1) is a ruled surface whose elements are lines a distance a form the X axis, making an angle α with this axis.

This is a one-fold hyperboloid of revolution. Hence the name "hypoid" given to such gears.

One very important consequence of the A-K Theorem now follows.

Corollary 2.10.2 *Given three rigid bodies in motion, A, B and C, there exists an instant axis of pure rotation (i.a.p.r.) of B with respect to C if, and only if, there exist i.a's.p.r. of both B and C with respect to A and these intersect, the i.a.p.r. of B with respect to C passing through the said intersection. Furthermore, all three axes are coplanar.*

Exercise 2.10.4 Prove Corollary 2.10.2.

As an application of Corollary 2.10.2, solve the following problem.

Example 2.10.2 (Kane (2.9)). A shaft, terminating in a truncated cone C of semivertical angle θ, see Fig 2.10.4, is supported by a thrust bearing consisting of a fixed race R and four identical spheres S of radius r. When the shaft rotates about its axis, S rolls on R at both of its points of contact with R, and C rolls on S.

Proper choice of the dimension b allows to obtain pure rolling of C on S. Determine b

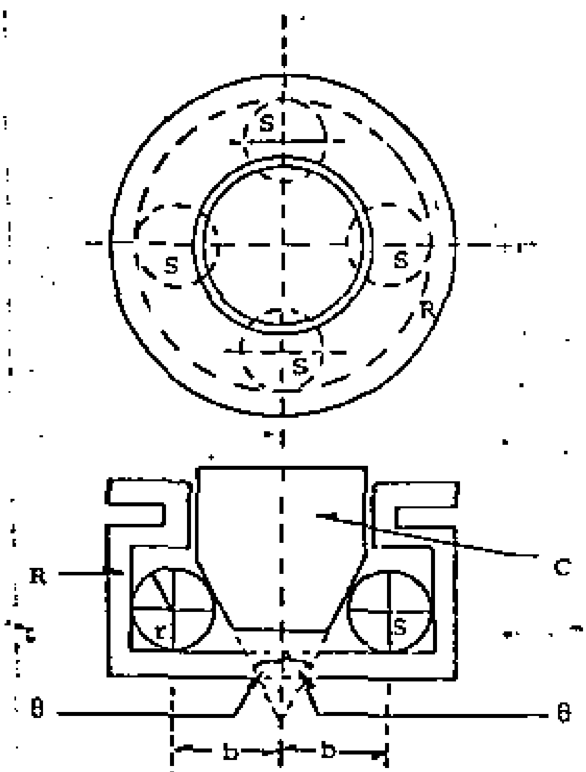

Fig 2.10.4 Shaft rotating on thrust bearings.

Solution:

From Corollary 2.10.2, if all C, S and R move with pure rolling relative motion, then the i.a's.p.r. all coincide at one common point. Clearly, the i.a.p.r. of C with respect to S is the cone element passing through the contact point (between C and S), whereas the i.a.p.r. of C with respect to R is the symmetry axis of C. The intersection of those two axes is the cone apex, which henceforth is referred to as point O. Length b is now determined by the condition that the i.a.p.r. of C with respect to R passes through O.

But two points of this axis are already known, namely, the two points of contact of S on R, henceforth referred to as points S_1 and S_2. Then, the geometry of Fig 2.10.5 follows

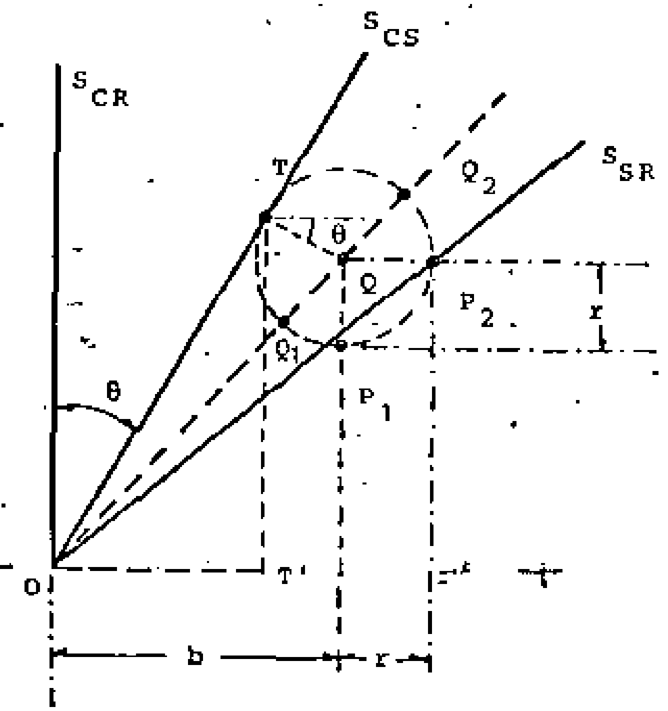

Fig 2.10.5 Instant axes of pure rotation of bodies C, S and R of Fig 2.10.4.

From Fig 2.10.5 it is clear that axis S_{SR} makes a 45° angle with axis S_{CR}. Let T be the contact point between C and S. From a well known theorem of plane geometry,

$$\overline{OT}^2 = \overline{OP}_1\,\overline{OP}_2 \qquad (2.10.13)$$

Appliying the Pythagorean Theorem to triangle OT'T,

$$\overline{OT}^2 = \overline{OT'}^2 + \overline{T'T}^2 \qquad (2.10.14)$$

But

$$\overline{OT'} = b - r\cos\theta \qquad (2.10.15a)$$

and

$$\overline{T'T} = b + r + r\sin\theta \qquad (2.10.15b)$$

Hence,

$$\overline{OT}^2 = 2r^2(1+\sin\theta)+2br(1+\sin\theta-\cos\theta)+2b^2 \qquad (2.10.16)$$

Also,

$$\overline{OP}_1 = \sqrt{2}b \qquad (2.10.17a)$$

$$\overline{OP}_2 = \sqrt{2}(b+r) \qquad (2.10.17b)$$

Substitution of eqs. (2.10.16) and (2.10.17 a and b) into eq. (2.10.13) yields

$$r(1+\sin\theta)+b(\sin\theta-\cos\theta)=0$$

from which,

$$b=r\frac{1+\sin\theta}{\cos\theta-\sin\theta} \qquad (2.10.18)$$

One more consequence of the A-K Theorem is summarized in the following

<u>*Corollary 2.10.3 (Three center Theorem)*</u>. *In plane motion the three instant axes (centers in this context) of three rigid bodies in motion lie on a line (2.10)*

2.11 ACCELERATION DISTRIBUTION IN A RIGID BODY MOVING ABOUT A FIXED POINT

It was shown in Section 2.7 that the velocity of a point of a rigid body moving about a fixed point is given by

$$\underline{v}(t)=\underline{\Omega}(t)\underline{y}(t) \qquad (2.11.1)$$

where $\underset{\sim}{\Omega}(t)$ is the rigid body angular velocity and $\underset{\sim}{y}(t)$ is the current position vector of the point under consideration.

The acceleration $\underset{\sim}{a}(t)$ of the said point is now obtained differentiating both sides of eq. (2.11.1) with respect to time; thus

$$\underset{\sim}{a}(t)=\dot{\underset{\sim}{\Omega}}(t)\underset{\sim}{y}(t)+\underset{\sim}{\Omega}(t)\dot{\underset{\sim}{y}}(t)$$

But $\dot{\underset{\sim}{y}}(t)$ is $v(t)$, the above equation taking on the form

$$\underset{\sim}{a}(t)=\left(\dot{\underset{\sim}{\Omega}}(t)+\underset{\sim}{\Omega}^2(t)\right)\underset{\sim}{y}(t) \qquad (2.11.2)$$

The matrix in brackets appearing in eq. (2.11.2) is referred to, by analogy with eq. (2.11.1), as "the angular acceleration matrix". The acceleration of the point under study is formed by two components, as appearing in eq. (2.11.2), namely, the "tangential acceleration", $\dot{\Omega}(t)y(t)$, and the "normal acceleration", $\Omega^2(t)y(t)$, the former being tangential and the latter being normal to the velocity.

Exercise 2.11.1 Show that $\dot{\Omega}(t)y(t)$ and $\Omega^2(t)y(t)$ are, respectively, parallel and normal to the velocity.

There is one implicit fact in the above result, namely, in the square matrix vector space; one scalar-product (See Section 1.7) can be defined as $Tr(\underset{\sim}{A}\underset{\sim}{B}^T)$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ being matrices of the same space.

In this context, matrices $\dot{\underset{\sim}{\Omega}}(t)$ and $\underset{\sim}{\Omega}^2(t)$ are orthogonal, i.e., its scalar product venishes.

Exercise 2.11.2 Show that $Tr(\dot{\underset{\sim}{\Omega}}\underset{\sim}{\Omega}^2)=0$

Result (2.11.2) can be expressed, in Gibbs' notation as

$$\underset{\sim}{a}(t)=\dot{\underset{\sim}{\omega}}(t)\times\underset{\sim}{r}(t)+\underset{\sim}{\omega}(t)\times\left(\underset{\sim}{\omega}(t)\times\underset{\sim}{r}(t)\right) \qquad (2.11.3)$$

thereby making the result of Exercise 2.11.1 apparent

2.12 ACCELERATION DISTRIBUTION IN A RIGID BODY UNDER GENERAL MOTION.

Consider now the most general case of rigid body motion, in which none of the points of the body remains fixed.

From eq. (2.9.2), the velocity of a point-whose position vector is $\underset{\sim}{y}(t)$- of a rigid body under general motion is

$$\underset{\sim}{v}(t)=\underset{\sim}{v}_P(t)+\underset{\sim}{\Omega}(t)\left(\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)\right) \qquad (2.12.1)$$

where $\underset{\sim}{y}_P(t)$ and $\underset{\sim}{v}_P(t)$ are the position vector and the velocity, both known, of a given point P of the rigid body. The acceleration of a general point, $\underset{\sim}{a}(t)$, of the body under consideration is next obtained differentiating both sides of eq. (2.12.1) with respect to time, i.e.

$$\underset{\sim}{a}(t)=\underset{\sim}{a}_P(t)+\dot{\underset{\sim}{\Omega}}(t)\left(\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)\right)+\underset{\sim}{\Omega}(t)\left(\dot{\underset{\sim}{y}}(t)-\dot{\underset{\sim}{y}}_P(t)\right) \qquad (2.12.2)$$

and, from eq. (2.12.1),

$$\dot{\underset{\sim}{y}}(t)-\dot{\underset{\sim}{y}}_P(t)=\underset{\sim}{v}(t)-\underset{\sim}{v}_P(t)=\underset{\sim}{\Omega}(t)\left(\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)\right) \qquad (2.12.3)$$

which, when substituted in eq. (2.12.2), leads to

$$\underset{\sim}{a}(t)=\underset{\sim}{a}_P(t)+\left(\dot{\underset{\sim}{\Omega}}(t)+\underset{\sim}{\Omega}^2(t)\right)\left(\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)\right) \qquad (2.12.4)$$

which, except for the term $\underset{\sim}{a}_P(t)$, is identical to eq. (2.11.2), with $\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)$ instead of $\underset{\sim}{y}(t)$ of that equation.

The relative acceleration, $\underset{\sim}{a}(t)-\underset{\sim}{a}_P(t)$, of the general point with respect to P is clearly given as

$$\underset{\sim}{a}(t)-\underset{\sim}{a}_P(t)=\left(\dot{\underset{\sim}{\Omega}}(t)+\underset{\sim}{\Omega}^2(t)\right)\left(\underset{\sim}{y}(t)-\underset{\sim}{y}_P(t)\right) \qquad (2.12.5)$$

which again, is seen to be composed of both a tangential and a normal component.

Parallelling previous sections, the set of points of minimum-magnitude acceleration is now determined. Thus, the function ϕ defined as

$$\phi(\underset{\sim}{y})=\underset{\sim}{a}\cdot\underset{\sim}{a} \qquad (2.12.6)$$

is now minimized over $\underset{\sim}{y}$. Applying the "chain rule" to it,

$$\phi'(\underset{\sim}{y}) = 2\left(\frac{\partial \underset{\sim}{a}}{\partial \underset{\sim}{y}}\right)^T \underset{\sim}{a} \qquad (2.12.7)$$

where, from eq. (2.12.4),

$$\frac{\partial \underset{\sim}{a}}{\partial \underset{\sim}{y}} = \dot{\underset{\sim}{\Omega}} + \underset{\sim}{\Omega}^2 \qquad (2.12.8)$$

Hence, the minimum-magnitude acceleration satisfies

$$(-\dot{\underset{\sim}{\Omega}} + \underset{\sim}{\Omega}^2)\underset{\sim}{a} = \underset{\sim}{0} \qquad (2.12.9)$$

i.e. the minimum-magnitude acceleration is in the null space of $\dot{\underset{\sim}{\Omega}} + \underset{\sim}{\Omega}^2$. If both $\dot{\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Omega}^2$ have the same null space, then that minimum-magnitude acceleration lies in that space. Since both $\dot{\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Omega}$ are skew symmetric, vectors $\dot{\underset{\sim}{\omega}}$ and $\underset{\sim}{\omega}$ lying in their null space, can be defined in such a way that, for any vector $\underset{\sim}{r}$,

$$\underset{\sim}{\Omega}\underset{\sim}{r} = \underset{\sim}{\omega} x \underset{\sim}{r}, \dot{\underset{\sim}{\Omega}}\underset{\sim}{r} = \dot{\underset{\sim}{\omega}} x \underset{\sim}{r} \qquad (2.12.10)$$

Hence it becomes clear that for $\dot{\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Omega}$ to have the same null space, $\underset{\sim}{\omega}$ and $\dot{\underset{\sim}{\omega}}$ should be parallel. Furthermore, $\underset{\sim}{\Omega}^2$ and $\underset{\sim}{\Omega}$ have the same null space (Prove it) and hence, for $\dot{\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Omega}^2$ to have the same null space, $\underset{\sim}{\omega}$ and $\dot{\underset{\sim}{\omega}}$ should be parallel. A simple case for which $\dot{\underset{\sim}{\Omega}}$ and $\underset{\sim}{\Omega}^2$ have the same null-space is that for which the rotation axis, $\underset{\sim}{e}$, has a constant direction. In fact, if this is so, then,

$$\underset{\sim}{\Omega}\underset{\sim}{e} = \underset{\sim}{0} \qquad (2.12.11)$$

for all time t. Differentiating the latter expresion with respect to time yields

$$\dot{\underset{\sim}{\Omega}}\underset{\sim}{e} + \underset{\sim}{\Omega}\dot{\underset{\sim}{e}} = \underset{\sim}{0}$$

But, since the magnitude of e is unity and its direction is constant, $\dot{\underset{\sim}{e}} = \underset{\sim}{0}$ and hence, the latter equation leads to

$$\dot{\underset{\sim}{\Omega}}\underset{\sim}{e} = \underset{\sim}{0} \qquad (2.12.12)$$

thereby showing that, under the conditions stated, if $\underline{e}$ is in the null space of $\underline{\Omega}$, then it is also in the null space of $\dot{\underline{\Omega}}$. Hence, when the instant screw axis has a contant direction, the minimum-magnitude acceleration is parallel to that direction. If the involved matrices do not have a common (non-empty) null space, then the only possibility of eq. (2.12.9) to hold is if $\underline{a}=\underline{0}$. The latter condition is equivalent to

$$\underline{a}_P+(\dot{\underline{\Omega}}+\underline{\Omega}^2)(\underline{y}_0-\underline{y}_P)=\underline{0}$$

or

$$(\dot{\underline{\Omega}}+\underline{\Omega}^2)\underline{y}_0=(\dot{\underline{\Omega}}+\underline{\Omega}^2)\underline{y}_P-\underline{a}_P \qquad (2.12.13)$$

which has a unique solution if $\dot{\underline{\Omega}}$ and $\underline{\Omega}^2$ do not have a common null space, for then, the sum of them becomes of full rank, i.e. in that case rank rank$(\dot{\underline{\Omega}}+\underline{\Omega}^2)=3$, and hence this sum is nonsigular. In this case, then, one single point of the body, located by the position vector $\underline{y}_0$, has a zero acceleration. This point is called an *acceleration pole* and its position vector is given as

$$\underline{y}_0=\underline{y}_P-(\dot{\underline{\Omega}}+\underline{\Omega}^2)^{-1}\underline{a}_P \qquad (2.12.14)$$

Example 2.12.1. For a rigid circular cone rolling without slipping on a plane, its acceleration pole is its apex (prove it)

Exercise 2.12.2 The system shown in Fig 2.12.1 is an inversion of the worm-gear mechanism and is composed of a rigid arm OA of lenght b that can rotate freely about the axis EE', this axis being normal to the plane of motion of OA. A rigid wheel is coupled to OA at A in such a way that the wheel can rotate freely about axis FF' passing though A; this axis is perpendicular to both OA and EE'. If OA rotates at a constant rate p and the wheel rotates about FF' at a constant rate q, show that the point of the disk on OA, a distance d from O, has zero acceleration, the distance d being given by

$$d = \frac{q^2}{p^2+q^2} b$$

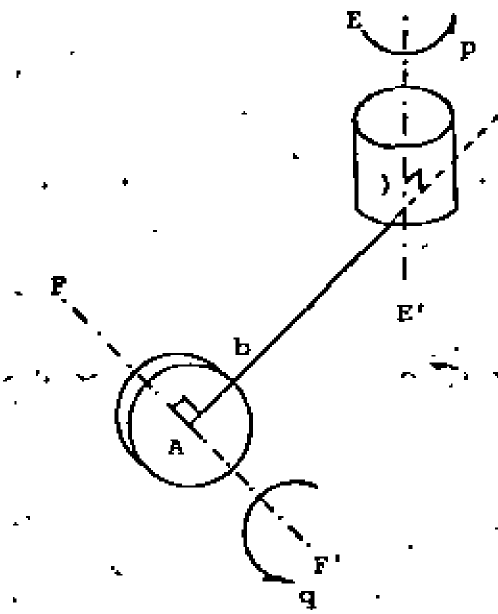

Fig 2.12.1 Inversion of the worm- gear mechanism

An extensive account of this topic is presented in (2.11)

2.13. ACCELERATION OF A MOVING POINT REFERRED TO A MOVING OBSERVER. CORIOLIS' THEOREM.

In Section 2.10 it was shown that the velocity $(\underline{v})^*_F$ of a moving point, referred to a fixed observer, given in terms of its position vector, $(\underline{\xi})_M$, referred to a moving observer is given by

$$(\underline{v})_F=(\underline{v}_o)_F+(\underline{\Omega})_F\ (\underline{\xi})_F+(\underline{Q})_F(\dot{\underline{\xi}})_M \quad (2.13.1)$$

where $\underline{Q}$ and $\underline{\Omega}$ are the rotation and the angular velocity matrices, respectively, of the moving axes with respect to the fixed ones.

The acceleration, $(\underline{a})_F$, of the moving point, in the fixed observer, is now obtained differentiating eq. (2.13.1) with respect to time, namely

$$(\underline{a})_F=(\underline{a}_o)_F+(\dot{\underline{\Omega}})_F(\underline{\xi})_F+(\underline{\Omega})_F(\dot{\underline{\xi}})_F+(\dot{\underline{Q}})_F(\dot{\underline{\xi}})_M+(\underline{Q})_F(\ddot{\underline{\xi}})_M \quad (2.13.2)$$

where

$$(\underline{a}_o)_F=(\dot{\underline{v}}_o)_F \quad (2.13.3)$$

But

$$(\underline{\xi})_F=(\underline{Q})_F\ (\underline{\xi})_M \quad (2.13.4)$$

Hence

$$(\dot{\underline{\xi}})_F=(\dot{\underline{Q}})_F\ (\underline{\xi})_M+(\underline{Q})_F(\dot{\underline{\xi}}) \quad (2.13.5)$$

Substitution of eqs. (2.13.4) and (2.13.5) into eq. (2.13.2) yields

$$(\underline{a})_F=(\underline{a}_o)_F+(\dot{\underline{\Omega}})_F(\underline{\xi})_F+(\underline{\Omega})_F((\dot{\underline{Q}})_F(\underline{\xi})_M+(\underline{Q})_F(\dot{\underline{\xi}})_M)+(\dot{\underline{Q}})_F(\dot{\underline{\xi}})_M+(\underline{Q})_F(\ddot{\underline{\xi}})_M \quad (2.13.6)$$

But

$$(\dot{\underline{Q}})_F(\underline{\xi})_M=(\dot{\underline{Q}})_F(\underline{Q}^T)_F(\underline{Q})_F(\underline{\xi})_M=(\underline{\Omega})_F(\underline{Q})_F(\underline{\xi})_M$$

---

* All vectors and matrices appearing in this section are functions of time, but for simplicity the argument (t) has been dropped.

and

$$(\Omega)_F(\dot{Q})_F(\xi)_M = (\Omega)_F(\dot{Q})_F(Q^T)_F(Q)_F(\xi)_M = (\Omega^2)_F(Q)_F(\xi)_M = (\Omega^2)_F(\xi)_F$$

Substituting the two latter expressions into eq. (2.13.6) one obtains

$$(a)_F = (a_o)_F + (\dot{\Omega}+\Omega^2)_F(Q)_F(\xi)_M + (Q)_F(\ddot{\xi})_M + 2(\Omega)_F(\dot{\xi})_M \qquad (2.13.7)$$

which is an expression for the acceleration of a point in terms of measurements of its position, velocity and acceleration, taken by a moving observer. The first two terms of eq. (2.13.7) are identical to the right hand side of eq. (2.12.4) with $y-y_p$ substituted for ξ; hence, the two said terms constitute the acceleration of a point fixed in the moving observer, coincident with the moving point under study, at a particular time. The third term stands for the acceleration of the moving point, as measured by the moving observer, and the fourth term is an acceleration term arising from the rotation of the moving observer, as is apparent from eq. (2.13.7); this term is known as "Coriolis' acceleration". Equation (2.13.7) constitutes, then, the Theorem of Coriolis. (2.11)

Exercise 2.13.1 The mechanism shown in Fig 2.13.1 is a component of a quick-return mechanism used in a crank shaper. Assuming that disk 2 rotates at a constant angular velocity ω_2=1800 rpm, determine graphically the angular acceleration of link 3, for the given configuration.

Hint: Two points, B2 and B3, coincide at B. Find the acceleration of B3 via eq. (2.13.7), referred to an observer fixed in 2.

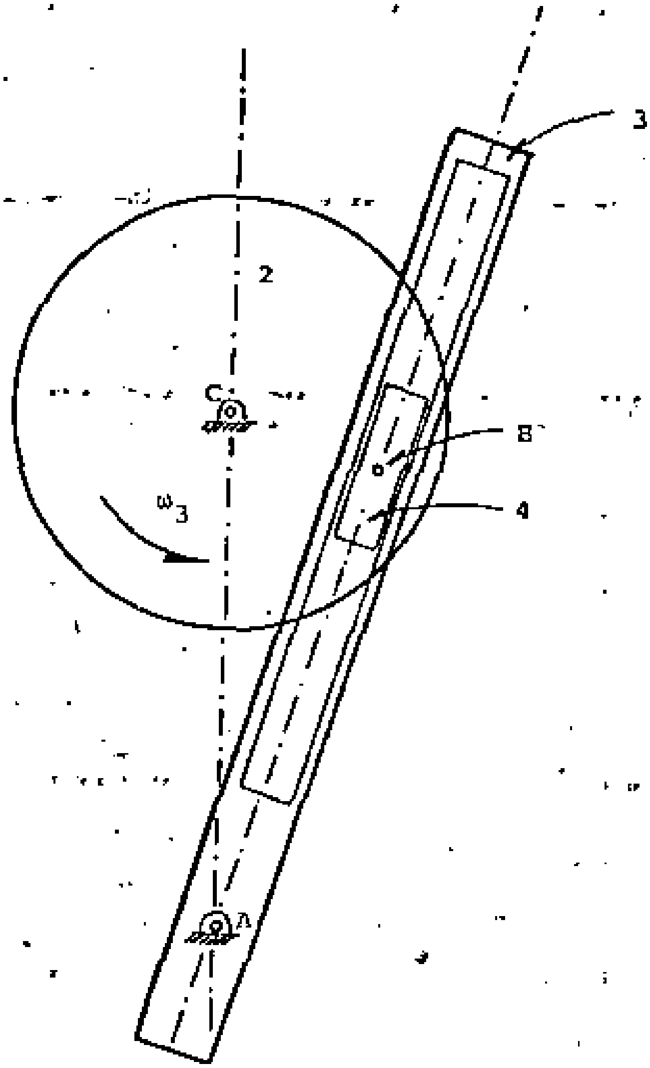

Fig 2.13.1 Driving system of a quick-return mechanism

<u>Exercise 2.13.2</u> The rectangular plate shown in Fig 2.13.2 is displaced from configuration 1 to configuration 2. Determine the locus of the points of the plate that undergo a displacement of minimum magnitude from 1 to 2. What is the value of this displacement?

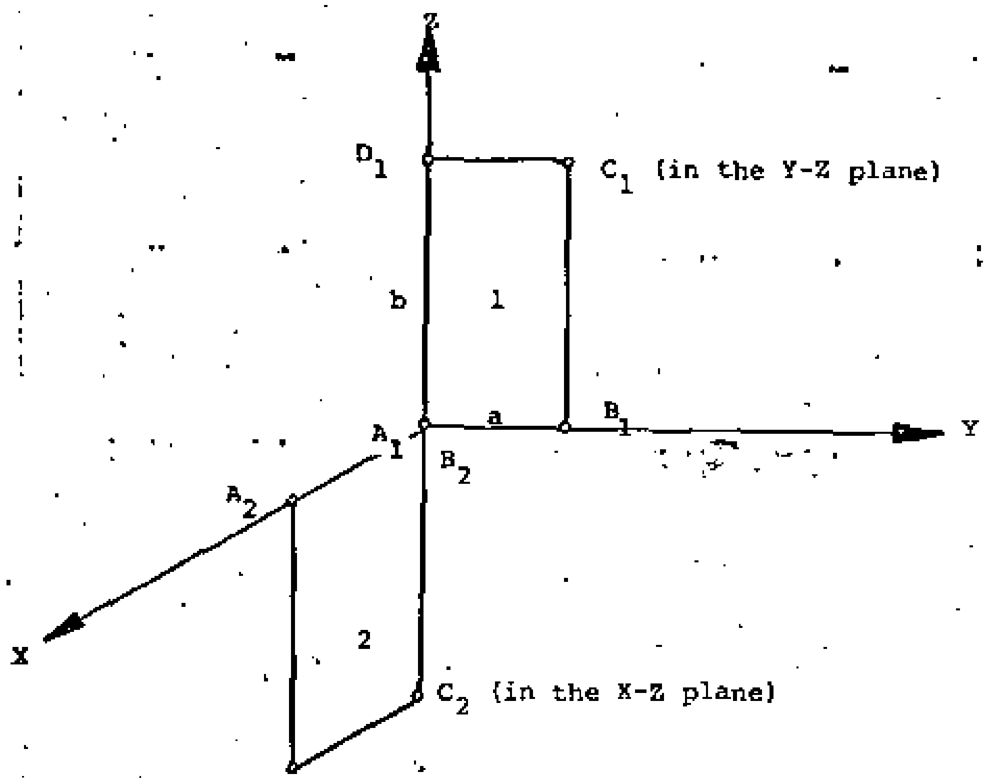

Fig. 2.13.2 Rigid plate undergoing a general displacement.

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

3. GENERALITIES ON LOWER - PAIR MECHANISMS

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3. GENERALITIES ON LOWER - PAIR MECHANISMS.

3.1 INTRODUCTION

The term mechanism has multiple meanings, depending on the context in which it is found. In the present context, a mechanism is a connection of elements intended to produce a certain action, generally related to the transmission of either power or information. For instance, power transmission is the main objective of a mechanism such as the universal joint or the differential gear train of a vehicle whereas information transmission is the main good of a Watt regulator mechanism. In any case, the basic idea is that of motion transformation.

The interest for the study of mechanisms arose originally in mechanical engineering. The underlying theory, however, has been found to embrace other areas such as biomechanics, and so, it finds wide applications in the study of some living entities like the locomotion systems of humans and animals. The wide variety of mechanisms as defined previously, can be divided into two classes, namely, lower-pair and upper-pair mechanisms. These terms are discussed in the present chapter, together with other related terms such as degree of freedom, kinematic pair and kinematic chain.

3.2 KENEMATIC PAIRS.

A kinematic pair is the coupling of two mechanical elements. If these two elements are rigid bodies, the pair can be either one of two kinds: i) lower pair or ii) upper pair. A lower pair exists when one element is coupled to the other via a wrapping action and contact takes place along a surface. If contact takes place along a line or a point, the resulting coupling is referred to as upper-pair. When a set of elements is connected in such a way that each element is coupled to at least two elements, a kinematic chain is formed. A link supplied with one kinematic pair at each of two ends is referred to as a dyad.

3.3 DEGREE OF FREEDOM.

The degree of freedom of a mechanical system is defined as an integer number corresponding to the minimum number of generalized {3.1} coordinates required to specify geometrically a configuration of the system. If the degree of freedom of a system is positive, it constitutes a mechanism; if it is zero, it corresponds to a statically determined structure, whereas, if it is negative, it corresponds to a statically undertermined (hyperstatic) structure, the negative of its degree of freedom being referred to as its redundancy. The degree of freedom of a rigid body free to move in space is thus equal to six, namely, a translation along each of three non-coplanar directions, plus a rotation (e.g. Euler's angles) about each of three non-coplanar directions (not necessarily corresponding to the three previous directions). Constraints are imposed on rigid bodies to make their motion useful for a certain purpuse, thus diminishing their degree of freedom. Kinematic pairs are in fact constraints imposed on the involved bodies.

3.4 TYPES OF LOWER PAIRS.

All possible lower pairs can be put into one of six different types, namely, i) Revolute (R), ii) Prismatic (P), iii) Screw (H), iv) Cylindric (C), v) Spheric (S) and vi) Planar (E), physical models of which are shown in Fig 3.4.1, their kinematic models appearing in Fig 3.4.2.

The revolute pair only allows rotation about one axis, therefore imposing five constraints: prevention of translation along three directions and of rotation about two axes.

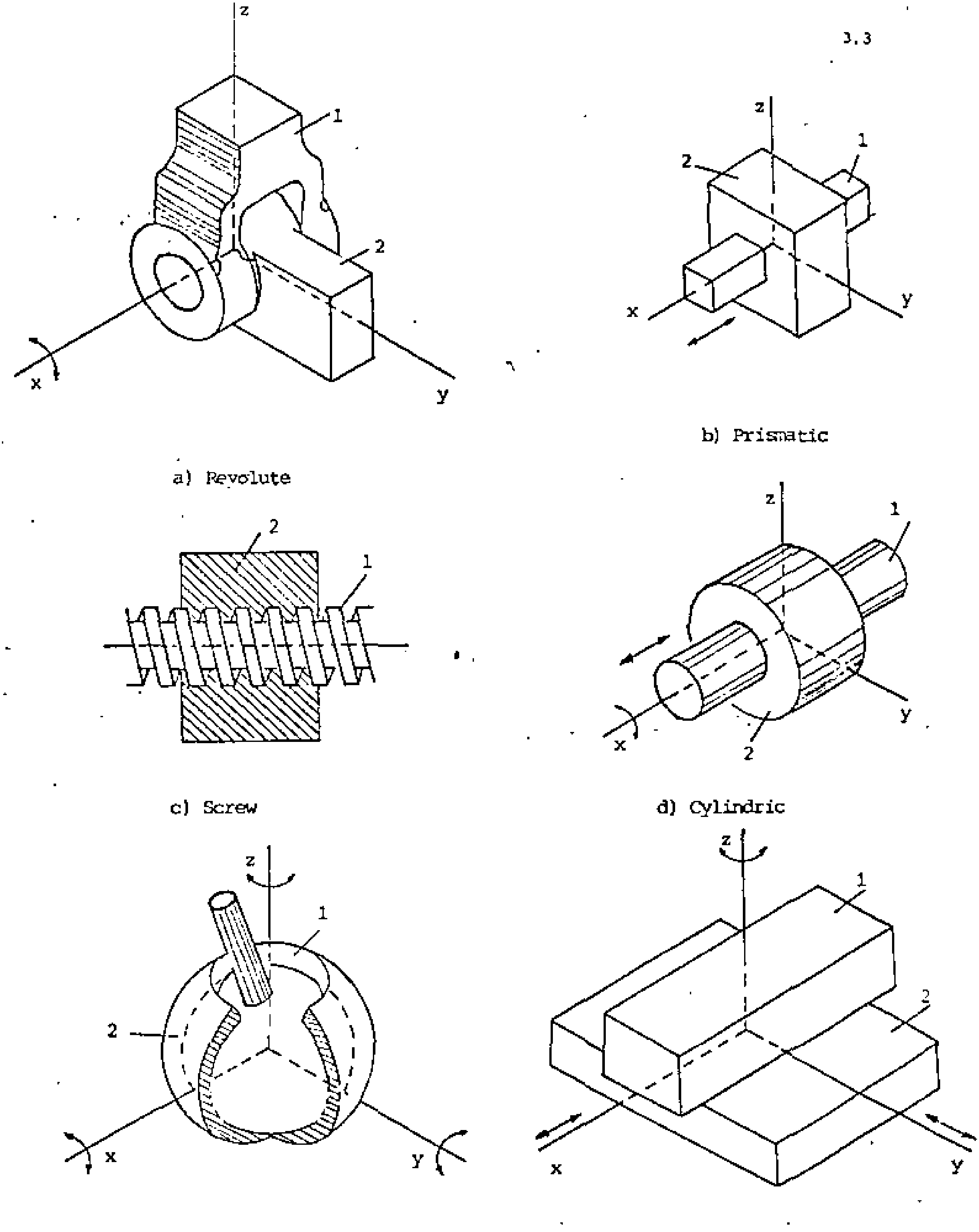

Fig. 3.4.1 Lower Pairs

3.4

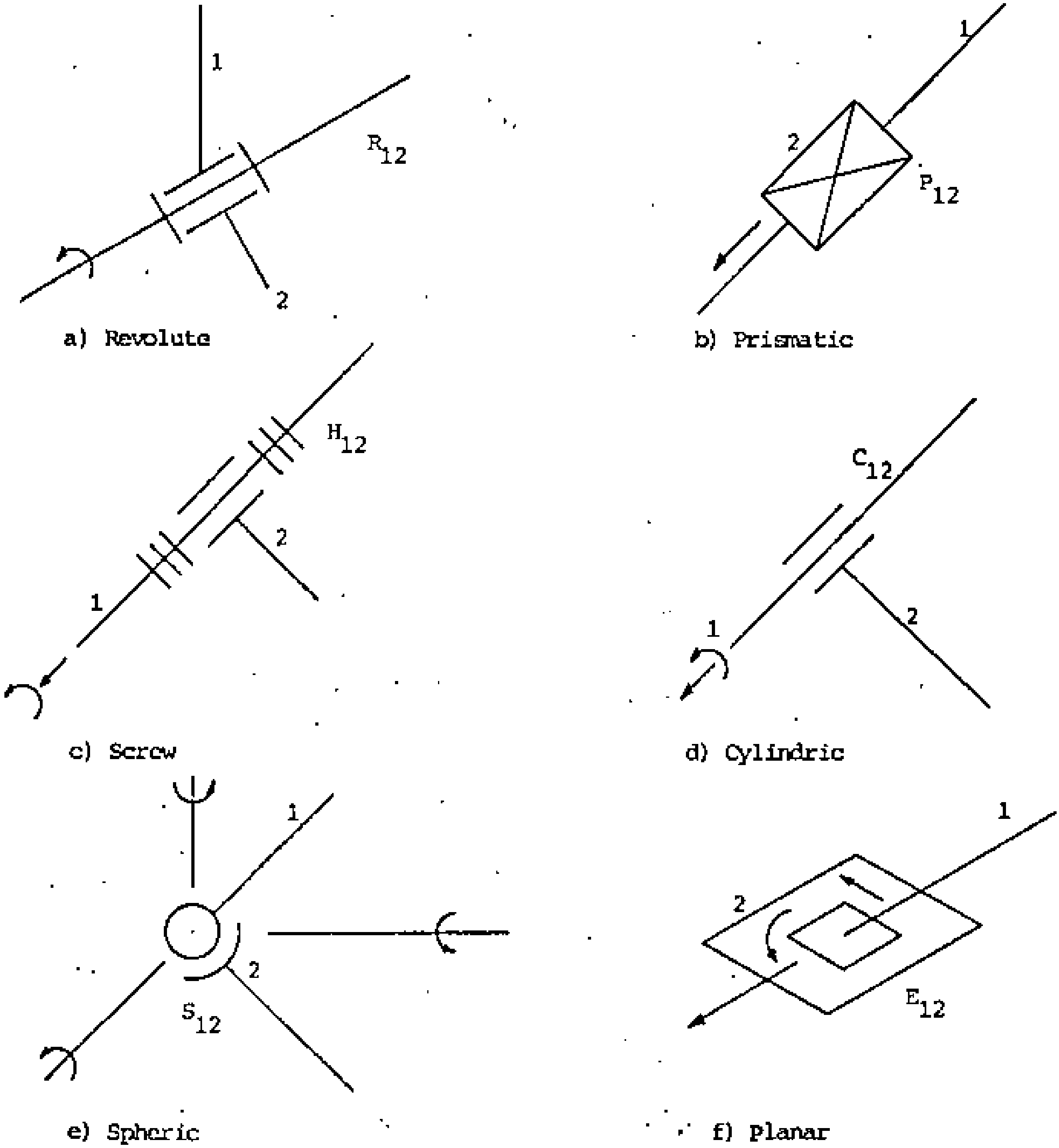

Fig. 3.4.2 Kinematic models of lower pairs

The prismatic pair only allows translation along one direction, also imposing five constraints: prevention of translation along two directions and of rotation about three axes.

The screw pair allows translation along one direction and rotation about the same direction, both being related. Thus, five constraints are also imposed by this pair, namely, prevention of translation along three (or, alternatively, two) directions and of rotation about two (or, alternatively, three) axes.

The cylindrical pair allows two independent motions, namely, translation about one axis and rotation about the same axis. This pair imposes clearly, four contraints.

The spherical pair allows rotation about three non-coplanar axes, thus imposing three constraints: prevention of translation along three non-coplanar directions.

The planar pair allows translation along two independent directions and rotation about one axis perpendicular to the plane of those directions. Thus, this pair imposes three constraints.

A dyad supplied with a revolute pair at each of two ends is referred to as an R-R dyad. Similarly are R-P, R-S, C-S, etc. dyads defined.

3.5 DEGREE OF FREEDOM OF A LOWER-PAIR MECHANISM. THE KUTZBACH-GRÜBLER FORMULA.

With the foregoing backgroud it is possible now to obtain an expression for the degree of freedom of a mechanism composed of only lower pairs, also called a "linkage". Let a linkage be composed of n rigid links.

Since only the relative motion of the links, with respect to a given one, is of interest, one link will be arbitrarily assumed to be fixed (to the observer under consideration). Hence, the degree of freedom of the set of links previously to any coupling is $6(n-1)$. Now, if the links are coupled via p_5 pairs of any of the three types, revolute, prismatic or screw, p_4 cylindrical pairs and p_3 pairs of any of the last two types, then the number of constraints imposed is

$$c=\sum_{3}^{5} ip_i$$

Hence, the degree of freedom of the linkage is given by

$$f=6(n-1)-\sum_{3}^{5} ip_i \qquad (3.5.1)$$

which is the so called "Kutzbach-Grübler formula". The topopogy (i.e. the description of the different couplings, regardless of the dimensions involved) of a linkage is usually referred to as a listing of the different pairs involved. To ilustrate this description, different types of mechanisms, RRRR, RCSR, RSSR, are shown in Fig 3.5.1.

Applying the Kutzbach-Grübler formula to the mechanisms of Fig 3.5.1, it is readily obtained that, whereas the degree of freedom of the RCSR type is one, that of the RRRR type is -2 and that of the RSSR type is +2. However, if the four revolute axes of the RRRR linkage are made parallel, then a planar four-bar linkage is obtained, which is a well known mechanism with degree of freedom + 1. Alternatively, if these four axes are made to intersect in one single point, a spherical linkage is obtained, one particular case of which is the familiar universal joint, also known to have a degree of freedom + 1. The reason why the K-G formula does not apply in these instances is that, in the first case (all revolute axes being parallel), all the points of the linkage move in one plane, whereas in the second case, they move on a sphere. In any of the previous instances, however, a rigid body before coupling has a triple degree of freedom, i.e. two translations and one rotation; hence Grübler's formula for plane mechanisms (3.2) should

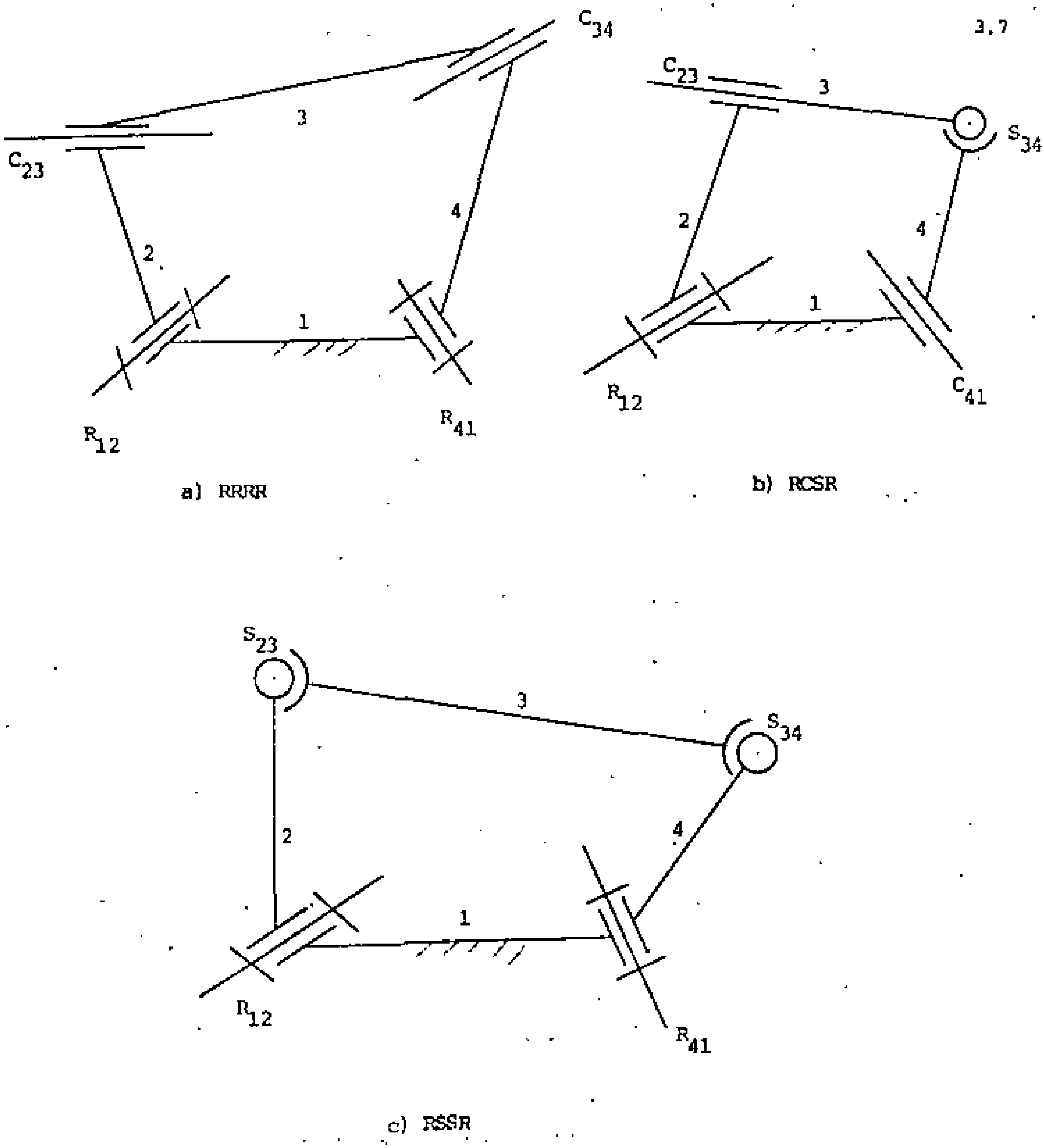

Fig 3.5.1 Some types of linkages

be applied. One more interesting case of linkage whose degree of freedom cannot be obtained from the K-G formula is the Bennett mechanism. This is a one-degree of freedom RRRR linkage that is neither plane nor spherical, but its links have particular proportions and its revolute axes have particular orientations (3.3), as is discussed with more detail in section 5.3. Regarding the RSSR linkage, it is apparent from Fig 3.5.1(c) that, if the rotation of the crank 2 is specified, the rotation of the follower 4 is uniquely defined, thus contradicting the result obtained by application of the Kutzbach-Grübler formula, which claims that (due to the double degree of freedom of the mechanism) two variables of the linkage motion should be specified in order to render the motions of the other links determined. This result arises from the fact that an indeterminacy exists in the motion of the coupler link 3 which, because of being coupled to the other links via two spherical pairs, shows the peculiarity that its motion is undetermined because it is a line rather than a body, i.e. no three noncollinear points can be defined on this link. This indeterminary arises from Theorem 2.6.5, after which the motion of three noncollinear points determine the motion of a rigid body. More on the motion of bodies defined by only two points can be found in (3.4)

3.6 LINKAGE PROBLEMS MEANT TO BE SOLVED BY APPLIED KINEMATICS.

Broadly speaking, there are two kinds of problems in Applied Kinematics regarding lower-pair mechanisms or linkages, namely,

i) Analysis

and

ii) Synthesis

The analysis problem is concerned with obtaining the different variables of interest involved in the motion of a given linkage. What is meant by the latter term is that not only the topology of the mechanism (whether it is RRRR or RSCR, etc.) is given, but also its geometry, i.e. its relevant dimensions. The problem is said to be solved when all the variables of interest (output) of the linkage motion are obtained in terms of other arbitrarily assigned variables (input). If the linkage has a degree of freedom one, then the input of the linkage contains one single variable, sufficient to render the other variables determined. If the linkage has a double degree of freedom, then the input contains two variables, and so on. The variables of interest can be obtained via different means, namely, i) a system of algebraic* equations expressing explicitly the output in terms of the input (This is very seldom obtained), ii) a discrete set of digital values of the variables involved** (most commonly being the case), iii) an analog readout (oscillogram), i.e. the cathode ray tube graph obtained in the oscilloscope of an analog computer, or iv) the actual plot obtained by mechanically recording the output of an existing linkage. In the first three cases the output does not exactly correspond to the mechanism itself, but to its mathematical model; the output is thus obtained via simulation. In the fourth case, the output can correspond to the actual mechanism if it is accessible for measurements, or if it is not accessible, then to its physical model. An example of the latter arises in the case of trying to measure joint motions in a living being, as is recorded in {3.5} where an experiment was made to measure rotations of the human subtalar and ankle-joint complex throgh the rotations of its physical model, an RRRR spherical linkage.

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* Algebraic equations as opposed to differential or integral equations since no inertia is involved in a purely kinematic analysis.

** This set could be obtained in tabular form or, if a mechanical plotter is used, in graphical form.

The synthesis problem is concerned with obtaining the relevant dimensions of a linkage of a *given topology* to perform a given operation. In this respect, the synthesis problem can also be thought of as one of system identification (3.6), since it is intended to obtain the parameters defining it starting with an input-output relationship. The synthesis problem can in turn be subdivided into two wide categories, namely, i) exact synthesis and ii) approximate synthesis. In the first case the obtained system of equations is meant to be solved exactly, whereas in the second one it is intended to obtain a "solution" satisfying the system with a minimum error. In either case, three basic synthesis problems can be defined, namely,

i) Function generation

ii) Rigid-body guidance

iii) Path generation

The function-generation synthesis problem is one of finding a linkage (of a given topology) such that its input- and output- links have given coordinate motions. The rigid body guidance synthesis problem is concerned with finding a linkage (of a given topology) such that one of its links follows a prescribed set of configurations. Finally, in the path-generation problem, a mechanism of a given topology is sought, with the property that one of its links contains one point passing through a prescribed set of positions.

In any case, the term finite is associated to the synthesis problem if the data are finitely separated. Otherwise, the synthesis problem is said to be of an infintesimal character, as would be the case when trying to satisfy conditions imposed on velocities or on accelerations. The analysis problem is discussed in Chapter 4, whereas the three synthesis problems both exact and approximate, are discussed in Chapters 5 and 6, respectively.

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

4. ANALYSIS OF LINKAGE MOTIONS

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AGOSTO, 1980

4. ANALYSIS OF LINKAGE MOTIONS

4.1 INTRODUCTION.

Two methods of analysis of linkage motions are presented in this chapter, both of them based on matrix computations. Methods based on Cartesian vector algebra or descriptive geometry can also be applied, as shown in (4.1). However, these are usually *ad hoc* methods and become cumbersome in many instances. The main aim of this chapter is to establish the methods required to obtain a (usually implicit) relationship between the input and the output variables of single-degree of freedom linkages.

4.2 THE METHOD OF DENAVIT AND HARTENBERG (4.5).

This method first appeared in (4.2) to (4.4) and is based on a closure relationship of successive affine transformations. An affine transformation is a change of coordinates involving a translation of the origin and a rotation of axes. Let X_1, Y_1, Z_1 and X_2, Y_2, Z_2 be two sets of coordinates related by an affine transformation, as appears in Fig 4.2.1

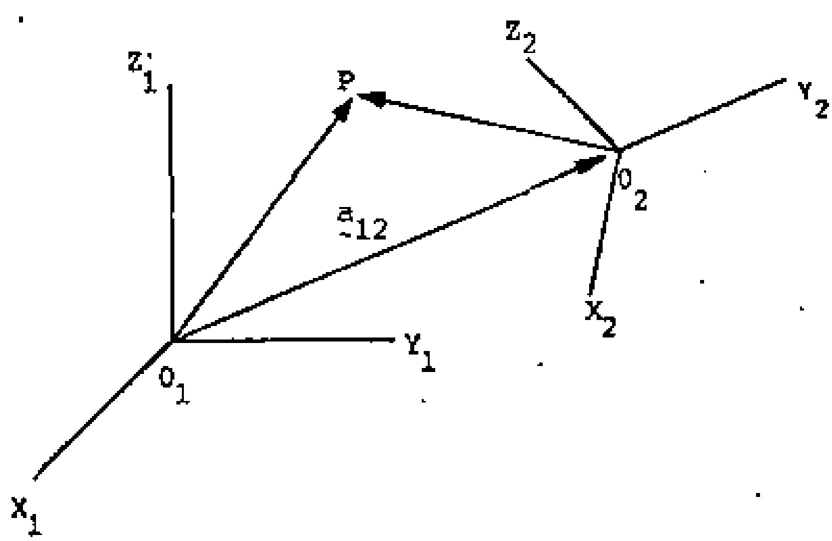

Fig 4.2.1 Translation and rotation of coordinate axes.

Thus, the position vector of any point P, referred to coordinates 1 and 2, can be expressed as

$$(\underset{\sim}{p})_1=(\underset{\sim}{a}_{12})_1+(\underset{\sim}{Q}_{12})_1(\underset{\sim}{p})_2 \tag{4.2.1}$$

where $\underset{\sim}{a}_{12}$ and $\underset{\sim}{Q}_{12}$ are the translation vector and the rotation matrix, from axes 1 to axes 2. Eq. (4.2.1) indicates the general form of an affine transformation. Symbolically, the transformation of (4.2.1) can be written as*

$$(\underset{\sim}{p})_1=(\underset{\sim}{T}_{12})_1(\underset{\sim}{p})_2 \tag{4.2.2}$$

Affine transformations contitute a group under the composition operation defined as

$$\underset{\sim}{T}_{13}=\underset{\sim}{T}_{23}\underset{\sim}{T}_{12} \tag{4.2.3}$$

$\underset{\sim}{T}_{23}$ is given through vector $\underset{\sim}{a}_{23}$ and matrix $\underset{\sim}{Q}_{23}$ as

$$(\underset{\sim}{p})_2=(\underset{\sim}{a}_{23})_2+(\underset{\sim}{Q}_{23})_2(\underset{\sim}{p})_3 \tag{4.2.4}$$

and $\underset{\sim}{T}_{13}$ is given through $\underset{\sim}{a}_{13}$ and $\underset{\sim}{Q}_{13}$, correspondingly, as

$$(\underset{\sim}{p})_1=(\underset{\sim}{a}_{13})_1+(\underset{\sim}{Q}_{13})_1(\underset{\sim}{p})_3 \tag{4.2.5}$$

Substitution of eq. (4.2.4) into eq. (4.2.1) yields

$$(\underset{\sim}{p})_1=(\underset{\sim}{a}_{12})_1+(\underset{\sim}{Q}_{12})_1((\underset{\sim}{a}_{23})_2+(\underset{\sim}{Q}_{23})_2(\underset{\sim}{p})_3)=$$

$$=(\underset{\sim}{a}_{12})_1+(\underset{\sim}{Q}_{12})_1(\underset{\sim}{a}_{23})_2+(\underset{\sim}{Q}_{12})_1(\underset{\sim}{Q}_{23})_2(\underset{\sim}{p})_3 \tag{4.2.6}$$

Hence

$$(\underset{\sim}{a}_{13})_1=(\underset{\sim}{a}_{12})_1+(\underset{\sim}{Q}_{12})_1(\underset{\sim}{a}_{23})_2 \tag{4.2.7}$$

which, alternatively, can be written as

$$(\underset{\sim}{a}_{13})_1=(\underset{\sim}{a}_{12})_1+(\underset{\sim}{a}_{23})_1 \tag{4.2.7a}$$

---

* $(\underset{\sim}{T}_{ij})_i$ should not be mistaken as a matrix. It is in fact a nonlinear operator.

in agreement with the geometrical meaning, that is to say, the vector connecting the origin O_1 to the origin O_3 equals the sum of that connecting O_1 to O_2 plus that connecting O_2 to O_3. Also, from eq. (4.2.6),

$$(\underset{\sim}{Q}_{13})_1=(\underset{\sim}{Q}_{12})_1(\underset{\sim}{Q}_{23})_2 \quad (4.2.8)$$

thereby showing that the composition of two affine transformations is also affine. Let the <u>identity affine transformation</u> $\underset{\sim}{T}_{ii}$, be defined as

$$(\underset{\sim}{x})_i=(\underset{\sim}{T}_{ii})_i(\underset{\sim}{x})_i \quad (4.2.9)$$

i.e., $\underset{\sim}{T}_{ii}$ is the coordinate transformation from coordinates i into themselves. Clearly, its vector, $\underset{\sim}{a}_{ii}$, is the zero vector, and its matrix, $\underset{\sim}{Q}_{ii}$, is the identity matrix. To show that affine transformations in fact constitute a group under the composition previously noted, all that remains to be established is the existence of an <u>inverse transformation</u>, $\underset{\sim}{T}_{ij}^{-1}$ such that

$$(\underset{\sim}{T}_{ij}^{-1})_j(\underset{\sim}{T}_{ij})_i(\underset{\sim}{x})_j = (\underset{\sim}{T}_{ij})_i(\underset{\sim}{T}_{ij}^{-1})_j(\underset{\sim}{x})_j=(\underset{\sim}{x})_j \quad (4.2.10)$$

where

$$(\underset{\sim}{T}_{ij})_i(\underset{\sim}{x})_j=(\underset{\sim}{a}_{ij})_i+(\underset{\sim}{Q}_{ij})_i(\underset{\sim}{x})_j$$

and

$$(\underset{\sim}{T}_{ij}^{-1})_j(\underset{\sim}{x})_i=(\underset{\sim}{T}_{ji})_j(\underset{\sim}{x})_i=(\underset{\sim}{a}_{ji})_j+(\underset{\sim}{Q}_{ji})_j(\underset{\sim}{x})_i \quad (4.2.11)$$

Thus

$$(\underset{\sim}{T}_{ij}^{-1})_j(\underset{\sim}{T}_{ij})_i(\underset{\sim}{x})_j=(\underset{\sim}{a}_{ji})_j+(\underset{\sim}{Q}_{ji})_j((\underset{\sim}{a}_{ij})_i+(\underset{\sim}{Q}_{ij})_i(\underset{\sim}{x})_j)=$$

$$=(\underset{\sim}{a}_{ji})_j+(\underset{\sim}{Q}_{ji})_j(\underset{\sim}{a}_{ij})_i+(\underset{\sim}{Q}_{ji})_j(\underset{\sim}{Q}_{ij})_i(\underset{\sim}{x})_j \quad (4.2.12)$$

Substituting eq. (4.2.12) into eq. (4.2.10) one obtains

$$(\underset{\sim}{a}_{ji})_j+(\underset{\sim}{Q}_{ji})_j(\underset{\sim}{a}_{ij})_i=\underset{\sim}{0} \quad (4.2.13a)$$

and

$$(\underset{\sim}{Q}_{ji})_j(\underset{\sim}{Q}_{ij})_i=\underset{\sim}{I} \quad (4.2.13b)$$

Hence

$$(\mathbf{Q}_{ji})_j=(\mathbf{Q}_{ij})_i^T \tag{4.2.14a}$$

and

$$(\mathbf{a}_{ji})_j=-(\mathbf{Q}_{ij})_i^T(\mathbf{a}_{ij})_i \tag{4.2.14b}$$

Next, a general composition law for n transformations is derived. Assuming that expressions similar to (4.2.7) and (4.2.8) hold for k transformations, it will be shown that they hold also for k+1, thereby obtaining general relationships by induction. Thus,

$$(\mathbf{a}_{1k})_1=(\mathbf{a}_{12})_1+(\mathbf{a}_{23})_1+\ldots+(\mathbf{a}_{k-1,k})_1 \tag{4.2.15}$$

$$(\mathbf{Q}_{1k})_1=(\mathbf{Q}_{12})_1(\mathbf{Q}_{23})_2\cdots(\mathbf{Q}_{k-1,k})_{k-1} \tag{4.2.16}$$

Then,

$$(\mathbf{a}_{1,k+1})_1=(\mathbf{a}_{1k})_1+(\mathbf{a}_{k,k+1})=(\mathbf{a}_{12})_1+(\mathbf{a}_{23})_1+\ldots+(\mathbf{a}_{k-1,k})_1+(\mathbf{a}_{k,k+1})_1$$

and

$$(\mathbf{Q}_{1,k+1})_1=(\mathbf{Q}_{k,k+1})_1(\mathbf{Q}_{1k})_1$$

Introducing a similary transformation to refer $\mathbf{Q}_{k,k+1}$ to k-coordinates,

$$(\mathbf{Q}_{1,k+1})_1=(\mathbf{Q}_{1k})_1(\mathbf{Q}_{k,k+1})_k(\mathbf{Q}_{1k})_1^T(\mathbf{Q}_{1k})_1=(\mathbf{Q}_{1k})_1(\mathbf{Q}_{k,k+1})_k=$$

$$=(\mathbf{Q}_{12})_1(\mathbf{Q}_{23})_2\cdots(\mathbf{Q}_{k-1,k})_{k-1}(\mathbf{Q}_{k,k+1})_k$$

Thus, in general

$$(\mathbf{a}_{1n})_1=\sum_{1}^{n-1}(\mathbf{a}_{i,i+1})_1 \tag{4.2.17}$$

and

$$(\mathbf{Q}_{1n})_1=(\mathbf{Q}_{12})_1(\mathbf{Q}_{23})_2\cdots(\mathbf{Q}_{n,n-1})_n \tag{4.2.18}$$

which are useful relationships because they enable the analyst, first, to compute the final rotation, from 1 to n, referred to coordinate system 1, in term of <u>successive</u> rotations, from i to i+1, <u>referred to coordinate system i</u>.

In general, however, $a_{i,i+1}$ will be more readily referred to coordinate system i, but the transformation to system 1 is easily performed as

$$(a_{i,i+1})_1=(Q_{1i})\ (a_{i,i+1})_i$$

and $[Q_{1i}]_1$ can be obtained from (4.2.18) with n=i. The method of Denavit and Hartenberg (MDH) is based on the closure equation

$$T_n \cdots T_{23}T_{12}=T_{11} \qquad (4.2.19)$$

or, equivalently,

$$(a_{12})_1+(a_{23})_1+\dots+(a_{n-1,n})_1+(a_{n1})_1=0 \qquad (4.2.20)$$

together with

$$(Q_{12})_1(Q_{23})_2\cdots(Q_{n-1,n})_{n-1}(Q_{n1})_n=I \qquad (4.2.21)$$

where, according to the relationship (4.2.14a),

$$(Q_{n1})_n=(Q_{1n})_n^T \qquad (4.2.22)$$

Next, the MDH is applied to a linkage composed of n links (rigid bodies) coupled by any of the six lower pairs (R,P,H,C,S or E) introduced in Chapter 3. Let the axis of the pair be defined as

i) the axis of rotation, if the pair is R,

ii) the direction of traslation, if the pair is P,

iii) the axis of rotation, which is identical to the direction of traslation, if the pair is H or C.

iv) the direction of the normal to the plane of contact if the pair is E.

Only if the pair is spheric no single axis can be defined. In this case, there is freedom to choose the axis of the pair and so, the analyst can choose it to his best convenience.

To implement the MDH, number the links successively 1,2,...,n. Then

a) Let Z_1 be the axis of the pair coupling the first link (the fixed one) with the second one (the driving or input link), in such a way that the variable denoting the input be positive along Z_1.

b) In general, let Z_i be the axis of the pair connecting links i and i+1.

c) Let X_i be the common perpendicular to Z_{i-1} and Z_i, directed from Z_{i-1} to Z_i

d) Let a_i be the distance between Z_i and Z_{i+1}, always positive.

e) Let s_i be the coordinate of the intersection of axis X_{i-1} with axis Z_i, in frame X_i-Y_i-Z_i. Since it is a coordinate, its sign can be plus or minus, depending on the position of the said intersection. The absolute value of s_i is the distance between X_i and X_{i+1}.

f) Let α_i be the angle between Z_i and Z_{i+1}, measured along the positive direction of X_{i+1}

g) Let θ_i be the angle between X_i and X_{i+1}, measured along the positive direction of Z_i.

h) Construct the translation vectors $(\underline{a}_{i,i+1})_i$ and the rotation matrices $(\underline{Q}_{i,i+1})_i$, as is described next.

i) Apply the closure conditions (4.2.20) and (4.2.21) and from them obtain the sought input-output relationship.

In order to construct the translation vectors $(\underline{a}_{i,i+1})_i$, it is necessary first to construct the rotation matrices $(\underline{Q}_{i,i+1})_i$, which is done next. The relationship between coordinate systems i and i+1 is shown in Fig 4.2.1, in agreement with the notation of f) and g).

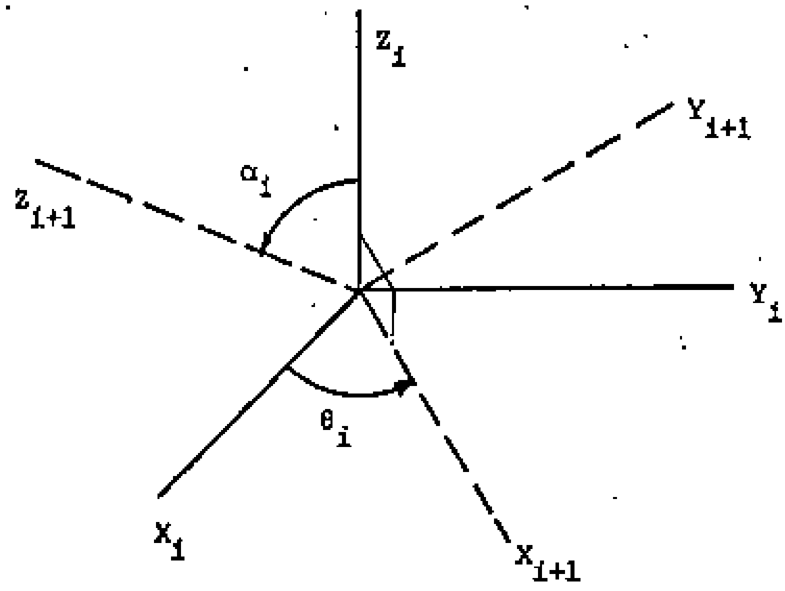

Fig 4.2.2. Relative position of coordinate systems i and i+1

Since X_{i+1} is perpendicular to Z_i (by definition), X_i and X_{i+1} lie in a plane perpendicular to Z_i. Thus, X_i can be made coincident with X_{i+1} by means of a rotation through an angle θ_i about Z_i, as shown in Fig 4.2.3, where X_i', Y_i', Z_i' are the original axes X_i, Y_i, Z_i after the said rotation, so that Z_i' coincides with Z_i.

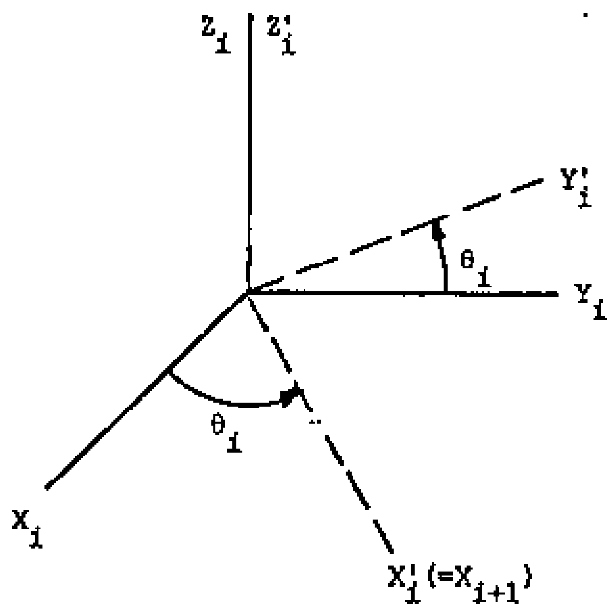

Fig 4.2.3 Rotation through an angle θ_i about axis Z_i.

Hence, from Section 2.3,

$$(\underset{\sim}{Q}_{ii'})_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (4.2.23)$$

Next notice that X_{i+1} (i.e. X_i') is perpendicular to Y_i', Y_{i+1}', Z_i (i.e. Z_i') and Z_{i+1}. Hence, Y_i' and Z_i' can be made coincident with Y_{i+1} and Z_{i+1} by means of a rotation through an angle α_i about X_i'. The relative position of axes i' and i+1 is shown in Fig 4.2.4.

From Fig 4.2.4 and section 2.3,

$$(\underset{\sim}{Q}_{i',i+1})_{i'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix} \qquad (4.2.24)$$

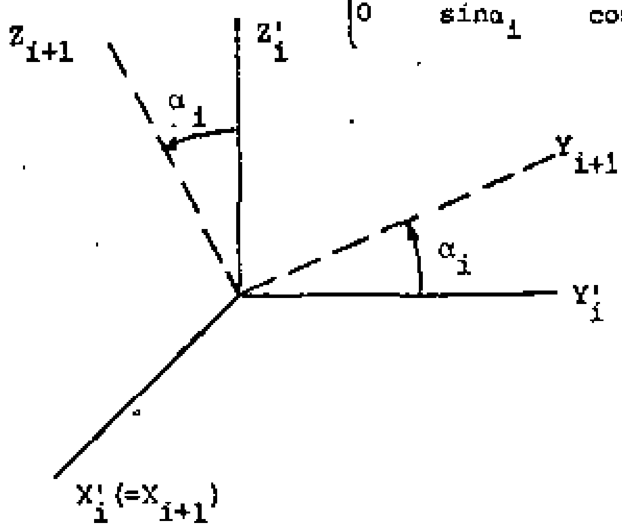

Fig 4.2.4 Relative position of axes i' and axes i+1

Finally, from eq. (4.2.18),

$$(\underset{\sim}{Q}_{i,i+1})_i = (\underset{\sim}{Q}_{i,i'})_i (\underset{\sim}{Q}_{i',i+1})_{i'} \qquad (4.2.25)$$

the desired matrix is obtained as

$$\left(\underset{\sim}{Q}_{i,i+1}\right)_i = \begin{pmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{pmatrix} \quad (4.2.26)$$

Now it is possible to construct vectors $\left(\underset{\sim}{a}_{i,i+1}\right)_i$. The relative configuration of three successive links appears in Fig 4.2.5, where the notation of a) to g) has been followed.

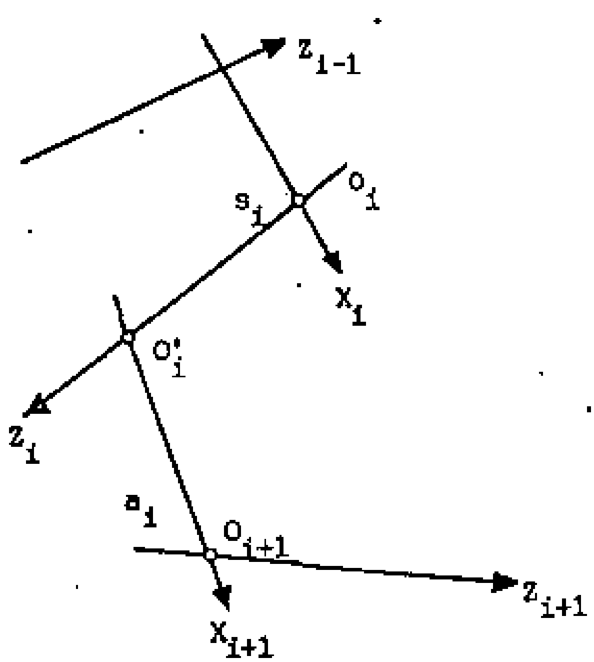

Fig 4.2.5 Three successive links of a linkage

From Fig 4.2.5,

$$\left(\underset{\sim}{a}_{i,i+1}\right)_i = \left(\overrightarrow{O_iO_{i'}}\right)_i + \left(\underset{\sim}{Q}_{i,i+1}\right)_i \left(\overrightarrow{O_{i'}O_{i+1}}\right)_{i+1} \quad (4.2.27)$$

with

$$\left(\overrightarrow{O_iO_{i'}}\right)_i = \left(0,0,s_i\right)^T \quad (4.2.28)$$

and

$$\left(\overrightarrow{O_i'O_{i+1}}\right)_{i+1} = \left(a_i,0,0\right)^T \quad (4.2.29)$$

$(\underset{\sim}{Q}_{i,i+1})_i$ being as given by eq. (4.2.26).

Substituting (4.2.26), (4.2.28) and (4.2.29) into (4.2.27),

$$(\underset{\sim}{a}_{i,i+1})_i = (a_i \cos\theta_i, a_i \sin\theta_i, s_i)^T \qquad (4.2.30)$$

Expressions (4.2.26) and (4.2.30) enable the analyst to systematically construct the affine transformations required to establish the closure condition (4.2.19).

As is shown in Examples 4.2.1 and 4.2.2, it is not always necessary to apply both closure conditions (4.2.20) and (4.2.21) arising from (4.2.19), since only one suffices.

Example 4.2.1. Analysis of the universal joint. The layout of a universal (or Hooke's) joint appears in Fig 4.2.6, where the DH notation has been used. The universal joint is a special class of RRRR spherical linkages. Obtain an input-output relationship of the form $f(\theta_1, \theta_4) = 0$.

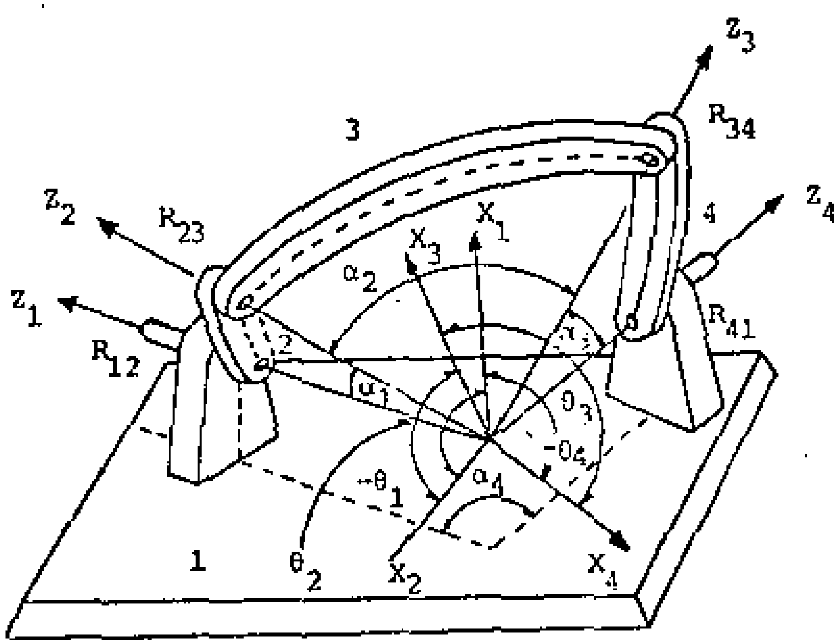

Fig 4.2.6 Universal Joint

Since all coordinate axes involved in this linkage have a common origin, the closure condition on the translation vectors is irrelevant and only that on the rotation matrices will be employed. The rotation matrices appearing in the present analysis are constructed from the fact that, in this case,

$$\alpha_1=\alpha_2=\alpha_3=90°$$

Constructing the rotation matrices according to eqs. (4.2.26) and (4.2.30), one obtains

$$(\underset{\sim}{Q}_{14})_1=(\underset{\sim}{Q}_{12})_1(\underset{\sim}{Q}_{23})_2(\underset{\sim}{Q}_{34})_3$$

Thus,

$$(\underset{\sim}{Q}_{12})_1(\underset{\sim}{Q}_{23})_2(\underset{\sim}{Q}_{34})_3=\begin{bmatrix} c\theta_1c\theta_2c\theta_3+s\theta_1s\theta_3 & c\theta_1s\theta_2 & c\theta_1c\theta_2s\theta_3-s\theta_1c\theta_3 \\ s\theta_1c\theta_2c\theta_3-c\theta_1s\theta_3 & s\theta_1s\theta_2 & s\theta_1c\theta_2s\theta_3+c\theta_1c\theta_3 \\ s\theta_2c\theta_3 & -c\theta_2 & s\theta_2s\theta_3 \end{bmatrix} \quad (4.2.31)$$

On the other hand, from (4.2.26) and noticing that, for i=n, i+1=1,

$$(\underset{\sim}{Q}_{41})_4=\begin{bmatrix} c\theta_4 & -s\theta_4c\alpha_4 & s\theta_4s\alpha_4 \\ s\theta_4 & c\theta_4c\alpha_4 & -c\theta_4s\alpha_4 \\ 0 & s\alpha_4 & c\alpha_4 \end{bmatrix}$$

But, from (4.2.14a),

$$(\underset{\sim}{Q}_{14})_1=(\underset{\sim}{Q}_{41})_4^T$$

Hence,

$$(\underset{\sim}{Q}_{14})_1=\begin{bmatrix} c\theta_4 & s\theta_4 & 0 \\ -s\theta_4c\alpha_4 & c\theta_4c\alpha_4 & s\alpha_4 \\ s\theta_4s\alpha_4 & -c\theta_4s\alpha_4 & c\alpha_4 \end{bmatrix} \quad (4.2.32)$$

Equating the (1,2)-and the (2,2)-entries of both forms of $(Q_{14})_1$ -eqs. (4.2.31) and (4.2.32) - one obtains

$$\cos\theta_1 \sin\theta_2 = \sin\theta_4 \tag{4.2.33a}$$

$$\sin\theta_1 \sin\theta_2 = \cos\theta_4 \cos\alpha_4 \tag{4.2.33b}$$

Eliminating θ_2 in the above expressions,

$$\tan\theta_1 = \cos\alpha_4 \cot\theta_4 \tag{4.2.34}$$

which is the input-output relationship meant to be obtained.

