centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

V CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

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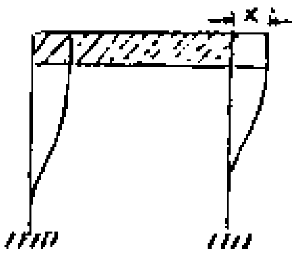
DINAMICA ESTRUCTURAL

DR. OCTAVIO A RASCON CH.

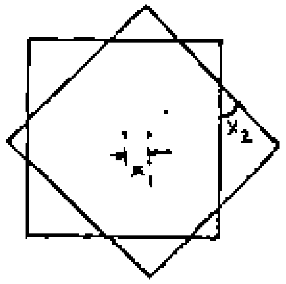
DEFINICION.

GRADOS DE LIBERTAD = NUMERO DE COORDENADAS GENERALIZADAS (DESPLAZAMIENTOS O GIROS) QUE SE REQUIEREN PARA DEFINIR LA POSICION DEL SISTEMA EN CUALQUIER INSTANTE.

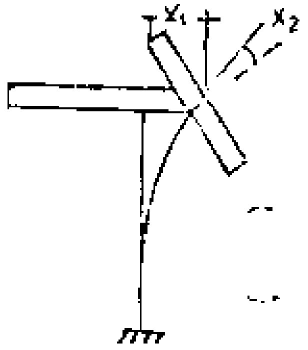
EJEMPLOS



UN GRADO DE
LIBERTAD



DOS GRADOS DE
LIBERTAD



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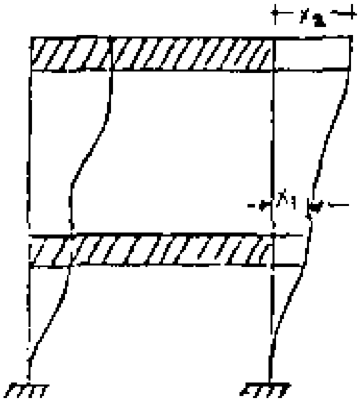

DOS GRADOS DE
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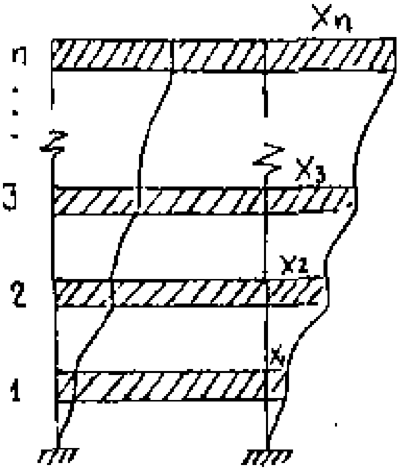

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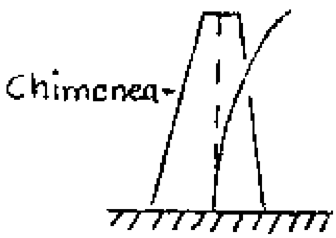
LIBERTAD

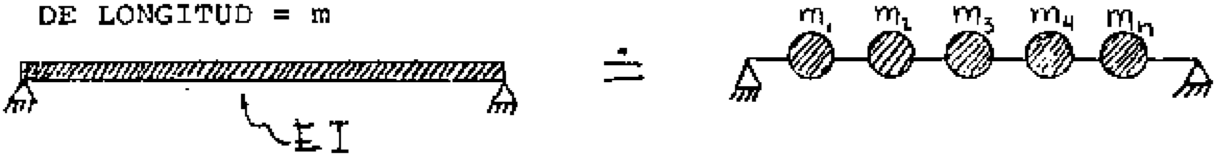


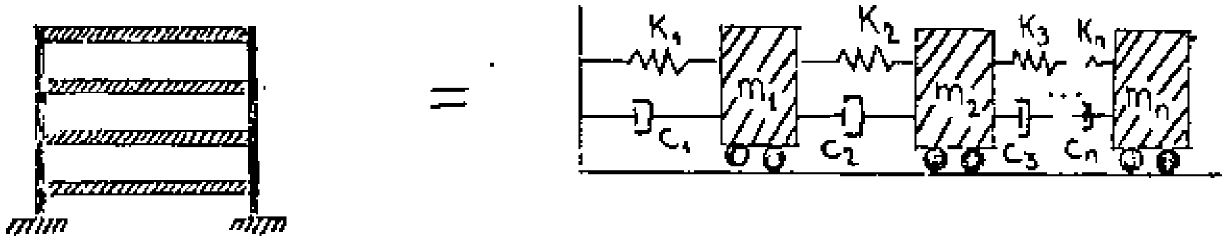
INFINITO NUMERO DE
GRADOS DE LIBERTAD

METODOS DE DISCRETIZACION DE SISTEMAS CONTINUOS

1. POR CONCENTRACION DE MASAS

MASA POR UNIDAD
DE LONGITUD = m



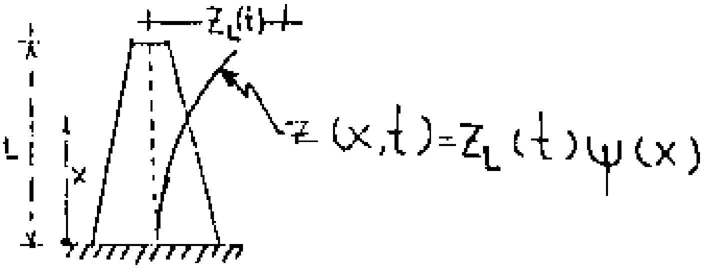


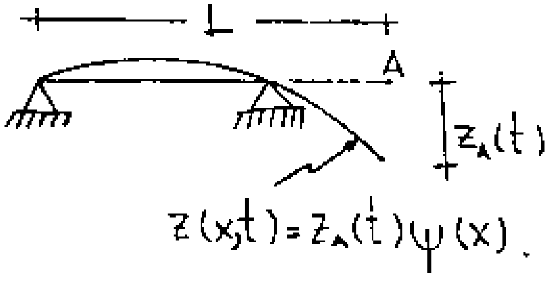
2. EXPRESANDO LA CONFIGURACION DE VIBRACION DE LA ESTRUCTURA COMO UNA SERIE DE FUNCIONES ESPECIFICADAS. POR EJEMPLO, SI ESTAS FUNCIONES SON ARMONICAS:

$$Z(x,t) = \sum_{i=1}^{N} b_i \text{ sen } \frac{i\pi x}{L}$$

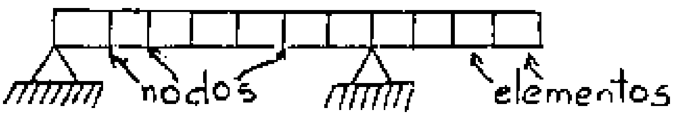
EN GENERAL, PARA CUALQUIER TIPO DE FUNCION $\psi(x)$:

$$z(x,t) = \sum_{i=1}^{N} z_i(t)\psi_i(x)$$





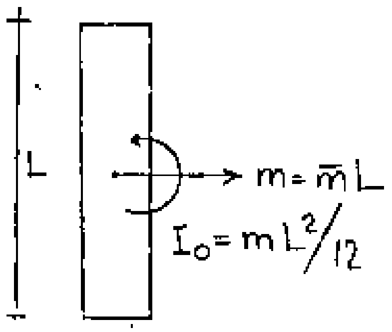
3. MEDIANTE ELEMENTOS FINITOS



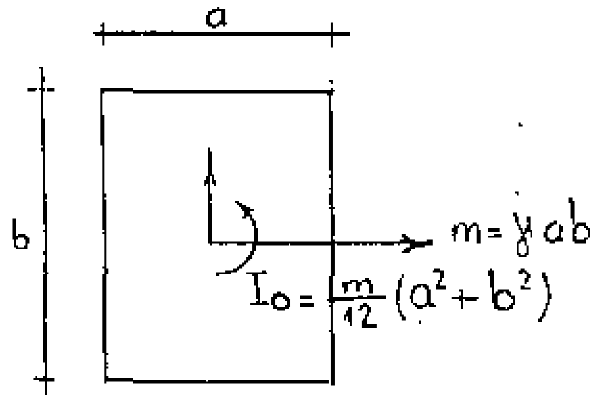
AL PLANTEAR LAS ECUACIONES DE EQUILIBRIO DE CUERPOS RIGIDOS ES A MENUDO NECESARIO CONOCER LOS MOMENTOS DE INERCIA DE MASA. A CONTINUACION SE PRESENTAN ALGUNOS CASOS:

$\bar{m}$ = MASA POR UNIDAD DE LONGITUD

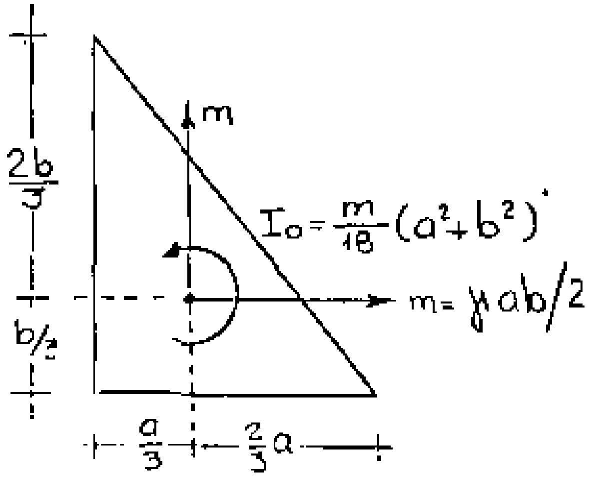
γ = MASA POR UNIDAD DE AREA



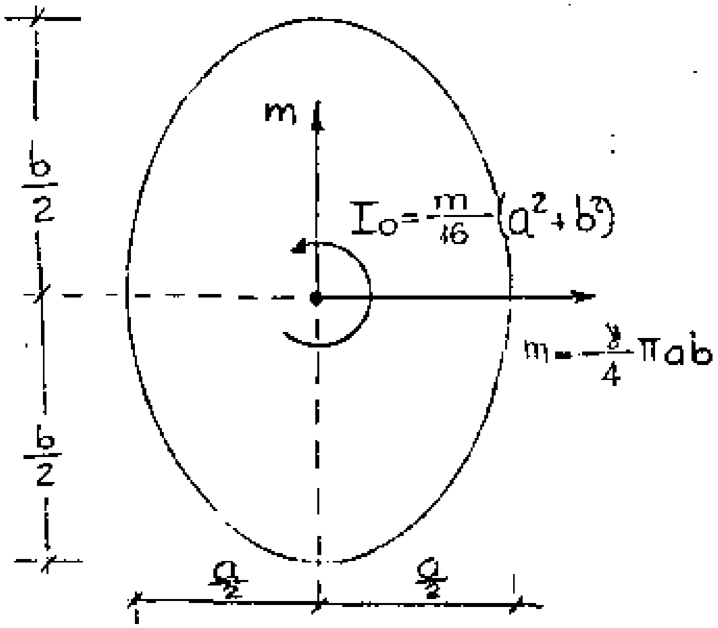
BARRA UNIFORME



PLACA UNIFORME RECTANGULAR

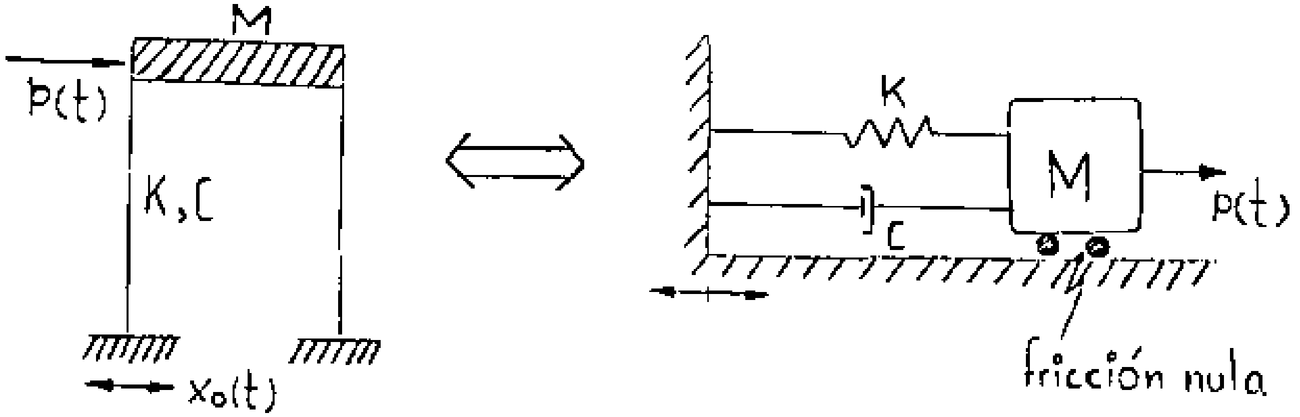


PLACA UNIFORME TRIANGULAR



PLACA UNIFORME ELIPTICA

RESPUESTA DINAMICA DE SISTEMAS ELASTICOS LINEALES DE UN GRADO DE LIBERTAD CON AMORTIGUAMIENTO VISCOSO



t = TIEMPO

M = MASA

K = RIGIDEZ

C = AMORTIGUAMIENTO

f(t) = FUERZA EXTERNA

$X_o(t)$ = DESPLAZAMIENTO DEL SUELO

EL AMORTIGUAMIENTO VISCOSO ES TAL QUE PRODUCE UNA FUERZA DE RESTAURACION PROPORCIONAL A LA VELOCIDAD RELATIVA DE LA MASA RESPECTO AL SUELO.

EL AMORTIGUAMIENTO SE DEBE PRINCIPALMENTE A LA FRICCION INTERNA ENTRE LOS GRANOS O PARTICULAS DEL MATERIAL DE LA ESTRUCTURA, Y A FRICCION EN LAS JUNTAS Y CONEXIONES DE LA MISMA. ES EL ELEMENTO DEL SISTEMA QUE DISCIPA ENERGIA.

2a. LEY DE NEWTON:

"LA RAPIDEZ DE CAMBIO DEL MOMENTUM DE CUALQUIER MASA, m, ES IGUAL A LA FUERZA QUE ACTUA SOBRE ELLA"

$$p(t) = \frac{d}{dt}\left(m\frac{dx}{dt}\right) = \frac{d}{dt}(m\dot{x})$$

$p(t)$ = FUERZA ACTUANTE

x = DESPLAZAMIENTO

t = TIEMPO

SI m ES CONSTANTE: $p(t) = m\ddot{x}$

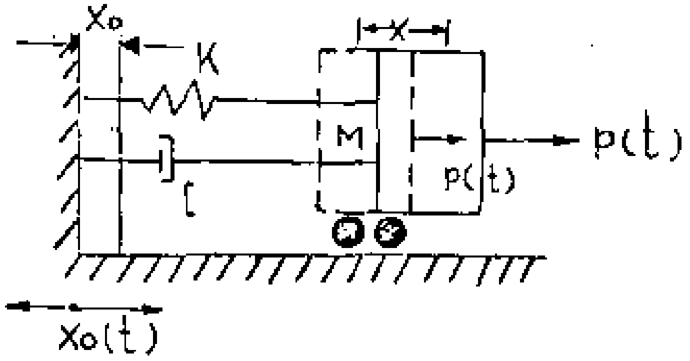
PRINCIPIO DE D'ALAMBERT

SI LA 2a. LEY DE NEWTON LA ESCRIBIMOS COMO

$$p(t) - m\ddot{x} = 0$$

AL SEGUNDO TERMINO DE LA ECUACION SE LE CONOCE COMO *FUERZA DE INERCIA*; EL CONCEPTO DE QUE UNA MASA DESARROLLA UNA FUERZA DE INERCIA PROPORCIONAL A SU ACELERACION Y QUE SE OPONE A ELLA SE CONOCE COMO PRINCIPIO DE D'ALAMBERT, Y PERMITE QUE LAS ECUACIONES DE MOVIMIENTO SE EXPRESEN COMO ECUACIONES DE *EQUILIBRIO DINAMICO*.

ECUACION DE EQUILIBRIO



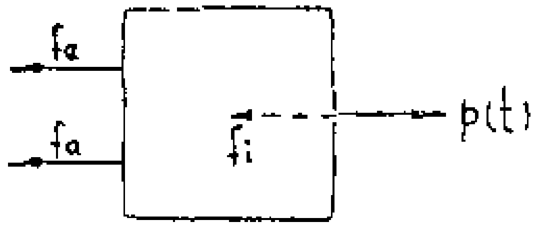


DIAGRAMA DE CUERPO LIBRE

EQUILIBRIO: $f_e + f_a + f_i = p(t)$ (1)

PARA UN SISTEMA ELASTICO: $f_e = K(x - x_o) = ky$

PARA AMORTIGUAMIENTO VISCOSO: $f_a = c(\dot{x} - \dot{x}_o) = C\dot{y}$ (2)

POR EL PRINCIPIO DE D'ALAMBERT: $f_i = m\ddot{x} = m(\ddot{y} + \ddot{x}_o)$

SUSTITUYENDO LAS ECS. 2 EN LA EC. 1 SE OBTIENE:

$$m(\ddot{y} + \ddot{x}_o) + c\dot{y} + k\dot{y} = p(t)$$

DE DONDE

$$\boxed{M\ddot{y} + c\dot{y} + Ky = p(t) - M\ddot{x}_o} \qquad (3)$$

DIVIDIENDO ENTRE M AMBOS MIEMBROS DE LA EC. 3:

$$\ddot{y} + \frac{C}{M} y + \frac{K}{M} y = \frac{p(t)}{M} - \ddot{x}_o$$

SI $\frac{C}{M} = 2h$, y $\frac{K}{M} = \omega^2$, DONDE ω = FRECUENCIA CIRCULAR NATURAL, EN RAD/SEG:

$$\boxed{\ddot{y} + 2h\,\dot{y} + \omega^2 y = \frac{p(t)}{M} - \ddot{x}_o} \qquad (4)$$

CUANDO SE TIENEN EXCITACIONES EN EL SISTEMA SE TRATA DE UN PROBLEMA DE VIBRACIONES FORZADAS; EN CASO CONTRARIO EL PROBLEMA ES DE VIBRACIONES LIBRES.

VIBRACIONES LIBRES

EN ESTE CASO LA ECUACION DIFERENCIAL DE EQUILIBRIO RESULTA SER

$$\ddot{y} + 2h\,\dot{y} + \omega^2 y = 0$$

CUYA SOLUCION ES

$$y(t) = e^{-ht}(C_1 \text{ sen } \omega' t + C_2 \cos \omega' t) \qquad (5)$$

DONDE $\omega' = \sqrt{\omega^2 - h^2}$ = FRECUENCIA CIRCULAR NATURAL AMORTIGUADA

Y C_1 Y C_2 SON CONSTANTES QUE DEPENDEN DE LAS CONDICIONES INICIALES

(EN t=0) DE DESPLAZAMIENTO Y VELOCIDAD QUE TENGA LA MASA DEL SISTEMA.