<u>Example 4.2.2. Analysis of an RSRC linkage</u>. A typical RSCR linkage is shown in Fig 4.2.7, where the DH notation has also been used. The parameters and variables have the values

$$\alpha_1 = \alpha_1 \quad \alpha_2 = \alpha_2 \quad \alpha_3 = -\frac{\pi}{2} \quad \alpha_4 = \frac{\pi}{2}$$

$$\theta_1 = -\phi \quad \theta_2 = \theta_2 \quad \theta_3 = \theta_3 \quad \theta_4 = \theta_4$$

$$a_1 = a \quad a_2 = b \quad a_3 = 0 \quad a_4 = 0$$

$$s_1 = c \quad s_2 = 0 \quad s_3 = 0 \quad s_4 = -s$$

For the analysis of this linkage, only the closure condition of the translation vectors will be needed. Since these vectors, as given by eq. (4.2.30), have to be expressed in one single coordinate system, the rotation matrices will also be constructed. Thus,

$$(Q_{12})_1 = \begin{pmatrix} c\phi & s\phi c\alpha_1 & -s\phi s\alpha_1 \\ -s\phi & c\phi c\alpha_1 & -c\phi s\alpha_1 \\ 0 & s\alpha_1 & c\alpha_1 \end{pmatrix} \tag{4.2.35a}$$

$$(Q_{23})_2 = \begin{pmatrix} c\theta_2 & -s\theta_2 c\alpha_2 & s\theta_2 s\alpha_2 \\ s\theta_2 & c\theta_2 c\alpha_2 & -c\theta_2 s\alpha_2 \\ 0 & s\alpha_2 & c\alpha_2 \end{pmatrix} \tag{4.2.35b}$$

$$(Q_{34})_3 = \begin{pmatrix} c\theta_3 & 0 & -s\theta_3 \\ s\theta_3 & 0 & c\theta_3 \\ 0 & -1 & 0 \end{pmatrix} \quad (4.2.35c)$$

$$(Q_{41})_4 = \begin{pmatrix} c\theta_4 & 0 & s\theta_4 \\ s\theta_4 & 0 & -c\theta_4 \\ 0 & 1 & 0 \end{pmatrix} \quad (4.2.35d)$$

and

$$(a_{12})_1 = (ac\phi, -as\phi, +c)^T \quad (4.2.36a)$$

$$(a_{23})_2 = (bc\theta_2, bs\theta_2, 0)^T \quad (4.2.36b)$$

$$(a_{34})_3 = (0, 0, 0)^T \quad (4.2.36c)$$

$$(a_{41})_4 = (0, 0, -s)^T \quad (4.2.36d)$$

The closure condition on the translation vectors requires that vectors (4.2.36) be all expressed in terms of the X_1, Y_1, Z_1 coordinate system, to yield

$$(a_{12})_1 + (a_{23})_1 + (a_{34})_1 + (a_{41})_1 = (0)_1 \quad (4.2.37)$$

Performing the corresponding trasformations,

$$(a_{23})_1 = (Q_{12})_1 (a_{23})_2 = \begin{pmatrix} b(c\phi c\theta_2 + s\phi s\theta_2 c\alpha_1) \\ b(-s\phi c\theta_2 + c\phi s\theta_2 c\alpha_1) \\ bs\theta_2 s\alpha_1 \end{pmatrix} \quad (4.2.38a)$$

Since $(a_{34})_3$ vanishes,

$$(a_{34})_1 = (0) \quad (4.2.38b)$$

$$(a_{41})_1 = (Q_{14})_1 (a_{41})_4 = \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix} \quad (4.2.38c)$$

Substituting (4.2.36a) and (4.2.38a) - (4.2.38c) into eq. (4.2.37), the following scalar equations are obtained

$$a\cos\phi + b(\cos\phi\cos\theta_2 + \sin\phi\cos\alpha_1\sin\theta_2) = 0 \quad (4.2.39a)$$

$$-a\sin\phi+b(-\sin\phi\cos\theta_2+\cos\phi\cos\alpha_1\sin\theta_2)-s=0 \qquad (4.2.39b)$$

$$c+b\sin\alpha_1\sin\theta_2=0 \qquad (4.2.39c)$$

which, for every value of the input angle $\phi(t)$, yields a nonlinear algebraic system for the corresponding three unknowns $\alpha_1(t)$, $\theta_2(t)$ and $s(t)$.

One method to solve the foregoing system is via the Newton-Raphson method, as shown in Section 1.13, by application of the NRDAMP subroutine. However, in this particular case it is not necessary to spend so much computer time for, by introducing suitable trigonometric identities, the system (4.2.39) can be reduced to

$$s(t) = a\sin\phi + \sqrt{b^2 - c^2 - a^2\cos^2\phi} \qquad (4.2.40)$$

which explicity provides the set of values of s for each value of ϕ.

A table of values for s(t) was obtained from a digital computer output, for the following values of mechanism parameters:

$a = 1.00$ m, $b = 3.00$ m, $c = 2.00$ m

for a value of $\dot{\phi}$ constantly equal to 1500 rpm. Similar tables for velocity and acceleration values were obtained via differentiation of s(t) by application of second central differences. Curves appearing in Fig 4.2.8 were obtained from the said tables.

Notice that eq. (4.2.40) could have also been obtained from the geometry of the linkage of Fig 4.2.7, due to the simplicity of the linkage.

In more general instances, however, the geometry is not so simple and the MDH becomes essential to perform the analysis.

In addition to the digital-computer method previously outlined, to obtain the output $s(t)$, $\dot{s}(t)$ and $\ddot{s}(t)$ out of eq. (4.2.40), analog computer methods can also be applied.

An analog computer is a (usually electrical) physical system whose behaviour is governed by the same mathematical model governing the system

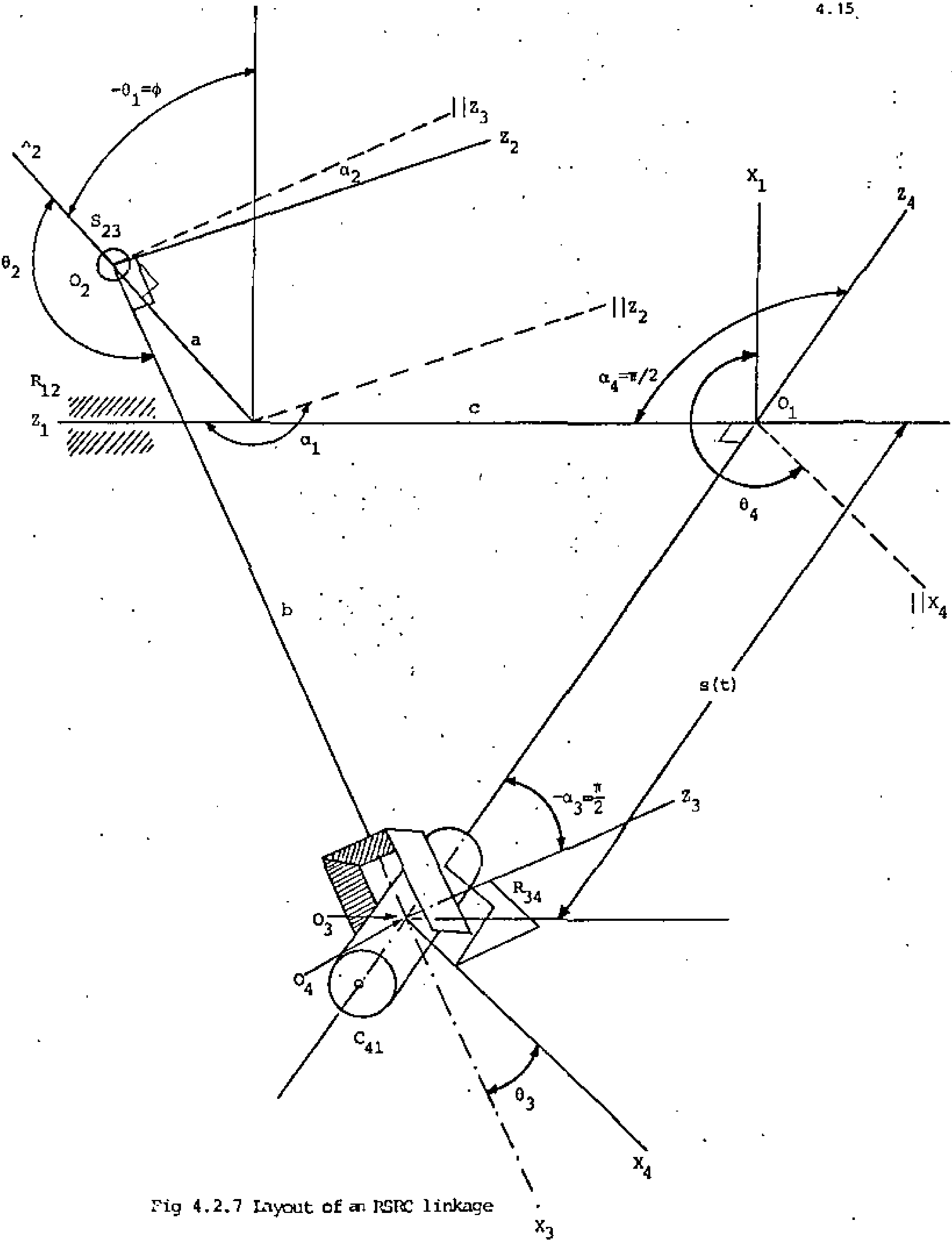

Fig 4.2.7 Layout of an RSRC linkage

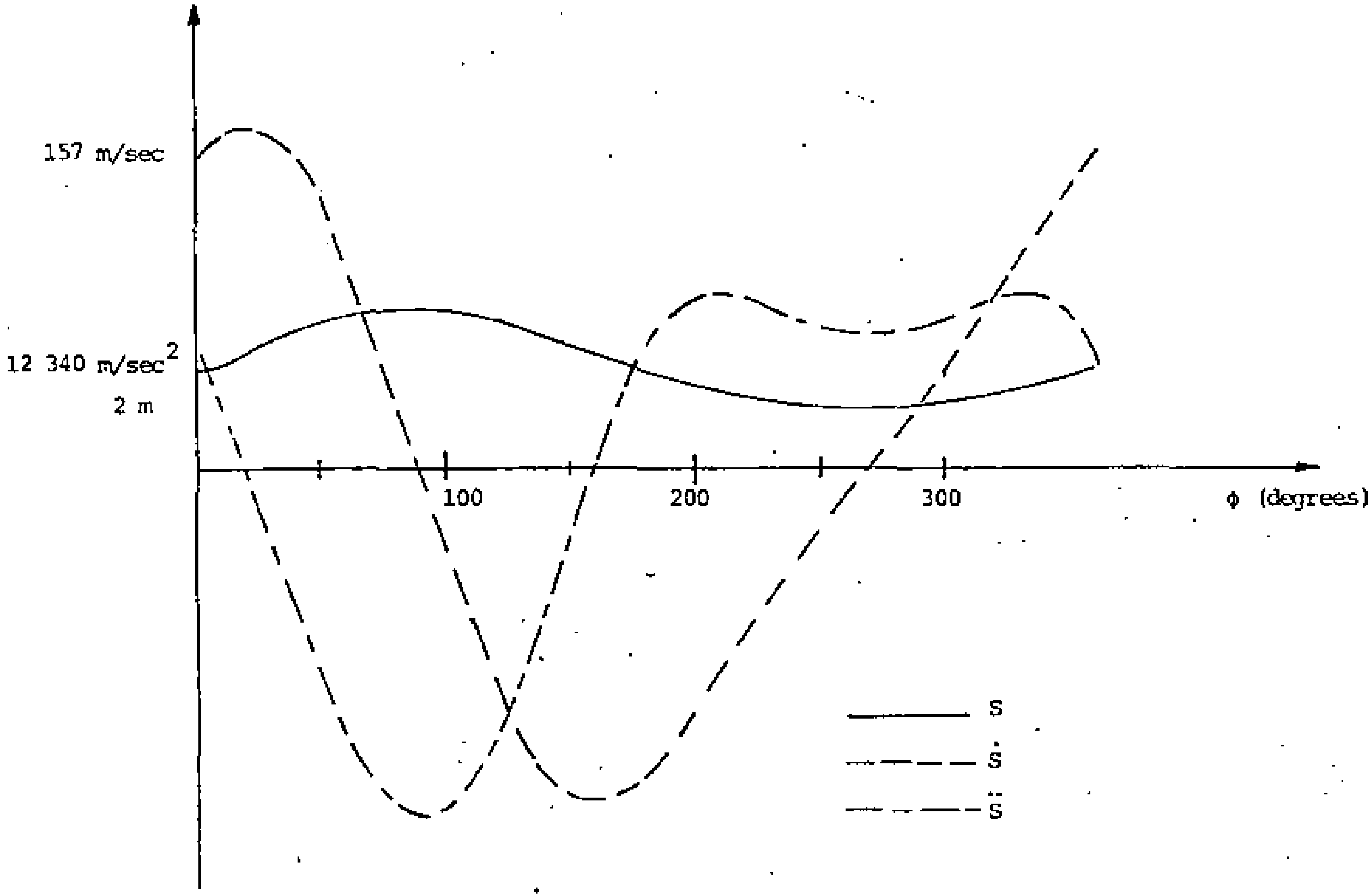

Fig 4.2.8 Displacement, velocity and acceleration curves of an RSRC linkage.

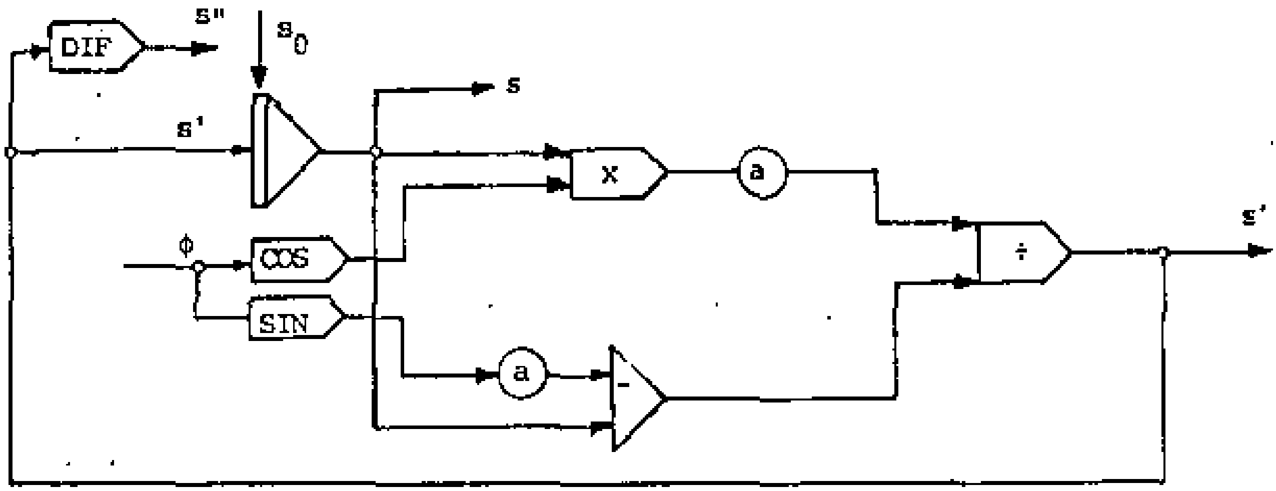

Fig 4.2.10 Analog realization of eq. (4.2.41)

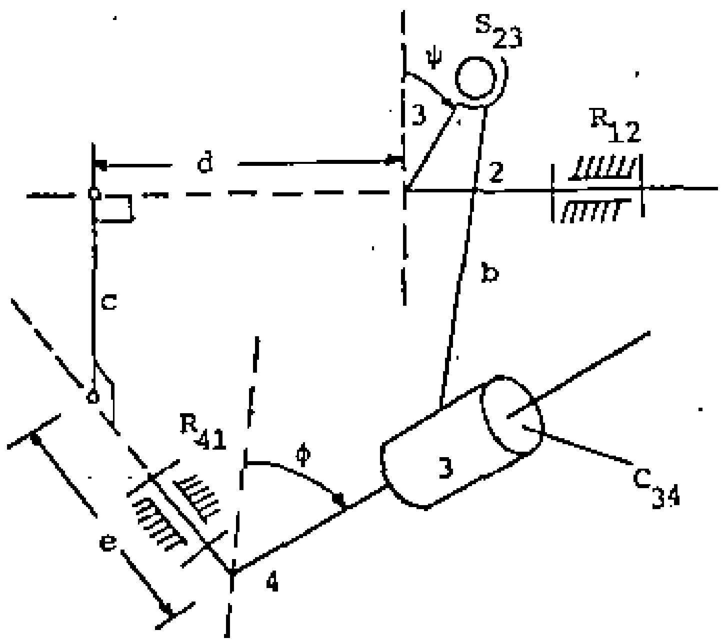

Fig 4.2.11 An RSCR linkage

under analysis. The elements constituting an analog computer perform the usual mathematical operations appearing in mathematical models, i.e., algebraic addition, multiplication, division, integration and differentiation. Besides these operations, the analog computer is also supplied with function generators. All these elements are symbolically represented as appearing in Fig 4.2.9. An analog computer representation of a mathematical model is usually called "a realisation" of the model, because via that representation the model is taken into physical reality.

Equation (4.2.40) could readily be realized in an analog computer, and two differentiations would have to be performed to obtain the acceleration $\ddot{s}(t)$. However, due to the noise present in every physical system, and the fact that a differentiator is a noise amplifier, it becomes undesirable to perform more than one differentiation. To avoid the second differentiation to obtain $\ddot{s}(t)$, then, eq. (4.2.40) is first differentiated to obtain

$$s'(\phi)=\frac{ds}{d\phi}=\frac{\dot{s}}{\dot{\phi}}=\frac{as\cos\phi}{s-a\sin\phi} \tag{4.2.41}$$

which has to be integrated once, with initial conditions $s=s_0$ at $\phi=\phi_0$, and differentiated only once to obtain $\ddot{s}(t)$. The analog realization of eq. (4.2.41) appears in Fig 4.2.10 ■

Details about analog simulation of linkages can be found in (4.6). Concerning simulation of mechanical systemsin general, the reader can see (4.7).

A digital computer-oriented algorithm, based on an iterative procedure, to obtain the history of all variables of a linkage is presented in (4.8).

Exercise 4.2.10 Given the RSCR linkage of Fig 4.2.11, obtain an input-output relationship $f(\psi,\phi)$, an analog realization yielding $\phi,\dot{\phi}$ and $\ddot{\phi}$ and curves ϕ vs. t, $\dot{\phi}$ vs. t and $\ddot{\phi}$ vs. t, for the following values:

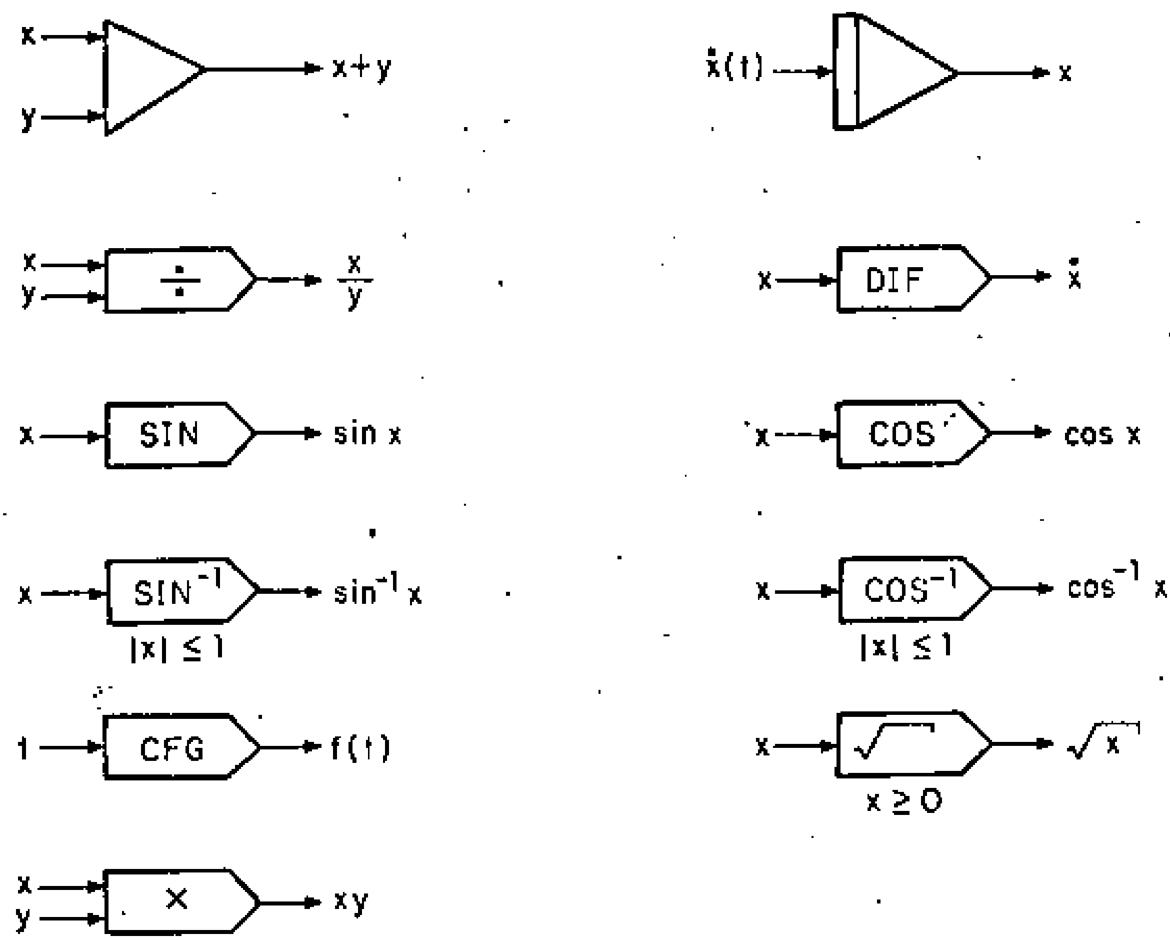

Fig 4.2.9. Elements for analog realisations

|a=0.5m, b=1.0m, c=0.25m, d=0.75m, e=0.5m, $\dot{\phi}$=1200rpm (const)

4.3 AN ALTERNATE METHOD OF ANALYSIS. Using the MDH one does not necessarily obtain one single relationship for the input and the output variables, but usually a system of nonlinear algebraic equations involving all the different linkage variables which appear strongly coupled, as occurs with eqs. (4.2.39). If the system is not very complicated, then it can happen that, after introducing appropriate trigonometric identities, one can obtain a single input-output relationship where the only variables appearing are the input and the output. If it is not obvious how to obtain the said single relationship, then one is forced to solve a system, instead of one single nonlinear equation. Gupta (4.10) has presented a method which does not require all the apparatus of the MDH and yields a single relationship between the input and the output. The method is based on an equation establishing the constancy of either a link length or a link angle, throughout the linkage motion. The linkage is assumed to have a single-loop which, for simplicity, is assumed to be composed of just four links*: the fixed link, the input link, the coupler link and the output link. Let A be a point on the input axis; B, a point of the pair connecting the input and the coupler links; C, a point of the pair connecting the coupler and the output links; and D, a point of the output axis. Furthermore, let $\underset{\sim}{a}_0$, $\underset{\sim}{b}_0$, $\underset{\sim}{c}_0$, $\underset{\sim}{d}_0$ and $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$, denote the position vectors of the corresponding points in both, a reference configuration and a current configuration, respectively.

Let also $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ be the rotation matrices carrying the input and the output links from the reference configuration to the current configuration. Then, from the lenght constancy of the coupler link,

*This linkage could be RSSR, RSCR or its inversions. Spherical linkages are dealt with next.

$$||\underset{\sim}{b} - \underset{\sim}{c}||^2 = ||\underset{\sim}{b}_0 - \underset{\sim}{c}_0||^2 \qquad (4.3.1)$$

where, clearly

$$\underset{\sim}{b} = \underset{\sim}{a}_0 + \underset{\sim}{Q}(\underset{\sim}{b}_0 - \underset{\sim}{a}_0) \qquad (4.3.2a)$$

and

$$\underset{\sim}{c} = \underset{\sim}{d}_0 + \underset{\sim}{R}(\underset{\sim}{c}_0 - \underset{\sim}{d}_0) \qquad (4.3.2b)$$

Hence,

$$||\underset{\sim}{b}-\underset{\sim}{c}||^2 = ||\underset{\sim}{a}_0+\underset{\sim}{Q}(\underset{\sim}{b}_0-\underset{\sim}{a}_0) - \underset{\sim}{d}_0 - \underset{\sim}{R}(\underset{\sim}{c}_0 - \underset{\sim}{d}_0)||^2$$

which yiels the desired input-output scalar equation which was to be obtained, after substitution into eq. (4.3.1), namely

$$||\underset{\sim}{a}_0 + \underset{\sim}{Q}(\underset{\sim}{b}_0 - \underset{\sim}{a}_0) - \underset{\sim}{d}_0 - \underset{\sim}{R}(\underset{\sim}{c}_0 - \underset{\sim}{d}_0)||^2 = ||\underset{\sim}{b}_0 - \underset{\sim}{c}_0||^2 \qquad (4.3.3)$$

Equation (4.3.3) contains only two unknowns, the input and the output angles, thereby showing how one single scalar equation for these two variables can be obtained. The next example illustrates how to apply this method to spherical linkages.

Example 4.3.1 Analysis of the general RRRR spherical linkage.

The MDH can be applied, of course, to general spherical linkages, following the same procedure applied to the universal joint.

The fact that the MDH introduces other variables besides the input and the output, however, produces very cumbersome equations, algebraically difficult to handle. The alternate method proves, in this case, to be particulary helpful. Consider the RRRR spherical linkage appearing in Fig 4.3.1

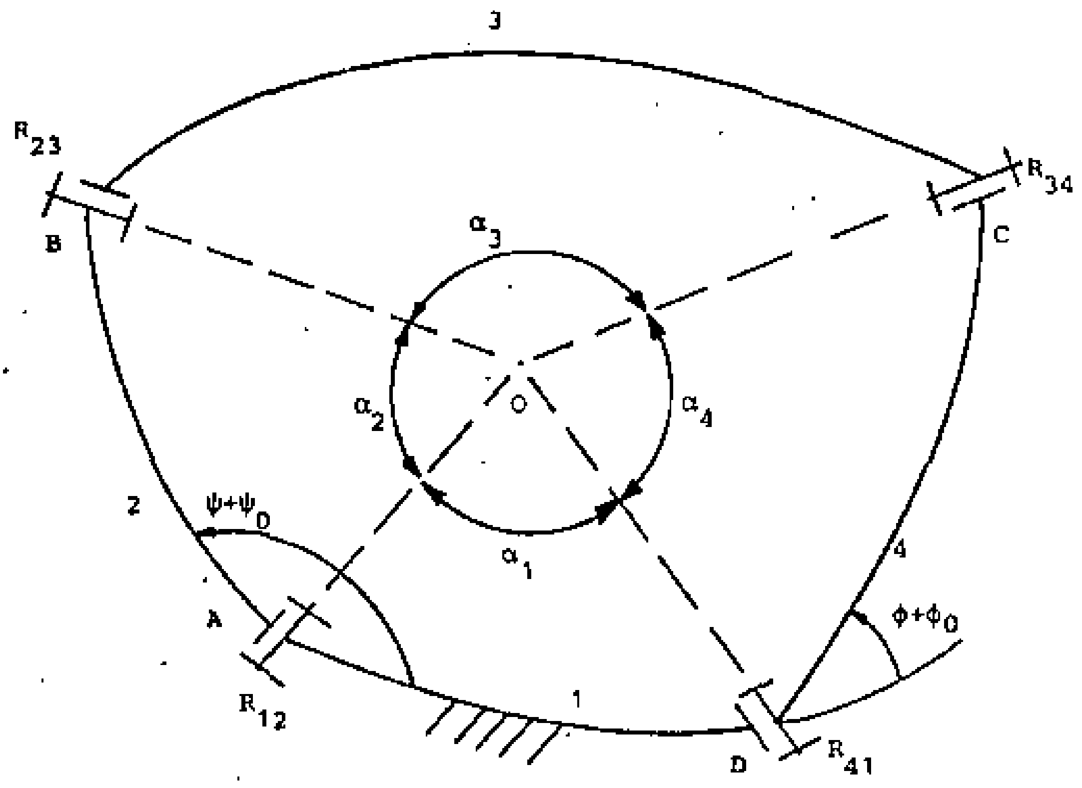

Fig 4.3.1 General RRRR spherical linkage

Let $\underset{\sim}{a}_0$, $\underset{\sim}{b}_0$, $\underset{\sim}{c}_0$, $\underset{\sim}{d}_0$ and $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c},\underset{\sim}{d}$ be the position vectors of points A, B, C and D in a reference configuration and in a time-varying configuration, respectively. Furthermore, let $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ be the rotation matrices carruing links 2 and 4, respectively, from the reference to the current configurations. Thus,

$$\underset{\sim}{b}=\underset{\sim}{Q}\underset{\sim}{b}_0 \qquad (4.3.4)$$

and

$$\underset{\sim}{c}=\underset{\sim}{R}\underset{\sim}{c}_0 \qquad (4.3.5)$$

Clearly, $\underset{\sim}{a}=\underset{\sim}{a}_0$ and $\underset{\sim}{d}=\underset{\sim}{d}_0$. The cosine of α_3 in both the reference and the current configurations is

$$(\cos\alpha_3)_{ref} = \underset{\sim}{b}_0^T\underset{\sim}{c}_0 \qquad (4.3.6a)$$

and

$$(\cos\alpha_3)_{cur} = \underset{\sim}{b}^T\underset{\sim}{c} \qquad (4.3.6b)$$

where $||b_0||$ and $||c_0||$ are assumed unity, without loss of generality. Since link 3 is rigid, α_3 remains constant throughout the linkage motion and, since $||\underset{\sim}{b}|| = ||\underset{\sim}{b}_0||$ and $||\underset{\sim}{c}|| = ||\underset{\sim}{c}_0||$, one obtains

$$\underset{\sim}{b}^T\underset{\sim}{c} = \underset{\sim}{b}_0^T\underset{\sim}{c}_0 \qquad (4.3.7)$$

or, substituting the relations (4.3.4) and (4.3.5) in the above equation,

$$\underset{\sim}{b}_0^T\underset{\sim}{Q}^T\underset{\sim}{R}\underset{\sim}{c}_0 = \underset{\sim}{b}_0^T\underset{\sim}{c}_0 = \cos\alpha_3 \qquad (4.3.8)$$

which is the scalar input-output relationship meant to be obtained. The only variables appearing in eq. (4.3.8) are ψ, contained in $\underset{\sim}{Q}$, and ϕ, contained in $\underset{\sim}{R}$. Define the coordinate axes appearing in Fig 4.3.2, with X_i and X_o directed along the axes of R_{12} and R_{41}, respectively.

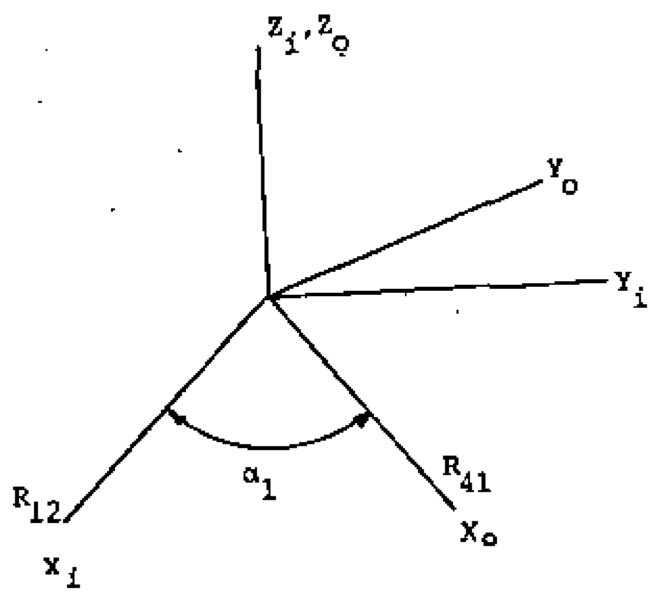

Fig 4.3.2 Fixed coordinate axes containing the axes of R_{12} and R_{41}

Matrices $\underset{\sim}{Q}$ and $\underset{\sim}{R}$, referred to i- and o- axes respectively, are given as

$$(\underset{\sim}{Q})_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}, \quad (\underset{\sim}{R})_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \qquad (4.3.9)$$

Vectors $\underset{\sim}{b}_0$ and $\underset{\sim}{c}_0$ are shown in Figs 4.3.3a and 4.3.3b

Hence,

$$\left(\underset{\sim}{b}_0\right)_i = \begin{pmatrix} \cos\alpha_2 \\ \sin\alpha_2\cos\psi_0 \\ \sin\alpha_2\sin\psi_0 \end{pmatrix} , \quad \left(\underset{\sim}{c}_0\right)_o = \begin{pmatrix} \cos\alpha_4 \\ \sin\alpha_4\cos\phi_0 \\ \sin\alpha_4\sin\phi_0 \end{pmatrix} \qquad (4.3.10)$$

In order to perform the products appearing in eq. (4.3.8) it is necessary to express all vectors and matrices in the same coordinate axes.

The transformation matrix carrying the i-axis into the o-axes, referred to the i-axes, is given as

$$\left(\underset{\sim}{S}\right)_i = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4.3.11)$$

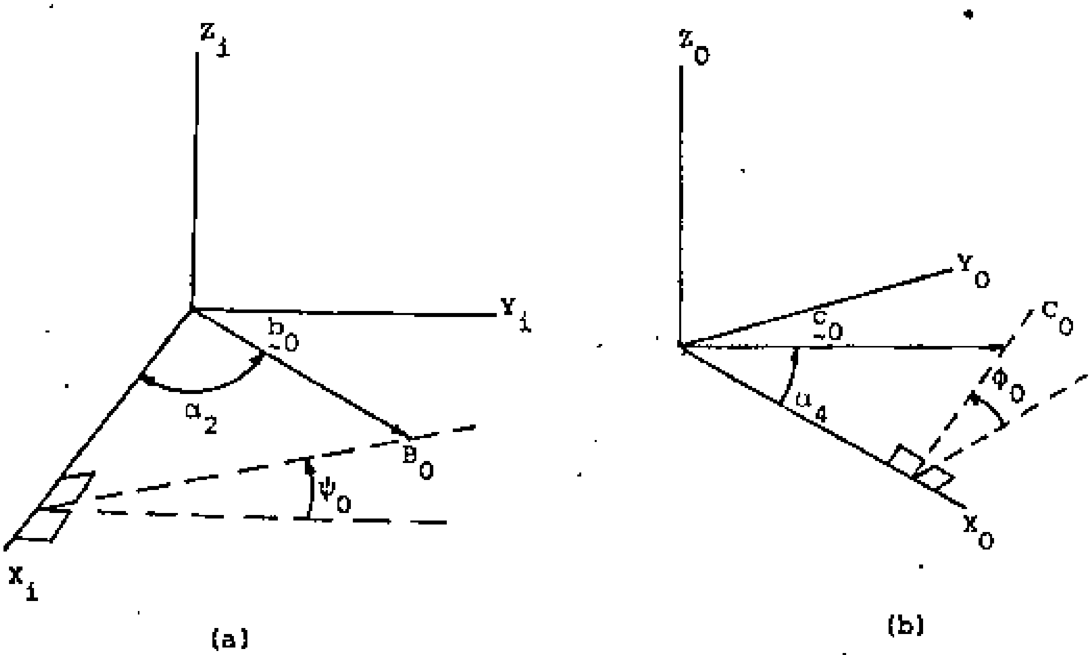

Fig 4.3.3 Reference configuration of points B and C.

The product $\underset{\sim}{b}_0^T \underset{\sim}{Q}^T \underset{\sim}{R}\, \underset{\sim}{c}_0$ needed in eq. (4.3.8) is next computed

$$\underset{\sim}{b}_0^T \underset{\sim}{Q}^T \underset{\sim}{R}\underset{\sim}{c}_0 = (\underset{\sim}{Q}\underset{\sim}{b}_0)_i^T \; (\underset{\sim}{R}\underset{\sim}{c}_0)_i = (\underset{\sim}{Q}\underset{\sim}{b}_0)_i^T \; (\underset{\sim}{S})_i (\underset{\sim}{R}\underset{\sim}{c}_0)_o \qquad (4.3.12)$$

which yiedls

$$\underset{\sim}{b}^T \underset{\sim}{Q}^T \underset{\sim}{R}\underset{\sim}{c}_0 = c\alpha_2\left(c\alpha_1 c\alpha_4 - s\alpha_1 s\alpha_4 c(\phi+\phi_0)\right) + s\alpha_2 c(\psi+\psi_0)\left(s\alpha_1 c\alpha_4 + c\alpha_1 s\alpha_4 c(\phi+\phi_0)\right) +$$
$$+ s\alpha_2 s\alpha_4 s(\psi+\psi_0) s(\phi+\phi_0) \qquad (4.3.13)$$

When eq. (4.3.13) is substituted into eq. (4.3.8) one obtaines the desired input-output equation

$$c\alpha_2\left(c\alpha_1 c\alpha_4 - s\alpha_1 s\alpha_4 c(\phi+\phi_0)\right) + s\alpha_2 c(\psi+\psi_0)\left(s\alpha_1 c\alpha_4 + c\alpha_1 s\alpha_4 c(\phi+\phi_0)\right) +$$
$$+ s\alpha_2 s\alpha_4 s(\psi+\psi_0) s(\phi+\phi_0) - c\alpha_3 = 0 \qquad (4.3.13a)$$

in which ψ and ϕ are measured from the reference values ψ_0 and ϕ_0, respectively, as defined previously. If the said angles are measured from the plane of the axes of R_{12} and R_{41}, instead, then the latter equation becomes

$$c\alpha_2(c\alpha_1 c\alpha_4 - s\alpha_1 s\alpha_4 c\phi) + s\alpha_2 c\psi(s\alpha_1 c\alpha_4 + c\alpha_1 s\alpha_4 c\phi) + s\alpha_2 s\alpha_4 s\psi s\phi - c\alpha_3 = 0 \qquad (4.3.14)$$

The input-output equation for the universal joint can now be obtained as a special case of eq. (4.3.14), letting $\alpha_2 = \alpha_3 = \alpha_4 = 90°$, thus obtaining

$$c\alpha_1 c\psi c\phi + s\psi s\phi = 0 \qquad (4.3.15)$$

or

$$\frac{c\alpha_1}{\tan\phi} + \tan\psi = 0$$

which is equivalent to eq. (4.2.34) previously obtained. In fact, α_4 of eq. (4.2.34) corresponds to α_1 in eq. (4.3.15) and

$$\psi = 180° - \theta_1, \phi = \theta_4$$

The method of Denavit and Hartenberg, as presented in Section 4.2, is a very well structured systematic procedure for the analysis of "single-loop linkages", i.e. Those whose links are all binary*; but problems arise when "multiple-loop linkages" are to be analyzed. Sheth and Uicker (4.9) have generalized the notation of Denavit and Hartenberg, however, to overcome the aforementioned situation and furthermore, to extend the application of the MDH to the analysis of higher-pair mechanisms.

* If a link is coupled to 2 other links, it is called binary; if it is coupled to 3, it is called ternary, and so on.

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

6. AN INTRODUCTION TO THE OPTIMAL SYNTHESIS OF LINKAGES

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6. AN INTRODUCTION TO THE OPTIMAL SYNTHESIS OF LINKAGES

6.1 INTRODUCTION. The problems of linkage synthesis outlined in Ch. 5 lead in general to nonlinear systems of algebraic equations. As discussed in Section 1.13, these systems can admit multiple solutions or none, the said solutions satisfying all the equations exactly, if roundoff errors are disregarded, which is done in this case since the numerical eccuracy of the solutions is irrelevant in the current discussion. The resulting solutions, however, were not evaluated regarding the "quality" of the linkage they produced, the said quality being previously defined, of course. The quality under consideration could be the overall "size" of the linkage (this size could be in turn defined as the sum of the lengths of all the links, for example), its minimum mechanical advantage or the maximum angular acceleration of one of its links.

The several possibilities that could arise, different to those studied in Ch. 5, are:

i) the number of prescribed conditions exceeds the maximum allowable, thus yielding an overdetermined problem which in general cannot be solved exactly

ii) the number of prescribed conditions is smaller than the maximum allowable, thus making it possible for the designer to select the best linkage (in some sense) from a variety of alternatives.

iii) disregarding the relationship between the number of prescribed conditions and the maximum available, one objective function of the linkage (size, mechanical advantage, etc.) is meant to be optimized.

All these three cases correspond to problems of linkage optimization and will be discussed in this Chapter.

6.2 THE OPTIMIZATION PROBLEM. The most general, hence most abstract optimization problem, can be stated as follows:

"Given a vector space V and a subset Ω of it, find the vector $\underset{\sim}{x}^* \subset \Omega$ such that a (also given) scalar function f defined over V, attains its optimal value at $\underset{\sim}{x}^*$"

The problem thus defined has not been solved*, but solutions have been found to some special cases of it, many of which are applicable to linkage synthesis problems.

To begin with, hte kind of problems that usually arise in the realm of linkage optimization are defined over a vector space V of finite dimension, i.e. the problems that will be handled in this context contain a finite number of independent variables. Problems containing an infinite (even more, a non-denumerable) set of unknowns arise in areas such as optimal contrll systems [6.1] where the vector space V is a Banach space [6.2], i.e. a vector space whose elements are continuous functions of one real variable (time) with some special properties that will not be listed here.

Furthermore, the subset Ω of the finite-dimensional vector spaces V that will be handled in this chapter, is defined by a set of either equality or inequality algebraic relationships (in the case of Banach spaces, the said subset is defined by differential equations). In addition, the objective function to optimize is a functional over V, i.e. a scalar algebraic function (as opposed to integral functional in the case of Banach spaces) of the n independent variables corresponding to the dimension of V. Hence, the optimization problem arising in linkage synthesis can be stated in the general form:

$$\text{optimize} \quad f=f(\underset{\sim}{x})$$

$$\text{subject to} \quad \underset{\sim}{g}(\underset{\sim}{x})=\underset{\sim}{0} \tag{6.2.1}$$

$$\text{and} \quad \underset{\sim}{h}(\underset{\sim}{x})\geq\underset{\sim}{0}$$

where $\underset{\sim}{x}$ is an n-dimensional vector of space V, f is a vector-valued scalar function, $\underset{\sim}{g}$ is an m-dimensional $(m<n)$ vector-valued vector function and $\underset{\sim}{h}$, a p-dimensional vector-valued vector function, and $m+p\geq n$.

Remark: Vector spaces are non-ordered sets [6.3], hence no inequality relation can be associated to them. Therefore, the inequality appearing in (6.2.1) should be understood as a compact symbol to mean

$$h_i(\underset{\sim}{x})\geq 0 \quad i=1,\ldots,p \tag{6.2.2}$$

*in all its generality

h_i being the i^{th} component of vector $\underset{\sim}{h}$. A similar symbology is used to express positive definite and positive semidefinite square matrices $\underset{\sim}{A}$, i.e.

$$\underset{\sim}{A} > \underset{\sim}{0} \text{ (positive definite)}$$

$$\underset{\sim}{A} \geq \underset{\sim}{0} \text{ (positive semidefinite)}$$

An extensive discussion of the formulation of the problem of linkage synthesis as a mathematical programming problem appears in [6.4]. In the abstract of this paper the author states:

"The synthesis of mechanisms thus ceases to be a narrow and isolated scientific field but becomes part of a broad sphere of science encompassing areas obviously very distant such as economy, military science, automation, cybernetics and others".

6.3 <u>OVERDETERMINED PROBLEMS OF LINKAGE SYNTHESIS</u>. It was shown in Section 5.2 that the synthesis of the RSSR linkage for function generation leads to a system of up to eight linear equations in eight unknowns, thus making it possible to satisfy up to eight conditions of the type $\phi_i = \phi_i(\psi_i)$ between the input and the output angles, ψ_i and ϕ_i, respectively*. In Section 5.3 it was shown that a rigid body can be guided through up to three successive configurations by means of an R-R dyad and reference was made to results appearing in the literature [5.11,12,17-19] showing that the said three-position-rigid-body guidance problem, when solvable, has two real solutions, which constitute a Bennett (RRRR-single-degree-of-freedom-spatial) linkage . The resulting synthesis equations were shown to be twelve nonlinear equations in twelve unknowns. Futher on, in Section 5.5 it was shown that, by means of an RRSS linkage, a spatial path can be traced that passes through up to eight points in space, the resulting equations being 39 (according to Suh [5.22]) nonlinear equations in 39 unknowns.

There can arise technical problems, however, that require to synthesize either: i) an RSSR linkage that satisfies a discrete input-output function

*It was also mentioned in that Section that, as Mohan Rao et al. [5.4] claim, it is possible to extend this synthesis problem to 10 precision points if scale factors between the synthesized function and input and output angles are not specified; but this possibility will not be discussed here because the synthesized function is assumed to be defined only at a discrete set of points.

over more than eight points, or ii) an RRRR spatial (Bennett) linkage that carries a rigid body through more than three configurations, or iii) an RRSS linkage that traces a spatial path passing through more than eight points. In all these cases the number of resulting synthesis equations will exceed that of available linkage parameters (unknowns), thus leading to an overdetermined problem. This problem could involve a system of linear equations, as in case i) above, or a system of nonlinear ones, as in the remaining two previously discussed cases.

The overdetermined arising problem does not have a solution in the "usual" sense, i.e. in the sense of satisfying the equations exactly. As was discussed in Section 1.11 for linear systems and in Section 1.13 for nonlinear ones, a solution vector $\underset{\sim}{x}_o$ is sought that yields the minimum error (in the approximation to the said equations), in this instance. The most common way of measuring such error is through the Euclidean norm of the vector whose components are the involved equations, ie. if this error is denoted by e and the system of equations is $\underset{\sim}{f}(\underset{\sim}{x})=\underset{\sim}{0}$ ($\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{b}=\underset{\sim}{0}$ in the linear case), then

$$e^2=||\underset{\sim}{f}(\underset{\sim}{x})||^2=\underset{\sim}{f}^T\underset{\sim}{f} \qquad (6.3.1a)$$

or

$$e^2=||\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{b}||^2=(\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{b})^T(\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{b}) \qquad (6.3.1b)$$

whether the system of equations is nonlinear or linear optimization resulting problem can be stated as

$$\text{"Minimize } \underset{\sim}{f}^T\underset{\sim}{f} \text{ over } \underset{\sim}{x}\text{"} \qquad (6.3.2)$$

If the system of equations is linear, $\underset{\sim}{f}$ should be replaced by $\underset{\sim}{A}\underset{\sim}{x}-\underset{\sim}{b}$ in (6.3.2), of course.

Notice that problem (6.3.2) is a particular case of problem (6.2.1), in which no constraints are present, i.e. this is a so-called "unconstrained optimization problem"; furthermore, due to its particular quadratic nature, it is referred to specifically as a least-square problem. The optimum $\underset{\sim}{x}_o$ is found, as outlined in section 1.12 for the linear case, by triangularization of matrix $\underset{\sim}{A}$ via Householder reflections and "back substitution", both of which are implemented in subroutines HECOMP and HOLVE appearing in Figs. 1.12.4 and 1.12.5. The nonlinear case, as discussed in Section 1.13, is solved iteratively by Newton-Raphson method, which requires the computation of the optimal correction at each stage, via the solution of a linear least-square problem, which is done, as already mentioned, by means of subroutines HECOMP and HOLVE. SUBROUTINE NEWRAMC, whose listing appears in Fig. 1.13.2, combines HECOMP and HOLVE to solve the nonlinear least-square problem (6.3.2).

The least-square problem arising in linkage syntheses has been already dealt with in the litterature. Suh and Mecklenburg [6.5] solved such problem by application of Powell's method. The said method is one of the so-called "direct search methods" and specifically it performs uni-directional searches, i.e. it optimizes the objective function once at each iteration, keeping fixed all the variables but one, at each time. More on direct search methods will be discussed in Section 6.5.

The linear least-square solution via Householder reflections is next illustrated with an example taken from [6.5] for comparisson.

Example 6.3.1. Synthesize an RSSR linkage so that its input ψ and its output ϕ be related according to Table 6.3.1. Referring to the nomenclature of Fig 5.2.1, assume

$$a_4=1, \alpha_4=90°$$

Solution:

Use is made of eq. (5.2.18) and definitions (5.2.19), thus obtaining a system of 19 linear equations in the six unknowns $k_1, k_2, \ldots, k_6$, of the form

$$\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{b} \qquad (6.3.3)$$

TABLE 6.3.1

THESE ARE THE SPECIFIED POINTS

	PSI(DEGREES)	PHI(DEGREES)
1	0.	0.
2	-.50000000E+01	.24000000E+01
3	-.10000000E+02	.51000000E+01
4	-.15000000E+02	.82000000E+01
5	-.20000000E+02	.11500000E+02
6	-.25000000E+02	.15200000E+02
7	-.30000000E+02	.19100000E+02
8	-.35000000E+02	.23300000E+02
9	-.40000000E+02	.27700000E+02
10	-.45000000E+02	.32300000E+02
11	-.50000000E+02	.37200000E+02
12	-.55000000E+02	.42300000E+02
13	-.60000000E+02	.47500000E+02
14	-.65000000E+02	.53000000E+02
15	-.70000000E+02	.58700000E+02
16	-.75000000E+02	.64600000E+02
17	-.80000000E+02	.70900000E+02
18	-.85000000E+02	.78000000E+02
19	-.90000000E+02	.90000000E+02

where $\tilde{A}$ is a 19x6 matrix, $\tilde{x}$ is a 6-dimensional vector whose i^{th} component is k_i and $\tilde{b}$ is a 19-dimensional vector whose i^{th} component is $c\phi_i c\psi_i + c\alpha_4 s\psi_i s\phi_i$. Both matrix $\tilde{A}$ and vector $\tilde{b}$ appear in Table 6.3.2.

Subroutines HECOMP and HOLVE were used to obtain the least-square solution $\tilde{x}_o$, i.e. the six values $k_1,...,k_6$ that approximate the 19 equations (6.3.3) with the minimum error, in the sense of the Euclidean norm. The optimal resulting values and the corresponding linkage parameters appear in Table 6.3.3

The error obtained with Powell's method, reported in [6.5], as well as that obtained with Householder reflections, are shown in Table 6.3.4. The overall error, for comparisson purposes is taken as

$$e=\sqrt{\frac{1}{n}\sum_1^n(\phi_i^*-\phi_i^{**})^2}$$

where ϕ_i^* and ϕ_i^{**} are the generated and the specified values of the output variable, respectively, corresponding to the value ψ_i of the input variable.

The nonlinear case solution is illustrated with the following example.

<u>Example 6.3.2 Synthesis of a plane RRRR linkage for rigid-body guidance through 17 specified configurations.</u>

The specified configurations are shown in Table 6.3.5. The synthesis equations are those derived in Appendix 4, namely, eqs. (A.4.6) which are next rewritten as

$$f_j=||(1-e^{i\theta j})a_0+b_0-r_j||^2-||b_0||^2=0, j=1,...,16 \quad (6.3.4)$$

The objective function is, then

$$\phi(a_0,b_0)=\tilde{f}^T\tilde{f}=\sum_1^{16} f_j^2 \quad (6.3.5)$$

Minimizing ϕ, as given by eq. (6.3.5) is, then, a nonlinear least-square problem, already discussed in case ii) of section 1.13. Thus, it can be solved using SUBROUTINE NERANC. The 17 specified configurations are shown in Table 6.3.5.

TABLE 6.3.2

MATRIX A						VECTOR B
1.000000	0.000000	-1.000000	0.000000	0.000000	-1.000000	1.000000
0.999123	0.041876	-0.996195	-0.087156	0.041716	1.000000	0.995321
0.996041	0.088894	-0.984808	-0.173648	0.087544	1.000000	0.980909
0.987776	0.142629	-0.965926	-0.258819	0.137769	1.000000	0.956050
0.979925	0.199368	-0.939693	-0.342020	0.197345	1.000000	0.920828
0.965016	0.262189	-0.906308	-0.422618	0.237624	1.000000	0.874602
0.944949	0.327218	-0.866025	-0.500000	0.283379	1.000000	0.818350
0.910446	0.395546	-0.819152	-0.573576	0.324012	1.000000	0.752347
0.885394	0.464842	-0.766044	-0.642788	0.356090	1.000000	0.678251
0.845262	0.534352	-0.707107	-0.707107	0.377844	1.000000	0.597690
0.796530	0.604599	-0.642788	-0.766044	0.388629	1.000000	0.512000
0.739631	0.673013	-0.573576	-0.819152	0.386024	1.000000	0.424235
0.675590	0.737277	-0.500000	-0.866025	0.368639	1.000000	0.337795
0.601815	0.798636	-0.422618	-0.906308	0.337518	1.000000	0.254348
0.519519	0.854459	-0.342020	-0.939693	0.292242	1.000000	0.177686
0.428935	0.903335	-0.258819	-0.965926	0.233800	1.000000	0.111017
0.327218	0.944749	-0.173648	-0.984808	0.164089	1.000000	0.056021
0.207912	0.978148	-0.087156	-0.996195	0.085251	1.000000	0.018133
-0.000000	1.000000	0.000000	-1.000000	-0.000000	1.000000	0.000000

TABLE 6.3.3

LINKAGE PARAMETERS

A1=	0.71126871
A2=	2.62059792
A3=	0.80357731
A4=	1.00000000
O1=	-1.18624029
SA=	-2.41755601
PHIO=	31.12706811

THESE ARE PARAMETERS K

K(1)=	0.081573
K(2)=	-3.489375
K(3)=	1.176454
K(4)=	-1.395550
K(5)=	0.386427
K(6)=	2.093812

TABLE 6.3.4

	APPROXIMATION ERROR USING POWELL'S METHOD (DEGREES)	APPROXIMATION ERROR USING HOUSEHOLDER'S METHOD (DEGREES)
1	0.00000000	0.01432265
2	-.00120000	[illegible]
3	-.03160000	-.00550843
4	0.02330000	0.0157[illegible]
5	-.02650000	-.02[illegible]
6	0.02250000	0.027[illegible]
7	0.01600000	0.01077[illegible]
8	0.03830000	[illegible]
9	[illegible]	[illegible]
10	-.03790000	[illegible]
11	-.00640000	-.00[illegible]3477
12	0.02270000	[illegible]
13	-.04220000	-.04[illegible]
14	-.00020000	-.0013[illegible]
15	0.03150000	0.03454250
16	[illegible]	[illegible]
17	-.00020000	0.00[illegible]
18	-.01250000	[illegible]0497
19	0.02980000	[illegible]

THE ROOT MEAN SQUARE ERROR OBTAINED BY:

POWELL'S METHOD IS 0.00165269 DEGREES

HOUSEHOLDER'S METHOD IS 0.00132079 DEGREES

TABLE 6.3.5 Specified configurations for rigid-body guidance

THESE ARE THE SPECIFIED CONFIGURATIONS

	X(CM)	Y(CM)	THETA (DEGREES)
0	7.880	-0.260	313.720
1	8.490	-7.290	332.330
2	7.680	2.820	349.930
3	6.300	4.210	353.180
4	4.580	4.950	359.870
5	2.740	5.010	355.840
6	1.010	4.410	356.300
7	0.259	3.880	3.900
8	-0.400	-3.090	3.670
9	0.250	-3.760	3.690
10	1.000	-4.290	4.150
11	2.730	-4.890	5.120
12	4.560	-4.830	6.810
13	6.280	-4.090	10.000
14	7.660	-2.700	13.000
15	8.440	-0.610	18.000
16	7.790	-2.690	46.270

A program was written to obtain the least-square solution to the prolbem, which turned out to have two different solutions, one for each side of the linkage, thus enabling the designer to construct the whole linkage. The two solutions are shown in Table 6.3.6

TABLE 6.3.6 Solutions to overdetermined rigid-body guidance problem of linkage synthesis

First solution	Second solution:
$A_0(5.124, 2.255)$	$A_0(1.444, -6.705)$
$B_0(0.549, -0.703)$	$B_0(6.371, -9.315)$

Thus, the linkage is composed of one fixed link with input-and output-links hinged to it at each of both points B shown in Table 6.3.6. The coupler link is hinged to the input- and output links at each of both points A_0 also shown in that table.

Since the problem is overdetermined, the solutions obtained do not zero each of the 16 fuctions f_j shown in (6.3.4). They minimize the quadratic norm (6.3.5), instead. Since each f_j is in turn quadratic in a_0 and b_0, ϕ is quartic in these complex numbers. Thus, ϕ has units of length raised to the fourth power. In order to obtain a dimensionless measure of the error in the approximation, the error is computed as

$$e = \frac{1}{16|b_0|} \sum_{1}^{16} \sqrt{|f_j|} \qquad (6.3.6)$$

This way, the error of the first solution resulted to be 38.74%, whereas that of the second one, 60.77%

An overdetermined problem of the RR spatial dyad for rigid-body guidance is presented in the next example.

<u>Example 6.3.3. Synthesis of the RR spatial dyad for rigid-body guidance.</u>

In this case, one fourth position of points A, B and C is added to those three already specified in Example 5.3.1. All four positions are the following:

$A_0(0,0,1)$ $B_0(0,1,0)$ $C_0(1,0,0)$
$A_1(0,0,0)$ $B_1(1,\sqrt{2},1)$ $C_1(\sqrt{6}/2,\sqrt{2}/2,0)$
$A_2(1,0,0)$ $B_2(1+\sqrt{2}/2,0,\sqrt{6}/2)$ $C_2(1+\sqrt{2},0,0)$
$A_3(0,0,0)$ $B_3(0,\sqrt{2},0)$ $C_3(0,\sqrt{2}/2,\sqrt{6}/2)$

The synthesis equations are those of Example 5.3.1, eqs. (5.3.53a) to (5.3.53h), except that eqs. (5.3.53a) to (5.3.53d) are now written for i=1,2,3, thus obtaining a system of 16 equations in 12 unknowns. The first two screws were already obtained in that example. The third one, corresponding to the fourth configuration, was also obtained via SUBROUTINE SCREW . All three screws are then

$$\underset{\sim}{s}_1=\begin{bmatrix}-0.769\\0.590\\-0.245\end{bmatrix},\underset{\sim}{a}_1=\begin{bmatrix}0.454\\0.787\\0.470\end{bmatrix},\ t_1=0.245,\theta_1=-56.600°$$

$$\underset{\sim}{s}_2=\begin{bmatrix}-0.906\\0.194\\0.375\end{bmatrix},\underset{\sim}{a}_2=\begin{bmatrix}0.305\\0.176\\0.645\end{bmatrix},t_2=-1.282,\theta_2=-129.736°$$

$$\underset{\sim}{s}_3=\begin{bmatrix}0.194\\-0.906\\0.375\end{bmatrix},\underset{\sim}{a}_3=\begin{bmatrix}0.176\\0.216\\0.430\end{bmatrix},\ t_3=-0.375,\theta_3=129.736°$$

The nonlinear least-square problem arising from this dyad-synthesis problem was solved using SUBROUTINE NERAMC. The solutions obtained are shown in Table 6.3.7

6.4. UNDERDETERMINDED PROBLEMS OF LINKAGE SYNTHESIS. In this Section the problem opposite to that presented in Section 6.3 is discussed i.e. problems that will be treated here contain an excess of the number of unknowns over that of equations; hence, they admit infinitely many solutions. If the system is linear, then a unique solution exists whose norm is a global minimum and can be found via the Moore-Penrose (See Section 1.11) pseudo-inverse matrix of the system. If the said system is nonlinear, then all local minima can be found, as discussed in Section 1.13 via the introduction of Lagrange multipliers. As Fox and Gupta [6.6] point it out, linear optimization problems arise raredy in mechanism synthesis, that is to say, although (as was shown in Section 5.2) linkage synthesis problems for function generation can be formulated as linear problems, in this case the linkage parameters do not appear directly in the synthesis equations in linearly. Hence, only the nonlinear case will be discussed in this Section and the solution procedure will be illustrated with one example.

Example 6.4.1. Synthesize an R-R dyad that conducts a rigid body through the first two configurations shown in Fig 5.4.3. Since the general number of equations for this problem is 4n+4 (See Section 5.4), and for two configurations n=1, the resulting number of equations is 8, but the number of unknowns is 12, thus obtaining an underdetermined system of equations.

Solution:

The data are:

$$\underset{\sim}{Q}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \underset{\sim}{r}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (6.4.1)$$

where components are referred to the $X_0Y_0Z_0$ coordinate frame.

The unknowns are;

$$\underset{\sim}{a}_0 = \begin{bmatrix} a_{01} \\ a_{02} \\ a_{03} \end{bmatrix}, \quad \underset{\sim}{b}_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \end{bmatrix}, \quad \underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \underset{\sim}{v}_0 = \begin{bmatrix} v_{01} \\ v_{02} \\ v_{03} \end{bmatrix} \qquad (6.4.2)$$

Equation (5.4.5), written for j=1, is

$$(a_{01}-a_{03})^2+4a_{02}^2+(-a_{02}+a_{03})^2+2b_{01}(a_{01}-a_{03})+$$

$$+4(b_{02}-1)a_{02}+2b_{03}(-a_{01}+a_{03})-2b_{02}+1=0 \qquad (6-4.3a)$$

Equation (5.4.8), written for j=0,1 leads to

$$u_1b_{01}+u_2b_{02}+u_3b_{03}=0 \qquad (6.4.3b)$$

$$u_1(a_{01}-a_{03})+2u_2a_{02}+u_3(-a_{01}+a_{03})-u_2=0 \qquad (6.4.3c)$$

Equation (5.4.9), written for j=0,1 leads to

$$v_{01}b_{01}+v_{02}b_{02}+v_{03}b_{03}=0 \qquad (6.4.3d)$$

$$v_{01}(a_{01}-a_{03})+2v_{02}a_{02}+v_{03}(-a_{01}+a_{03})-v_{02}=0 \qquad (6.4.3e)$$

Equation (5.4.10), written for j=1 is

$$u_1v_{03}-u_2v_{02}+u_3v_{01}=u_1v_{01}+u_2v_{02}+u_3v_{03} \qquad (6.4.3f)$$

Finally, eqs. (5.4.11) lead to

$$u_1^2+u_2^2+u_3^2=1 \qquad (6.4.3g)$$

$$v_{01}^2+v_{02}^2+v_{03}^2=1 \qquad (6.4.3h)$$

Equations (6.4.3a-h) constitute a system of 8 equations in the 12 unknowns: $a_{01}, a_{02}, a_{03}, b_{01}, b_{02}, b_{03}, u_1, u_2, u_3, v_{01}, v_{02}, v_{03}$. The objective function that can now be introduced, meant to be minimized, is

$$\phi(\mathbf{a}_0,\mathbf{b}_0)= a_{01}^2+{}_{02}^2+a_{03}^2+b_{01}^2+b_{02}^2+b_{03}^2 \qquad (6.4.4)$$

Hence, the optimization problem can be sated as:

"Minimize $\phi(\underset{\sim}{a}_0,\underset{\sim}{b}_0)$ as given by eq. (6.4.4), subject to eqs. (6.4.3a-h)" The solution to the foregoing problem proceeds as follows: Let $\underset{\sim}{f}$ be an 8-dimensional vector whose components are the left-hand sides of eqs. (6.4.3) when all the right-hand sides are set equal to zero, the non-zero terms having been transfered to the left hand side, of course. Let $\underset{\sim}{x}$ be a 12-dimensional vector whose components are $[a_{01},a_{02},a_{03},b_{01},b_{02},b_{03},u_1,u_2,u_3,v_{01},v_{02},v_{03}]^T$. Define a new objective function ψ as

$$\psi=\phi+\underset{\sim}{\lambda}^T\underset{\sim}{f} \qquad (6.4.5)$$

where $\underset{\sim}{\lambda}=[\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8]^T$ is a vector containing the Lagrange multipliers of the system. The stationarity condition (See Section 1.10) applied to function ψ is

$$\frac{\partial\psi}{\partial\underset{\sim}{x}}\equiv \frac{\partial\phi}{\partial\underset{\sim}{x}} + \frac{\partial\underset{\sim}{f}}{\partial\underset{\sim}{x}}\underset{\sim}{\lambda}= \underset{\sim}{0} \qquad (6.4.6)$$

which yields a system of 12 additional equations. System (6.4.6), together with system (6.4.3) constitute a system of 20 equations in 20 unknowns, which are the 12 components of vector $\underset{\sim}{x}$ + the 8 components of vector $\underset{\sim}{\lambda}$. This is a determined system whose roots can be found via the method of Newton-Raphson.

Exercise 6.4.1 Solve system (6.4.3a-h),(6.4.6)

6.5 LINKAGE OPTIMIZATION SUBJECT TO INEQUALITY CONSTRAINTS. The class of optimization problems discussed in Section 6.3. does not include any constraints, hence these are called "unconstrained optimization problems". Problems presented in Section 6.4 were formulated as quadratic optimization problems subject to equality constraints. There are several methods of tackling this class of problems,but in that Section only the classical approach, i.e. the Lagrange multipliers method, was presented.

Optimization problems subject to inequality constraints arise very frequently in linkage syntheses. For example if the RRRR plane linkage shown in Fig 6.5.1 is synthesized so that its input link be

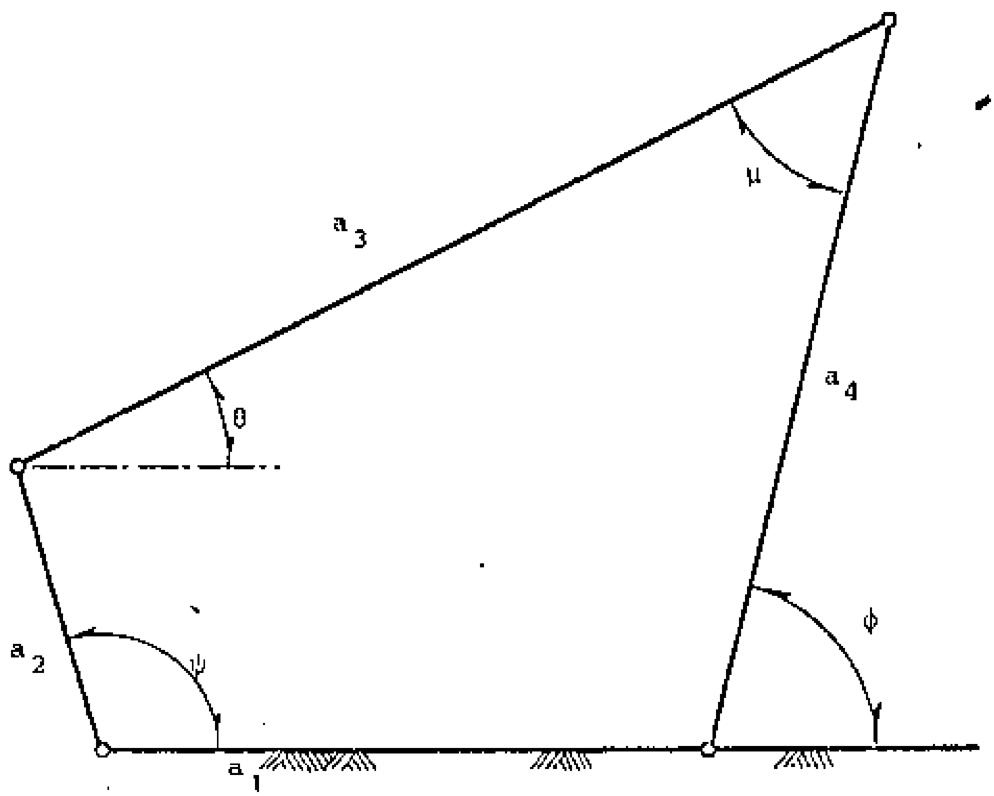

Fig. 6.5.1 Plane RRRR linkage

a crank, this condition (possibility of 360° rotation) is expressed as [6.7]

$$a_3+a_4>a_2-a_1$$

and

$$a_1+a_2<a_3+a_4$$

As another example, assume the transmision angle of the same linkage, μ, defined as the angle between the coupler and the output links, is constrained to lie between μ_a and μ_b, then

$$\mu_a<\mu<\mu_b$$

would be the said inequality constraint. The transmission angle μ is clearly given by

$$\mu=\phi-\theta$$

where ϕ and θ are defined in Fig 6.5.1, except that in this instance ϕ is measured from a line containing the fixed link, i.e. with $\phi_0=0$. Angles ϕ and θ are obtained from Freudentein's equation as

$$k_1-k_2\cos\phi+k_3\cos\psi+\cos(\phi-\psi)=0$$

and

$$r_1+r_2\cos\theta+r_3\cos\psi-\cos(\theta-\psi)=0$$

the involved coefficients being defined as

$$k_1=\frac{a_3^2-a_1^2-a_2^2-a_4^2}{2a_2a_4},\quad k_2=\frac{a_1}{a_2},\quad k_3=\frac{a_1}{a_4}$$

and

$$r_1=\frac{a_4^2-a_1^2-a_2^2-a_3^2}{2a_2a_3},\quad r_2=\frac{a_1}{a_2},\quad r_3=\frac{a_1}{a_3}\ .$$

As a "rule of thumb", μ is usually constrained to lie between 40° and 140°. This class of problems, then, is of the general type (6.2.1). An extensive account of different methods to solve this problem appears in [6.6]. Fox and Gupta in this reference classify the different mehtods of solution to this problem as: i) penalty function methods and ii) direct methods. These are outlined next. i) Letting $\phi(\underset{\sim}{x})$ be the original objective function, a new objective function ψ is defined, which "penalizes" violation of the inequality constraints thus ensuring that the optimization search be carried on within the feasible region, i.e, within the subset Ω where all constraints are satisfied. The new objective function can then be constructed either via an "interior" penalty function or via an "exterior" one. In the first case, it is defined as

$$\psi=\phi(\underset{\sim}{x})-r\sum_{1}^{p}\frac{1}{h_i(\underset{\sim}{x})}$$

whereas in the second case, as

$$\psi=\phi(\underset{\sim}{x})+r\sum_{1}^{p}\left[\max\{0,h_i(\underset{\sim}{x})\}\right]^2$$

where the value of the scalar r is varied at each iteration in a way that it leads to the optimum, namely, assume that ϕ is to be minimized; then, if an interior penalty function is used, r is decreased. Otherwise, r is increased. An algorithm called "Sequential Unconstrained Minimization Technique" (SUMT), implementing the interior penalty function method, has been published and extensively tested. The corresponding computer program, implementing this algorithm as developed by Fiacco and McCormick, appears in [6.8]. ii) Direct methods are those that handle inequalities as such. There are several of this kind, but the most utilized methods in optimal mechanical design are: <u>the method of the feasible directions</u>, <u>the gradient projection method</u> and <u>the complex method</u>.

The method of feasible directions, developed by Zoutendijk [6.9] is intended in principle to handle linear inequality constraints, but nonlinear ones can also be handled if a previous suitable linearization is performed. This method requires first a search direction $\underset{\sim}{e}$ and, second, a search distance, s, so that when both are determined, given a current value $\underset{\sim}{x}_k$ of the vector argument, the next value, $\underset{\sim}{x}_{k+1}$, is given by

$$\underset{\sim}{x}_{k+1}=\underset{\sim}{x}_k+s_k\underset{\sim}{e}_k \tag{6.5.1}$$

where $$||\underset{\sim}{e}_k||=1.$$

To find the direction $\underset{\sim}{e}_k$, assume that the inequality constraints are linear, i.e. of the form

$$\underset{\sim}{A}\underset{\sim}{x}+\underset{\sim}{b}\geq\underset{\sim}{0} \tag{6.5.2}$$

$\underset{\sim}{A}$ being an mxn matrix (m<n). There are two possibilities, namely, either $\underset{\sim}{x}_k$ lies within the feasible region or it lies on its boundary. If it lies within it, then $\underset{\sim}{e}_k$ is just taken as the gradient of the objective function, i.e.

$$\underset{\sim}{e}_k=\pm\frac{\nabla\phi}{||\nabla\phi||} \tag{6.5.3}$$

where the + sign is taken if ϕ is to be maximized; otherwise, the - sign should be chosen. If, however, $\underset{\sim}{x}_k$ lies on the boundary of the feasible region, then some, say m', of the m inequalities (6.5.2) become equations. If the new value $\underset{\sim}{x}_{k+1}$ is not going to violate the constraints (6.5.2), then the search direction and distance should satisfy the inequalities

$$\underset{\sim}{A}\underset{\sim}{x}_k+s_k\underset{\sim}{A}\underset{\sim}{e}_k+\underset{\sim}{b}\geq\underset{\sim}{0} \tag{6.5.4}$$

which constitute a set of m inequalities. However, m' of the components of vector $\underset{\sim}{A}\underset{\sim}{x}_k+\underset{\sim}{b}$ vanish, as assumed before. Hence, taking s_k positive, if (6.5.4) is to be satisfied for all its components, then the search direction $\underset{\sim}{e}_k$ should satisfy the following inequality

$$\underset{\sim}{A}\underset{\sim}{e}_k\geq\underset{\sim}{0} \tag{6.5.5}$$

Since vector $\underset{\sim}{e}_k$ must satify a linear inequality, it is most convenient to use a linear norm for it, instead of the Euclidean one, i.e. let

$$||\underset{\sim}{e}_k||_1=\max_i\{[e_{ik}]\}=1 \qquad (6.5.6)$$

where e_{ik} is the i^{th} component of vector $\underset{\sim}{e}_k$. In order to turn the finding of vector $\underset{\sim}{e}_k$ into a standard linear programming problem, which requires the sought vector to meet a nonnegativity condition, the following change of variable is introduced

$$\underset{\sim}{t}_k=\underset{\sim}{e}_k+\underset{\sim}{f} \qquad (6.5.7)$$

where $\underset{\sim}{f}$ is an m-dimensional vector whose components are all unity. Thus, $\underset{\sim}{e}_k$ is found from the following linear program:

$$\text{Max}(\nabla\phi_k)^T(\underset{\sim}{t}_k-\underset{\sim}{f})$$

subject to

$$-\underset{\sim}{t}_k+2\underset{\sim}{f}\geq 0$$

and

$$\underset{\sim}{t}_k\geq 0$$

where the 2n involved inequality constraints arise from (6.5.6). The foregoing procedure to determine $\underset{\sim}{e}_k$ is due to Glass and Cooper [6.10]. If the inequality constraints are nonlinear, then a local linearization should be performed to find matrix $\underset{\sim}{A}$ of (6.5.2) as

$$A=\left.\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{x}}\right|_{\underset{\sim}{x}=\underset{\sim}{x}_k}$$

which yields good results if the feasible region is convex.

Once the direction search is found, the distance search s_k is determined performing a unidirectional optimization in the direction of $\underset{\sim}{e}_k$. There can arise two possibilities: the first is that the optimizing s_k carries $\underset{\sim}{x}$ to a point within the feasible region, in which case no difficulty is present; the other possibility appears when $\underset{\sim}{x}_{k+1}$ happens to fall out of the feasible region. In this case, inequality (6.5.4) does not hold any longer for all its components. Assuming that m' of the m inequalities (6.5.4) are violated for the previously found optimizing s_k, then there exist s_{ik} values smaller than s_k $(i=1,\ldots,m')$ for which the corresponding relations (6.5.4) become equations, i.e. for which

$$a_{ij}x_{jk}+s_{ik}a_{ij}e_{jk}+b_i=0,\ i=1,2,\ldots,m' \qquad (6.5.8)$$

where a renumbering of the said relations might have needed be performed. Solving for s_{ik} from (6.5.8),

$$s_{ik}=-\frac{a_{ij}x_{jk}+b_i}{a_{ij}e_{jk}} \qquad (6.5.9)$$

Hence, the distance search is then taken as

$$s_k=\min\{s_{ik}\}_1^{m'}$$

thus completing one iteration of the whole procedure.

The gradient projection method, developed by Rosen [6.11,12], performs the search (for the optimum) initially along the boundaries of the feasible region. The procedure is, as outlinead by Beveridge and Schechter [6.13], the following:

1. Given a point $\underset{\sim}{x}_p$ where r of the inequality constraints become equations, evaluate $\nabla\phi$ at $\underset{\sim}{x}_p$. The said set of equations is then written as

$$\underset{\sim}{g}(\underset{\sim}{x}_p)=\underset{\sim}{0} \qquad (6.5.10)$$

2. Clculate the projection of the gradient onto the plane tangent to the surface (6.5.9). This projection defines a direction $\underset{\sim}{e}$ in space lying in the said tangent plane. Vector $\underset{\sim}{e}$ is determined as follows:
Let $\frac{d\phi}{ds}$ be the directional derivative of ϕ along direction $\underset{\sim}{e}$, thus

$$\frac{d\phi}{ds}=(\nabla\phi)^T\underset{\sim}{e} \qquad (6.5.11)$$

Since $\underset{\sim}{e}$ is contained in a plane tangent to (6.5.9),

$$\frac{d\underset{\sim}{g}}{ds}(\nabla\underset{\sim}{g})^T\underset{\sim}{e}=\underset{\sim}{0} \qquad (6.5.12)$$

vector $\underset{\sim}{e}$ being defined as of magnitude unity, i.e.

$$\underset{\sim}{e}^T\underset{\sim}{e}=1 \qquad (6.5.13)$$

Define the objective function

$$\psi=\frac{d\phi}{ds}+\lambda_o(1-\underset{\sim}{e}^T\underset{\sim}{e})+\underset{\sim}{\lambda}^T(\nabla\underset{\sim}{g})^T\underset{\sim}{e} \qquad (6.5.14)$$

which accounts for constraints (6.5.11) and (6.5.12). Substitution of of (6.5.10) into (6.5.13) together with the stationarity condition for ψ yield

$$\frac{d\psi}{d\underset{\sim}{e}}=\nabla\phi+\nabla\underset{\sim}{g}\underset{\sim}{\lambda}-2\lambda_o\underset{\sim}{e}=\underset{\sim}{0} \qquad (6.5.15)$$

from which

$$\underset{\sim}{e}=\frac{\nabla\underset{\sim}{g}\underset{\sim}{\lambda}+\nabla\phi}{2\lambda_o} \qquad (6.5.16)$$

which, when substituted in (6.5.11), yields an equation for $\underset{\sim}{\lambda}$, namely

$$(\nabla\underset{\sim}{g})^T\nabla\underset{\sim}{g}\underset{\sim}{\lambda}=-(\nabla\underset{\sim}{g})^T\nabla\phi \qquad (6.5.17)$$

λ_o being computed from

$$\lambda_o=\frac{1}{2}\sqrt{(\nabla\phi)^T\nabla g\lambda+||\nabla\phi||^2} \qquad (6.5.18)$$

which follows from the substitution of eq. (6.5.15) into eq. (6.5.12) Hence, e is given as

$$e=-\frac{\nabla g\lambda+\nabla\phi}{\sqrt{(\nabla\phi)^T\nabla g\lambda+||\nabla\phi||^2}} \qquad (6.5.19)$$

3. At this stage two possibilities can arise: either e vanishes or it does not. If it does not vanish, intiate a search in this direction until a constraint boundary is found. Let x_p' be the intersection of the line going from x_p in the direction of e, with the boundary. Then, if $\phi(x_p')$ is better than $\phi(x_p)$, one iteration is completed and the procedure is restarted. If, on the other hand, $\phi(x_p')$ is not better than $\phi(x_p)$, perform a unidirectional search along the line connecting x_p and x_p'. Call x_{p+1} the sought optimizing value, and return to step 1 with x_{p+1} being set equal to x_p.
4. In case e vanishes, then it could happen that the components of λ have all the same sign or some are zero, thus complying with the Kuhn-Tucker conditions [6.13,p.282]. If this sign is nonnegative, then a minimum has been reached; if nonpositive, then a maximum.

If not all components of λ bear the same sign, then some of these components are deleted from it, keeping only those λ_i $(i=1,2,\ldots,q<r)$ with the same sign. Next, drop the last r-q components of $g(x)$, i.e., those corresponding to the λ's with a sign different from those of the λ's that have been kept. Then restart the procedure at step 2. The PROJG program [6.8pp.399-411] implements Rosen's algorithm. To describe the complex method, due to Box [6.14], some definitions are in order: Given a vector space V of dimension n, the (closed)

polyhedron imbedded in it with the smallest number of vertices (n+1) is called a simplex. A polyhedron with a larger number of vertices is called a complex. Simplexes in two and three dimensions are the triangle and the tetrahedron, respectively. Complexes in these spaces could be the quadrilateron and the hexahedron, respectively. The complex method of Box proceeds as follows:

1. A complex of 2n vertices is defined within the feasible region. There exist some techniques to define these vertices in such a way that they guarantee that all of them fall into the said region, but they are applicable to only a few particular cases. In the absence of a criterion to choose the said vertices, it is advised to assign them randomly, rejecting those that fall out of the feasible region, until the complex is completed.

2. Evaluate the objective function at each of the 2n vertices and let $\underset{\sim}{x}_w$ be the vertex where this function attains its worst value.

3. Compute the centroid, $\underset{\sim}{x}_c$, of the remaining vertices, i.e.

$$\underset{\sim}{x}_c=\frac{1}{2n-1}\left(\sum_{1}^{2n} \underset{\sim}{x}_i-\underset{\sim}{x}_w\right) \tag{6.5.20}$$

4. Replace the vertex at $\underset{\sim}{x}_w$ by a new one, $\underset{\sim}{x}'_w$ in the following way

$$\underset{\sim}{x}'_w=\underset{\sim}{x}_c-r(\underset{\sim}{x}_w-\underset{\sim}{x}_c) \tag{6.5.21}$$

where r is a real positive number whose value is recommended by Box to be taken as 1.3

5. Two possibilities can arise at this stage: either $\underset{\sim}{x}'_w$ lies within the feasible region or not. If it does, a new iteration can be restarted at stage 2. If it does not, then a new value, $\underset{\sim}{x}''_w$, is defined as

$$\underset{\sim}{x}''_w=\frac{1}{2}(\underset{\sim}{x}_c+\underset{\sim}{x}'_w)$$

If $\underset{\sim}{x}''_w$ again happens to lie outside of the feasible region, a new value is again defined as the middle point between $\underset{\sim}{x}_c$ and $\underset{\sim}{x}''_w$, until the said new value falls within the feasible region. Once the rejected vertex has been regenerated, the procedure can be restarted at step 2. If the procedure converges, the complex becomes so small that it can be considered to sink to one single point, which is then taken as the optimizing value $\underset{\sim}{x}_o$. In practice, the procedure is stopped when the complex has sunk to a size less than a prescribed finite value.

Box's algorithm has been implemented in some computer programs, for example, the one appearing in [6.8,pp.368-385] and the OPTIM package [6.15,16]. The latter has been applied very successfully at the University of Mexico in several kinds of problems of mechanical design. When the number of decision variables (dimension of vector $\underset{\sim}{x}$) goes beyond 5, however, it presents convergence difficulties, in which case other method should be used.

<u>Exercise 6.5.1.</u> Repeat the synthesis of Example 6.3.1 imposing the constraint that the transmission angle (See Section 5.2) lie between 40° and 140°

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

7. COMPUTER - AIDED ANALYSIS AND SYNTHESIS OF CAM MECHANISMS

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AGOSTO, 1980

"COMPUTER-AIDED ANALYSIS AND SYNTHESIS OF CAM MECHANISMS"

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ABSTRACT. The applicability of both analog and digital techniques to the analysis and the design of cam-follower pairs is presented. It is assumed that either an analog or a digital computer is available for the analysis or the synthesis of follower motions produced by cams, for the corresponding equations are so numerically involved that they are impossible to be solved by any other means (e.g. geometrical methods). It is shown that optimal designs of cams are possible to be obtained by the methods here presented, these designs being optimal in the sense of providing a cam of minimum size for the maximum allowable values of certain parameters such as the pressure angle.

Symbols of operators for analog realizations are defined in the Appendix.

INTRODUCTION. The litterature concerned with the analysis and the design of cams mainly deals with geometrical methods ([1]* to [5]) and only a few ([6] to [8]) introduce analytical techniques (i. e. techniques dealing with equations). To the knowledge of the author, little attention has been devoted to obtain constraint equations of motion

*Numbers in brackets designate references at end of paper.

for any cam-follower pair, which allow the analyst or the designer to use the modern techniques available to solve the class of mathematical problems arising from these analyses. It is the aim of this paper to obtain the said constraint equations for two different types of followers and solve them via analog or digital methods. Both the analysis and the design processes are discussed and, as to the latter, a procedure to obtain optimal designs is outlined.

It is pointed out that, contrary to the case of linkage synthesis, where the design parameters form a finite set, in the cam synthesis process, the parameters to be obtained not only form an infinite set, but actually constitute a continuum, i. e. the totality of values of $\rho(\theta)$, for θ contained in the closed interval $[0, 2\pi]$, $\rho = \rho(\theta)$ being the polar equation of the cam profile, as shown in Fig. 1. Thus, in the case of cams, the design can be obtained via a continuous process, as the one provided by analog computers (of course, the applicability of digital computers is not discarded by this fact). The importance of this is two-fold: i) The output of an analog computer is instantaneously read out via a plotter or an oscilloscope, thus allowing to adjust design parameters continuously to obtain optimal designs. ii) Many digital simulators of dynamical systems are programmed from an analog computer realization, thus simplifying the programming labor and saving time and effort.

On the other hand, the numerical output obtained from a digital computer can be directly fed into a numerically controlled machine tool to produce the optimal design obtained via a digital technique.

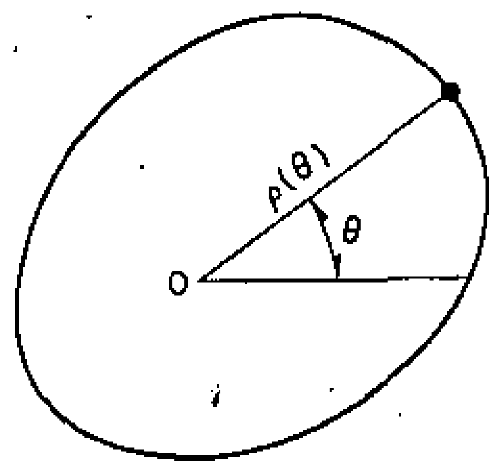

Figure 1. Polar coordinates of the points on the profile of a cylindrical cam

ANALYSIS AND DESIGN OF A CAM WITH A KNIFE-EDGED FOLLOWER.

Consider the cylindrical cam shown in Fig. 2, where e is the offsetting of the follower path.

From this figure, the relations

$$s = \rho \sin (\theta + \psi) \qquad (1)$$

$$e = \rho \cos (\theta + \psi) \qquad (2)$$

follow. The variable $s(\psi)$ is the displacement of the follower, the output of the mechanism. In the same figure, C is a line fixed to the cam, thus revolving counterclockwise at the rate $\dot{\psi}$, ψ being the angle between lines C and F, the latter being fixed to the frame of the layout. For each value of ψ contained in $[0, 2\pi]$, equations (1) and (2) constitute a system of nonlinear algebraic equations in the two unknowns θ and s (algebraic equations as opossed to other kinds such as differential or integral equations), if the cam profile $\rho = \rho(\theta)$ is known, as is the case in the analysis of a given mechanism. The unknowns are θ and ρ if the desired output $s(\psi)$ is given, as occurs in the design of a cam to provide a given motion of the follower.

The analog computer realization of system (1), (2) for analysis is shown in Fig. 3, whereas for synthesis of a cam profile, it is shown in Fig. 4.

On the other hand, eqs. (1) and (2) can be solved numerically via Newton-Raphson's method [9] or any other one to solve nonlinear algebraic systems. The algorithm suggested for analysis is the following:

Choose a set $\{\psi^i\}$ of values of ψ in $[0, 2\pi]$, preferably equally

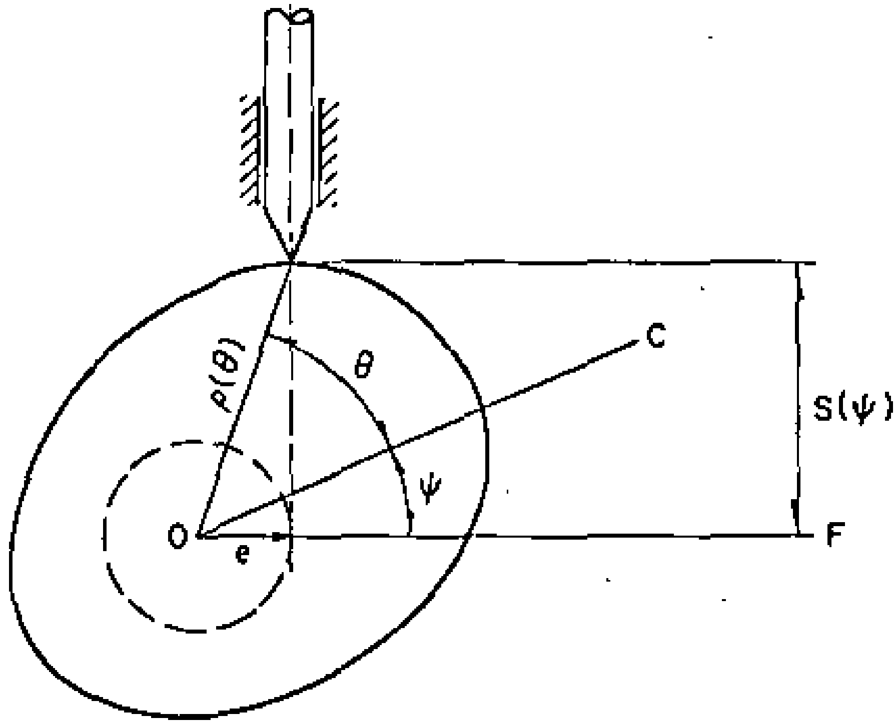

Figure 2. Cylindrical cam with a knife-edged follower

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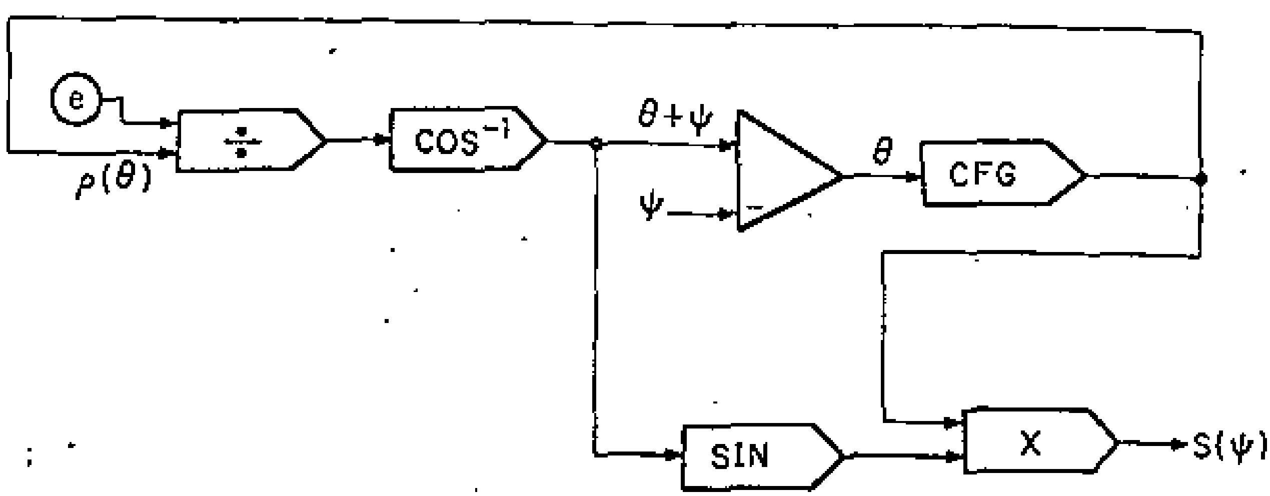

Figure 3. Analog realization of the equations for the analysis of the displacement of the knife-edged follower of a cylindrical cam

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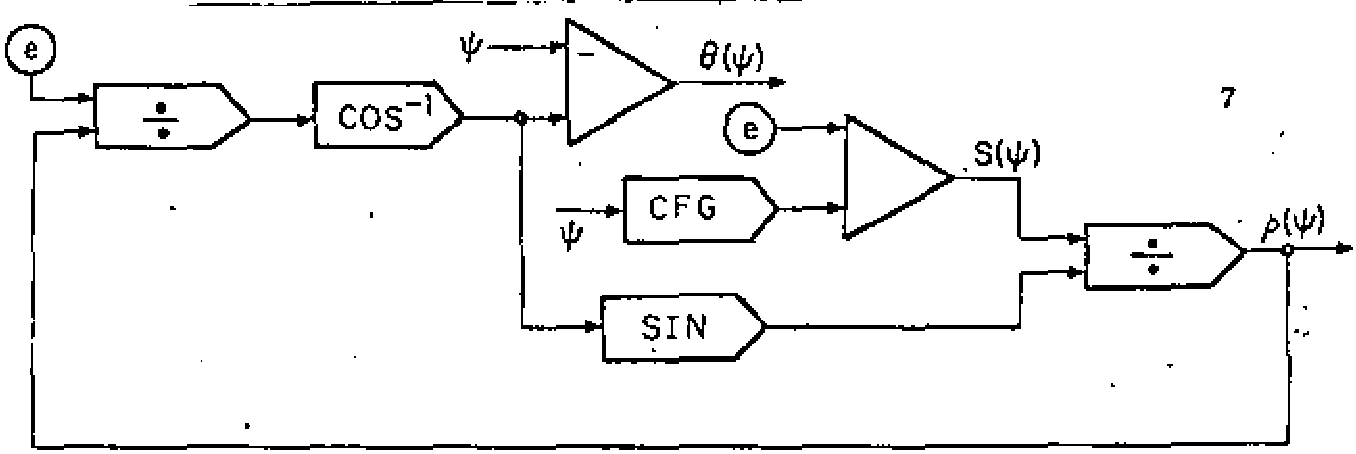

Figure 4. Analog realization of the synthesis equations of the cam profile for a knife-edged follower

spaced, but there is no reason to impose this condition. For each value ψ^i, solve equation (2) via Newton-Raphson's method, thus obtaining the set $\theta^i = \theta(\psi^i)$. In order to accelerate the process, choose the initial guess θ_0^i to start the iteration at $\psi = \psi^i$ as the value $\theta^{i-1} = \theta(\psi^{i-1})$, obtained in the previous iteration cycle. Because of the continuity of θ with respect to ψ, θ^i is close to θ^{i-1} if ψ^i and ψ^{i-1} are reasonably close.

Applying Newton-Raphson's algorithm, the iterative scheme is

$$\theta_{k+1} = \theta_k - \frac{\rho(\theta_k)\cos(\theta_k + \psi^i) - e}{\rho'(\theta_k)\cos(\theta_k + \psi^i) - \rho(\theta_k)\sin(\theta_k + \psi^i)} \qquad (3)$$

with $\theta_0(\psi^i)$ given. The follower displacement $s^i = s(\psi^i)$ is then obtained from eq. (1) as

$$s^i = \rho(\theta^i)\sin(\theta^i + \psi^i) \qquad (4)$$

Example 1. Obtain the displacement $s(\psi)$ of the knife-edged follower of a cylindrical cam having the profile given by the cardioid $\rho = 2 - \cos\theta$, with $e = 0.5$, both ρ and e having the same units of length.

Equations (3) and (4) were implemented in a digital computer, thus obtaining the values shown in Table 1, the corresponding plot appearing in Fig. 5.

TABLE I ANALYSIS OF A KNIFE-EDGED FOLLOWER CAM

INITIAL GUESS

PSI = 0. S = 0. THETA = .900E+02

VARIABLE TAN IS THE ERROR SIZE

PSI (DEGREES)	THETA (DEGREES)	S (U. OF LENGTH)	TAN (DIMENSIONLESS)	ITER
0.	.729608E+02	.163224E+01	0.	5
.100000E+02	.666638E+02	.142489E+01	.511160E-06	4
.200000E+02	.479131E+02	.123216E+01	.722019E-06	4
.300000E+02	.349340E+02	.106904E+01	.434302E-06	4
.400000E+02	.222710E+02	.951190E+00	.522305E-07	4
.500000E+02	.105462E+02	.885485E+00	.430376E-09	4
.600000E+02	-.362028E-09	.866025E+00	.566727E-07	3
.700000E+02	-.954894E+01	.881986E+00	.304641E-06	3
.800000E+02	-.184032E+02	.924607E+00	.545697E-11	4
.900000E+02	-.265333E+02	.988406E+00	.345608E-10	4
.100000E+03	-.350415E+02	.107623E+01	.100044E-09	4
.110000E+03	-.431716E+02	.116619E+01	.198270E-09	4
.120000E+03	-.513226E+02	.126694E+01	.314685E-09	4
.130000E+03	-.595610E+02	.140719E+01	.422006E-09	4
.140000E+03	-.679289E+02	.154537E+01	.494765E-09	4
.150000E+03	-.764497E+02	.169543E+01	.494765E-09	4
.160000E+03	-.851338E+02	.184175E+01	.436557E-09	4
.170000E+03	-.939617E+02	.200613E+01	.320142E-09	4
.180000E+03	-.102958E+03	.216783E+01	.189175E-09	4
.190000E+03	-.112143E+03	.232374E+01	.101863E-09	4
.200000E+03	-.121437E+03	.247149E+01	.291038E-10	4
.210000E+03	-.130058E+03	.260667E+01	.145519E-10	4
.220000E+03	-.140397E+03	.272499E+01	0.	4
.230000E+03	-.150046E+03	.282248E+01	0.	4
.240000E+03	-.159797E+03	.289562E+01	.147847E-07	3
.250000E+03	-.169647E+03	.294153E+01	.184940E-06	3
.260000E+03	-.179594E+03	.295601E+01	.417640E-08	3
.270000E+03	-.189640E+03	.294372E+01	.419110E-06	3
.280000E+03	-.199709E+03	.289613E+01	0.	4
.290000E+03	-.210049E+03	.282164E+01	.145519E-10	4
.300000E+03	-.220433E+03	.271552E+01	.160071E-09	4
.310000E+03	-.230960E+03	.258169E+01	.785803E-09	4
.320000E+03	-.241656E+03	.242372E+01	.312866E-08	4
.330000E+03	-.252557E+03	.224474E+01	.100121E-07	4
.340000E+03	-.263711E+03	.204944E+01	.336149E-07	4
.350000E+03	-.275178E+03	.184314E+01	.953587E-07	4
.360000E+03	-.287031E+03	.163224E+01	.242660E-06	4

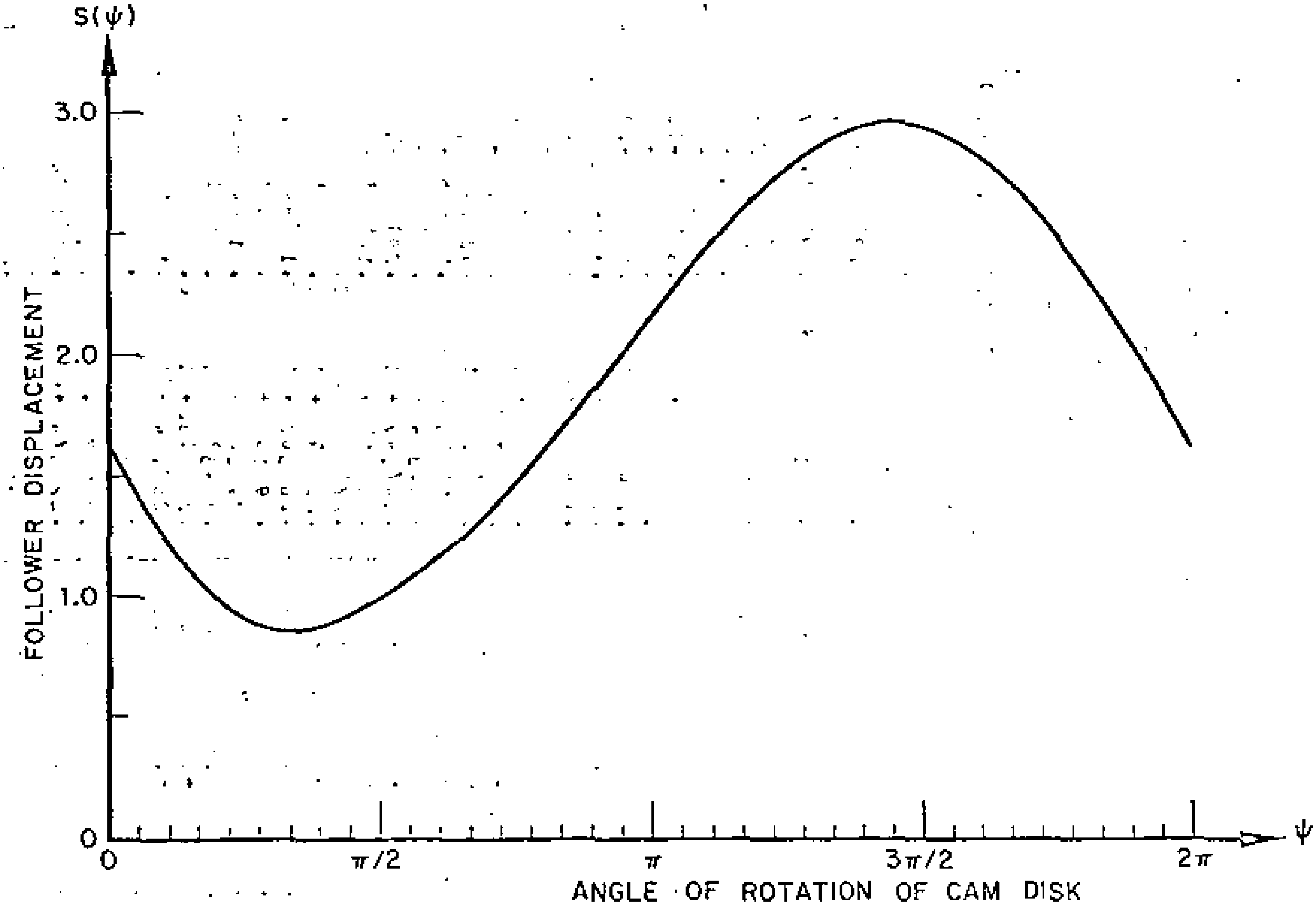

Figure 5. Displacement of the knife-edged follower of the cylindrical cam given by the profile $\rho = 2 - \cos\theta$, with offsetting $e = 0.5$

<u>OPTIMAL SYNTHESIS OF A KNIFE-EDGED FOLLOWER CAM.</u> In the synthesis process the unknown variables are $\theta(\psi)$ and $\rho(\psi)$ which provide $\rho = \rho(\theta)$, the polar equation of the cam profile. In this case, Newton-Raphson's iterative Scheme [9] for a nonlinear algebraic system $\underline{f}(\underline{x}) = \underline{0}$ is applied. Letting $\underline{x}_k$ be the k <u>th</u> value of vector $\underline{x}$, the (k + 1) <u>st</u> approximated value of the roots of the vector equation $\underline{f}(\underline{x}) = \underline{0}$ - where $\underline{f}$ and $\underline{x}$ are of the same dimension - is

$$\underline{x}_{k+1} = \underline{x}_k - \underline{J}^{-1}(\underline{x}_k)\,\underline{f}(\underline{x}_k), \qquad (5)$$

$\underline{J}(\underline{x})$ being the Jacobian matrix of $\underline{f}$ with respect to $\underline{x}$, i.e. the J_{lm} element of $\underline{J}$ is given by

$$J_{lm} = \frac{\partial f_l}{\partial x_m} \qquad (6)$$

For the particular system under consideration, let

$$x_1 = \theta, \qquad (7)$$

$$x_2 = \rho, \qquad (8)$$

$$f_1 = \rho \sin(\theta + \psi) - s, \qquad (9)$$

$$f_2 = \rho \cos(\theta + \psi) - e, \qquad (10)$$

in eq. (5), thus obtaining the following iterative scheme.

$$\theta_{k+1} = \theta_k + \frac{s(\psi_i)}{\rho_k}\cos(\theta_k + \psi_i) - \frac{e}{\rho_k}\sin(\rho_k + \psi_i), \qquad (11)$$

$$\rho_{k+1} = s(\psi_i)\sin(\theta_k + \psi_i) + e\cos(\theta_k + \psi_i) \qquad (12)$$

Again, as in the case of the analysis of a given mechanism, the initial guess (θ_0 , ρ_0) for $\psi = \psi^i$ is taken as (θ^{i-1} , ρ^{i-1}), the solution to system (1), (2) for $\psi = \psi^{i-1}$, in order to accelerate the convergence.

The important parameter in this design is the pressure angle α , defined as the angle between the follower path and line N, normal to the cam profile, as shown in Fig. 6. Notice that the inclination of line N with respect to line OF can be greater or less than 90°, the sign of α thus changing accordingly. Since the absolute value of α is the relevant parameter, the pressure angle is given by

$$\alpha = |\phi + \theta + \psi - \pi| . \tag{13}$$

From a well known result in analytic geometry [10], the following relations are obtained

$$\phi = \tan^{-1} \frac{\rho}{\rho'(\theta)} , \tag{14}$$

or

$$\phi = \tan^{-1} \frac{\rho\theta'(\psi)}{\rho'(\psi)} , \tag{14a}$$

where the chain rule has been applied. Equations (13) and (14a) allow one to compute $\alpha(\theta)$ and hence control this variable, except that $\rho'(\theta)$ is not known in the synthasis process. However, this value can be obtained in terms of θ, ρ, ψ, and $s(\psi)$, as is next shown.

Differentiation of eqs. (1) and (2) with respect to θ and elimination of $\psi'(\theta)$ leads to

$$\rho'(\theta) = \frac{\rho s'(\psi) \sin(\theta + \psi)}{s'(\psi)\cos(\theta + \psi) - \rho} , \tag{15}$$

which is the desired expression, needed to compute $\alpha(\theta)$.

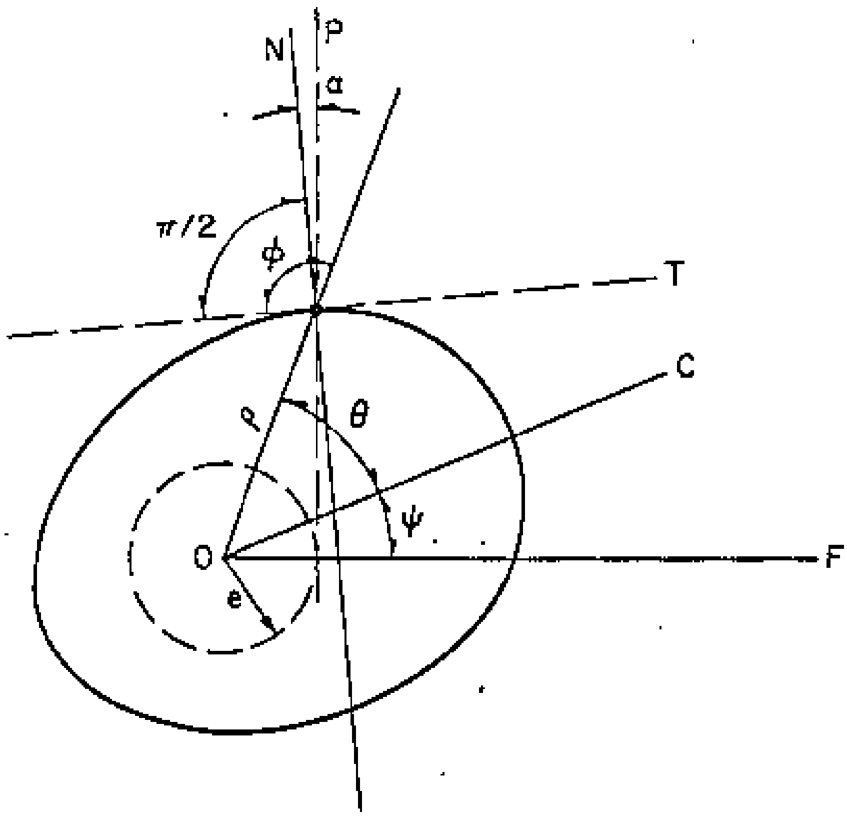

Figure 6. Pressure angle of a cylindrical cam with a knife-edged follower

In cam design practice it is customarily recommended to keep the value of α below 30° in order to ensure a good mechanical advantage of the mechanism. Experienced designers know that if the maximum value of α becomes too large, a way of decreasing this value is to increase the radius of the "base circle" [11] of the cam. How much to increase this radius is something that is left to their expertise and, to the knowledge of the author, there are no systematic means of performing an optimal design. Thus, the designer has to proceed by trial and error geometric designs which, in addition to being tedious, are very inaccurate.

An alternative to geometrical methods of design is next proposed:

i) Let (θ_0^1, ρ_0^1) be the initial guess to start the iterates for $\psi = \psi^1 = 0$. Determine the cam profile for this guess, recording the corresponding values of α.

ii) If the maximum value of α becomes too large, increase ρ_0^1, keeping θ_0^1 as it is. In case the said maximum value of α becomes too small, the cam size is too big, thus a reduction of ρ_0^1 is possible, provided α is not made too large.

iii) Proceed - of course, automatically in a digital computer - by trial and error until the maximum value of α is 30°, thus obtaining the cam of minimum size for which the maximum value attained by the pressure angle is allowable.

The advantage of the foregoing method is that it can easily be implemented in a digital computer so that the trial and error procedure is automated. One more systematic way to proceed is to differentiate the pressure angle and equate its derivative to zero, thus obtaining, with the other constraint equations, a nonlinear algebraic system of dimension three, for unknown values ψ^*, θ^*, ρ^*, at which α equals α_M, the maximum allowable value of α. Since a nonlinear algebraic system has multiple roots, it may become very difficult to find the useful one, i.e. one which is kinematically acceptable. This procedure, though highly systematic, will not be necessarily faster and more accurate than the one previously proposed.

Example 2. Synthesize the cam profile that yields the following displacement of its knife-edged follower:

$$s(\psi) = c + \sin^2 \frac{\psi}{2}, \tag{16}$$

keeping the pressure angle below 30° and the cam size as small as possible. In eq. (16) c is the radius of the base circle, a parameter to be adjusted to yield the optimal desing.

The iterative scheme was implemented in a digital computer with the initial guess:

For $\psi = 0$,

$$\theta = 1 \text{ rad} \tag{17}$$

$$\rho = 1, 2, 3, \ldots, 10. \tag{18}$$

It was found that the optimal design laid between $\rho = 1.0$ and $\rho = 2.0$. Then a search was performed between 1.0 and 2.0, with an interval length of 0.1, the result being that the said optimizing value of ρ laid between

1.6 and 1.7. At each search, one digit was gained in the optimizing value of ρ, having finally obtained the optimizing value $\rho = 1.6234$. The output of the procedure is shown in Table 2 and the corresponding cam profile in Fig. 7.

ANALYSIS AND SYNTHESIS OF A FLAT-FACED FOLLOWER CAM.

Consider the mechanism shown in Fig. 8, comprised of a disk cam and a flat-faced follower, with offsetting e. Let lines C and F be fixed to the cam and to the frame of the layout, respectively. Angles ψ and θ have the same meaning as in Fig. 2. Angle ϕ is that formed by the radius vector OA and the tangent to the cam profile at A - i.e. the flat face of the follower. Finally, $s(\psi)$ is the displacement of the follower.

From Fig. 8 it is clear that

$$s = \rho \sin(\theta + \psi), \tag{19}$$

$$\phi = \pi - (\theta + \psi), \tag{20}$$

$$\rho'(\theta) \tan \phi = \rho(\theta) \tag{21}$$

Substituting eq. (20) into eq. (21), one obtains

$$\rho'(\theta) \sin(\theta + \psi) + \rho(\theta) \cos(\theta + \psi) = 0. \tag{22}$$

Equations (19) and (22) are the constraint equations for both analysis and synthesis. Consider first the analysis of a given cam and follower mechanism, with $\rho = \rho(\theta)$ known. As in the analysis of a kinfe-edged follower cam, for each value ψ^i in $[0, 2\pi]$, eq. (22) is a nonlinear algebraic equation in the unknown $\theta^i = \theta(\psi^i)$; hence, Newton-Raphson's method can also be applied. Substitution of the computed values θ^i into

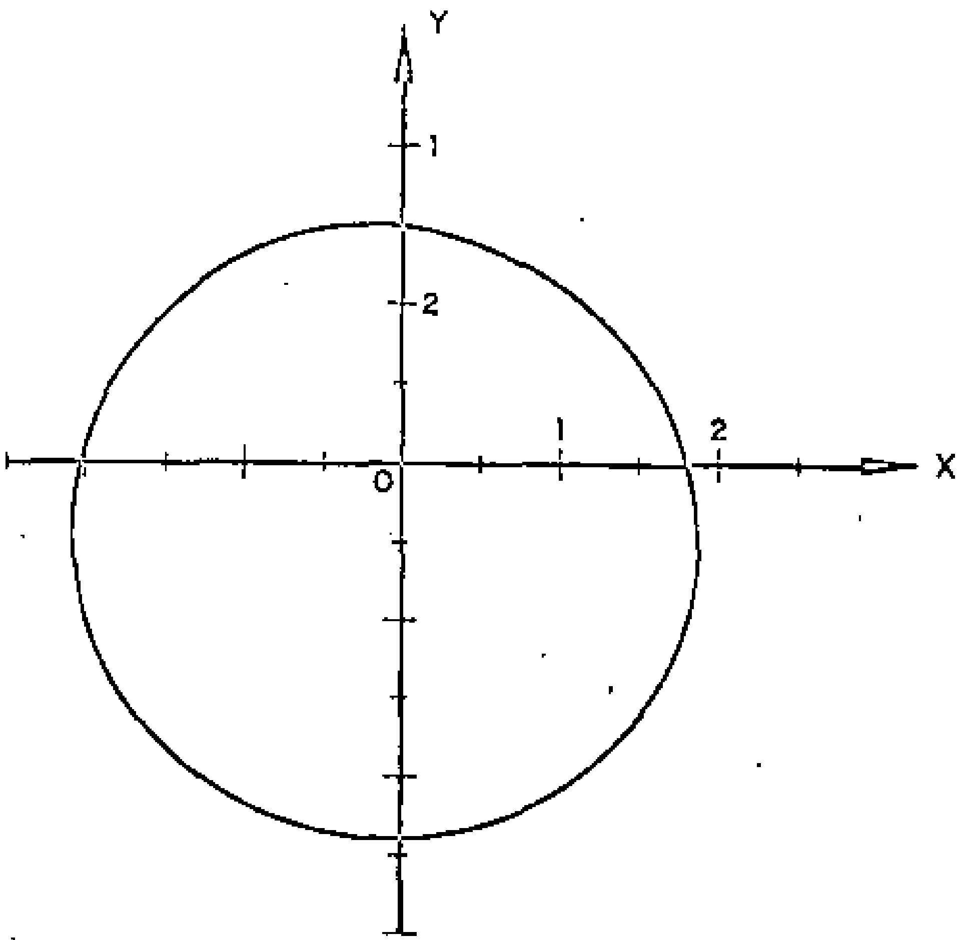

Figure 7. Cam profile that generates the displacement $s = \sin^2 \psi/2 + C$ of a knife-edged follower, with a maximum pressure angle of 30°

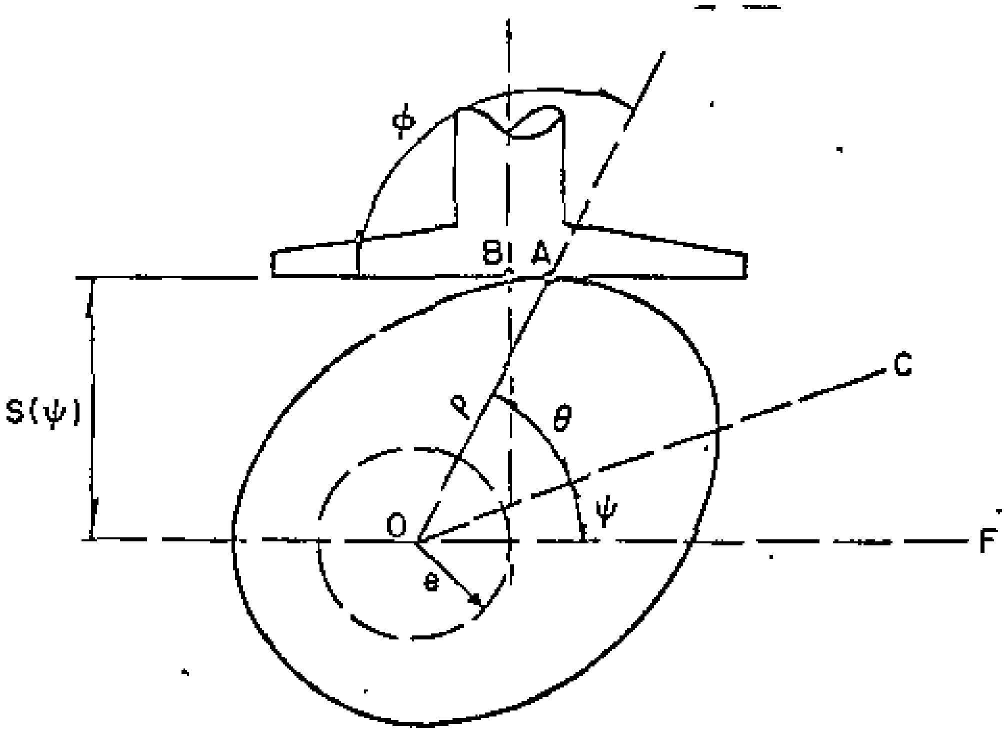

Figure 8. Cylindrical cam with a flat-faced follower

TABLE 2 SYNTHESIS OF A KNIFE-EDGED-FOLLOWER-CAM

INITIAL GUESS

PSI = 0. RHO = .162340E+01 THETA = .572958E+02

VARIABLE TAU IS THE ERROR SIZE

PSI (DEGREES)	RHO (LENGTH UNITS)	THETA (DEGREES)	ALPHA (DEGREES)	TAU (DIMENSIONLESS)	NO. ITER
0.	.145467E+01	.690963E+02	.201037E+02	.327525E-06	4
.100000E+02	.146181E+01	.599087E+02	.167407E+02	.212474E-06	3
.200000E+02	.148303E+01	.502968E+02	.132589E+02	.299610E-09	4
.300000E+02	.151775E+01	.407656E+02	.989596E+01	.124064E-08	4
.400000E+02	.156504E+01	.313685E+02	.686728E+01	.277136E-08	4
.500000E+02	.162356E+01	.220633E+02	.433080E+01	.515865E-08	4
.600000E+02	.169163E+01	.128031E+02	.237363E+01	.965999E-08	4
.700000E+02	.176724E+01	.356503E+01	.101915E+01	.323894E-07	4
.800000E+02	.184814E+01	-.569653E+01	.244615E+00	.234683E-07	4
.900000E+02	.193187E+01	-.149999E+02	.833763E-09	.112373E-07	4
.100000E+03	.201586E+01	-.243611E+02	.222864E+00	.770273E-08	4
.110000E+03	.209752E+01	-.337908E+02	.848064E+00	.565103E-08	4
.120000E+03	.217431E+01	-.432946E+02	.181320E+01	.405188E-08	
.130000E+03	.224385E+01	-.528754E+02	.306109E+01	.278822E-08	
.140000E+03	.230397E+01	-.625338E+02	.454053E+01	.174179E-08	4
.150000E+03	.235280E+01	-.722697E+02	.620597E+01	.937965E-09	4
.160000E+03	.238880E+01	-.820819E+02	.801688E+01	.377701E-09	4
.170000E+03	.241087E+01	-.919697E+02	.993679E+01	.543237E-10	4
.180000E+03	.241830E+01	-.101932E+03	.119324E+02	.120348E-10	4
.190000E+03	.241087E+01	-.111970E+03	.139725E+02	.248885E-09	4
.200000E+03	.238880E+01	-.122082E+03	.160273E+02	.852651E-09	4
.210000E+03	.235280E+01	-.132270E+03	.180674E+02	.183382E-08	4
.220000E+03	.230397E+01	-.142534E+03	.200630E+02	.318241E-08	4
.230000E+03	.224385E+01	-.152875E+03	.219829E+02	.493279E-08	4
.240000E+03	.217431E+01	-.163295E+03	.237938E+02	.697645E-08	4
.250000E+03	.209752E+01	-.173791E+03	.254592E+02	.920656E-08	4
.260000E+03	.201586E+01	-.184361E+03	.269387E+02	.114461E-07	4
.270000E+03	.193187E+01	-.195000E+03	.281865E+02	.134362E-07	4
.280000E+03	.184814E+01	-.205697E+03	.291517E+02	.148723E-07	4
.290000E+03	.176724E+01	-.216435E+03	.297769E+02	.154466E-07	4
.300000E+03	.169163E+01	-.227192E+03	.299997E+02	.148792E-07	4
.310000E+03	.162356E+01	-.237937E+03	.297551E+02	.131171E-07	4
.320000E+03	.156504E+01	-.248632E+03	.289805E+02	.103840E-07	4
.330000E+03	.151775E+01	-.259234E+03	.276260E+02	.714960E-08	4
.340000E+03	.148303E+01	-.269703E+03	.256689E+02	.407720E-08	4
.350000E+03	.146181E+01	-.280001E+03	.231324E+02	.172395E-08	4
.360000E+03	.145467E+01	-.290104E+03	.201037E+02	.398761E-09	4

eq. (19) yields the desired displacement $s^i = s(\psi^i)$. The iterative scheme for eq. (22) is the following:

$$\theta_{k+1} = \theta_k - \frac{\rho'(\theta_k)\sin(\theta_k + \psi^i) + \rho(\theta_k)\cos(\theta_k + \psi^i)}{[\rho''(\theta_k) - \rho(\theta_k)]\sin(\theta_k + \psi^i) + 2\rho'(\theta_k)\cos(\theta_k + \psi^i)} \quad (23)$$

The initial guess θ_0^1 is given by the analyst and the subsequent guesses θ_0^i are given by the previously found values $\theta(\psi^{i-1})$, as in the case of the knife-edged follower.

Once $\theta(\psi^i)$ is found, the displacement $s(\psi^i)$ is computed as given by eq. (19), i.e. as

$$s(\psi^i) = \rho(\theta^i)\sin(\theta^i + \psi^i), \quad (24)$$

where

$$\theta^i = \theta(\psi^i) \quad (25)$$

Alternatively, eqs. (19), and (22) can be realized in an analog computer diagram, as shown in Fig. 10, for the analysis of the motion of the flat-faced follower.

Example 3. Obtain the displacement $s(\psi)$ of the flat-faced follower of an eccentric circular cam of radius $a = 1.0$ and accentricity $e = 0.5$ (both a and e have the same units of length), whose profile is given by the equation

$$\rho(\theta) = e\cos\theta + \sqrt{a^2 - e^2\sin^2\theta} \quad (26)$$

The iterative scheme (23) was implemented in a digital computer, the corresponding results being shown in Table 3 and Fig. 9.

TABLE 3 ANALYSIS OF A FLAT-FACED FOLLOWER CAM
THESE ARE THE VARIABLES PSI,THETA,TOL,MAX, IN THIS ORDER
0. .150000E+01 .100000E-05 99
VARIABLE TAU IS THE ERROR SIZE

PSI (DEGREES)	THETA (DEGREES)	S (U. OF LENGTH)	TAU (NUMBER)	ITER (NUMBER)
0.	.759638E+02	.100000E+01	.132807E-08	3
.100000E+02	.667234E+02	.104341E+01	.437353E-09	3
.200000E+02	.577885E+02	.108551E+01	.187562E-09	3
.300000E+02	.491066E+02	.112500E+01	.679146E-10	3
.400000E+02	.406308E+02	.116070E+01	.153904E-10	3
.500000E+02	.323190E+02	.119151E+01	.967423E-11	3
.600000E+02	.241333E+02	.121651E+01	.863706E-11	3
.700000E+02	.160392E+02	.123492E+01	0.	3
.800000E+02	.800488E+01	.124620E+01	0.	3
.900001E+02	-.458365E-04	.125000E+01	.886167E-06	8
.100000E+03	-.800493E+01	.124620E+01	.526419E-06	2
.110000E+03	-.160393E+02	.123492E+01	.930035E-07	2
.120000E+03	-.241333E+02	.121651E+01	.134911E-07	2
.130000E+03	-.323190E+02	.119151E+01	.307985E-06	2
.140000E+03	-.406308E+02	.116070E+01	0.	3
.150000E+03	-.491067E+02	.112500E+01	.848931E-11	3
.160000E+03	-.577885E+02	.108550E+01	0.	3
.170000E+03	-.667235E+02	.104341E+01	.124958E-10	3
.180000E+03	-.759638E+02	.100000E+01	.548790E-10	3
.190000E+03	-.855668E+02	.956588E+00	.136416E-09	3
.200000E+03	-.955930E+02	.914495E+00	.270382E-09	3
.210000E+03	-.106102E+03	.875000E+00	.447913E-09	3
.220000E+03	-.117146E+03	.839303E+00	.548030E-09	3
.230000E+03	-.128758E+03	.808489E+00	.459754E-09	3
.240000E+03	-.140935E+03	.783494E+00	.224805E-09	3
.250000E+03	-.153623E+03	.765077E+00	.434166E-10	3
.260000E+03	-.166704E+03	.753798E+00	0.	3
.270000E+03	-.180000E+03	.750000E+00	.150541E-08	2
.280000E+03	-.193296E+03	.753798E+00	.296404E-06	2
.290000E+03	-.206377E+03	.765077E+00	.121200E-10	3
.300000E+03	-.219065E+03	.783494E+00	.205525E-09	3
.310000E+03	-.231242E+03	.808489E+00	.753569E-09	3
.320000E+03	-.242854E+03	.839303E+00	.128745E-08	3
.330000E+03	-.253898E+03	.875000E+00	.132339E-08	3
.340000E+03	-.264407E+03	.914495E+00	.958613E-09	3
.350000E+03	-.274433E+03	.956588E+00	.537749E-09	3
.360000E+03	-.284036E+03	.100000E+01	.249510E-09	3

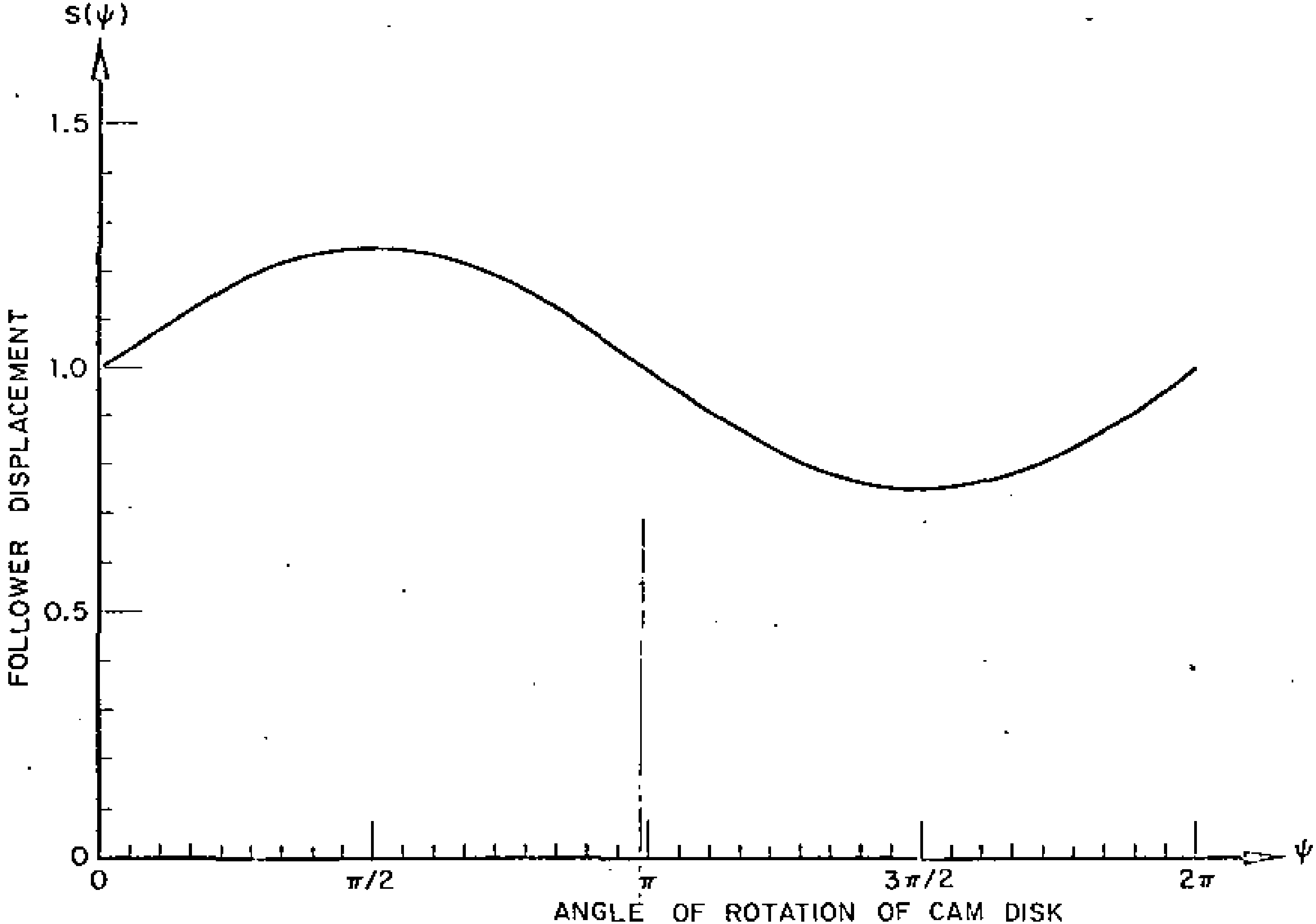

Figure 9. Displacement of the flat-faced follower of the cylindrical cam given by the profile $\rho = 0.25\cos\theta + \sqrt{1-(0.25\sin\theta)^2}$, without offsetting

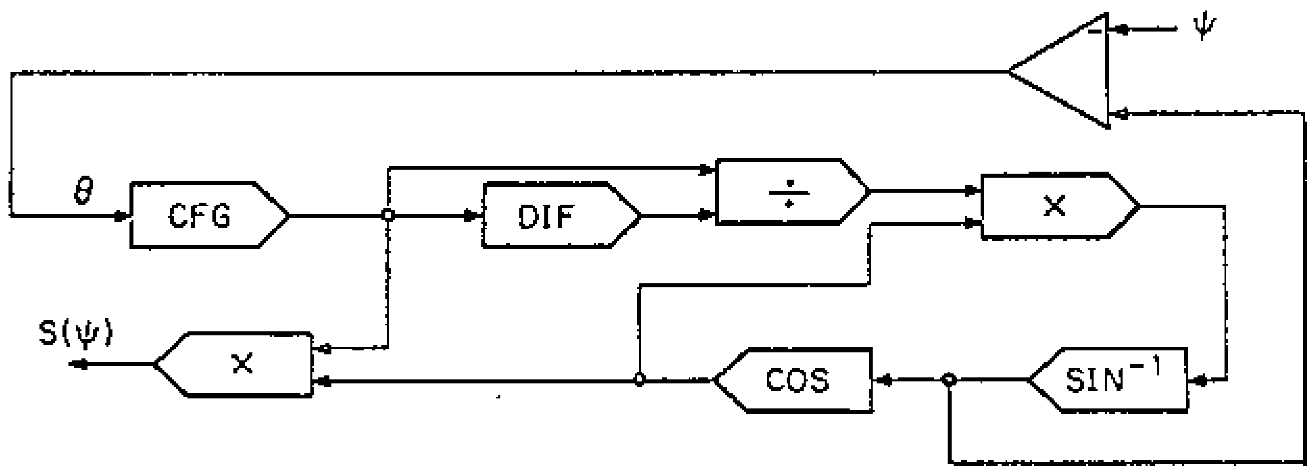

Figure 10. Analog realization of the equations for the analysis of the displacement of the flat-faced follower of a cylindrical cam

<u>OPTIMAL SYNTHESIS OF A FLAT-FACED FOLLOWER CAM.</u> The synthesis process for this type of cams is essentially different from its analysis, for the synthesis equations (19) and (22) are not algebraic in both θ and ρ, since eq. (22) contains $\rho'(\theta)$. Hence, Newton-Raphson's method is not applicable any more, and a routine for the integration of ordinary differential systems is to be used. For this purpose, it is necessary to express eqs. (19) and (22) in standard form [12], which is done next.

Differentiation of eq. (19) with respect to ψ, together with eq. (22) leads to

$$s'(\psi) = \rho(\theta)\cos(\theta+\psi) \qquad (27)$$

Differentiating the latter equation with respect to ψ once again, together with eq. (22), yields

$$\theta'(\psi) = \frac{s''(\psi) + \rho\sin(\theta+\psi)}{\rho'(\theta)\cos(\theta+\psi) - \rho\sin(\theta+\psi)} \qquad (28)$$

From eqs. (19), (22) and (28),

$$\theta'(\psi) = -\frac{[s(\psi) + s''(\psi)]\sin(\theta+\psi)}{\rho(\theta)} \qquad (28a)$$

$$\rho'(\psi) = [s(\psi) + s''(\psi)]\cos(\theta+\psi). \qquad (29)$$

Equations (28a) and (29) constitute a nonlinear ordinary differential system of dimension two, with the initial values

$$\theta_0 = \theta(\psi_0),\ \rho_0 = \rho(\psi_0), \qquad (30)$$

given. It is in the standard form of state variables to be solved by a suitable routine or to be realized in an analog computer.

The synthesis equations, (28) and (29), are realizad in Fig. 11.

The pressure angle for this type of follower is zero for any configuration, thus, it is not a relevant design variable. The counterpart of the pressure angle, in this case, is the offset $\overline{BA}$, of Fig. 8, between the point of contact A and the axis of the follower path. Denoting this length by x, it is given by

$$x = \rho \cos(\theta + \psi) - e. \tag{31}$$

For an optimal design it is required to keep the (absolute) value of x below certain allowable maximum x_M, minimizing the cam size. In this case, if x is made too small, the cam size is too big; thus, the cam size cannot be diminished without any constraint for, if it is made too small, the offset x becomes too big. For this reason, the optimal design is that for which the maximum absolute value of the offset x is x_M.

The procedure to obtain the optimal design for this type of follower is suggested to be similar to the case of a knife-edged follower, i.e.

i) Integrate system (28a), (29) with the initial values θ_0, ρ_0 and record the value x.

ii) If $\max\limits_{0 \le \theta \le 2\pi} |x| > x_M$, increase ρ_0, keeping θ_0 as it is. If that maximum value is below x_M, decrease ρ_0, keeping the same initial value θ.

iii) Proceed by trial and error until $\max\limits_{0 \le \theta \le 2\pi} |x| = x_M$.

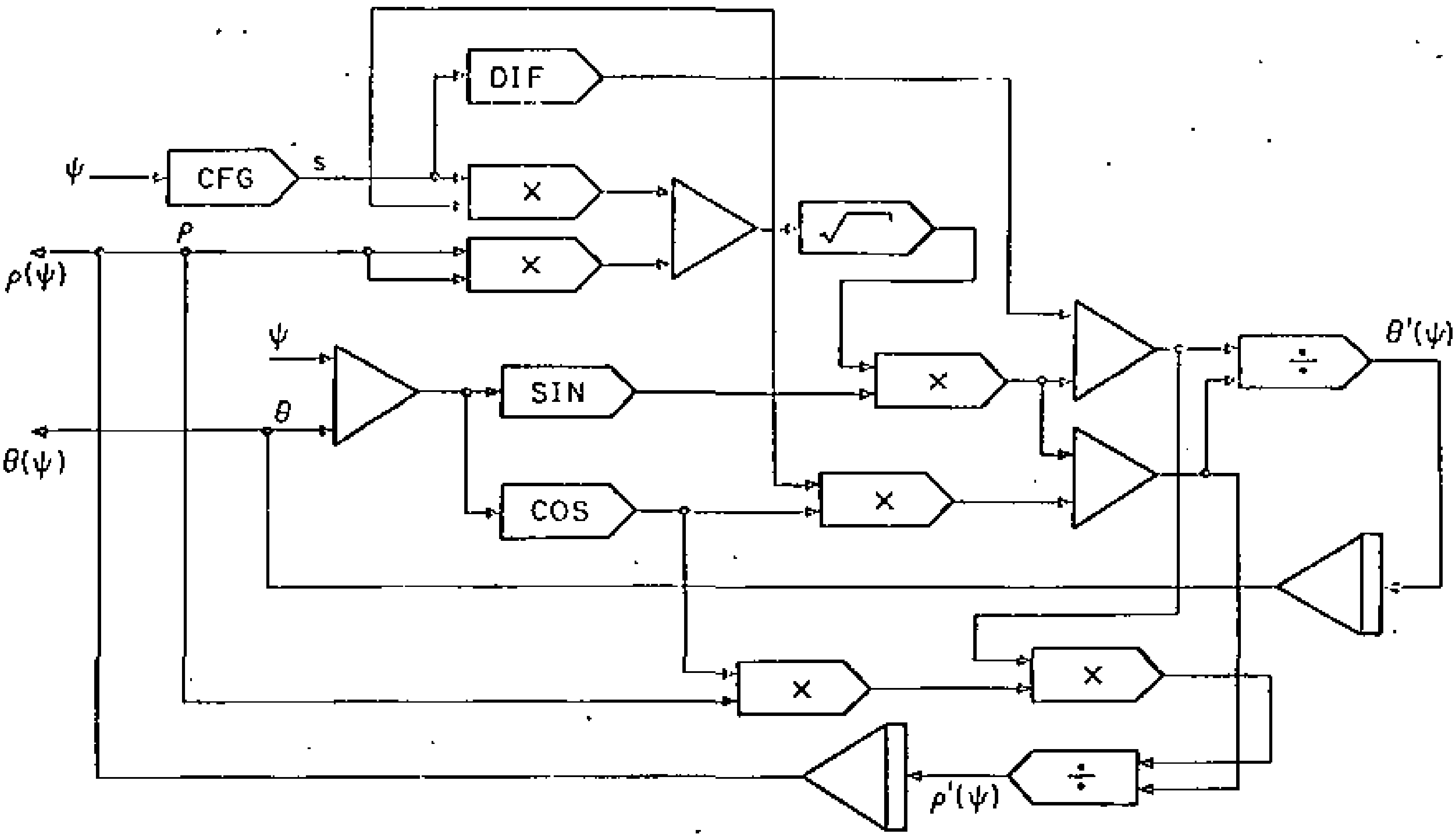

Figure 11. Analog realization of the equations for the synthesis of the cam profile for a flat-faced follower

Example 4. Design the profile of a flat-faced follower cam that yields the follower displacement given by

$$s(\psi) = c + \sin^2 \frac{\psi}{2} ,$$

so that this design is of minimum size and the offset x attains a maximum (absolute) value of 10% of the constant c. This constant is the radius of the base circle of the cam. Assume e = 0.

The cam profile synthesis was performed via digital computation for the following initial values:

$$\psi_0 = 0^\circ , \quad \theta_0 = 90^\circ , \quad \rho_0 = 1, 2, \ldots , 10$$

It was found that the optimizing value of ρ_0 is 5.0. The numerical results are shown in Table 4, the corresponding profile, which turned out to be a circle -as it should be- appearing in Fig. 12.

CONCLUSIONS. In Tables 1, 2 and 3 it is seen that the procedure converges very quickly, in at most five iterations (except for a singularity at $\psi = 90^\circ$, in Table 3 of Example 3), when Newton-Raphson's algorithm is used. Column headed ITER gives the number of iterations required to obtain convergence, where the convergence criterion was takes as

$$\| x_{k+1} - x_k \| < 10^{-6} \| x_k \| ,$$

the symbol $\| \cdot \|$ meaning the norm [13] of the argument. This norm was taken as the sum of the absolute value of the components of the vector under consideration. It is called TAM in the program.

AKNOWLEDGEMENTS. This research project was completed under the sponsorship of the Graduate Division (División de Estudios Superiores) and

TABLE 4 SYNTHESIS OF A FLAT-FACED FOLLOWER CAM.

PSI (DEGREES	THETA (DEGREES)	RHO (U. OF LENGTH)	OFFSET (PERCENT)
0.	.9000000E+02	.5000000E+01	.2563344E-09
.1000000E+02	.7900660E+02	.5008349E+01	.1736483E+01
.2100000E+02	.6696111E+02	.5036398E+01	.3563682E+01
.3100000E+02	.5609310E+02	.5077950E+01	.5150385E+01
.4100000E+02	.4553605E+02	.5133137E+01	.6560595E+01
.5100000E+02	.3471444E+02	.5199679E+01	.7771465E+01
.6100000E+02	.2424524E+02	.5275751E+01	.8746204E+01
.7100000E+02	.1393807E+02	.5358113E+01	.9455193E+01
.8000000E+02	.4802450E+01	.5435526E+01	.9848085E+01
.9000000E+02	-.5194432E+01	.5522681E+01	.1000001E+02
.1000000E+03	-.1503665E+02	.5608482E+01	.9848085E+01
.1100000E+03	-.2473617E+02	.5690441E+01	.9396933E+01
.1200000E+03	-.3430662E+02	.5766282E+01	.8660260E+01
.1300000E+03	-.4376439E+02	.5833981E+01	.7660450E+01
.1400000E+03	-.5312700E+02	.5891795E+01	.6427881E+01
.1500000E+03	-.6241285E+02	.5938278E+01	.5000004E+01
.1600000E+03	-.7164083E+02	.5972296E+01	.3420204E+01
.1700000E+03	-.8083010E+02	.5993034E+01	.1736483E+01
.1800000E+03	-.9000000E+02	.6000001E+01	.1015889E-06
.1900000E+03	-.9916990E+02	.5993034E+01	-.1736483E+01
.2000000E+03	-.1083592E+03	.5972296E+01	-.3420204E+01
.2100000E+03	-.1175871E+03	.5938278E+01	-.5000004E+01
.2200000E+03	-.1268730E+03	.5891795E+01	-.6427881E+01
.2300000E+03	-.1362356E+03	.5833981E+01	-.7660450E+01
.2400000E+03	-.1456934E+03	.5766282E+01	-.8660260E+01
.2500000E+03	-.1552638E+03	.5690441E+01	-.9396933E+01
.2600000E+03	-.1649632E+03	.5608482E+01	-.9848085E+01
.2700000E+03	-.1748056E+03	.5522681E+01	-.1000001E+02
.2800000E+03	-.1848025E+03	.5435526E+01	-.9848085E+01
.2900000E+03	-.1949614E+03	.5349663E+01	-.9396933E+01
.3000000E+03	-.2052850E+03	.5267827E+01	-.8660260E+01
.3100000E+03	-.2157700E+03	.5192752E+01	-.7660450E+01
.3200000E+03	-.2264060E+03	.5127061E+01	-.6427881E+01
.3300000E+03	-.2371754E+03	.5073151E+01	-.5000004E+01
.3400000E+03	-.2480529E+03	.5033060E+01	-.3420204E+01
.3500000E+03	-.2590067E+03	.5008349E+01	-.1736483E+01
.3600000E+03	-.2700000E+03	.5000000E+01	-.1720596E-07

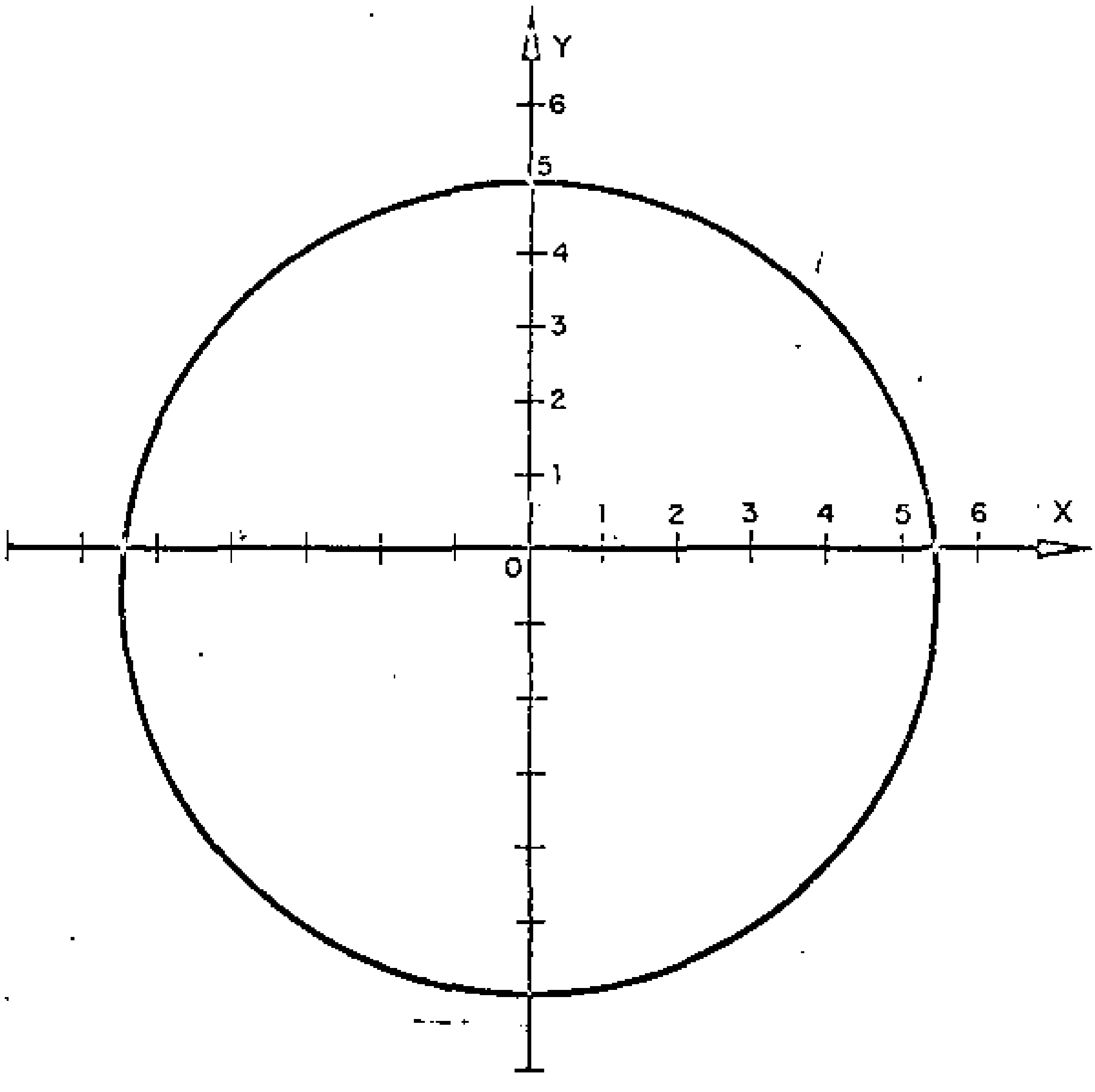

Figure 12. Cam profile that generates the displacement $s = \sin^2 \psi/2 + C$ of a flat-faced follower, so that the contact point has a maximum offsetting of 10 %. C was found to be 5.0 (u. of length)

the Department of Mechanical and Electrical Engineering of the School of Engineering of the University of Mexico (Universidad Nacional Autónoma de México).

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APPENDIX

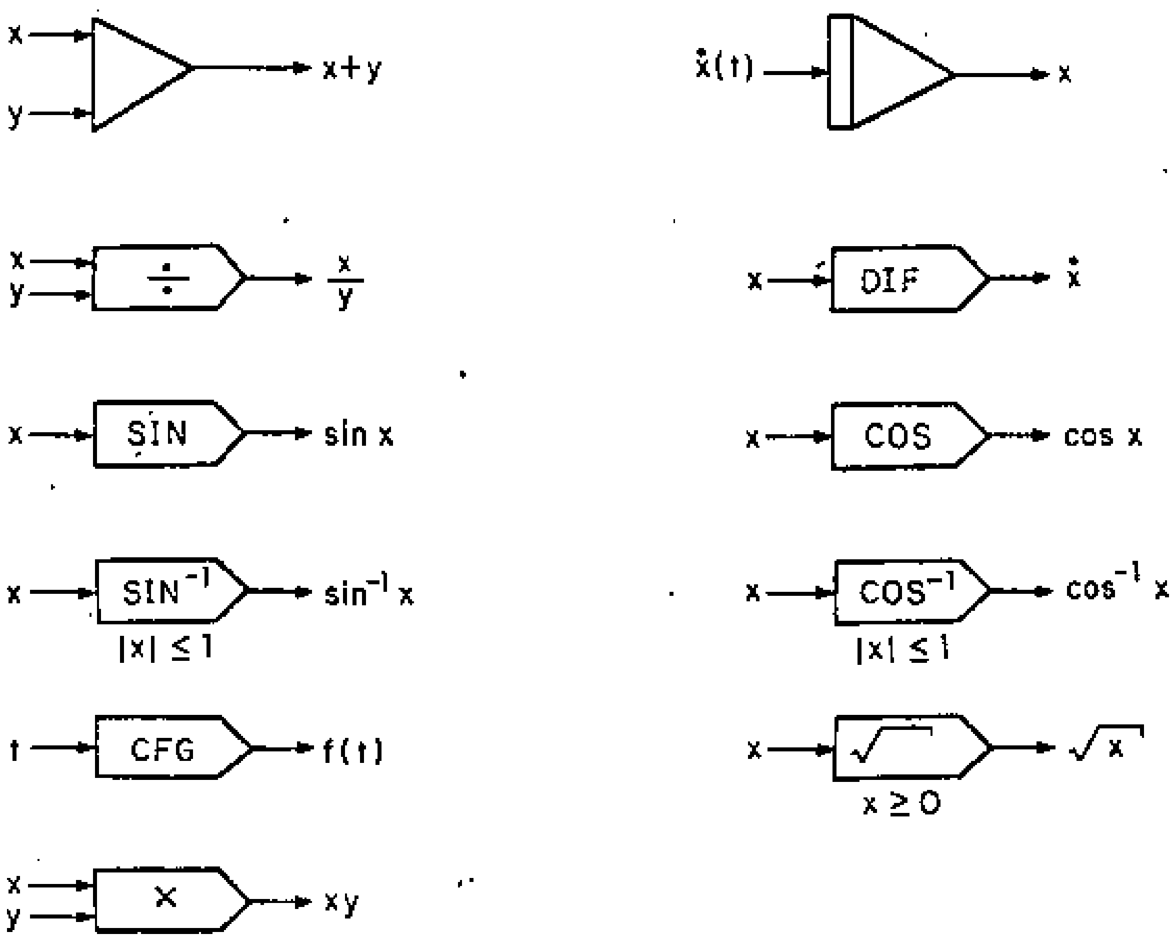

Figure 13. Definition of symbols for operators in analog realizations

"Analyse und Synthese von Kurvenmechanismen mittels Rechners"

Kurzfassung: Es werden die Anwendungsmöglichkeiten von Analogie und Digitalrechnern bei der Lösung von Gleichungen vorgestellt, die sich aus der Analyse und Synthese von aus Nockenscheibe und Stössel gebildeten kinematischen Paaren ergeben. Es wird einleuchtend dargelegt, dass der Schaltkreis eines Analogierechners oder ein Unterprogramm, das nicht-lineare algebraische und/oder nicht-lineare Differentialgleichungen zu lösen vermag, ausreicht, um die Bewegung des Stössels von beliebigem Überpaartyp zu berechnen oder um das Profil der Nockenscheibe, die eine gewünschte Bewegung des Stössels hervorbringen soll, zu bestimmen.