ESTAS RESULTAN SER

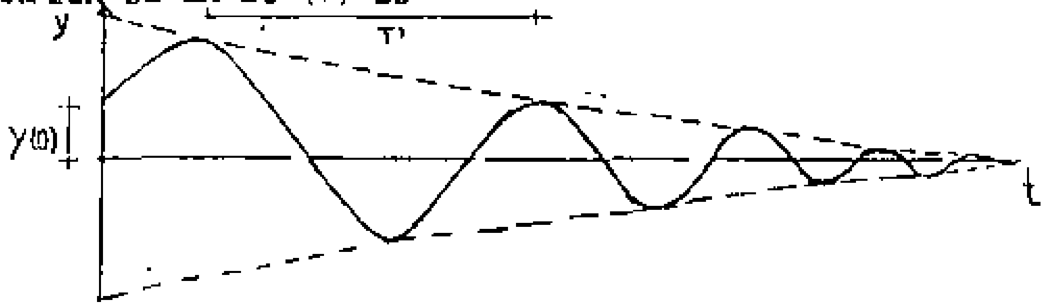
$$\boxed{C_1 = \frac{\dot{y}(0) + hy(0)}{\omega'}} \quad Y \quad \boxed{C_2 = y(0)} \qquad (6)$$

LA EC (5) SE PUEDE ESCRIBIR TAMBIEN COMO:

$$y(t) = Ae^{-ht} \cos(\omega' t - \theta) \qquad (7)$$

DONDE $A = \sqrt{C_1^2 + C_2^2}$ Y $\theta = \tan^{-1} \frac{C_1}{C_2}$ = ANGULO DE FASE

LA GRAFICA DE LA EC (7) ES



$T' = \frac{2\pi}{\omega'}$ = PERIODO NATURAL AMORTIGUADO, SEG

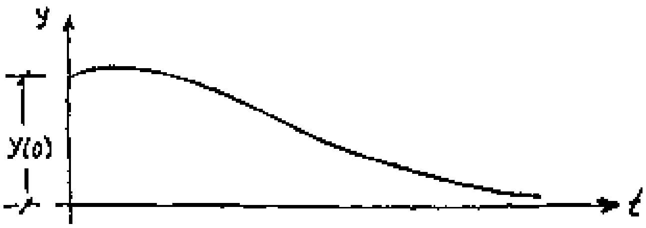
$f' = \frac{1}{T'}$ = FRECUENCIA NATURAL AMORTIGUADA, cps

VEAMOS EL CASO ESPECIAL DE LA EC. (5) EN QUE $h \to \omega$. EN TAL CASO, $\omega' = \sqrt{\omega^2 - h^2} \to 0$, cos $\omega' t \to 1$ Y sen $\omega' t \to \omega' t$, CON LO CUAL LA EC. (5) SE REDUCE A

$$y(t) = e^{-\omega t} ([(\dot{y}(0) + hy(0))/\omega'](\omega' t) + y(0))$$

$$= e^{-\omega t}[\dot{y}(0)t + (1 + \omega t)y(0)]$$

LA GRAFICA DE ESTA ECUACION ES



Y OBVIAMENTE NO REPRESENTA UN MOVIMIENTO OSCILATORIO, POR LO CUAL

SI $h = \omega$ SE DICE QUE SE TIENE AMORTIGUAMIENTO CRITICO. EN TAL CASO:

$$h_{cr} = \omega = \frac{C_{cr}}{2M} = \sqrt{\frac{K}{M}}$$

DE DONDE $C_{cr} = 2\sqrt{KM}$. (8)

A LA RELACION $\zeta = C/C_{cr}$ SE LE LLAMA FRACCION DEL AMORTIGUAMIENTO CRITICO.

DESPEJANDO A M DE LA EC. (8) Y SUSTITUYENDOLA EN LA EC. $h = C/(2M)$ SE OBTIENE:

$$h = \frac{C}{2\,\frac{C_{cr}^2}{4K}} = \frac{C}{C_{cr}}\;\frac{2K}{2\sqrt{KM}} = \zeta\sqrt{\frac{K}{M}} = \zeta\omega$$

ADEMAS:

$$\omega' = \sqrt{\omega^2 - h^2} = \sqrt{\omega^2 - \omega^2\zeta^2} = \omega\sqrt{1-\zeta^2}$$

$$\omega' = \omega\sqrt{1 - \zeta^2} \qquad (9)$$

LOS VALORES USUALES EN ESTRUCTURAS QUE ASUME ζ VARIAN ENTRE 2 Y 5%. EN ESTE INTERVALO ω' Y ω SON CASI IGUALES; VEAMOS, POR EJEMPLO, EL CASO EN QUE $\zeta = 0.1$

$$\omega' = \omega \sqrt{1 - 0.01} = 0.995\omega$$

OTRA FORMA DE MEDIR EL GRADO DE AMORTIGUAMIENTO QUE TIENE UNA ESTRUCTURA ES MEDIANTE EL DECREMENTO LOGARITMICO, EL CUAL SE DEFINE COMO EL LOGARITMO DEL COCIENTE DE DOS AMPLITUDES CONSECUTIVAS

$$L = \ln \frac{y(t)}{y(t + T')} = \ln \frac{Ae^{-ht}\cos(\omega't-\theta)}{Ae^{-h(t+T')}\cos[\omega'(t+T')-\theta]}$$

$$= \ln\{\frac{e^{-ht}}{e^{-h(t+T')}} \frac{\cos(\omega't - \theta)}{\cos(\omega't + \omega'T'-\theta)}\}$$

$$= \ln\{\frac{e^{-ht}}{e^{-ht}\, e^{-hT'}} \frac{\cos(\omega't - \theta)}{\cos(\omega't - \theta + 2\pi)}\}$$

$$= \ln e^{+hT'} = hT' = \zeta\omega T' = \zeta\omega \frac{2\pi}{\omega\sqrt{1-\zeta^2}}$$

$$\boxed{L = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad (10)$$

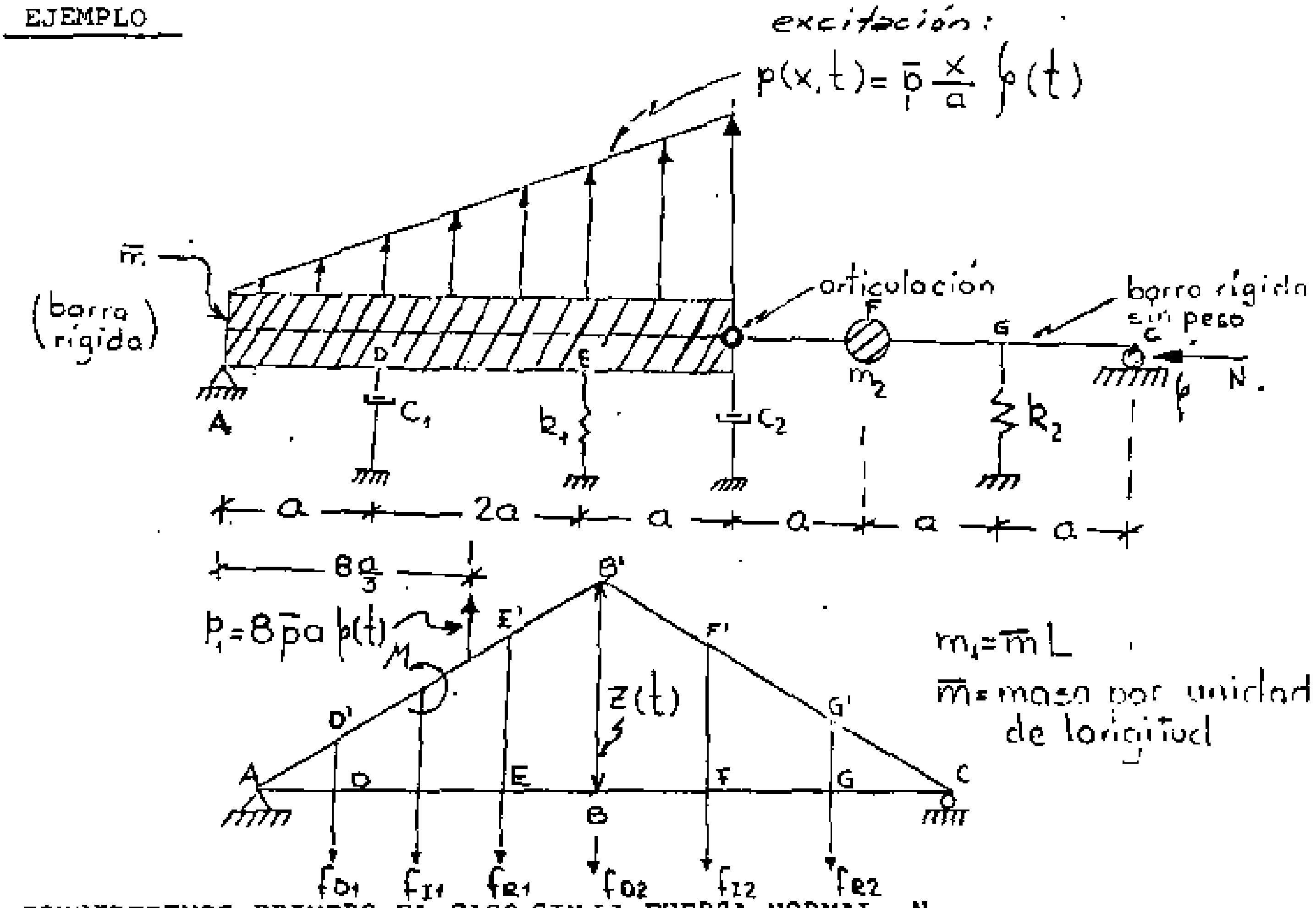
SI ζ ES PEQUEÑO,

$$\boxed{L \doteq 2\pi\zeta} \qquad (11)$$

ECUACION DE MOVIMIENTO GENERALIZADA.

HAY PROBLEMAS QUE APARENTEMENTE CORRESPONDE A VIBRACIONES DE SISTEMAS DE VARIOS GRADOS DE LIBERTAD PERO QUE EN REALIDAD SON DE UN GRADO SOLAMENTE.

EJEMPLO



CONSIDEREMOS PRIMERO EL CASO SIN LA FUERZA NORMAL, N.

TOMANDO COMO COORDENADA GENERALIZADA A Z(t):

$$f_{R1} = k_1(EE') = k_1\frac{3}{4}Z(t) \; ; \quad f_{R2} = k_2(GG') = K_2\frac{1}{3}Z(t)$$

$$f_{D1} = C_1\left(\frac{d}{dt}DD'\right) = C_1\frac{1}{4}\dot{Z}(t); \; f_{D2} = C_2\dot{Z}(t)$$

$$f_{I1} = m_1\frac{1}{2}\ddot{Z}(t) = \bar{m}L\frac{1}{2}\ddot{Z}(t) = 2a\bar{m}\ddot{Z}(t)$$

$$f_{I2} = m_2\frac{2}{3}\ddot{Z}(t)$$

$$M = I_0 \frac{1}{4a} \ddot{Z}(t) = \frac{\bar{m}L}{4a} \frac{L^2}{3} \ddot{Z}(t) = \frac{4}{3} a^2 \bar{m} \ddot{Z}(t)$$

$$p_1 = 8\bar{p}a\zeta(t)$$

LA ECUACION DE MOVIMIENTO DEL SISTEMA SE PUEDE ESTABLECER IGUALANDO A CERO EL TRABAJO VIRTUAL REALIZADO POR TODAS LAS FUERZAS AL DARLE AL SISTEMA UN DESPLAZAMIENTO VIRTUAL EN EL PUNTO B IGUAL A δZ. EN TAL CASO

$$\delta W = -k_1 \frac{3}{4} Z(t)(\frac{3}{4}\delta Z) - K_2 \frac{1}{3} Z(t)(\frac{1}{3}\delta Z) - C_1 \frac{\dot{Z}(t)}{4} (\frac{\delta Z}{4}) -$$

$$- C_2 Z(t)(\delta Z) - 2a\tilde{m} \ddot{Z}(t)(\frac{\delta Z}{2}) - m_2 \frac{2\ddot{Z}(t)}{3}(\frac{2}{3}\delta Z) -$$

$$- \frac{4}{3} a^2 \tilde{m} \ddot{Z}(t)(\frac{\delta Z}{4a}) + 8\bar{p}a\zeta(t)(\frac{2}{3}\delta Z) = 0$$

SIMPLIFICANDO SE OBTIENE

$$\Big[(a\bar{m} + \frac{a\bar{m}}{3} + \frac{4m_2}{9}) \ddot{Z}(t) + (\frac{C_1}{16} + C_2) \dot{Z}(t) + (\frac{9}{16} k_1 + \frac{k_2}{9}) Z(t) -$$

$$- \frac{16}{3} \bar{p}\bar{a}\zeta(t)\Big] \delta Z = 0 \qquad (A)$$

COMO EL DESPLAZAMIENTO VIRTUAL δZ NO ES CERO, SE DEBE CUMPLIR QUE EL TERMINO ENTRE PARENTESIS ES CERO. EN TAL CASO:

$$\boxed{\tilde{m} \ddot{Z}(t) + \tilde{C} \dot{Z}(t) + \tilde{k} Z(t) = \tilde{p}(t)}$$

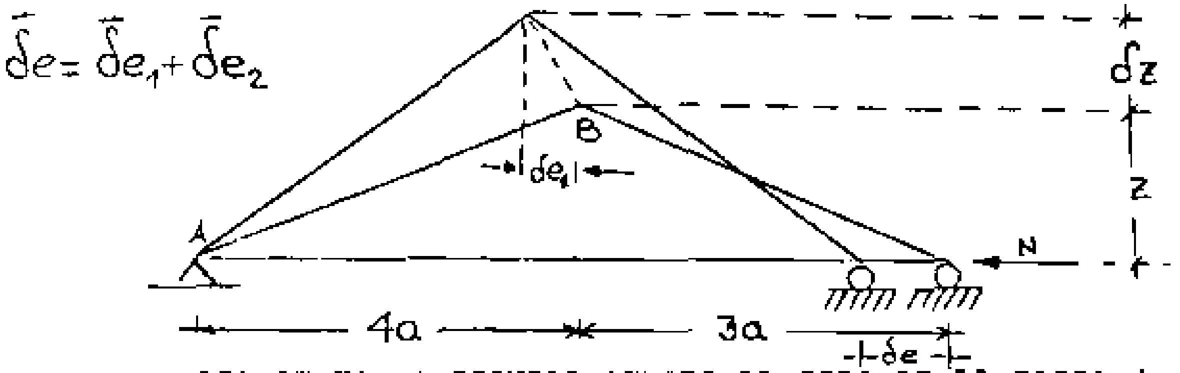
EN DONDE

$$\tilde{m} = \frac{4}{3} \; \bar{m}a + \frac{4}{9} m_2 \quad ; \quad \tilde{c} = \frac{C_1}{16} + C_2$$

$$\tilde{k} = \frac{9}{16} k_1 + \frac{k_2}{9} \quad ; \quad \tilde{p}(t) = \frac{16}{3} \bar{p}a\zeta(t)$$

ESTOS PARAMETROS SE DENOMINAN MASA, AMORTIGUAMIENTO, RIGIDEZ Y FUERZA GENERALIZADAS, RESPECTIVAMENTE.

CONSIDEREMOS AHORA EL CASO DE LA FUERZA NORMAL N SOLAMENTE:



δe_i = DESPLAZAMIENTO HORIZONTAL DEBIDO AL GIRO DE LA BARRA i

$$\delta e_1 = \frac{Z}{4a} \delta Z \; ; \; \delta e_2 = \frac{Z}{3a} \delta Z$$

$$\therefore \quad \delta e = \frac{7}{12} \; \frac{Z}{a} \; \delta Z$$

EL TRABAJO VIRTUAL ES:

$$\delta W = N\delta e = \frac{7}{12} \; \frac{NZ}{a} \; (\delta Z)$$

COMO EL SISTEMA ES LINEAL SE PUEDE SUMAR ESTE TRABAJO VIRTUAL AL DE LA ECUACION (A), CON LO CUAL LA RIGIDEZ GENERALIZADA SE MODIFICA, QUEDANDO EN LA FORMA:

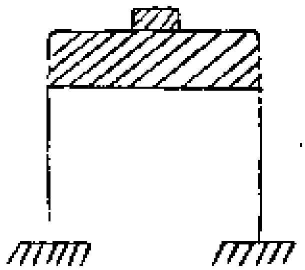
$$\tilde{k} = \frac{9}{16}k_1 + \frac{1}{9}k_2 \boxed{-\frac{7}{12}\frac{N}{a}}$$

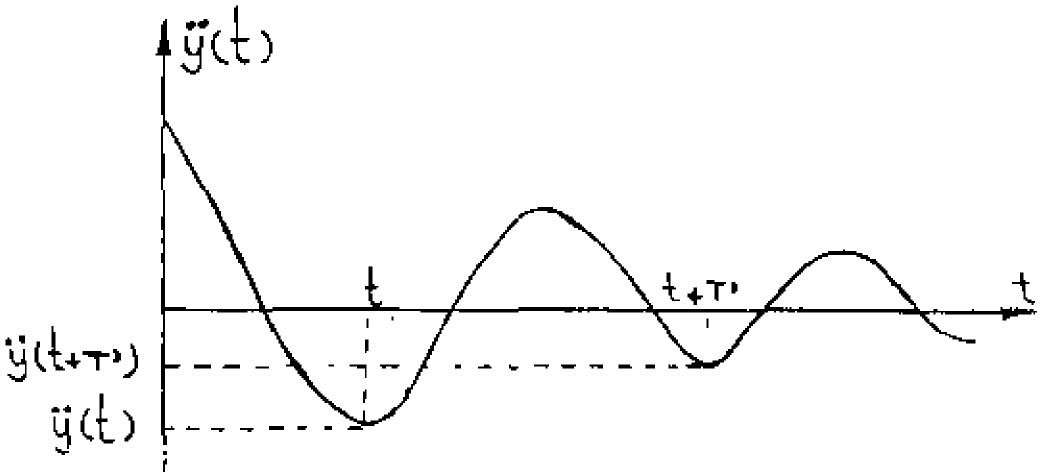
DE ESTA RIGIDEZ SE PUEDE SACAR, DE PASO, LA CARGA CRITICA DE PANDEO HACIENDO $\tilde{k} = 0$:

$$N_{cr} = (\frac{27}{28}k_1 + \frac{4}{21}k_2)a$$

DETERMINACION EXPERIMENTAL DE ζ EN ESTRUCTURAS REALES O EN MODELOS

SI SE REALIZA UN EXPERIMENTO EN EL CUAL SE SACA A LA ESTRUCTURA DE SU POSICION SE SACA A LA ESTRUCTURA DE SU POSICION DE EQUILIBRIO ESTATICO Y SE DEJA VIBRANDO LIBREMENTE, EL REGISTRO DE LAS ACELERACIONES QUE SE REGISTREN EN LA MASA TENDRA LA MISMA FORMA QUE LA GRAFICA DE LA EC. 7.



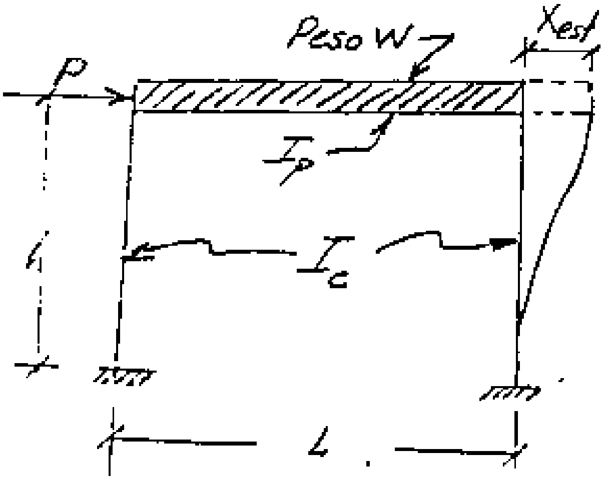


SI DE DICHO REGISTRO SE MIDEN $\ddot{y}(t + T')$ y $\ddot{y}(t)$ SE PUEDE OBTENER L Y, DE LA EC. (11), DESPEJAR A ζ

$$\zeta = \frac{L}{2\pi}$$

Ejemplo

Calcular el periodo natural de vibración de la estructura mostrada en la siguiente figura:



$$K = \frac{P}{X_{est}}$$

P = carga estática

X_{est} = desplazamiento producido por P

I_c = Momento de inercia de las columnas

I_p = Momento de inercia del sistema de piso

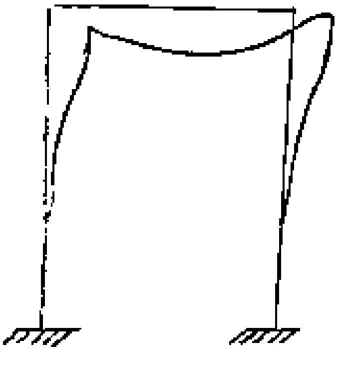
Mediante el análisis estático del marco se encuentra que

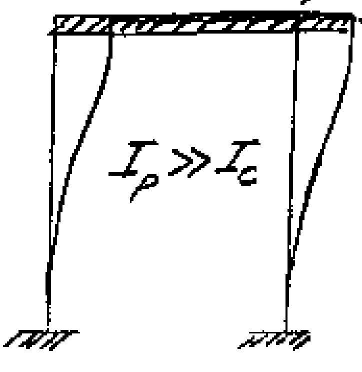
$$X_{est} = \frac{Ph^3}{6EI_c} \; \frac{\frac{3}{2} + \frac{I_c}{I_p}\frac{L}{h}}{6 + \frac{I_c}{I_p}\frac{L}{h}} \Rightarrow K = \frac{6EI}{h^3} \; \frac{6 + \frac{I_c}{I_p}\frac{L}{h}}{\frac{3}{2} + \frac{I_c}{I_p}\frac{L}{h}}$$

$$\text{Periodo natural} = T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{K}{m}}} = 2\pi\sqrt{\frac{W}{gK}}$$

$$T = 2\pi\sqrt{\frac{Wh^3}{g6EI} \; \frac{\frac{3}{2} + \frac{I_c}{I_p}\frac{L}{h}}{6 + \frac{I_c}{I_p}\frac{L}{h}}} \text{, en seg}$$

Si $I_p \gg I_c \quad (I_p \to \infty)$, $K = \frac{24EI}{h^3}$

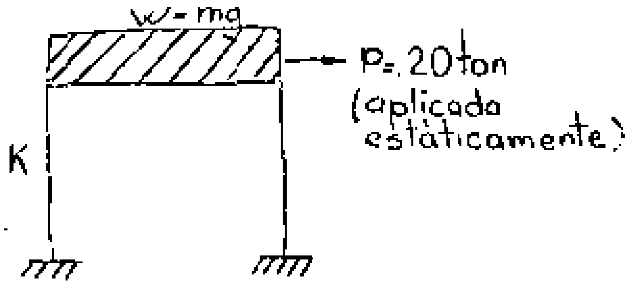


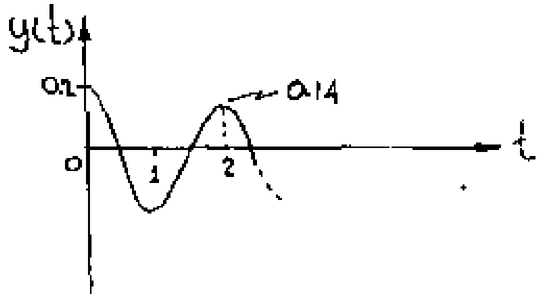


Estructura de CORTANTE: Cuando las deformaciones ocurren principalmente debido a la fuerza cortante de entrepiso.