Die hier vorgestellten Methoden zur Erlangung von Synthesen führen unmittelbar zu optimalen Synthesen, optimal in dem Sinne, dass sie gewisse parameter, wie z.B. den Druckwinkel, unterhalb eines gewissen, für eine Nockenscheibe mit minimalen Dimensionen zulässigen Oberwertes halten.

Die Methoden werden an zwei spezifischen Paartypen von Nockenscheibe-Stössel illustriert: Analyse und Synthese von Nockenscheiben mit Spitzstössel und Stössel mit stumpfer Stirnfläche. Die sich ergebenden Gleichungen werden per Digitalrechner errechnet, und die erlangten Synthesen sind von optimalen Dimensionen. Die Ergebnisse werden sowohl in tabellarischer als auch in graphischer Form gegeben.

Aus den hier vorgestellten Fakten kann der Leser schlussfolgern, dass diese Methoden auf andere Typen von Nockenscheibe-Stössel-Paaren unmittelbar übertragbar sind.

Abschliessend sei hier betont, dass mit diesen Methoden möglich ist, jene Präzision und Schnelligkeit optimal auszunützen, die im Produktionsprozess Werkmaschinen mit numerischer Kontrolle bei der Massenproduktion von Kurvenmechanismen bieten.

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

8. SINTESIS DE UN SISTEMA DE SUSPENSION PARA VEHICULOS DE TRANSPORTE MASIVO CON COMPORTAMIENTO DINAMICO PRESCRITO

DR. JORGE ANGELES ALVAREZ

AGOSTO, 1980

SINTESIS DE UN SISTEMA DE SUSPENSION PARA VEHICULOS DE TRANSPORTE MASIVO CON COMPORTAMIENTO DINAMICO PRESCRITO.

JORGE ANGELES[1]

ISMAEL ESPINOSA[2]

RESUMEN

Se presenta un procedimiento de síntesis para obtener recomendaciones de rediseño de la suspensión de vagones de transporte masivo. El objetivo de este estudio es mejorar el funcionamiento del sistema de suspensión bajo condiciones dinámicas de operación de tal manera que las velocidades críticas queden fuera del intervalo más frecuente de operación. Los resultados obtenidos indicaron que sólamente una de las dos secciones de la suspensión requieren amortiguamiento, cuyo valor se obtuvo a través del método de análisis del lugar geométrico de las raíces.

NOMENCLATURA

a, b, c, a', b', c' = coeficientes de polinomios del denominador

a_i = coeficientes del polinomio característico

$\underset{\sim}{A}$ = matriz de coeficientes de 6x6 de la ecuación de estado

b_1 = amortiguamiento de la suspensión primaria, en N s/m.

b_2 = amortiguamiento de la suspensión secundaria, en N s/m.

b_{ij} = elementos de la matriz de amortiguamiento $\underset{\sim}{B}$

$\underset{\sim}{B}$ = matriz de amortiguamiento de 3x3

d, e, f, d', e', f' = coeficientes de polinomios del numerador

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d = diámetro de la rueda en m.

D = función de disipación

$\underset{\sim}{I}$ = matriz de identidad

k_1 = rigidez de los resortes que soportan los ejes, N/m

k_2 = rigidez de los resortes que soportan el puente motor diferencial, N/m.

k_3 = rigidez de los resortes que soportan la mitad de la masa del cuerpo del vagón, N/m.

k_4 = rigidez del resorte del neumático, N/m.

k_{ij} = elementos de la matriz de rigidez $\underset{\sim}{K}$

$\underset{\sim}{K}$ = matriz de rigidez de 3x3

m_1 = masa del chasis (H), N s^2/m.

m_2 = masa de cada puente motor diferencial, N s^2/m

m_3 = mitad de la masa del cuerpo del vagón, N s^2/m.

$\underset{\sim}{M}$ = matriz de inercia de 3x3

T = energía cinética

v = velocidad crítica en m/s

V = energía potencial

$\underset{\sim}{x}$ = vector de coordenadas generalizadas

$\dot{\underset{\sim}{x}}$ = velocidad generalizada

$\underset{\sim}{y}$ = vector característico asociado con el valor característico ω^2.

ω^2 = valor característico

ω = frecuencia natural, en s^{-1}

$|s\underset{\sim}{I}-\underset{\sim}{A}|$ = determinante de la matriz $s\underset{\sim}{I}-\underset{\sim}{A}$

INTRODUCCION

El sistema de transporte masivo en estudio ha mostrado dos velocidades críticas abajo del valor máximo de operación, de las cuales, la más alta se encuentra dentro del intervalo de operación más frecuente, lo que produce condiciones de vibración indeseables que afectan tanto la comodidad del pasajero como la vida de la estructura de la suspensión.

Algunas de las laternativas que deben tomarse en cuenta para mejorar el funcionamiento del sistema son las siguientes: (1) cambiar únicamente la rigidez de los resortes, (2) sin modificar dichas rigideces, introducir amortiguadores adecuados y, (3) usar una combinación de amortiguadores y nuevos valores de rigidez de resortes, lo cual equivaldría a obtener un diseño totalmente nuevo. Por esta razón, sólo se considerará la tercera alternativa en caso de que las dos primeras no fueran factibles.

El primer paso en el procedimiento de síntesis es el de modelar con precisión el sistema actual, como se describe a continuación:

MODELADO DEL SISTEMA

Cada tren está compuesto por nueve vagones (Fig. 1) de los cuales, seis son de tracción y los demás, de arrastre. Cada vagón, ya sea de tracción o de arrastre, está suspendido sobre dos carros localizados en sus extremos, llamados comúnmente "bogies". Cada bogie tiene dos ejes con dos neumáticos en cada eje (Fig. 2). A la vez, cada eje está acoplado al chasís del bogie, denominado "H", por medio de una suspensión llamada "suspensión primaria", compuesta por ocho resortes idénticos, a razón de cuatro por cada eje, de rigidez k_1, y cuatro más de rigidez k_2, de los cuales cada par soporta el "puente del motor diferencial". El "bogie" completo está esquemáticamente representado en la Fig. 3. El cuerpo del vagón se acopla a la "H" por medio de una "suspensión secundaria" compuesta de dos resortes idénticos de rigidez k_3. Por otra parte, la rigidez del resorte de los neumáticos es k_4. Con excepción del amortiguamiento interno del hule en el cual están vulcanizados los resortes, el sistema no cuenta con ninguna otra forma de amortiguamiento. Con referencia a la Fig. 3,

m_1 = masa de la "H"

m_2 = masa de cada puente motor diferencial

m_3 = mitad de la masa del cuerpo del vagón

El modelo icónico que corresponde al esquema de la Fig. 3 se muestra en la Fig. 4, donde

$$\underset{\sim}{x} = [x_1, x_2, x_3]^T$$

es el vector de coordenadas generalizadas.

Se tomaron medidas de campo [1,2][1], de las cuales se obtuvieron dos velocidades críticas. La primera tiene un valor medio 5.5 m/s, y la segunda se encuentra entre 17.5 y 18.9 m/s, de forma tal que la última se localiza dentro del intervalo de operación más frecuente, entre 5.56 m/s y 22.22 m/s. Para propósitos de diseño, este intervalo de velocidad se denominará "región prohibida".

El modelo matemático que corresponde a la Fig. 4 tiene la forma

$$\underset{\sim}{M}\ddot{\underset{\sim}{x}} + \underset{\sim}{K}\underset{\sim}{x} = \underset{\sim}{0} \qquad (1)$$

donde $\underset{\sim}{M}$ y $\underset{\sim}{K}$ son, respectivamente, las matrices de inercia y de rigidez. La masa y la rigidez de los elementos involucrados es supuestamente constante, por lo cual las matrices que aparecen en la ecuación (1) se pueden obtener como las matrices hessianas de la energía cinética con respecto a la velocidad generalizada $\dot{\underset{\sim}{x}}$, y de la energía potencial con respecto a las coordenadas generalizadas $\underset{\sim}{x}$, respectivamente, o sea,

$$\underset{\sim}{M} = \frac{\partial^2 T}{\partial \dot{\underset{\sim}{x}}^2}, \quad \underset{\sim}{K} = \frac{\partial^2 V}{\partial \underset{\sim}{x}^2} \qquad (2)$$

---

[1] Los numeros en paréntesis rectangular indican las referencias al final del informe.

siendo T y V las energías cinética y potencial, respectivamente. Estas son

$$T = \frac{1}{2} m_1\dot{x}_1^2 + \frac{1}{2} (2m_2)\dot{x}_2^2 + \frac{1}{2} m_3\dot{x}_3^2 \qquad (3)$$

y

$$V = \frac{1}{2} (8k_1 + 4k_2 + 4k_4)x_1^2 + \frac{1}{2} (8k_1 + 4k_2 + 2k_3)x_2^2 +$$
$$+ \frac{1}{2} 2k_3x_3^2 - (8k_1 + 4k_2)x_1x_2 - 2k_3x_2x_3 \qquad (4)$$

Entonces,

$$\underset{\sim}{M} = \begin{bmatrix} 2m_2 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \qquad \underset{\sim}{K} = \begin{bmatrix} k_{11} & k_{12} & 0 \\ k_{12} & k_{22} & k_{23} \\ 0 & k_{23} & k_{33} \end{bmatrix} \qquad (5)$$

donde

$$k_{11} = 8k_1 + 4k_2 + 4k_4, \qquad k_{12} = -8k_1 - 4k_2$$
$$k_{22} = 8k_1 + 4k_2 + 2k_3, \qquad k_{23} = -k_{33} - 2k_3 \qquad (6)$$

Las velocidades críticas se obtienen del valor característico ω^2 derivado de la ecuación (1). El problema de valores caracte rísticos correspondiente es, entonces,

$$\underset{\sim}{K}\underset{\sim}{y} = \omega^2 \underset{\sim}{M}\underset{\sim}{y} \qquad (7)$$

en el cual $\underset{\sim}{y}$ es el vector característico asociado al valor ca racterístico ω^2. La relación entre las frecuencias naturales ω y las velocidades críticas v es

$$v = 0.5\, d\omega \qquad (8)$$

donde v está dado en m/s y ω en s^{-1}, siendo d el diámetro de las ruedas en m.

Para determinar qué rigidez de resorte deberá cambiarse, se hizo un análisis de sensibilidad de las velocidades críticas relativo a la rigidez de cada resorte. Los cambios de velocidad crítica se obtuvieron por medio de un modelo de computadora de la ecuación (1) en donde se utilizaron los siguientes valores nominales

$$k_1 = 4.9 \times 10^6 \text{ N/m}$$
$$k_2 = 3.43 \times 10^6 \text{ N/m}$$
$$k_3 = 8.37 \times 10^5 \text{ N/m}$$
$$k_4 = 1.783 \times 10^6 \text{ N/m}$$
$$m_1 = 1.971 \times 10^3 \text{ N s}^2/\text{m}$$
$$m_2 = 1.628 \times 10^3 \text{ N s}^2/\text{m}$$
$$m_3 = 1.578 \times 10^4 \text{ N s}^2/\text{m}$$

Las figuras 5 a 7 muestran las curvas de influencia de la rigidez de cada resorte en cada valor de velocidad crítica. Estas curvas se obtuvieron utilizando un paquete IBM de sub-rutinas [3].

DETERMINACION DE LOS VALORES DE AMORTIGUAMIENTO PARA CAMBIAR LAS VELOCIDADES CRITICAS.

Los resultados obtenidos del análisis de sensibilidad establecieron que la simple modificación de rigidez no sería suficiente para cambiar las velocidades críticas en forma substancial, por lo que es indispensable adicionar amortiguadores.

A continuación, se analiza el nuevo modelo icónico, mostrado en la Fig. 8, en forma semejante a lo hecho para el modelo de la Fig. 4; pero con la inclusión de amortiguadores b_1 y b_2 en las suspensiones primaria y secundaria respectivamente.

De acuerdo con la Fig. 8, el modelo matemático toma la forma

$$\underset{\sim}{M}\ddot{\underset{\sim}{x}} + \underset{\sim}{B}\dot{\underset{\sim}{x}} + \underset{\sim}{K}\underset{\sim}{x} = \underset{\sim}{0} \tag{9}$$

donde las matrices $\underset{\sim}{M}$ y $\underset{\sim}{K}$ son las mismas que aparecen en la ecuación (5) y $\underset{\sim}{B}$ es la matriz de amortiguamiento

$$\underset{\sim}{B} = \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{12} & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{bmatrix} \tag{10}$$

obtenida como la matriz hessiana con respecto a $\dot{\underset{\sim}{x}}$ de la función de disipación D dada como

$$D = \frac{1}{2} b_1(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} b_2 (\dot{x}_3 - \dot{x}_2)^2 \tag{11}$$

Los elementos de $\underset{\sim}{B}$ son

$$\begin{aligned} &b_{11} = b_1,\ b_{12} = -b_1,\ b_{22} = b_1 + b_2 \\ &b_{23} = -b_2,\ b_{33} = b_2 \end{aligned} \tag{12}$$

siendo b_1 y b_2 el amortiguamiento de las suspensiones primaria y secundaria, respectivamente.

La etapa inicial del proceso de solución fue tratar de determinar los valores óptimos de amortiguamiento; peró esto hizo notar que uno de los amortiguadores tenía poco o ningún efecto en las velocidades críticas. Por esta razón, se decidió analizar el efecto simple de cada uno de los amortiguadores. Para hacer esto, se consideró conveniente utilizar el método del lugar geométrico de las raíces [4] .

Para efectuar tal análisis, la ecuación (9) se escribe en la forma usual de variables de estado

$$\dot{\underset{\sim}{x}} = \underset{\sim}{A}\underset{\sim}{x} \tag{13}$$

donde

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{I} \\ \underset{\sim}{M}^{-1}\underset{\sim}{K} & \underset{\sim}{M}^{-1}\underset{\sim}{B} \end{bmatrix} \tag{14}$$

e $\underset{\sim}{I}$ es la matriz de identidad de 3x3.

El polinomio característico asociado con la ecuación (13) es

$$\left|s\underset{\sim}{I} - \underset{\sim}{A}\right| = s^6 + a_5 s^5 + a_4 s^4 + a_3 s^2 + a_1 s + a_0 \tag{15}$$

donde

$$a_5 = \frac{b_{11}}{2m_2} + \frac{b_{33}}{m_3} + \frac{b_{22}}{m_1}$$

$$a_4 = \frac{b_{11}\, b_{33}}{m_1 m_3} + \frac{b_{11}}{2m_2}\left(\frac{b_{33}}{m_1} + \frac{b_{33}}{m_3}\right) + \frac{k_{11}}{2m_2} + \frac{k_{22}}{m_1} + \frac{k_{33}}{m_3}$$

$$a_3 = \frac{b_{23}\, k_{12} + b_{11}\, k_{33}}{m_1 m_3} + \frac{k_{11} + k_{12}}{2m_2}\left(\frac{b_{22}}{m_1} + \frac{b_{33}}{m_3}\right) +$$

$$+ \frac{k_{12}}{2m_2}\left(\frac{b_{23}}{m_1} + \frac{b_{23}}{m_3}\right) + \frac{b_{11}}{2m_2}\left(\frac{k_{33}}{m_1} + \frac{k_{33}}{m_3}\right)$$

$$a_2 = \frac{k_{12} k_{33}}{m_1 m_3} + \frac{k_{11} + k_{12}}{2m_2}\ \frac{k_{22}}{m_1} + \frac{k_{33}}{m_3} - \frac{k_{12}}{2m_2}\left(\frac{k_{33}}{m_1} + \frac{k_{33}}{m_3}\right) +$$

$$+ \frac{b_{11} b_{33}(k_{11} + k_{12})}{2m_1 m_2 m_3}$$

$$a_1 = \frac{(b_{11}k_{33} + b_{23}k_{12})\ (k_{11} + k_{12})}{2m_1m_2m_3}$$

$$a_0 = \frac{-(k_{11} + k_{12})\ k_{12}k_{33}}{2m_1m_2m_3}$$

Entonces, la ecuación característica es

$$s^6 + a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 = 0 \qquad (16)$$

y puede ser manipulada para que adquiera una forma adecuada para el análisis del lugar geométrico de las raices. Se considerarán tres casos: (1) $b_1 = 0$, $b_2 \neq 0$, (2) $b_1 \neq 0$, $b_2 = 0$, y (3) $b_1 \neq 0$, $b_2 \neq 0$.

Para el caso (1), la ecuación (16) se puede escribir como

$$\frac{b_{33}s\ (ds^4 + es^2 + f)}{s^6 + as^4 + bs^2 + c} = -1 \qquad (17)$$

donde

$$a = \frac{k_{11}}{2m_2} + \frac{k_{22}}{m_1} + \frac{k_{33}}{m_3}$$

$$b = \frac{k_{12}k_{33}}{m_1m_3} + \frac{k_{11} + k_{12}}{2m_2}\ \frac{k_{22}}{m_1} + \frac{k_{33}}{m_3} - \frac{k_{12}}{2m_2}\ (\frac{k_{33}}{m_1} + \frac{k_{33}}{m_3})$$

$$c = \frac{-(k_{11} + k_{12})\ k_{12}k_{33}}{2m_1m_2m_3}$$

$$d = \frac{1}{m_1} + \frac{1}{m_3}$$

$$e = \frac{-k_{12}}{m_1m_3} + (\frac{k_{11} + k_{12}}{2m_2})\ (\frac{1}{m_1} + \frac{1}{m_3}) - \frac{k_{12}}{2m_2}\ (\frac{1}{m_1} + \frac{1}{m_3})$$

$$f = \frac{-k_{12}\,(k_{11} + k_{12})}{2m_1 m_2 m_3}$$

En la ecuación (17), usando $b_{33} = b_2$ como un parámetro que varía de 0 a ∞, se obtienen las curvas del lugar geométrico de las raíces, que se muestran en las Figs. 3 y 10.

Para el caso (2), la ecuación (16) se puede escribir como

$$\frac{b_{11} s\,(d' s^4 + e' s^2 + f')}{s^6 + a' s^4 + b' s^2 + c'} = -1$$

donde

$$a' = a$$

$$b' = b$$

$$c' = c$$

$$d' = \frac{1}{2m_2} + \frac{1}{m_1}$$

$$e' = \frac{k_{33}}{m_1 m_3} + \frac{k_{11} + k_{12}}{2m_2} \quad \frac{1}{m_1} + \frac{1}{2m_2}\left(\frac{k_{33}}{m_1} + \frac{k_{33}}{m_3}\right)$$

$$f' = \frac{k_{33}\,(k_{11} + k_{12})}{2m_1 m_2 m_3}$$

Ahora, $b_{33} = b_1$ es el parámetro, y la curva del lugar geométrico de las raíces se muestra en la Fig. 11.

Para hacer la curva del caso (3) se requiere de los llamados contornos de las raices [5], puesto que hay dos parámetros que considerar. Sin embargo, el resultado en el caso (2) indica que el amortiguador b_1 casi no tiene ningún efecto en la modificación de las velocidades críticas. Por tal razón, se le dio un valor fijo a b_1 ($b_1 = 1.632 \times 10^3$ N s/m), y b_2 se tomó como el parámetro, de lo cual se obtuvo la curva que se muestra en la Fig. 12.

CONCLUSIONES

Las curvas que aparecen en las Figs. 5 a 7 muestran que cambios considerables en las rigideces de los resortes no modifican substancialmente el comportamiento dinámico del sistema. De hecho, las rigideces k_1 y k_2 no tienen ningún efecto en tal comportamiento. Por esta razón, se decidió considerar la segunda alternativa.

Los lugares geométricos de las raíces, dibujados en las Figs. 9 a 12, muestran la variación de las raíces características al cambiar los valores del amortiguamiento. De los tres ramales que se muestran, sólo dos son de interés: los que se encuentran dentro de la región prohibida. Estas raíces están directamente relacionadas con las velocidades críticas por medio de la ecuación (8) y, por tanto, la posibilidad de mover las raíces características del eje imaginario a otras posiciones en el plano complejo, implica la modificación de las velocidades críticas.

El amortiguamiento de la suspensión primaria (caso 2) no tiene ningún efecto en la modificación de las velocidades críticas, ya que el sistema se vuelve incontrolable y las raíces de interés permanecen en la misma posición (ver la región prohibida en Fig. 11).

Por otro lado, el amortiguamiento en la suspensión secundaria (caso 1) genera una solución comprometida puesto que las raíces de interés se mueven en direcciones opuestas, de tal manera que mientras un ramal se aleja de la región prohibida, el otro se acerca a ella (ver Fig. 10). Por tal motivo, se escogió solamente un valor de amortiguamiento como la solución más adecuada, siendo este: $b_2 = 2.471 \times 10^6$ N s/m. Usando este valor de amortiguamiento, las raíces de interés permanecen fuera de la región prohibida y, consecuentemente, generan velocidades críticas adecuadas.

Con el uso de amortiguamiento en las suspensiones primaria y secundaria (caso 3) los resultados no se alteran puesto que, como se puede ver en la Fig. 12, los ramales de interés en el lugar geométrico de las raíces no sufren ninguna variación apreciable, dado que solamente el tercer ramal, que se encuentra fuera de la región prohibida, cambia su forma. Es fácil inferir que este resultado es válido para todos los valores de b_1 en el caso (3), puesto que se requiere de valores muy altos de b_1 para obtener modificaciones muy ligeras del lugar geométrico correspondiente. Consecuentemente, la suspensión primaria no requiere amortiguamiento.

En trenes de alta velocidad, el amortiguamiento no elimina la condición vibratoria, pero se han determinado valores óptimos para disminuirla [6].

Sin embargo, en el sistema de transporte aquí presentado, el problema no es la alta velocidad y por tanto, el comportamiento dinámico del tren podría ser mejorado satisfactoraimente mediante la inclusión de amortiguadores en la suspensión secundaria. No obstante, al igual que en [6], también sería conveniente estudiar las propiedades del riel y su alojamiento, así como su influencia en la dinámica vertical del tren para poder obtener un mejor conocimiento del sistema, que permita recomendar soluciones más adecuadas.

RECONOCIMIENTO

Este proyecto de investigación fue costeado conjuntamente por el Sistema de Transporte Colectivo del Distrito Federal y la División de Estudios Superiores (hoy de Posgrado) de la Facultad de Ingeniería de la Universidad Nacional Autónoma de México.

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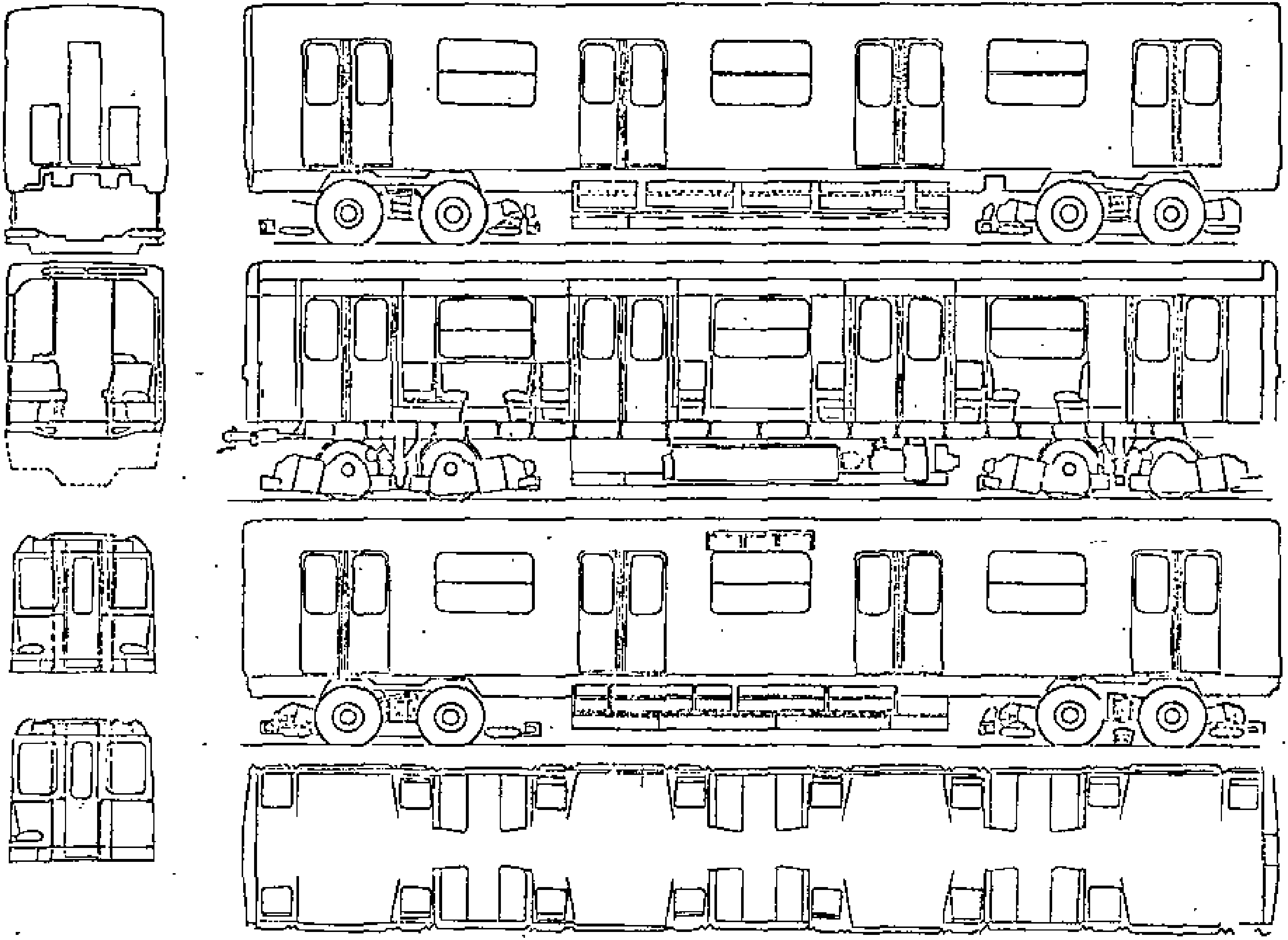

FIG. 1 VAGON DE TRANSPORTE MASIVO

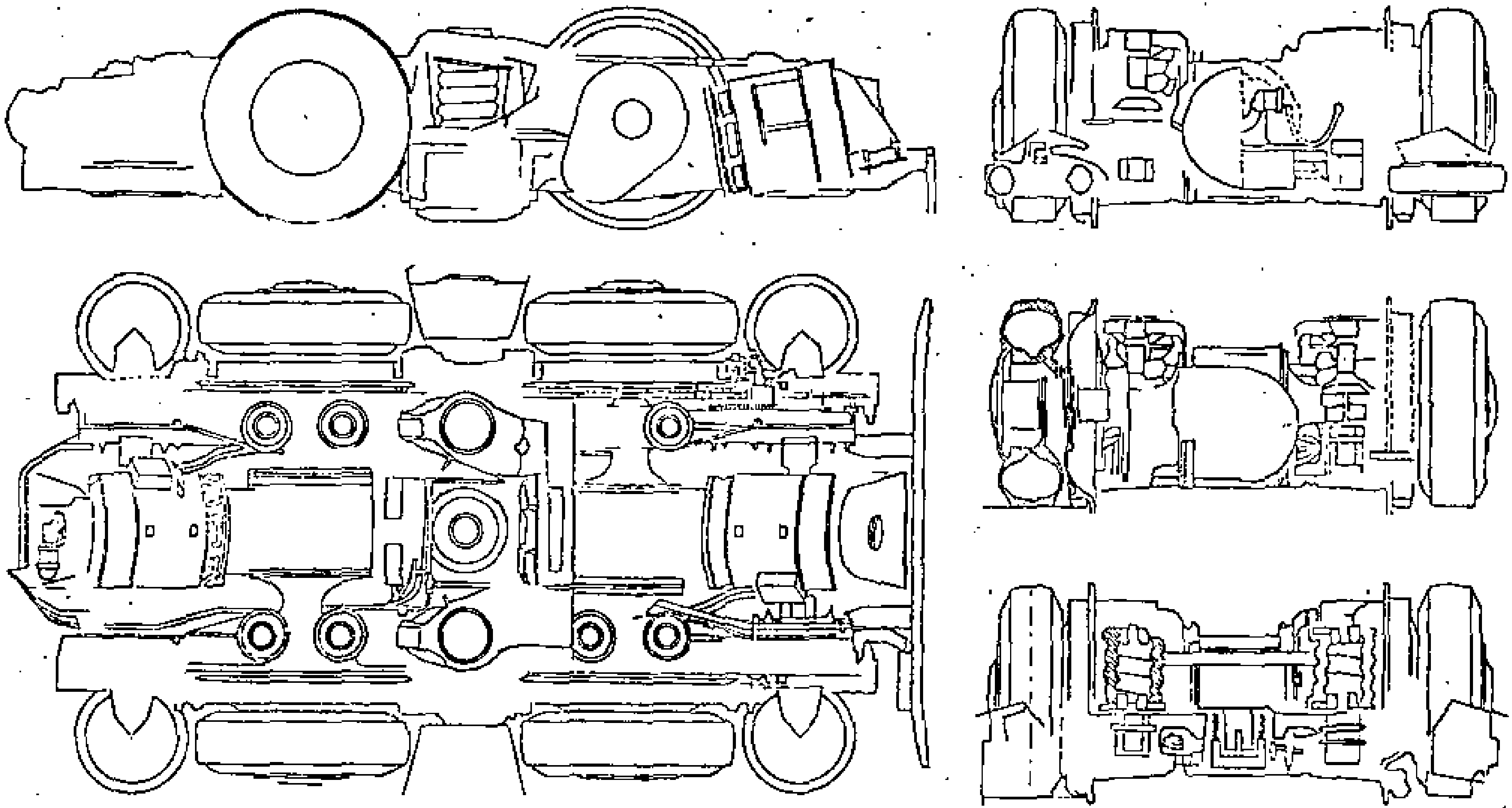

FIG. 2 "BOGIE" DE LA SUSPENSION

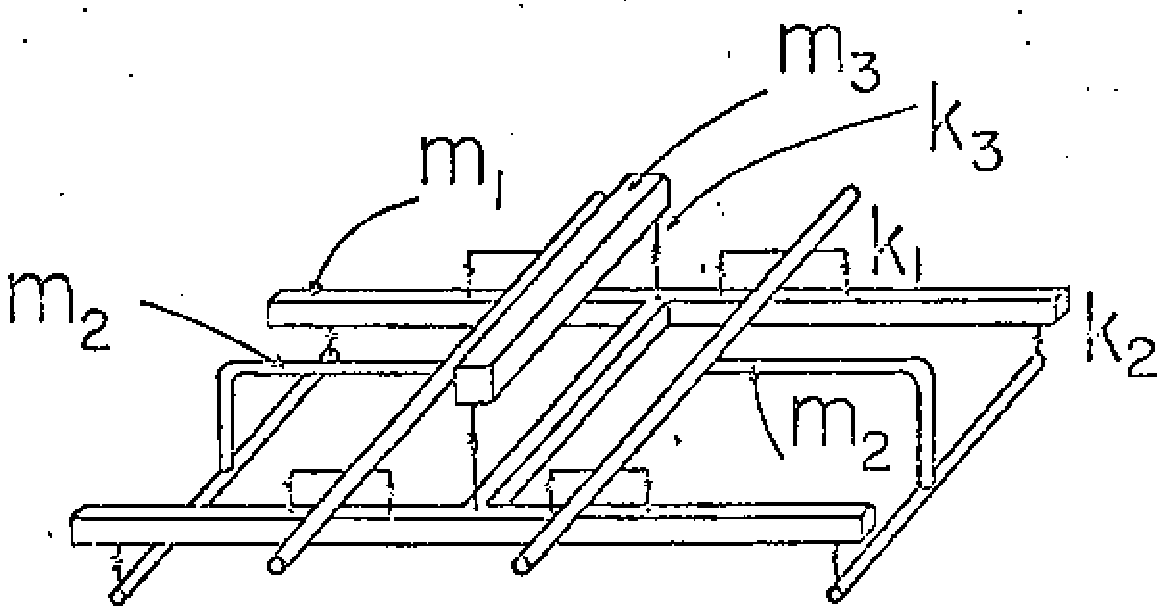

FIG. 3 DISPOSICION DE LOS ELEMENTOS DEL SISTEMA DE SUSPENSION

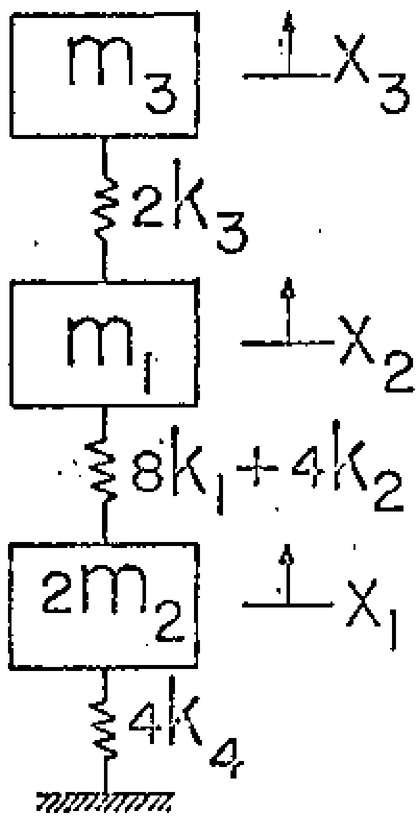

FIG. 4 MODELO ICONICO DEL SISTEMA DE SUSPENSION SIN AMORTIGUAMIENTO

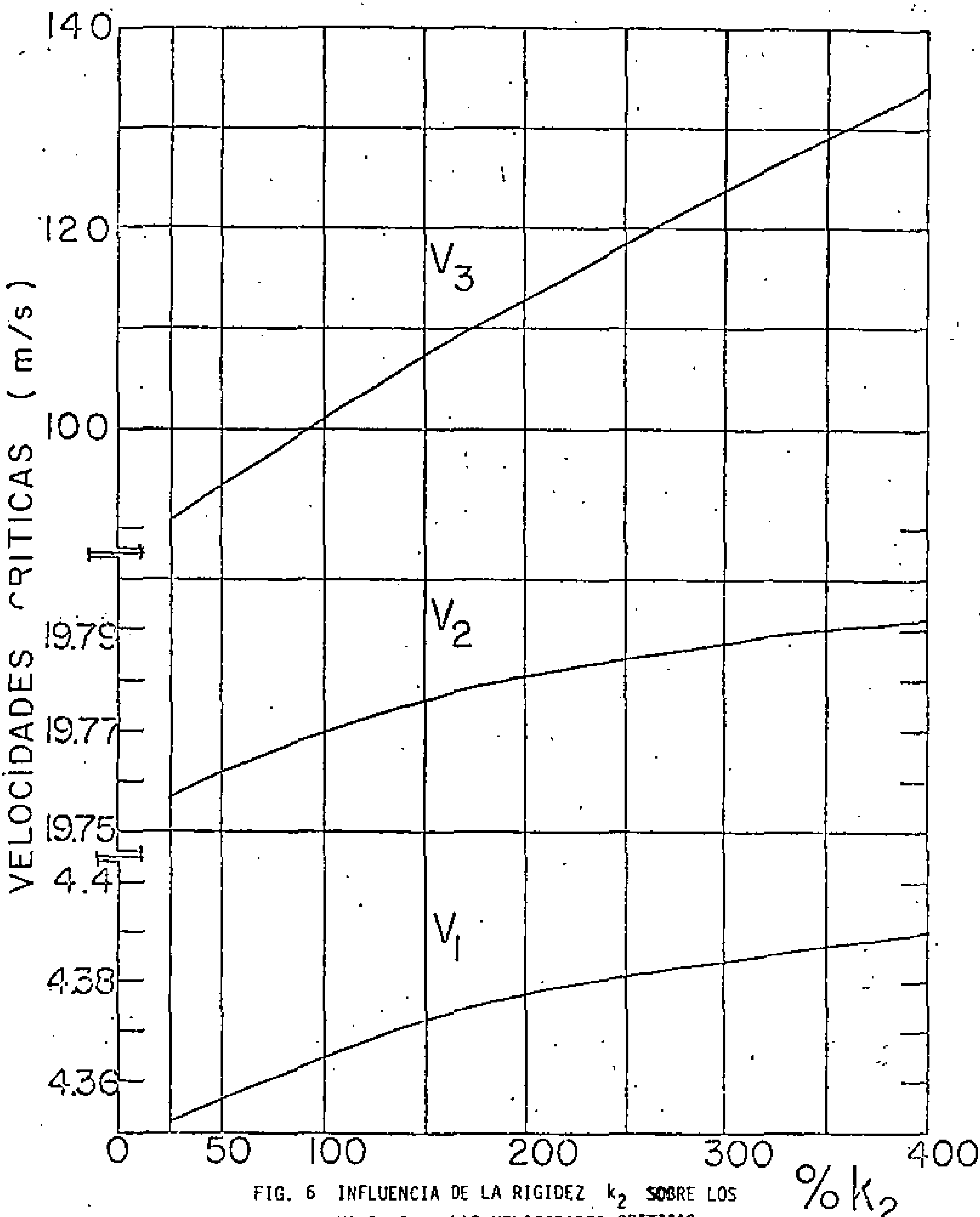

FIG. 6 INFLUENCIA DE LA RIGIDEZ k_2 SOBRE LOS VALORES DE LAS VELOCIDADES CRITICAS

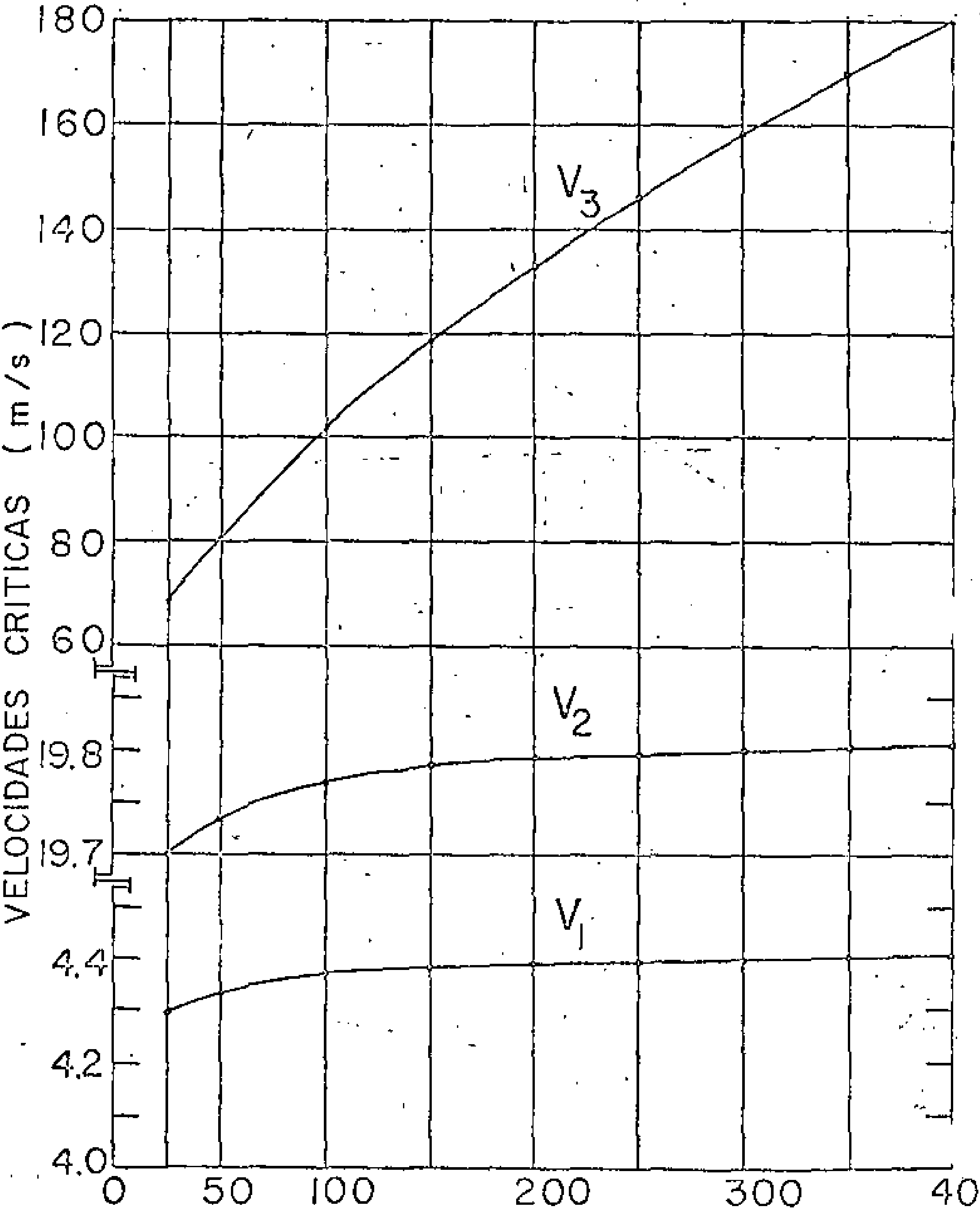

FIG. 5 INFLUENCIA DE LA RIGIDEZ k_1 SOBRE LOS VALORES DE LAS VELOCIDADES CRITICAS

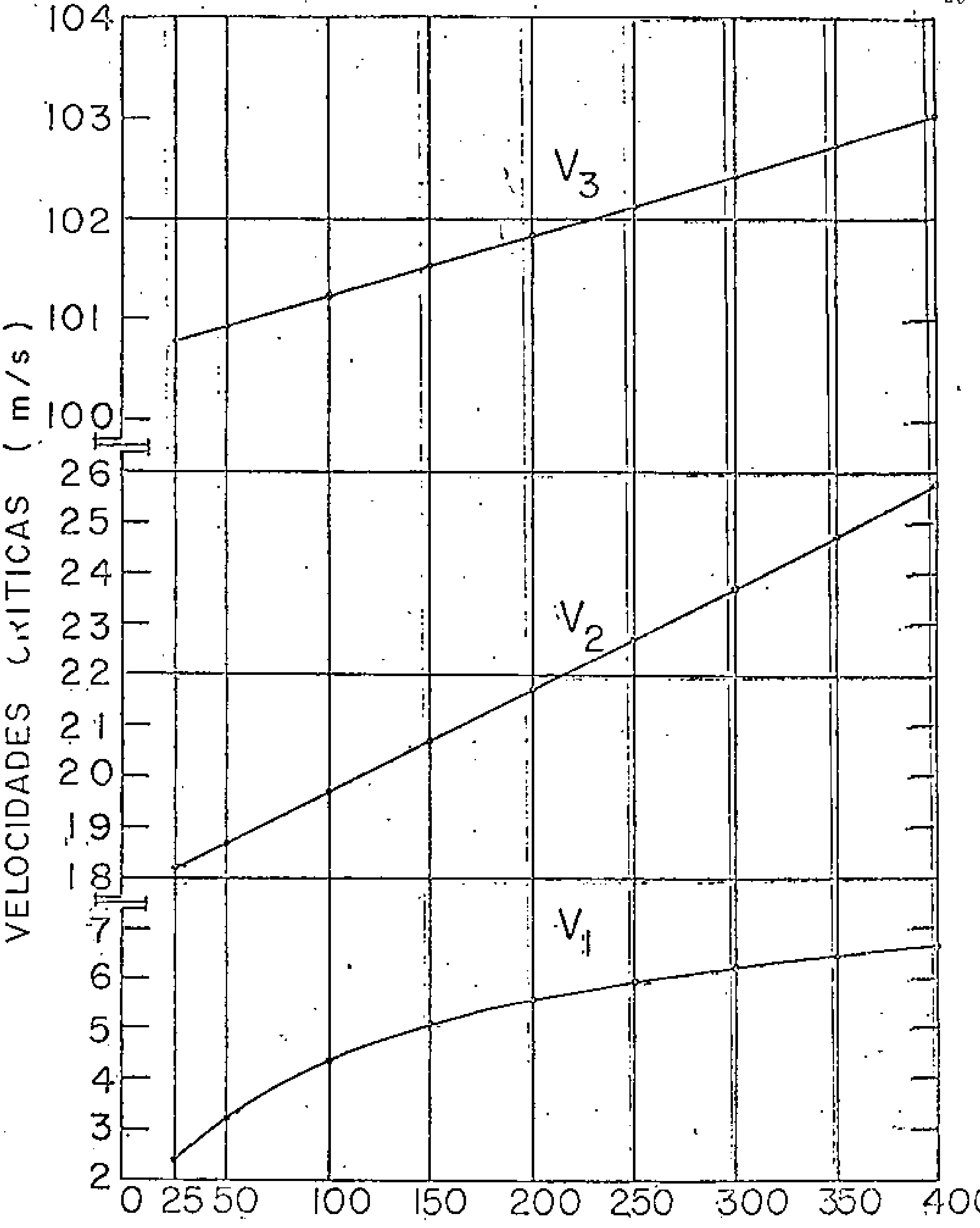

FIG. 7 INFLUENCIA DE LA RIGIDEZ k_3 SOBRE LOS VALORES DE LAS VELOCIDADES CRITICAS

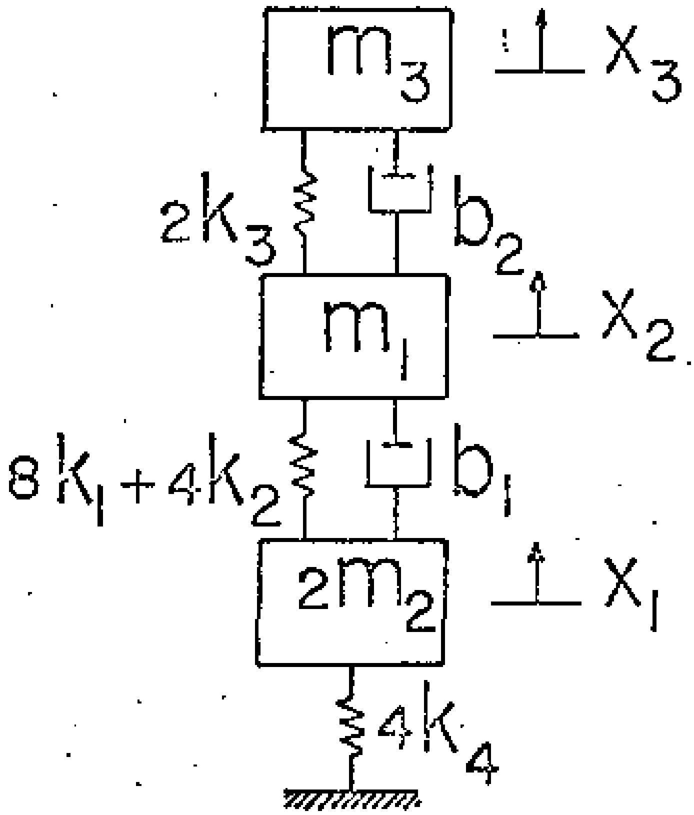

FIG. 8 MODELO ICONICO DEL SISTEMA DE SUSPENSION AMORTIGUADO

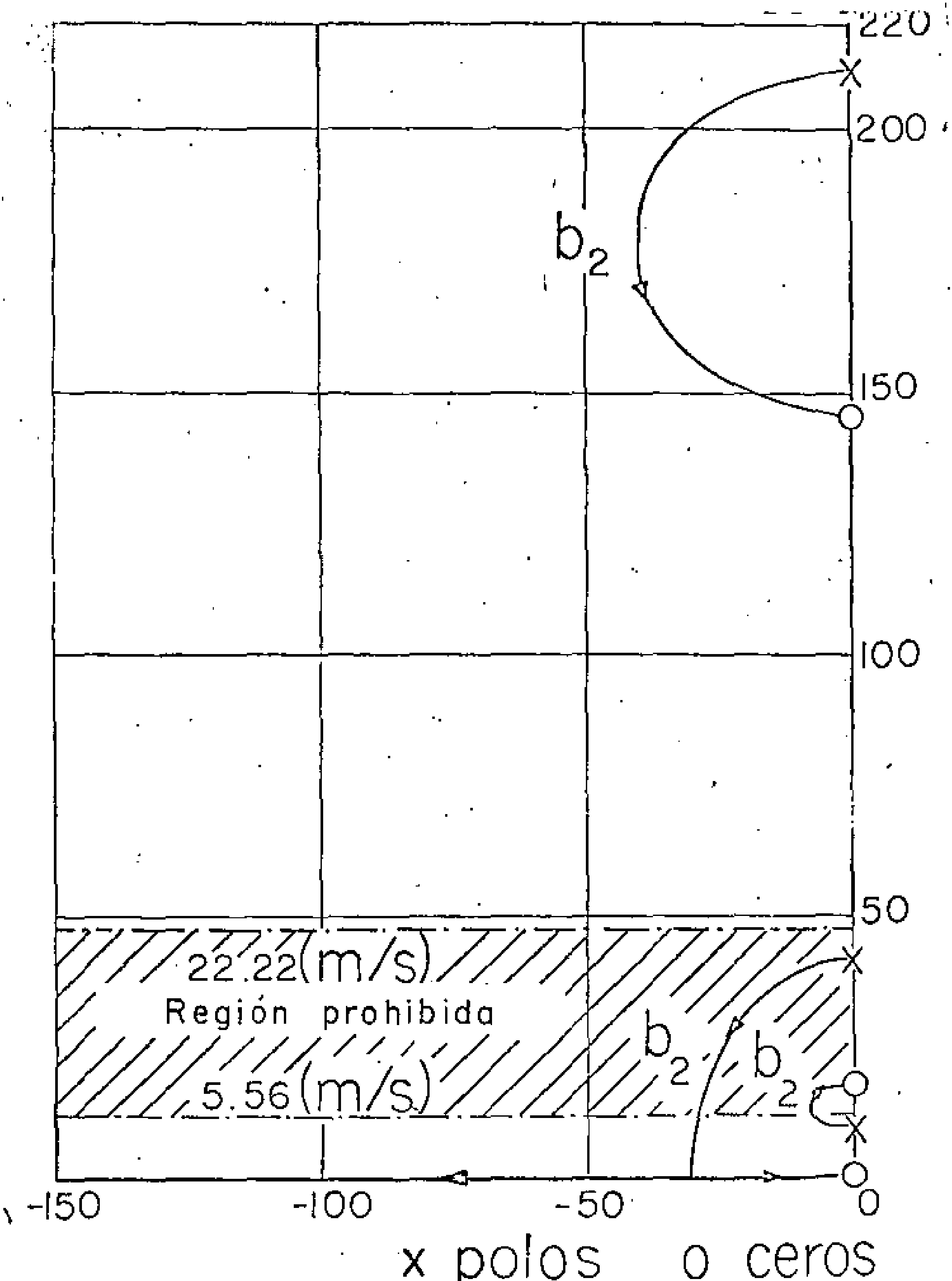

FIG. 9 LUGAR GEOMETRICO DE LAS RAICES DE LA EC (17) CON $b_1 = 0$ y $0 \leq b_2 < \infty$

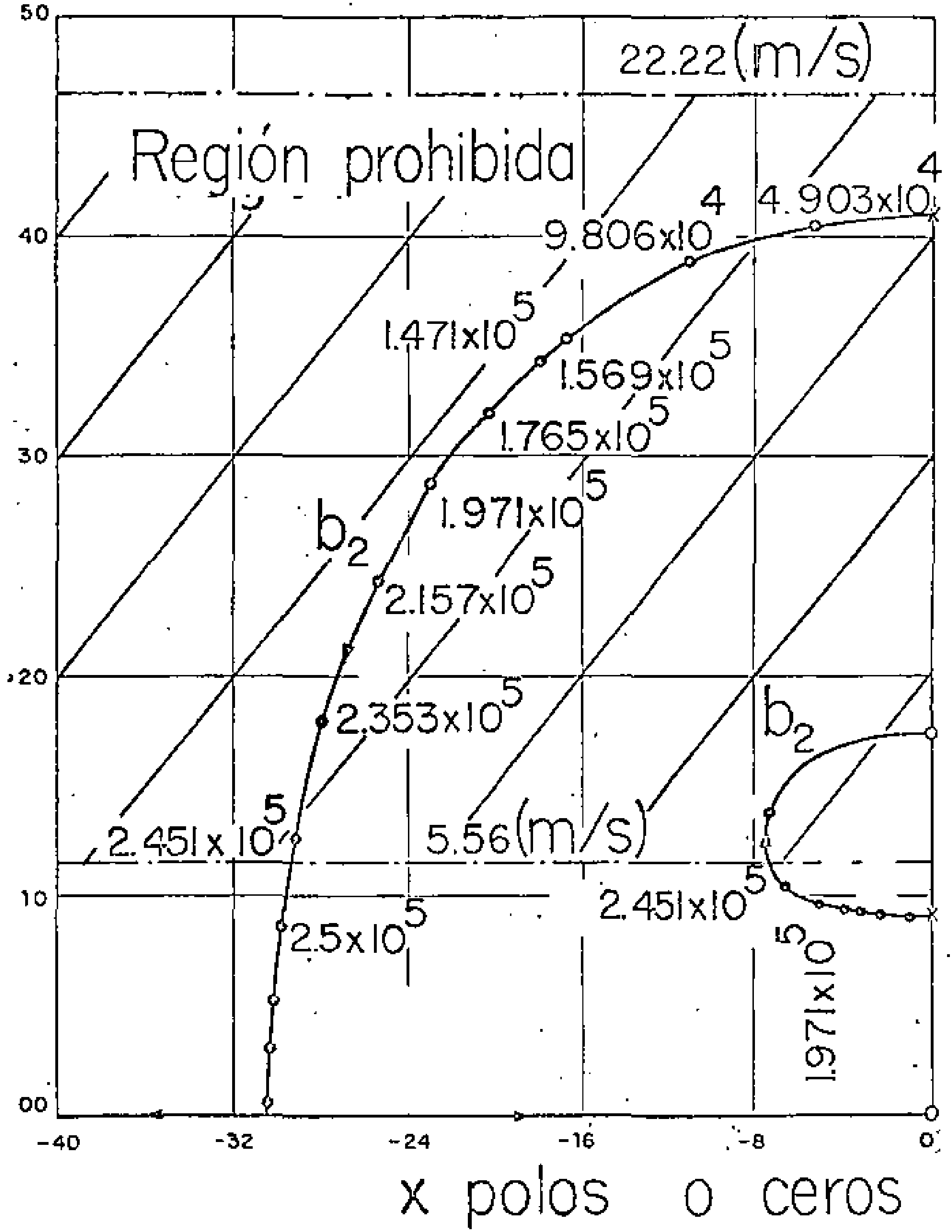

FIG. 10 DETALLE DEL LUGAR GEOMETRICO DE LA FIG. 9

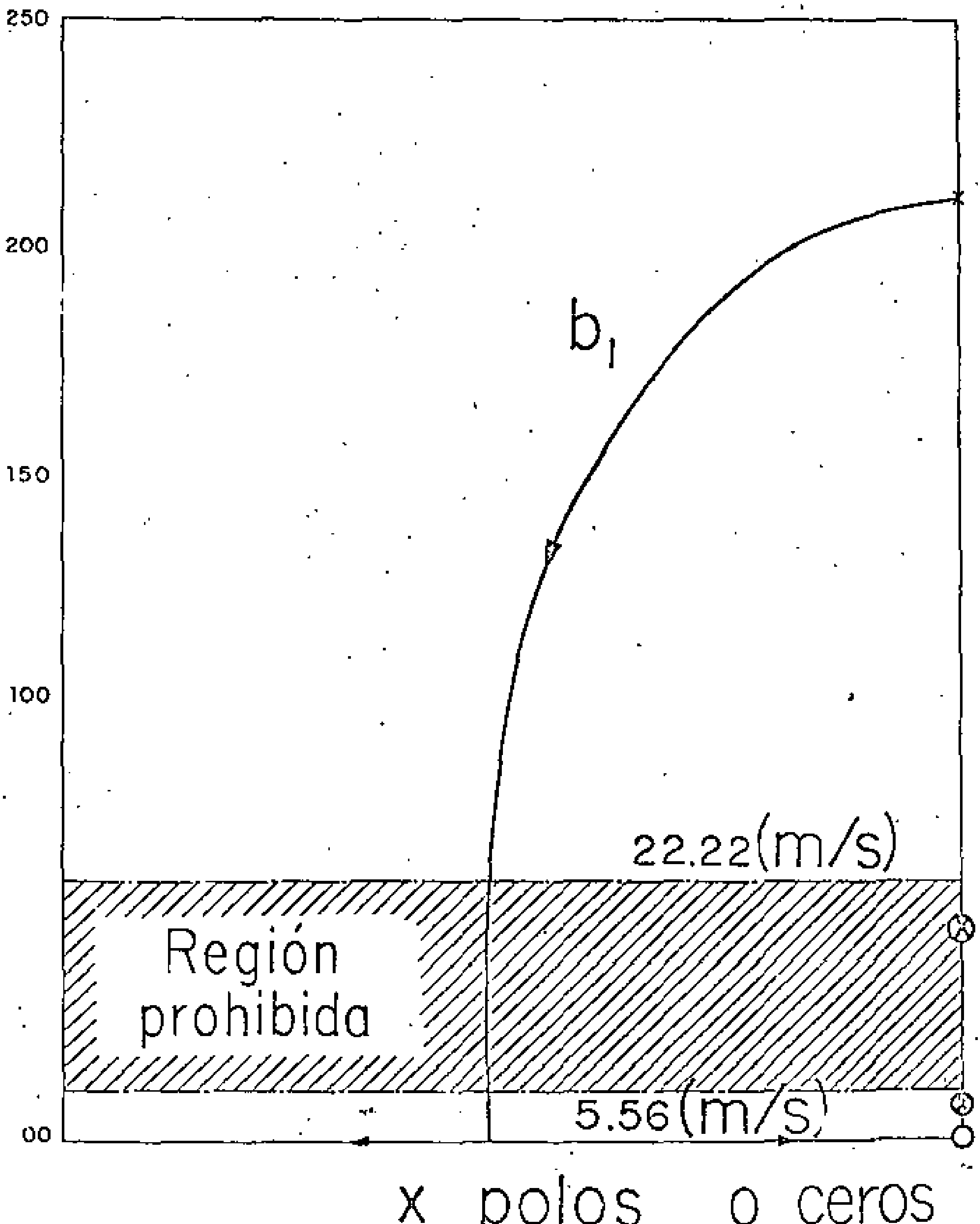

FIG. 11 LUGAR GEOMETRICO DE LAS RAICES DE LA EC. (18) CON $0 \leq b_1 < \infty$ Y $b_2=0$.

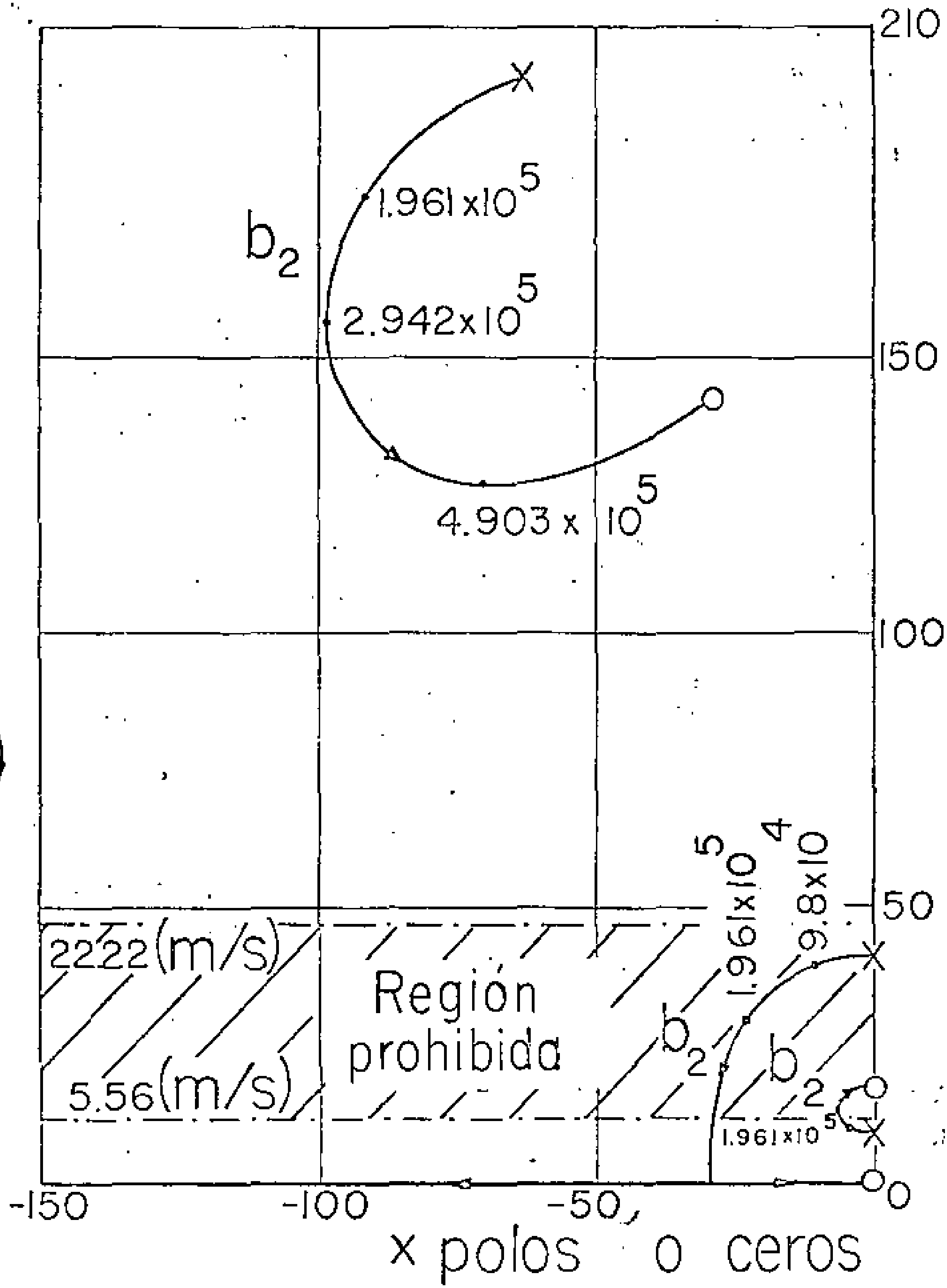

FIG. 12 LUGAR GEOMETRICO DE LAS RAICES PARA $b_1 = 1.569 \times 10^5$ Y $0 \leq b_2 < \infty$

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ANALISIS SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

5. SYNTHESIS OF LINKAGES

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5. SYNTHESIS OF LINKAGES

5.1 INTRODUCTION. The problem of linkage synthesis of Applied Kinematics was outlined in Chapter 3. In the present chapter, the problem of exact synthesis is discussed and current methods of synthesis are presented. The three usual problems of synthesis are discussed, namely

i) funtion generation

ii) rigid body guidance

iii) path generation

and exact solutions to the resulting design equations are meant to be obtained, these solutions being exact up to round-off and/or measuring errors. Chapter 6 deals with the problem where no exact solution can be found, in which case the best approximation is sought, in the sense of rendering the minimum quadratic error in the approximation.

5.2 SYNTHESIS FOR FUNCTION GENERATION

Due to the fact that a linkage is a coupling of rigid bodies, a finite number of parameters (like those of the notation of Denavit and Hartenverg, Ch. 3) defines it. Hence, the set of design equations is of an algebraic character, i.e. no derivatives of the design parameters appear in them, and the number of these parameters is finite. Hence, no linkage can be obtained to produce an arbitrarily prescribed input/output function pointwise in the whole continuum of values of the input where the function is prescribed. The said function can only be produced at a finite set of input values, the number of this set being equal to the number of independent design parameters. Thus, the problem of linkage synthesis for function generation can be stated as:

"Given a function $f=f(x_i)$ $(i=1,2,...,n)$, defined over a discrete set $\{x_i\}_1^n$, find the relevant dimensions of a linkage of a given topology* to produce an input-output relationship that coincides with the function f at the given discrete set $\{x_i\}_1^n$ of input values*. The method to solve this problem consits of two stages, namely

i) Derivation of the input-output relationship for the prescribed topology, and

ii) Determination of the linkage parameters from the above relationship.

The first stage is now discussed. In ch. 4 it was shown that the MDH** can be applied to obtain an input-output relationship that, hopefully, does not contain other variables than the input and the output. It was also shown that an alternate method, less complex than that of Denavit and Hartenberg, guarantees that only the input and the output variables will appear in the input-output relationship. That method, however, is restricted to single-loop mechanisms, whereas that of DH can be extended to multiple-loop mechanisms. Hence, either method can be applied in the first stage of this problem, for single-loop linkages. The second stage is carried out by two different approaches, which are next discussed. It is assumed that an input-outout implicit function has been obtained, this function having the general form

$$f(x_i, y_i, \underline{p}) = 0, i = 1, 2, \ldots, n \qquad (5.2.1)$$

where $\{(x_i, y_i)\}_1^n$ is a set of n pairs of values relating the ith value of the input x to the ith value of the output y, and $\underline{p}$ is an n-dimensional vector containing the parameters of the linkage under consideration. Notice that (5.2.1) represents in fact a system of n synthesis equations

*See Section 3.6
**Method of Denavit and Hartenberg

in the n unknowns that constitute vector $\underline{p}$. Hence, the synthesis problem can be solved through eq. (5.2.1). However, the system (5.2.1) is, in general, nonlinear, and no unique solution is guaranteed to exist; even more, the system might have no solution at all. If the system has one or more solutions, these can be obtained via the method of Newton-Raphson, as shown in Section 1.13. This is the first approach to the solution of the synthesis equations.

The second approach introduces a nonlinear transformation of the synthesis parameters which transforms the synthesis equations into a linear system. Let

$$\underline{q}=\underline{q}(\underline{p}) \tag{5.2.2}$$

be this tranformation. When (5.2.2) is introduced into eq. (5.2.1), the following linear system is obtained

$$\underline{A}\underline{q}=\underline{b} \tag{5.2.3}$$

System (5.2.3) can be solved very efficiently via the LU decomposition algorithm, as was shown in Section 1.12. Once the unique solution $\underline{q}_u$ to (5.2.3) has been obtained, this is introduced into eq. (5.2.2), which is nonlinear if the original system of synthesis equations was nonlinear as well, and, if this tranformation -eq. (5.2.2)- is chosen in such a way that the synthesis parameters appear in it very weakly coupled, then the linkage parameter vector $\underline{p}$ can be obtained without the need of a numerical method. The synthesis procedure is illustrated by means of the following example.

Example 5.2.1 Given the RSSR linkage appearing in Fig 5.2.1, determine its geometric parameters $a_1, a_2, a_3, a_4, s_1, s_4$ and α_4 that produce a given input-output relationship

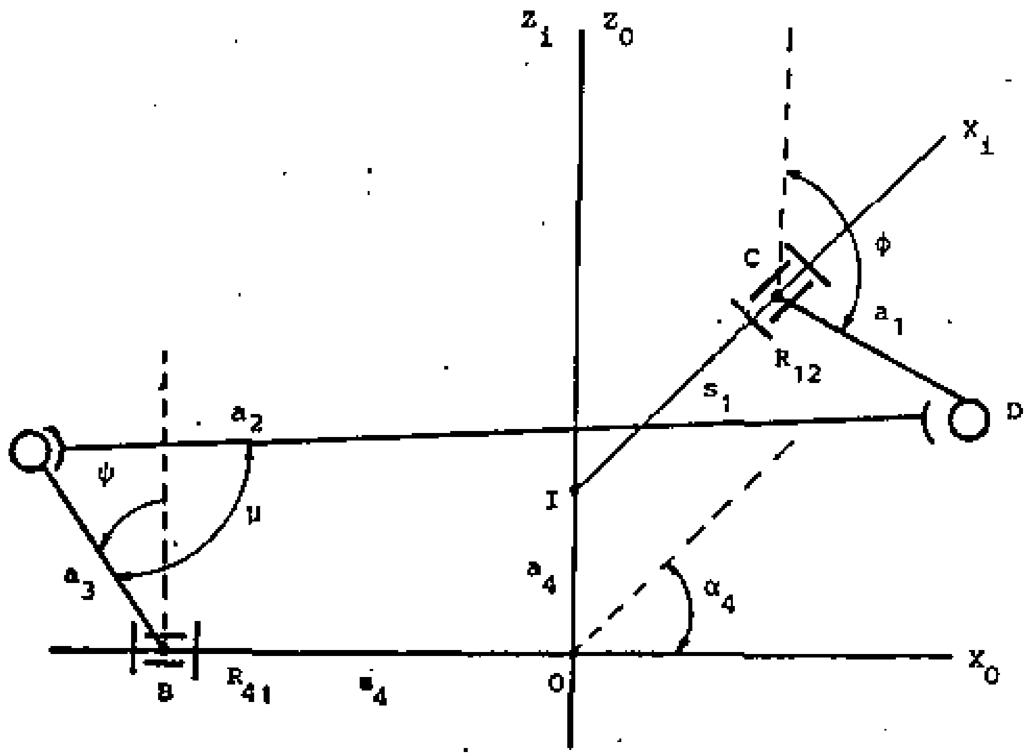

Fig 5.2.1 RSSR Function generating linkage

Using the method of Section 4.3, define sets of axes X_i Y_i Z_i and X_o Y_o Z_o whose X-axes coincide with the axes of R_{12} and R_{41} respectively, its Z-axes being parallel to the common normal OI, and define their positive directions from O to I. Finnally, the Y-axes form with the previous ones right-hand rectangular sets of coordinate axes, as shown in Fig 5.2.2.

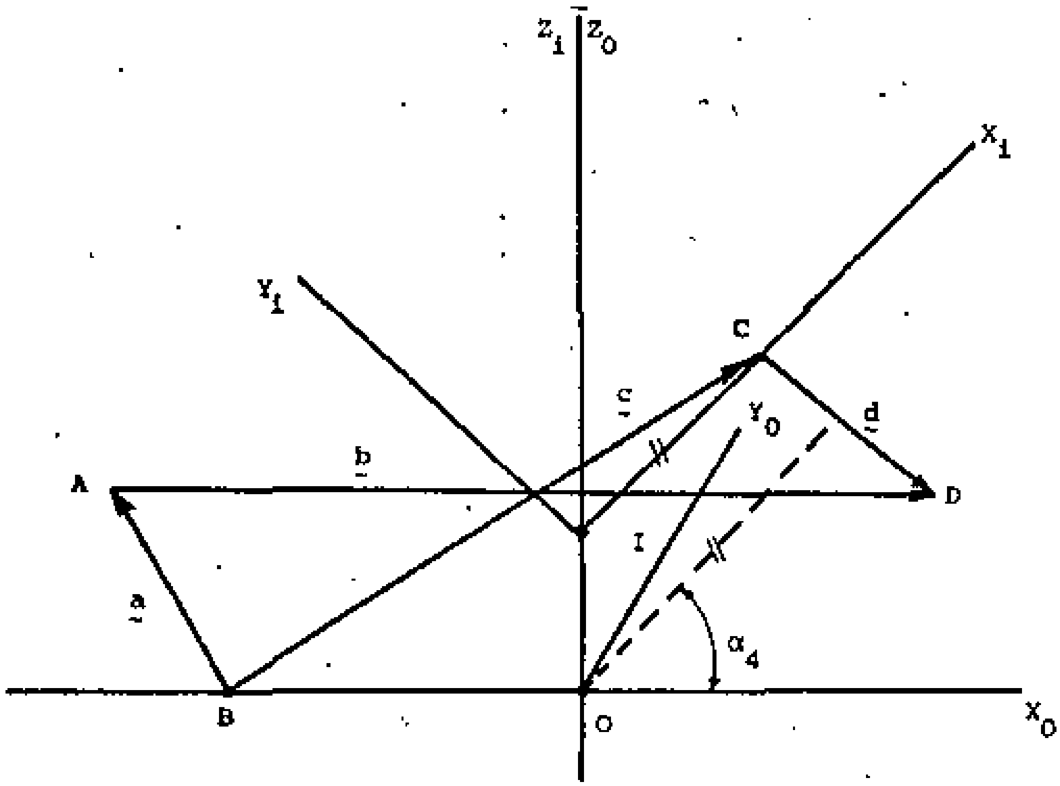

Fig 5.2.2 Coordinate axes fixed at input and output axes

Define:

$\underline{a}$ = vector directed from B to A

$\underline{b}$ = vector directed from A to D

$\underline{c}$ = vector directed from B to C

$\underline{d}$ = vector directed from C to D

From the geometry of the linkage, then

$$\underline{a}+\underline{b}=\underline{c}+\underline{d} \tag{5.2.4}$$

Hence,

$$\underline{b}=\underline{c}+\underline{d}-\underline{a}, \tag{5.2.5}$$

and

$$\underline{b}^T=\underline{c}^T+\underline{d}^T-\underline{a}^T \tag{5.2.6}$$

Let $\underline{a}_j$, $\underline{b}_j$ and $\underline{d}_j$ $(j=0,1,2,\ldots,n)$ be the values attained by vectors $\underline{a}$, $\underline{b}$ and $\underline{d}$ respectively, at the successive configurations j.

Substituting the values of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{d}$ at their current configuration j in eqs. (5.2.5a and b) and multiplying termwise the resulting equations leads to

$$||\underset{\sim}{b}_j||^2=||\underset{\sim}{c}||^2+||\underset{\sim}{d}_j||^2+||\underset{\sim}{a}_j||^2+2\underset{\sim}{c}^T\underset{\sim}{d}_j-2\underset{\sim}{c}^T\underset{\sim}{a}_j-2\underset{\sim}{d}_j^T\underset{\sim}{a}_j \qquad (5.2.7)$$

where, due to the rigid-body condition,

$$||\underset{\sim}{a}_j||^2=a_3^2 \qquad (5.2.8a)$$

$$||\underset{\sim}{b}_j||^2=a_2^2 \qquad (5.2.8b)$$

$$||\underset{\sim}{d}_j||^2=a_1^2 \qquad (5.2.8c)$$

From fig 5.2.2, vector $\underset{\sim}{c}$ is given as

$$\underset{\sim}{c} = \vec{BO} + \vec{OI} + \vec{IC} \qquad (5.2.9)$$

where, with reference to i-coordinates,

$$\vec{BO} = \left(s_4c\alpha_4,\ -s_4s\alpha_4,\ 0\right)^T \qquad (5.2.10a)$$

$$\vec{OI} = \left(0,\ 0,\ a_4\right)^T \qquad (5.2.10b)$$

where the notation cx, sx has been introduced to represent cosx and sinx, respectively, whenever the variable x has been previously defined as an angle. Thus, eq. (5.2.9) leads to

$$\left(\underset{\sim}{c}\right)_i = \left(s_4c\alpha_4+s_1,-s_4s\alpha_4,a_4\right)^T \qquad (5.2.10c)$$

Furthermore,

$$\left(\underset{\sim}{d}\right)_i = \left(0,-a_1\sin\phi,a_1\cos\phi\right)^T \qquad (5.2.11a)$$

$$\left(\underset{\sim}{a}\right)_0 = \left(0,-a_3\sin\phi,a_3\cos\phi\right)^T \qquad (5.2.11b)$$

Define $\underset{\sim}{S}$ as the rotation matrix carrying axes labelled i into those labelled o. Thus

$$\left(\underset{\sim}{S}\right)_i = \begin{bmatrix} \cos\alpha_4 & \sin\alpha_4 & 0 \\ -\sin\alpha_4 & \cos\alpha_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (5.2.12)$$

and so

$$(\underset{\sim}{a})_i = (\underset{\sim}{S})_i(\underset{\sim}{a})_o = a_3 \begin{pmatrix} -\sin\alpha_4\sin\psi \\ -\cos\alpha_4\sin\psi \\ \cos\phi \end{pmatrix} \qquad (5.2.13)$$

Other terms appearing in eq. (5.2.7), besides those of eqs. (5.2.8), can now be computed. These are

$$||\underset{\sim}{c}||^2 = s_1^2 + s_4^2 + 2s_1s_4\cos\alpha_4 + a_4^2 \qquad (5.2.14a)$$

$$\underset{\sim}{c}^T\underset{\sim}{d}_j = a_1(s_4\sin\alpha_4\sin\phi_j + a_4\cos\phi_j) \qquad (5.2.14b)$$

$$\underset{\sim}{c}^T\underset{\sim}{a}_j = a_3(a_4\cos\psi_j - s_1\sin\alpha_4\sin\psi_j) \qquad (5.2.14c)$$

$$\underset{\sim}{d}_j^T\underset{\sim}{a}_j = a_1a_3(\cos\alpha_4\sin\phi_j\sin\phi_j + \cos\psi_j\cos\phi_j) \qquad (5.2.14d)$$

Substituting expressions (5.2.8) and (5.2.14) into eq. (5.2.7) one obtains

$$\begin{aligned} a_2^2 = {} & s_1^2 + s_4^2 + 2s_1s_4\cos\alpha_4 + a_4^2 + a_1^2 + a_3^2 + \\ & + 2a_1(a_4\cos\phi_j + s_4\sin\alpha_4\sin\phi_j) \\ & - 2a_3(a_4\cos\phi_j - s_1\sin\alpha_4\sin\phi_j) \\ & - 2a_1a_3(\cos\alpha_4\sin\phi_j\sin\phi_j + \cos\psi_j\cos\phi_j) \end{aligned} \qquad (5.2.15)$$

which is the desired input-output relationship. Now, if angle ϕ is measured from certain reference ϕ_0 by letting

$$\phi_j = \phi_0 + p_j \qquad (5.2.16a)$$

$$k_1 = \frac{a_4 + s_4\sin\alpha_4\tan\phi_0}{a_3} \qquad (5.2.16b)$$

$$k_2 = \frac{s_4\sin\alpha_4 - a_4\tan\phi_0}{a_3} \qquad (5.2.16c)$$

$$k_3 = -\frac{a_4}{a_1\cos\phi_0} \qquad (5.2.16d)$$

$$k_4 = \frac{s_1\sin\alpha_4}{a_1\cos\phi_0} \qquad (5.2.16e)$$

$$k_5 = \tan\phi_0 \qquad (5.2.16f)$$

$$k_6 = \frac{a_1 - a_2^2 + a_3^2 + a_4^2 + s_1^2 + s_4^2 + 2s_1s_4\cos\alpha_4}{2a_1a_3\cos\phi_0} \qquad (5.2.16g)$$

eq. (5.2.15) becomes

$$k_1 \cos p_j + k_2 \sin p_j + k_3 \cos\psi_j + k_4 \sin\psi_j +$$
$$+ k_5 (\cos\phi_j \sin p_j - \cos\alpha_4 \sin\phi_j \cos p_j) +$$
$$+ k_6 = \cos\phi_j \cos p_j + \cos\alpha_4 \sin\phi_j \sin p_j, \quad j=1,\ldots,6 \tag{5.2.17}$$

Eqs. (5.2.17) constitute then a linear algebraic system of six equations in six unknowns $(k_1,\ldots,k_6)$. This system can be solved efficiently by application of subroutines DECOMP and SOLVE, of Sect. 1.12. In order to compute the linkage parameters a_1, a_2, a_3, a_4, s_1, s_4, $\cos\alpha_4$ and $\cos\phi_0$, however, the nonlinear system (5.2.16) has to be solved for the said parameters. The equations of this system, though nonlinear, are weakly coupled, for which reason its solution can be performed without having to resort to a numerical method. The aforementioned nonlinear system, nevertheless, contains a surplus of two unknowns. One of these unknowns can be eliminated through division by it, the surplus of unknowns thus reducing to one. In fact, the solution $a_2 = 0$ is ruled out, for this would lead to a topologically different layout, namely a conpling of three, instead of four, links. Scaling the linkage by a factor $1/a_2$, i.e. setting $a_2 = 1$ does not alter the input-output relationship, for this does not depend upon the absolute but upon the relative lengths of the different links. The remaining unknown in excess can be eliminated by assigning a value to it.

For the linkage appearing in Fig 5.2.1 let

i	1	2	3	4	5	6
ϕ_i	0	45°	60°	90°	60°	30°
ψ_i	0	30°	45°	60°	90°	180°

$a_2 = 1$, $\alpha_4 = 120°$

The arising linear system of equations was solved using subroutines DECOMP and SOLVE. The following results were obtained:

$k_1 = -0.840173$, $k_2 = 0.303087$, $k_3 = -1.350247$

$k_4 = -0.036659$, $k_5 = -1.140727$, $k_6 = 0.489926$

$a_1 = -0.175003$, $a_2 = 1.000000$, $a_3 = -0.724996$

$a_4 = 0.155768$, $s_1 = 0.007284$, $s_4 = -0.684495$

$\phi_0 = -0.851042$ rad

which is the solution to Example 5.2.1.

Mohan Rao et al. (5.2) have extended Denavit and Hartenberg's idea up to ten accuracy-point synthesis. For seven-point synthesis, they introduce the zero location of the input dial, ψ_0 , as one additional unknown in the above formulation, ending up with the following synthesis equations:

$$k_1 c\psi_j + k_2 s\psi_j + k_3 c\phi_j + k_4 s\phi_j - k_5 (c\alpha_4 c\phi_j s\psi_j - c\psi_j s\phi_j) - k_6 (s\phi_j c\psi_j c\alpha_4 - c\phi_j s\psi_j) +$$
$$+ k_7 - k_8 (c\alpha_4 c\phi_j c\psi_j + s\phi_j s\psi_j) = c\alpha_4 s\phi_j s\psi_j + c\phi_j c\psi_j , j = 1, 2, \ldots, 7 \qquad (5.2.20)$$

where

$$k_1 = \frac{s_1 s\alpha_4 \tan\psi_0 - a_4}{a_1 c\phi_0} \qquad (5.2.21a)$$

$$k_2 = \frac{a_4 \tan\phi_0 + s_1 s\alpha_4}{a_1 c\phi_0} \qquad (5.2.21b)$$

$$k_3 = \frac{a_4 + s_4 s\alpha_4 \tan\phi_0}{a_3 c\psi_0} \qquad (5.2.21c)$$

$$k_4 = \frac{s_4 s\alpha_4 - a_4 \tan\phi_0}{a_3 c\psi_0} \qquad (5.2.21d)$$

$$k_5 = \tan\phi_0 \qquad (5.2.21e)$$

$$k_6 = \tan\psi_0 \qquad (5.2.21f)$$

$$k_7 = \frac{a_1^2-1+a_3^2+a_4^2+s_1^2+s_2^2+2s_1s_4c\alpha_4}{2a_1a_3c\phi_0c\psi_0} \tag{5.2.21g}$$

$$K_8 = \tan\phi_0\tan\psi_0 \tag{5.2.21h}$$

Eqs. (5.2.20) constitute a linear system of seven equations in eight unknowns. However, the eight k_i are not independent, for they are related by

$$k_8 = k_5k_6 \tag{5.2.22}$$

Thus, the synthesis equations comprise the seven equations (5.2.20) plus eq. (5.2.22), i.e. a system of eight equations in eight unknowns, out of which, seven are linear and one is nonlinear.

To solve this system, the aforementioned authors proposed a method based on the principle of superposition of linear systems*, i.e. the principle under which if $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are solutions to $\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{b}$ and $\underset{\sim}{A}\underset{\sim}{x}=\underset{\sim}{c}$, respectively, then $\beta\underset{\sim}{x}_1+\gamma\underset{\sim}{x}_2$ is the solution to $\underset{\sim}{A}\underset{\sim}{x}=\beta\underset{\sim}{b}+\gamma\underset{\sim}{c}$. The method is next outlined:

i) Write eqs. (5.2.20) in the form

$$k_1c\phi_j+ k_2s\psi_j+k_3c\phi_j+k_4s\phi_j-k_5(c\alpha_4c\phi_js\psi_j-c\psi_js\phi_j)-k_6(s\phi_jc\psi_jc\alpha_4-c\phi_js\psi_j)+$$
$$+k_7=(c\alpha_4s\phi_js\psi_j+c\phi_jc\psi_j)+k_8(c\alpha_4c\phi_jc\psi_j+s\phi_js\psi_j),\; j=1,2,\ldots,7. \tag{5.2.20a}$$

ii) Define the following vectors

$$\underset{\sim}{x}=(k_1,k_2,\ldots,k_7)^T$$
$$\underset{\sim}{b}=(b_1,b_2,\ldots,b_7)^T$$
$$\underset{\sim}{c}=(c_1,c_2,\ldots,c_7)^T$$

where

$$b_j=c\alpha_4s\phi_js\psi_j+c\phi_jc\psi_j,\; j=1,2,\ldots,7$$
$$c_j=c\alpha_4c\phi_jc\psi_j+s\phi_js\psi_j,\; j=1,2,\ldots,7$$

* See Section 1.11

iii) Rewrite system (5.2.20a) in the form

$$\underline{A}\underline{x}=\underline{b}+k_8\underline{c} \tag{5.2.20b}$$

iv) Solve for $\underline{x}_1$ and $\underline{x}_2$ from

$$\underline{A}\underline{x}_1=\underline{b}, \underline{A}\underline{x}_2=\underline{c}$$

v) Write the solution to system (5.2.20b) as

$$\underline{x}=\underline{x}_1+k_8\underline{x}_2 \tag{5.2.23}$$

where k_8 is not known as yet

vi) Letting λ_1 and μ_1 be the 5th and 6th components of $\underline{x}_1$, respectively, and defining λ_2 and μ_2 analogously, one obtains from (5.2.23),

$$k_5=\lambda_1+k_8\lambda_2 \tag{5.2.24}$$

$$k_6=\mu_1+k_8\mu_2 \tag{5.2.25}$$

vii) Substitute (5.2.24) and (5.2.25) into (5.2.22), thus obtaining

$$\lambda_2\mu_2k_8^2+(\lambda_1\mu_2+\lambda_2\mu_1-1)k_8+\lambda_1\mu_1=0 \tag{5.2.26}$$

from which two values for k_8 can be obtained

viii) With one real value of k_8, obtain the values of $k_1, k_2, \ldots, k_7$ from eq. (5.2.23)

ix) The linkage parameters can now be obtained from eqs. (5.2.21), thus completing the proposed synthesis.

By leaving α_4 unspecified in the above formulation, the same authors, (5.2) state the 8-point synthesis problem as

$$k_1c\psi_j+k_2s\psi_j+k_3c\phi_j+k_4s\phi_j-k_5c\phi_js\psi_j-k_6c\phi_js\phi_j-k_7s\phi_js\psi_j+k_8=c\psi_jc\phi_j,$$
$$j=1,2,\ldots,8 \tag{5.2.27}$$

thus obtaining a <u>linear system</u> of eight equations in eight unknowns, which can readily be solved via the LU algorithm. In the above system

$$k_1=\frac{s_1\tan\psi_0 s\alpha_4-a_4}{a_1c\phi_0(1+\tan\phi_0\tan\psi_0c\alpha_4)} \tag{5.2.28a}$$

$$k_2 = \frac{a_4 \tan\psi_0 + s_1 s\alpha_4}{a_1 c\phi_0 (1+\tan\phi_0 \tan\psi_0 c\alpha_4)} \qquad (5.2.28b)$$

$$k_3 = \frac{a_4 + s_4 \tan\phi_0 s\alpha_4}{a_3 c\psi_0 (1+\tan\phi_0 \tan\psi_0 c\alpha_4)} \qquad (5.2.28c)$$

$$k_4 = \frac{s_4 s\alpha_4 - a_4 \tan\phi_0}{a_3 c\psi_0 (1+\tan\phi_0 \tan\psi_0 c\alpha_4)} \qquad (5.2.28d)$$

$$k_5 = \frac{\tan\phi_0 c\alpha_4 - \tan\psi_0}{1+\tan\phi_0 \tan\psi_0 c\alpha_4} \qquad (5.2.28e)$$

$$k_6 = \frac{\tan\psi_0 c\alpha_4 - \tan\phi_0}{1+\tan\phi_0 \tan\psi_0 c\alpha_4} \qquad (5.2.28f)$$

$$k_7 = \frac{c\alpha_4 + \tan\phi_0 \tan\psi_0}{1+\tan\phi_0 \tan\psi_0 c\alpha_4} \qquad (5.2.28g)$$

$$k_8 = \frac{a_1^2+a_2^2+a_3^2+a_4^2+s_1^2+s_4^2 2s_1 s_4 c\alpha_4}{2a_1 a_3 c\phi_0 c\psi_0 (1+\tan\phi_0 \tan\psi_0 c\alpha_4)} \qquad (5.2.28h)$$

The obtention of the linkage parameters from eqs. (5.2.28) is not a simple matter, for these appear strongly coupled in those equations. A method to solve for the said parameters is also presented in [5.2]. In the same paper, the authors propose that the RSSR linkage synthesis can be extended up to 10 points if scale parameters for the function intended to be generated are introduced.

Since the RSSR linkage lends itself very suitably to be used as a function generator, it has received much attention. Luck presents in [5.3] a method of synthesis of this linkage that allows for optimization by the introduction of a free parameter. Referring to Fig 5.2.3, this author writes the input-output function of the RSSR linkage in the form

$$f_1(\phi,\psi) = A_0 c\psi + B_0 s\phi + C_0 = 0 \qquad (5.2.29)$$

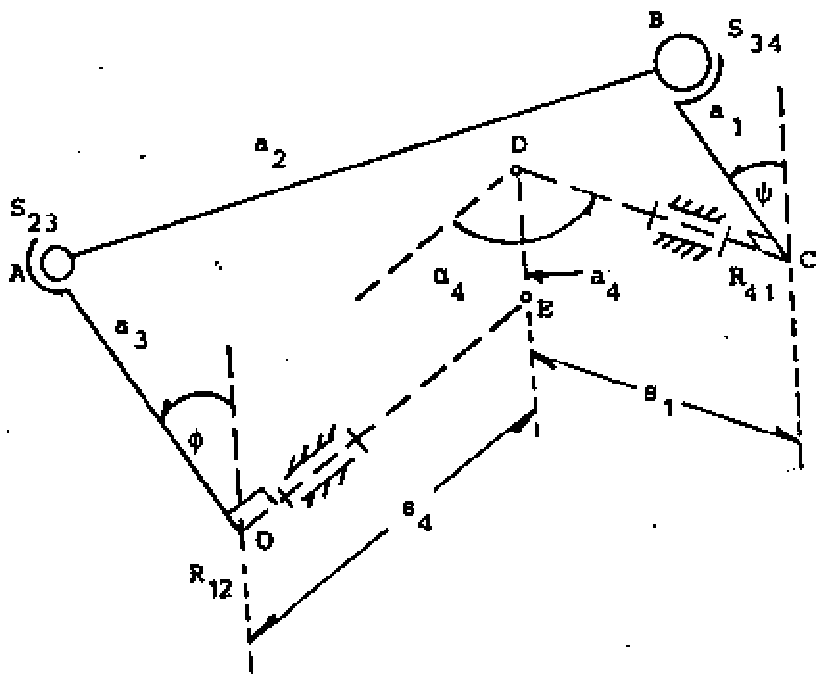

Fig. 5.2.3. RSSR function generating linkage

where

$$A_0=2a_1(a_4 s\alpha_4-a_3 c\alpha_4 c\phi) \tag{5.2.30a}$$

$$B_0=2a_1(a_4-a_3 s\phi) \tag{5.2.30b}$$

$$C_0=s_1^2+s_4^2+a_1^2-a_2^2+a_3^2a_4^2-2(s_1 s_4 c\alpha_4+s_1 a_3 s\alpha_3 c\phi+a_3 a_4 s\phi) \tag{5.2.30c}$$

Normalizing the linkage lengths with respect to s_4, the following variables are defined

$$x_1=\frac{a_3}{s_4},\ x_2=\frac{a_2}{s_4},\ x_3=\frac{a_1}{s_4},\ \lambda=\frac{s_1}{s_4},\ e=\frac{a_4}{s_4},\ s_4=1 \tag{5.2.31}$$

The input-output function is then transformed into

$$f\ (\phi,\psi)=x_1^2-x_2^2+x_3^2+1+\lambda^2-2\lambda c\alpha_4+e\ -2x_1(\lambda s\alpha_4 c\phi+es\phi)+2x_3(es\psi+s\alpha_4 c\psi)$$
$$-2x_1x_3(s\psi s\phi+c\alpha_4 c\psi c\phi)=0 \tag{5.2.32}$$

Moreover, define

$$\phi_j=\alpha+\Delta\phi_j,\ \psi_j=\beta+\Delta\psi_j,\ j=1,2,\ldots,n \tag{5.2.33}$$

with

$$\Delta\psi_0=\Delta\phi_0\equiv 0$$

Subtracting $f(\phi_0,\psi_0)=0$ from $f(\phi_j,\psi_j)=0$ $(j=1,2,\ldots,n)$ leads to the following linear homogenous system

$$f(\Delta\phi_j,\Delta\psi_j)\equiv f(\phi_j,\psi_j)-f(\phi_0,\psi_0)=A_j u_1+B_j u_2+C_j u_3+D_j u_4+E_j u_5+F_j u_6=0 \tag{5.2.34}$$

where

$$u_1=\frac{e}{x_1},\ u_2=\frac{s\alpha_4}{x_1},\ u_3=-\frac{e}{x_3} \tag{5.2.35a}$$

$$u_4=-\frac{\lambda s\alpha_4}{x_3},\ u_5=-c\alpha_4,\ u_6=1 \tag{5.2.35b}$$

$$A_j=s(\beta+\Delta\psi_j)-s\beta \tag{5.2.36a}$$

$$B_j=c(\beta+\Delta\psi_j)-c\beta \tag{5.2.36b}$$

$$C_j = s(\alpha\Delta\phi_j) - s\alpha \qquad (5.2.36c)$$

$$D_j = c(\alpha+\Delta\phi_j) - c\alpha \qquad (5.2.36d)$$

$$E_j = c(\alpha+\Delta\phi_j)c(\beta+\Delta\psi_j) - c\alpha c\beta \qquad (5.2.36e)$$

$$F_j = -s(\alpha+\Delta\phi_j)s(\beta+\Delta\psi_j) + s\alpha s\beta \qquad (5.2.36f)$$

System (5.2.34) has non-trivial solutions if and only if its determinant vanishes. In the problem of synthesis for function generation, the values of $\Delta\psi_j$ and $\Delta\phi_j (j=1,2,\ldots,n)$ are given. Hence, the determinant δ is a function of α and β only. This determinant has, in fact, the following form

$$\begin{vmatrix} A_2 & B_2 & \cdots & F_2 \\ A_3 & B_3 & \cdots & F_3 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_7 & B_7 & \cdots & F_7 \end{vmatrix} = \delta(\alpha,\beta) \qquad (5.2.37)$$

The function $\delta(\alpha,\beta)$ vanishes along the curve $\beta=\beta(\alpha)$ defined over the plane α-β, as shown in Fig 5.2.4

Luck proposes in (5.3) a numerical method to find pairs of values (α_i,β_i) along which δ vanishes, this method being based on the "regula-falsi" algorithm. A different method is proposed here.

For a given value of β, say β_i, δ can be regarded as a function of one real variable, namely, α, i.e.

$$\delta_i = \delta_i(\alpha)$$

To find the roots of δ_i, the method of Newton-Raphson is applied, as follows

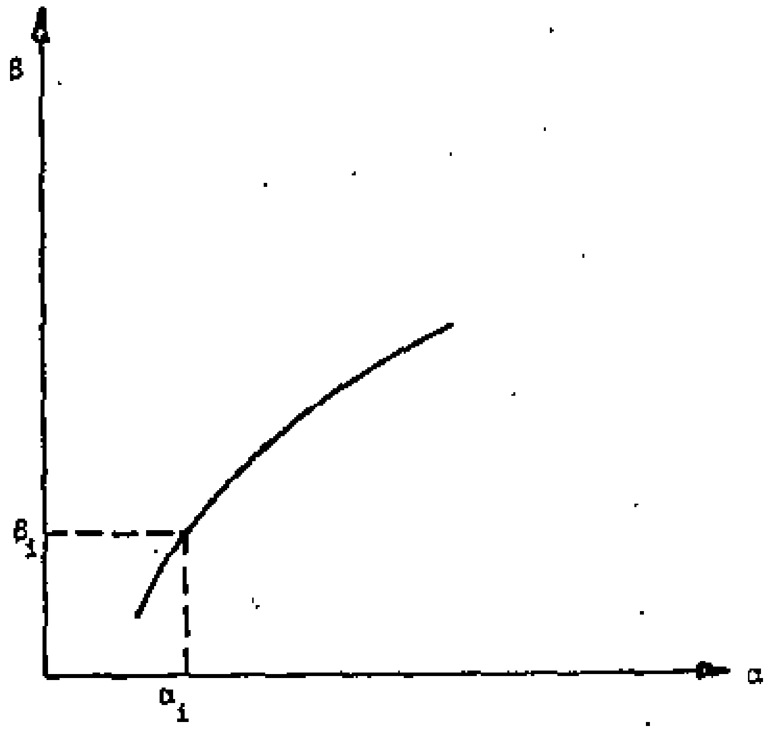

Fig 5.2.4 Set of values of α and β along which δ vanishes.

1.i) Estimate (or guess) a starting value of α, say α^o, to begin the iterative procedure

1.ii) From the current value α^k of the sought root, α_i, of δ_i, compute an improved value of α_i, say α^{k+1} as

$$\alpha^{k+1}=\alpha^k-\frac{\delta_i(\alpha^k)}{\delta_i'(\alpha^k)} \tag{5.2.38}$$

1.iii) If the correction value $\delta_i(\alpha^k)/\delta_i'(\alpha^k)$ is greater than a given tolerence ε, return to Step 1.ii. Otherwise, verity if $|\delta_i(\alpha^k)|\leq\varepsilon$. If so, stop the procedure and accept the current value α^k as the value of the sough root, α_i. If not, repeat the procedure starting at Step 1.i, with a different value of α^o.

Expression (5.2.38) greaty simplifies when the formula for the derivative of a determinant is introduced, namely

$$\delta_i'(\alpha)=\delta_i(\alpha)\mathrm{Tr}\left\{\underline{M}^{-1}(\alpha)\underline{M}'(\alpha)\right\} \tag{5.2.39}$$

Where $\underline{M}$ is the matrix whose determinant is δ and $\underline{M}'(\alpha)$ is the matrix whose entries are the derivatives, with respect to α, of the corresponding entries of $\underline{M}$. Since formula (5.2.39) is not very popular, its derivation is presented in Appendix 2, Substitution of eq. (5.2.39) into eq. (5.2.38) leads to

$$\alpha^{k+1}=\alpha^k-\frac{1}{\mathrm{Tr}\left(\underline{M}^{-1}(\alpha^k)\underline{M}'(\alpha^k)\right)} \tag{5.2.40}$$

to compute $\underline{M}^{-1}(\alpha^k)\underline{M}'(\alpha^k)$, let

$$\underline{M}^{-1}(\alpha^k)\underline{M}'(\alpha^k)=\underline{N}$$

i.e., after dropping the argument for shortness,

$$\underline{M}\underline{N}=\underline{M}' \tag{5.2.41}$$

In eq. (5.2.41), let $\underline{n}_p$ and $\underline{m}_p'$ denote vectors identical to the p<u>th</u> columns of matrices $\underline{N}$ and $\underline{M}'$, respectively. Hence, matrix $\underline{N}$ can be computed by solving the 6 linear systems

$$\underline{M}\underline{n}_p=\underline{m}_p', \quad p=1\ 2,\ldots,6 \tag{5.2.42}$$

which can be done via the LU decomposition algorithm. At this stage there are two simplifications, namely,

2.i) Subroutine DECOMP need only once be applied at the k<u>th</u> iteration for, once the LU decomposition of $\underline{M}$ is obtained, this can be used to solve the 6 systems appearing in (5.2.42)

2.ii) Since A_j and B_j do not contain α explicitly, the first two columns of $\underline{M}'$ vanish. Therefore, $\underline{n}_1=\underline{n}_2=\underline{0}$ and subroutine SOLVE need be applied only four times. Funthermore,

2.iii) If at any of the iterations DECOMP detects matrix $\underline{M}$ to be singular, the system of (5.2.42) cannot be solved, but this is not necessary any more, for precisely what one is seeking is that value of α_i for a given value of β_i, that renders matrix $\underline{M}$ singular, i.e. that makes δ zero.

Once the value α_i that, for a given β_i makes the determinant vanish, has been found, a new value of β_i, say β_{i+1}, is introduced and the process starting in 1.i. is repeated, except that a new starting value α^o need not be guessed, for a good estimate for α_{i+1} is, of course, α_i, provided β_i and β_{i+1} are sufficiently close to each other. This way, a set of discrete values $\{\alpha_i, \beta_i\}$ that zero δ, can be given in tabular form; however, if a continuous function $\beta=\beta(\alpha)$ is necessary, this can be approximated by interpolation. The most efficient way of interpolating such a function is by means of spline functions [5.4].

SUBROUTINE ZERDET, appearing in Fig 5.2.5 implements the algorithm outlined in 1.i-1.iii. This subroutine was used to find the graph β vs. α arising from the following problem for seven-point accuracy synthesis (Example 1 of [5.3]).

$\Delta\phi_2=30°$, $\Delta\psi_2=-16.1°$

$\Delta\phi_3=75°$, $\Delta\psi_3=-11.5°$

$\Delta\phi_4=135°$, $\Delta\psi_4= 22.5°$

$\Delta\phi_5=195°$, $\Delta\psi_5= 53.5°$

$\Delta\phi_6=240°$, $\Delta\psi_6= 57.9°$

$\Delta\phi_7=300°$, $\Delta\psi_7= 41.3°$

This graph appears in Fig 5.2.6

Given one pair of values of the aforementioned set $\{\alpha_i, \beta_i\}$ and recalling that $u_6=1$, the following linear inhomogeneous system is obtained

$$\underline{G}\underline{u}=-\underline{f} \qquad (5.2.43)$$

where $\underline{G}$ is the submatrix of $\underline{M}$ containing the first five rows and columns of it, $\underline{u}=(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_5)^T$ and $\underline{f}=(\underline{F}_2, \underline{F}_3, \ldots, \underline{F}_6)^T$. Vector $\underline{u}$ can be obtained

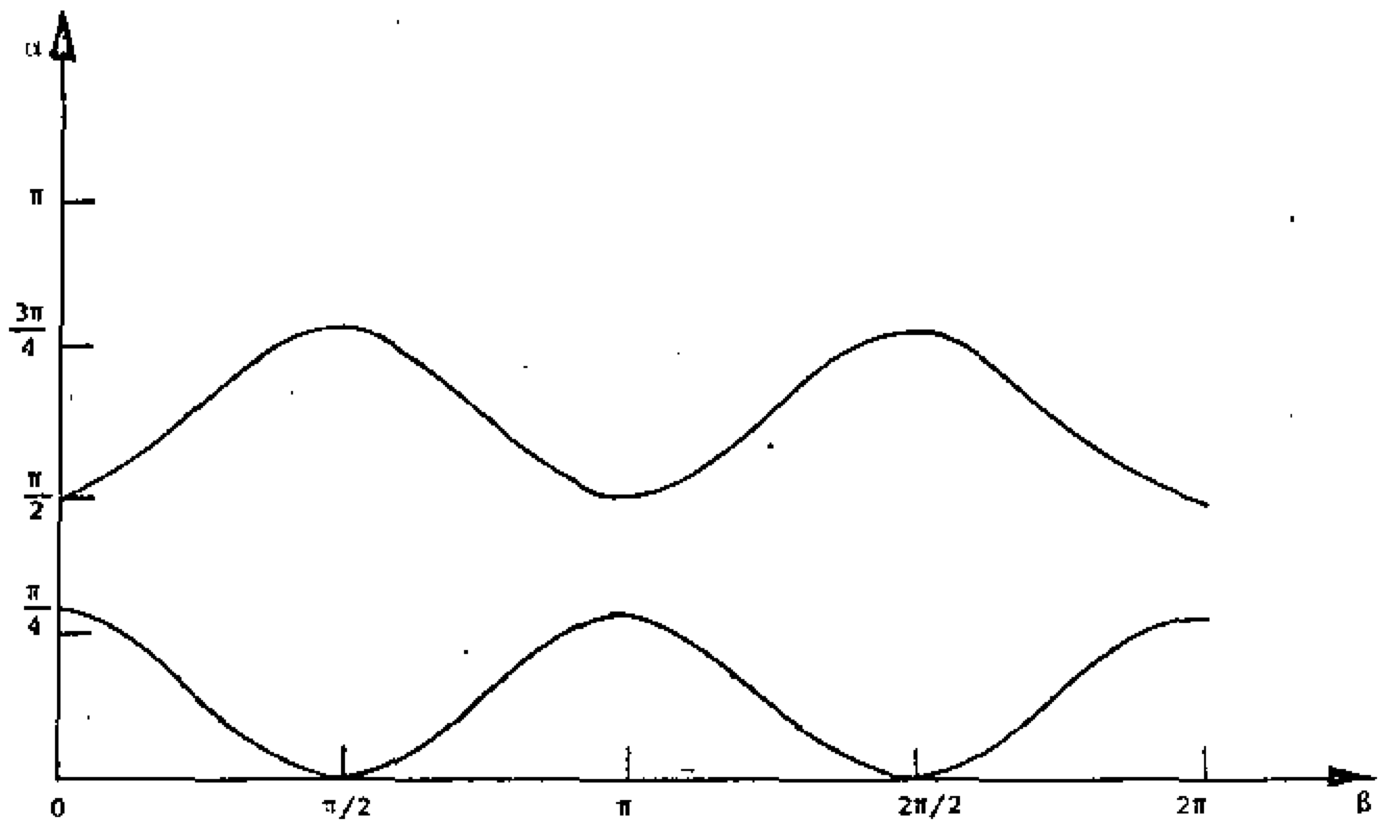

Fig 5.2.6 Free-parameter relationship for the synthesis of a function generator linkage with seven specified points.

from eq. (5.2.43) via the LU decomposition algotithm. With $\underline{u}$ known, the normalized lengths x_1, x_3, λ, e and the angle α_4 can be obtained by substitution into eqs. (5.2.35). The remaining length, x_2, is readily obtained from eq. (5.2.32) when, for convenience, the values of ψ_1 and ϕ_1 are introduced in that equation. Thus, the synthesis problem is totally solved.

Two observations are in order: i) This method allows to synthesize a function generator for up to seven precision points and ii) since infinitely many combinations of α and β (that turn the determinant δ zero) exist, that combination rendering the best transmission angle can be used, thus allowing for optimization.

So far no velocity nor acceleration nor higher-derivative conditions have been considered. The introduction of such conditions, rendering a synthesis problem for infinitesimally-separated positions is, however, straightforward. In fact, all that need be done is to differentiate the input-output function with respect to time as many times as necessary, substitute in the resulting equations the prescribed values of the input, the output and their derivatives and form a system of synthesis equations similar to that appearing in eqs. (5.2.17). This is next illustrated with an example.

Example 5.2.2 Synthesis of the RSSR linkage for function generation with prescribed dead-points.

It is required to determine the dimensions of the linkage shown in Fig 5.2.1, for a given value of angle α_4, to produce an oscillation of link BC of 90° in such a way that the return motion (under no load) be performed twice as faster as the first half (under full load). The input link should be a crank, i.e. it should rotate through 360°

Solution:

In the synthesis equation 5.2.17, let

$$k_1 = \frac{s_1}{a_1}\ ,\ k_2 = \frac{a_4}{a_1},\ k_3 = \frac{a_4}{a_3},\ k_4 = \frac{s_4}{a_3} \tag{5.2.44a}$$

$$k_5 = -\frac{a_1^2 - a_2^2 + a_3^2 + a_4^2 + s_1^2 + s_4^2 + 2s_1 s_4 c\alpha_4}{2a_1 a_3} \tag{5.2.44b}$$

thus obtaining the suitable synthesis equation in the form

$$k_1 s\alpha_4 s\phi - k_2 c\phi + k_3 c\psi + k_4 s\alpha_4 s\psi + k_5 - (c\phi c\psi + c\alpha_4 s\phi s\psi) = 0 \tag{5.2.45}$$

To meet the problem conditions, assume that the no-load motion is performed during a 120° rotation of the input crank, the load motion being executed during the remaining 240° rotation of the said crank. Thus, the following conditions can be imposed

$$\phi_1 = 0;\ \psi_1 = 0,\ \dot{\psi}_1 = 0 \tag{5.2.46a}$$

$$\phi_2 = 120°,\ \psi_2 = 90°,\ \dot{\psi}_2 = 0 \tag{5.2.46b}$$

These conditions do not suffice for the present problem for, even if they are met, the output linkage could rotate through an angle of 270° and not one of 90°, as required. To ensure the proper motion to be performed by the linkage, the additional following condition is imposed

$$\phi_3 = 180°,\ \psi_3 = 67.5° \tag{5.2.46c}$$

which arises from the assumption that, when the input link has rotated through an angle of 60° ($= \phi_3 - \phi_2$) of the load motion (1/4 of this motion), the output link has rotated through an angle of 22.5° ($= \psi_2 - \psi_3$) of the same motion (1/4 of this motion, also).

To specify the velocity conditions, the synthesis equation (5.2.45) is differentiated with respect to time, thus obtaining

$$k_1 \dot{\phi} s\alpha_4 c\phi + k_2 \dot{\phi} s\phi - k_3 \dot{\psi} s\psi + k_4 \dot{\psi} s\alpha_4 c\psi - (-\dot{\phi} s\phi c\psi - \dot{\psi} c\phi s\psi$$

$$+ \dot{\phi} c\alpha_4 c\phi s\psi + \dot{\psi} c\alpha_4 s\phi c\psi) = 0 \tag{5.2.47}$$

It is then noticed that five synthesis equations can be obtained to produce five unknowns, thereby justifying the use of a synthesis equation of the form of eq. (5.2.45). The said five synthesis equations are:

For $\phi_1=0$ and $\psi_1=0$, eq. (5.2.45) leads to

$$-k_2+k_3+k_5-1=0 \qquad (5.2.48a)$$

For $\phi_1=0$, $\psi_1=0$ and $\dot{\psi}_1=0$, with $\dot{\phi}_1\neq 0$, which can then be dropped, eq. (5.2.47) leads to

$$k_1 s\alpha_4=0 \qquad (5.2.48b)$$

For $\phi_2=120°$ and $\psi_2=90°$, eq. (5.2.45) leads to

$$\frac{\sqrt{3}}{2}s\alpha_4 k_1+\frac{1}{2}k_2+s\alpha_4 k_4+k_5-\frac{\sqrt{3}}{2}c\alpha_4=0 \qquad (5.2.48c)$$

For $\phi_2=120°$, $\psi_2=90°$ and $\dot{\psi}_2=0$, with $\dot{\phi}_2\neq 0$ eq. (5.2.47) leads to

$$-\frac{1}{2}s\alpha_4 k_1+\frac{\sqrt{3}}{2}k_2+\frac{1}{2}c\alpha_4=0$$

Finally, from condition (5.2.46c), eq. (5.2.45) leads to

$$k_2+c67.5°k_3+s67.5°s\alpha_4 k_4+k_5+c67.5°=0 \qquad (5.2.48d)$$

Solving for the five unknowns in the foregoing system (5.2.48), one obtains

$$k_1 = 0$$

$$k_2 = -\frac{\sqrt{3}}{3}c\alpha_4$$

$$k_3 = \frac{1}{1-c}\left(1 + c + \frac{2\sqrt{3}}{3}c\alpha_4(s-1)\right)$$

$$k_4 = \frac{2\sqrt{3}}{3}\cot\alpha_4$$

$$k_5 = 1 - \frac{\sqrt{3}}{3}c\alpha_4 - \frac{1}{1-c}\left(1 + c + \frac{2\sqrt{3}}{3}c\alpha_4(s-1)\right)$$

where $c\equiv\cos 67.5°$, $s\equiv\sin 67.5°$

Putting the above expressions into a computer and computing them for different values of α_4 yields numerous linkages, out of which the best (in a given sense, e.g., the one with the best transmission angle) can be selected.

The transmission angle, μ, defined as angle BAD of the linkage appearing in Fig 5.2.1, is a measure of the mechanical advantage of the linkage, i.e. the ratio of output torque (M_ϕ) to input torque (M_ψ). From Fig 5.2.1, the definition of vectors $\underline{a}, \underline{b}, \underline{c}$ and $\underline{d}$ following it, and eqs. (5.2.5), (5.2.11), (5.2.14) and (5.2.16c), it follows that

$$\cos\mu = -\frac{\underline{b}^T\underline{a}}{a_2 a_3} = -\frac{(\underline{c}+\underline{d}-\underline{a})^T\underline{a}}{a_2 a_3}$$

from which

$$\cos\mu = -\frac{a_4 c\psi - s_1 s\alpha_4 s\psi + a_1 c\alpha_4 s\psi s\phi + a_1 c\psi c\phi - a_3}{a_2} \qquad (5.2.49)$$

The reason why the transmission angle is a measure of the mechanical advantage of the linkage is the following: disregarding the inertia forces of the links, the input torque, M_ϕ, is transmitted from link CD to link AB (Fig 5.2.1) through the coupler link AD by means of a force collinear with line AD. If this force, F, is resolved into two components, one parallel to line AB and the other perpendicular to it, the working component is that perpendicular to AB and its value is F sinμ, whereas the nonworking component is that parallel to AB and its value is F cosμ. Thus, for a "good" force transmission, the parallel component of F should be as small as possible, which is attained if cosμ lies close to zero, i.e. if μ lies close to either 90° or 270°.

An expression for the mechanical advantage of the RSSR linkage can be obtained via a static analysis of the linkage, thus finding an expression for M_ϕ in terms of the linkage parameters, ψ,ϕ and M_ψ. This approach would be too tedious. A different approach is next pressented.

Disregarding the effect of friction forces for simplicity, the output power is equated to the input power, i.e.

$$M_\phi \dot{\phi} = M_\psi \dot{\psi} \tag{5.2.50}$$

from which, the mechanical advantage, m, is obtained as

$$m = \frac{M_\psi}{M_\phi} = \frac{\dot{\phi}}{\dot{\psi}} \tag{5.2.51}$$

From eq. (5.4.47),

$$m = \frac{k_3 s\psi - k_4 s\alpha_4 c\psi - s\psi c\phi + c\alpha_4 c\phi s\psi}{k_1 s\alpha_4 c\phi + k_2 s\phi + c\psi s\phi - c\alpha_4 c\phi s\psi} \tag{5.2.52}$$

An analogous procedure can be followed to determine the mechanical advantage of other types linkages.

Exercise 5.2.1 Determine the mechanical advantage of the following linkages: Spherical RRRR, RSRP, RSRC

Exercise 5.2.2. Using the results of Example 5.2.2, find the dimensions of the linkage whose transmission angle lies between 40° and 140° (or between 220° and 1320°) throughout the performance of the linkage.

Mobility analysis

In synthesising function- generating linkages one is interested in producing a link with certain mobility conditions. These conditions refer to the range of motion of either the input- or the output link or of both. With this respect, one of these links is to be either a crank (possibility of rotation through 360°) or a rocker (possibility of rotation through a fraction of a complete turn). The conditions under which these mobility specifications are met are next discussed.

Referring to Fig 5.2.1 the following analysis is performed to establish tho conditions under which the input link, CD, is a crank.

For a given position of the input link, i.e. for a given value of ϕ, the relative position of the pairs D and A is depicted in Fig 5.2.7. In that Figure, the locus of A is the circle centered at B with a radius a_3, $\underline{n}$ being the unit normal to the plane of the circle, i.e. the vector paralel to the axis of the the revolute pair R_{41}.

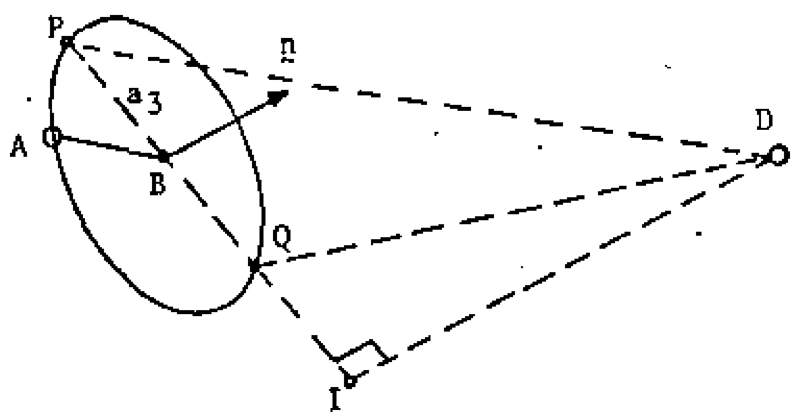

Fig 5.2.7 Relative position of pairs A and D of Fig 5.2.1

In what follows the following Theorem will be resorted to

<u>THEOREM 5.2.1</u> *Given a circle and a point not lying on its circumference and not necessarily in the plane of the circle, the points on the circumference lying the closest and the farthest from the point, Q and P, respectively, have the property that lines QD and PD are perpendicular to the tangent to the circle passing through Q and through P.*

<u>Exercise 5.2.3</u> Using the method of the Lagrange multipliers, prove Theorem 5.2.1

From Theorem 5.2.1 it follows that P and Q are determined as the intersections of the circle with line IB. Line IB is in turn determined by the center of the circle, B, and the intersection of the plane of the circle with its perpendicular from D. Let $\ell(\phi)$ and $L(\phi)$ be the lenght of segments DQ and DP, respectively. Hence, the condition for the input link to be a crank is that a_2, the lenght of the coupler link, lie within the maximum value of ℓ and the minimum value L, i.e.

$$\ell_{max} < a_2 < L_{min}$$

Exercise 5.2.4 Derive the following expressions.

$$\ell(\phi) = \{ \left(\sqrt{a_1^2 + s_1^2 + a_4^2 + 2a_4 a_1 \cos\phi - (s_1 \cos\alpha_4 + a_1 \operatorname{sen}\alpha_4 \operatorname{sen}\phi)^2} - a_3 \right)^2$$

$$+ (s_4 + s_1 \cos\alpha_4 + a_1 \operatorname{sen}\alpha_4 \operatorname{sen}\phi)^2 \}^{1/2}$$

$$L(\phi) = \{ \left(\sqrt{a_1^2 + s_1^2 + a_4^2 + 2a_4 a_1 \cos\phi - (s_1 \cos\alpha_4 + a_1 \operatorname{sen}\alpha_4 \operatorname{sen}\phi)^2} + a_3 \right)^2$$

$$+ (s_4 + s_1 \cos\alpha_4 + a_1 \operatorname{sen}\alpha_4 \operatorname{sen}\phi)^2 \}^{1/2}$$

5.3 MECHANISM SYNTHESIS FOR RIGID-BODY GUIDANCE.

A complete account of the theory and applications of this subject appears in (5.5-5.12). Different approaches to this problem are presented, all of them regarding the calculation of geometric parameters of one dyad* at a time, but the most unified treatment is that of Tsai and Roth (5.11). The method introduced by these authors is based on formulae relating the different screw angles and displacements of composed motions (carrying the rigid body intended to be guided from its reference configuration to its n specified successive configurations) to the directions and positions of the screw axes involved, as is shown next.

Let a rigid body B occupy configuration B_0 initially (henceforth called "the reference configuration") and assume it is intended to conduct this rigid body through n successive configurations B_j $(j=1,2,...,n)$, all of them being finitely separated, i.e. the screw motions relating one of these configurations to each other and to the reference one contain parameters (angle and displacement) with only finite values (Fig 5.3.1)

The motion carrying B from B_0 to B_j can be regarded as the composition of two motions: one, given by a screw $\underset{\sim}{M}_j$** carrying B from B_0 to B'_j, followed by a second one, given by a screw $\underset{\sim}{F}_j$, carriying B from B'_j to B_j. The axes of both screws $\underset{\sim}{M}_j$ and $\underset{\sim}{F}_j$ are lines M_j and F_j, shown in Fig 5.3.1, where F_j is fixed in space, i.e. its position relative to B_0 does not change after the motion is completed, whereas line M_j is a moving one. Let u_j and w_j be the scalar displacements of $\underset{\sim}{M}_j$ and $\underset{\sim}{F}_j$, respectively, and α_j and γ_j their

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*See Section 3.2.

**$\underset{\sim}{M}_j$ denotes a screw, whereas M_j, its axis.

respective angles. The screw S_j resulting from the composition of M_j and F_j

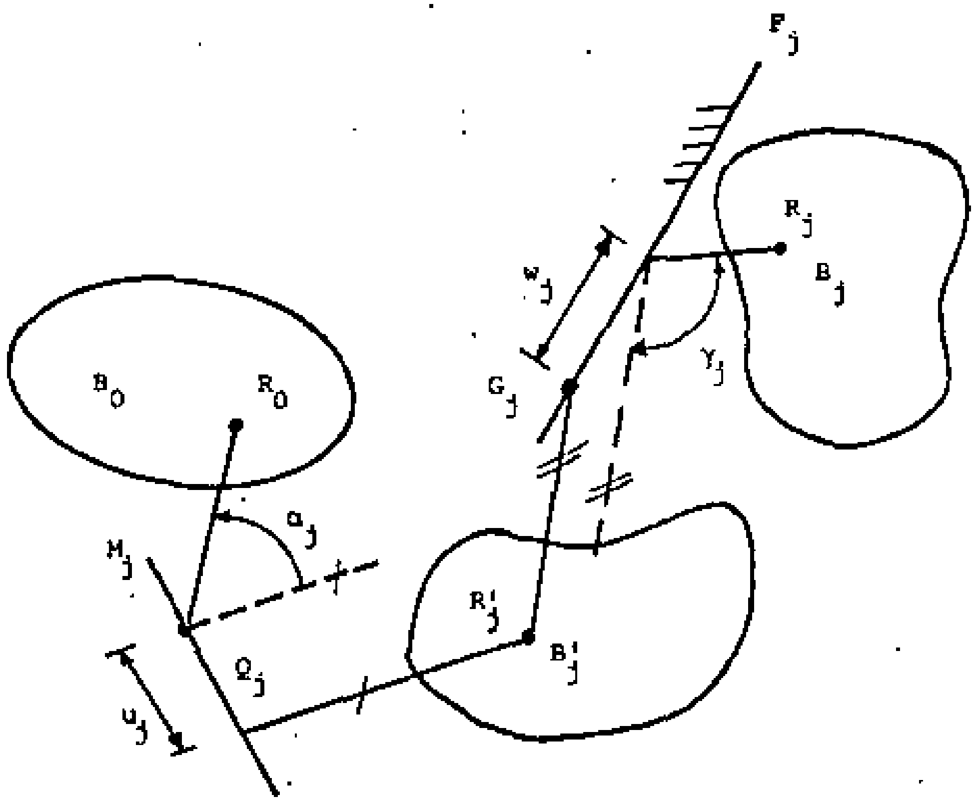

Fig 5.3.1 Successive configurations of a rigid body

has an axis S_j passing through point A_j, its displacement being denoted by t_j and its angle by θ_j. All three screws form what Roth calls a "screw triangle", Next, relationships among the screw displacements, angle and axes are obtained.

In what follows, let $\underline{m}_j$, $\underline{f}_j$ and $\underline{s}_j$ be unit vectors parallel to the axes of $\underline{M}_j$, $\underline{F}_j$ and $\underline{S}_j$. Moreover, let $\underline{q}_j$, $\underline{g}_j$, and $\underline{a}_j$ be the position vectors of points Q_j, G_j and A_j, located on M_j, F_j and S_j, respectively. For short, the indices are dropped from the screw parameters in the following derivations.

Starting with Rodrigues' formula (2.5.3) the following relation is readily obtained (5.13,5.14)

$$\tan\frac{\theta}{2}\mathbf{s} = \frac{\tan\frac{\alpha}{2}\mathbf{m}+\tan\frac{\gamma}{2}\mathbf{f}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\mathbf{f}\times\mathbf{m}}{1-\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\mathbf{m}\cdot\mathbf{f}} \tag{5.3.1}$$

Multiplying both sides of the above equations times $\times\mathbf{f}$ yields

$$\tan\frac{\theta}{2}\mathbf{s}\times\mathbf{f} = \frac{\tan\frac{\alpha}{2}\mathbf{m}\times\mathbf{f}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}(\mathbf{f}\times\mathbf{m})\times\mathbf{f}}{1-\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\mathbf{m}\cdot\mathbf{f}} \tag{5.3.2}$$

Multiplying both sides of the above equation times $\cdot\mathbf{s}$, the left hand side clearly vanishes thus leading to

$$\tan\frac{\alpha}{2}\left(\mathbf{m}\times\mathbf{f}\cdot\mathbf{s}+\tan\frac{\gamma}{2}(\mathbf{f}\times\mathbf{m})\times\mathbf{f}\cdot\mathbf{s}\right)=0 \tag{5.3.3}$$

which, for non-zero values of α, vanishes only if the term in brackets does, i.e. if

$$\tan\frac{\gamma}{2}= \frac{\mathbf{m}\times\mathbf{f}\cdot\mathbf{s}}{(\mathbf{m}\times\mathbf{f})\cdot(\mathbf{f}\times\mathbf{s})} \tag{5.3.4}$$

which is one of the relations sought.

To stablish a second relation, multiply both sides of eq. (5.3.1) times $\times\mathbf{m}$, thus obtaining

$$\tan\frac{\theta}{2}\mathbf{s}\times\mathbf{m}= \frac{\tan\frac{\gamma}{2}\mathbf{f}\times\mathbf{m}+\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}(\mathbf{f}\times\mathbf{m})\times\mathbf{f}}{1-\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}\mathbf{m}\cdot\mathbf{f}} \tag{5.3.5}$$

Multiplying both sides of the latter equation times $\cdot\mathbf{s}$ renders its left hand side zero, thus leading to

$$\tan\frac{\gamma}{2}\left(\mathbf{f}\times\mathbf{m}\cdot\mathbf{s}+\tan\frac{\alpha}{2}(\mathbf{f}\times\mathbf{m})\times\mathbf{m}\cdot\mathbf{s}\right)=0 \tag{5.3.6}$$

which, for non-zero values of γ, vanishes only if the term in brackets does, i.e. if

$$\tan\frac{\alpha}{2} = \frac{\underset{\sim}{f}\cdot\underset{\sim}{s}\times\underset{\sim}{m}}{(\underset{\sim}{s}\times\underset{\sim}{m})\cdot(\underset{\sim}{m}\times\underset{\sim}{f})} \qquad (5.3.7)$$

thereby obtaining one second relationship

To obtain a third relationship, one connecting θ to $\underset{\sim}{m}, \underset{\sim}{f}$ and $\underset{\sim}{s}$, proceed as follows:

Eq. (5.3.1) was obtained by first rotating B through an angle α about an axis parallel to $\underset{\sim}{m}$, and then through an angle γ about an axis parallel to $\underset{\sim}{f}$. The resulting rotation is equivalent to a single one through an angle θ about an axis parallel to $\underset{\sim}{s}$. If now configuration B_j' is regarded as the reference one, B_0 can be reached from B_j' through B_j. B_j can be reached from B_j' in exactly the same way as described previously, i.e. by means of a rotation through an angle γ about an axis parallel to $\underset{\sim}{f}$; but now B_0 can be reached from B_j by means of the inverse of the rotation carrying B_0 to B_j, i.e., via a rotation through an angle $-\theta$ abount an axis parallel to $\underset{\sim}{s}$. Writing the equation equivalent to (5.3.1) for this composition of rotations

$$\tan(-\frac{\alpha}{2})\underset{\sim}{m} = \frac{\tan\frac{\gamma}{2}\underset{\sim}{f} - \tan\frac{\gamma}{2}\underset{\sim}{s} - \tan\frac{\theta}{2}\tan\frac{\gamma}{2}\underset{\sim}{s}\times\underset{\sim}{f}}{1+\tan\frac{\gamma}{2}\tan\frac{\theta}{2}\underset{\sim}{f}\cdot\underset{\sim}{s}} \qquad (5.3.8)$$

Proceeding in a fashion analogous to the one used to obtain (5.3.4) and (5.3.7), i.e. multiplying both sides of eq. (5.3.8) times $\times\underset{\sim}{s}\cdot\underset{\sim}{m}$, one finally obtains

$$\tan\frac{\theta}{2} = -\frac{\underset{\sim}{f}\cdot\underset{\sim}{s}\times\underset{\sim}{m}}{(\underset{\sim}{f}\times\underset{\sim}{s})\cdot(\underset{\sim}{s}\times\underset{\sim}{m})} \qquad (5.3.9)$$

as the third relationship sought.

In the following, expressions for u, w and t are obtained. The composition of screws $\underset{\sim}{M}$ and $\underset{\sim}{F}$ is shown in Fig. 5.3.2. In This figure, $\underset{\sim}{r}$ is the original position vector of a point P of a rigid body undergoing first a screw motion of scalar displacement u, parallel to vector $\underset{\sim}{m}$, and rotation α about an axis passing through Q and parallel to $\underset{\sim}{m}$. After this screw motion the position vector of P is $\mathbf{r}'$. Next, the rigid body undergoes one second screw motion of scalar displacement w parallel to $\mathbf{f}$, and rotation γ about an axis passing through G and parallel to $\mathbf{f}$. The final position vector of point P is $\mathbf{r}''$.

The composition of both screws is equivalent to one single screw of scalar displacement t parallel to $\mathbf{s}$, and rotation θ about an axis passing through A and parallel to $\mathbf{s}$. This screw is shown in Fig 5.3.3.

Let $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be the rotations of screws $\mathbf{M}$ and F, respectively, $\mathbf{Q}_3$ being that of the equivalent screw $\mathbf{S}$. Thus, applying eqs. (2.6.18) and (2.6.19),

$$\mathbf{r}''=\mathbf{r}'+w\mathbf{f}+(\mathbf{Q}_2-\mathbf{I})(\mathbf{r}'-\mathbf{g}) \qquad (5.3.10)$$

where, applying the same result again,

$$\mathbf{r}'=\mathbf{r}+u\mathbf{m}+(\mathbf{Q}_1-\mathbf{I})(\mathbf{r}-\mathbf{q}) \qquad (5.3.11)$$

Substitution of (5.3.11) into (5.3.10) yields, after cancellations and rearrangement of terms,

$$\mathbf{r}''=\mathbf{r}+w\mathbf{f}+u\mathbf{Q}_2\mathbf{m}+(\mathbf{Q}_3-\mathbf{I})\mathbf{r}-(\mathbf{Q}_2-\mathbf{I})\mathbf{g}-\mathbf{Q}_3\mathbf{q}+\mathbf{Q}_2\mathbf{q} \qquad (5.3.12)$$

in which $\mathbf{Q}_3=\mathbf{Q}_2\mathbf{Q}_1$, has been substituted.

On the other hand, using the equivalent screw to compute $\mathbf{r}''$,

$$\mathbf{r}''=\mathbf{r}+t\mathbf{s}+(\mathbf{Q}_3-\mathbf{I})(\mathbf{r}-\mathbf{a}) \qquad (5.3.13)$$

Equating the right hand sides of (5.3.12) and (5.3.13), one obtains

$$w\mathbf{f}+u\mathbf{Q}_2\mathbf{m}-t\mathbf{s}+(\mathbf{Q}_3-\mathbf{I})(\mathbf{a}-\mathbf{q})+(\mathbf{Q}_2-\mathbf{I})(\mathbf{q}-\mathbf{g})=\mathbf{0} \qquad (5.3.14)$$

from (5.13,p.5),

$$\mathbf{Q}_2\mathbf{m}=\mathbf{m}+(1-c\gamma)\mathbf{f}\times(\mathbf{f}\times\mathbf{m})+s\gamma\mathbf{f}\times\mathbf{m} \qquad (5.3.15)^*$$

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*$c\gamma\equiv\cos\gamma$, $s\gamma\equiv\sin\gamma$

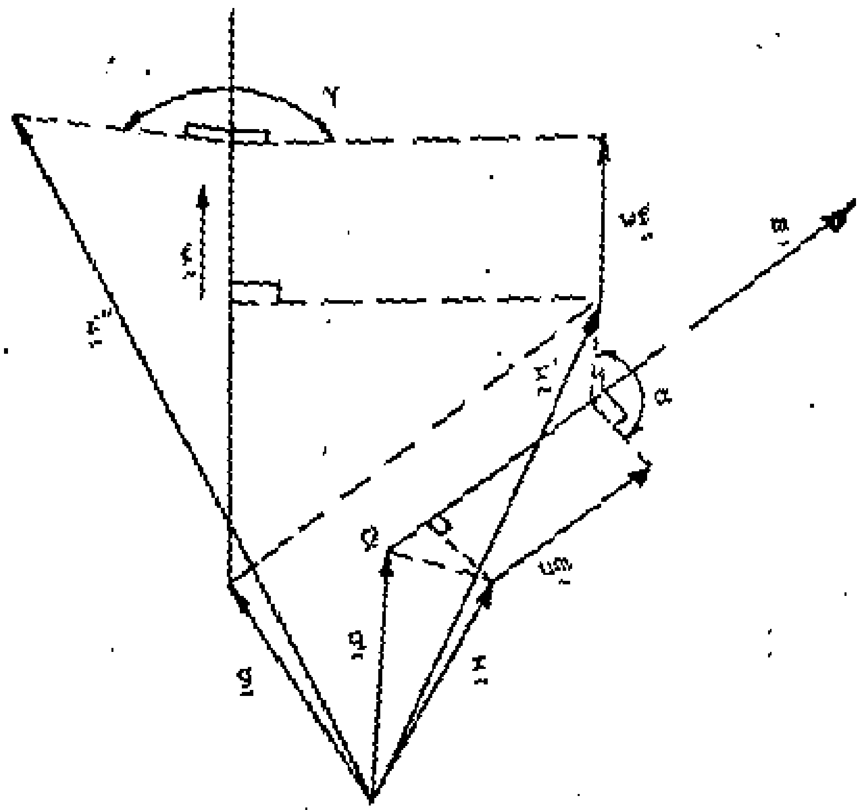

Fig 5.3.2 Composition of screws M and P

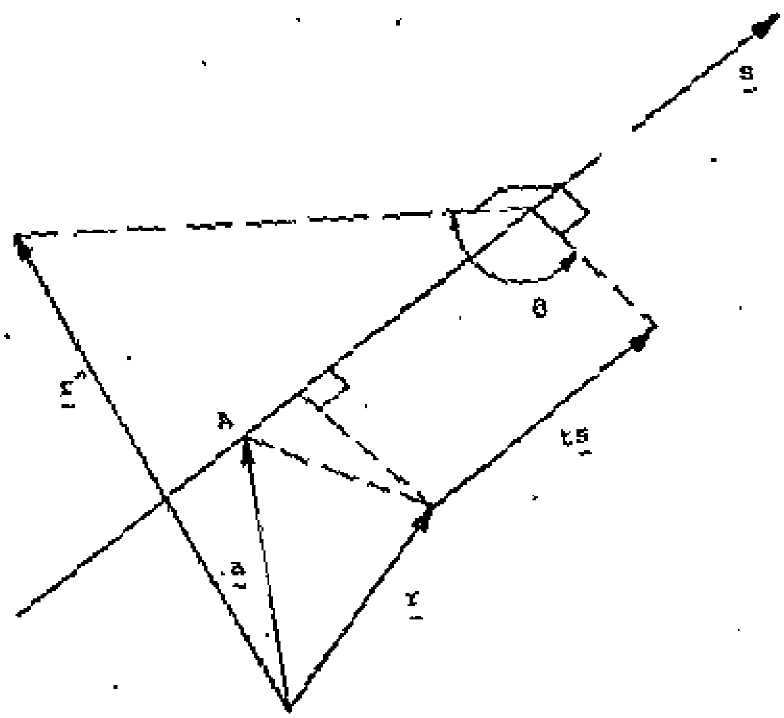

Fig 5.3.3. Equivalent screw

$$(Q_2-I)(q-g)=(1-c\gamma)f\times\left(f\times(q-g)\right)+s\gamma f\times(q-g) \tag{5.3.16}$$

$$(Q_2-I)(a-q)=(1-c\theta)s\times\left(s\times(a-q)\right)+s\theta s\times(a-q) \tag{5.3.17}$$

Substitution of (5.3.15)-(5.3.17) into (5.3.14) yields

$$wf-ts+um+u(1-c\gamma)f\times(f\times m)+us\gamma f\times m+(1-c\theta)s\times\left(s\times(a-q)\right)+s\theta s\times(a-q)$$
$$+(1-c\gamma)f\times\left(f\times(q-g)\right)+s\gamma f\times(q-g)=0 \tag{5.3.18}$$

Multiplying both sides of eq. (5.3.18) times $xf.s$ yields

$$um\times f.s+u(1-c\gamma)\left(f\times(f\times m)\right)\times f.s+us\gamma(f\times m)\times f.s+$$
$$+(1-c\theta)\left(s\times\left(s\times(a-q)\right)\right)\times f.s+s\theta\left(s\times(a-q)\right)\times f.s+$$
$$+(1-c\gamma)\left(f\times\left(f\times(q-g)\right)\right)\times f.s+s\gamma\left(f\times(q-g)\right)\times f.s=0 \tag{5.3.19}$$

Define

$$p_1=m\times f.s;\ p_2=(m\times f).(f\times s);\ p_3=(f\times s).(s\times m) \tag{5.3.20}$$

From eq. (5.3.4), the corresponding trigonometric identities and definitions (5.3.20), one obtains

$$c\gamma=\frac{p_2^2-p_1^2}{p_1^2+p_2^2},\quad 1-c\gamma=\frac{2p_1^2}{p_1^2+p_2^2},\quad s\gamma=\frac{2p_1p_2}{p_1^2+p_2^2} \tag{5.3.21}$$

$$c\theta=\frac{p_3^2-p_1^2}{p_1^2+p_3^2},\quad 1-c\theta=\frac{2p_1^2}{p_1^2+p_3^2},\quad s\theta=\frac{2p_1p_3}{p_1^2+p_3^2} \tag{5.3.22}$$

Substituting eqs. (5.3.20)-(5.3.22) into eq. (5.3.19),

$$u(p_1+\frac{2p_1^2}{p_1^2+p_2^2}\left(f\times(f\times m)\right)\times f.s+\frac{2p_1p_2}{p_1^2+p_2^2}(f\times m)\times f.s+$$

$$+\frac{2p_1^2}{p_1^2+p_2^2}\left(f\times\left(f\times(q-g)\right)\right)\times f.s+\frac{2p_1p_2}{p_1^2+p_2^2}\left(f\times(q-g)\right)\times f.s+$$

$$+\frac{2p_1^2}{p_1^2+p_3^2}\left(s\times\left(s\times(a-q)\right)\right)\times f.s-\frac{2p_1p_3}{p_1^2+p_3^2}\left(s\times(a-q)\right)\times f.s=0 \tag{5.3.23}$$

where

$$\left(f\times(f\times m)\right)\times f\cdot s=-m\times f\cdot s=-p_1 \qquad (5.3.24)$$

and

$$(f\times m)\times f\cdot s=(f\times m)\cdot(f\times s)=-p_2 \qquad (5.3.25)$$

Let A be the coefficient of u. Then,

$$A=p_1-\frac{2p_1^3}{p_1^2+p_2^2}-\frac{2p_1p_2^2}{p_1^2+p_2^2}=p_1\left(1-\frac{2p_1^2}{p_1^2+p_2^2}-\frac{2p_2^2}{p_1^2+p_2^2}\right)=$$

$$=p_1\,\frac{p_1^2+p_2^2-2(p_1^2+p_2^2)}{p_1^2+p_2^2}=-p_1 \qquad (5.3.26)$$

Let B be the sum of the next two terms in (5.3.23). Then

$$B=\frac{2p_1}{p_1^2+p_2^2}\left(p_1 s\times f+p_2(f\times s)\times f\right)\cdot(q-g) \qquad (5.3.27)$$

Let C be sum of the last two terms in (5.3.23),

$$C=-\frac{2p_1}{p_1^2+p_3^2}\left(p_1 f\times s+p_3(f\times s)\times s\right)\cdot\ (a-g) \qquad (5.3.28)$$

Substituting (5.3.26)-(5.3.28) into eq. (5.3.23), dropping the common factor p_1 in it and solving for u/2, one obtains

$$\frac{u}{2}=c_1\cdot(q-g)-c_2\cdot(q-q) \qquad (5.3.29)$$

where

$$c_1=\frac{1}{p_1^2+p_2^2}\left(p_1 s\times f+p_2(f\times s)\times f\right),\ c_2=\frac{1}{p_1^2+p_3^2}\left(p_1 f\times s+p_3(f\times s)\times s\right) \qquad (5.3.30)$$

To simplify expression (5.3.29), expand the terms in the denominators and those in the brackets of expressions (5.3.30).

$$p_1^2+p_2^2=(m\times f\cdot s)^2+\left((m\times f)\cdot(f\times s)\right)^2=(m\times f\cdot s)^2+||m\times f||^2||f\times s||^2\cos^2(m\times f,f\times s) \qquad (5.3.31)$$

But

$$\cos^2(\underline{m}x\underline{f},\underline{f}x\underline{s})=1-\sin^2(\underline{m}x\underline{f},\underline{f}x\underline{s})=1-\frac{||(\underline{m}x\underline{f})x(\underline{x}\underline{s})||^2}{||\underline{m}x\underline{f}||^2||\underline{f}x\underline{s}||^2} \quad (5.3.32)$$

Substituting expression (5.3.32) into (5.3.31), one obtains

$$p_1^2+p_2^2=(\underline{m}x\underline{f}.\underline{s})^2+||\underline{m}x\underline{f}||^2||\underline{f}x\underline{s}||^2-||(\underline{m}x\underline{f})x(\underline{f}x\underline{s})||^2 \quad (5.3.33)$$

where

$$||\underline{m}x\underline{f}||^2=(\underline{m}x\underline{f}).(\underline{m}x\underline{f})=(\underline{m}x\underline{f})x\underline{m}.\underline{f}=(\underline{f}-\underline{f}.\underline{m}\underline{m}).\underline{f}=1-(\underline{m}.\underline{f})^2 \quad (5.3.34)$$

and, similarly,

$$||\underline{f}x\underline{s}||^2=1-(\underline{s}.\underline{f})^2 \quad (5.3.35)$$

Furthermore,

$$\begin{aligned}||(\underline{m}x\underline{f})x(\underline{f}x\underline{s})||^2&=||(\underline{m}x\underline{f}).\underline{s}\underline{f}-(\underline{m}x\underline{f}).\underline{f}\underline{s}||^2\\&=||(\underline{m}x\underline{f}.\underline{s})\underline{f}||\\&=(\underline{m}x\underline{f}.\underline{s})^2||\underline{f}||^2=(\underline{m}x\underline{f}.\underline{s})^2\end{aligned} \quad (5.3.36)$$

Substitution of eqs. (5.3.34)-(5.3.36) into eq. (5.3.33) leads to

$$p_1^2+p_2^2=\left(1-(\underline{m}.\underline{f})^2\right)\left(1-(\underline{s}.\underline{f})^2\right) \quad (5.3.37)$$

and, similarly

$$p_1^2+p_3^2=\left(1-(\underline{s}.\underline{f})^2\right)\left(1-(\underline{s}.\underline{m})^2\right) \quad (5.3.38)$$

Moreover,

$$p_1\underline{s}\times\underline{f}+p_2(\underline{f}\times\underline{s})\times\underline{f}=p_1\underline{s}\times\underline{f}+p_2(\underline{s}-\underline{s}.\underline{f}\underline{f})=\underline{m}\times\underline{f}.\underline{s}\underline{s}\times\underline{f}+(\underline{m}\times\underline{f}).(\underline{f}\times\underline{s})\left(\underline{s}-\underline{s}.\underline{f}\underline{f}\right)$$

$$=\underline{m}.\underline{f}\times\underline{s}\underline{s}\times\underline{f}+\underline{m}.\underline{f}\times(\underline{f}\times\underline{s})\underline{s}.(\underline{1}-\underline{f}\underline{f})$$

$$=\underline{m}.\left\{\underline{f}\times\underline{s}\underline{s}\times\underline{f}+(\underline{f}\underline{f}-\underline{1}).\underline{s}\underline{s}.(\underline{1}-\underline{f}\underline{f})\right\} \qquad (5.3.39)^*$$

Each of the two terms in the brackets of expression (5.3.39) are next expanded

Let

$$\underline{f}\times\underline{s}=\underline{a},\underline{s}\times\underline{f}=\underline{b} \qquad (5.3.40)$$

Thus, the dyadic $\underline{f}\times\underline{s}\underline{s}\times\underline{f}$ can be written as**

$$\underline{f}\times\underline{s}\underline{s}\times\underline{f}=\underline{a}\underline{b}$$

---

* $\underline{1}$ is the identity dyadic, i.e. a dyadic that is isomorphic to the 3x3 identity matrix. Thus, in matrix notation, if the components of $\underline{f}$ are $f_1 f_2$ and f_3, then

$$\left(\underline{1}-\underline{f}\underline{f}\right)=\begin{pmatrix} 1-f_1^2 & -f_1f_2 & -f_1f_3 \\ -f_1f_2 & 1-f_2^2 & -f_1f_3 \\ -f_1^2f_3 & -f_2f_3 & 1-f_3^2 \end{pmatrix}$$

** For a short account on the algebra of dyadics see Appendix 1.

Introducing the usual index notation (5.15),

$$(\mathbf{f}\times\mathbf{s}\mathbf{s}\times\mathbf{f})_{ij}=a_i b_j \tag{5.3.41}$$

where, from definitions (5.3.40),

$$a_i=\varepsilon_{ik\ell}f_k s_\ell \ , \ b_j=\varepsilon_{jmn}s_m f_n \tag{5.3.42}$$

Hence

$$(\mathbf{f}\times\mathbf{s}\mathbf{s}\times\mathbf{f})_{ij}=\varepsilon_{ik\ell}\varepsilon_{jmn}f_k s_\ell s_m f_n \tag{5.3.43}$$

where, as is shown in Appendix 3,

$$\varepsilon_{ik\ell}\varepsilon_{jmn}=\delta_{ij}(\delta_{km}\delta_{\ell n}-\delta_{kn}\delta_{\ell m})+\delta_{jk}(\delta_{m\ell}\delta_{in}-\delta_{im}\delta_{\ell n})$$
$$+\delta_{j\ell}(\delta_{in}\delta_{kn}-\delta_{km}\delta_{in}), \tag{5.3.44}$$

Substituting (5.3.44) into (5.3.43), one obtains finally

$$\mathbf{f}\times\mathbf{s}\mathbf{s}\times\mathbf{f}=\mathbf{1}\left((\mathbf{f}.\mathbf{s})^2-1\right)+\mathbf{f}\mathbf{f}+\mathbf{s}\mathbf{s}-\mathbf{s}.\mathbf{f}(\mathbf{s}\mathbf{f}+\mathbf{f}\mathbf{s}) \tag{5.3.45}$$

Substitution of the latter expression into (5.3.39) and simplifying the résulting expression leads to

$$p_1\mathbf{s}\times\mathbf{f}+p_2(\mathbf{f}\times\mathbf{s})\times\mathbf{f}=\mathbf{m}.\left(1-(\mathbf{f}.\mathbf{s})^2\right)(\mathbf{f}\mathbf{f}-\mathbf{1}) \tag{5.3.46}$$

Similarly,

$$p_1\mathbf{f}\times\mathbf{s}+p_3(\mathbf{f}\times\mathbf{s})\times\mathbf{s}=\mathbf{m}.\left(1-(\mathbf{f}.\mathbf{s})^2\right)(\mathbf{1}-\mathbf{s}\mathbf{s}) \tag{5.3.47}$$

Substituting (5.3.37), (5.3.38), (5.3.46) and (5.3.47) into (5.3.30) and the corresponding expressions for $\mathbf{c}_1$ and $\mathbf{c}_2$ into (5.3.29), one obtains.

$$\frac{u}{2}=\frac{\mathbf{m}.(\mathbf{1}-\mathbf{f}\mathbf{f})}{1-(\mathbf{m}.\mathbf{f})^2}:(\mathbf{g}-\mathbf{q})+\frac{\mathbf{m}.(\mathbf{1}-\mathbf{s}\mathbf{s})}{1-(\mathbf{s}.\mathbf{m})^2}\cdot(\mathbf{a}-\mathbf{q}) \tag{5.3.48}$$

which is identical to the corresponding expression obtained by Roth and Tsai (5.11).

Multiplying eq. (5.3.14) times $\mathbf{Q}_2^T$ one obtains

$$w\underset{\sim}{f}+u\underset{\sim}{m}-t\underset{\sim}{Q}_2^T\underset{\sim}{s}+(\underset{\sim}{Q}_1-\underset{\sim}{Q}_2^T)(\underset{\sim}{a}-\underset{\sim}{q})-(\underset{\sim}{Q}_2^T-\underset{\sim}{I})(\underset{\sim}{q}-\underset{\sim}{g})=\underset{\sim}{0} \quad (5.3.49)$$

Multiplying the latter equation times $x\underset{\sim}{f}.\underset{\sim}{m}$ and proceeding in a manner similar to that leading to eq. (5.3.48) starting from eq. (5.3.15), one obtains

$$\frac{t}{2}=-\frac{\underset{\sim}{s}-(\underset{\sim}{s}.\underset{\sim}{m})\underset{\sim}{m}}{1-(\underset{\sim}{s}.\underset{\sim}{m})^2}.(\underset{\sim}{q}-\underset{\sim}{a})+\frac{\underset{\sim}{s}-(\underset{\sim}{s}.\underset{\sim}{f})\underset{\sim}{f}}{1-(\underset{\sim}{s}.\underset{\sim}{f})^2}.(\underset{\sim}{g}-\underset{\sim}{a}) \quad (5.3.50)$$

Now multiplying eq. (5.3.14) times $\underset{\sim}{Q}_3^T$ leads to

$$w\underset{\sim}{Q}_3^T\underset{\sim}{f}+u\underset{\sim}{m}-t\underset{\sim}{s}-(\underset{\sim}{Q}_3^T-\underset{\sim}{I})(\underset{\sim}{a}-\underset{\sim}{q})+(\underset{\sim}{Q}_1^T-\underset{\sim}{Q}_3^T)(\underset{\sim}{q}-\underset{\sim}{g})=\underset{\sim}{0} \quad (5.3.51)$$

Finally, multiplying eq. (5.3.51) times $x\underset{\sim}{m}.\underset{\sim}{s}$ and proceeding as before, one obtains

$$\frac{w}{2}=\frac{\underset{\sim}{f}-(\underset{\sim}{f}.\underset{\sim}{s})\underset{\sim}{s}}{1-(\underset{\sim}{f}.\underset{\sim}{s})^2}.(\underset{\sim}{a}-\underset{\sim}{g})-\frac{\underset{\sim}{f}-(\underset{\sim}{f}.\underset{\sim}{m})\underset{\sim}{m}}{1-(\underset{\sim}{f}.\underset{\sim}{m})^2}.(\underset{\sim}{q}-\underset{\sim}{g}) \quad (5.3.52)$$

Equations (5.3.4), (5.3.7), (5.3.9), (5.3.48), (5.3.50) and (5.3.52) are the <u>synthesis equations</u> which were meant to be obtained.