EJEMPLO

A UNA ESTRUCTURA DE UN PISO SE LE APLICA UNA CARGA HORIZONTAL DE 20 TON EN SU MASA, OBSERVANDOSE UN DESPLAZAMIENTO ESTATICO DE 0.2 CM. AL SOLTAR SUBITAMENTE LA FUERZA SE REGISTRA UN PERIODO DE OSCILACION DE 0.2 SEG, Y QUE LA AMPLITUD EN EL SEGUNDO CICLO ES DE 0.14 CM.





CALCULAR ω, ω', f', L y ζ

1. DE $T' \doteq \frac{2\pi}{\omega} = \frac{\pi 2}{\sqrt{\frac{K}{M}}} = \frac{2\pi\sqrt{W}}{\sqrt{Kg}} = 0.2$ Y $K = \frac{2.0}{0.2} = 100 \frac{TON}{CM}$

SE OBTIENE

$$W \doteq T'^2 \, Kg/4\pi^2 = (0.2)^2 \times 100 \times 981/4\pi^2 = \frac{0.04 \times 100 \times 981}{4 \quad 9.87}$$

$$W \doteq 99.4 \text{ TON}$$

2. $\omega' = \frac{2\pi}{T'} = \frac{2\pi}{0.2} = 10\pi \frac{RAD}{SEG}$; $f' = \frac{1}{T'} = \frac{1}{0.2} = 5$ cps

3. $L = \ln \frac{0.2}{0.14} = \ln 1.43 = 0.357$

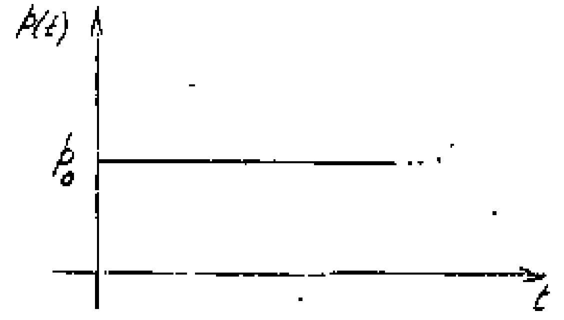
$$\zeta \doteq \frac{L}{2\pi} = \frac{0.357}{2\pi} = 0{,}0568 \quad O \quad \zeta = 5.68\ \%$$

$$C = \zeta C_{cr} = \zeta 2\sqrt{KM} = 0.1132 \sqrt{100 \times 99.4/981}$$

$$= 1{,}132 \times 0{,}318 = 0{,}36 \text{ TON SEG/CM}$$

EJEMPLO

CALCULAR LA RESPUESTA DE UN SISTEMA DE UN GRADO DE LIBERTAD SUJETO A LA SIGUIENTE EXCITACION:



CON $C = 0$

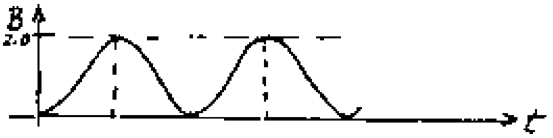
$m\ddot{x} + kx = p_o$

$x = C_1 \operatorname{sen}\omega t + C_2 \cos\omega t + p_o/k$

SI EN $t = 0$, $x = 0$ Y $\dot{x} = 0$:

$C_2 = -F_o/k$ Y $C_1 = 0$

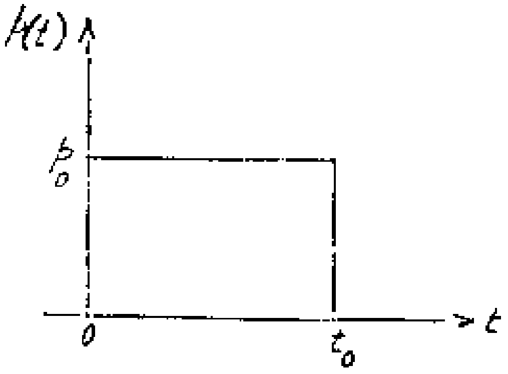
$\therefore \quad x = \frac{p_o}{k}(1 - \cos\omega t);$



B = FACTOR DE AMPLIFICACION DINAMICA $= \frac{x}{(\frac{p_o}{k})} = (1 - \cos\omega t)$

$B_{MAX} = 2$, EN $t = T/2, 3T/2...$

AHORA, SI LA EXCITACION ES DE DURACION t_o:



SI $t < t_o$:

$x = \frac{p_o}{k}(1 - \cos\omega t)$

$\dot{x}(t) = \frac{\omega p_o}{k} \operatorname{sen}\omega t$

EN $t = t_o$:

$$\left.\begin{aligned} x(t_o) &= \frac{p_o}{k}(1 - \cos\omega t_o) \\ \dot{x}(t_o) &= \frac{\omega p_o}{k} \operatorname{sen}\omega t_o \end{aligned}\right\} \text{CONDICIONES INICIALES PARA } t>t_o$$

SI $t > t_o$, $x = A \cos\omega t' + B \operatorname{sen}\omega t'$, CON $t' = t - t_o$.

EN $t' = 0$ ($t = t_o$), SE DEBEN CUMPLIR LAS CONDICIONES INICIALES ANTERIORES, LO CUAL CONDUCE A

$$A = \frac{p_o}{k}(1 - \cos\omega t_o) \quad Y \quad B = \frac{p_o}{k} \operatorname{sen}\omega t_o$$

POR LO QUE $\quad x = \frac{p_o}{k}(1 - \cos\omega t_o)\cos\omega t' + \frac{p_o}{k}\operatorname{sen}\omega t_o \operatorname{sen}\omega t'$

$$= \frac{p_o}{k}\sqrt{(1 - \cos\omega t_o)^2 + \operatorname{sen}^2\omega t_o}\ \operatorname{sen}(\omega t' - \theta)$$

O

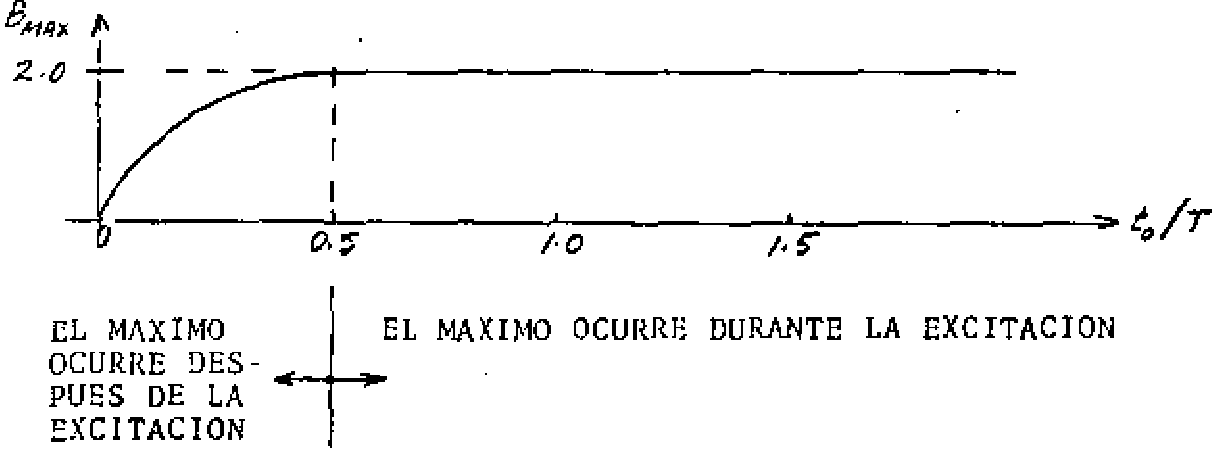
$$x = \frac{p_o}{k}\sqrt{2(1 - \cos t_o)}\ \operatorname{sen}(\omega t' - \theta)$$

$$= \frac{p_o}{k}\underbrace{(2 \operatorname{sen}\frac{\omega t_o}{2})\ \operatorname{sen}(\omega t' - \theta)}$$

B= FACTOR DE AMPLIFICACION

$$B_{MAX} = 2 \operatorname{sen}\frac{\omega t_o}{2} = 2 \operatorname{sen}(\pi\frac{t_o}{T})$$

CUANDO $\frac{\pi t_o}{T} = \frac{\pi}{2}$, $B_{MAX} = 2$



EL MAXIMO OCURRE DESPUES DE LA EXCITACION

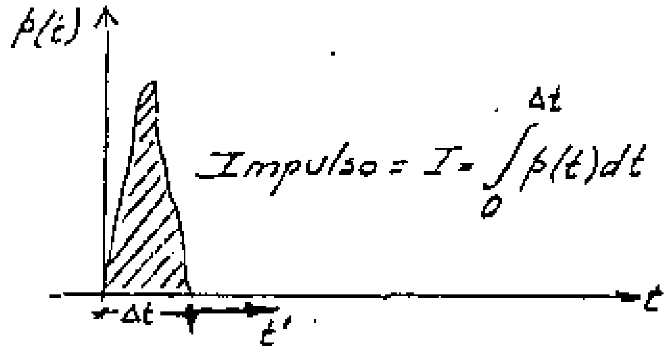
EL MAXIMO OCURRE DURANTE LA EXCITACION

SI t_o/T ES MUY PEQUEÑO, $\operatorname{sen}\frac{\pi t_o}{T} = \pi t_o/T$

- Y $$x_{MAX} = \frac{2p_o}{k}\,\frac{\pi t_o}{T} = \frac{2p_o}{m\frac{k}{m}}\,\frac{\omega t_o}{2} = \frac{p_o t_o}{m\omega} = \frac{1}{m\omega}$$

EN DONDE $i = p_o t_o$ = AREA BAJO LA EXCITACION

EJEMPLO: EXCITACION DADA POR UN IMPULSO.-SEA UN IMPULSO APLICADO DURANTE UN INTERVALO DE TIEMPO Δt MUY PEQUEÑO, TAL QUE $\Delta t/T << 1$:



POR EL PRINCIPIO IMPULSO - MOMENTO SE TIENE QUE

$$I = \int_0^{\Delta t} p(t)dt = m\dot{x}$$

EN DONDE $\dot{x}$ ES LA VELOCIDAD QUE EL IMPULSO LE IMPRIME A LA MASA DEL SISTEMA. DESPUES DE Δt EL SISTEMA QUEDA VIBRANDO LIBREMENTE CON VELOCIDAD INICIAL $\dot{x}(0) = \frac{I}{m}$, MIDIENDO EL TIEMPO EN LA ESCALA DE t', Y CON DESPLAZAMIENTO INICIAL QUE PUEDE CONSIDERARSE NULO, DEBIDO A QUE EN EL CORTO INTERVALO DE TIEMPO Δt LA MASA ADQUIERE UN DESPLAZAMIENTO DE MAGNITUD DESPRECIABLE. EN TAL CASO LA RESPUESTA RESULTA SER

$$x(t') = \frac{\dot{x}(o)}{\omega}\,\text{sen}\,\omega t' = \frac{I}{m\omega}\text{sen}\,\omega t'$$

SI EL SISTEMA TIENE AMORTIGUAMIENTO,

$$x(t') = \frac{I}{m\omega}\;e^{-\zeta\omega t'}\;\text{sen}\,\omega' t'$$

PRINCIPIO DE HAMILTON

$$\int_{t_1}^{t_2}\delta(T-V)dt + \int_{t_1}^{t_2}\delta W_{nc}dt = 0$$

DONDE

T = ENERGIA CINETICA TOTAL

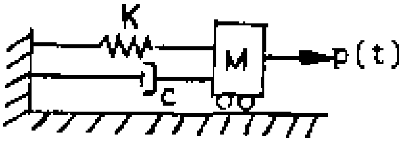
V= ENERGIA POTENCIAL TOTAL, INCLUYENDO ENERGIA DE DEFORMACION Y ENERGIA POTENCIAL DE LAS FUERZAS CONSERVATIVAS

W_{nc}= TRABAJO REALIZADO POR LAS FUERZAS NO CONSERVATIVAS (TALES COMO LAS DE AMORTIGUAMIENTO)

δ= VARIACION TOMADA DURANTE EL INTERVALO DE TIEMPO DE t_1 A t_2

EN ESTE PRINCIPIO SE ASUME QUE LA VARIACION, δx, DEL DESPLAZAMIENTO EN LOS INSTANTES t_1 Y t_2 ES NULO.

EJEMPLO



$T = \frac{1}{2} m\dot{x}^2$; $V = \frac{1}{2} kx^2$ (ES LA ENERGIA DE DEFORMACION, UNICAMENTE)

$$\delta W_{nc} = p(t)\delta x - c\dot{x}\delta x$$

$$\int_{t_1}^{t_2}\delta(\frac{1}{2} m\dot{x}^2 - \frac{1}{2}kx^2)dt + \int_{t_1}^{t_2}(p(t)\delta x - c\dot{x}\delta x)dt$$

$$\int_{t_1}^{t_2}(mx\delta x - kx\delta x)dt + \int_{t_1}^{t_2}(p(t) - c\dot{x})\delta x\ dt = 0$$

$$\int_{t_1}^{t_2}[m\dot{x}\delta\dot{x} - (c\dot{x} + kx - p(t))\delta x]dt = 0$$

INTEGRANDO POR PARTES EL PRIMER TERMINO DE ESTA INTEGRAL;

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}\ dt = m\dot{x}\delta x\Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{x}\ \delta\ x\ dt$$

$$=-\int_{t_1}^{t_2} m\ddot{x}\delta x\ dt$$

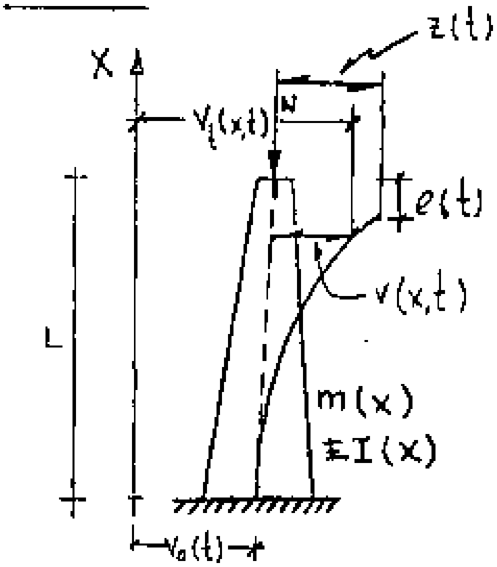
POR LO QUE

$$\int_{t_1}^{t_2} [- m\ddot{x} - c\dot{x} - kx + p(t)]\delta x\ dt = 0$$

PUESTO QUE δx ES ARBITRARIA, LA ECUACION ANTERIOR SE SATISFACE EN GENERAL SOLO SI

$$m\ddot{x} + c\dot{x} + kx - p(t) = 0$$

EJEMPLO



$$v(x,t) = \psi(x)z(t)$$

APLICANDO EL PRINCIPIO DE HAMILTON:

$$T= \frac{1}{2}\int_0^L m(x)(\dot{v}_t(x,t))^2 dx$$

ENERGIA POTENCIAL POR DEFORMACION:

$$V= \frac{1}{2}\int_0^L EI(x)(v''(x,t))^2 dx$$

$$e(t) = \frac{1}{2}\int_0^L [v'(x,t)]^2 dx$$

ENERGIA POTENCIAL DEBIDA A LA FUERZA NORMAL:

$$V_N = -\frac{N}{2}\int_0^L [v'(x,t)]^2 dt$$

EN ESTAS ECUACIONES: $\dot{v} = dv/dt$; $v' = dv/dx$

$v'' = d^2v/dx^2$

PUESTO QUE NO HAY FUERZAS DINAMICAS EXTERNAS, Y SI CONSIDERAMOS AMORTIGUAMIENTO NULO, ENTONCES $\delta W_{nc} = 0$, POR LO QUE

$$\int_{t_1}^{t_2} \delta(T-V)dt = 0 \qquad o$$

$$\int_{t_1}^{t_2} [\int_0^L m(x)\dot{v}_t(x,t)\ \delta\dot{v}_t dx - \int_0^L EI(x)v''(x,t)\ \delta v'' dx +$$

$$+ N\int_0^L v'(x,t)\ \delta v' dx] = 0$$

TOMANDO EN CUENTA QUE

$$\dot{v}_t = \dot{v} + \dot{v}_o\ ,\ v'' = \psi''Z,\ v' = \psi'Z \quad \dot{v} = \psi\dot{Z}$$

$$\delta\dot{v}_t = \delta\dot{v}\ ,\ \delta v'' = \psi''\delta Z,\ \delta v' = \psi'\delta z,\ \delta\dot{v} = \psi\delta\dot{z}$$

SE OBTIENE

$$\int_{t_1}^{t_2} [\dot{z}\delta\dot{z} \int_0^L m(x)\ \psi^2\delta x + \delta\dot{z}\dot{v}_o(t) \int_0^L m(x)\psi\delta x -$$

$$- z\delta z \int_0^L EI(x)(\psi'')^2\delta x + Nz\delta z \int_0^L (\psi'')^2\delta x|\ dt = 0$$

INTEGRANDO POR PARTES LAS PRIMERAS DOS INTEGRALES Y HACIENDO

$\tilde{m} = \int_0^L m(x)\psi^2\ dx$: MASA GENERALIZADA

$$\tilde{K} = \int_0^L EI(x)(\psi'')^2dx =$$ RIGIDEZ GENERALIZADA SIN CONSIDERAR FUERZA NORMAL

o $$\bar{K} = \int_0^L EI(x)(\psi'')^2dx - N\int_0^L(\psi')^2dx =$$ RIGIDEZ GENERALIZADA CON N

$$\tilde{p}(t) = \text{FUERZA GENERALIZADA EFECTIVA} = -\ddot{v}_o\int_0^L m(x)\psi\delta x$$

SE OBTIENE LA ECUACION

$$\int_{t_1}^{t_2}[\bar{m}\ddot{z} + \bar{k}z - \bar{p}(t)]\delta z\,dt = 0$$

POR LO QUE

$$\bar{m}\ddot{z} + \bar{k}z = \bar{p}(t)$$

<u>CASO PARTICULAR.</u> CONSIDEREMOS EI=cte Y m=cte = $\bar{m}$

SEA $\psi(x) = 1 - \cos\frac{\pi x}{2L}$; EN TAL CASO:

$$\tilde{m} = \int_0^L \bar{m}(\psi)^2dx = \bar{m}\int_0^L(1 - \cos\frac{\pi x}{2L})^2dx = 0.228\,\bar{m}L$$

<u>SI N=0:</u>

$$\tilde{k} = \int_0^L EI(\psi'')^2dx = EI\int_0^L(\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L})^2dx = \frac{\pi^4}{32}\frac{EI}{L^3}$$

$$\bar{p}(t) = -\ddot{v}_o(t)\int_0^L\bar{m}\psi\,dx = -\bar{m}\ddot{v}_o(t)\int_0^L(1 - \cos\frac{\pi x}{2L})\,dx = -0.364\,\bar{m}L\ddot{v}_o(t)$$