<u>Remarks about the synthesis equations</u>:

i) They are useful to synthesise spatial linkages, but not plane ones, because in the latter case, $\underset{\sim}{f}$, $\underset{\sim}{m}$ and $\underset{\sim}{s}$ are parallel. Hence angles α, γ and θ are undetermined in eqs. (5.3.4), (5.3.7) and (5.3.9) and displacements u, t and w are undertermined in eqs. (5.3.48), (5.3.50) and (5.3.52). Thus, to synthesise plane linkages, other equations should be used, like those developed by Suh [5.16] or by Angeles [5.17]. Equations appearing in [5.17] are derived in the Appendix.

ii) They enable the designer to synthesise dyads of any combination of the six kinematic lower pairs introduced in Ch. 3, except for the planar one. Notice, however, that since the axis of one of the two pairs of the dyad is fixed and the other one is moving, the dyad is not symmetric, for which reason the R-S dyad, for instance, is different from the S-R dyad. Hence, the total number of dyads that can be designed with the foregoing equations is $5^2=25$. The syntheses of all these dyads, except for the P-P one, are discussed in (5.11). As Roth and Tsai point out in (5.11), unless the guided rigid body undergoes pure translation, in general a P-P dyad does not exist for an arbitrary rigid body motion.

iii) If the different configurations of a rigid body meant to be guided are specified, not via their screws (referred to a common original configuration), but via the successive positions of a set of three non-collinear points of the rigid body, then the corresponding screws must first be computed. This can be done with the computer subroutine SCREW, whose listing appears in Fig 2.6.6.

iv) The path-generation problem of synthesis, discussed in Section 5.5, can also be treated using the equations under consideration.

The outlined synthesis procedure is illustrated by means of an example regarding the design of an R-R dyad for rigid-body guidance. The synthesis of this dyad has been studied extensively. It was first shown that the maximum number of specified configurations of the rigid body is three (See Section 5.4), if exact solutions are to be obtained, these solutions being exact up to round-off and/or measuring errors. This statement can be readily proved by the reader; besides, it is reported in several papers, e.g. in (5.11,p.94). Roth (5.18) showed that the aforementioned exact synthesis problem has no more than 24 real solutions, whereas Suh (5.19) showed that these solutions always come in pairs, each pair forming a Bennett mechanism (5.20), i.e. an RRRR spatial linkage with degree of freedom 1*. Finally, Roth and Tsai (5.12) showed that this problem has only one pair of real solutions, which constitute a Bennett mechanism.

Example 5.3.1 Synthesise an RRRR spatial linkage to guide a rigid body whose points A,B and C attain the following successive positions:

$A_0(0,0,1)$	$A_1(0,0,1)$	$A_2(1,0,0)$
$B_0(0,1,0)$	$B_1(1,\sqrt{2},1)$	$B_2(1+\sqrt{2}/2,0,\sqrt{6}/2)$
$C_0(1,0,0)$	$C_1(\sqrt{6}/2,\sqrt{2}/2,0)$	$C_2(1+\sqrt{2},0,0)$

The two involved screws, $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$, transporting the rigid body from configuration 0 to configurations 1 and 2, respectively were obtained with the aid of SUBROUTINE SCREW, the resulting parameters being those obtained from the computer printout of the program that was written for this purpose. These are the following:

$$\underset{\sim}{s}_1=\begin{bmatrix}-0.769\\0.590\\-0.245\end{bmatrix},\ \underset{\sim}{a}_1=\begin{bmatrix}0.454\\0.787\\0.470\end{bmatrix},t_1=0.245,\theta_1=-56.600°$$

$$\underset{\sim}{s}_2=\begin{bmatrix}-0.906\\0.194\\0.375\end{bmatrix}\ \ \underset{\sim}{a}_2=\begin{bmatrix}0.305\\0.176\\0.645\end{bmatrix},t_2=-1.282,\theta_2=-129.736°$$

In this problem, θ_j, $\underset{\sim}{s}_j$, $\underset{\sim}{a}_j$, u_j and w_j $(j=1,2)$ are known. In fact, $u_j=w_j=0\ (j=1,2)$, for the corresponding motions are produced by revolute pairs, thereby not allowing for any sliding; the reamaining aforementioned parameters are those obtained via SUBROUTINE SCREW. The synthesis equations that are applied are eqs. (5.3.9), (5.3.48), (5.3.50) and (5.3.52), all of

* If the Grübler-Kutzbach 3.5 formula is applied to this linkage, it is found that its degree of freedom is -2

them taken twice, once for each value of j.

Additionally, the unity-magnitude (i.e. normality) condition on vectors $\underset{\sim}{f}$ and $\underset{\sim}{m}$ yields two more equations. Finally, specifying the location of points G and Q, on the axes of the $\underset{\sim}{F}$ and $\underset{\sim}{M}$ screws, respectively, in such a way as to render GQ perpendicular to both axes, one obtains two additional equations. Summarizing, the problem leads to a system of 12 nonlinear algebraic equations in 12 unknowns. Roth and Tsai (5.12) introduced an algorithm that reduces the problem to finding the unique positive real root of a cubic equation. In the course of the description of their algorithm, they show that the system admits exactly two real different solutions which, as Suh (5.19) proved before, are bound to constitute a Bennett mechanism. Roth and Tsai also show that spurious solutions to the said system also appear, which contain, as solutions to $\underset{\sim}{f}$ or $\underset{\sim}{m}$, either $\underset{\sim}{s}_1$ or $\underset{\sim}{s}_2$. The author has found, following a different approach, that other spurious solutions can also appear, these solutions being inadmissible in the sense of yielding, for instance, point Q of the second solution identical to point G of the first one, thus making impossible the construction of the linkage.

The solution to the present problem is now obtained following a different procedure as that proposed by Roth and Tsai. In fact, although the algorithm proposed by these authors can be implemented even with a desk calculator (not necessarily a programmable one), the author could not see how to extend it to overdetermined problems, i.e. problems that involve the guidance of a rigid body through more than three different configurations, thus not allowing for an exact solution, in which case the best approximation (See Section 6.3) should be sought. The procedure followed here is based on the use of SUBROUTINE NRDAMP. First the synthesis equations are simplified considering that vector $\underset{\sim}{g}-\underset{\sim}{q}$ is perpendicular to both $\underset{\sim}{f}$ and $\underset{\sim}{m}$. Thus, the synthesis equations reduce to

For i=1,2:

$$F_i=\tan\frac{\theta_i}{2}\underline{f}^T(\underline{I}-\underline{s}_i\underline{s}_i^T)\underline{m}+\det(\underline{f},\underline{m},\underline{s}_i)=0 \quad (5.3.53a)$$

$$F_{i+2}=\underline{f}^T(\underline{I}-\underline{s}_i\underline{s}_i^T)(\underline{a}_i-\underline{g})=0 \quad (5.3.53b)$$

$$F_{i+4}=\underline{m}^T(\underline{I}-\underline{s}_i\underline{s}_i^T)(\underline{a}_i-\underline{g})=0 \quad (5.3.53c)$$

$$F_{i+6}=\underline{s}_i^T(\underline{g}-\underline{q})-\frac{t_i}{2}=0 \quad (5.3.53d)$$

$$F_9=\underline{f}^T\underline{f}-1=0 \quad (5.3.53e)$$

$$F_{10}=\underline{m}^T\underline{m}-1=0 \quad (5.3.53f)$$

$$F_{11}=\underline{f}^T(\underline{g}-\underline{q})=0 \quad (5.3.53g)$$

$$F_{12}=\underline{m}^T(\underline{g}-\underline{q})=0 \quad (5.3.53h)$$

The Jacobian matrix of the system, which is needed by NRDAMP, is now formed by computing the partial derivatives of the above functions with respect to the unknowns $\underline{f},\underline{m},\underline{g}$ and $\underline{q}$. Thus,

For i=1,2:

$$\frac{\partial F_i}{\partial \underline{f}} = \tan\frac{\theta_i}{2}(\underline{I}-\underline{s}_i\underline{s}_i^T)\underline{m}+\underline{m}\times\underline{s}_i \quad (5.3.54a)$$

$$\frac{\partial F_i}{\partial \underline{m}} = \tan\frac{\theta_i}{2}(\underline{I}-\underline{s}_i\underline{s}_i^T)\underline{f}-\underline{f}\times\underline{s}_i \quad (5.3.54b)$$

$$\frac{\partial F_i}{\partial \underline{g}} = \underline{0},\ \frac{\partial F_i}{\partial \underline{q}} = \underline{0} \quad (5.3.54c)$$

$$\frac{\partial F_{i+2}}{\partial \underline{f}} = (\underline{I}-\underline{s}_i\underline{s}_i^T)(\underline{a}_i-\underline{g}),\ \frac{\partial F_{i+2}}{\partial \underline{m}} = \underline{0} \quad (5.3.54d)$$

$$\frac{\partial \mathbf{F}_{i+2}}{\partial \mathbf{g}} = -(\mathbf{I}-\mathbf{s}_i\mathbf{s}_i^T)\mathbf{f}, \quad \frac{\partial \mathbf{F}_{i+2}}{\partial \mathbf{q}} = \mathbf{0} \qquad (5.3.54e)$$

$$\frac{\partial \mathbf{F}_{i+4}}{\partial \mathbf{f}} = \mathbf{0}, \quad \frac{\partial \mathbf{F}_{i+4}}{\partial \mathbf{m}} = (\mathbf{I}-\mathbf{s}_i\mathbf{s}_i^T)(\mathbf{a}_i-\mathbf{q}) \qquad (5.3.54f)$$

$$\frac{\partial \mathbf{F}_{i+4}}{\partial \mathbf{g}} = \mathbf{0}, \quad \frac{\partial \mathbf{F}_{i+4}}{\partial \mathbf{q}} = -(\mathbf{I}-\mathbf{s}_i\mathbf{s}_i^T)\mathbf{m} \qquad (5.3.54g)$$

$$\frac{\partial \mathbf{F}_{i+6}}{\partial \mathbf{f}} = \mathbf{0}, \quad \frac{\partial \mathbf{F}_{i+6}}{\partial \mathbf{m}} = \mathbf{0}, \quad \frac{\partial \mathbf{F}_{i+6}}{\partial \mathbf{g}} = \mathbf{s}_i, \quad \frac{\partial \mathbf{F}_{i+6}}{\partial \mathbf{q}} = -\mathbf{s}_i \qquad (5.3.54h)$$

$$\frac{\partial F_9}{\partial \mathbf{f}} = 2\mathbf{f}, \quad \frac{\partial F_9}{\partial \mathbf{m}} = \mathbf{0}, \quad \frac{\partial F_9}{\partial \mathbf{g}} = \mathbf{0}, \quad \frac{\partial F_9}{\partial \mathbf{q}} = \mathbf{0} \qquad (5.3.54i)$$

$$\frac{\partial F_{10}}{\partial \mathbf{f}} = \mathbf{0}, \quad \frac{\partial F_{10}}{\partial \mathbf{m}} = 2\mathbf{m}, \quad \frac{\partial F_{10}}{\partial \mathbf{g}} = \mathbf{0}, \quad \frac{\partial F_{10}}{\partial \mathbf{q}} = \mathbf{0} \qquad (5.3.54j)$$

$$\frac{\partial F_{11}}{\partial \mathbf{f}} = \mathbf{g}-\mathbf{q}, \quad \frac{\partial F_{11}}{\partial \mathbf{m}} = \mathbf{0}, \quad \frac{\partial F_{11}}{\partial \mathbf{g}} = \mathbf{f}, \quad \frac{\partial F_{11}}{\partial \mathbf{q}} = -\mathbf{f} \qquad (5.3.54k)$$

$$\frac{\partial F_{12}}{\partial \mathbf{f}} = \mathbf{0}, \quad \frac{\partial F_{12}}{\partial \mathbf{m}} = \mathbf{g}-\mathbf{q}, \quad \frac{\partial F_{12}}{\partial \mathbf{g}} = \mathbf{m}, \quad \frac{\partial F_{12}}{\partial \mathbf{q}} = -\mathbf{m} \qquad (5.3.54l)$$

Subroutines FUN and DFDX, required by NRDAMP, compute functions F_i $(i=1,\ldots 12)$ and the Jacobian matrix. For this purpose, a vector $\mathbf{x}$ of twelve components is defined in the folowing manner:

$$x_i = f_i, \; i=1,2,3$$

$$x_{i+3} = m_i, \; i=1,2,3$$

$$x_{i+6} = g_i, \; i=1,2,3$$

$$x_{i+9} = q_i, \; i=1,2,3$$

Analogously, a vector $\mathbf{p}$ of given parameters is defined as

For i=1,2,3,

$p_i = s_{1i}$

$p_{i+3} = s_{2i}$

$p_{i+6} = a_{1i}$

$p_{i+9} = a_{2i}$

for i=1,2,

$p_{i+12} = \theta_i$

$p_{i+14} = t_i$

In order to automate the calculations, the following arrays are defined:

VE(I,J), for I=1,2,...,7 and J=1,2,3, is the Jth component of the Ith vector of the array, where

$\underset{\sim}{f}$= 1st vector of the array

$\underset{\sim}{m}$= 2nd vector of the array

$\underset{\sim}{a}_1 - \underset{\sim}{q}$= 3rd vector of the array

$\underset{\sim}{a}_2 - \underset{\sim}{q}$= 4th vector of the array

$\underset{\sim}{a}_1 - \underset{\sim}{q}$= 5th vector of the array

$\underset{\sim}{a}_2 - \underset{\sim}{q}$= 5th vector of the array

$\underset{\sim}{g} - \underset{\sim}{a}$ = 7th vector of the array

MA(I,J,K), for I=1,2, and J,K=1,2,3, is the (J,K)th entry of the Ith matrix of the array, where

$\underset{\sim}{I} - \underset{\sim}{s}_i \underset{\sim}{s}_i^T$= the ith matrix of the array.

Other subsidiary subroutines are the following:

SUBROUTINE MAVE (I,MA,J,VE,PROV) computes the product of matrix I of MA times vector J of VE and stores this product in array PROV.

SUBROUTINE VEVE(I,VE,V,PROS) computes the inner product of vector I of VE times V and stores this product in PROS.

SUBROUTINE CROSS (I,X,J,P,CROS) Computes the cross product of either $\underset{\sim}{f}$(I=1) or $\underset{\sim}{m}$(I=2), stored in X, times either $\underset{\sim}{s}_1$(J=1) or $\underset{\sim}{s}_2$(J=2), both stored in P. It stores the product in array CROS. The listings of the aforementioned subroutines appear in Figs 5.3.4, 5.3.6, 5.3.7 and 5.3.8

The two meaningful solutions that were obtained, using the Newton-Raphson method, are

$$\underset{\sim}{f}_1 = \begin{pmatrix} 0.819 \\ -0.569 \\ 0.073 \end{pmatrix}, \underset{\sim}{m}_1 = \begin{pmatrix} 0.083 \\ -0.349 \\ -0.933 \end{pmatrix}, \underset{\sim}{g}_1 = \begin{pmatrix} 1.048 \\ 0.317 \\ 0.658 \end{pmatrix}, \underset{\sim}{q}_1 = \begin{pmatrix} 0.246 \\ -0.793 \\ 1.002 \end{pmatrix}$$

$$\underset{\sim}{f}_2 = \begin{pmatrix} -0.660 \\ -0.744 \\ 0.107 \end{pmatrix}, \underset{\sim}{m}_2 = \begin{pmatrix} -0.866 \\ 0.480 \\ -0.140 \end{pmatrix}, \underset{\sim}{g}_2 = \begin{pmatrix} 1.039 \\ 0.121 \\ -0.761 \end{pmatrix}, \underset{\sim}{q}_2 = \begin{pmatrix} 0.963 \\ 0.388 \\ 0.624 \end{pmatrix}$$

The conditions for the above linkage to constitute a Bennett mechanism (5.20 are, according to Fig 5.3.9,

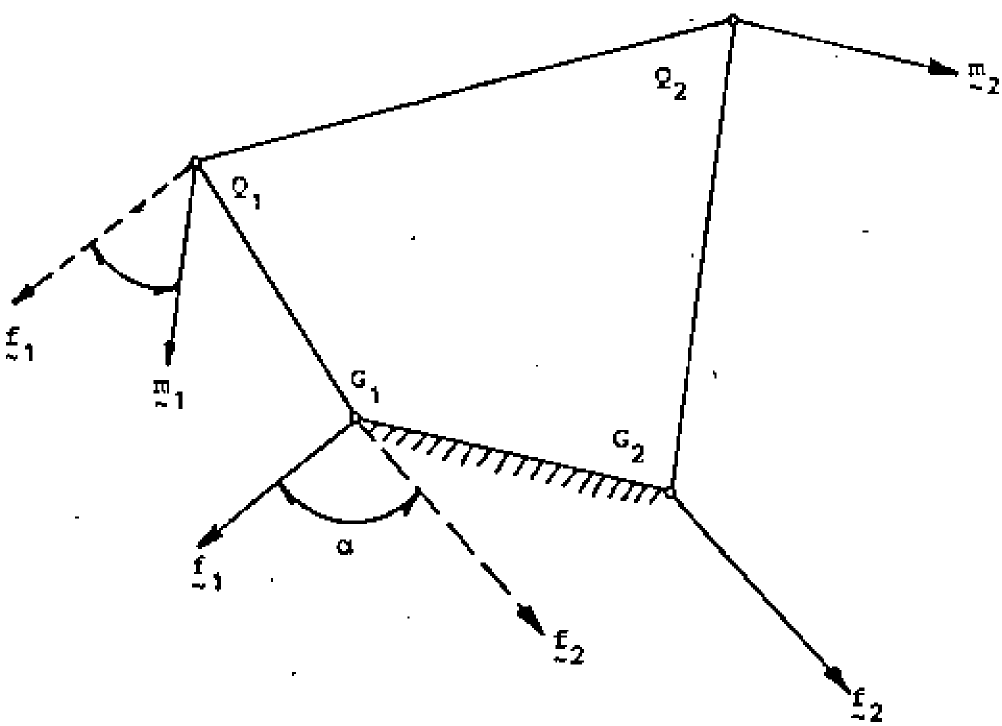

Fig 5.3.9 A Bennett Mechanism

$$\overline{G_1Q_1}=\overline{G_2Q_2}=a$$

$$\overline{G_1G_2}=\overline{Q_1Q_2}=b$$

$$\left(\underset{\sim}{f}_1^T\underset{\sim}{f}_2\right)=\left(\underset{\sim}{m}_1^T\underset{\sim}{m}_2\right)$$

$$\left(\underset{\sim}{f}_1^T\underset{\sim}{m}_1=\underset{\sim}{f}_2^T\underset{\sim}{m}_2\right)$$

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$$

It can be readily proved that the linkage just designed satisfies the above conditions and thus constitutes a Bennett mechanism.

The algorithm of Roth and Tsai requires referring the problem to a particular set of coordinate axes, in such a way that, letting $\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k}$ denote unit vectors along new X,Y and Z axes, respectively, these vectors are defined as

$$\underset{\sim}{k}=\underset{\sim}{s}_1$$

$$\underset{\sim}{j}=\frac{\underset{\sim}{s}_1\times\underset{\sim}{s}_2}{||\underset{\sim}{s}_1\times\underset{\sim}{s}_2||}$$

$$\underset{\sim}{i}=\underset{\sim}{j}\times\underset{\sim}{k}$$

Furthermore, the origin of coordinates is to be placed at the intersection of the common perpendicular to the axes of screws $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ with the axis of $\underset{\sim}{S}_1$. This way, in the new coordinates,

$$\underset{\sim}{s}_1=\begin{pmatrix}0\\0\\1\end{pmatrix},\underset{\sim}{a}_1=\begin{pmatrix}0\\0\\0\end{pmatrix},\underset{\sim}{s}_2=\begin{pmatrix}u\\0\\v\end{pmatrix},\underset{\sim}{a}_2=\begin{pmatrix}0\\h\\0\end{pmatrix}$$

where h is the distance between the axes of $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ times the sign of $\underset{\sim}{s}_1\times\underset{\sim}{s}_2\cdot(\underset{\sim}{a}_2-\underset{\sim}{a}_1)$. The solution obtained using the data in the new coordinate axes must be then transformed into the original coordinates. This is done via the following affine transformation:

$$(\underset{\sim}{f})_o=(\underset{\sim}{R})_o(\underset{\sim}{f})_n$$

$$\{\underline{m}\}_o = (\underline{R})_o \{\underline{m}\}_n$$

$$\{\underline{g}\}_o = \{\underline{a}_1\}_o + (\underline{R})_o \{\underline{g}\}_n$$

$$\{\underline{q}\}_o = \{\underline{a}_1\}_o + (\underline{R})_o \{\underline{q}\}_n$$

where subscripts o and n refer to original and new coordinates, respectively. Matrix $(\underline{R})_o$ can be obtained via Definition 1.2.1, as

$$(\underline{R})_o = \left(\underline{j}\times\underline{k} \;\vdots\; \frac{\underline{s}_1\times\underline{s}_2}{\|\underline{s}_1\times\underline{s}_2\|} \;\vdots\; \underline{s}_1\right)^T$$

where the transpose of the given matrix should be taken because the matrix taken as such (without transposing) represents the rotation of the original axes the new ones■.

<u>Exercise 5.3.1</u> Check the solution given above to Example 5.3.1, applying Tsai and Roth's algorithm (5.12).

5.4 A DIFFERENT APPROACH TO THE SYNTHESIS PROBLEM FOR RIGID-BODY GUIDANCE.

For dyads containing any combination of revolute and spherical pairs, this problem can be formulated in an alternative way, which formally resembles the plane linkage synthesis problem (5.17) discussed in Appendix 4. Let $X_0Y_0Z_0$ be coordinate axes attached to the rigid body intended to guide, in its reference configuration, and $X_jY_jZ_j$ those attached to it in its jth configuration. Furthermore, let S_{12} and S_{23} be spherical pairs connecting the dyad to synthesize to the frame of the mechanism (linkage 1) and to the rigid body (linkage 3). Then, from Fig 5.4.1, the next relationship follows

$$\underset{\sim}{r}_j+\underset{\sim}{a}_j+\underset{\sim}{b}_j=\underset{\sim}{a}_0+\underset{\sim}{b}_0 \tag{5.4.1}$$

$$\underset{\sim}{a}_j=\underset{\sim}{Q}_j\underset{\sim}{a}_0 \tag{5.4.2}$$

$\underset{\sim}{Q}_j$ being the rotation matrix of the screw carrying the rigid body from its reference to its $j^{\underline{th}}$ configuration.

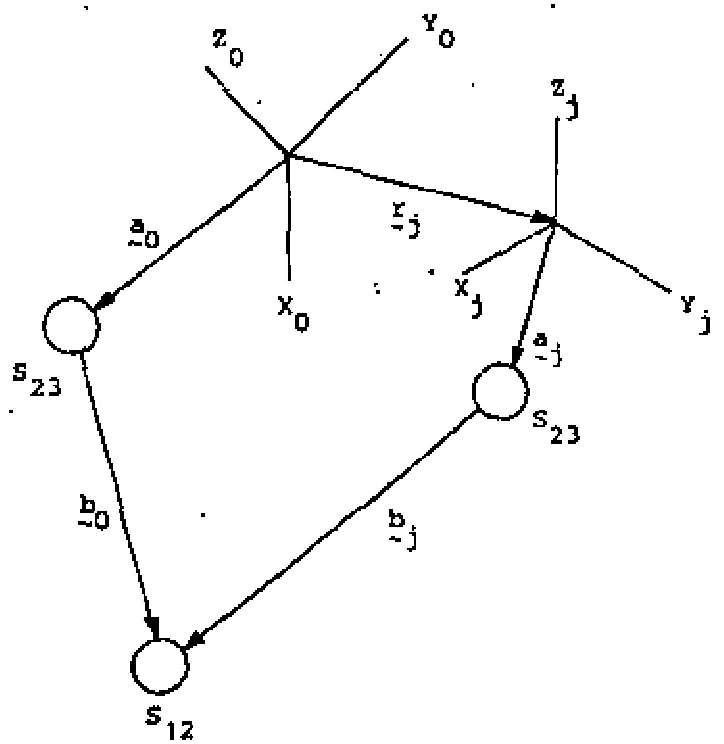

Fig 5.4.1 S-S dyad conducting a rigid body

Introducing eq. (5.4.2) into eq. (5.4.1) and solving for b_j one obtains

$$b_j = (I - Q_j) a_0 + b_0 - r_j \tag{5.4.3}$$

Since the dyad is rigid,

$$||b_j|| = ||b_0|| \tag{5.4.4}$$

Substituting eq. (5.4.3) into eq. (5.4.4) and squaring both sides of the resulting equation yields

$$||(I - Q_j) a_0 + b_0 - r_j||^2 = ||b_0||^2, j=1,2,\ldots,n \tag{5.4.5}$$

which constitute, in general, a nonlinear system of n synthesis equations to solve for the unknowns a_0 and b_0. Since these unknowns contain 6 scalar components, for exact synthesis the largest value that n can attain is 6, and hence, with an S-S dyad, a rigid body can be conducted through 7 configurations, counting the reference configuration as the seventh one.

Exercise 5.4.1 If it is intended to conduct the rigid body through 4 configurations only, then 3 unknowns must be specified to render the problem determined. These specified unknowns can be any combination of the six involved coordinates. Show that, if the three components of a_0 are specified, the system of equations becomes linear.

Regarding the four-configuration synthesis problem discussed in Exercise 5.4.1, if a_0 happens to be specified in such a way that for the $k^{\underline{th}}$ configurations the next holds.

$$r_k = (I - Q_k) a_0 \tag{5.4.6}$$

then the $k^{\underline{th}}$ synthesis equation becomes an identity, thus leading to an underdetermined system of equations, which can be rendered determined by either specifying one fifth configuration or by specifying one of the components of b_0. An alternative way of handling this situation, amply discussed in Ch. 6, is to leave the system of equations undertermined but imposing some optimality condition like, for instance, to minimize $||b_0||$, thus enabling the designer to obtain the minimum-length dyad.

From the above discussion it is apparent that it is highly desirable to be able to specify a_0 so as to satisfy eq. (5.4.6). However, it is not possible to solve for a_0 from this equation, for the matrix involved is singular, as discussed in Section 2.6. In that section it was also mentioned that in a system of equations like the one "defining" a_0 in eq. (5.4.6), two of the three equations are linearly dependent. Hence, this system is underdetermined and, properly speaking, <u>does not</u> actually <u>define</u> a_0. This vector can, however, be defined by imposing the condition that its magnitude be a minimum, which can be done via the pseudoinverse matrix of the system of 2 (linearly independent) equations in three unknowns obtained from eq. (5.4.6).

<u>Exercise 5.4.2</u> Show that if r_k is parallel to the axis of the screw carrying the rigid body from its reference configuration to its k^{th} configuration, then it is not possible to find a vector a_0 satisfying eq. (5.4.6).

If vector r_k is not on the null space of $I-Q_k$(Sect. 1.3), i.e., if it is not parallel to the axis of rotation of Q_k, then a minimum-norm vector a_0 can be obtained from the underdetermined system of the two linearly independent equations contained in eq. (5.4.6). This is a very useful result, because, as the length of vector a_0 is made the shortest possible, the weight and the dimensions of the linkage are minimized, thus allowing to construct a linkage with the smallest amount of material, driving it with the least possible amount of power and placing it in the narrowest space. More on linkage optimization will be discussed in Ch. 6

<u>Synthesis of R-S dyads.</u> In this case a configuration analogous to the one shown in Fig 5.4.1 is obtained, except that pair S_{12} is replaced by pair R_{12}. Similarly, for n+1 rigid-body configurations, the same synthesis equations (5.4.5) follow. Since this dyad can only rotate about the axis

of R_{12}, suitable constraints must be introduced in order to insure the normality between vectors $\underline{b}_j (j=0,1,\ldots,n)$ and the unit vector $\underline{c}$, directed along the axis of R_{12}. This normality condition is equivalent to the coplanarity condition among all vectors $\underline{b}_j$. Algebraically, the latter condition can be stated as

$$\det(\underline{b}_0,\underline{b}_1,\underline{b}_s)=0, s=2,\ldots,n \quad (5.4.7)$$

thus obtaining n-1 additional synthesis equations. The number of synthesis parameters (unknowns) in this case is again six, i.e. the three scalar components of vectors $\underline{a}_0$ and $\underline{b}_0$. Hence, in order to match the number of equations to that of unknowns, one must have

$$2n-1=6$$

i.e.

$$n=\frac{7}{2}$$

which, unfortunately is not an integer.

From the foregoing discussion it follows that some parameters should be specified beforehand. If m parameters are specified, then n must have the value

$$n=\frac{7-m}{2}$$

Thus, letting $N=n+1$ be the number of prescribed rigid body configurations, the following table is obtained

m	N
1	4
3	3
5	2

It is pointed out, again, that if three parameters are specified, these can be chosen to be the three components of $\underline{a}_0$ satisfying the two linearly independent equations of (5.4.6), for one k, such that the magnitude of

$\underset{\sim}{a}_0$ is a minimum. If no constraint equations (5.4.7) were to be present in this case, this would allow one to prescribe one fourth configuration of the rigid body under guidance. The presence of the said equations, however, does not prevent the introduction of such an extra configuration, for, from equations (5.4.3) and (5.4.6), $\underset{\sim}{b}_k=\underset{\sim}{b}_0$, for that k for which (5.4.6) holds, the $k^{\underline{th}}$ equation of (5.4.7) thereby being identically satisfied. In conclusion, the R-S dyad can be synthesised for a maximum of 4 prescribed configurations of a rigid body in such a way that $||\underset{\sim}{a}_0||$ is minimised, therefore allowing for a minimum-weight synthesis. The latter result could also be concluded from the discussions of the same problem appearing in (5.11) and (5.21), where the authors show that the said rigid-body guidance problem leads to an underdetermined system of equations. The synthesis of S-R dyads is essentially the same as that of R-S dyads

<u>Synthesis of R-R dyads</u>. Again the picture shown in Fig 5.4.1 relating the reference-and the $j^{\underline{th}}$ configurations for an S-S dyad, still holds for R-R dyads, except that pairs S_{12} and S_{23} are replaced by R_{12} and R_{23}, respectively, as shown in Fig 5.4.2. In addition to the synthesis equations (5.4.5), constraint equations for these pairs should be introduced. In fact, if both revolute pairs are connected by means of a link of axis parallel to the common perpendicular to the axis of both revolute pairs, this perpendicularity should be observed throughout the different specified configurations.

The perpendicularity of the axis of the link with the axis of R_{12} can be stated as

$$\underset{\sim}{u}^T\underset{\sim}{b}_j=0,\ j=0,1,\ldots,n \qquad (5.4.8)$$

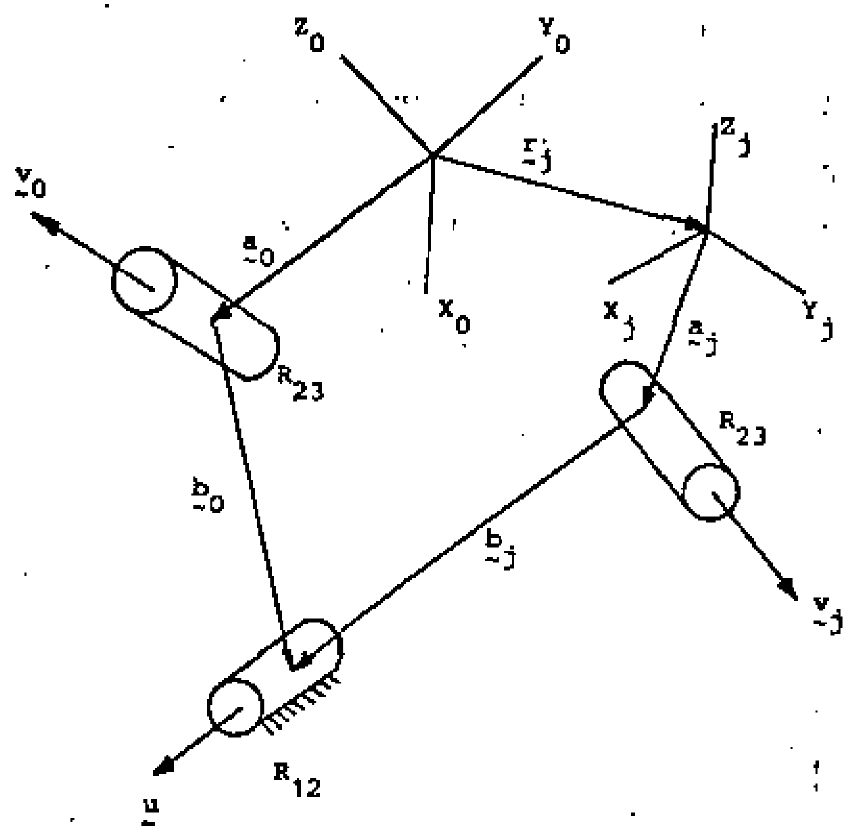

Fig 5.4.2 Rigid-body guidance by means of an R-R dyad

Additionally, the perpendicularity of the said link axis with the axis of R_{23} can be stated as

$$v^T b_j = 0, j=0,1,\ldots,n$$

or, introducing eq. (5.4.3), and the fact that

$$v_j = Q_j v_0, j=0,1,\ldots,n$$

the latter perpendicularity condition appears as

$$v_0^T Q^T \left((I-Q) a_0 + b_0 - r_j \right) = 0, j=0,1,\ldots,n \qquad (5.4.9)$$

where

$$Q_0 \equiv I, r_0 \equiv 0$$

From Fig 5.4.2, it is clear that vectors u and v belong to the same rigid body, namely the R-R link to be synthesised.

Equations (5.4.8) and (5.4.9), however, do not guarantee that these vectors undergo a rigid body motion from the reference to the j^{th} configuration, In fact, a 180-degree rotation of vector v_j about the link axis (vector b_j) -thus producing a reflexion of the link- could not be detected from eq. (5.4.9), for this would still hold after such a reflection. Hence, an additional constraint should be imposed, namely the constancy of the angle between vectors u and v, i.e.

$$u^T v_j = u^T v_0, j=1,2,\ldots,n \qquad (5.4.10)$$

Finally, the normality of vectors u and v_0 leads to

$$||u|| = 1, ||v_0|| = 1 \qquad (5.4.11)$$

Summarizing, eqs. (5.4.5), (5.4.8-11) constitute a system of 4n+4 equations to compute the 12 unknown components of vectors a_0, b_0, u and v_0. To match the number of equations to that of unknowns, one must have

$4n+4=12$, i.e. $n=2$

Hence, by means of an R-R dyad, a rigid body can be guided through 3 configurations (counting the reference as the third configuration), a result which was mentioned in Section 5.3. As in the two foregoing cases, it is possible to realize an optimal synthesis of this linkage. In fact, instead of obtaining the unknown vectors from the 12 derived equations, find $\underset{\sim}{a}_0$ from eq. (5.4.6) such that its magnitude be a minimum. Hence, the number of unknowns reduces to nine, but the number of equations to satisfy is nine, too, thus making it possible to obtain one or several exact solutions to the problem. In fact, find $\underset{\sim}{a}_0$ from eq. (5.4.6) for $\underset{\sim}{r}_j=\underset{\sim}{r}_2$; then the system of equations is: eq. (5.4.5) for $j=1$; additional equations are eq. (5.4.9a) - next derived- for $j=0,1$, eq. (5.4.10) for $j=1,2$ and the two eqs. (5.4.11). Eqs. (5.4.5) (5.4.8) and (5.4.9a) are identically satisfied for $j=2$ if $\underset{\sim}{a}_0$ is obtained from eq. (5.4.6). Eq. (5.4.9a) is an alternate form of the perpendicularity condition between $\underset{\sim}{b}_j$ and $\underset{\sim}{v}_j$ -eq. (5.4.9)-, which can be stated as the perpendicularity between $\underset{\sim}{b}_j$ and $\underset{\sim}{v}_0$, i.e.

$$\underset{\sim}{v}_j^T\left((\underset{\sim}{I}-\underset{\sim}{Q}_j)\underset{\sim}{a}_0+\underset{\sim}{b}_0-\underset{\sim}{r}_j\right)=0,\ j=0,1,\ldots,n \qquad (5.4.9a)$$

<u>Exercise 5.4.3</u> Show that, if $\underset{\sim}{a}_0$ is chosen so as to satisfy eq. (5.4.6), with $\underset{\sim}{r}_j=\underset{\sim}{r}_2$, then eqs. (5.4.5), (5.4.8) and (5.4.9a) are identically satisfied for $j=2$.

Using the latter approach to the synthesis problem under study would yield two of the specified configurations of the body corresponding to two conjugate configurations of the linkage, as can be readily proved. Regarding this approach, a word of caution is in order, however: As is proposed to show in Exercise 5.4.2, it is possible that situations arise where it is not possible to find $\underset{\sim}{a}_0$ from eq. (5.4.6)

Example 5.4.1 Synthesis an R-R dyad to conduct a rigid body -to which axes $X_iY_iZ_i$ (i=0,1,2) are attached- throught the three positions shown in Fig 5.4.3

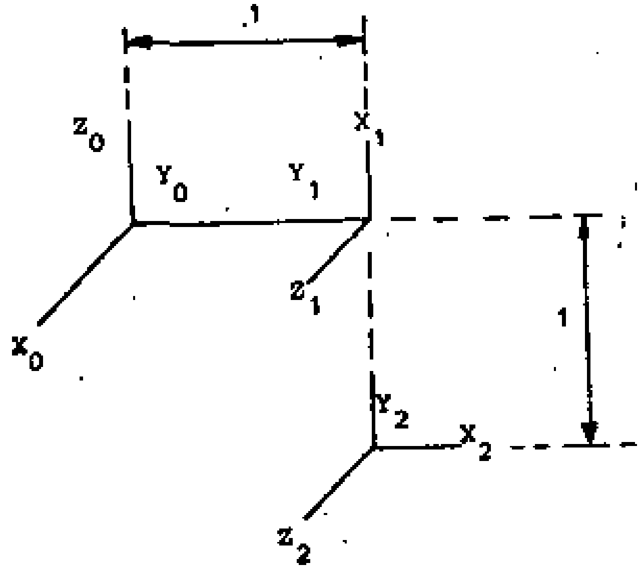

Fig 5.4.3 Rigid-body guidance throught three configurations.

Solution:

In what follows all vectors and matrices are referred to $X_iY_iZ_i$ axes. Matrices $\underline{Q}_1$ and $\underline{Q}_2$ rotating axes from the reference configuration 0 to configurations 1 and 2 are obtained using Definition 1.2.1, as is shown in Section 2.3. Hence,

$$\underline{Q}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \underline{Q}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (5.4.12)$$

Then $\underline{a}_0$ is determined from the following equation, so that its lenght is a minimum

$$(\underline{I}-\underline{Q}_2)\underline{a}_0=\underline{r}_2 \tag{5.4.13}$$

As shown in Section 2.6, $\underline{I}-\underline{Q}_2$ is singular of rank 2; hence eq. (5.4.13) contains exactly 2 linearly independent equations, these being

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{01} \\ a_{02} \\ a_{03} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

whose minimum norm solution is readily obtained via the Moore-Penrose pseudo-inverse (See Section 1.12). The computed solution is

$$\underline{a}_0=\left(-\frac{1}{3},\frac{2}{3},-\frac{1}{3}\right)^T$$

Let the unknown vectors $\underline{b}_0$, $\underline{u}$ and $\underline{v}_0$ have the following components:

$$\underline{b}_0=\begin{pmatrix} b_{01} \\ b_{02} \\ b_{03} \end{pmatrix}, \quad \underline{u}=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \underline{v}_0=\begin{pmatrix} v_{01} \\ v_{02} \\ v_{03} \end{pmatrix} \tag{5.4.16}$$

Eq. (5.4.5) for j=1 takes on the form

$$\frac{2}{3}\,b_{02} - \frac{4}{3}\,b_{03} = -\frac{5}{9} \tag{5.4.17a}$$

Vectors $\underline{b}_1$ and $\underline{b}_2$ are computed from eq. (5.4.3), from which it can readily be checked that $\underline{b}_2=\underline{b}_0$, thereby satisfying eq. (5.4.5) for j=2 identically. Eq. (5.4.8) for j=0,1 then, takes on the forms

$$u_1b_{01}+u_2b_{02}+u_3b_{03}=0 \tag{5.4.17b}$$

$$u_1b_{01}+u_2\left(\frac{1}{3}+b_{02}\right)+u_3\left(-\frac{2}{3}+b_{03}\right)=0 \tag{5.4.17c}$$

Eq. (5.4.9a), for j=0,1 leads to

$$u_{01}b_{01}+v_{02}b_{02}+v_{03}b_{03}=0 \tag{5.4.17d}$$

$$v_{01}b_{01}+v_{02}(\frac{1}{3}+b_{02})+v_{03}(-\frac{2}{3}+b_{03})=0 \tag{5.4.17e}$$

Eq. (5.4.10), for j=1,2 yields

$$u_1v_{03}-u_2v_{02}+u_3v_{01}=u_1v_{01}+u_2v_{02}+u_3v_{03} \tag{5.4.17f}$$

$$u_1v_{03}+u_2v_{01}+u_3v_{02}=u_1v_{01}+u_2v_{02}+u_3v_{03} \tag{5.4.17g}$$

where $\underline{v}_j$ was computed performing the product $\underset{\sim}{Q}_j\underline{v}_0$. Finally, eqs. (5.4.11) take on the form

$$u_1^2+u_2^2+u_3^2=1 \tag{5.4.17h}$$

$$v_{01}^2+v_{02}^2+v_{03}^2=1 \tag{5.4.17i}$$

The solutions to eqs. (5.4.17) provide the different possible dyads that guide the rigid body through the different configurations shown in Fig 5.4.2. This system of nine equations in nine unknowns, though nonlinear, can be solved without the aid of a digital computer, as is shown next. To refer to the foregoing equations, only the letter attached to each is mentioned, for shortness.

Subtracting (b) from (c),

$$u_2-2u_3=0 \tag{a'}$$

Subtracting (f) from (g),

$$u_2(v_{01}+v_{02})+v_3(v_{02}-v_{01})=0 \tag{b'}$$

Substituting (a') into (b'),

$$u_3(v_{01}+3v_{02})=0 \tag{c'}$$

First solution:

To satisfy eq. (c') there are two possibilities. One is

$$u_3=0 \quad \text{(d')}$$

(d') in (a') leads to

$$u_2=0 \quad \text{(e')}$$

(d') and (c') in h yields

$$u_1=\pm 1 \quad \text{(f')}$$

In what follows, this solution is developed taking only the plus sign of eq. (f'). Substitution of (f') into (f) yields

$$v_{03}=v_{01} \quad \text{(g')}$$

Subtracting (d) from (e),

$$v_{02}-2v_{03}=0 \quad \text{(h')}$$

Solving for v_{01} and v_{02} in (g') and (h'), in terms of v_{03} and substituting the resulting expressions into (i), one obtains

$$6v_{03}^2=1$$

from which

$$v_{03}=\frac{\sqrt{6}}{6} \quad \text{(i')}$$

and

$$v_{01}=\frac{\sqrt{6}}{6} \quad \text{(j')}$$

$$v_{02}=\frac{2\sqrt{6}}{6} \quad \text{(k')}$$

Substituting (d'),(e') and (f') into eq. (b) leads to

$$b_{01}=0 \quad \text{(l')}$$

Substituting (i'),(j') and (k') into eq. (d) leads to

$$2b_{02}+b_{03}=0 \quad \text{(m')}$$

Solving for b_{02} and b_{03} from eqs. (a) and (m'),

$$b_{02}=-\frac{1}{6} \qquad (n')$$

$$v_{03}=\frac{1}{3} \qquad (p')$$

Hence, the first solution to this problem is

$$\underset{\sim}{b}_0=\begin{pmatrix}0\\-1/6\\1/3\end{pmatrix},\ \underset{\sim}{u}=\begin{pmatrix}1\\0\\0\end{pmatrix},\ \underset{\sim}{v}_0=\begin{pmatrix}\sqrt{6}/6\\\sqrt{6}/3\\\sqrt{6}/6\end{pmatrix} \qquad (5.4.'9)$$

<u>Second solution:</u>

If now u_3 is assumed to have a nonzero value, it is possible to divide eq. (c') by it and obtain

$$v_{01}+3v_{02}=0 \qquad (a")$$

From (h') and (a") it is possible to solve for v_{01} and v_{03} in terms of v_{02}.

$$v_{01}=-3v_{02} \qquad (b")$$

$$v_{03}=\frac{1}{2}v_{02} \qquad (c")$$

Substitution of (b") and (c") into (i) yields

$$v_{02}=\frac{2\sqrt{41}}{41} \qquad (d")$$

Hence

$$v_{01}=-\frac{6\sqrt{41}}{41} \qquad (e")$$

$$v_{03}=\frac{\sqrt{41}}{41} \qquad (f")$$

Substituting (d"),(e") and (f") into eqs. (f) and (g) leads to

$$u_1-2u_2-6u_3=-6u_1+2u_2+u_3 \qquad (g")$$

$$u_1-6u_2+2u_3=-6u_1+2u_2+u_3 \qquad (h")$$

from which one can solve for u_1 and u_2 in terms of u_3, thus obtaining

$$u_1 = \frac{15}{7} u_3 \qquad (i")$$

$$u_2 = 2u_3 \qquad (j")$$

Substitution of (i") and (j") into eq. (h) leads to

$$u_3 = \frac{7\sqrt{470}}{470} \qquad (k")$$

Hence,

$$u_1 = \frac{15\sqrt{470}}{470} \qquad (l")$$

$$u_2 = \frac{14\sqrt{470}}{470} \qquad (m")$$

Substituting (k"), (l") and (m") into eq. (b) one obtains

$$15b_{01}+14b_{02}+7b_{03}=0 \qquad (n")$$

Performing a similar substitution into eq. (d), one obtains

$$-6b_{01}+2b_{02}+b_{03}=0 \qquad (p")$$

Solving eqs. (n") and (p") for b_{01} and b_{02} in terms of b_{03},

$$b_{01}=0 \qquad (q")$$

$$b_{02}=\frac{1}{2} b_{03} \qquad (r")$$

Substitution of (r") into eq. (a) leads finally to

$$b_{02}=\frac{5}{18} \qquad (s")$$

$$b_{03}=\frac{5}{9} \qquad (t")$$

Thus, the second solution is

$$\underset{\sim}{b}_0=\begin{pmatrix}0\\5/18\\5/9\end{pmatrix},\quad \underset{\sim}{u}=\begin{pmatrix}7\\15\\14\end{pmatrix}\alpha,\quad \underset{\sim}{v}_0=\begin{pmatrix}-6\\2\\1\end{pmatrix}\beta \qquad (5.4.19)$$

where

$$\alpha=\frac{\sqrt{470}}{470},\quad \beta=\frac{\sqrt{41}}{41}$$

Both dyads defined by $\underset{\sim}{a}_0$ as given by (5.4.13) and both expressions for $\underset{\sim}{b}_0$, (5.4.16) and (5.4.17), are optimal in the sense that $\underset{\sim}{a}_0$ is of minimum length. However, with respect to $\underset{\sim}{b}_0$, the first solution is the best.

In conclusion, regarding R-R dyad synthesis, it has been shown how to obtain an optimal dyad for a problem usually known to have only two real solutions, thus not allowing for any optimisation.

5.5 LINKAGE SYNTHESIS FOR PATH GENERATION. This problem can be stated as: "Synthesise a linkage of a given topology such that a point R of one of its links attains successively the specified positions $R_0, R_1, \ldots, R_n$". This problem can be regarded as one of rigid-body guidance with incompletely specified displacements (5.22), for the motion of one single point of a rigid body does not define the body motion (its orientation thus remains undefined). To formulate the synthesis equations, use is made of eq. (5.3.12), written in the form

$$\mathbf{r}_j = \mathbf{r}_0 + w_j \mathbf{f} + u_j \mathbf{Q}_{2j} \mathbf{m} + (\mathbf{Q}_{3j} - \mathbf{I})(\mathbf{r}_0 - \mathbf{q}) + (\mathbf{Q}_{2j} - \mathbf{I})(\mathbf{q} - \mathbf{g}), \quad j = 1, 2, \ldots, n \tag{5.5.1}$$

where $\mathbf{r}_0, \mathbf{r}_j$ $(j=1,\ldots,n)$ are the position vectors of points R_0, R_j $(j=1,\ldots,n)$, respectively. Other variables indexed with j are parameters of the involved screws -defined in Section 5.3- relating the j^{th} configuration of the rigid body under study to its reference configuration, indexed with 0. Notice that vectors $\mathbf{f}, \mathbf{m}, \mathbf{q}$ and $\mathbf{g}$ are not indexed, this being due to the fact that they remain the same throughout the rigid-body motion, except when dealing with spherical pairs, as is shown later.

Differently from what occurs with the rigid-body guidance problem, in this case the dyads, constituting the linkage intended to design, are not synthesised separately; the linkage is synthesised as a whole, instead. In fact, since in this case the rigid-body rotations involved are not specified, the composed-screw-axis directions, $\mathbf{s}_j$, as well as the corresponding rotations, θ_j, are unknown. These are not independent, once the directions of the screw-axes $\mathbf{f}$ and $\mathbf{m}$ are specified, for then θ_j can be obtained from eq. (5.3.9), in terms of $\mathbf{s}_j$, $\mathbf{f}$ and $\mathbf{m}$. Hence, only the n $\mathbf{s}_j$ vectors remain as independent unknowns. These vectors, however, remain the same, no matter what dyad is being synthesised, i.e. although in general the variables

$\underline{w}_j, \underline{f}, \underline{u}_j, \underline{m}, \underline{q}$, and $\underline{g}$ change from one dyad to the other, the $\underline{s}_j$ vectors are common. The synthesis procedure is illustrated with one example.

Example 5.5.1 Synthesis of an RRSS path generator linkage.

With no loss of generality, it can be assumed that $\underline{r}_0'=\underline{0}$ in eqs. (5.5.1). Then, the R-R dyad synthesis is first formulated, although, as mentioned previously, this synthesis cannot be performed independently from the one of the S-S dyad. This is due to the fact that the synthesis equations for both dyads are coupled via the $\underline{s}_j$ vectors. In performing the R-R dyad synthesis, the different variables have the following meanings (See Fig 5.3.1 and its description): u_j and w_j are the displacements associated with the fixed and the moving R pairs of the dyad, respectively. Hence, they vanish.

$\underline{f}$ and $\underline{m}$ are unit vectors pointing in the direction of the axes of the fixed and the moving R pairs of the dyad. $\underline{s}_j$ is the unit vector pointing in the direction of the axis of the screw carrying the body from its reference- to its $j^{\underline{th}}$ configuration.

$\underline{g}$ and $\underline{q}$ are position vectors of points located along the axes of the screws $\underline{F}_j$ and $\underline{M}_j$, respectively. As in Section 5.3, those points will be located at the intersections of the said axes with their common normal.

Under these cirucumstances, eqs. (5.5.1) take on the form

$$\underline{r}_j=-(\underline{Q}_{3j}-\underline{I})\underline{q}+(\underline{Q}_{2j}-\underline{I})(\underline{q}-\underline{g})=\underline{r}_j(\underline{f},\underline{m},\underline{s}_j,\underline{q},\underline{g}) \qquad (5.5.2)$$

where, clearly

$$\underline{Q}_{2j}=\underline{Q}_{2j}(\underline{f},\gamma_j),\gamma_j=\gamma_j(\underline{f},\underline{m},\underline{s}_j)$$

$$\underline{Q}_{3j}=\underline{Q}_{3j}(\underline{s}_j,\theta_j),\theta_j=\theta_j(\underline{f},\underline{m},\underline{s}_j)$$

The normality of vectors $\underline{f}$, $\underline{m}$ and $\underline{s}_j(j=1,2,\ldots,n)$ is expressed as

$$||\underline{f}||=1,\ ||\underline{m}||=1,\ ||\underline{s}_j||=1,\ j=1,\ldots,n \qquad (5.5.3)$$

whereas the specific locations of points G and Q along the screw axes are expressed as

$$(\underset{\sim}{q}-\underset{\sim}{g})^T\underset{\sim}{f}=0, \quad (\underset{\sim}{q}-\underset{\sim}{g})^T\underset{\sim}{m}=0 \qquad (5.5.4)$$

Thus, the synthesis of this dyad leads to a system of 3n+12 unknowns: $\underset{\sim}{f}$, $\underset{\sim}{m}$, $\underset{\sim}{s}_j$ (3n unknown scalars), $\underset{\sim}{q}$ and $\underset{\sim}{g}$, in the 4n+4 equations: 3n equations (5.5.2) (one for each direction of the physical space)+ n+2 equations (5.5.3) + 2 equations (5.5.4).

The synthesis of the S-S dyad is formulated in the following way: Referring to Fig 5.5.1, let S_{12} and S_{23} be the spherical pairs of the dyad under consideration. Let B_0 and B_j be the reference- and the $j^{\underline{th}}$ configurations of body B, and B_j' an intermediate configuration. Now, the guidance of B from B_0 through B_j is performed in two stages: first, B_0 is rotated about point Q through a rotation of axis $\underset{\sim}{m}_j$ and angle α_j (it will be seen next that neither $\underset{\sim}{m}_j$ nor α_j appear explicitly in the resulting synthesis equations for this dyad). Let B_j' be the configuration of B after this rotation. Next, fixing B_j to the dyad, rotate the whole assembly as one single rigid body about point G through an axis $\underset{\sim}{f}_j$ and an angle γ_j, B_j being the final configuration of the rigid body.

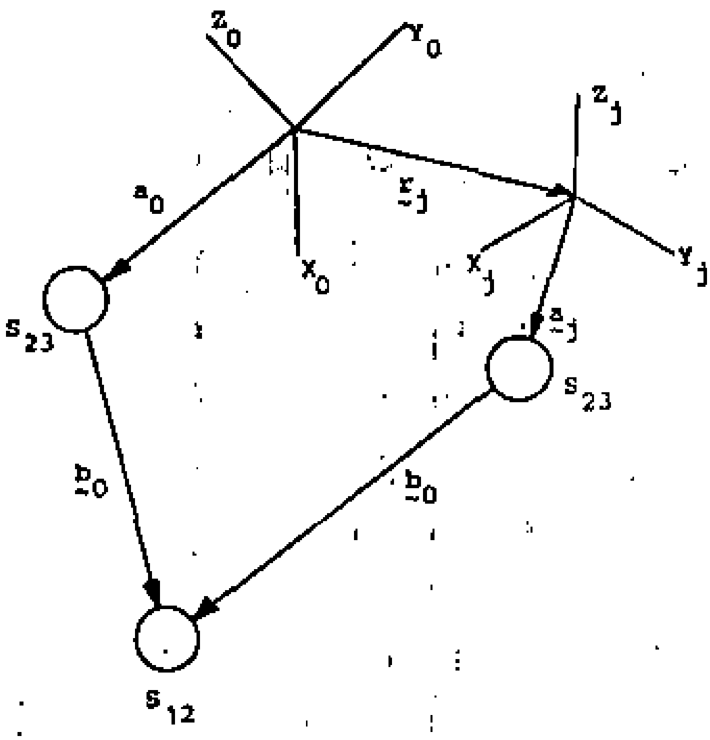

Fig 5.5.1 Displacement of a rigid by means of an S-S dyad

Since in this case also $\mathbf{u}_j=\mathbf{w}_j=0$, and $\mathbf{r}_0$ can be equated to 0, eq. (5.5.1) takes on the form

$$\mathbf{r}_j=-(\mathbf{Q}_{3j}-\mathbf{I})\mathbf{g}+(\mathbf{Q}_{2j}-\mathbf{I})(\mathbf{q}-\mathbf{g})=\mathbf{r}_j(\mathbf{f}_j,\mathbf{m}_j,\mathbf{s}_j,\mathbf{q},\mathbf{g}) \qquad (5.5.5)$$

where

$$\mathbf{Q}_{2j}=\mathbf{Q}_{2j}(\mathbf{f}_j,\gamma_j),\gamma_j=\gamma_j(\mathbf{f}_j,\mathbf{m}_j,\mathbf{s}_j)$$

$$\mathbf{Q}_{3j}=\mathbf{Q}_{3j}(\mathbf{s}_j,\theta_j),\theta_j=\theta_j(\mathbf{f}_j,\mathbf{m}_j,\mathbf{s}_j)$$

Clearly, vectors $\mathbf{f}_j$ and $\mathbf{m}_j$ are different from $\mathbf{f}$ and $\mathbf{m}$ appearing in eqs. (5.5.2)-(5.5.4), which is made clear because the latter vectors are not indexed. Vectors $\mathbf{s}_j$, appearing in eqs. (5.5.2)-(5.5.3), however, are identical to those appearing in eqs. (5.5.5); hence they do not add extra unknowns to the synthesis problem of the overall linkage.

Other synthesis equations are the following:

The normality of vectors $\underset{\sim}{f}_j$ and $\underset{\sim}{m}_j$ is expressed as

$$||\underset{\sim}{f}_j||=1,\ ||\underset{\sim}{m}_j||=1,\ j=1,2,\ldots,n \qquad (5.5.6)$$

Since the S-S dyad can rotate about axis GQ without changing either configuration B_j' or B_j, it can be assumed that the axes of rotation, given by unit vectors $\underset{\sim}{f}_j$ and $\underset{\sim}{m}_j$, are both perpendicular to axis GQ, i.e.

$$(\underset{\sim}{q}-\underset{\sim}{g})^T\underset{\sim}{f}_j=0,\ j=1,2,\ldots,n \qquad (5.5.7)$$

$$(\underset{\sim}{q}-\underset{\sim}{g})^T\underset{\sim}{m}_j=0,\ j=1,2,\ldots,n \qquad (5.5.8)$$

Thus, the synthesis of the S-S dyad adds the following unknowns to the overall linkage synthesis problem: $\underset{\sim}{f}_j$, $\underset{\sim}{m}_j$ $(j=1,\ldots,n)$, $\underset{\sim}{q}$ and $\underset{\sim}{g}$, i.e. 6n+6 additional unknowns (Vectors $\underset{\sim}{s}_j$ are not counted as new unknowns, since these were already taken into account when discussing the synthesis of the R-R dyad). The additional equations are: 3n equations (5.5.5) + 2n equations (5.5.6) +n equations (5.5.7)+n equations (5.5.8). Summarizing, then, the RRSS-linkage-synthesis problem for path generation gives rise to system of 11n+4 equations in 9n+18 unknowns. Matching the number of equations to that of unknowns, it is soon realized that n should have the value 7, i.e. it is possible to generate a path passing through 8 specified points (counting the reference- as the 8th position), which coincides with the result obtained by Suh [5.23].

Remarks on the formulation of the RRSS linkage synthesis problem for path generation.

i) In the previous linkage-synthesis problem angles γ and θ appear in eq. (5.5.1) and angles γ_j and θ_j do in eq. (5.5.2). These angles, however, are not counted as additional unknowns, since they can be computed, via eqs. (5.5.4) and (5.5.9), with the values of $\underset{\sim}{f}$, $\underset{\sim}{m}$ (or $\underset{\sim}{f}_j$ and $\underset{\sim}{m}_j$) and $\underset{\sim}{s}_j$.

ii) The problem, as just stated for 8-point synthesis, leads to a system of 81 equations in 81 unknowns which, because of being nonlinear, is diffecult to handle numerically. Suh's formulation (5.23) leads to a system of 39 equations in 39 unknowns, in this instance. Roth and Tsai (5.11) recommend that, for dyads containing revolute, prismatic, helical or cylindrical pairs, one point on the pair axis (G or Q of the previous formulation) be taken as the intersection of the axis with, say, the X-Y plane, thus diminishing the number of unknowns (one component of the $\underline{f}$ and of the $\underline{m}$ vectors is thus zero). This way, the number of equations and of unknowns is reduced to 65 which, anyway, is about twice that of Suh's. Thus, it might seem more advantageous to use the latter approach.

Since Suh's formulation is amply discused in (5.23), it is not presented here. An alternative approach, based on Section 5.4, is presented instead.

<u>An alternative approach to the problem formulation for the RRSS linkage synthesis problem.</u>

Referring to Figs 5.4.1 and 5.4.2, the synthesis equations are next written. To distinguish the position vectors of points A and B, in their reference configurations, belonging to the R-R dyad, from those belonging to the S-S dyad, the latter are starred. The synthesis equations for the R-R dyad are those derived in Section (5.4), i.e.

$$||(\underline{I}-\underline{Q}_j)\underline{a}_0+\underline{b}_0-\underline{r}_j||^2=||\underline{b}_0||^2, j=1,\ldots,n \qquad (5.5.9)$$

where it is convenient to express $\underline{Q}_j$ as appearing in eq. (2.5.4), i.e.

$$\underline{Q}_j=-\underline{P}_j^2\cos\theta_j+\underline{P}_j\sin\theta_j+\underline{R}_j \qquad (5.5.10)$$

Calling $\underset{\sim}{e}_j$ the unit vector parallel to the axis of $\underset{\sim}{Q}_j$ and $(\alpha_j, \beta_j, \gamma_j)^T$ its components (in the $X_0Y_0Z_0$ axes), the matrices appearing at the right-hand side of eq. (5.5.10) are, from eq. (2.5.4a),

$$\underset{\sim}{P}_j = \begin{bmatrix} 0 & -\gamma_j & \beta_j \\ \gamma_j & 0 & -\alpha_j \\ -\beta_j & \alpha_j & 0 \end{bmatrix}, \quad \underset{\sim}{R}_j = \begin{bmatrix} \alpha_j^2 & \alpha_j\beta_j & \alpha_j\gamma_j \\ \alpha_j\beta_j & \beta_j^2 & \beta_j\gamma_j \\ \alpha_j\gamma_j & \beta_j\gamma_j & \gamma_j^2 \end{bmatrix} \qquad (5.5.11)$$

Additional equations for the R-R dyad are:

$$\underset{\sim}{u}^T \underset{\sim}{b}_j = 0, \; j=1,\ldots,n \qquad (5.5.12)$$

$$\underset{\sim}{v}_0^T \underset{\sim}{b}_j = 0, \; j=1,\ldots,n \qquad (5.5.13)$$

$$\underset{\sim}{u}^T \underset{\sim}{v}_j = \underset{\sim}{u}^T \underset{\sim}{v}_0, \; j=1,2,\ldots,n \qquad (5.5.14)$$

$$||\underset{\sim}{u}||=1, \; ||\underset{\sim}{v}_0||=1 \qquad (5.5.15)$$

$$||\underset{\sim}{e}_j||=1, \; j=1,2,\ldots,n \qquad (5.5.16)$$

The synthesis equations for the S-S dyad are

$$||(\underset{\sim}{I}-\underset{\sim}{Q}\,)\underset{\sim}{a}_0^* + \underset{\sim}{b}_0^* - \underset{\sim}{r}_j||^2 = ||\underset{\sim}{b}_0^*||^2, \; j=1,\ldots,n \qquad (5.5.17)$$

Summarizing, the problem of an RRSS linkage synthesis for path generation leads to a system of 6n + 4 equations in 4n + 18 unknowns. The equations are: n equations (5.5.9)+(n+1) equations (5.5.12)+(n+1) equations (5.5.13)+n equations (5.5.14)+2 equations (5.5.15)+n equations (5.5.516)+n equations (5.5.17). The unknowns are:

$$\underset{\sim}{a}_0, \underset{\sim}{b}_0, \underset{\sim}{e}_j\,(j=1,\ldots,n), \theta_j, \underset{\sim}{u}, \underset{\sim}{v}_0, \underset{\sim}{a}_0^* \text{ and } \underset{\sim}{b}_0^*.$$

If the number of equations is to be matched with that of unknowns, it is readily concluded that the linkage thus synthesised can trace a spatial path passing through 8 prescribed points, which result coincides with that of Suh's Via the latter approach, however, the number of equations and unknowns (for 8-point synthesis) is 46, a number still greater than that of Suh's, yet smaller than that obtained using Roth and Tsai's equations.

5.6. EPILOGUE. The subject of linkage synthesis is far more extense than has been presented here. Topics that were not covered are, amongst others, linkage synthesis for rigid-body guidance and for path generation with infinitesimally-separated positions, i.e. to meet prescribed conditions on velocities and higher derivatives. The subject is treated in [5.9], [5.10] and [5.11], but was not included here due to space limitations.

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ANALISIS, SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

NOTAS COMPLEMENTARIAS

DR. JORGE ANGELES ALVAREZ

ACOSTO, 1980

Terminología y notación

$\underset{\sim}{x}, \underset{\sim}{v}$, etc. : vector de dimensión n.

$\underset{\sim}{A}, \underset{\sim}{B}$, etc. : matriz de $m \times n$ (m → renglones, n → columnas)

x_i, v_i, etc. : i^{a} componente de $\underset{\sim}{x}, \underset{\sim}{v}$, etc.

a_{ij}, b_{ij}, etc. : elemento en la j^{a} columna y i^{o} renglón de $\underset{\sim}{A}, \underset{\sim}{B}$, etc.

j^{a} columna ↓

$$\underset{\sim}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \leftarrow i^{o} \text{ renglón}$$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$b_i = a_{ij} x_j \equiv a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n$$

$$A\,\underset{\sim}{B} = \underset{\sim}{C}$$

$$c_{ij} = a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

f: escalar, $\underset{\sim}{x}$: vector de dim n

$$f = f(\underset{\sim}{x}) = f(x_1, x_2, \ldots, x_n)$$

$$\frac{\partial f}{\partial \underset{\sim}{x}} \equiv \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right]^T$$

$$\left(\frac{\partial f}{\partial \underset{\sim}{x}}\right)_i \equiv \frac{\partial f}{\partial x_i}$$

$\underset{\sim}{f}$: vector de dim m, $\underset{\sim}{x}$: vector de dim n

$$\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{x})$$

$$f_1 = f_1(x_1, x_2, \ldots, x_n)$$

$$f_2 = f_2(x_1, x_2, \ldots, x_n)$$

$$\vdots$$

$$f_m = f_m(x_1, x_2, \ldots, x_n)$$

$$\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \ldots & \partial f_1/\partial x_n \\ \vdots & & \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 \ldots & \partial f_m/\partial x_n \end{bmatrix}$$

$$\underset{\sim}{g} = \underset{\sim}{g}(\underset{\sim}{x}) = \frac{\partial f}{\partial \underset{\sim}{x}} \quad \text{(gradiente)}$$

$$\Rightarrow \frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{x}} = \frac{\partial^2 f}{\partial \underset{\sim}{x}^2} = \underset{\sim}{H} \quad \text{(matriz Hessiana)}$$

$$\frac{\partial^2 f}{\partial \underset{\sim}{x}^2} = \begin{bmatrix} \partial^2 f_1/\partial x_1^2 & \cdots & \partial^2 f_1/\partial x_1 \partial x_n \\ \vdots & & \vdots \\ \partial^2 f_m/\partial x_1 \partial x_n & \cdots & \partial^2 f_m/\partial x_n^2 \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$\left(\frac{\partial^2 f}{\partial \underset{\sim}{x}^2}\right)^T = \frac{\partial^2 f}{\partial \underset{\sim}{x}^2}$$

$f = bx \Rightarrow \frac{\partial f}{\partial x} = b$

$f = ax^2 \Rightarrow \frac{\partial f}{\partial x} = 2ax$

si $f = \underset{\sim}{b}^T \underset{\sim}{x} \Rightarrow \frac{\partial f}{\partial \underset{\sim}{x}} = \underset{\sim}{b}$

Si $f = \underset{\sim}{x}^T \underset{\sim}{A} \underset{\sim}{x} \Rightarrow \frac{\partial f}{\partial \underset{\sim}{x}} = 2 \underset{\sim}{A} \underset{\sim}{x}$

$\frac{\partial^2 f}{\partial \underset{\sim}{x}^2} = 2 \underset{\sim}{A}$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$m \times n \quad n \quad m$$

i) $m = n$ ya se estudió

ii) $m > n$

iii) $m < n$

$m = 2$, $n = 1$; p. ej.