<u>SI N≠0:</u>

$$\tilde{k} = \frac{\pi^4}{32}\frac{EI}{L^3} - N\int_0^L(\psi')^2dx = N\int_0^L(\frac{\pi}{2L}\,\text{sen}\frac{\pi x}{2L})^2dx$$

$$\tilde{k} = \frac{\pi^4}{32}\frac{EI}{L^3} - \frac{N\pi^2}{8L}$$

PARA CARGA DE PANDEO: $\frac{\pi^4}{32}\frac{EI}{L^3} - \frac{N_{cr}\pi^2}{8L} = 0 \Rightarrow N_{cr} = \frac{\pi^2}{4}\ \frac{EI}{L^2}$

CON LO QUE $\tilde{K} = \frac{\pi^4 EI}{32L^3}(1 - \frac{N}{N_{cr}})$ Y LA ECUACION DE EQUILIBRIO QUEDA EN LA FORMA:

$$\boxed{0.228\ \bar{m}L\ddot{z}(t) + \frac{\pi^4 EI}{32L^3}(1 - \frac{N}{N_{cr}})\ z(t) = 0.364\bar{m}L\ddot{v}_o(t)}$$

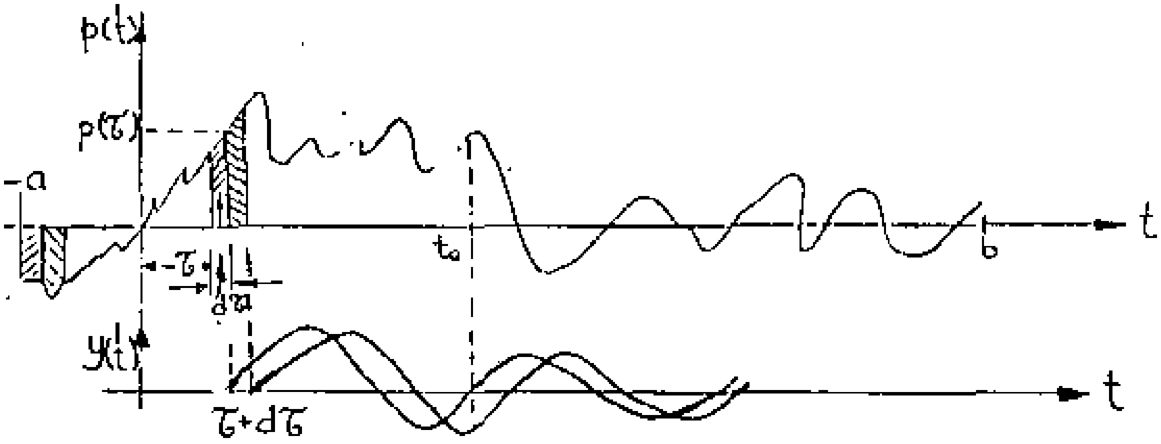
LA FRECUENCIA CIRCULAR NATURAL CORRESPONDIENTE ES

$$\boxed{\omega = \sqrt{\frac{\pi^4 EI(1 - \frac{N}{N_{cr}})}{7.296\ \bar{m}L^4}}}$$

SOLUCION AL PROBLEMA DE VIBRACIONES FORZADAS

A. FUERZA EXTERNA

VEAMOS PRIMERO EL CASO EN QUE EXISTE p(t) Y QUE $\ddot{x}_o(t) = 0$, SIENDO p(t) ARBITRARIA



PUESTO QUE $d\tau << T$, LA FUERZA APLICADA EN $t= \tau$ PRODUCIRA UN INCREMENTO INSTANTANEO EN LA VELOCIDAD DE LA MASA IGUAL A

$$\dot{y} = \frac{p(\tau)d\tau}{M}$$

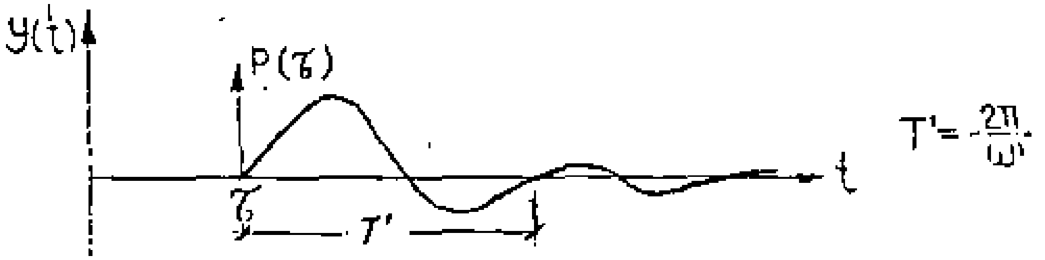
Y UN INCREMENTO INSTANTANEO NULO EN EL DESPLAZAMIENTO, ES DECIR, $y=0$. TOMANDO ESTOS INCREMENTOS COMO CONDICIONES INICIALES EN $t=\tau$, LA EC. 5 DA COMO RESULTADO

$$y(t) = \frac{p(\tau)d\tau}{M\omega'} \text{ sen } \omega'(t-\tau) \; e^{-h(t-\tau)} \quad ; \; t \geq \tau$$

PUESTO QUE EL SISTEMA ES LINEAL ES POSIBLE SUPERPONER LOS EFECTOS OCASIONADOS POR LOS IMPULSOS APLICADOS EN CADA τ QUE HAYAN OCURRIDO ANTES DEL INSTANTE t DE INTERES; ES DECIR,

$$\boxed{y(t) = \frac{1}{M\omega'}\int_{-\infty}^{t} p(\tau)e^{-h(t-\tau)}\operatorname{sen}\omega'(t-\tau)d\tau} \qquad (12)$$

LA FUNCION $\frac{1}{M\omega'}\,e^{-h(t-\tau)}\operatorname{sen}\omega'(t-\tau)$,QUE ES LA RESPUESTA A UN IMPULSO INSTANTANEO UNITARIO DE FUERZA, SE LE CONOCE COMO FUNCION DE TRANSFERENCIA DEL SISTEMA.



LA SOLUCION DADA EN LA EC. (12) SE DENOMINA INTEGRAL DE DUHAMEL. ESTA CONSTITUYE LA SOLUCION PARTICULAR DE LA ECUACION DIFERENCIAL DE EQUILIBRIO; LA SOLUCION GENERAL ES:

$$\boxed{y(t) = Ae^{-ht}\cos(\omega' t-\theta) + \frac{1}{M\omega'}\int_{-\infty}^{t} p(\tau)e^{-h(t-\tau)}\operatorname{sen}\omega'(t-\tau)d\tau}$$

EN DONDE A y θ DEPENDEN DE LAS CONDICIONES INICIALES DE DESPLAZAMIENTO Y VELOCIDAD, $y(0)$ Y $\dot{y}(0)$, RESPECTIVAMENTE. EN GENERAL LA PARTE DE LA RESPUESTA DADA POR LA SOLUCION PARTICULAR ES LA MAS IMPORTANTE, YA QUE LA OTRA PARTE SE AMORTIGUA RAPIDAMENTE.

B. MOVIMIENTO DEL SUELO

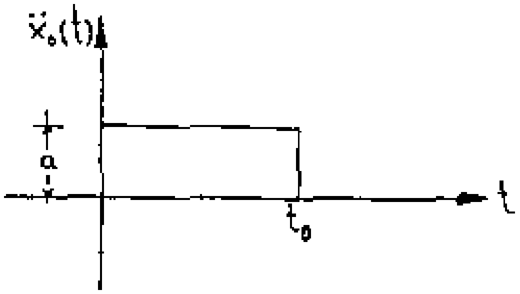
PARA ESCRIBIR LA SOLUCION PARTICULAR DE LA ECUACION DIFERENCIAL DE EQUILIBRIO PARA EL CASO DE VIBRACION FORZADA POR MOVIMIENTO DE LA BASE DE LA ESTRUCTURA, BASTA CAMBIAR $p(\tau)/M$ DE LA EC. (12) POR $-\ddot{x}_o$, YA QUE EN DICHA ECUACION APARECE EN EL MIEMBRO DERECHO $p(t)/M$ CUANDO LA EXCITACION ES $P(t)$ Y APARECE $-\ddot{x}_o$ CUANDO LA EXCITACION ES POR MOVIMIENTO DEL SUELO. EN ESTE CASO

LA SOLUCION PARTICULAR ES, ENTONCES

$$\boxed{y(t) = \frac{-1}{\omega'}\int_{-\infty}^{t} \ddot{x}_o(\tau)e^{-h(t-\tau)}\ \mathrm{sen}\,\omega'(t-\tau)d\tau} \tag{14}$$

EJEMPLO

CALCULAR LA RESPUESTA DE UN SISTEMA DE UN GRADO DE LIBERTAD CON AMORTIGUAMIENTO NULO, CUANDO LA EXCITACION ES LA SIGUIENTE:



$$\zeta = 0$$

$$\ddot{x}_o(t) = a,\ \mathrm{SI}\ 0 \leqslant t \leqslant t_o$$

$$\ddot{x}_o(t) = 0,\ \mathrm{SI}\ t<0\ \text{Ó}\ t>t_o$$

CONSIDERESE QUE $y(0)=0$ Y $\dot{y}(0)=0$. PUESTO QUE LAS CONDICIONES INICIALES SON NULAS SE TIENE QUE A=0 (UTILIZANDO LA EC. (13) Y LA SOLUCION PARTICULAR QUE SIGUE, EC. (A)):

$$y(t) = \frac{-1}{\omega}\int_{-\infty}^{t} a\ \mathrm{sen}\,\omega(t-\tau)d\tau = \frac{-a}{\omega}\int_{0}^{t} \mathrm{sen}\ \omega(t-\tau)d\tau$$

$$= \frac{-a}{\omega^2}\ (1-\cos\omega t) \qquad \mathrm{SI}\quad 0\leq t\leq t_o \tag{A}$$

PARA FINES DE DISEÑO ESTRUCTURAL ES IMPORTANTE CONOCER LA RESPUESTA MAXIMA; ESTA OCURRE CUANDO $\cos\omega t = -1$, O SEA, CUANDO

$$\omega t = \pi \quad \mathrm{O} \quad t = \frac{\pi}{\omega} = \frac{\pi}{\frac{2\pi}{T}} = \frac{T}{2}$$

Y VALE

$$\text{MAX } (|y(t)|) = \frac{2a}{\omega^2} = \frac{a}{2\pi^2} T^2, \quad \text{SI} \quad 0 \leq \frac{T}{2} \leq t_o \quad \text{O} \quad 0 \leq T \leq 2t_o$$

PARA $t > t_o$, O SEA, PARA $T/2 > t_o$ ES NECESARIO OBTENER LA RESPUESTA EN VIBRACION LIBRE CON LAS CONDICIONES INICIALES DE VELOCIDAD Y DESPLAZAMIENTO CORRESPONDIENTES A $t = t_o$:

$$y(t_o) = \frac{-a}{\omega^2} (1 - \cos\omega t_o) \quad ; \quad \dot{y}(t_o) = \frac{-a}{\omega} \operatorname{sen}\omega t_o$$

APLICANDO LAS ECS. (5) Y (6) OBTENEMOS:

$$y(t) = \frac{-a}{\omega^2} \left[\operatorname{sen}\omega t_o \operatorname{sen}\omega t' - (1 - \cos\omega t_o) \cos\omega t'\right]$$

$$= \frac{-a}{\omega^2} \sqrt{\operatorname{sen}^2\omega t_o + (1 - \cos\omega t_o)^2} \operatorname{sen} (\omega t' - \varnothing)$$

$$y(t) = \frac{-2a}{\omega^2} \operatorname{sen}\frac{\omega t_o}{2} \operatorname{sen}(\omega t' - \varnothing)$$

DONDE $t' = t - t_o$ Y $\varnothing = \tan^{-1}\left(\frac{1 - \cos\omega t_o}{\operatorname{sen}\omega t_o}\right)$

EL VALOR MAXIMO DE LA RESPUESTA EN ESTE INTERVALO ES

$$\text{MAX}(|y(t)|) = \frac{2a}{\omega^2} \left|\operatorname{sen}\frac{\omega t_o}{2}\right| , \quad \text{SI} \quad t > t_o \quad \text{O} \quad T > 2t_o$$

EXCITACION ARMONICA

CONSIDEREMOS AHORA EL CASO EN QUE LA ESTRUCTURA ES EXCITADA POR LA FUERZA ARMONICA

$$p(t) = p_o \text{ sen}\Omega t$$

DE DURACION INDEFINIDA.

LA SOLUCION DE ESTE PROBLEMA SE PUEDE ENCONTRAR SUSTITUYENDO A $p(t) = p_o \text{ sen}\Omega t$ EN LA INTEGRAL DE DUHAMEL Y OBTENIENDO SU SOLUCION. SIN EMBARGO, EL RESULTADO LO OBTENDREMOS DE LA CONSIDERACION DE QUE PARA QUE EL MIEMBRO DERECHO DE LA ECUACION DIFERENCIAL DE EQUILIBRIO APAREZCA UN TERMINO ARMONICO ES NECESARIO QUE EN EL IZQUIERDO SE TENGAN COMBINACIONES DE TERMINOS TAMBIEN ARMONICOS. CONSIDEREMOS, POR LO TANTO, LA SOLUCION

$$y(t) = A \text{ sen}\Omega t + B \cos\Omega t \qquad (14')$$

Y DETERMINEMOS LOS VALORES QUE DEBEN TENER A Y B PARA SATISFACER LA ECUACION DIFERENCIAL DE EQUILIBRIO, PARA LO CUAL HAY QUE SUSTITUIR A $y(t)$, $\dot{y}(t)$ Y $\ddot{y}(t)$ EN LA ECUACION DIFERENCIAL. HACIENDO ESTO Y FACTORIZANDO:

$$(-A\Omega^2 - 2h\Omega B + \omega^2 A) \text{ sen}\Omega t +$$

$$(-B\Omega^2 + 2hA\Omega + \omega^2 B) \cos\Omega t = \frac{p_o}{M} \text{ sen}\Omega t + 0 \times \cos\Omega t$$

PARA QUE ESTA IGUALDAD SE CUMPLA SE REQUIERE QUE

$$-A\Omega^2 - 2h\Omega B + \omega^2 A = \frac{p_o}{M}$$

$$-B\Omega^2 + 2h\Omega A + \omega^2 B = 0$$

RESOLVIENDO ESTE SISTEMA DE ECUACIONES SE OBTIENE:

$$A = \frac{\frac{P_o}{M}\,(\Omega^2 - \omega^2)}{(\omega^2 - \Omega^2)^2 + 4h^2\Omega^2}$$

$$B = \frac{-2h\Omega\,\frac{P_o}{M}}{(\omega^2 - \Omega^2)^2 + 4h^2\Omega^2}$$

SUSTITUYENDO A Y B EN LA EC.(14'):

$$y(t) = \frac{\frac{P_o}{M}}{(\omega^2 - \Omega^2)^2 + 4h^2\Omega^2}\{(\Omega^2 - \omega^2)\ \mathrm{sen}\Omega t - 2h\Omega\ \cos\Omega t\} \qquad (15)$$

O, TAMBIEN

$$y(t) = \frac{\frac{P_o}{M}}{\sqrt{(\omega^2 - \Omega^2)^2 + 4h^2\Omega^2}}\ \mathrm{sen}(\Omega t - \phi) \qquad (16)$$

EN DONDE $\phi = \text{ANG TAN } (\frac{-B}{A}) = \text{TAN}^{-1}\ \frac{2h\Omega}{\omega^2 - \Omega^2}$ = ANGULO DE FASE (17)

DIVIDIENDO NUMERADOR Y DENOMINADOR DE LAS ECS. (16) Y (17) ENTRE ω^2 SE OBTIENE:

$$\boxed{y(t) = \frac{\frac{P_o}{k}}{\sqrt{(1 - \frac{\Omega^2}{\omega 2}) + (2\zeta -)^2}}\ \mathrm{sen}(\Omega t - \phi)} \qquad (18)$$

$$\boxed{\phi = \text{TAN}^{-1}\ \frac{2\zeta\frac{\Omega}{\omega}}{1 - \frac{\Omega^2}{\omega^2}}} \qquad (19)$$

SI SE TIENE EXCITACION ARMONICA EN LA BASE DE LA ESTRUCTURA $x_o(t)$ = asenΩt, O SEA, $\ddot{x}_o = a\Omega^2$senΩt, BASTA CAMBIAR A p_o/M EN LA EC. (16) POR $-a\Omega^2$; HACIENDO ESTO SE OBTIENE

$$y(t) = \frac{(\Omega/\omega)^2}{\sqrt{(1 - \frac{\Omega^2}{\omega^2})^2 + (2\zeta\frac{\Omega}{\omega})^2}}\ a\ \text{sen}(\Omega t-\emptyset) \qquad (20)$$

FACTOR DE AMPLIFICACION DINAMICA DE DESPL. = $B_d = \text{MAX}|\frac{y(t)}{a}|$

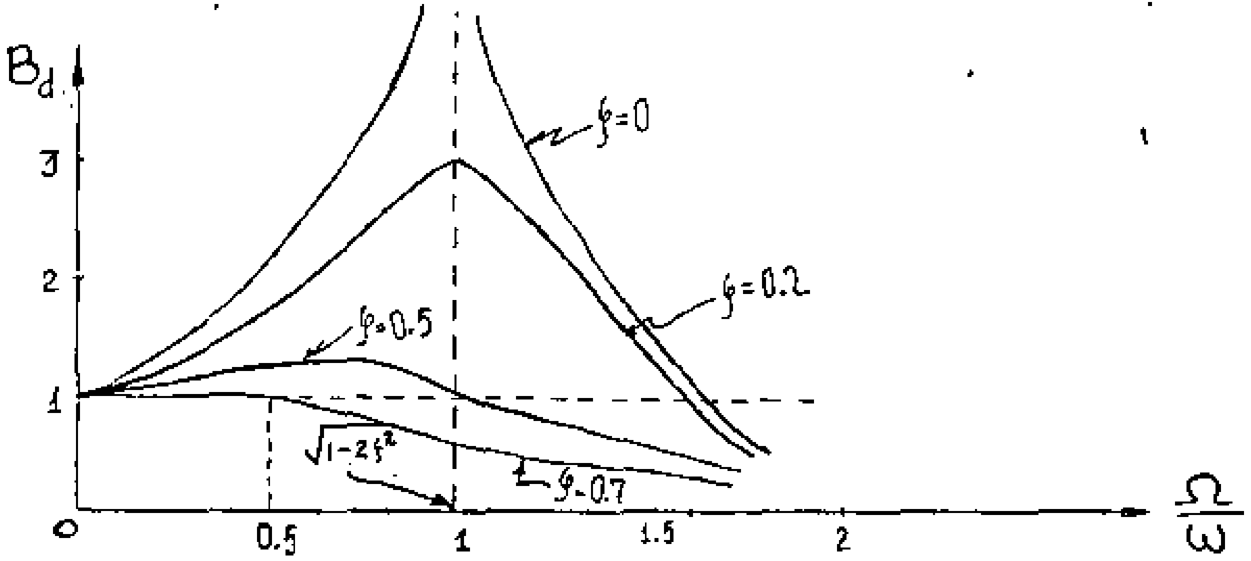


FIG. 1. CURVAS DE AMPLIFICACION DINAMICA PARA EL CASO DE FUERZA EXTERNA

$$\boxed{B_d = \frac{1}{\sqrt{(1 - \frac{\Omega^2}{\omega^2})^2 + (2\zeta\frac{\Omega}{\omega})^2}}} \qquad (21)$$

LOS FACTORES DE AMPLIFICACION DINAMICA DE VELOCIDAD Y ACELERACION SE SE PUEDEN OBTENER DERIVANDO RESPECTO A t LA EC. (16) O LA (20), SEGUN SEA EL CASO. LOS RESULTADOS SON, RESPECTIVAMENTE,

$$\text{MAX}|\frac{\dot{y}(t)}{a\omega}| = B_v = \frac{\Omega}{\omega}B_d \qquad Y \qquad B_a = (\frac{\Omega}{\omega})^2 B_d = \text{MAX}|\frac{\ddot{y}(t)}{a\omega^2}| \quad (22)$$

EJEMPLO

CON UNA MAQUINA VIBRATORIA PORTATIL QUE PRODUCE FUERZAS ARMONICAS SE PROBO UNA ESTRUCTURA, AJUSTANDO LA MAQUINA EN LAS FRECUENCIAS $\Omega_1 = 16 \frac{RAD}{SEG}$ Y $\Omega_2 = 25 \frac{RAD}{SEG}$, CON UNA FUERZA MAXIMA DE 500 LB EN CADA CASO. LAS AMPLITUDES Y ANGULOS DE FASE DE LA RESPUESTA QUE SE MIDIERON FUERON:

$$\rho_1 = 7.2 \times 10^{-3} in,\ \phi_1 = 15° \ (\cos\phi_1 = 0.966 \ ;\ sen\phi_1 = 0.259)$$

$$\rho_2 = 14.5 \times 10^{-3} in,\ \phi_2 = 55° (\cos\phi_2 = 0.574;\ sen\phi_2 = 0.819)$$

EVALUAR LAS PROPIEDADES DINAMICAS DEL SISTEMA.

HACIENDO:

$$\rho_i = \frac{P_o}{k} B_{d_i} = \frac{P_o}{k} \frac{1}{1-\beta^2} \underbrace{\left(\frac{1}{1+\left[2\zeta\beta/(1-\beta^2)\right]^2}\right)_i^{1/2}}_{\cos\phi_i}$$

$$\rho_i = \frac{P_o}{k} \frac{\cos\phi_i}{1-\beta^2} \quad ; \ \beta = \Omega/\omega$$

O

$$k - k\beta^2 = \frac{P_o \cos\phi_i}{\rho_i} = k - \Omega^2 m \qquad (23)$$

SUSTITUYENDO LOS VALORES EXPERIMENTALES DE LAS DOS PRUEBAS:

$$\left.\begin{array}{l} k - (16)^2 m = \dfrac{500\ (0.966)}{7.2 \times 10^{3}} \\ k - (25)^2 m = \dfrac{500\ (0.574)}{14.5 \times 10^{-3}} \end{array}\right\} \rightarrow \begin{array}{l} k = 100\ 000\ \dfrac{lb}{in} \\ m = 128.5\ \dfrac{lb\ SEG^2}{in} \\ \quad\Downarrow \\ \omega = \sqrt{\dfrac{k}{m}} = 27.9\ \dfrac{RAD}{SEG} \end{array}$$

USANDO LAS ECS. (17) Y (23) SE OBTIENE:

$$\zeta = \frac{P_o \ \text{sen}\O_i}{2\beta_i \ \ k\rho_i} \ ; \text{ DE DONDE } \zeta \ \frac{500 \ (0.259)}{2\frac{16}{27.9} \ 100\ 000(7.2 \times 10^{-3})} = 15.7\%$$

RESONANCIA

CUANDO LA EXCITACION TIENE FRECUENCIA IGUAL A LA NATURAL DEL SISTEMA, SE DICE QUE SE PRESENTA EL CASO DE RESONANCIA. DE LA EC.(20) ES EVIDENTE QUE SI $\beta=\Omega/\omega=1$ SE TIENE

$$y(t) = \underbrace{\frac{1}{2\zeta}}_{B_d} a \ \text{sen}(\Omega t-\O)$$

O $(B_d)_{res} = \frac{1}{2\zeta}$ EN CASO DE MOVIMIENTO DEL SUELO Y DE FUERZA EXTERNA

SIN EMBARGO, AUNQUE ESTA RESPUESTA ES CASI IGUAL A LA MAXIMA, ESTA OCURRE CUANDO $\Omega = \omega\sqrt{1-2\zeta^2}$. EN EL CASO DE $\dot{y}(t)$ y $\ddot{y}$ (t), EL MAXIMO OCURRE, RESPECTIVAMENTE, CUANDO

$\Omega = \omega$ Y $\Omega = \frac{\omega}{\sqrt{1-2\zeta^2}}$ · SI $\zeta \leq 20\%$, LOS VALORES DE ESTAS Ω NO DIFIEREN EN MAS DE 2%.

EL MAXIMO VALOR DE B_d (PARA $\Omega = \omega\sqrt{1-2\zeta^2}$) ES

$$(B_d)_{MAX} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{O} \qquad (B_d)_{MAX} = \frac{(\Omega/\omega)^2}{2\zeta\sqrt{1-\zeta^2}}$$

SI SE TIENE FUERZA EXTERNA O MOVIMIENTO DEL SUELO, RESPECTIVAMENTE.

SE OBSERVA EN ESTAS ECUACIONES QUE SI $\zeta=0$, $(B_d)_{MAX} = \infty$.

SI SE ANALIZA LA SOLUCION GENERAL DE LA ECUACION DIFERENCIAL DE MOVIMIENTO PARA EL CASO DE CONDICIONES INICIALES NULAS Y $\beta=1$ SE TIENE QUE:

$$y(t) = e^{-ht}(A \text{ sen } \omega' t + B \cos \omega' t) - \frac{p_o}{k} \; \frac{\cos\omega t}{2\zeta}$$

$$y(0) = B - p_o/(2\zeta k) = 0$$

DE DONDE, HACIENDO $y(0)=0$ Y $\dot{y}(0)=0$, SE OBTIENEN:

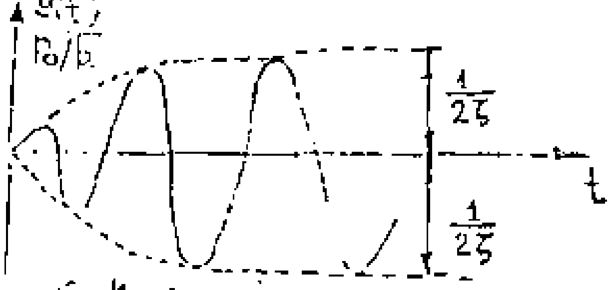
$$A = \frac{p_o}{k} \; \frac{\omega}{2\omega'} = \frac{p_o}{k} \; \frac{1}{2\sqrt{1-\zeta^2}} \quad ; \quad B = \frac{p_o}{k} \; \frac{1}{2\zeta}$$

POR LO QUE

$$y(t) = \frac{1}{2\zeta} \; \frac{p_o}{k} \left[e^{-ht} \left(\frac{\zeta}{\sqrt{1-\zeta^2}} \text{ sen}\omega' t + \cos\omega' t\right) - \cos\omega t\right]$$

PARA AMORTIGUAMIENTOS PEQUEÑOS:

$$\frac{y(t)}{p_o/k} \doteq \frac{1}{2\zeta} \; (e^{-ht}-1)\cos\omega t$$

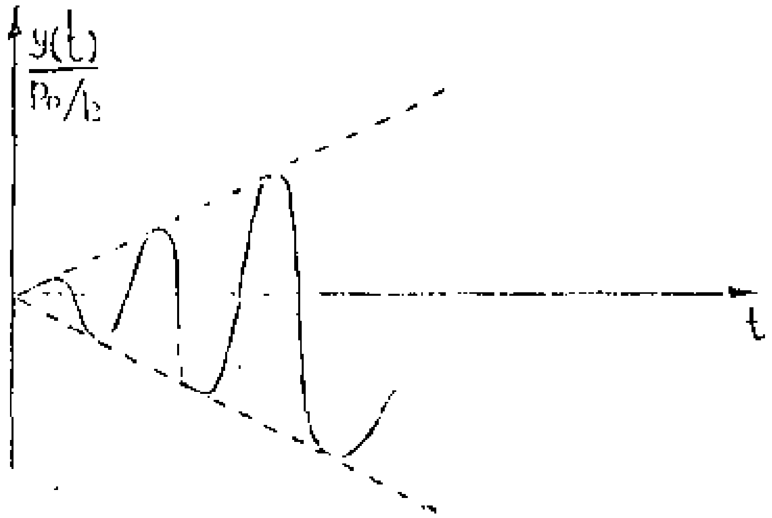


Si $\zeta>0$ y $\beta=1$

SI $\zeta=0$, APLICANDO LA REGLA DE L'HOSPITAL, SE OBTIENE:

$$\frac{y(t)}{p_o/k} = \frac{1}{2} \; (\text{sen}\omega t - \omega t \cos\omega t)$$

O SEA, EL MAXIMO DE LA RESPUESTA TIENDE A INFINITO GRADUALMENTE.



CARACTERISTICAS DINAMICAS DE LOS REGISTRADORES DE SISMOS.

SI LA ACELERACION DE LA BASE DE UN INSTRUMENTO ES ARMONICA, DADA POR LA ECUACION

$$\ddot{x}_o(t) = a \operatorname{sen}\Omega t$$

EL FACTOR DE AMPLIFICACION RESULTA SER

$$\bar{B}_d = \frac{1}{\sqrt{(1-\frac{\Omega^2}{\omega^2})^2 + (2\zeta\frac{\Omega}{\omega})^2}} \quad \frac{1}{\omega^2} = \frac{B_d}{\omega^2}$$

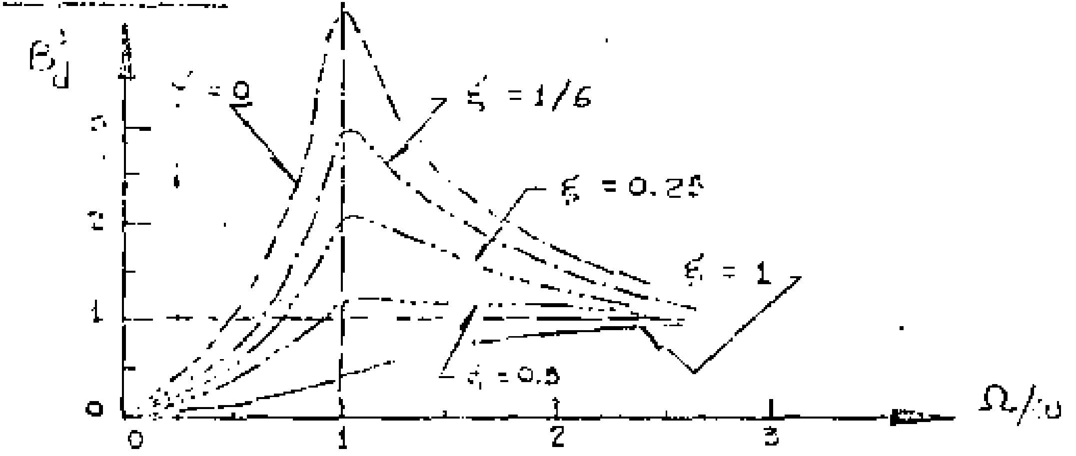
PUESTO QUE LA FIG I CORRESPONDE A B_d, Y EN ELLA SE OBSERVA QUE PARA $\zeta = 0.7$ SE TIENE $B_d \doteq 1$ PARA $0 \leq \Omega/\omega \leq 0.6$, SE CONCLUYE QUE EL DESPLAZAMIENTO DE LA MASA DE UN SISTEMA ES PROPORCIONAL A LA ACELERACION DE SU BASE, SI ESTE TIENE AMORTIGUAMIENTO DEL 70% Y SI LAS EXCITACIONES QUE SE TRATAN DE REGISTRAR TIENEN FRECUENCIAS INFERIORES AL 60% DE LA FRECUENCIA NATURAL DEL SISTEMA. SI ESTO SE CUMPLE, EL APARATO RESULTA SER UN ACELEROMETRO.

EN INGENIERIA SISMICA LA MAXIMA FRECUENCIA DE INTERES ES DEL ORDEN DE 10 CPS (T = 0.1 SEG), POR LO QUE LOS ACELEROMETROS TIENEN FRECUENCIA NATURAL DE 16 A 20 CPS.

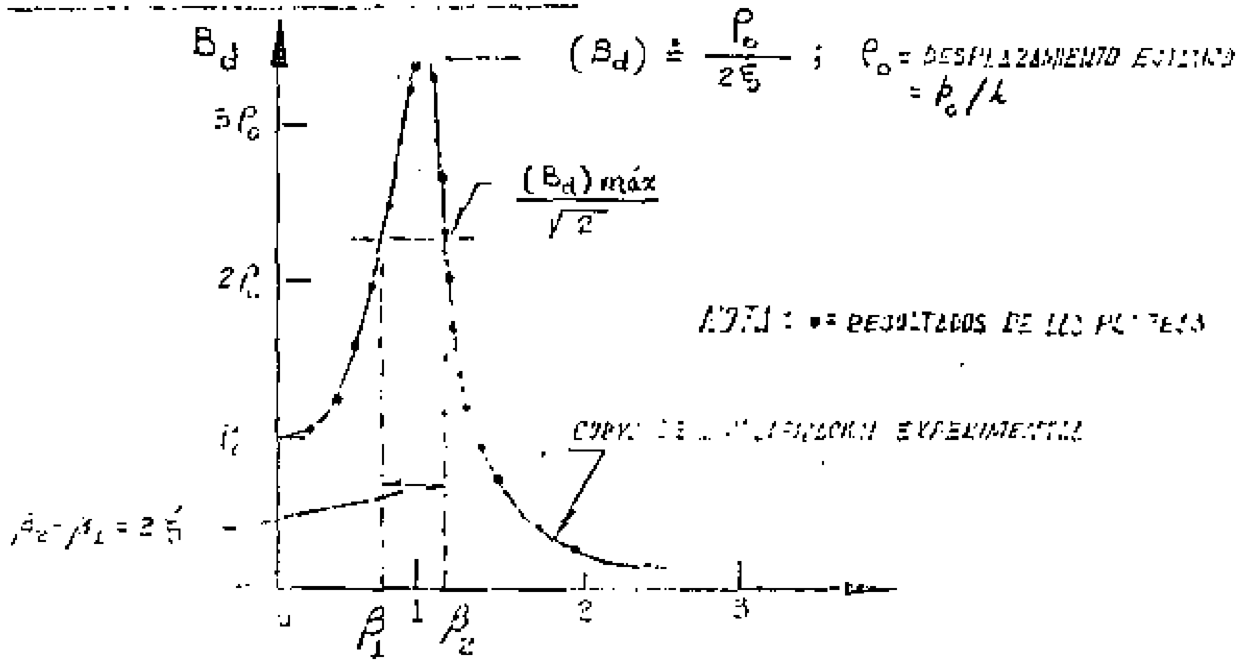
POR OTRA PARTE SI LA EXCITACION DEL SUELO ES $x_o = a\ \text{sen}\Omega t$, O SEA, $\ddot{x} = -a\ \Omega^2 \text{sen}\Omega t$, ENTONCES EL FACTOR DE AMPLIFICACION RESULTA SER EL SEÑALADO EN LA ECUACION (20), ES DECIR,

$$B'_d = \frac{(\Omega/\omega)^2}{\sqrt{(1-(\Omega/\omega)^2) + (2\zeta\Omega/\omega)^2}}.$$

EN LA GRAFICA CORRESPONDIENTE SE OBSERVA QUE SI $\zeta=0.5$ Y $\Omega>\omega$ EL DESPLAZAMIENTO DE LA MASA ES PROPORCIONAL AL DEL SUELO; SI ESTO SE CUMPLE, EL APARATO, CONSTITUYE UN DESPLAZOMETRO, CONOCIDO TAMBIEN COMO SISMOMETRO.



DETERMINACION EXPERIMENTAL DEL AMORTIGUAMIENTO DE UNA ESTRUCTURA MEDIANTE VIBRACIONES FORZADAS ARMONICAS

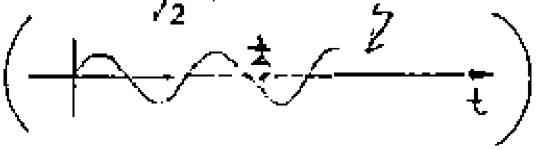


SI SE DETERMINA B_d EXPERIMENTALMENTE MEDIANTE UNA SERIE DE PRUEBAS DE VIBRACION FORZADA CON FUERZAS ARMONICAS, Y ADEMAS SE DETERMINA p_o, ENTONCES

$$\boxed{\zeta \doteq \frac{p_o}{2(B_d)_{MAX}}} \qquad (24)$$

OTRO METODO PARA DETERMINAR ζ CON BASE EN LA CURVA EXPERIMENTAL DE B_d SE CONOCE CON EL NOMBRE DE "METODO DEL ANCHO DE BANDA DE LA MITAD DE POTENCIA". ESTE SE BASA EN DETERMINAR LAS FRECUENCIAS QUE CORRESPONDEN AL VALOR rms DE LA AMPLITUD EN RESONANCIA, EL CUAL VALE $(B_d)_{MAX}/\sqrt{2}$; SEAN β_2 Y β_1 ESTAS FRECUENCIAS. DE LA ECUACION DE B_d SE OBTIENE:

rms $= \frac{A}{\sqrt{2}}$ = RAIZ CUADRADA DEL VALOR MEDIO CUADRATICO



$$\frac{1}{\sqrt{2}} \quad \frac{p_o}{2\zeta} = p_o/\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}$$

ELEVANDO AL CUADRADO AMBOS MIEMBROS:

$$\frac{1}{8\zeta^2} = \frac{1}{(1-\beta^2)^2 + (2\zeta\beta)^2}$$

DE DONDE $\beta^2 = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$

DE AQUI, DESPRECIANDO EL TERMINO ζ^2 DEL RADICAL, SE OBTIENE

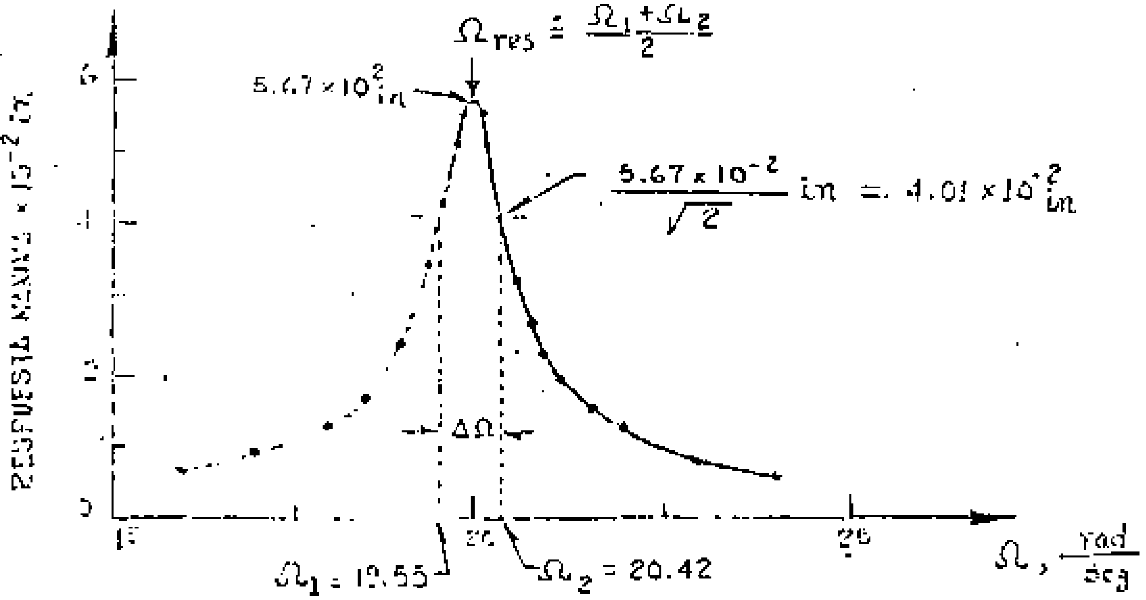
$$\beta_1^2 \doteq 1 - 2\zeta - 2\zeta^2 \quad ; \quad \beta_1 \doteq 1 - \zeta - \zeta^2$$

$$\beta_2^2 \doteq 1 + 2\zeta - 2\zeta^2 \quad ; \quad \beta_2 \doteq 1 + \zeta - \zeta^2$$

$$\beta_2 - \beta_1 \doteq 2\zeta$$

DE DONDE $$\boxed{\zeta \doteq \frac{\beta_2 - \beta_1}{2}} \qquad (25)$$

EJEMPLO



DE LA EC (25) $\quad A\Omega = \Omega_2 - \Omega_1 = 0.87 \frac{\text{RAD}}{\text{SEG}}$

$$\zeta \doteq \frac{\beta_2 - \beta_1}{2} = \frac{\frac{\Omega_2}{\Omega_{res}} - \frac{\Omega_1}{\Omega_{res}}}{2} = \frac{\Omega_2 - \Omega_1}{\Omega_2 + \Omega_1} = \frac{0.87}{39.97} \approx 2.18\%$$

METODO NUMERICO β DE NEWMARK PARA RESOLVER EL PROBLEMA DE VIBRACIONES FORZADAS.

EL METODO QUE A CONTINUACION SE DESCRIBE ES ADAPTABLE A SISTEMAS NO LINEALES CON VARIOS GRADOS DE LIBERTAD.