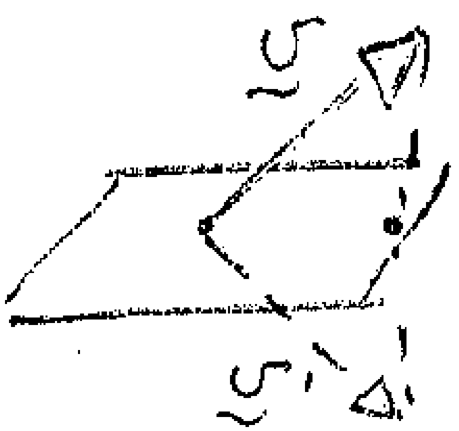
$$\begin{cases} x_1 = 3 \\ x_1 = 5 \end{cases}$$

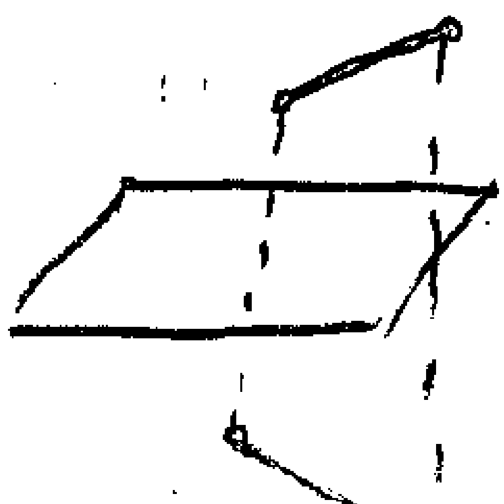
$$\underset{\sim}{e} = \underset{\sim}{A}\,\underset{\sim}{x} - \underset{\sim}{b}$$

$$\min_{\underset{\sim}{x}} \|\underset{\sim}{e}\|^2$$

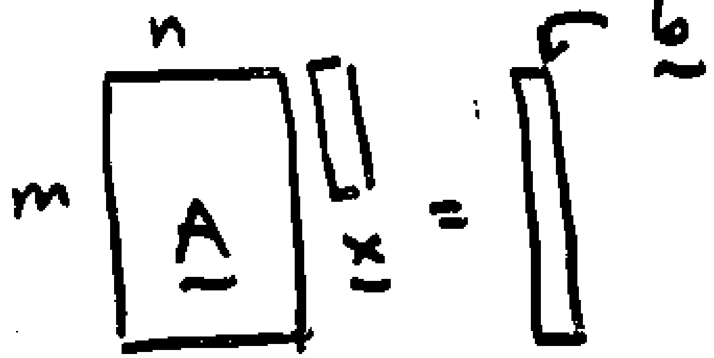
Reflexiones de Householder

$$\underset{\sim}{v}' = \underset{\sim}{H}\,\underset{\sim}{v}$$



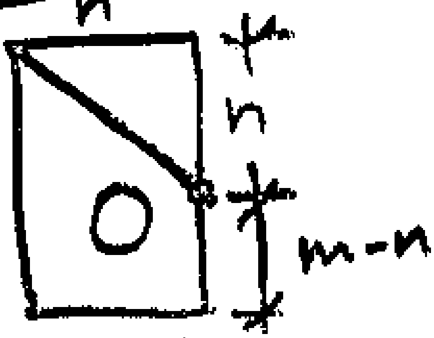


$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$



$$\underbrace{\underset{\sim}{H}_n \underset{\sim}{H}_{n-1} \cdots \underset{\sim}{H}_1 \underset{\sim}{A}}\,\underset{\sim}{x} = \underbrace{\underset{\sim}{H}_n \underset{\sim}{H}_{n-1} \cdots \underset{\sim}{H}_1 \underset{\sim}{b}}_{\underset{\sim}{b}^*}$$

$\underset{\sim}{A}^* =$



$$\underset{\sim}{A}^* = \begin{bmatrix} \underset{\sim}{A}^*_1 \\ \hline \underset{\sim}{A}^*_2 \end{bmatrix} \begin{matrix} n \\ m-n \end{matrix}, \quad \underset{\sim}{b}^* = \begin{bmatrix} \underset{\sim}{b}^*_1 \\ \hline \underset{\sim}{b}^*_2 \end{bmatrix} \begin{matrix} n \\ m-n \end{matrix}$$

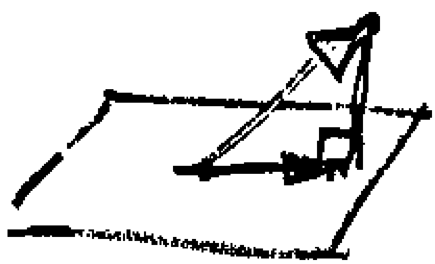
$$\underset{\sim}{A}^*_2 = \underset{\sim}{O}^{\,n}$$

HECOMP
HOLVE

$$\Rightarrow \quad \underset{\sim}{A}^*_1 \underset{\sim}{x} = \underset{\sim}{b}^*_1 \longrightarrow n \times n \Rightarrow \underset{\sim}{x}_0$$

$$\underset{\sim}{O}\,\underset{\sim}{x} = \underset{\sim}{b}^*_2 \neq \underset{\sim}{0}$$

$$\underset{\sim}{b}^*_2 = \underset{\sim}{e}$$



$$\underset{m}{\underline{f}}(\underset{n}{\underline{x}}) = \underset{m}{\underset{\sim}{0}} \qquad \frac{\partial \underline{f}}{\partial \underline{x}} : m \times n$$

ii) $m > n$

$$\text{Min } \|\underset{\sim}{e}\|^2 = \underline{f}^T \underset{\sim}{f} = \phi$$

$$\frac{\partial \phi}{\partial \underset{\sim}{x}} = 0 \Rightarrow \underbrace{\left(\frac{\partial \underline{f}}{\partial \underline{x}}\right)^T}\frac{\partial \phi}{\partial \underset{\sim}{f}} = 2\underbrace{\left(\frac{\partial \underset{\sim}{f}}{\partial \underline{x}}\right)^T}_{n \times m}\underset{\sim}{f} = \underset{\sim}{0}$$

Condición de minimalidad: $\underset{\sim}{J}^T \underset{\sim}{f} = \underset{\sim}{0}$

$\underset{\sim}{x}_0$, $\Delta \underset{\sim}{x}_1$

$$\underbrace{\underset{\sim}{f}(\underset{\sim}{x}_0 + \Delta \underset{\sim}{x}_1)}_{\underset{\sim}{0}} = \underset{\sim}{f}(\underset{\sim}{x}_0) + \underset{\sim}{J}(\underset{\sim}{x}_0)\,\Delta \underset{\sim}{x}_1 + \cancel{\underset{\sim}{e}}\ \underset{\sim}{b}$$

$$\Rightarrow \underbrace{\underset{\sim}{J}(\underset{\sim}{x}_0)}_{m \times n} \Delta \underset{\sim}{x}_1 = -\underset{\sim}{f}(\underset{\sim}{x}_0)$$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \underset{\sim}{x}_0 = (\underset{\sim}{A}^T \underset{\sim}{A})^{-1} \underset{\sim}{A}^T \underset{\sim}{b} = \underset{\sim}{A}^I \underset{\sim}{b}$$

$$\Rightarrow \Delta \underset{\sim}{x}_1 = -(\underset{\sim}{J}^T \underset{\sim}{J})^{-1} \underset{\sim}{J}^T \underset{\sim}{f}$$

$$\Delta \underset{\sim}{x}_k = -\left[\underset{\sim}{J}^T(\underset{\sim}{x}_k)\, \underset{\sim}{J}(\underset{\sim}{x}_k)\right]^{-1} \underset{\sim}{J}^T(\underset{\sim}{x}_k)\, \underset{\sim}{f}(\underset{\sim}{x}_k)$$

$\|\Delta \underset{\sim}{x}_k\| \to \underline{0}$, se alcanzan la converg.

iii) $m < n$ $\|\underset{\sim}{x}\|^2 = \underset{\sim}{x}^T \underset{\sim}{x}$

7

$$\underset{\sim}{A} \underset{\sim}{x} = \underset{\sim}{b} \quad (*)$$

$$\underset{\underset{\sim}{x}}{\text{Min}} \|\underset{\sim}{x}\|^2, \text{ s. a. } (*)$$

$$\phi = \|\underset{\sim}{x}\|^2 + \underset{\sim}{\lambda}^T (\underset{\sim}{A} \underset{\sim}{x} - \underset{\sim}{b})$$

$$\frac{\partial \phi}{\partial \underset{\sim}{x}} = 0 \Rightarrow \frac{\partial f}{\partial \underset{\sim}{x}} = 2 \underset{\sim}{x} + \underset{\sim}{A}^T \underset{\sim}{\lambda} = \underset{\sim}{0}$$

$$\Rightarrow \underset{\sim}{x} = -\frac{1}{2} \underset{\sim}{A}^T \underset{\sim}{\lambda} \quad (**)$$

$(**)$ en $(*) \Rightarrow -\frac{1}{2} \underset{\sim}{A} \underset{\sim}{A}^T \underset{\sim}{\lambda} = \underset{\sim}{b}$

$$\Rightarrow \underset{\sim}{\lambda} = -2 (\underset{\sim}{A} \underset{\sim}{A}^T)^{-1} \underset{\sim}{b}$$

$$\Rightarrow \underset{\sim}{x} = \underset{\sim}{A}^T (\underset{\sim}{A} \underset{\sim}{A}^T)^{-1} \underset{\sim}{b} = \underset{\sim}{A}^{II} \underset{\sim}{b}$$

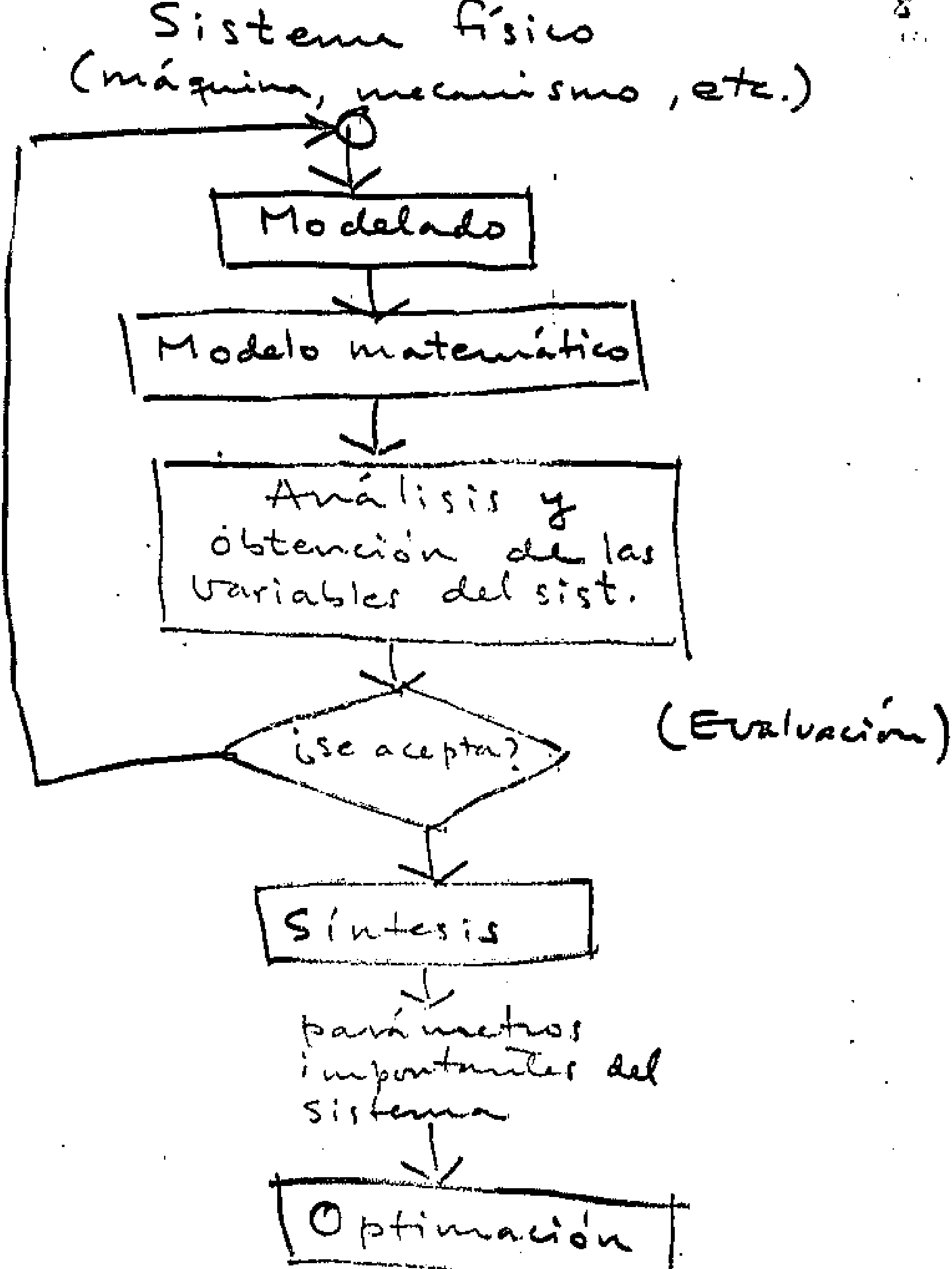
No lineal

Min $\|\underset{\sim}{x}\|^2$ s. a. $\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0}$ (m ecs)

$$\phi = \|\underset{\sim}{x}\|^2 + \underset{\sim}{\lambda}^T \underset{\sim}{f}$$

$$\frac{\partial \phi}{\partial \underset{\sim}{x}} = \underset{\sim}{0} = 2 \underset{\sim}{x} + \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{x}}\right)^T \underset{\sim}{\lambda} = \underset{\sim}{0} \quad \text{(n ecs.)}$$

m+n ecs. no lineales

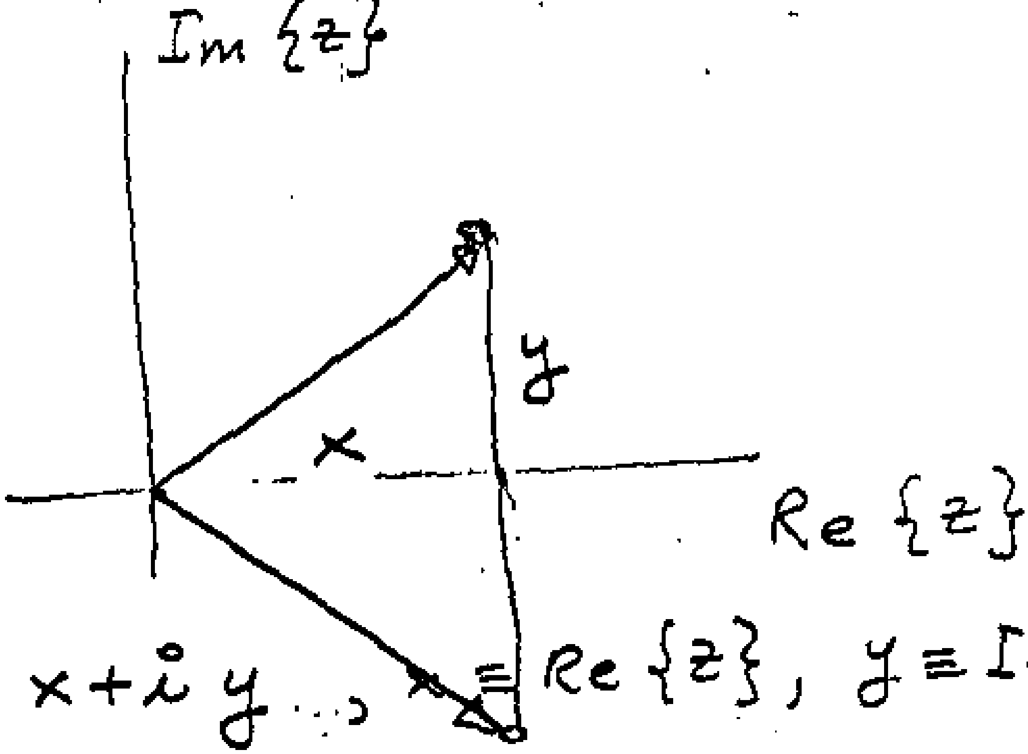

Sistema físico
(máquina, mecanismo, etc.)
Modelado
Modelo matemático
Análisis y obtención de las variables del sist.
¿se acepta?
(Evaluación)
Síntesis
parámetros importantes del sistema
Optimación

Modelos matemáticos

Sistemas de ecuaciones

1. Algebraicos
2. Diferenciales ordinarias
3. Diferenciales parciales

} { lineales / N.L.



$z = x + i y$, $x \equiv Re\{z\}$, $y \equiv Im\{z\}$

$i \equiv \sqrt{-1}$, $i^2 = -1$

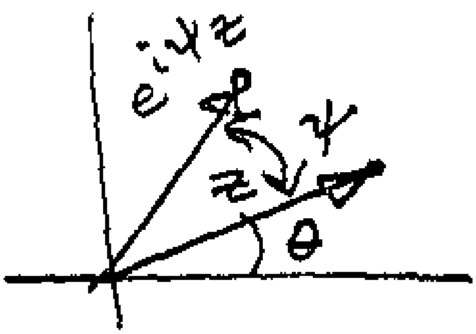
$\|z\| = \sqrt{x^2 + y^2} = \sqrt{z \bar{z}} = r$

$\bar{z} \equiv x - i y$

$\theta = \tan^{-1} \frac{y}{x} = \arg z$

$z \equiv r e^{i\theta}$

$e^{i\psi} z = r e^{i(\theta + \psi)}$



$$\Rightarrow \| a_j - b \| = \| a_0 - b \| \ , \ j = 0, 1, \ldots, n$$

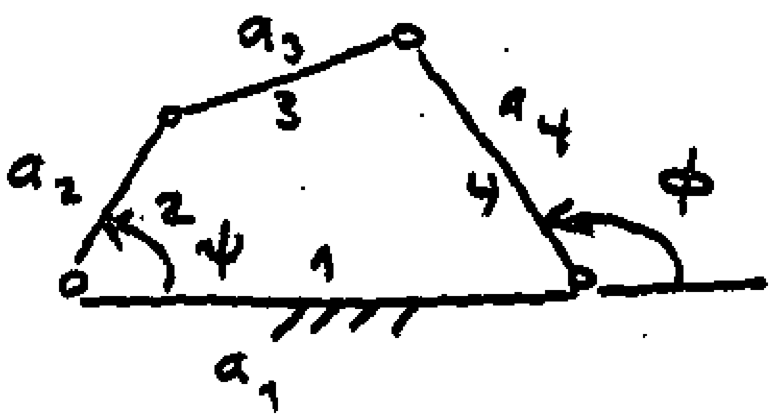
$$a_j = r_j + z_j \ , \ z_j = e^{i\theta_j'} z_0 \ , \ \theta_j' = \theta_j - \theta_0$$

$$a_j = r_j + e^{i\theta_j'} z_0 \ , \ z_0 = a_0 - r_0$$

$$\Rightarrow a_j = r_j + e^{i\theta_j'}(a_0 - r_0)$$

$$a_j - b = r_j + e^{i\theta_j'}(a_0 - r_0) - b$$

$$\Rightarrow \| e^{i\theta_j'}(a_0 - r_0) + r_j - b \|^2 = \| a_0 - b \|^2 , \ j = 1, \ldots, n$$

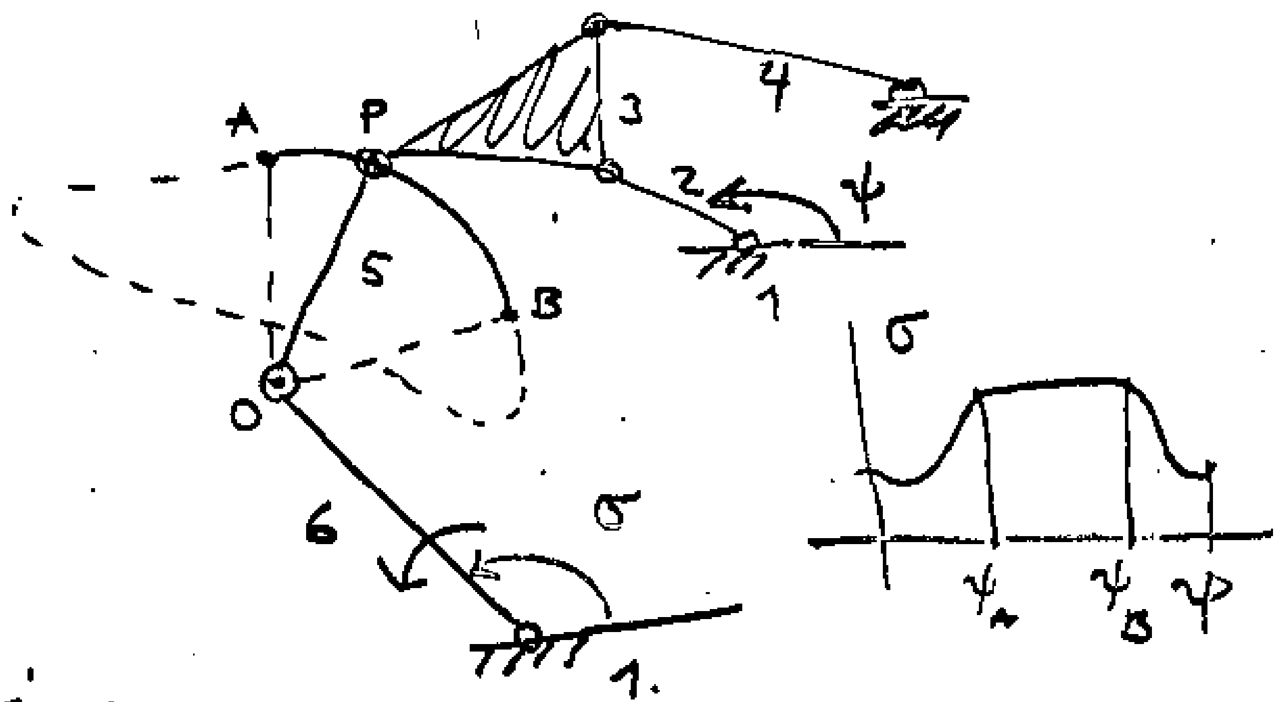


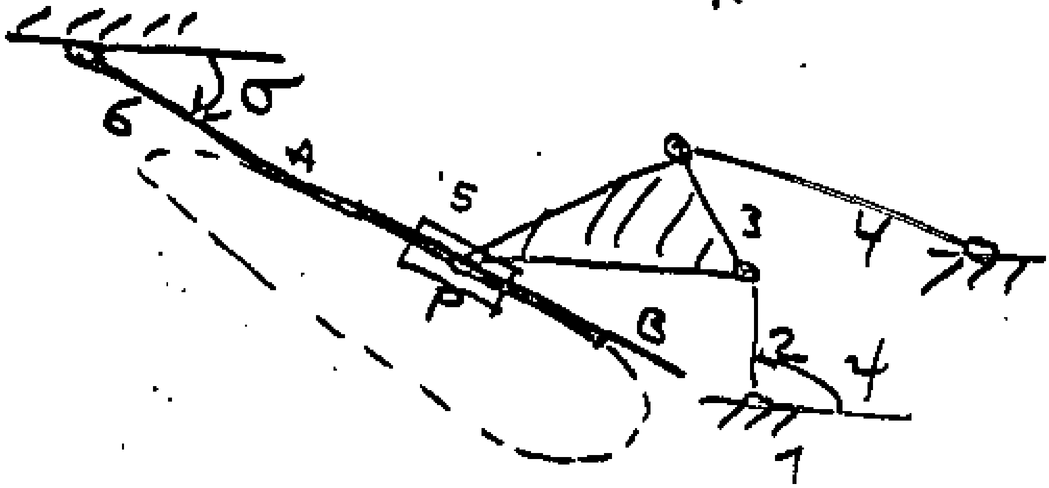
Problema: Determinar los valores de a_1, a_2, a_3 y a_4 tales que produzcan el conjunto de pares $\{\psi_i, \phi_i\}_1^n$

$$k_1 - k_2 \cos\phi_i + k_3 \cos\psi_i + \cos(\phi_i - \psi_i) = 0,$$

$i = 1, \ldots, n$

$$k_1 = \frac{a_3^2 - a_1^2 - a_2^2 - a_4^2}{2 a_2 a_4}, \ k_2 = \frac{a_1}{a_2}, \ k_3 = \frac{a_1}{a_4}$$


A
P
3
4
2
ψ
5
B
1
O
6
σ
1.
σ
ψ_A
ψ_B
ψ


6
σ
A
5
3
4
P
B
2
ψ
1

Lado izq.

n incóg: θ'_j, $j=1,\dots,n$

n ecs.

Lado der.

2+n incóg. x_A^*, y_A^*, θ'_j, $j=1,\dots,n$

n ecs.

$\Rightarrow$ 2. n ecs.

$n+(2+n)-n = 2+n$ incóg.

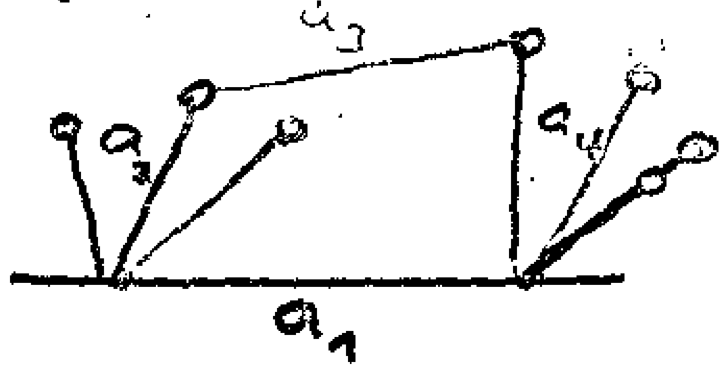
$2n = 2+n$

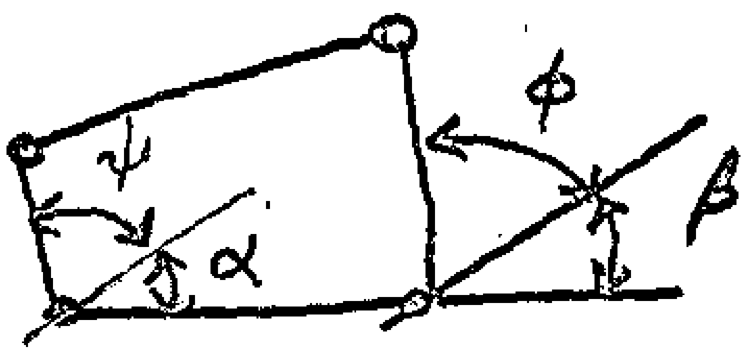
$\Rightarrow$ $\boxed{n=2}$

$$\underset{\sim}{A}\,\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & -\cos\phi_1 & \cos\psi_1 \\ 1 & -\cos\phi_2 & \cos\psi_2 \\ 1 & -\cos\phi_3 & \cos\psi_3 \end{bmatrix}, \quad \underset{\sim}{x} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} -\cos(\phi_1 - \psi_1) \\ -\cos(\phi_2 - \psi_2) \\ -\cos(\phi_3 - \psi_3) \end{bmatrix}$$

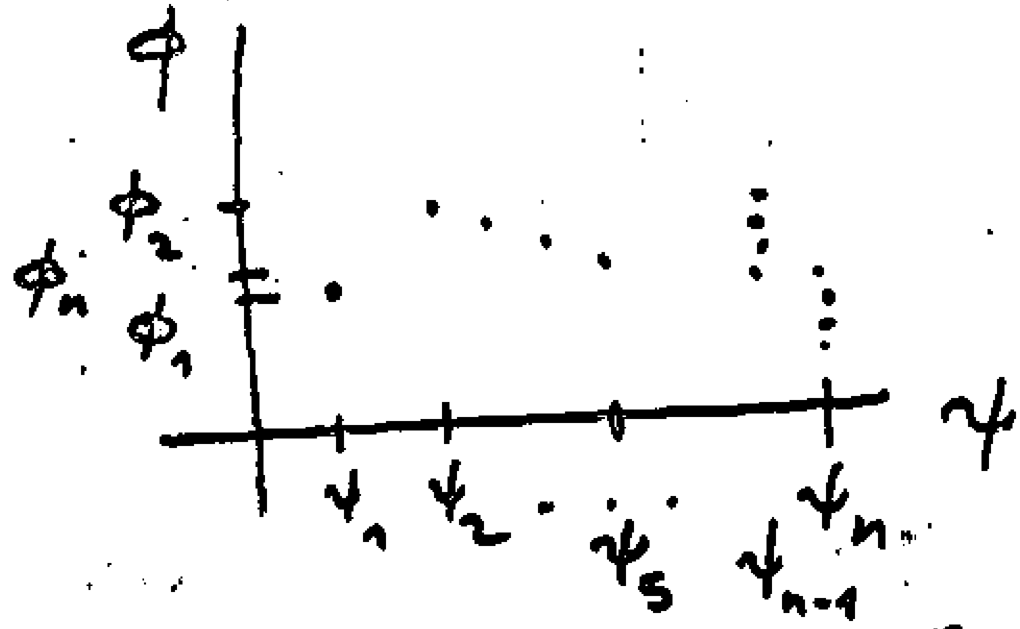




$$k_1 - k_2\cos(\phi+\beta) + k_3\cos(\psi+\alpha) + \cos(\phi-\psi+\beta-\alpha) = 0$$

$$\underset{\sim}{A} = \underbrace{\begin{bmatrix} 1 & -\cos\phi_1 & \cos\psi_1 \\ 1 & -\cos\phi_2 & \cos\psi_2 \\ \vdots & \vdots & \vdots \\ 1 & -\cos\phi_n & \cos\psi_n \end{bmatrix}}_{3} \updownarrow n, \quad \underset{\sim}{b} = \begin{bmatrix} -\cos(\phi_1 - \psi_1) \\ -\cos(\phi_2 - \psi_2) \\ \vdots \\ -\cos(\phi_n - \psi_n) \end{bmatrix}$$

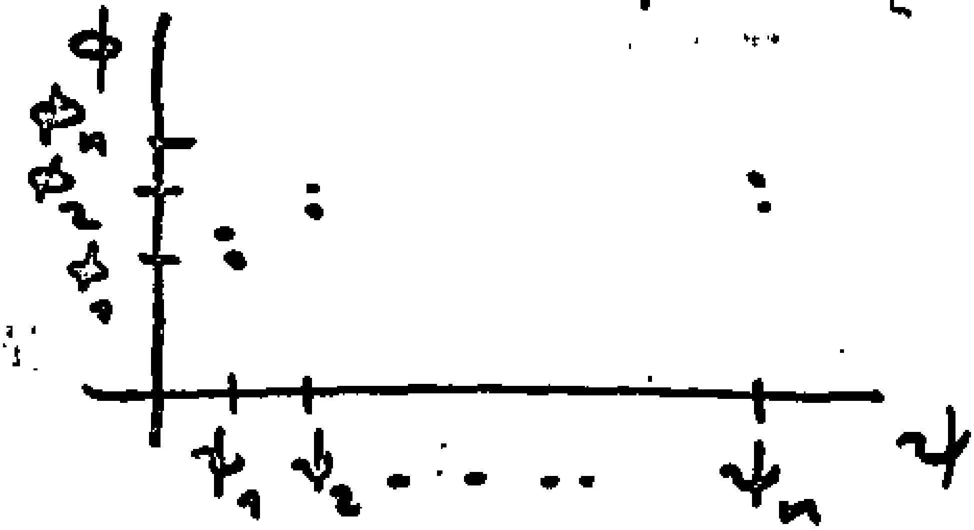
NRDAMG



$e_6, e_7, \ldots, e_n$ | $e^2 = \sum_6^n e_i^2$

$e_6, e_7, \ldots, e_n$ | $e^2 = \sum_6^n e_i^2$

$e_6, e_7, \ldots, e_n$ | $e^2 = \sum_6^n e_i^2$



$$e^2 = \sum_1^n e_i^2$$

$$e = \sum_6^n |e_i| \quad , \quad e = \max_i |e_i|$$

$$\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$$

$$\underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d} - \underset{\sim}{a} \Rightarrow \underset{\sim}{b}^T = \underset{\sim}{c}^T + \underset{\sim}{d}^T - \underset{\sim}{a}^T$$

$$\underbrace{\|\underset{\sim}{b}\|^2}_{a_2^2} = \|\underset{\sim}{c}\|^2 + \underbrace{\|\underset{\sim}{d}\|^2}_{a_1^2} + \underbrace{\|\underset{\sim}{a}\|^2}_{a_3^2} + 2\underset{\sim}{c}^T\underset{\sim}{d} - 2\underset{\sim}{c}^T\underset{\sim}{a} - 2\underset{\sim}{d}^T\underset{\sim}{a}$$

$$\underset{\sim}{c} = \vec{BO} + \vec{OI} + \vec{IC} \longrightarrow [s_1, 0, 0]^T$$

$$\vec{BO} \downarrow [s_4 c\alpha_4, -s_4 s\alpha_4, 0]^T \qquad \vec{OI} \downarrow [0, 0, a_4]^T$$

$$[\underset{\sim}{c}]_1 = [s_4 c\alpha_4 + s_1, -s_4 s\alpha_4, a_4]^T$$

$$[\underset{\sim}{d}]_1 = [0, -a_1 s\phi, a_1 c\phi]^T$$

$$[\underset{\sim}{a}]_0 = [0, -a_3 s\psi, a_3 c\psi]^T$$

$$[\underset{\sim}{S}]_1 = \begin{bmatrix} c\alpha_4 & s\alpha_4 & 0 \\ -s\alpha_4 & c\alpha_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; [\underset{\sim}{a}]_1 = [\underset{\sim}{S}]_1 [\underset{\sim}{a}]_0$$

$$[\underset{\sim}{a}]_1 = a_3[-s\alpha_4 s\psi, -c\alpha_4 s\psi, c\psi]^T$$

$$\phi_j \leftarrow \phi, \quad \psi_j \leftarrow \psi$$

$$\underset{\sim}{c}^T \underset{\sim}{d}_j = a_1(s_4 s\alpha_4 s\phi_j + a_4 c\phi_j)$$

$$\underset{\sim}{c}^T \underset{\sim}{a}_j = a_3(a_4 c\psi_j - s_1 s\alpha_4 s\psi_j)$$

$$\underset{\sim}{d}_j^T \underset{\sim}{a}_j = a_1 a_3 (c\alpha_4 s\psi_j s\phi_j + c\psi_j c\phi_j)$$

$$a_2^2 = s_1^2 + s_4^2 + 2 s_1 s_4 c\alpha_4 + a_4^2 + a_1^2 + a_3^2 +$$
$$+ 2a_1(a_4 c\phi_j + s_4 s\alpha_4 s\phi_j) - 2a_3(a_4 c\psi_j - s_1 s\alpha_4 s\psi_j) - 2a_1 a_3(c\alpha_4 s\psi_j s\phi_j + c\psi_j c\phi_j)$$

$$\phi_j = \phi_0 + \rho_j$$

$$k_1 = \frac{a_4 + s_4 s\alpha_4 \tan\phi_0}{a_3}, \quad k_2 = \frac{s_4 s\alpha_4 - a_4 \tan\phi_0}{a_3}$$

$$k_3 = -\frac{a_4}{a_1 c\phi_0}, \quad k_4 = \frac{s_1 s\alpha_4}{a_1 c\phi_0}, \quad k_5 = \tan\phi_0$$

$$k_6 = \frac{a_1^2 - a_2^2 + a_3^2 + a_4^2 + s_1^2 + s_4^2 + 2 s_1 s_4 c\alpha_4}{2 a_1 a_3 c\phi_0}$$

$$k_1 c\rho_j + k_2 s\rho_j + k_3 c\psi_j + k_4 s\psi_j + k_5(c\psi_j s\rho_j - c\alpha_4 s\psi_j c\rho_j)$$
$$+ k_6 = c\psi_j c\rho_j + c\alpha_4 s\psi_j s\rho_j, \quad j = 1, \ldots, 6$$

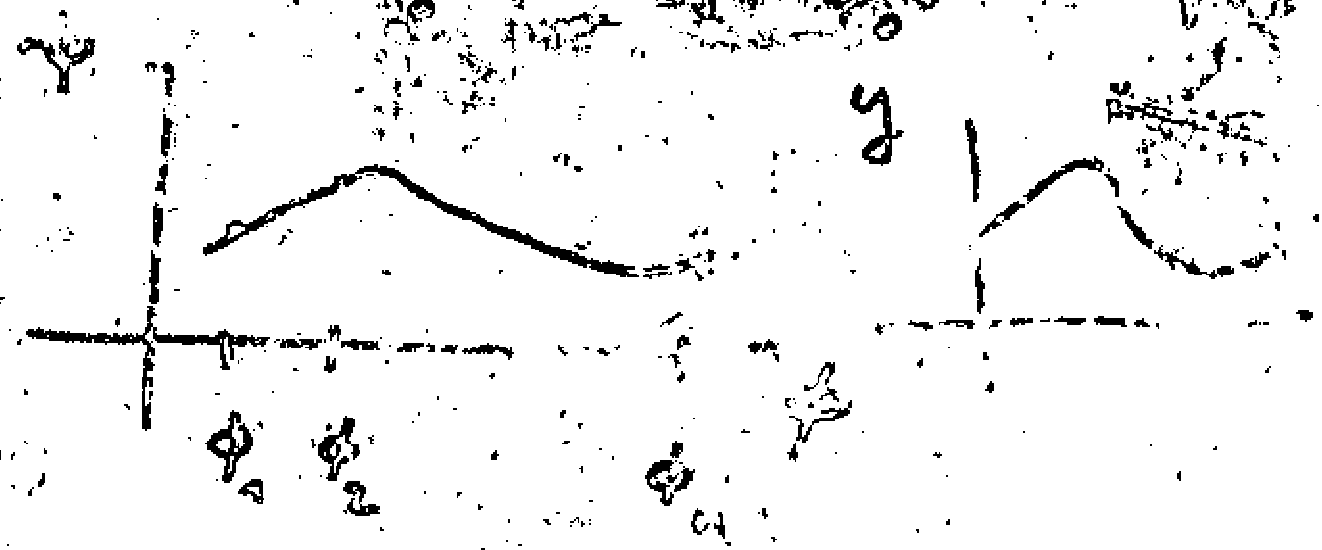
$$\underset{\sim}{A}\, \underset{\sim}{x} = \underset{\sim}{b}$$

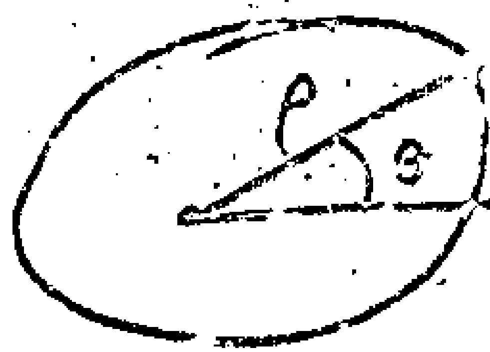
$$\underset{\sim}{A} = \begin{bmatrix} c\rho_1 & s\rho_1 & c\psi_1 & s\psi_1 & (\ )_1 & 1 \\ c\rho_2 & s\rho_2 & c\psi_2 & s\psi_2 & (\ )_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c\rho_6 & s\rho_6 & c\psi_6 & s\psi_6 & (\ )_6 & 1 \end{bmatrix}, \quad \underset{\sim}{b} = \begin{bmatrix} c\psi_1 s\psi_1 + c\alpha_4 s\psi_1 s\rho_1 \\ c\psi_2 s\psi_2 + c\alpha_4 s\psi_2 s\rho_2 \\ \vdots \\ c\psi_6 s\psi_6 + c\alpha_4 s\psi_6 s\rho_6 \end{bmatrix}$$

$$\underset{\sim}{x} = [k_1, \ldots, k_6]^T$$

$y = y(x)$

$y = a\psi + b, \quad x = c\varphi + d$





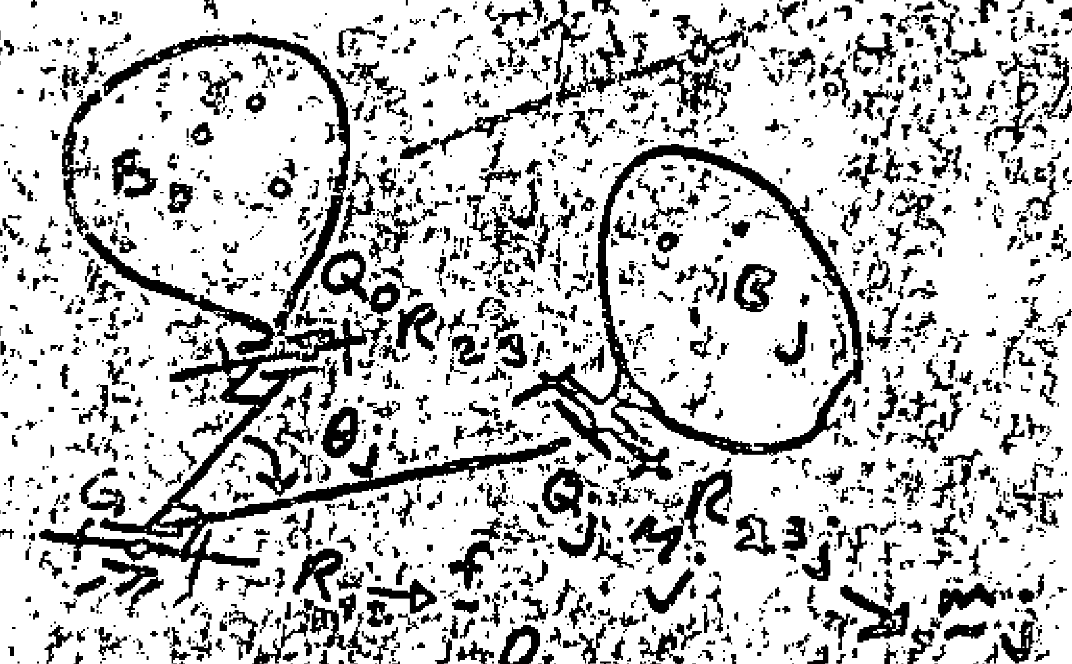
$\rho = \rho(\theta)$

Teorema de Chasles

$P, \ldots, \theta$

$\|z\| = 1$

R R SC...



$(Q_j, \underline{m}_j, \alpha_j, \gamma_j), (G_j, \underline{f}_j, \gamma_j, \gamma_j),$

$(A_j, \underline{a}_j, \theta_j, t_j)$

$\tan \frac{\gamma_j}{2} = \frac{\underline{m}_j \cdot \underline{f} \cdot \underline{a}}{(\underline{m}_j \cdot \underline{f}) \cdot (\underline{f} \cdot \underline{a})}$

$\tan \frac{\alpha_j}{2} = \frac{\underline{f} \cdot \underline{a} \cdot \underline{m}_j}{(\underline{a} \cdot \underline{m}_j)(\underline{m}_j \cdot \underline{f})}$ (7)

$\tan \frac{\theta_j}{2} = \frac{\underline{f} \cdot \underline{a} \cdot \underline{m}_j}{(\underline{f} \cdot \underline{a})(\underline{a} \cdot \underline{m}_j)}$

$\underline{u} = \frac{\underline{m} \cdot (\underline{a} - \underline{a} \cdot \underline{f})}{1 - (\underline{m} \cdot \underline{f})^2} (\underline{g} - \underline{q}) + \frac{\underline{m} \cdot (\underline{a} - \underline{a}\, \underline{a})}{1 - (\underline{a} \cdot \underline{m})^2} (\underline{a} - \underline{q})$ (3)

$\underline{w} = \frac{\quad}{\quad} (\underline{a} - \underline{g}) + \frac{\quad}{\quad} (\underline{g} - \underline{q})$

$\underline{v} = \frac{\quad}{\quad} (\underline{q} - \underline{a}) + \frac{\quad}{\quad} (\underline{g} - \underline{a})$

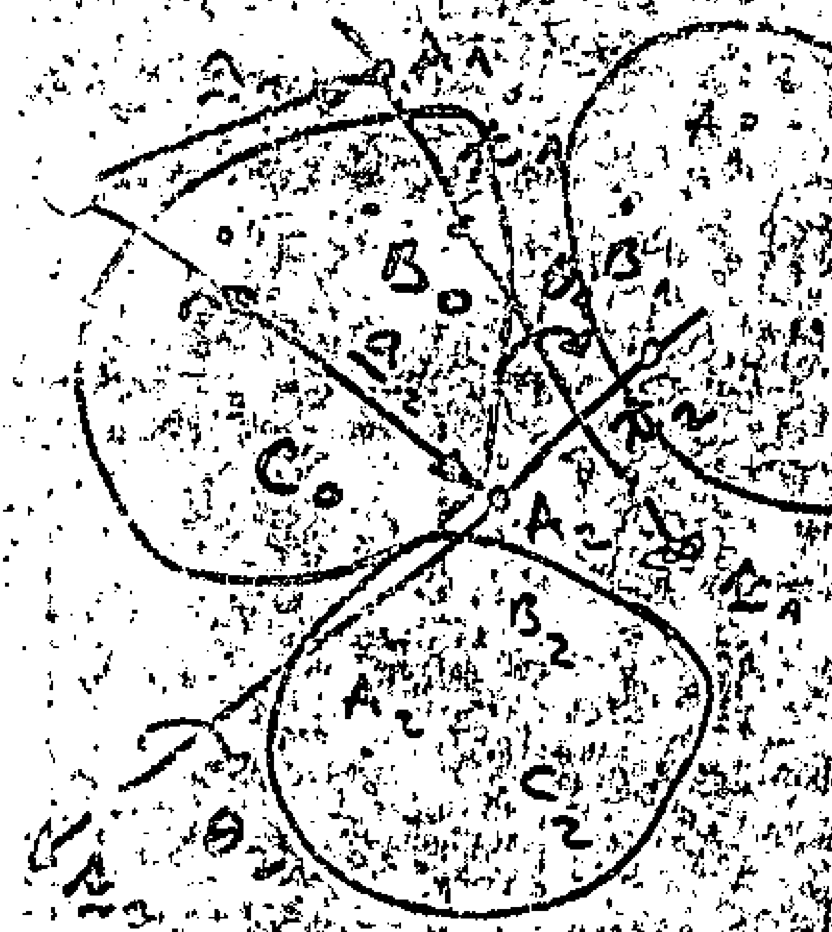
2

RR

RRRR

$l = 3 \cdot 6 -$ [illegible]

$l = 3 \cdot 3 -$ [illegible] $2 =$ [illegible]



Análisis dinámico de sistemas mecánicos.

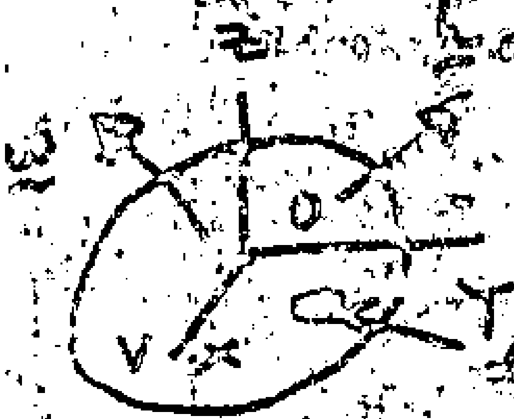
Mecánica newtoniana	Mecánica Lagrangiana
Vectores cartesianos	Vectores en R^n
2ª Ley de Newton	P.pios. de mínimo
D.C.L.	Sistemas conservativos
En Ley de Newton:	P.pio de mínima acción
$\underset{\sim}{F} = m\ddot{\underset{\sim}{r}}$	$\int L(q, \dot{q}, t)\, dt \to$ mín.
$\underset{\sim}{L}_0 = \frac{d\underset{\sim}{K}_0}{dt}$	
$\underset{\sim}{K}_0 = \underset{\sim}{r}_0 \times ...$	

$$\underset{\sim}{I}_0 = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{bmatrix}$$

$I_x = \int_V (y^2+z^2)\,dm$

$I_y = \int_V (x^2+z^2)\,dm$

$I_z = \int_V (x^2+y^2)\,dm$



$I_{xy} = -\int_V xy\,dm$

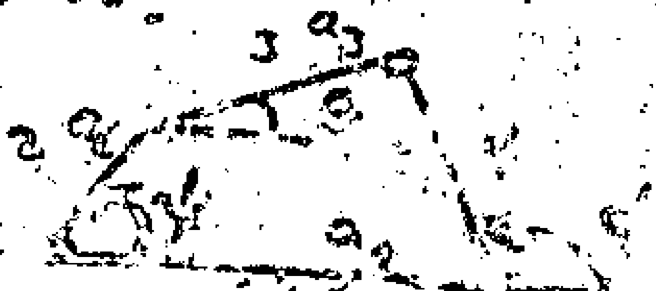
$I_{xz} = -\int_V xz\,dm$

$I_{zy} = I_{yz} = -\int_V yz\,dm$

$\underset{\sim}{\omega} = [\omega_x, \omega_y, \omega_z]^T$

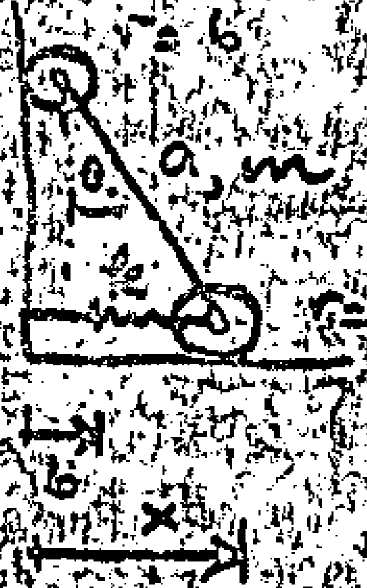
$$\underset{\sim}{h}_0 = \underset{\sim}{I}_0\,\underset{\sim}{\omega} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{Bmatrix} I_x\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ I_{xy}\omega_x + I_y\omega_y + I_{yz}\omega_z \\ I_{xz}\omega_x + I_{yz}\omega_y + I_z\omega_z \end{Bmatrix}$$

Coord. gen.



q_1, θ_1, ϕ

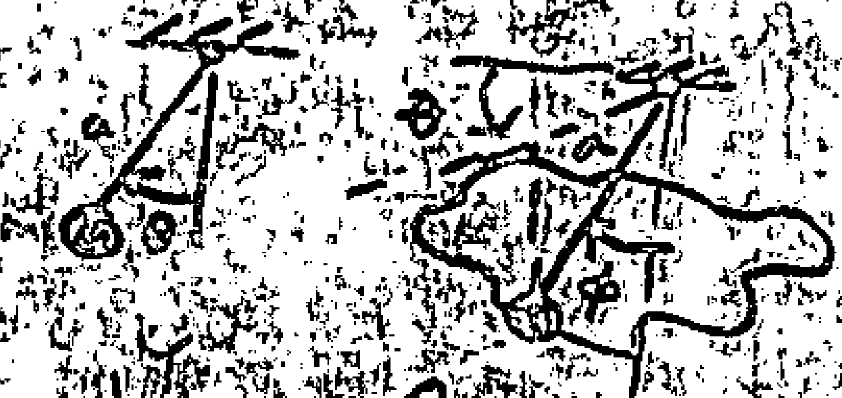
$$= 4 \times 3 - 5 \times 2 = 12 - 10 = 2$$



C.G.: x, θ

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}$$

$$x = a \operatorname{sen} \theta$$



[illegible] (q_1, q_2, q_3) [illegible]

[illegible] Lagrangian [illegible]

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$ [illegible]

[illegible]

[illegible]

$T = $ [illegible] $, \quad V = $ [illegible]

$= $ [illegible] $, \quad K = $ [illegible]

$= $ [illegible]

$= M$ [illegible] $\Rightarrow$ [illegible]

$= -K q$

[illegible] Lagrangian: [illegible]

12cm. [illegible]

5cm. [illegible]

[illegible]

[illegible] → [illegible]

[illegible]

centro de educación continua
división de estudios de posgrado
facultad de ingeniería unam

ANALISIS, SINTESIS Y OPTIMIZACION EN INGENIERIA MECANICA

Una Simbología Generalizada en TMM

Dr. Justo Nieto Nieto

Agosto 1980.

PROPUESTA DE UNA SIMBOLOGIA GENERALIZADA PARA LA SINTESIS ESTRUCTURAL DE MECANISMOS

Por: Salvador BRESO y Justo NIETO

RESUMEN

En este trabajo se propone una simbología generalizada para la representación de las cadenas cinemáticas, de utilidad en el Análisis y Síntesis de Mecanismos.

1. INTRODUCCION

La topología de las cadenas cinemáticas es un tema abierto a la investigación. La razón por la que los cinemáticos sienten por el mismo una especial predilección se debe, aparte de lo anterior, a que es el tema de más contenido lógico-intuitivo de la T.M.M., teniendo, a demás, conexiones con las teorías de Grupos y Grafos. En este trabajo se pretende:

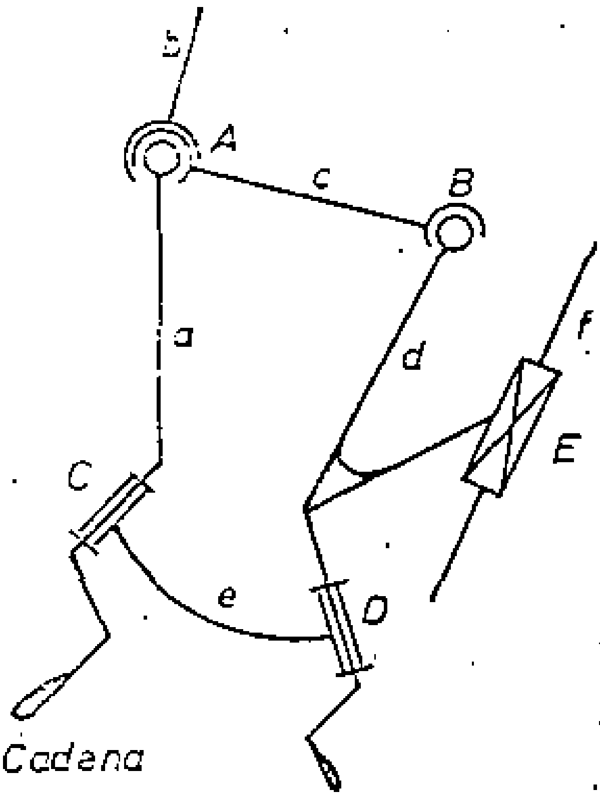
- Sacar a luz una nomenclatura general para cadenas cinemáticas con barras i-arias, y nudos i-arios, que tengan barras rígidas y flexibles.

- Encontrar a partir de esta nomenclatura, todas las posibles configuraciones de nudos y todas las posibles configuraciones de cadenas cinemáticas.

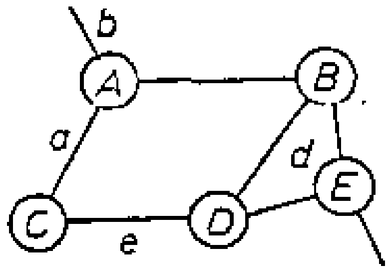
- Determinar métodos de análisis de la movilidad de estas cadenas generalizadas, empleando estos resultados del análisis en la síntesis extructural de cadenas cinemáticas.

2. SIMBOLOGIA PROPUESTA PARA NUDOS

Se asimila el nudo a una "caja negra". Se empléa el término de nudo en ver de par, para distinguir precisamente este aspecto de caja negra, en donde lo que importa son las "salidas" que permite, o sea su movilidad; sin que importe la configuración interna de este, que hace posible esta movilidad. Otra razón es que se hace uso de concepto del grafo directo asociado a una cadena cinemática, en donde los nudos (vérticas) del grafo se corresponde con los pares generalizados, y las aristas del grafo a las barras.



Cadena Cinemática



Grafo directo

	a	b	c	d	e	f
A	1	1	1			
B			1	1		
C	1				1	
D				1	1	
E				1.		1

Matriz de incidencia

De cada grafo directo se obtiene su matriz de incidencia nudos-barras, de términos a_{ij}, unos, si el nudo de la fila i conecta a la barra de la columna j, cero en caso contrario.

Algunas de las propiedades de esta matriz son:

- La suma de los elementos de una fila es el orden del nudo.
- La suma de los elementos de una columna es el orden de la barra
- La suma de todos los elementos de las filas, o de las columnas, de la matriz, o número total de incidencias es igual a:

$$S = 2n_2 + 3n_3 + 4n_4 \ldots\ldots\ldots = b_1 + 2b_2 + 3b_3 \ \ldots\ldots$$

siendo:

$$N = \text{nº total de nudos} = n_2 + n_3 + n_4 \ldots\ldots$$

$$B = \text{nº total de barras} = b_1 + b_2 + b_3 \ldots\ldots$$

b_i = nº total de barras i-arias (que poseen o conectan, i nudos)

n_i = nº total de nudos i-arios (que conectan i barras)

Cada unión de dos barras puede poseer alguno o todos de los siguientes movimientos independientes.

- Tres rotaciones R_1 , R_2 , R_3
- Tres traslaciones T_1 , T_2 , T_3
- Tres giros G_1 , G_2 , G_3
- Tres desplazamientos D_1 , D_2 , D_3

De este modo, cada nudo de dos barras puede poseer, como máximo, cinco de seis movimientos de cuerpo rígido y otros cinco como máximo, de seis cualitativamente distintos que, inicialmente, se asignan a movimientos de barras flexibles.

Los subíndices 1, 2, 3 representan direcciones cualesquiera pudiendo ser estas ortogonales o no y / o concurrentes.

Por tanto, cada nudo de dos barras se representará por un conjunto de letras R, T, G, D y con los subíndices correspondientes, tantas - como movilidad tenga el nudo. Por ejemplo un par R sería R_1 , un par C sería R_1T_1. Como los nudos pueden tener conexionadas más de dos barras, en estos casos, en cada nudo i-ario, formado por las barras l, m, n, p, ... es suficiente y necesario, para fijar su movilidad, distinguir las libertades de las barras dos a dos, o sea, las lm, mn, np,... Estos i-1 subconjuntos llevan un subíndice que indican las dos barras que representan.

En general los nudos tendrán móvilidad, procedentes de conexionar barras flexibles y rígidas. A estos les llamamos MIXTOS, para distinguir los de los nudos RIGIDOS, que son los pares clásicos con barras rígidas, y de los FLEXIBLES que son aquellos pares con movilidad de flexibilidad únicamente.

Con esta idea es posible encontrar rápidamente todas las posibles configuraciones de nudos. Se tendra en cuenta que:

a) Las rotaciones paralelas de dos barras son redundantes, e igual ocurre con las traslaciones, desplazamientos, etc.

b) Dos barras solo pueden tener cinco grados de libertad relativos como máximo.

Como consecuencia de todo lo anterior surge la siguiente enumeración de pares.

1

- NUDOS
  - Rigidos
    - Binarios
      - G = 1
        - R
        - T
      - G = 2
        - RR
        - RT
        - TT
      - G = 3
        - RRR
        - RRT
        - RTT
        - TTT
      - G = 4
        - RRRT
        - RRTT
        - RTTT
      - G = 5
        - RRRTT
        - RRTTT
    -
      - G = 1.1
        - R.R
        - R.T
        - T.T
      - G = 1.2
        - R. RR
        - R. RT
        - R. TT
        - T. TT
        - T. TR
        - T. RR

cont.

- NUDOS
  - Rigidos
    - Ternarios
      - G = 1.3
        - R . RRR
        - R . RRT
        - R . RTT
        - R . TTT
        - T . TTT
        - T . TTR
        - T . TRR
        - T . RRR
      - G = 1.4
        - R . RR
        - R . RR
        - R . RT
        - T . RT
        - T . RR
        - T . RR
      - G = 1.5
        - R . RR
        - R . RR
        - T . RR
        - T . RR
      - G = 2.3
        - RR . R
        - RR . R
        - RR . RT
        - RR . TT
        - RT . R
        - RT . R
        - RT . R
        - RT . T
        - TT . R
        - TT . R
        - TT . R
        - TT . T

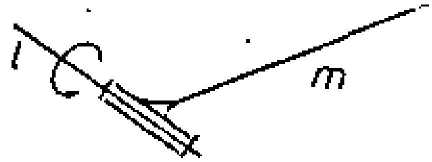
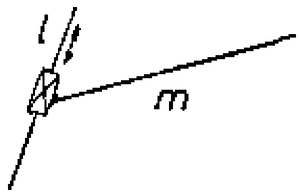
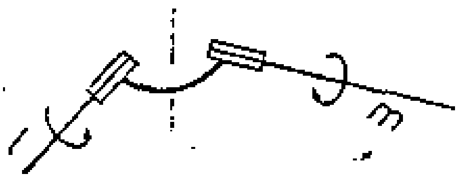
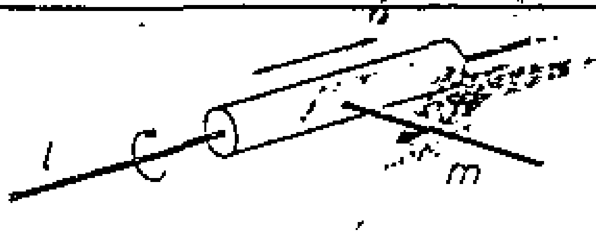
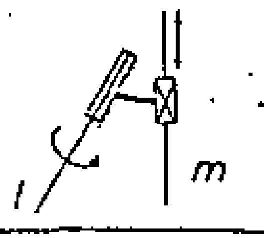
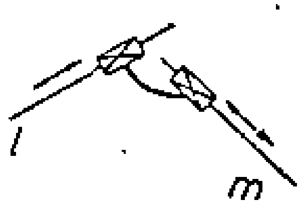
cont.

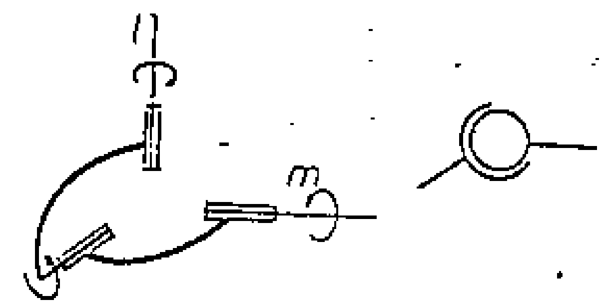
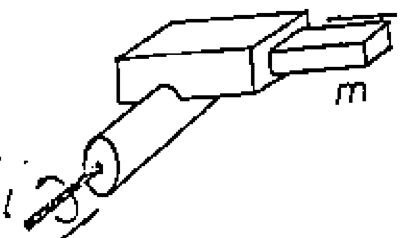
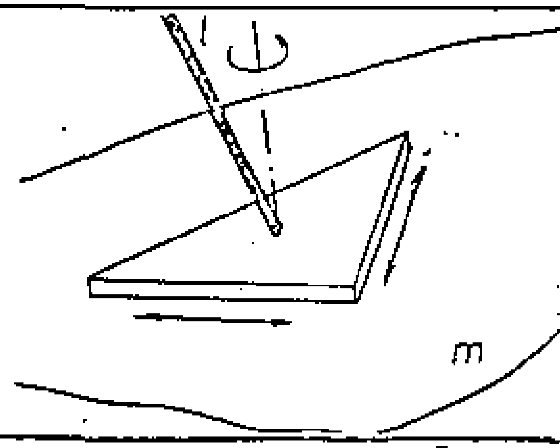
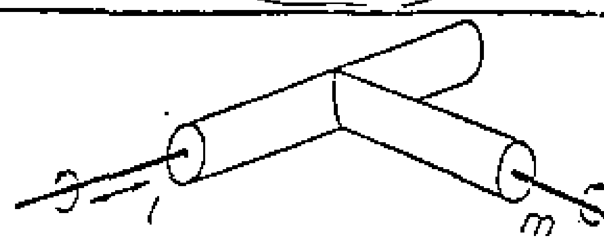
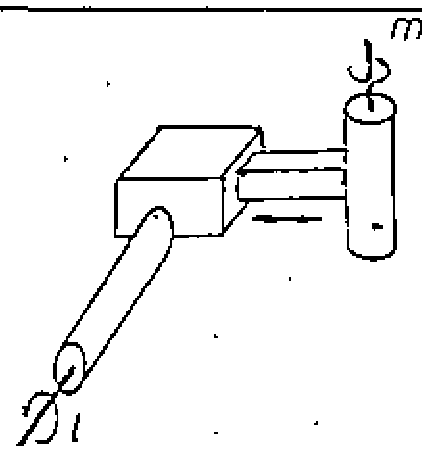
Flexibles	Ternarios	$G = 4.5$	RRRT RRRTT RRRT RRTTT RRTT RRRTT RRTT RRTTT RTTT RRRTT RTTT RRTT-T
		$G = 5.5$	RRRTT RRRTT RRRTT RRTTT RRTTT RRTTT
	Binarios	$\hat{G} = 1$	G D
		$\hat{G} = 2$	G G G D D D
Mixtos	Binarios	$G ; \hat{G} = 1 ; 1$	R.G R.D T.G T.D
		$G ; \hat{G} = 1 ; 2$	R . GG R . GD R . DD T . GG T . GD T . DD
		$G ; \hat{G} = 2 ; 1$	RR . G RR . D RT . G RT . D TT . G TT . D

En cada combinación de libertades de los nudos pueden existir varias soluciones correspondientes a la situación de los ejes.
A continuación se muestran algunas de estas soluciones.

Enumeracion y constitucion de nudos

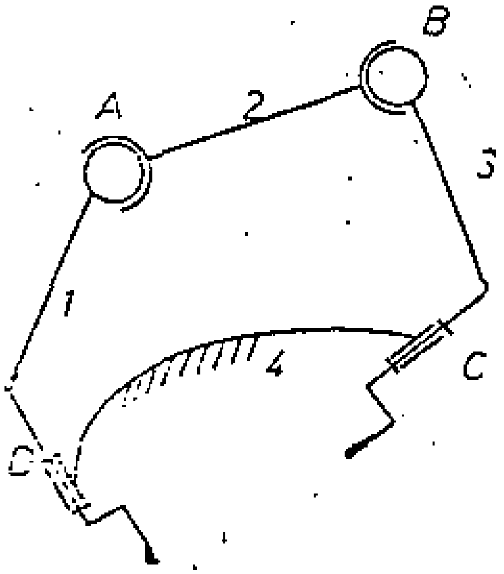
Nudos rigidos binarios r^2

g.d.l.	nº	tipo	simbolo	esquema
1	1	r^2-1	R_1	
1	2	r^2-1	T_1	
2	1	r^2-2	R_1 R_2	
2	2	r^2-2	R_1 T_1	
2	3	r^2-2	R_1 T_2	
2	4	r^2-2	T_1 T_2	

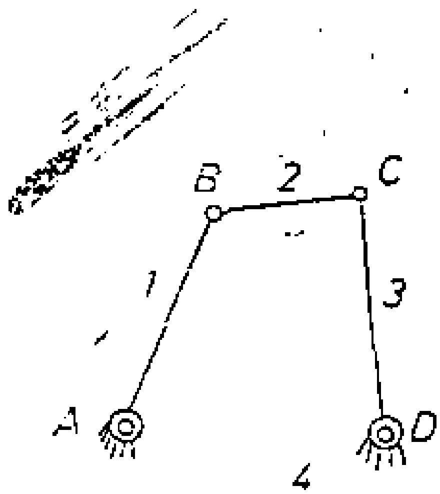
3	1	r^2-3	$R_1 R_2 R_3$	
3	2	r^2-3	$R_1\ T_1\ T_2$	
3	3	r^2-3	$R_1\ T_2\ T_3$	
3	4	r^2-3	$R_1\ R_2\ T_2$	
3	5	r^2-3	$R_1\ R_2\ T_3$	

3. SIMBOLOGIA PROPUESTA PARA CADENAS

La representación que se propone para las cadenas cinemáticas consiste en la expresión simbólica de los nudos junto a la matriz de incidencias. Se verá con algunos ejemplos.

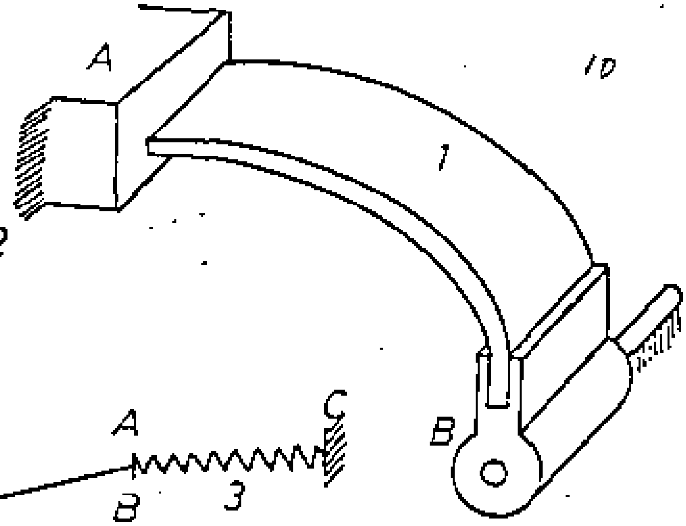


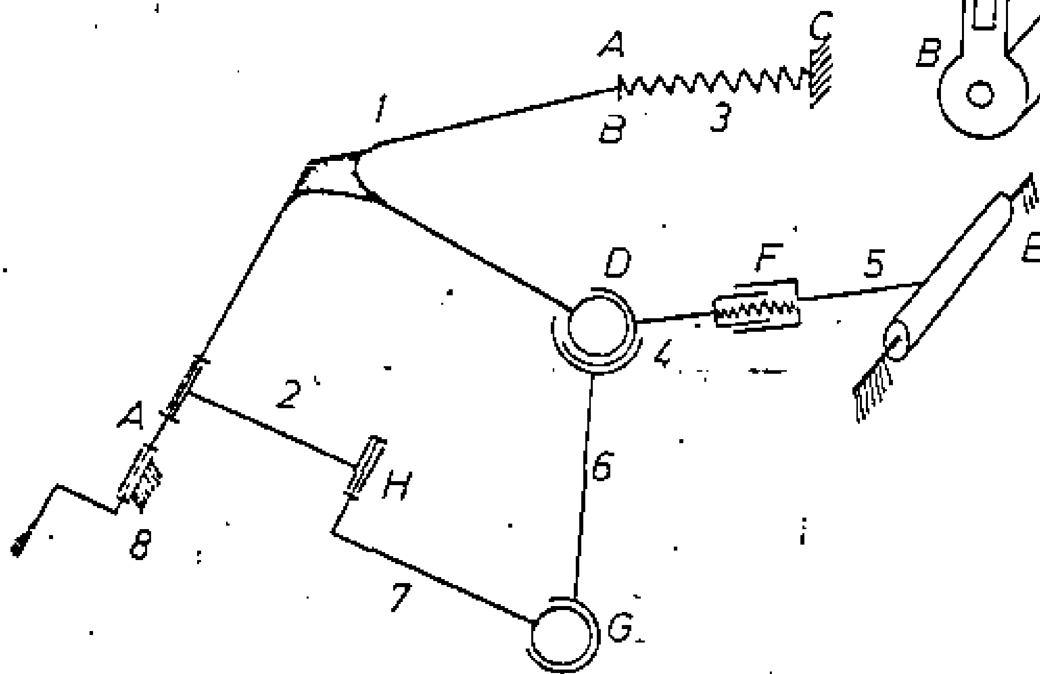
	Nudo \ Barras	1	2	3	4
R_1 R_2 R_3	A	1	1		
R_1 R_2 R_3'	B		1	1	
R_1	C			1	1
R_2	D				1



	Nudos \ Barras	1	2	3	4
R_1	A	1			1
R_1	B	1	1		
R_1	C		1	1	
R_1	D			1	1

Nudos \ Barras		1	2
G_4	A	1	1
$R\ G_1$	B	1	1





		1	2	3	4	5	6	7	8
$(R_4)'' - (R_1)''$	A	1	1						1
$G_1\ G_2\ G_3\ D_1$	B	1		1					
$G_1\ G_2\ G_3\ D_1$	C			1					1
$(R_1 R_2 R_3)'' - (R_1 R_2 R_3)''$	D	1			1		1		
$R_4\ \ T_1$	E					1			1
$G_2\ \ D_1$	F				1	1			
$R_1\ R_2\ R_3$	G						1	1	
R_1	H		2 → 1						1

4. MOVILIDAD DE LAS CADENAS CINEMATICAS

Se verá en primer lugar el caso de cadenas con nudos rígidos.

Se introduce la siguiente terminología adicional a la expuesta en el apartado 2.

M = Movilidad de la cadena cinemática = G + 6

G = Grados de libertad del mecanismo

n_i^j = Nº de nudos i-arios con movilidad total j

La movilidad total de un nudo se define como suma de las libertades que permiten las barras que conectan.

Por tanto, se puede hacer el siguiente esquema que representa las posibles movilidades de los nudos i-arios, y para una movilidad dada el tipo de pares parciales que la hacen posible.

mov	binarios	ternarios	cuaternarios	pentarios +
1				
2	pares clasicos de la clase I a la V	I.I		
3		I.II	I.I.I	...
4		I.III II.II	I.I.II	I.I.I.I
5		I.IV II.III	I.I.III I.II.II	~ ~
6		I.V II.IV III.III	I.I.IV I.II.III II.II.II	~ ~
7		II.V III.IV	~ ~	~ ~

Dada una configuración arbitraria de nudos y barras la movilidad M de la misma se puede obtener como:

$$M = A_1 - A_2 + A_3$$

en donde:

A_1 = Grados de libertad de los nudos supuestos libres

A_2 = Grados de libertad que restringen las barras

A_3 = Grados de libertad adicionales en virtud de geometría especial (dimensional y direccional)

Para determinar A_1 se tiene que:

Cada nudo binario de movilidad uno necesita 6+1 parámetros para quedar definido. Luego si existen n_2^1 nudos de esta clase el total de parámetros será $7n_2^1$

Cada nudo binario de movilidad dos necesita 6+2 parámetros para quedar definido. Luego si existen n_2^2 de esta clase el número total será $8n_2^2$

Cada nudo binario de movilidad tres necesita 6+3 parámetros para quedar definido. Luego si existen n_2^3 de esta clase el número total será $9n_2^3$

Cada nudo binario de movilidad cuatro necesita 6+4 parámetros para quedar definido. Luego si existen n_2^4 de esta clase el número total será $10\,n_2^4$

Cada nudo binario de movilidad cinco necesita 6+5 parámetros para quedar definido. Luego si existen n_2^5 de esta clase el número total será $11\,n_2^5$

Cada nudo ternario de movilidad dos necesita 6+2 parámetros para quedar definido. Luego si existen n_3^2 de esta clase el número total será $8n_3^2$

Cada nudo ternario de movilidad tres necesita 6+3 parámetros para quedar definido. Luego si existen n_3^3 de esta clase el número total será $9n_3^3$

Cada nudo ternario de movilidad cuatro necesita 6+4 parámetros para quedar definido. Luego si existen n_3^4 de esta clase el número total será $10n_3^3$

Cada nudo cuaternario de movilidad tres necesita 6+3 parámetros para quedar definido. Luego si existen n_4^3 de esta clase el número total será $9n_4^3$

Por tanto, el número de grados de libertad de los nudos supuestos libres es:

$$A_1 = 7n_2^1 + 8n_2^2 + 9n_2^3 + 10n_2^4 + 11n_2^5$$

$$8n_3^2 + 9n_3^3 + 10n_3^4 + 11n_3^5 + \ldots + 16n_3^{10} \quad 2.5$$

$$9n_4^3 + 10n_4^4 + 11n_4^5 + \ldots\ldots + 21n_4^{15} \quad 3.5$$

$$- - - - - - - - - - - - - - -$$

$$- - - - - - - - - - - - - - -$$

Para determinar A_2 se tiene que:

Cada barra monaria no restringe los grados de libertad del nudo al que está conectada, por tanto no hay que tenerlas en cuenta.

Cada barra binaria restringe en 6 el número de libertades que tendrían los nudos supuestos libres. Luego si existen b_2 de estas barras, la reducción será $6b_2$

Cada barra ternaria restringe en 12 el número de libertades que tendrían los nudos supuestos libres. Luego si existen b_3 de estas barras la reducción será $12b_3$

Cada barra cuaternaria, pentaria, ... restinge en 18, 24, ..., el número de libertades de los nudos que conectan supuestos libres.

Por tanto:

$$A_2 = 6\,(b_2 + 2b_3 + 3b_4 + 4b_5 \ldots\ldots)$$

Por tanto, la expresión de la movilidad de una cadena, prescindiendo del término A_3 será:

$$M = 7n_2^1 + 8(n_2^2 + n_3^2) + 9(n_2^3 + n_3^3 + n_4^3) + \ldots\ldots$$

$$- 6\,(b_2 + 2b_3 + 3b_4 + \ldots\ldots)$$

que puede quedar como:

$$M = \sum_{i=1} (6+i)\, n^i - \sum_{k=2} 6(k-1)\, b_k$$

siendo n^i el número de nudos de movilidad i, y b_k el número de barras de orden k

Otras expresiones para M pueden deducirse teniendo en cuenta las relaciones ya utilizadas:

$$N = n_2 + n_3 + n_4 + \dots = (n_2^1 + n_2^2 + n_2^3) + (n_3^2 + n_3^3 + \dots) + \dots = n^1 + n^2 + n^3 +$$

$$B = b_1 + b_2 + b_3 + b_4 + \dots$$

$$n_2 = n_2^1 + n_2^2 + n_2^3 + \dots$$

$$n_3 = n_3^2 + n_3^3 + n_3^4 + \dots$$

$$n^1 = n_2^1$$

$$n^2 = n_2^2 + n_3^2$$

$$n^3 = n_2^3 + n_3^3 + n_4^3$$

$$2n_2 + 3n_3 + 4n_4 + \dots = b_1 + 2b_2 + 3b_3 + \dots$$

$$2N + n_3 + 2n_4 + \dots = B + b_2 + 2b_3 + \dots$$

El valor de la movilidad M así obtenido, representa un valor mínimo de la movilidad, ya que factores de geometría especial introducen grados de libertad adicionales.

Estos factores pueden ser de tipo direccional como ocurre en:

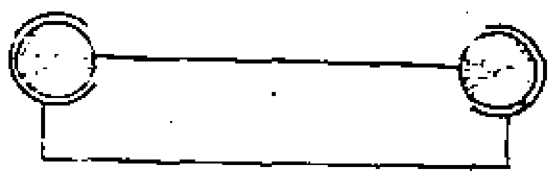
- el clásico cuatro - barras plano que sólo es móvil por que tiene los cuatro ejes de los nudos paralelos. (al aplicar la fórmula da G = -2)
- el cuatro - barras esférico, que sólo es móvil por que los cuatro

ejes de los nudos se cortan en un punto, centro de la esfera.

- el mecanismo de Bennet
- todos los mecanismos planos.

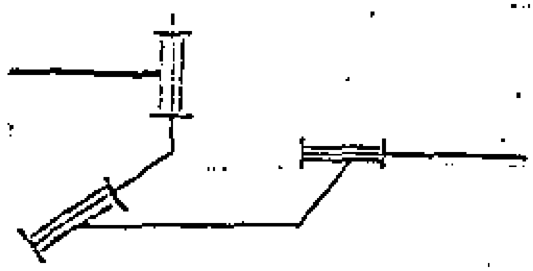
o de tipo dimensional, por ejemplo, cuando existen barras de igual longitud (doble paralelogramo de cinco barras)

Un caso típico de direccionalidad es en la cadena EE

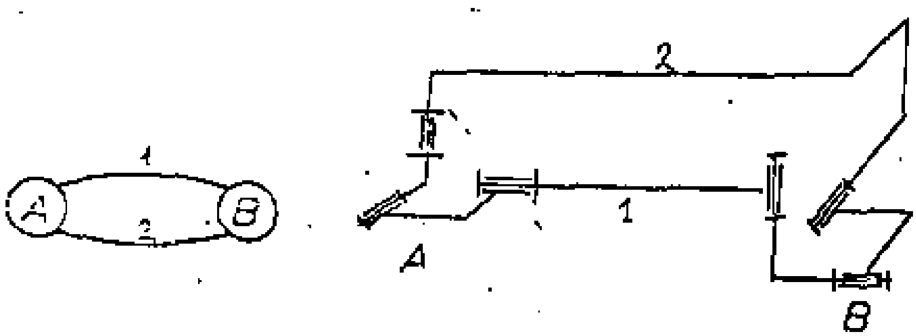


que al ser tratados sus nudos como binarios con movilidad tres la movilidad de la cadena resulta ser 6, cuando en realidad tiene 7

La explicación de esta anomalía se debe a que un nudo binario en tres grados de libertad sería, en realidad, el de la figura



y evidentemente, al conexionar dos pares de este tipo para formar la cadena



la aplicación de la fórmula es correcta, es decir, el número de grados de libertad del mecanismo es cero. Sin embargo, en los casos de los dos pares esféricos, siempre existe un eje de rotación común (alineado) a ambos pares, que es el que permite este grado de libertad adicional. Análogamente ocurre para dos pares planos directamente unidos.

Si la cadena está formada por nudos flexibles o mixtos la expresión que se propone para la movilidad generalizada estará formada por dos términos diferenciados, uno correspondiente a los movimientos de cuerpo rígido y otro correspondiente a las flexibilidades.

$$M_G = M_r M^*$$

5. SINTESIS ESTRUCTURAL DE CADENAS CINEMATICAS GENERALIZADAS

Las posibilidades, caminos, de síntesis estructural son las siguientes:

a) Si todos los nudos son rígidos

- a.1. A partir de la expresión de la movilidad buscando soluciones enteras de las variables que intervienen, cuando se fija el nº de barras, movilidad, etc ....
- a.2. Para cualquiera de las soluciones anteriores del caso a.1. cada n admite diversas soluciones tanto en el orden del nudo como en la clase de las conexiones parciales que configuran el nudo
- a.3. A su vez cada solución a.2. admite más variantes solución, según la tabla de enumeración de nudos ya vista en el apartado 2 según los subíndices 1.2.3. del par.
- a.4. A partir de una matriz de incidencias especificada según convenga, se obtiene el grafo, y a partir de las consideraciones a.2. y a.3. se pueden obtener otras soluciones.

b) Si todos los nudos no son rígidos

Valen las mismas observaciones que en el caso anterior; solo que, la expresión de la movilidad generalizada, tiene una complejidad que la hace poco tratable. En este caso es preferible usar la matriz de incidencia y el grafo asociado como elemento de síntesis.