PROCEDIMIENTO:

1. SEAN y_i, $\dot{y}_i$, y $\ddot{y}_i$, CONOCIDOS EN EL INSTANTE t_i, y $t_{i+1} = t_i + \Delta t$. SUPONGAMOS EL VALOR DE $\ddot{y}_{i+1}$

2. CALCULEMOS $\dot{y}_{i+1} \doteq \dot{y}_i + (\ddot{y}_i + \ddot{y}_{i+1})\Delta t/2$ (26)

3. CALCULEMOS $y_{i+1} \doteq y_i + \dot{y}_i \Delta t + (\frac{1}{2} - \beta) \ddot{y}_i (\Delta t)^2 + \beta \ddot{y}_{i+1} (\Delta t)$ (27)

4. CALCULEMOS UNA NUEVA APROXIMACION PARA $\ddot{y}_{i+1}$ A PARTIR DE LA ECUACION DIFERENCIAL DE EQUILIBRIO:

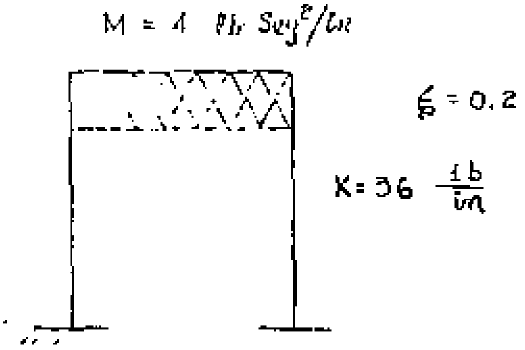
$$\ddot{y}_{i+1} \doteq -2\zeta\omega\dot{y}_{i+1} - \omega^2(y_{i+1} - y_{est}) - (\ddot{x}_o)_{i+1} \qquad (29)$$

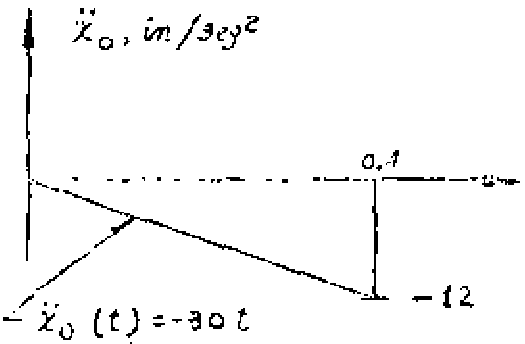
DONDE $y_{est} = p(t_{i+1})/k$

5. REPITAMOS LAS ETAPAS 2 A 4 EMPEZANDO CON EL NUEVO VALOR $\ddot{y}_{i+1}$ HASTA QUE EN DOS CICLOS CONSECUTIVOS SE TENGAN VALORES DE $\ddot{y}_{i+1}$ CASI IGUALES.

SE RECOMIENDAN VALORES DE β DE <u>1/6 A 1/4 Y $\Delta t \doteq 0.1T$</u> PARA ASEGURAR CONVERGENCIA Y ESTABILIDAD.

EJEMPLO





CALCULAR LA RESPUESTA DE LA ESTRUCTURA APLICANDO EL METODO β DE NEWMARK

$$\omega = \sqrt{K/M} = \sqrt{36/4} = 3 \; \frac{\text{RAD}}{\text{SEG}}$$

$$h = \zeta\omega = 0.2 \times 3 = 0.6 \quad ; \quad T = \frac{2\pi}{3} = 2.09 \text{ SEG}$$

TOMAREMOS β=0.2 Y Δt = 0.2 ($\doteq$ 0.1T) SUSTITUYENDO EN LAS ECS. (26), (27) y (28):

$$\dot{y}_{i+1} \doteq y_i + 0.1 \; (\ddot{y}_i + \ddot{y}_{i+1})$$

$$y_{i+1} \doteq y_i + 0.2\dot{y}_i + 0.012\ddot{y}_i + 0.008\ddot{y}_{i+1}$$

$$\ddot{y}_{i+1} = -1.2\dot{y}_{i+1} - 9y_{i+1} - (\ddot{x}_o)_{i+1}$$

EN t=0 SABEMOS QUE SE TIENE y=0, $\dot{y}$=0 Y $\ddot{y}$=0

EN t=0 + Δt = 0.2 SEG; SUPONGAMOS $\ddot{y}_{i+1}$ = 5.0 IN/SEG^2; $\ddot{x}_o$ =-6

$y_i = 0$

$\dot{y}_i = 0$

1er CICLO

$\dot{y}_{i+1} = 0 + 0.1\ (0 + 5) = 0.5$; $y_{i+1} = 0 + 0 + 0 + 0.008 \times 5 = 0.04$

$\ddot{y}_{i+1} = -1.2 \times 0.5 - 9 \times 0.04 - (-30 \times 0.2) = 5.04$

2º CICLO

$\dot{y}_{i+1} \doteq 0 + 0.1\ (0 + 5.04) = 0.504$; $y_{i+1} \doteq 0 + 0 + 0 + 0.008 \times 5.04 =$

$= 0.04032$

$\ddot{y}_{i+1} = -1.2 \times 0.504 - 9 \times 0.4032 - (-6) = 5.033$ IN/SEG^2

ESTOS CALCULOS SE PUEDEN ORGANIZAR MEDIANTE UNA TABLA COMO LA SIGUIENTE:

t SEG	$\ddot{x}_o$ IN/SEG^2	$\ddot{y}$ ING/SEG^2	$\dot{y}$ ING/SEG	y IN
0	0	0	0	0
0.2	-6	5.0000	0.5000	0.04000
		5.040	0.5040	0.04032
		5.033	0.5033	0.04026
		5.034	0.5034	0.04027
0.4	-12	8.0000	1.8078	0.26536
		7.442	1.7510	0.26079
		7.534	1.7602	0.26163
		7.533	1.7601	0.26162
0.4^+	0	-4.467	1.7601	0.26162
0.6	0	-6.000	0.7134	0.51204
		-5.464	0.7670	0.51633
		-5.550	0.7584	0.51564
		.	.	.
		.	.	.
		.	.	.

EN $t = 0.2 + \Delta t = 0.4$ SEG: $\dot{x}_o = -30 \times 0.4 = -12$

$\ddot{y}_i = 5.034$, $\dot{y}_i = 0.5034$, $y_i = 0.04027$

SUPONIENDO $\ddot{y}_{i+1} = 8{,}000$ SE OBTIENE:

1^{er} CICLO

$$\begin{cases} \dot{y}_{i+1} = 0.5034 + 0.1\ (5.034 + 8.000) = 1.8068 \\ y_{i+1} = 0.04027 + 0.2 \times 0.5034 + 0{,}012 \times 5.034 + 0.008 \times 8 = 0.26538 \\ \ddot{y}_{i+1} = -1{,}2 \times 1{,}8068 - 9 \times 0.26536 - (-12) = 7.442 \text{ IN/SEG}^2 \end{cases}$$

EN $t = 0.4^+$ SOLO CAMBIA $\ddot{y}$: $\ddot{y}_{0.4+} = \ddot{y}_{0.4-} + \ddot{x}_o = 7.533 - 12 = -4.467$

EN $t = 0.6$, $\ddot{y}_i = -4.467)$ $\dot{y}_i = 1.7601$; $y = 0.26162$

ESPECTROS DE RESPUESTA ESTRUCTURAL

RECORDEMOS QUE LA SOLUCION DEL PROBLEMA DE VIBRACIONES FORZADAS CON EXCITACION SISMICA ES

$$Y(t) = \frac{-1}{\omega'} \int_{-\infty}^{t} \ddot{x}_o(t-\tau)e^{-\zeta\omega(t-\tau)} \text{sen } \omega'(t-\tau)d\tau$$

DE LA OBSERVACION DE ESTA ECUACION SE CONCLUYE QUE EL DESPLAZAMIENTO RELATIVO, $Y(t)$, ES FUNCION DEL TIEMPO, t, EL AMORTIGUAMIENTO, ζ, Y LA FRECUENCIA CIRCULAR NATURAL, ω (O DEL PERIODO NATURAL):

$y(t) = f(t,\omega,\zeta)$

FIJEMOS UN VALOR DE ζ, POR EJEMPLO $\zeta=0$, Y LUEGO ASIGNEMOS VALORES A ω, POR EJEMPLO 0.1, 0.2. 0.3, ETC, HASTA CUBRIR UN INTERVALO DE INTERES, Y PARA CADA CASO CALCULEMOS LA FUNCION RESULTANTE DE APLICAR LA ECUACION ANTERIOR. CON ESTA OBTENEMOS

$y_1(t) = f_1(t,\ 0.1,\ 0) = f_1(t)$

$y_2(t) = f_2(t,\ 0.2,\ 0) = f_2(t)$

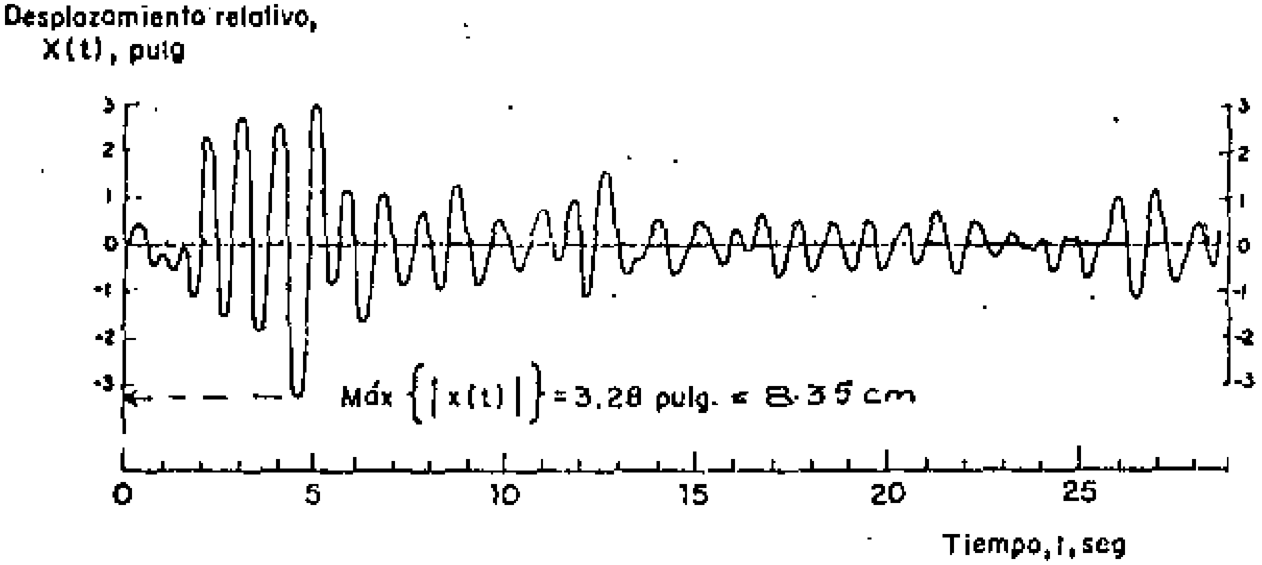
$y_3(t) = f_3(t,\ 0.2,\ 0) = f_3(t)$

SEAN $D_1 = MAX|y_1(t)| = D(\omega_1,\zeta)$

$D_2 = MAX|y_2(t)| = D(\omega_2,\zeta)$

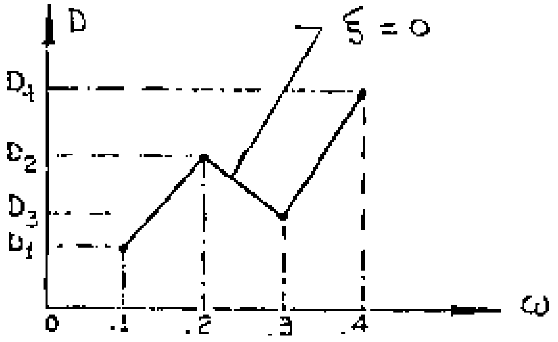
$D_3 = MAX|y_3(t)| = D(\omega_3,\zeta)$

$\vdots$



Respuesta de un sistema amortiguado simple con $T_1 = 1.0$ seg y $\zeta = 0.10$, al sismo de El Centro, Cal., 1940, componente N-S

EN TAL CASO, LA GRAFICA



ES EL *ESPECTRO DE RESPUESTA DE DESPLAZAMIENTOS* PARA $\zeta=0$. SI ESTE PROCESO DE REPITE FIJANDO OTROS VALORES DE ζ. POR EJEMPLO, $\zeta=0.02$, 0.05, 0.1, 0.2, ETC, SE OBTENDRAN LOS ESPECTROS DE DESPLAZAMIENTOS CORRESPONDIENTES.

DE MANERA ANALOGA SE PUEDEN OBTENER LOS ESPECTROS PARA OTROS TIPOS DE RESPUESTA, TALES COMO *VELOCIDAD RELATIVA*, *ACELERACION ABSOLUTA*, ETC, QUE SON, RESPECTIVAMENTE

$$V = \text{MAX}\left|\dot{y}(t)\right|_{\zeta,\omega} \;; \quad A = \text{MAX}\left|\ddot{x}(t)\right|_{\zeta,\omega} \qquad (29)$$

PSEUDO - ESPECTROS

ESTADISTICAMENTE SE HA ENCONTRADO QUE

$$S_V = \omega D \doteq V \qquad (30)$$

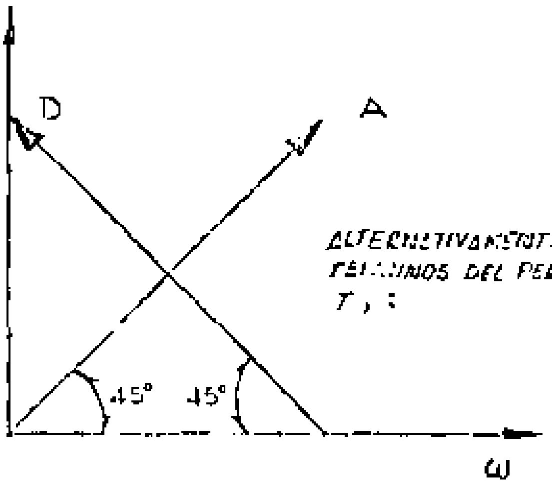
$$S_A = \omega^2 D \doteq A \doteq \omega V \qquad (31)$$

A S_V Y S_A SE LES LLAMA *PSEUDOESPECTROS*.

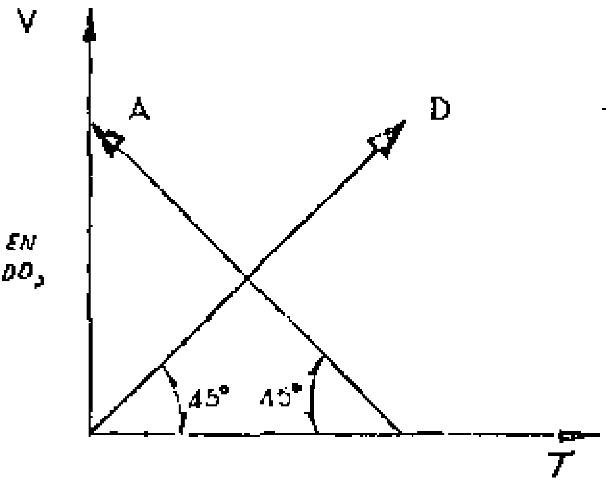
DE LA EC. (30): $\log D = \log V - \log \omega = \log V + \log T - \log 2\pi$

DE LA EC. (31): $\log A = \log V + \log \omega = \log V - \log T + \log 2\pi$

ESTAS ECUACIONES CORRESPONDEN A LINEAS RECTAS EN PAPEL LOGARITMICO; LA PRIMERA CON PENDIENTE -1 Y LA SEGUNDA CON PENDIENTE +1, SI SE USA ω COMO VARIABLE INDEPENDIENTE; SI SE USA T, LA PRIMERA TENDRA PENDIENTE + 1, Y LA SEGUNDA, -1.

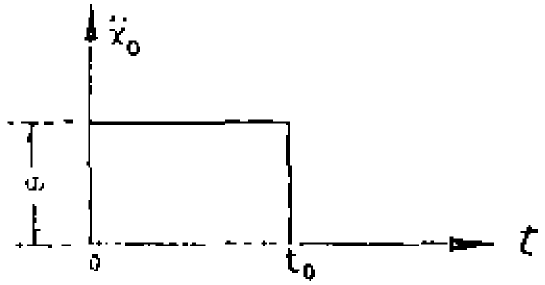


ALTERNATIVAMENTE, EN TERMINOS DEL PERIODO, T, :



EJEMPLO

CALCULAR EL ESPECTRO CORRESPONDIENTE A LA EXCITACION (CONSIDERESE $\zeta=0$)



EN UN EJEMPLO ANTERIOR SE OBTUVO

$$y(t) = \frac{-a}{\omega 2}(1 - \cos\omega t), \text{ SI } 0\le t\le t_o$$

$$D = \text{MAX}|Y(t)| = \frac{2a}{\omega 2} \quad ; \; 0\le \frac{T}{2}\le t_o, \quad (0\le T\le 2t_o)$$

$$S_V = \omega D = \frac{2a}{\omega} \quad , \; S_A = \omega V = 2a \; .$$

Y $$D = \text{MAX}|y(t)| = \frac{2a}{\omega^2} \text{ sen } \frac{\omega t_o}{2} \quad , \quad \text{SI } T>2\ t_o$$

$$S_V = \omega D = \frac{2a}{\omega} \left|\text{sen } \frac{\omega t_o}{2}\right| \; ; \quad S_A = \omega V = 2a\left|\text{sen}\frac{\omega t_o}{2}\right|$$

$$\lim_{\omega\to 0} S_V = \lim_{\omega\to 0} \left\{at_o \frac{\text{sen}\frac{\omega t_o}{2}}{\frac{\omega t_o}{2}}\right\} = at_o$$

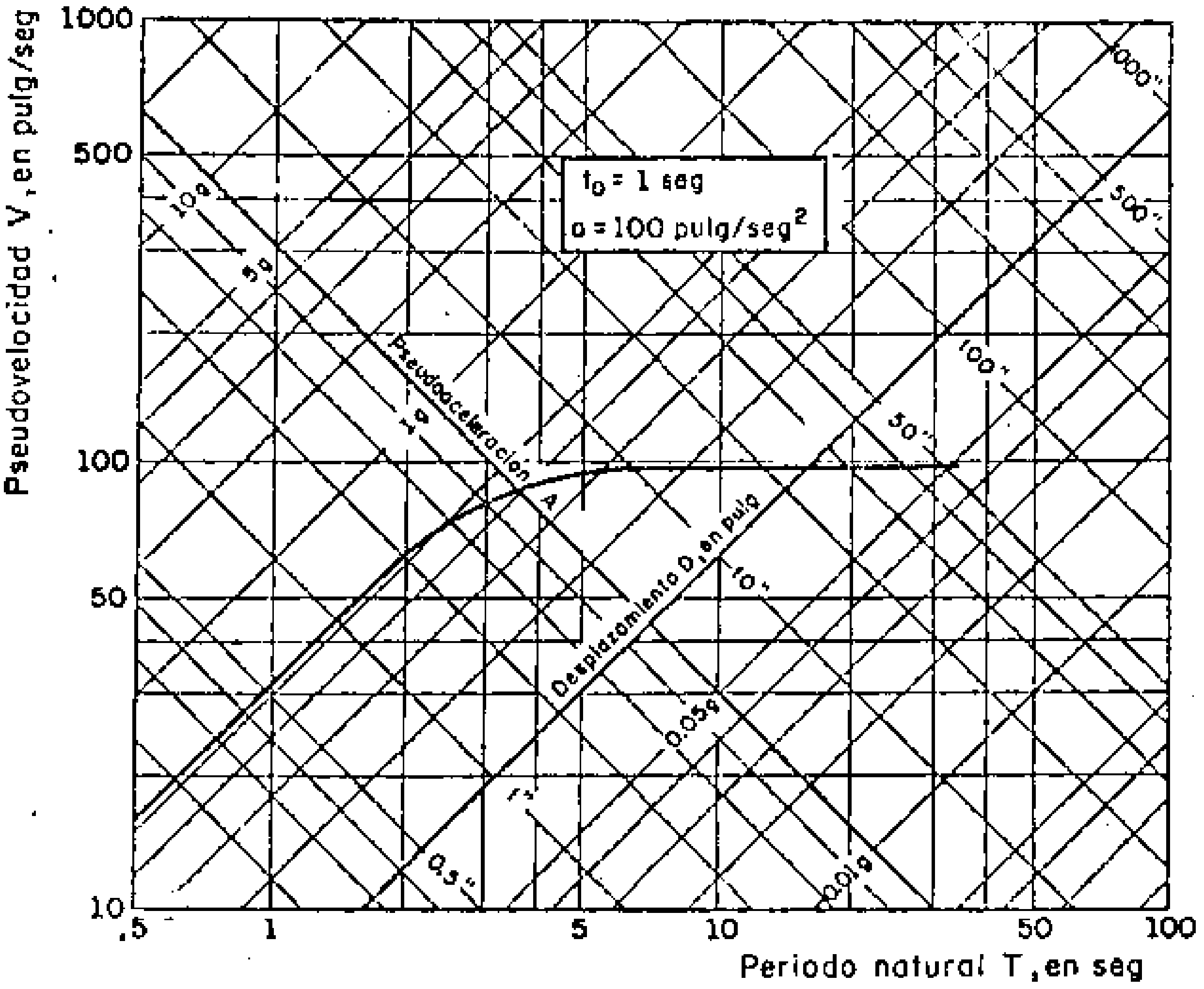
CASO PARTICULAR: SI t_o = 1 SEG y a = 100 IN/SEG^2

$$S_V = \frac{2 \times 100}{\frac{2\pi}{T}} = \frac{100}{\pi} T \text{ , SI } 0 \le T \le 2 \text{ SEG}$$

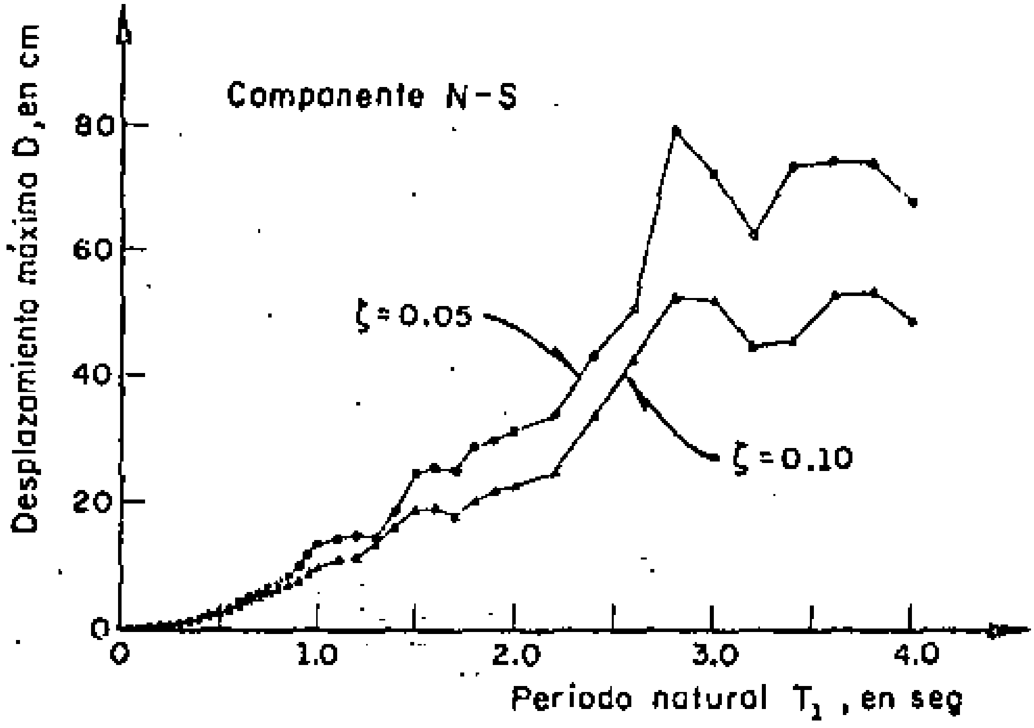
$$S_V = \frac{100T}{\pi}\left|\text{sen}\frac{\frac{2\pi}{T}x1}{2}\right| =$$

$$= \frac{100T}{\pi}\left|\text{sen}\ \frac{\pi}{T}\right| \text{ SI } T>2 \text{ SEG}$$

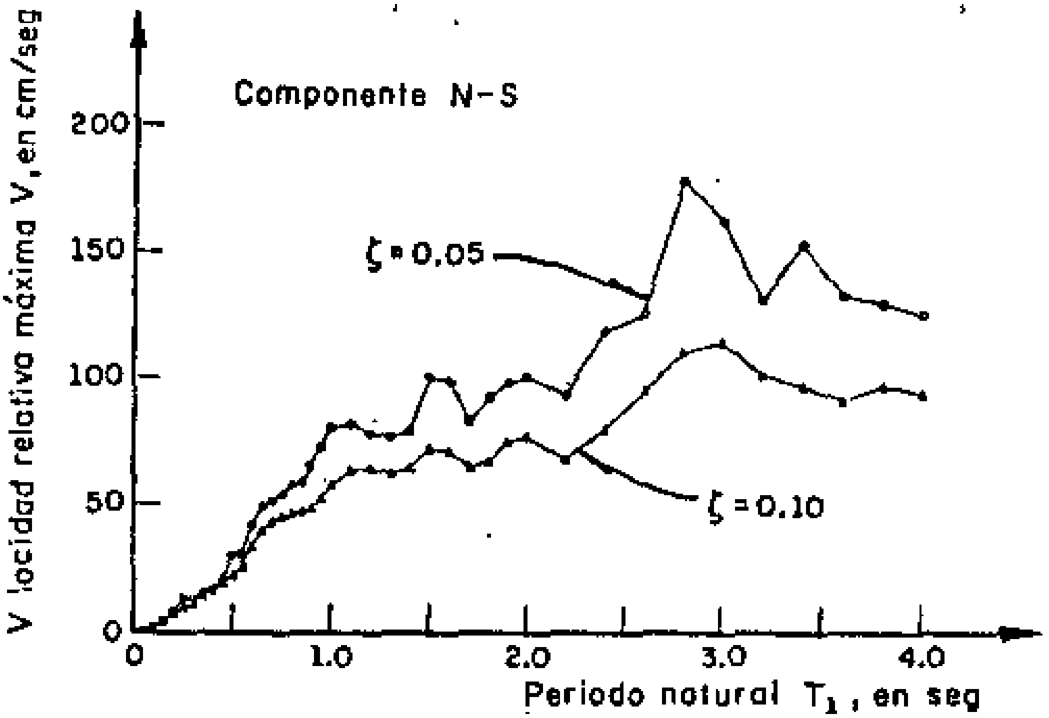
$$\lim_{T \to \infty} S_V = 100 \text{ IN/SEG}$$

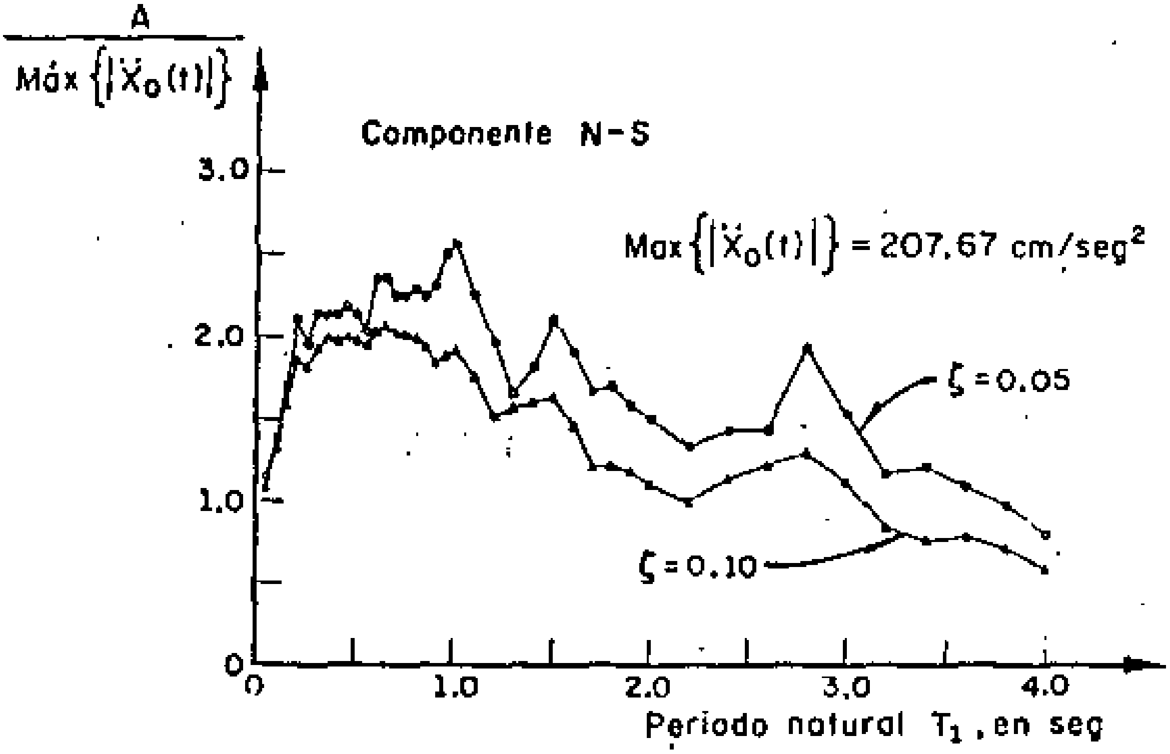


Espectro no amortiguado correspondiente a un pulso rectangular de aceleraciones. Según N. Newmark y E. Rosenblueth, ref 1

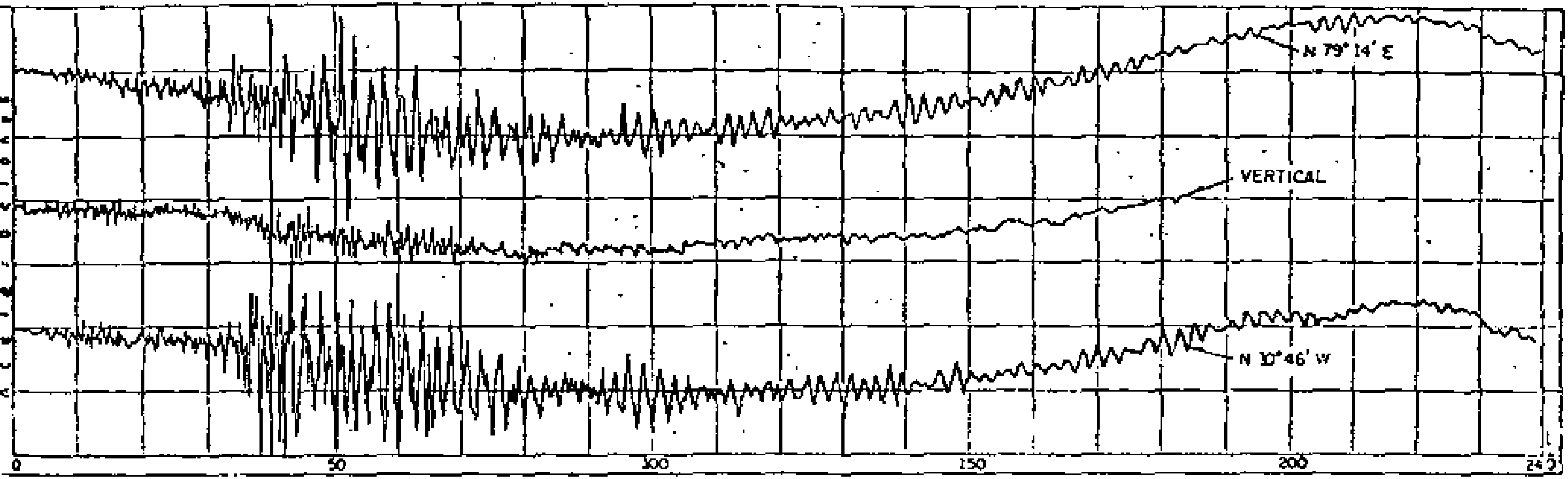


Espectro de desplazamientos. Sismo de Tokachi-Oki, Japón (1968). Según H. Tsuchida, E. Kurata y K. Sudo, ref 4



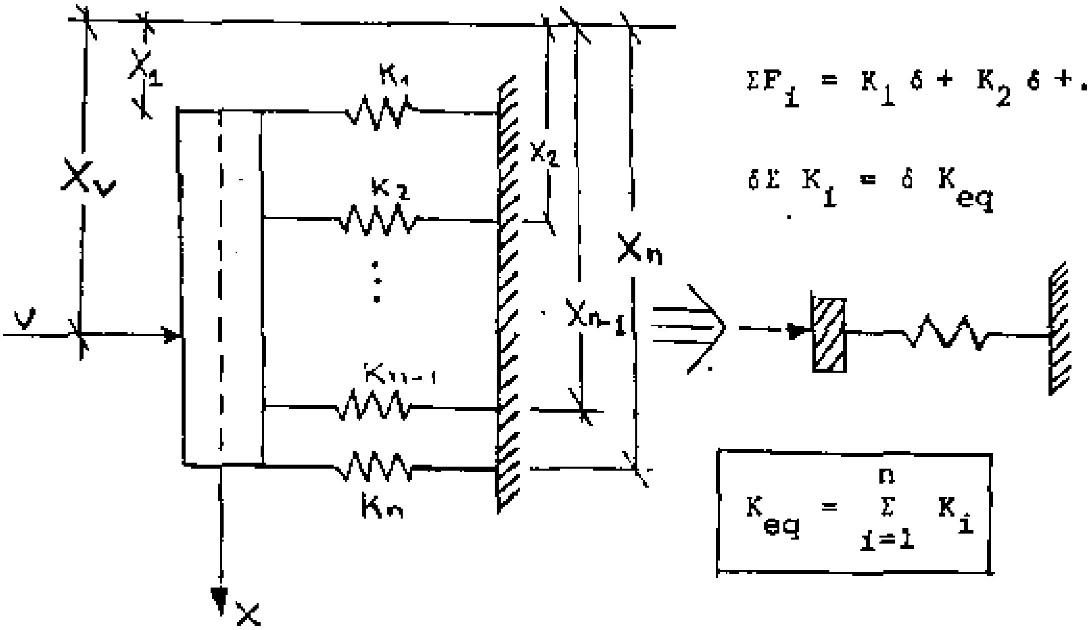


Espectros de velocidades y de aceleraciones. Sismo de Tokachi-Oki, Japón (1968). Según H. Tsuchida, E. Kurata y K. Sudo, ref 4



Acelerogramas originales del sismo registrado el 11-V-1962, en la ALAMEDA CENTRAL, Mex. D. F.

DISTRIBUCION DE LAS FUERZAS CORTANTES EN UN ENTREPISO



$$\Sigma F_i = K_1\delta + K_2\delta + \ldots + K_n\delta = V = K_{eq}\delta$$

$$\delta\Sigma K_i = \delta K_{eq}$$

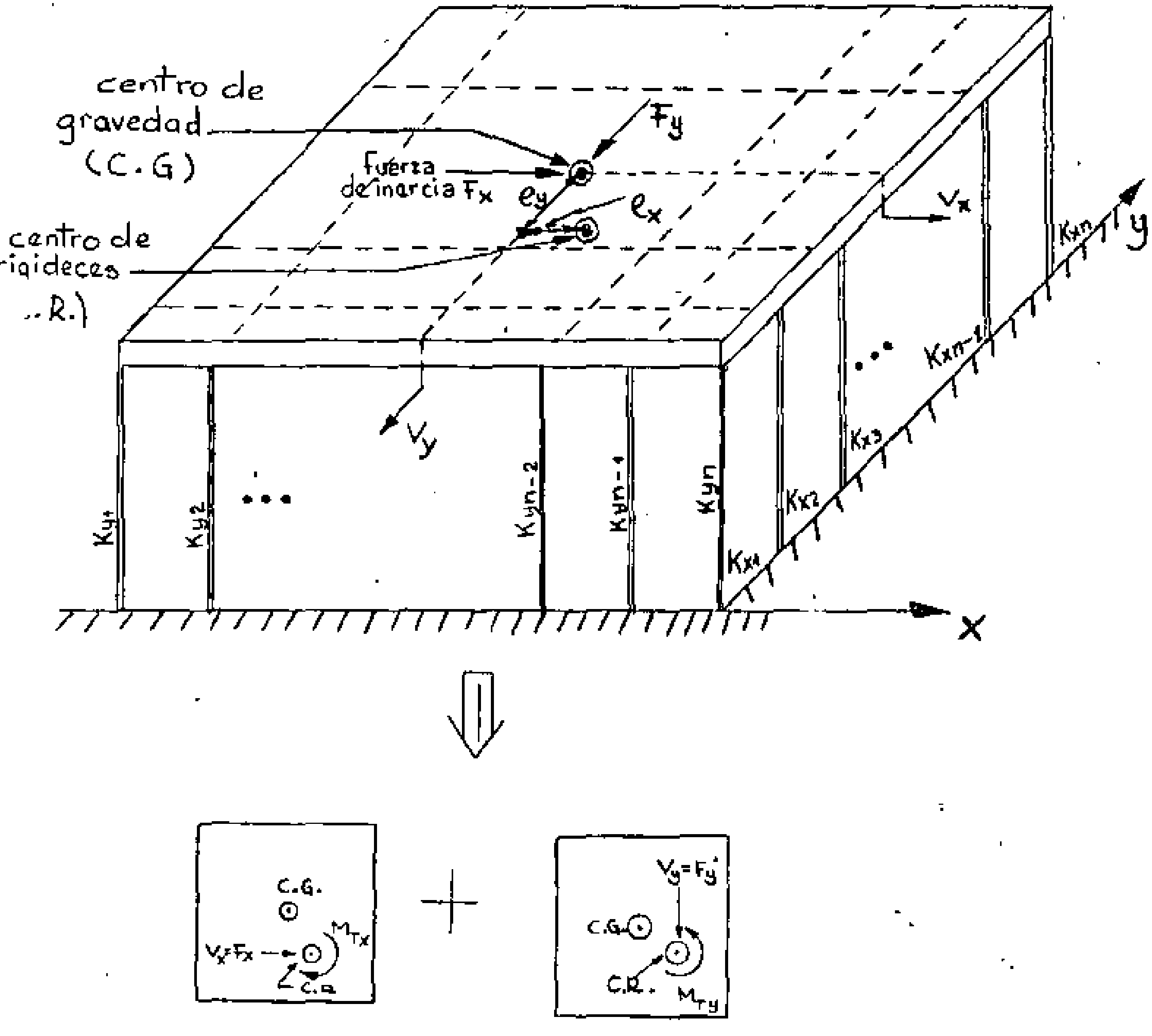
$$\boxed{K_{eq} = \sum_{i=1}^{n} K_i}$$

$$\Sigma M_i = \Sigma F_i X_i = \Sigma K_i\delta X_i = \delta\Sigma K_i X_i = VX_v = K_{eq}\delta X_v$$

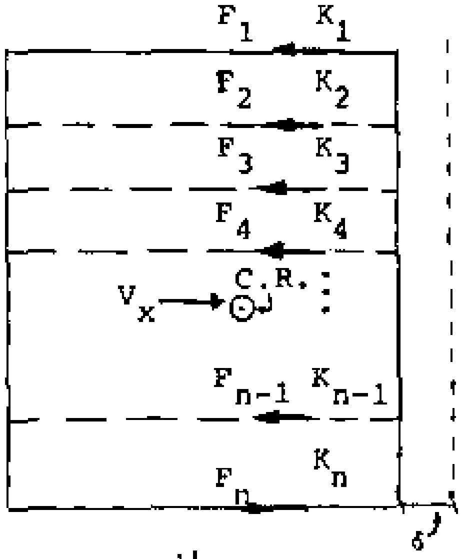
$$\boxed{X_v = \frac{\sum\limits_{i=1}^{n} K_i X_i}{\sum\limits_{i=1}^{n} K_i}}$$