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SOBRE LA SITUACION DE LOS CIA EN MOVIMIENTO PLANO.

Por: IGNACIO CUADRADO Y JUSTO NIETO

RESUMEN

En este trabajo se obtienen algunas propiedades asociadas a la distribución de los CIA en el movimiento plano de un mecanismo articulado de cuatro barras. En particular se encuentra: a) la situación de los mismos para entrada variable, que viene expresada por una transformación bilineal, b) la expresión del coeficiente B de influencia en barras contiguas.

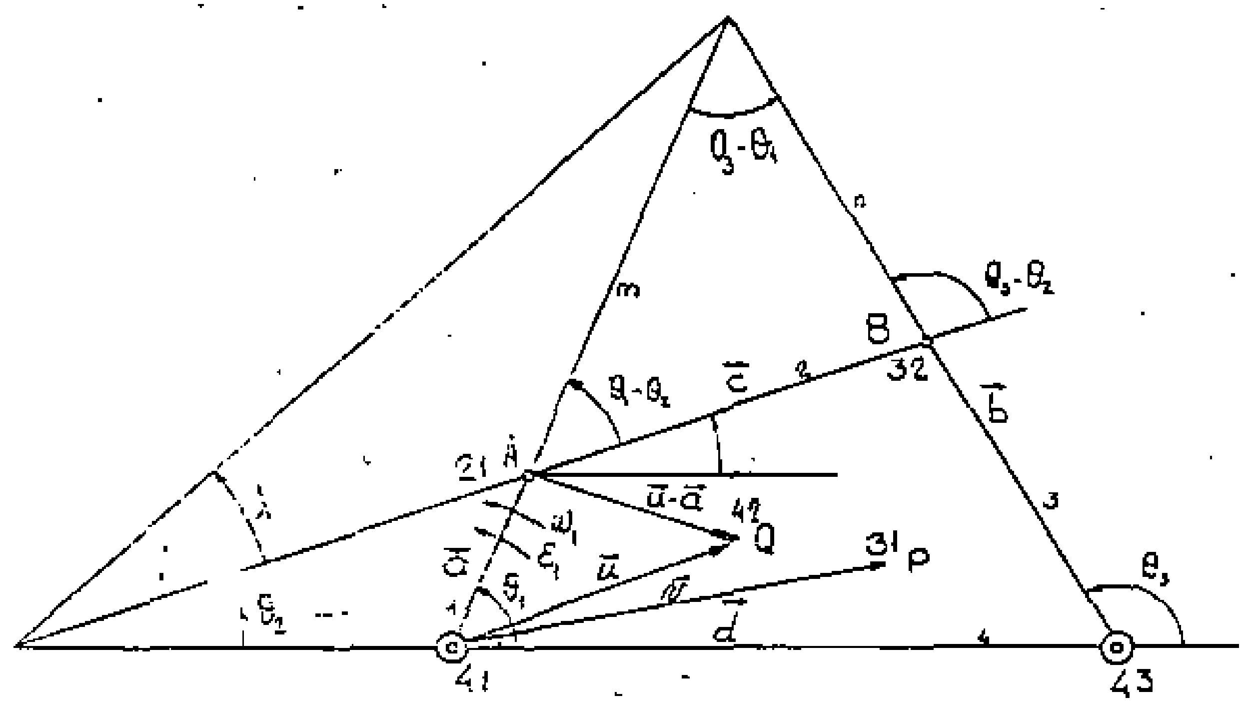
1. INTRODUCCION

La creencia de que las propiedades de la distribución de los centros instantáneos de aceleración, CIA, en el movimiento plano, no han sido suficientemente utilizados para el análisis cinemático de mecanismos, nos han hecho elaborar este trabajo, cuya finalidad no es otra que tutorial, es decir, de interés docente. El trabajo se dedica a un cuatro barras plano, pero, de modo análogo, podría ser extrapolado a otro mecanismo, p.e. el biela manivela.

Sea el 4.b. de la figura. Para nominar los CIA, a diferencia de los CIR, empleamos los dos números de las barras que representan, colocando en primer lugar el número mayor. La situación del centro instantáneo 42, se obtiene haciendo uso de la clásica expresión de las aceleraciones relativas:

$$\vec{a}_q = \bar{s} = \bar{a}_2 - \bar{a}_{q2} \tag{1}$$

Empleando la dualidad plano vectorial-plano complejo: $\vec{\varepsilon} \times \vec{a} = \varepsilon . a . i$



(con i unidad imaginaria), definiendo unos coeficientes de influencia A^*, B^*, para las barras 1 y 2 dados por

$$\begin{aligned} \omega_2 &= A^* \omega_1 \\ \varepsilon_2 &= A^* \varepsilon_1 + B^* \omega_1^2 \end{aligned} \tag{2}$$

y sacando a luz un parámetro de la entrada, con su signo dependiente de ε_1 dado por:

$$e = \frac{\varepsilon_1}{\omega_1^2} \tag{3}$$

se obtiene, sustituyendo en (1),

$$\bar{u} - \bar{a}' = \frac{1 - A^* i}{i(A^* e + B^*) - A^{*2}} \bar{a} \tag{4}$$

De igual forma, se puede proceder para el CIA 31. En este caso

$$\vec{v} = \frac{A^2 - i(Ae + B)}{A^2 - 1 + i(e - Ae - B)} \vec{d} \tag{5}$$

siendo A y B los coeficientes de influencia de las barras 1 y 3

2. CONSECUENCIAS

Algunas de las consecuencias obtenidas de las anteriores expresiones son las siguientes:

a- De (4) resulta evidente que $\vec{u}-\vec{a}$ y $\vec{v}$ no dependen, como es obvio, del sentido de ω_1. La distribución de $\vec{u}-\vec{a}$ y $\vec{v}$ no es simétrica con la entrada e.

b- Si $\omega_1 = cte \Rightarrow \varepsilon_1 = 0 \Longleftrightarrow e = 0$ entonces:

$$\vec{u}-\vec{a} = \frac{1}{i B^* - A^{*2}} \vec{a}$$

$$\vec{v} = \frac{A^2 - i B}{A^2 - 1 - i B} \vec{d}$$

que permite obtener la distribución de los CIA, en un caso bastante corriente.

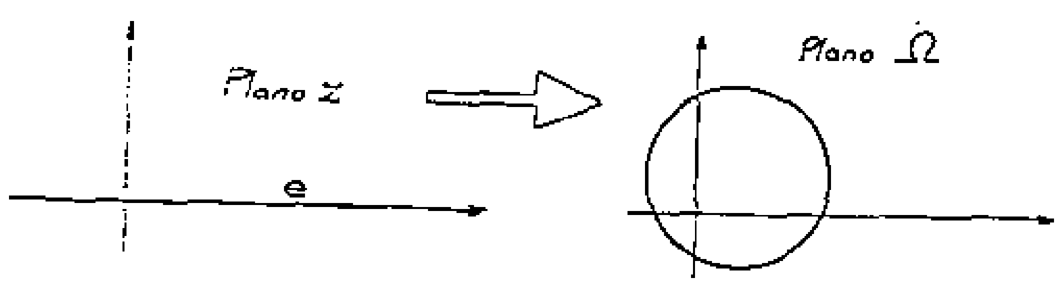
c- Si $\omega_1 = 0$; $\varepsilon_1 = cte$ hipótesis válida, solo en el instante inicial.

$$\vec{u}-\vec{a} = -\frac{1}{A^*} \vec{a}$$

$$\vec{v} = \frac{A}{A-1} \vec{d}$$

lo que indica la colinealidad de las barras con los CIA

d- Si la entrada e es variable y la posición constante, la distribución de los CIA viene expresada por una transformación bilineal



$$\Omega = \frac{\vec{u}-\vec{a}}{\vec{a}} = \frac{1 - e i}{i(A^* e + B^*) - A^{*2}}$$

$$\Omega = \frac{\vec{v}}{\vec{d}} = \frac{A^2 - i(Ae + B)}{A^2 - 1 + i(e - Ae - B)}$$

que transforma el eje real e en una circunferencia. Los puntos correspondientes a $e=0,\infty,\lambda$ etc, y los inversos del plano Ω, así como las intersecciones reales, tangentes, etc se obtienen fácilmente haciendo uso de dicha transformación bilineal, p.e. el otro valor real de Ω que corresponde a la colineación de $\vec{\mu}-\vec{a}$ y $\vec{a}$ se obtienen por

$$e = \frac{B^*}{A^*(A^*-1)}$$

$$\vec{v}-\vec{a} = -\left[\frac{A^{*2}(A^*-1)^2+B^{*2}}{(A^*-1)^2A^{*4}+A^{*2}B^{*2}}\right]\vec{a}$$

lo que dice que siempre esta opuesto a $\vec{a}$

e- Si la entrada es constante y la geometría A^*, B^* variable, la distribución de los CIA, viene dada, supuesto p.e. $\mathcal{E}_1=0$, por:

$$\vec{u}-\vec{a} = \frac{1}{iB^*-A^{*2}}\vec{a}$$

$$\vec{v} = \frac{A^2-iB}{A^2-1-iB}\vec{d}$$

3. NOTAS FINALES.

Los coeficientes geométricos de influencia vienen dados, para el 4.b. de la figura, por:

$$A = \frac{d\theta_2}{d\theta_1} = \frac{x}{x+a}$$

$$B = \frac{d^2\theta_2}{d\theta_1^2} = \frac{A(1-A)}{tg\,\lambda}$$

$$A^* = \frac{d\theta_3}{d\theta_1} = -\frac{a}{m}$$

$$B^* = \frac{d^2\theta_3}{d\theta_1^2} = \frac{A^*(1-A)}{tg(\theta_3-\theta_1)} + \frac{A^*(A^*-A)}{tg(\theta_3-\theta_2)}$$

(5)

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ANALISIS, SINTESIS Y OPTIMIZACION EN INGENIERIA MECANICA

Optimización de la Síntesis de Guiado de Cuerpo Rígido por el Método de la Métrica Variable.

DR. JUSTO NIETO NIETO

Agosto, 1980.

OPTIMIZACION DE LA SINTESIS DE GUIADO DE CUERPO RIGIDO POR EL METODO DE LA METRICA VARIABLE

Por Juan Ignacio CUADRADO, Javier FUENMAYOR y Justo NIETO

1. RESUMEN

Este trabajo trata de la aplicación de un método de optimización numérico, el de la Métrica Variable, a la Síntesis de guiado de cuerpo rígido.

Con dicho método, dada una serie de posiciones del cuerpo rígido, se encuentran puntos con propiedades de mínima distancia a rectas y circunferencias, pudiéndose por tanto guiar al cuerpo con deslizaderas o manivelas acopladas respectivamente a los mismos puntos.

2. INTRODUCCION

El guiado de cuerpo rígido, consiste en encontrar puntos especiales del mismo, tales que su posición se encuentre sobre trayectorias de fácil generación.

De éstas las más usuales son la recta y la circunferencia.

Las trayectorias rectas se podrán generar mediante deslizadores y las circunferencias mediante barras articuladas por una parte al punto que describe la trayectoria circular y por otra parte a un centro fijo que será el centro de la circunferencia.

El problema se plantea de la siguiente forma

Encontrar puntos pertenecientes al cuerpo rígido, que tengan una propiedad de distancia cero o mínima, a trayectorias determinadas tales como circunferencias o rectas.

[illegible], al optimizar minimizando una función error que se elige como la suma de los cuadrados de las distancias de un punto cualquiera del cuerpo rígido a la trayectoria, cuando el anterior va variando de posición.

Una vez definida la función error F, el problema se puede solucionar de dos formas diferentes.

[illegible] analítica, planteando las condiciones de mínimo

$$F = F(x_1, x_2, x_3, \dots)$$

$$\nabla F = 0$$

Matriz Hessiana definida positiva en los puntos anteriores.

La resolución de las ecuaciones anteriores, se obtendrán todos los mínimos de la función.

La desventaja de este método para este problema es la complejidad del sistema de ecuaciones que lo hace difícil de resolver.

o Optimización numérica

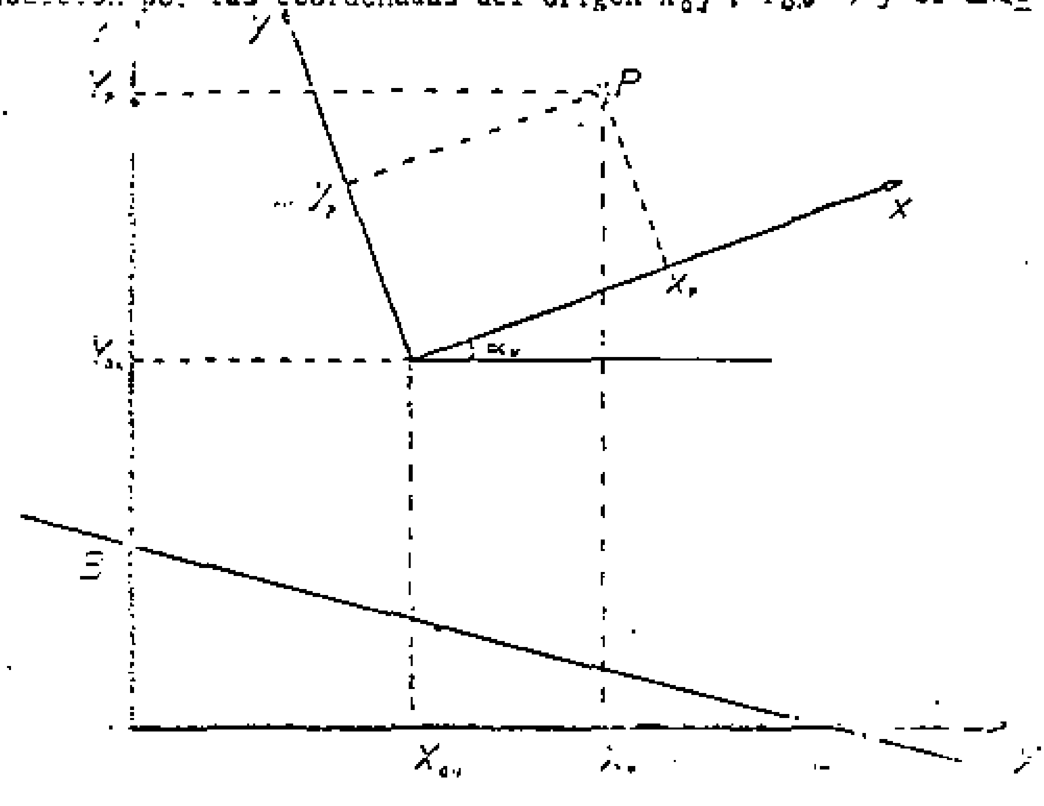
Este es el camino elegido para realizar este trabajo

Dado que la función error es no lineal, que se aborda el problema sin restricciones, y que tanto la función como las derivadas son relativamente fáciles de evaluar, se ha escogido como método el de la Métrica Variable o de Davidon - Fletcher - Powell

Posteriormente, en el Anexo se hace una descripción del método.

3. ... RECTILINEO

Dado un cuerpo rígido, sistema de referencia móvil (x, y), definido en su posición por las coordenadas del origen (X_{0v}, Y_{0v}) y el ángu



lo de inclinación del eje x con respecto al X, α_N , dado un punto P cualquiera del mismo, de coordenadas (x_p , y_p) con respecto al sistema móvil, y dada una recta definida por los puntos de corte con los ejes de referencia (A,B).

La ecuación de la recta en su forma canónica será

$$\frac{X}{A} + \frac{Y}{B} = 1$$

La distancia del punto P a la recta será

$$d(P, r) = \frac{B X_p + A Y_p - AB}{\sqrt{B^2 + A^2}}$$

donde según la figura anterior

$$X_p = X_{ON} + x_p \cos \alpha_N - y_p \operatorname{sen} \alpha_N$$

$$Y_p = Y_{ON} + x_p \operatorname{sen} \alpha_N + y_p \cos \alpha_N$$

La función error en este caso será

$$F = \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} \frac{(B X_{Pi} + A Y_{Pi} - AB)^2}{B^2 + A^2}$$

N es el número de posiciones del cuerpo rígido

En este caso los datos de partida son X_{ON} , Y_{ON} , α_N de cada posición, por tanto

$$F = F(x, y, A, B)$$

A partir de esta función hallada, se ha resuelto el problema mediante la aplicación del método anteriormente señalado, minimizando en cuatro dimensiones.

De la resolución del problema se obtienen las coordenadas del punto del cuerpo rígido (x,y) respecto al sistema móvil y los puntos de corte de la recta con los ejes (A,B)

4. GUIADO EN UNA CIRCUNFERENCIA

En las mismas condiciones del caso anterior, y según la figura, la circunferencia vendrá definida por las coordenadas de su centro (X_c, Y_c) y por su radio R

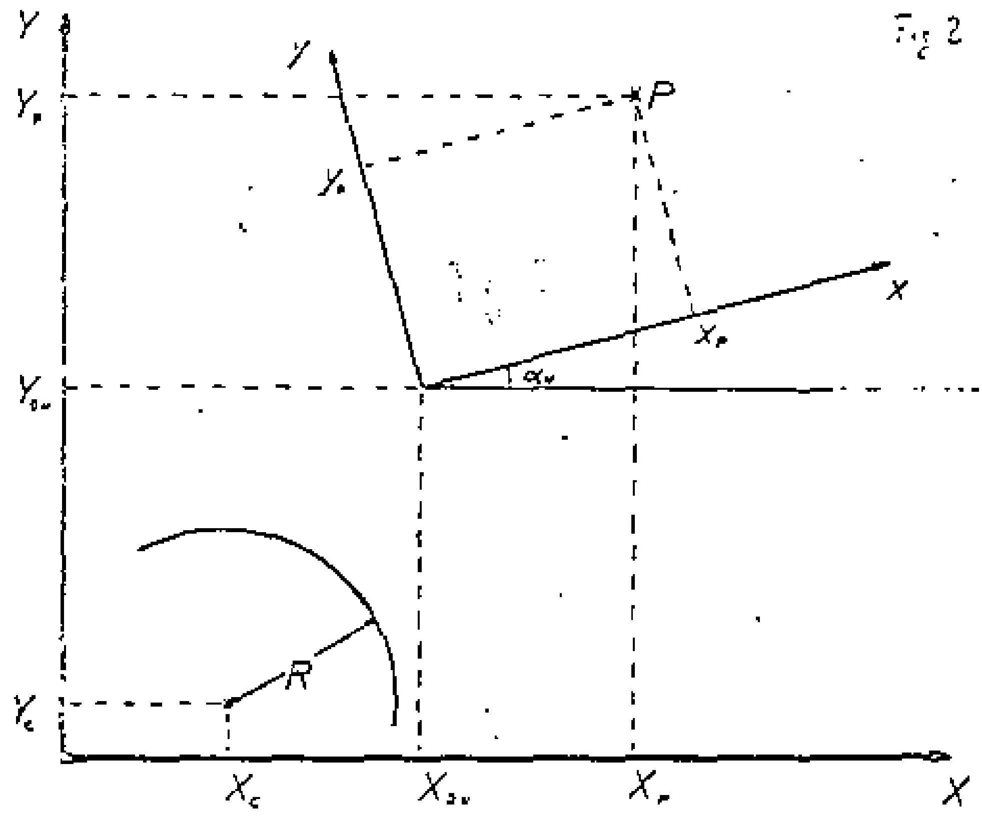


Fig 2

La distancia del punto P a una circunferencia será

$$d(P,c) = \sqrt{(X_p - X_c)^2 + (Y_p - Y_c)^2} - R$$

Los valores X_p p Y_p, pueden ponerse al igual que en el caso anterior en función de x_p, y_p, X_{OM}, Y_{OM} y α_M

La función error será

X son los diversos puntos de iteración
D son las direcciones de minimización
H es una matriz cuadrada N x N siendo N el número de dimensiones del problema

Esta matriz en la primera iteración puede ser la matriz identidad y a lo largo de las diversas iteraciones tiende a la inversa de la matriz Hessiana

En cada iteración a partir de un punto, se halla una dirección de minimización

$$D_i = - H \nabla F(x_i)$$

El siguiente punto de iteración será el mínimo de F en esta nueva iteración

En este trabajo para minimizar en una dirección se ha dividido un intervalo grande en dicha dirección, en subintervalos, en los cuales se ha evaluado la función cogiendo aquel en que se hace mínima; se ha buscado el mínimo dentro de él, mediante el método Golden.

Lo anterior se hace para evitar en lo posible que converja a un punto que sea mínimo relativo

Una vez hallado el nuevo punto de iteración, se observa su convergencia y en el caso de no ser suficiente, se vuelve a evaluar una nueva H.

$$H_{i+1} = H_i + \frac{z_i \, z_i'}{z_i \, Y_i} - \frac{(H_i \, y_i)\,(H_i \, y_i)'}{y_i \, H_i \, y_i}$$

donde

$$z_i = x_{i+1} - x_i \qquad ; \qquad y_i = \nabla F(x_{i+1}) - \nabla F(x_i)$$

Después se repite el proceso en cada iteración

$$F = \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} \left(\sqrt{(X_p - X_c)^2 + (Y_p - Y_c)^2} - R \right)^2$$

En este caso

$$F = F(x, y, X_c, Y_c, R)$$

Luego tendremos una minimización en cinco dimensiones

5. CONCLUSIONES.

En este trabajo se ha puesto a punto un programa en lenguaje BASIC procesado en máquina IBM-5100, que permite encontrar puntos de un cuerpo rígido que siguen trayectorias casi rectilíneas o casi circulares para diversas posiciones de este.

Se ha comprobado la efectividad del método operando con las funciones propuestas

ANEXO

Este anexo describe la forma de operación con el método de la Métrica Variable o de Davidon - Fletcher - Powell

El diagrama de bloques del cálculo por este método es el siguiente

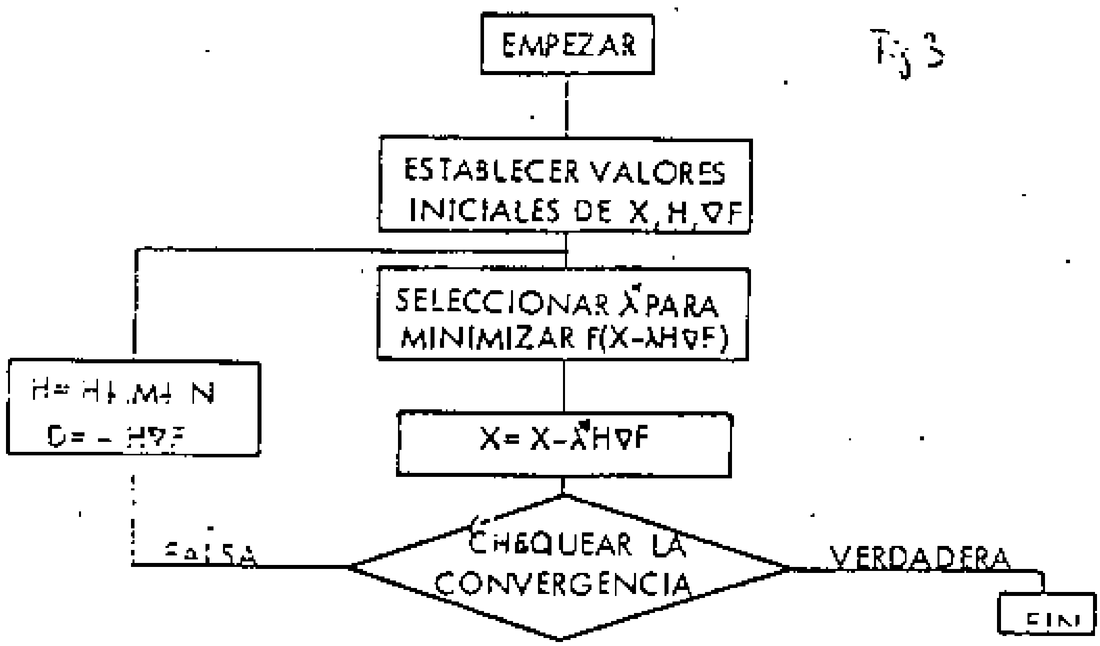


Fig 3

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ANALISIS, SINTESIS Y OPTIMIZACION EN INGENIERIA MECANICA

Algunos Resultados Teóricos para Bandas con Grandes Deformaciones

Dr. Justo Nieto Nieto

Agosto 1980.

4.1. ALGUNOS RESULTADOS TEORICOS PARA BANDAS CON GRANDES DEFORMACIONES

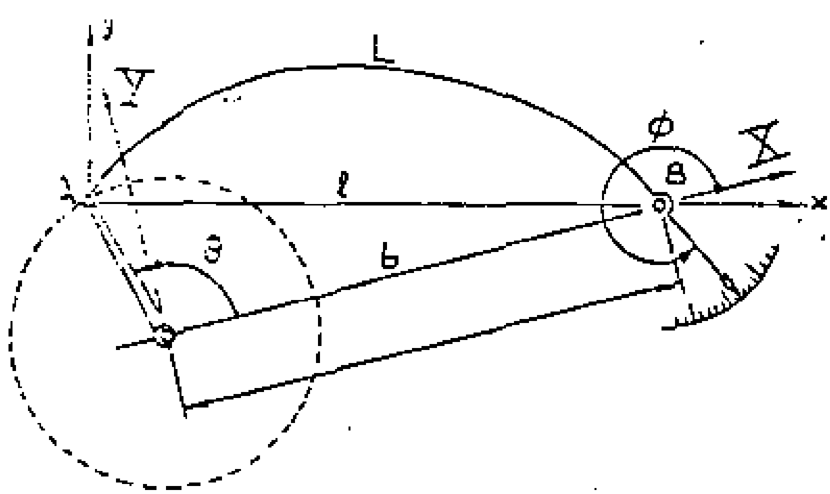
Por Salvador Bresó y Justo Nieto

RESUMEN

La obtención de algunas propiedades relativas a las deformadas elásticas en bandas con grandes deformaciones, así como las ecuaciones elastodinámicas son los resultados más significativos de este trabajo.

INTRODUCCION

En un trabajo previo se distinguieron cuatro casos en el estudio de una banda flexible, de espesor pequeño en relación con las otras dos dimenciones, cuando algunos de sus puntos se ven obligados a moverse a lo largo de trayectorias especificadas. En el caso del mecanismo de la figura, los cuatro casos son los siguientes:



a) CASO ELASTOESTATICO

Encuentra la posición del equilibrio elástico de la banda para cualquier posición prefijada de los puntos A y B

b) CASO ELASTODINAMICO

Encuentra la respuesta dinámica a la deformada anterior, es decir superpone al caso anterior el problema elastodinámico, (efectos inerciales, etc.)

c) CASO ELASTOCINEMATICO

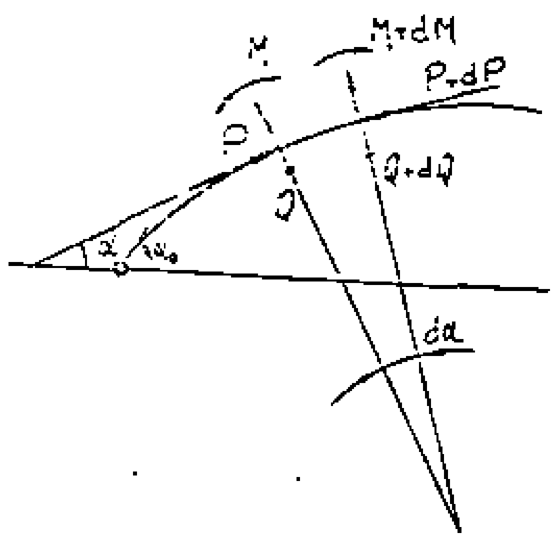
Consiste en obtener la sucesión de deformadas estáticas (caso a) cuando los puntos A y B tienen leyes de movimiento preestablecidas. (El punto A se mueve sobre una circunferencia y el punto B permanece fijo)

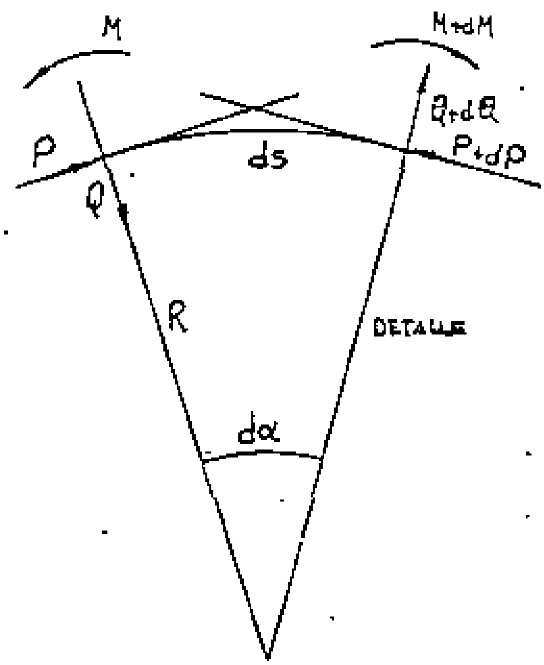
d) CASO GENERAL ELASTODINAMICO

Introduce los efectos simultáneos de los casos b) y c)

2. OBTENCION DE LAS ECUACIONES DE COMPORTAMIENTO

CASO ELASTOESTATICO





Las tres ecuaciones de equilibrio estático del elemento diferencial en ausencia de carga sobre el mismo son:

$$\begin{aligned} &\frac{dM}{dS} = Q \\ &- dP + Q\, d\alpha = 0 \\ &dQ + P\, d\alpha = 0 \end{aligned} \qquad (1)$$

que junto con la relación

$$M = \frac{EI}{R} \qquad (2)$$

forman un sistema de cuatro ecuaciones que permiten, entre otras posibilidades, la eliminación de P, Q, M, obteniendo una ecuación en dos cualesquiera de las cinco variables x, y, α, ρ, s, que permiten expresar la deformada. Por ejemplo, en las variables x, y la E.D. resultante es

$$\frac{d}{dx}\left(\frac{1+y'^2}{y''}\frac{d}{dx}\left(\frac{d}{(1+y'^2)^{1/2}\,dx}\left(\frac{y''}{(1+y'^2)^{3/2}}\right)\right)\right) = -\frac{y''}{(1+y'^2)^{3/2}}\frac{d}{dx}\left(\frac{y''}{(1+y'^2)^{3/2}}\right) \qquad (3)$$

en las R, α

$$\frac{1}{R}\, d\left(\frac{1}{R}\right) = d\left(-\frac{d}{d\alpha}\left(\frac{d}{R\, d\alpha}\left(\frac{1}{R}\right)\right)\right) \qquad (4)$$

o bien

$$d\left[\frac{1/R^2}{2} + \frac{d^2}{d\alpha^2}\left(\frac{1/R^2}{2}\right)\right] = 0 \qquad (5)$$

y en las R, S

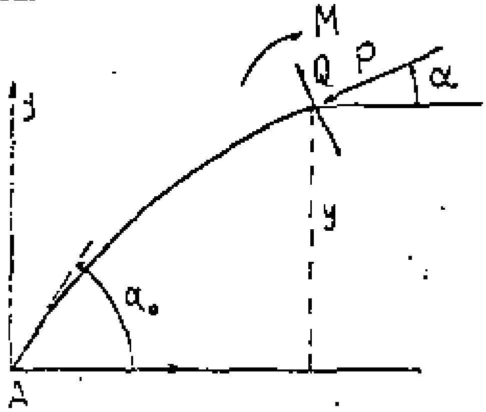
$$\frac{d}{dS}\left(R\,\frac{d^2}{dS^2}\left(\frac{1}{R}\right)\right) + \frac{1}{R}\,\frac{d}{dS}\left(\frac{1}{R}\right) = 0 \qquad (6)$$

Estas expresiones (3), (4), (5) y (6) se han obtenido en la hipótesis de E.I constantes.

Si esto no es así las anteriores se modifican ligeramente p.e. la (6) quedaría

$$\frac{d}{dS}\left(R\,\frac{d^2}{dS^2}\left(\frac{EI}{R}\right)\right) + \frac{1}{R}\,\frac{d}{dS}\left(\frac{EI}{R}\right) \qquad (7)$$

Por otra parte, si se considera la ausencia de cargas sobre la banda, EI constantes y articulación sin rozamiento en A, tiene interés usar un cuerpo libre diferente, con lo que las expresiones de las deformadas se simplifican



En este caso las ecuaciones a emplear son

$$F \cdot y = M$$

$$M = \frac{EI}{R} \qquad (8)$$

que originan las siguientes ecuaciones en dos de las cinco variables anteriormente consideradas que permiten obtener, cualquiera de ellos, la deformación estática

$$R \cdot y = \frac{EI}{F} \qquad (9)$$

$$-\frac{y''}{y(1+y'^2)^{3/2}} = \left[\frac{1}{L}\int_0^{\ell} \sqrt{1-y''/y}\; dx\right]^3 = \frac{F}{EI} = a \qquad (10)$$

$$\frac{1}{R^2} = 2a(\cos\alpha - \cos\alpha_0) \qquad (11)$$

$$y^2 = \frac{2}{a}(\cos\alpha - \cos\alpha_0) \qquad (12)$$

$$a \,\mathrm{sen}\,\alpha = \frac{d}{ds}\left(\frac{d\alpha}{ds}\right) \Leftrightarrow \left(\frac{d\alpha}{ds}\right)^2 = 2a(\cos\alpha - \cos\alpha_0) \qquad (13)$$

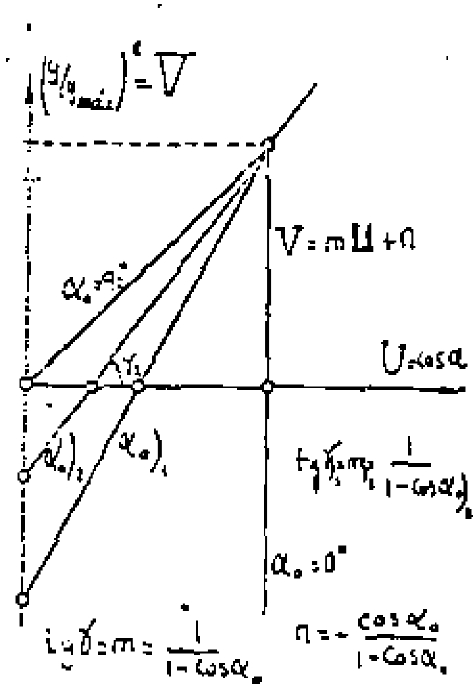
Observese de estas últimas que

a. Se han presentado solo cinco de las diez posibles expresiones
b. Que las diez, solo tres las (9), (11) y (12), pueden ser expresadas en forma finita, no diferencial, entre las variables, y las otra. Siete restantes tienen como solución integrales elípticas
c. De (9) y (12)

$$\left.\begin{aligned} R_{mín} \cdot y_{máx} &= \frac{1}{a} \\ \cos \alpha_{mín} &= 1 \end{aligned}\right\} \Rightarrow a = \frac{2(1-\cos\alpha_o)}{y_{máx}^2} \qquad (14)$$

d. De (12) y (14)

$$\left(\frac{y}{y_{máx}}\right)^2 = \frac{\cos\alpha - \cos\alpha_o}{1-\cos\alpha_o}$$



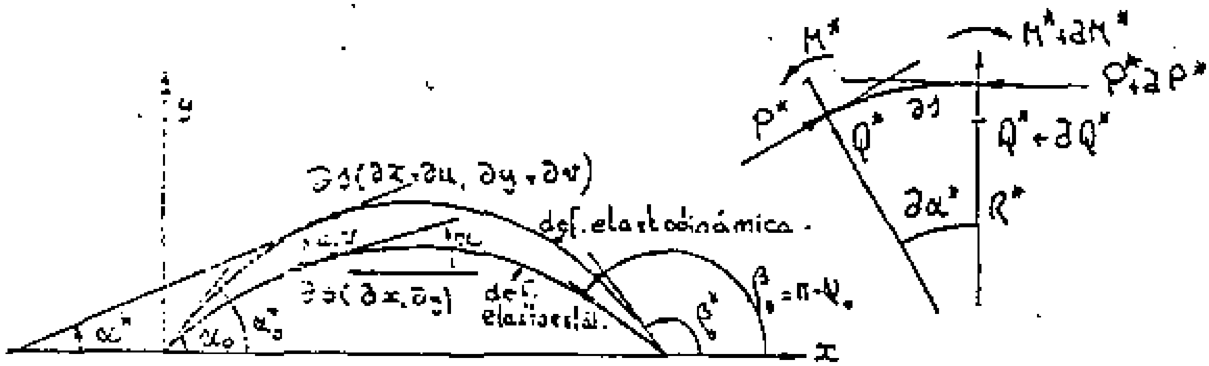
que permite obtener rápidamente las pendientes u ordenadas para cada posición inicial α_* . En particular, si se conoce Y max y α_*, se halla para cualquier y el α , y viceversa. Obsérvese que punto 1.1. del plano UV es de paso obligado para todo α_* y que V solo toma valores positivos y de valor unidad como máximo

e. La deformada elastoestática es independiente del producto E.I.. Es decir, varillas de igual L y l pero diferentes geometrías y material tienen la misma deformada

f. Evidentemente, las ecuaciones diferenciales anteriores están remitidas a las de contorno del problema

g. La deformada es simétrica

CASO ELASTODINAMICO



Las ecuaciones de equilibrio elastodinámico a considerar serían:

$$\frac{\partial M^*}{\partial s} = Q^* \qquad (16)$$

$$\left.\begin{aligned} -\partial P^* + Q^*\partial\alpha^* &= (\ddot{u}\cos\alpha^* + \ddot{v}\,\mathrm{sen}\,\alpha^*).A.\delta.\partial s \\ \partial Q^* + P^*\partial\alpha^* &= (\ddot{v}\cos\alpha^* - \ddot{u}\,\mathrm{sen}\,\alpha^*).A.\delta.\partial s \end{aligned}\right\} \qquad (17)$$

$$M^* = \frac{EI}{R^*} \qquad (18)$$

$$\varepsilon = \frac{\partial s - \partial S}{\partial S} = \frac{P^* - P}{AE} \qquad (19)$$

Observese que:

1. En las (16) se ha incluido la acción de inercia de la masa distribuida
2. Se ha omitido la existencia de
   - Cargas distribuidas
   - Inercia rotatoria

- Amortiguamiento estructural
- Efecto del cortante

3. La inclusión de los efectos anteriores es posible, asi por ejemplo: la inercia rotatoria se tendría en cuenta sustituyendo en la (16) M^* por:

$$M^* - I.A.\bar{d}\ \frac{\partial^2}{\partial t^2}\left(\text{arc. tg}\ \frac{y' + \frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} - \text{arc. tg}\ y'\right) \qquad (20)$$

el amortiguamiento estructural se tendría en cuenta sustituyendo en la (16) M^* por

$$M^* - c_d I \frac{\partial}{\partial t}\left(\frac{1}{R}\right) \qquad (21)$$

4. $$\partial \Delta = \sqrt{(1 + \partial u/\partial x)^2 + (y' + \partial v/\partial x)^2}\ \partial x \qquad (22)$$

$$\text{sen}\,\alpha^* = \left(y' + \frac{\partial v}{\partial x}\right)\left(\sqrt{(1 + \partial u/\partial x)^2 + (y' + \frac{\partial v}{\partial x})^2}\right)^{-1} \qquad (23)$$

$$\cos\alpha^* = (1 + \partial u/\partial x)\left(\sqrt{(1 + \partial u/\partial x)^2 + (y' + \partial v/\partial x)^2}\right)^{-1} \qquad (24)$$

$$R^* = \frac{\left[1 + \left(\frac{y' + \frac{\partial v}{\partial x}}{1 + \partial u/\partial x}\right)^2\right]^{\frac{3}{2}}}{-\left[\frac{\left(y'' + \frac{\partial^2 v}{\partial x^2}\right)\left(1 + \frac{\partial u}{\partial x}\right) - \left(y' + \frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)}{\left(1 + \frac{\partial u}{\partial x}\right)^3}\right]} \qquad (25)$$

5. La (19) puede tomar cualquiera de las cuatro formas siguientes según se utilice la

5.1. Hipótesis lineal

$$\varepsilon^2 \ll \varepsilon \qquad u_x^2 \ll u_x \ll 1 \qquad v_x^2 \ll v_x \ll 1$$

$$\varepsilon = \frac{u_x + y' v_x}{1 + y'^2} \qquad (26)$$

5.2. Hipótesis media

$$\varepsilon^2 \ll \varepsilon \qquad u_x^2 \ll u_x \ll 1$$

$$\varepsilon = \frac{u_x + y' v_x + v_x^2/2}{1 + y'^2} \qquad (27)$$

5.3. Hipótesis no lineal

$$\varepsilon^2 \ll \varepsilon$$

con lo que

$$\varepsilon = \frac{1}{2}\left[(1+\dot{u}_x)^2+(y'+v_x)^2\right]\Big/(1+y'^2)\ -\frac{1}{2} \qquad (28)$$

5.4. Hipótesis general

$$\varepsilon = \left[\sqrt{\frac{(1+\frac{\partial u}{\partial x})^2+(y'+v_x)^2}{1+y'^2}}\ -1\right] \qquad (29)$$

6. Las ecuaciones a considerar son CINCO: Tres relaciones de equilibrio estático del elemento diferencial, la relación momento curvatura y, la relación deformación desplazamientos, en las variables $u, v, \ell, P^*, Q^*, M^*$ Con lo que se puede obtener la deformada dinámica para cada instante t y cada punto x, y, ya que la deformada estática es conocida.

CASO ELASTOCINEMATICO

Las leyes función del tiempo para las variables x, y, P.Q.M. etc. se obtienen de (Fig. 1)

$$\ell = \sqrt{a^2+b^2-2ab\cos\theta}$$

$$\theta = f(t) \qquad \text{p.e. } \theta = \omega t$$

y en un sistema de referencia X.Y.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & \operatorname{sen}\phi \\ -\operatorname{sen}\phi & -\cos\phi \end{pmatrix}\begin{pmatrix} X - a\cos\theta \\ Y - a\operatorname{sen}\theta \end{pmatrix}$$

$$\operatorname{tg}\phi = -\frac{a\operatorname{sen}\theta}{b-a\cos\theta}$$

CASO GENERAL

Las ecuaciones a considerar son las mismas del caso elastodinámico en las condiciones del caso elastocinemático, introduciendo los valores absolutos para las aceleraciones, así, en las (17).

$$\ddot{u} \longrightarrow \ddot{u} + (-a\dot{\theta}^2\cos\theta) + (-\dot{\phi}^2)(x+u) + (-\ddot{\phi})(y+v) - 2\dot{v}\dot{\phi}$$

$$\ddot{v} \longrightarrow \ddot{v} + (-a\dot{\theta}^2\operatorname{sen}\theta) + (-\dot{\phi}^2)(y+v) + (\ddot{\phi})(x+u) + 2\dot{u}\dot{\phi}$$

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ANALISIS, SINTESIS Y OPTIMACION EN INGENIERIA MECANICA

COEFICIENTE DE SEGURIDAD Y FIABILIDAD DE SISTEMAS MECANICOS

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COEFICIENTE DE SEGURIDAD Y FIABILIDAD DE SISTEMAS MECANICOS

INTRODUCCION

El concepto de fiabilidad (reliability), expresa una medida de la capacidad de un equipo (desde un elemento al sistema más complicado) para funcionar sin fallos cuando está en servicio. En lenguaje sencillo, podríamos decir que una cosa es fiable cuando está bien hecha. Una definición de fiabilidad es: "La probabilidad de que un equipo funcione satisfactoriamente durante un período de tiempo dado, y bajo unas condiciones de funcionamiento especificadas".

Obsérvese que le fiabilidad se caracteriza por cuatro conceptos:

- es una probabilidad

  -- por ello, la herramienta matemática forma parte de la estadística aplicada, por tanto, no puede predecir sucesos discretos, sino probabilidades medias.

- realice una función satisfactoria

  -- La función exigida es un concepto que demanda ser cuidadosamente definido y depende exclusivamente del caso bajo estudio, por tanto, es un concepto subjetivo.

- en un tiempo dado

  -- Que no ha de expresarse necesariamente en horas (unidades de tiempo), sino, por ejemplo, en: No. de ciclos, No. de funciones, distancias, etc.

- condiciones de funcionamiento especificado

  -- Estas condiciones pueden ser de muy variada naturaleza, por ejemplo, temperaturas, corrosión, desgaste, tensiones de tracción, flexión, etc. estáticas o dinámicas.

Realmente, la fiabilidad no es algo que se haya inventado ahora; lo que es reciente es la sistematización de los medios y el desarrollo de técnicas para el estudio de la fiabilidad.

En el campo de la electrónica, en donde primero se desarrolló (desde 1960), la fiabilidad ha sido muy utilizada. La razón de ello es que un equipo electrónico es un conjunto de piezas elementales que realizan funciones de vida relativamente corta y sometidas a solicitaciones sencillas, en donde la fiabilidad de cada componente puede ser bien conocida y por ello, la fiabilidad del equipo puede ser bien estudiada.

En Mecánica no ocurre ésto. La Mecánica es de evolución más lenta que la electrónica, las piezas que forman los equipos pertenecen a series pequeñas, en donde las funciones que puede realizar un mismo componente son muy variadas (obsérvese, por ejemplo, la diferencia entre las funciones que puede realizar y solicitaciones que pueden afectarle, en caso de un tornillo y de una resistencia eléctrica). Las solicitaciones son más complejas en Mecánica que en Eléctrica y, como una consecuencia de la falta de normalización, no se dispone de datos estadísticos suficientes.

Otra razón de la lenta aplicación de la fiabilidad en Mecánica es que el Ingeniero Mecánico, es un hombre que le gusta la concreción, es escéptico en relación a la fiabilidad y no está dispuesto a juzgar por probabilidades si una pieza se rompe o desgasta. En Ingeniería Mecánica la experiencia juega un papel importante.

Por todo lo anterior, estamos en los balbuceos de la fiabilidad en Mecánica.

Pero ¿existe necesidad de introducir la fiabilidad en Mecánica?

Observando, por ejemplo, cualquier ecuación emanada de la resistencia de materiales, se comprueba en ella la existencia de tres ingredientes básicos: Propiedades del material, acciones (cargas), y geometría del elemento solicitado. Cualquiera de los tres ingredientes tiene naturaleza aleatoria, es decir, las propiedades del material varían según el lingote utilizado, las cargas presentan una variabilidad evidente, y las dimensiones de la pieza están sujetas a las aleatoriedades de las tolerancias. En consecuencia, cualquier concepto obtenido con estos ingredientes, por ejemplo, un coeficiente clásico de seguridad, tiene dicho carácter estadístico. Así pues, la Fiabilidad (probabilidad de que un equipo funcione, etc.) es una ampliación natural del modelo matemático que rige el fenómeno físico, aproximándolo a este fenómeno real, y en ningún caso es un concepto intrusista o de forzamiento de las condiciones que rodean al fenómeno.

En la determinación del coeficiente de seguridad clásico influye:

- Probabilidad de que un fallo pueda causar lesiones o pérdida de vidas humanas.

- Probabilidad de que el fallo sea de reparación costosa.

- La incertidumbre en el conocimiento de las cargas que se han tenido en cuenta.

- Las hipótesis hechas en el análisis, así como la determinación de los factores de concentración de esfuerzos, inducidos por fatiga e impacto.

- El conocimiento de las condiciones ambientales.

- El conocimiento de las tensiones residuales inducidas en el montaje y conformación.

- Influencia de la corrosión.

Valores típicos de n oscilan desde

$1{,}25 \div 1{,}50$ a $3 \div 4$

Analicemos con más detalle el coeficiente clásico (o determinista) de seguridad. De todas las fases del proyecto de máquinas (sistemas mecánicos) la elección del grado de seguridad ha sido la más importante que compete al proyectista, ya que de esta decisión dependen la economía del proyecto, riesgo de accidentes irreparables, etc. Este tradicional método de diseñar, involucra: 1o. un posible modo de fallo (teoría de rotura, de falla, etc.) 2o. un posible valor de la acción aplicada y 3o. un valor representativo de una relevante propiedad del material (por ejemplo, energía de distorsión, resistencia última, etc.), la cual gobierna la resistencia del elemento bajo la carga aplicada y con el modo de fallo supuesto.

Una expresión para tal factor de seguridad puede ser

$$n = \frac{\text{Tensión de rotura (o de fluencia)}}{\text{Tensión aplicada}} \quad \text{con } n > 1$$

Aunque con estos procedimientos los resultados hasta la fecha no han sido necesariamente malos (quizá un cierto sobredimensionado). Los hechos siguientes:

a. La variabilidad aleatoria de las propiedades del mateiral, carga aplicada, y dimensiones, es algo cierto. Estas variaciones se han intentado resolver introduciendo un coeficiente de seguridad mínimo dado por

$$n = \frac{S - \Delta S}{\sigma + \Delta\sigma} \quad \frac{\text{Tensión de rotura mínima}}{\text{Tensión aplicada máxima}}$$

evidentemente

$$S - \Delta S \geq \sigma + \Delta\sigma$$

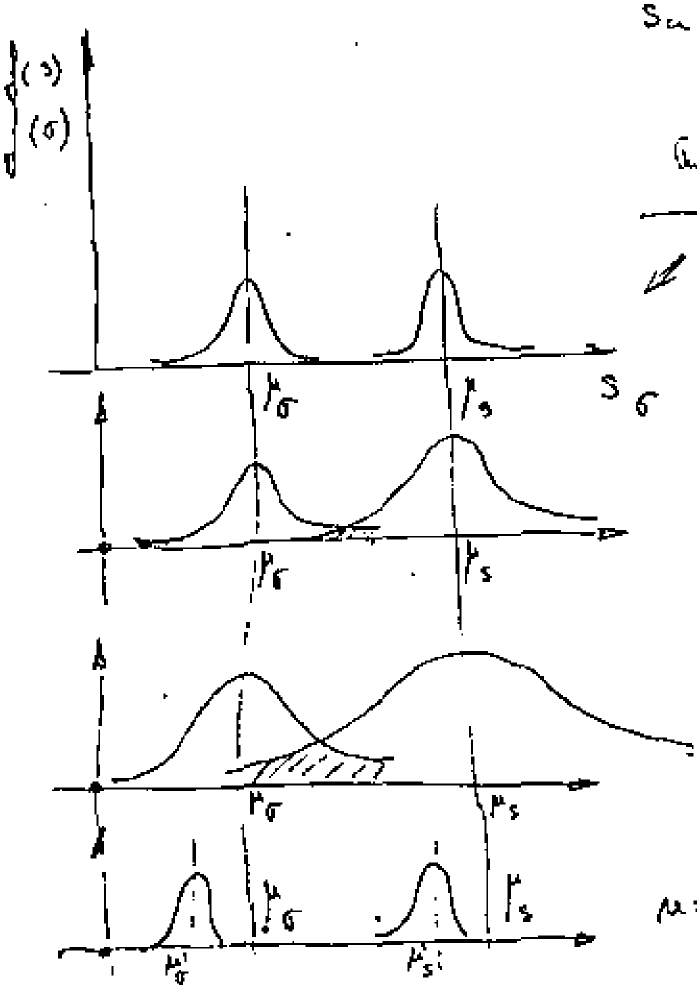
y el minimum minimorum.

$$S - \Delta S = \sigma + \Delta\sigma \implies \frac{S}{\sigma} = \frac{1 + \frac{\Delta\sigma}{\sigma}}{1 - \frac{\Delta S}{S}}$$

Otro coeficiente de seguridad usado es el cociente de los valores medios de las respectivas distribuciones de resistencia y carga

$$n = \frac{\mu_S}{\mu_\sigma} \quad \text{(coeficiente central de seguridad)}$$

evidentemente, en este caso, manteniendo constante n los resultados pueden ser muy diferentes para las distribuciones de tensión y resistencias que se indican



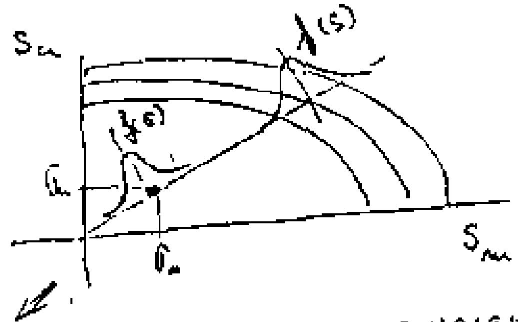


DIAGRAMA DE HAIGH

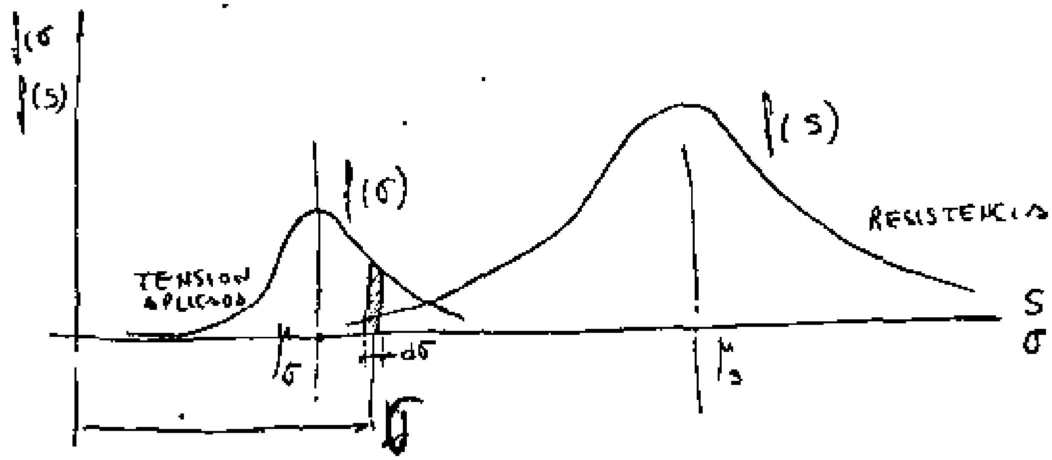
$$\mu = \frac{\mu'_s}{\mu'_\sigma} = \frac{K\mu_s}{K\mu_\sigma}$$

La fiabilidad corresponde al caso de que $S > 0$

es decir

$$R = Prob(m > 1) \Leftrightarrow Prob(S > \sigma) \Leftrightarrow Prob(S - \sigma > 0)$$

El valor de R depende del grado de solapamiento de las distribuciones y se halla:



La probabilidad de que una tensión se encuentre en el área rayada

$$Prob\left(\sigma - \frac{d\sigma}{2} \leq \sigma \leq \sigma + \frac{d\sigma}{2}\right) = f(\sigma)\, d\sigma$$

Las probabilidades de que la resistencia S sea mayor que σ_1 es

$$Prob(S > \sigma_1) = \int_{\sigma}^{\infty} f(s)\, ds$$

La probabilidad de tener una tensión σ en el intervalo $d\sigma$ y que la resistencia S sea mayor que σ es el producto de las dos probabilidades, y esto es por tanto la fiabilidad relativa a la posibilidad de una tensión σ

$$dR = f(\sigma)\, d\sigma \int_{\sigma}^{\infty} f(s)\, ds$$

La fiabilidad es la probabilidad de que la resistencia S sea mayor a todos los posibles valores de σ es decir:

$$R = \int_{-\infty}^{+\infty} f(\sigma) \left(\int_{\sigma}^{\infty} f(s)\, ds \right) d\sigma$$

Se puede razonar de la misma forma considerando una resistencia S y la probabilidad de que la tensión σ sea inferior. En este caso

$$R = \int^{\infty} f(s) \left(\int^{S} f(\sigma)\, d\sigma \right) ds$$

b. Diseños con relativamente grandes coeficientes de seguridad, a veces fallan en servicios, indicando esto que la incertidumbre, asociada con las entradas de diseño no puede ser ignorada.

A menudo se piensa que un factor de seguridad mayor que una cantidad no origina fallos. Realmente, con altos factores de seguridad la probabilidad de fallo puede variar de un valor satisfactorio a uno indeseable. Por ejemplo: Un coeficiente de seguridad de uno, según el esquema clásico, indicaría que el fallo ocurre en un 100% de los casos, porque no hay seguridad, en cambio, si las distribuciones son normales el fallo ocurre en un 50% de los casos.

c. El método tradicional es incapaz de predecir el riesgo implicado por el coeficiente de seguridad, o la fracción de fallos de los elementos en servicio de una máquina.

Estos hechos, como decimos, han originado una insatisfacción con el uso de este coeficiente clásico de seguridad.

La teoría de la fiabilidad permite abordar problemas tales como:

a. Obtener la fiabilidad de un componente o de un sistema, es decir

$$R(t) = P(t<T) = \text{probabilidad de no fallo en } t<T$$

b. Mejorar la fiabilidad de un equipo o sistema

- Reduciendo la complejidad del sistema al mínimo necesario, eliminando los componentes complejos e innecesarios y las configuraciones complejas que aumentan la probabilidad de que el sistema falle.
- Aumentando la fiabilidad de los componentes en el sistema.
- Por redundancia en caliente (paralelo), o en frío
- Mantenimiento preventivo y de reparación

c. Diseño para una fiabilidad dada (síntesis de fiabilidad). Maximizar la fiabilidad de un sistema para un peso tamaño o coste dado, o inversamente, para una fiabilidad dada obtener un peso mínimo, etc.

2. ALGUNAS BASES ESTADISTICAS DE LA FIABILIDAD

Si f(t) es la función de distribución de (densidad de probabilidad) de la probabilidad de fallo en un tiempo t (o mortalidad del componente)

$$R(t) = 1 - F(t) = 1 - \int_0^t f(t)\, dt \Rightarrow R'(t) = -f(t)$$

Evidentemente $\begin{matrix} R(0) = 1 \\ R(\infty) = 0 \end{matrix} \Rightarrow R(t)$ es una función monótona decreciente. y siendo $F_{(t)}$ la infiabilidad a una edad inferior a t

Tasa porcentaje de fallo instantáneo (relación de azar, probabilidad instantánea de fallo por componente)

Sean $N_{(t)}$ el número de dispositivos que sobreviven a un tiempo t (o que funcionan de forma satisfactoria) de N_0

$\frac{N(t)}{N_0}$ es un estimador de la fiabilidad de los dispositivos en el tiempo t. El límite de $\frac{N(t)}{N_0}$ cuando $N_0 \to \infty$ representa la probabilidad de sobrevivir en el instante t y por ello la fiabilidad de los dispositivos. En el intervalo dt, dn dispositivos habrán fallado. Se define la tasa de fallo o probabilidad instantánea de fallo por componente como el número de fallas en intervalo por unidad de tiempo y unidad de componente

$$\lambda(t) = \frac{-dN}{dt \cdot N}$$

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{N(t) - N(t+\Delta t)}{N(t) \cdot \Delta t} = \lim_{\Delta t \to 0} \frac{\frac{N(t) - N(t+\Delta t)}{N(0)}}{\frac{N(t)}{N(0)} \Delta t} = \frac{dR(t)}{R(t) \cdot dt} = \quad (1)$$

$$= \frac{R'(t)}{R(t)} = \frac{f(t)}{R(t)}$$

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{\Delta N(t)}{N(t)\,\Delta t}$$

y representa el coeficiente medio de extinción del colectivo expresado en tanto por uno.

Por integración de (1)

$$R(t) = e^{-\int \lambda(t)\,dt}$$

si se expresa en el intervalo t_1, t_2 se tiene

$$R(t_2, t_1) = \frac{R(t_2)}{R(t_1)} = e^{-\int_{t_1}^{t_2} \lambda(t)\,dt}$$

$$P(t_2/t_1) = \frac{N_2}{N_1} = \frac{N_2/N_0}{N_1/N_0}$$

Si $\lambda(t)$ es constante

$$R(t_1\ t_2) = e^{-\lambda(t_2 - t_1)}$$

$$R(t) = e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$f(t) = \lambda e^{-\lambda t}$$

Se llama tanto medio de fallo λt correspondiente a un periodo T.

$$\lambda t = \frac{1}{T} \int_0^T \lambda(t)\,dt.$$

Se llama tiempo medio para fallos (tiempo medio para el 1er. fallo).

$$\overline{T} = MTTF = \int_0^\infty t\, f(t)\, dt = -\int_0^\infty t\, \frac{dR(t)}{dt}\, dt = \int_0^\infty R(t)\, dt.$$

Se llama tiempo medio entre fallos.(Tiempo medio entre fallos consecutivos)

$$\overline{\overline{T}} = \int_0^\infty R(t)\, dt.$$

el MTBF se aplica a sistemas con n suficientes elevados

el MTTF se aplica a componentes

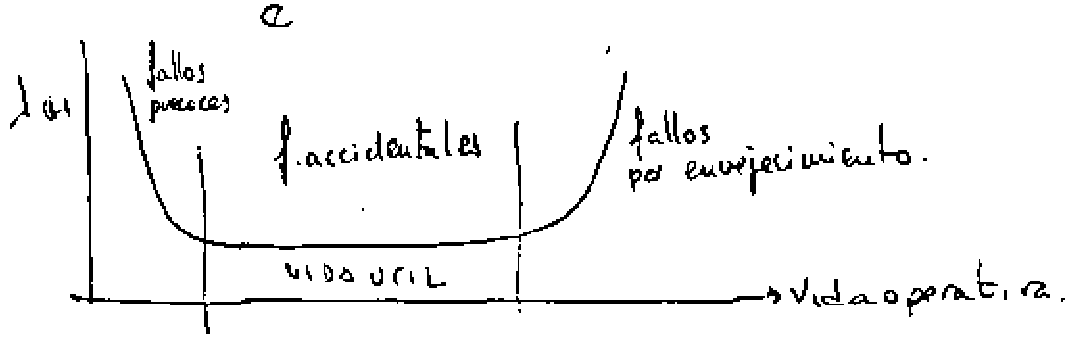
La relación entre ellos es

$$\frac{1}{MTBF} = \sum_{j=1}^{n} \frac{1}{\overline{T}_j}$$

Si el sistema opera con todos los componentes nuevos $\overline{T}$ y $\overline{\overline{T}}$ son idénticos.

MODOS DE FALLO

En Mecánica, como en Electrónica, se distinguen tres categorías de fallo, que aparecen en diferentes fases de la edad del dospositivo. Estas son: período de fallos precoces, período de fallos accidentales y período de fallos por envejecimiento.



Los fallos precoces son los que se producen en un período inicial de funcionamiento. Son debidos a fallos en unidades que pasaron indebidamente el control de calidad, o a defectos de fabricación. Este tipo de fallos hay que reducirlos al mínimo a través de END o del rodaje.

Los fallos accidentales o (catastróficos) son los que se producen aleatoriamente en cualquier momento del intevalo, sobrevienen de modo inesperado, motivados generalmente por un aumento violento de las tensiones o esfuerzos que actúan sobre las unidades. Las acciones "fuera de misión no se consideran", por todo ello, se puede suponer que

los fallos por envejecimiento son los debidos a las pérdidas de aptitud de las unidades por el uso, por ejemplo, desgastes, fatigas, modificaciones de estructura interna, etc. Puede afirmarse que el envejecimiento es una enfermedad que gravita sobre todas las unidades componentes o sistemas en funcionamiento.

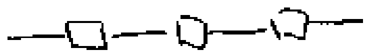
La tasa de fallos se obtiene como proposición de ambos tipos de fallos.

Los fallos por envejecimiento, aparecen en Mecánica más tarde, aunque más rápidamente que en la Electrónica. Si los END son eficaces para eliminar los fallos precoces y si las condiciones de funcionamiento son normales y no existen sobretensiones imprevistas, se puede esperar una tasa de fallos nulos, hasta la aparición de las primeras manifestaciones de envejecimiento.

3. FIABILIDAD DE SISTEMAS

SERIE

Sean n componentes independientes



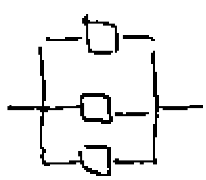
"El sistema falla si falla uno al menos" $\Rightarrow$ que el sistema es fiable si todos son fiables al mismo tiempo.

$$R_{ss} = \prod R_i = e^{-\int_0^t \Sigma\, \lambda_i(t)\, dt} \qquad \text{siendo} \left\{ \Sigma \lambda_i(t) = \lambda_{ss} \right.$$

Obsérvese la analogía entre esta ecuación y la expresión "Una cadena resiste lo que el eslabón más débil".

PARALELO

"El sistema funciona cuando uno al menos funciona"

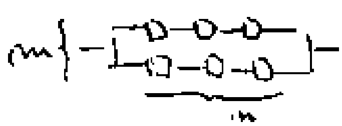


$$F_{sp} = \prod F_i(t) \Rightarrow$$

$$R_{sp} = 1 - F_{sp}(t) = 1 - \prod_i \left(1 - R_i(t)\right)$$

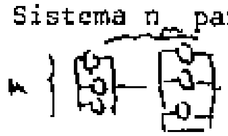
COMBINADOS SERIES PARALELO

- Sistema m series en paralelo



$$R = 1 - (1 - R^n)^m$$

- Sistema n paralelos en serie

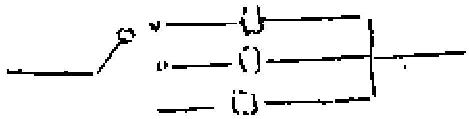


$$R = \left(1 - (1-R)^m\right)^n$$

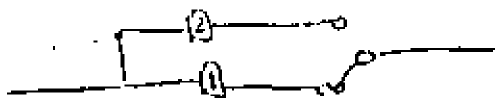
- mixtos serie paralelos. Se reducen a los anteriores

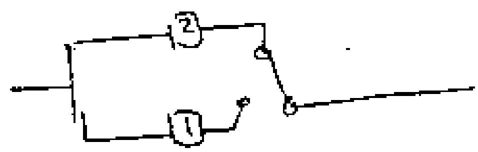
SISTEMAS Standby (fríos)

Estas unidades no funcionan hasta que ellas son llamadas a operar, en contraste con el sistema en paralelo en que todos operan simultáneamente.



Caso de dos





La fiabilidad R_{sb} del sistema es igual =

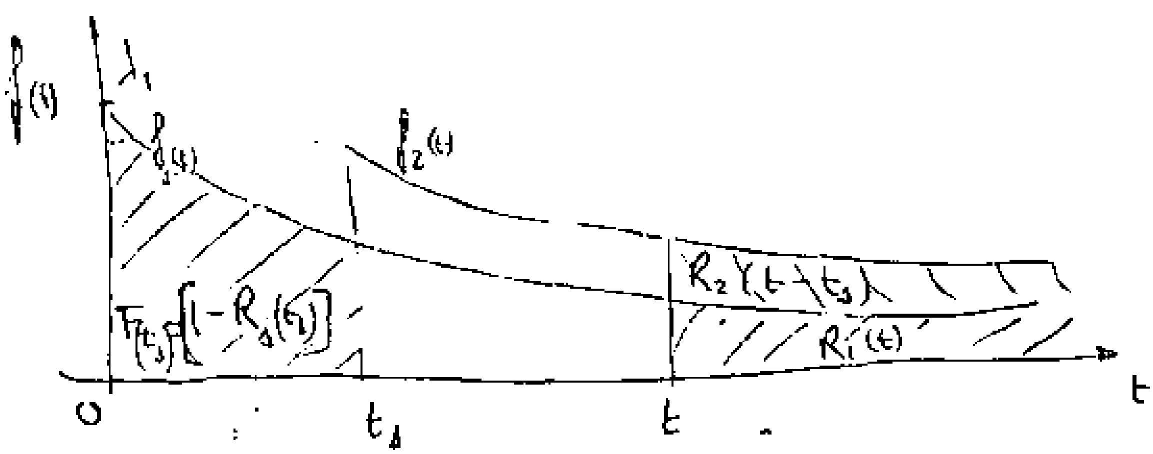
- El equipo (1) funcione hasta el tiempo t
- El equipo (1) falle a partir de t_1 y que (2) no falle a partir de t_1 hasta t.

(Suponiendo un 100% de habilidad en el sensor y contacto

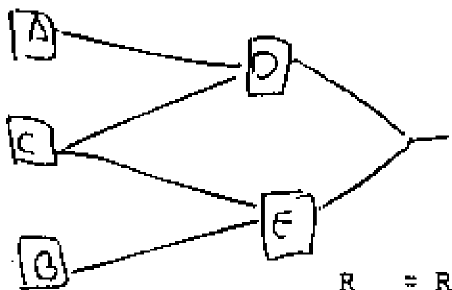
$$R_{sb} = R_1(t) + 1-R_1(t_1) \quad .R_2\ (t-t_1)$$

Suponiendo λ_i = constante

$$R_{sb} = e^{-\lambda_1 t} + \frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \left(e^{-(\lambda_1 - \lambda_2) t} \right)$$



SISTEMAS MIXTOS

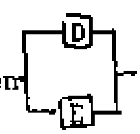


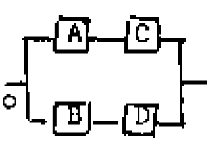
A y B no pueden considerarse en paralelo con D y E ya que, no participan en ambos.

Esto se resuelve a partir de la probabilidad condicional (teorema de Bayes)

$$R_{sm} = R_{s/c\ bien} \cdot R_{c\ bien} + R_{s/c\ malo} \cdot R_{c\ malo}$$

La probabilidad de un suceso (fiabilidad del sistema mixto) es igual a la probabilidad de funcionamiento supuesto que el componente C funciona por la probabilidad de que este componente funcione más la probabilidad del sistema supuesto que C no funcione por la infabilidad de éste.

$R_{s/c\ bien}$ 

$R_{s/c\ malo}$ 

$$R_{s/bien} = R_d \cdot R_e - R_e R_d$$

$$R_{s/c\ malo} = R_a R_c + R_b R_d - R_a R_c R_b R_d$$

luego

$$R_{sm} = \left(R_d R_e - R_e R_d\right) R_c + \left(R_a R_c + R_b R_d - R_a R_c R_b R_d\right)\left(1 - R_c\right)$$

SISTEMAS MULTIMODOS

- Hasta ahora se ha supuesto que los sistemas funcionan o no funcionan. Sin embargo, algunos componentes pueden funcionar de muchas maneras

La fiabilidad de un sistema de esta naturaleza puede determinarse por:

a) Calculat todas las posibles permutaciones de los M_i modos de funcionamiento para los N_i componentes

$$\text{Permutación} = \Sigma M_i^{N_i} = M_1^{N_1} \cdot M_2^{N_2} \cdot$$

b) Hallar (escribir) (por un procedimiento sistemático) todas las permutaciones.

c) Encontrar aquellos modos que hagan que el sistema funcione, es decir, se desechan los incompatibles con el funcionamiento.

d) Se calculan las probabilidades de todos los modos posibles.

e) La fiabilidad total es la suma de las fiabilidades.

4. APLICACIONES DE LA FIABILIDAD

4.1 Hallar el coeficiente de seguridad (fiabilidad) de dos distribuciones de resistencia y de tensión

a. Distribuciones cualesquiera

R = Probabilidad ($S > \sigma$) $\Leftrightarrow$ Prob. ($S - \sigma > 0$) ($\frac{S}{\sigma} > 1$)

Solución: 1 Por aplicación de las fórmulas del apartado (1)

2. Por palicación de las transformadas de MELLIN

$$\left.\begin{aligned} F &= \int_\sigma^\infty f(s)\,ds \\ G &= \int_\sigma^\infty f(\sigma)\,d\sigma \end{aligned}\right\} \Rightarrow \int_{-\infty}^{+\infty} f(\sigma) \left(\int_\sigma^\infty f(s)\,ds \right) d\sigma = \int_0^1 F\,dG$$

3. Método de Montecarlo

b. Distribuciones normales

Aplicando el álgebra de variables aleatorias independientes se tiene

dist. normales

Adición (x + y) $\mu_{x+y} = \mu_x + \mu_y \; ; \; \sigma_{x+y} = (\sigma_x^2 + \sigma_y^2)^{1/2}$

Resta (x - y) $\mu_{x-y} = \mu_x - \mu_y \; ; \; \sigma_{x-y} = (\sigma_x^2 + \sigma_y^2)^{1/2}$

Multiplic. x,y $\mu_{x \cdot y} = \mu_x \cdot \mu_y \; ; \; \sigma_{xy} = (\mu_x^2 \sigma_y^2 + \mu_y^2 \sigma_x^2 + \sigma_x^2 \sigma_y^2)^{1/2}$

División $\frac{x}{y}$ $\mu_{x/y} = \frac{\mu_x}{\mu_y} \; ; \; \sigma_{x/y} = \frac{1}{\mu_y} \left(\frac{\mu_x^2 \sigma_y^2 + \mu_y^2 \sigma_x^2}{\mu_y^2 + \sigma_y^2} \right)^{1/2}$

Cuadrado x^2 $\mu_{x^2} = \mu_x^2 + \sigma_x^2 \qquad \sigma_{x^2} = (4 \mu_x^2 \sigma_x^2 + 2 \sigma_x^4)^{1/2}$

Raíz cuadrada $\mu_{\sqrt{x}} = \left[\frac{1}{2} (4 \mu_x^2 - 2 \sigma_x^2 \right]^{1/2}$

$$\sigma_{\sqrt{x}} = \left[\mu_x - \frac{1}{2} (4 \mu_x^2 - 2 \sigma_x^2)^{1/2} \right]^{1/2}$$

$X = S - \sigma$

La media es

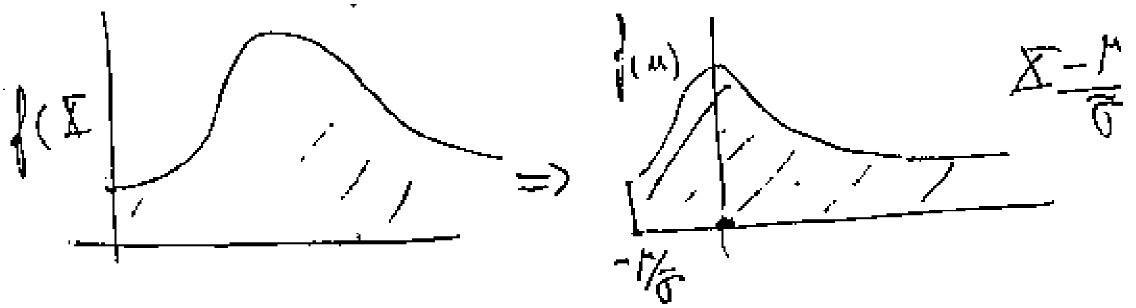
$$\mu = \mu_s - \mu_\sigma$$

La derivación típica es

$$\bar{\sigma} = \sqrt{\bar{\sigma}_S^2 - \bar{\sigma}_\sigma^2}$$

de aquí

$$R = \text{Prob}(\bar{X} > 0) = \text{Prob}\left(\frac{\bar{X} - \mu}{\bar{\sigma}} = u > -\frac{\mu}{\bar{\sigma}} = -\frac{\mu_s - \mu_\sigma}{\sqrt{\bar{\sigma}_s + \bar{\sigma}_\sigma^2}} \right)$$



Si se fija una fiabilidad, por ejemplo 0.99865 $\Rightarrow$ que $\mu = -\frac{\mu}{\bar{\sigma}} = -3.$

si se desea mayor fiabilidad entonces

$$\frac{\mu}{\sigma} = \frac{\mu_s - \mu_\sigma}{\sqrt{\tilde{\sigma}_s^2 + \tilde{\sigma}_\sigma^2}} > 3.$$ ecuación de síntesis de fiabilidad

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