⟺ POSICION DEL CENTRO DE RIGIDECES

DISTRIBUCION DE FUERZAS CORTANTES DIRECTAS Y POR TORSION



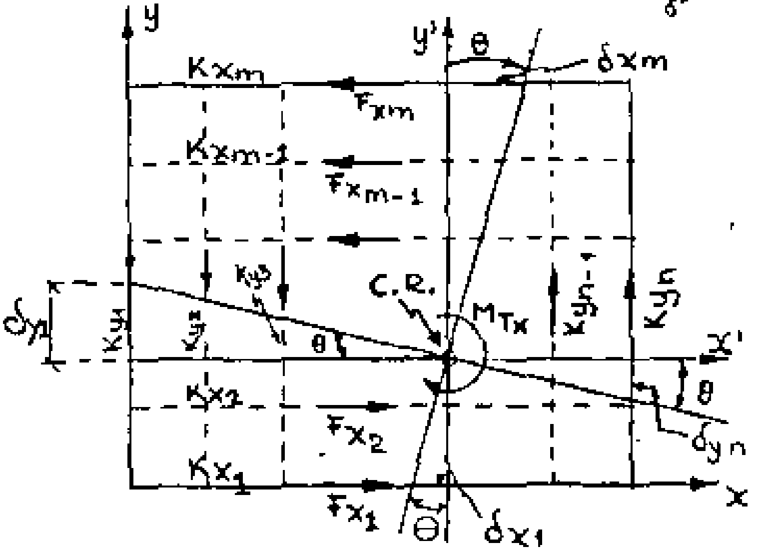
VEAMOS COMO SE DISTRIBUYEN LAS FUERZAS CORTANTES EN LOS MARCOS



$$F_i = K_i \delta$$

$$\Sigma F_i = \Sigma K_i \delta = V_x \therefore \delta = \frac{V_x}{\Sigma K_i}$$

$$\boxed{F_i = V_x \frac{K_i}{\sum\limits_{i=1}^{n} K_i}}$$



$$F_{x_i} = K_{x_i} \delta_{x_i} = K_{x_i} Y'_i \theta$$

$$F_{x_i} = K_{y_i} \delta_{y_i} = K_{y_i} X'_i \theta$$

$$\Sigma M_{C.R.} = \Sigma F_{x_i} Y'_i + \Sigma F_{y_i} X'_i$$

$$= \theta (\Sigma K_{x_i} Y'^2_i + \Sigma K_{y_i} X'^2_i)$$

$$= M_{T_x}$$

DE DONDE $\theta = \dfrac{M_T}{\Sigma K_{x_i} Y'^2_i + K_{y_i} X'^2_i}$

POR LO QUE

$$\boxed{F_{x_i} = M_{TX} \frac{K_{x_i} Y'_i}{\Sigma K_{x_i} Y'^2_i + K_{y_i} X'^2_i}}$$

$$\boxed{F_{y_i} = M_{TX} \frac{K_{y_i} X'_i}{\Sigma K_{x_i} Y'^2_i + \Sigma K_{y_i} X'^2_i}}$$

SISTEMAS NO LINEALES DE UN GRADO DE LIBERTAD

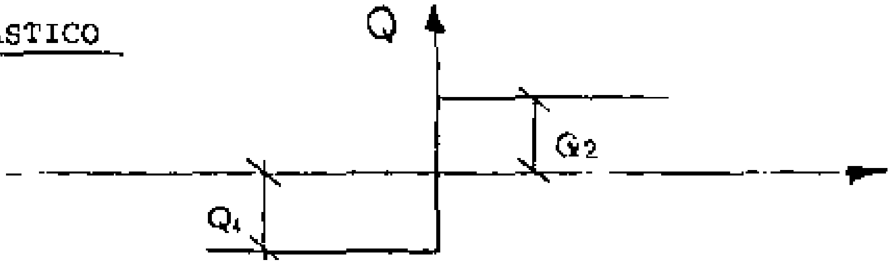
ECUACION DE MOVIMIENTO:

$$M\ddot{x} + Q(y,\dot{y}) = P(t) \quad ; \quad y = x-x_o = \text{DESPLAZAMIENTO RELATIVO}$$

SI $Q(y, \dot{y}) = KY + C\dot{Y}$ SE TIENE EL SISTEMA ELASTICO LINEAL

MODELOS PARTICULARES

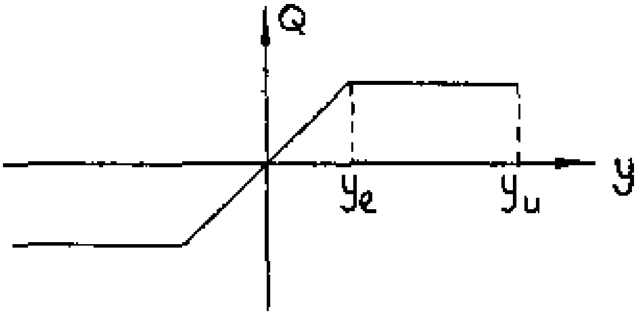
1. RIGIDO-PLASTICO



$Q = -Q_1 + C\dot{y}$, SI $\dot{y}<0$

$Q = Q_2 + C\dot{y}$, SI $\dot{y}<0$ EN DONDE C = CONSTANTE. SE HA EMPLEADO COMO MODELO EN EL ANALISIS DE TALUDES Y CORTINAS DE PRESAS DE TIERRA Y ENROCAMIENTO

2. ELASTO-PLASTICO



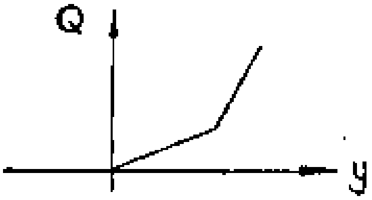
$Q = Q_1(y) + Cy$

SE EMPLEA COMO MODELO EN EL ANALISIS DE ESTRUCTURAS DUCTILES.

FACTOR DE DUCTILIDAD $= \mu = y_u/y_e$

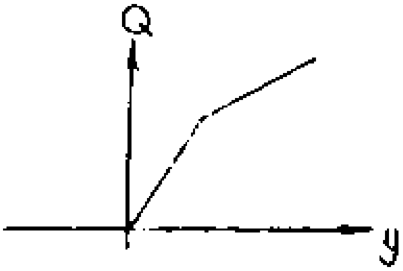
y_u = DESPLAZAMIENTO MAXIMO QUE PUEDE SOPORTAR EL SISTEMA SIN FALLAR.

3. SISTEMA BILINEAL



CON ENDURECIMIENTO

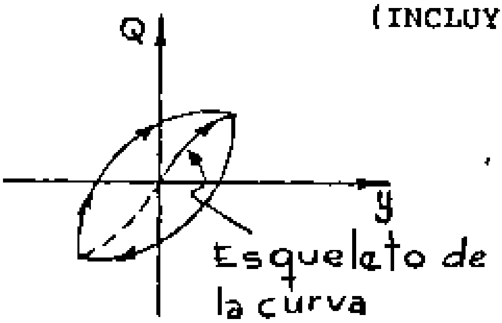
SE USA COMO MODELO PARA ANALISIS DE PUENTES COLGANTES



CON ABLANDAMIENTO

SE USA COMO MODELO DE SISTEMAS QUE SE DEGRADAN POR AGRIETAMIENTO (MUROS DE MAMPOSTERIA, POR EJEM).

4. TIPO MASING



(INCLUYE A LOS ANTERIORES COMO CASOS ESPECIALES)

$$\frac{Q - Q_o}{2} = Q_1\left(\frac{y - y_o}{2}\right)$$

Q_o = FUERZA EN $y = y_o$

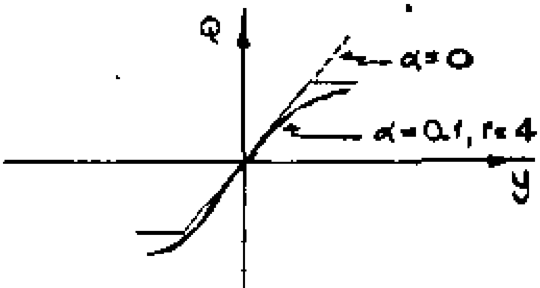
y_o = DESPLAZAMIENTO EN EL CUAL EL PROCESO SE INVIRTIO ($\dot{y}$ CAMBIO DE SIGNO) POR ULTIMA VEZ

CASO PARTICULAR DEL ESQUELETO

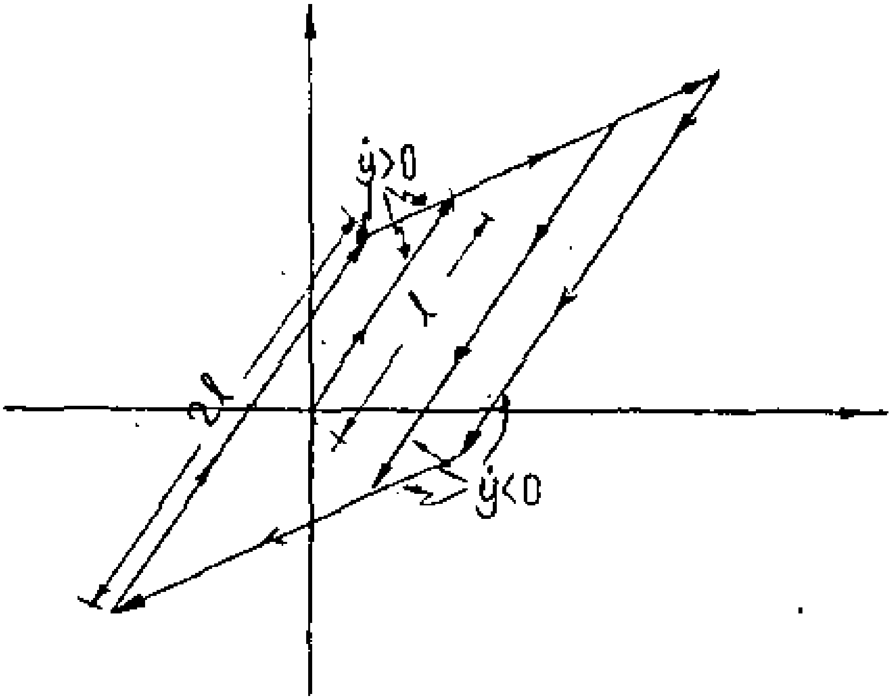
$$\frac{y}{y_1} = \frac{Q}{Q_1} + \alpha\left(\frac{Q}{Q_1}\right)^r$$

(MODELO RAMBER - OSGOOD)

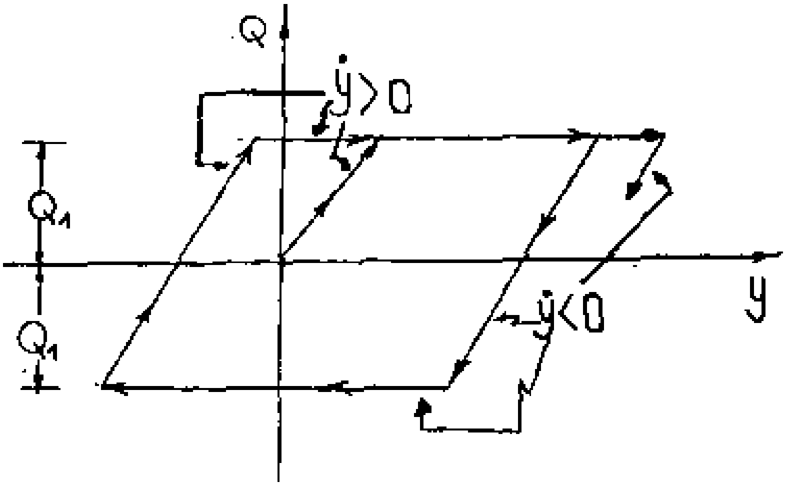
DONDE y_1, Q_1, α y r SON CONSTANTES POSITIVAS



EJEMPLO: CASO BILINEAL



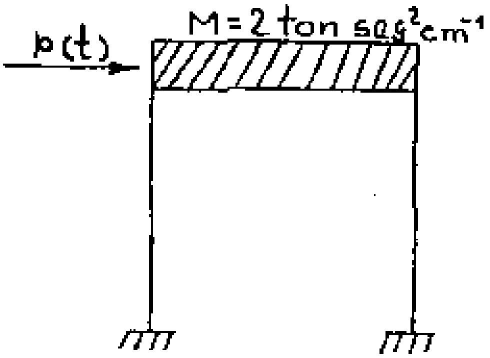
EJEMPLO CASO ELASTOPLASTICO.

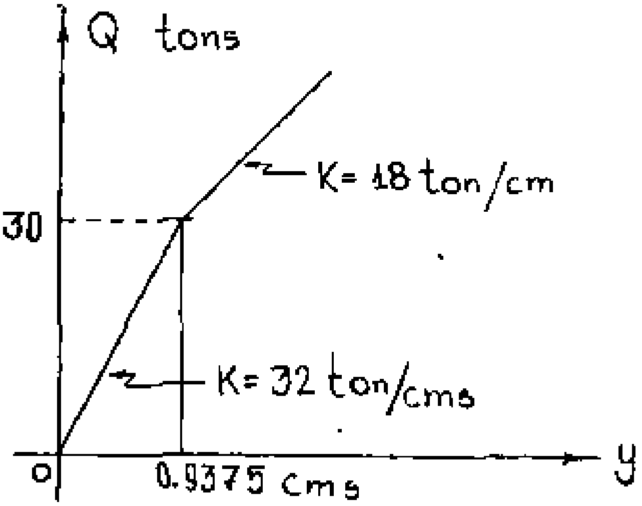


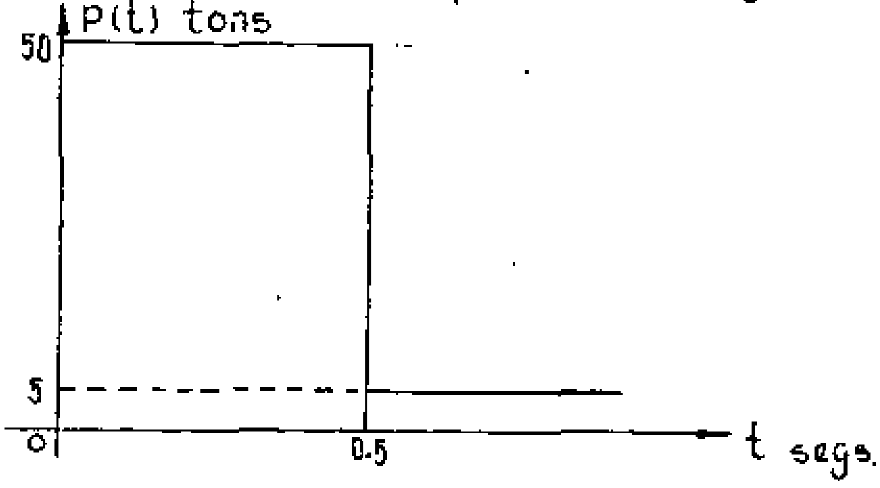
METODO β DE NEWMARK

PARA EL ANALISIS DE SISTEMAS NO LINEALES SE PUEDE USAR EL METODO β DE NEWMARK DESCRITO ANTERIORMENTE.

EJEMPLO







ECUACION DE EQUILIBRIO DINAMICO , $M\ddot{Y} + Q(Y) = P(t)$

$$\ddot{Y} = \frac{P(t) - Q(Y)}{M} = \frac{P(t) - Q(y)}{2} \qquad (I)$$

PARA LA APLICACION DEL METODO DE NEWMARK SE TIENEN LAS SIGUIENTES EXPRESIONES:

$$t_{i+1} = t_i + At$$

$$\dot{Y}_{i+1} = \dot{Y}_i + (\ddot{Y}_i + \ddot{Y}_{i+1})\ \Delta t/2$$

$$Y_{i+1} = Y_i + \dot{Y}_i\ At + (0.5 - \beta)\ddot{Y}_i\ (\Delta t)^2 + \beta\ \ddot{Y}_{i+1}\ (\Delta t)^2$$

CONSIDERANDO $\Delta t = 0.10$ SEG. Y $\beta = 1/6$ SE PUEDE ESCRIBIR:

$$\dot{Y}_{i+1} = \dot{Y}_i + \frac{1}{20}\,(\ddot{Y}_i + \ddot{Y}_{i+1}) \qquad \text{(II)}$$

$$Y_{i+1} = Y_i + \dot{Y}_i(0.10) + \frac{1}{600}\,(2\ddot{Y}_i + \ddot{Y}_{i+1}) \qquad \text{(III)}$$

EL PROCEDIMIENTO DE CALCULO ES COMO SIGUE:

SE ASUME $\ddot{Y}_{i+1}$

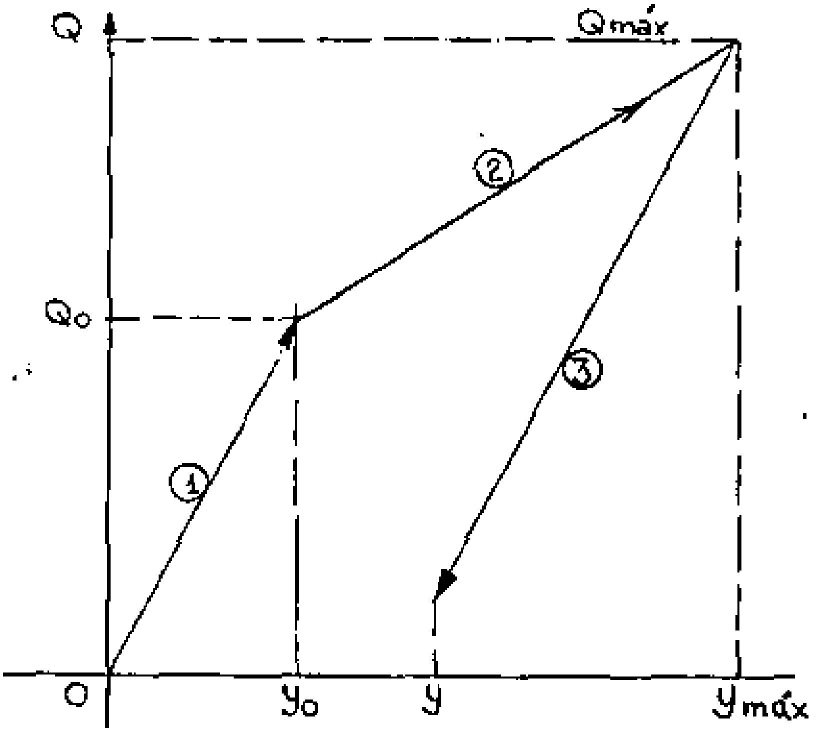
SE CALCULA $\dot{Y}_{i+1}$ CON LA ECUACION (II)

SE CALCULA Y_{i+1} CON LA ECUACION (III)

SE CALCULA UN MEJOR VALOR DE $\ddot{Y}_{i+1}$ CON LA ECUACION (I),

ETC.

PARA LA FUNCION DE RESISTENCIA Q SE TIENEN LOS SIGUIENTES CASOS:



1. COMPORTAMIENTO ELASTICO , $Q = 32\ Y$ TONS

2. CAMBIO DE RIGIDEZ , $Q = 30 + 18\ (Y - Y_o)$ TON

3. DESCARGA , $Q = Q_{\text{máx}} - 32(Y_{MAX} - Y)$ TONS

ESTA ULTIMA EXPRESION MANTIENE SU VALIDEZ HASTA QUE, $(Y_{MAX} - Y) \leq 2Y_o$

Y_o = 0.9375 CMS ; Q_o = 30.0 TON

PARA t = O, $\ddot{y} = \frac{P}{M} = \frac{50}{2} = 25$; $y = 0$; $\dot{y} = 0$

PARA t = 0.10, $y_i = \dot{y}_i = 0$; $\ddot{y}_i = 25$

1er. CICLO

SEA $\ddot{y}_{i+1} = 20$ COMO PRIMER TANTEO. EN TAL CASO

$$\dot{y}_{i+1} = 0 + \frac{1}{20}(0 + 25) = 2.25$$

$$y_{i+1} = \dot{0} + 0.10 \times 0 + \frac{1}{600}(2 \times 25 + 20) = 0.1167$$

$$Q = 32 \times 0.1167 = 3.7330$$

$$\ddot{y}_{i+1} = \frac{50 - 3.733}{2} = 23.134$$

2o. CICLO

$$\dot{y}_{i+1} = 23.134/2 = 16.567$$

$$y_{i+1} = 73.134/600 = 0.1219$$

$$Q = 32 \times 0.1219 = 3.9000$$

$$\ddot{y}_{i+1} = (50 - 3.9)/2 = 23.050$$

3er. CICLO

.

.

.

4o. CICLO

$\ddot{y}_{i+1} = 23.052$

$\dot{y}_{i+1} = 23.052/2 = 2.4026$

$y_{i+1} = 73,052/600 = 0.12175$

$Q = 32 \times 0.12175 = 3,8960$

$\ddot{y} = (50 - 3,8960)/2 = 23.052$ ... ETC.

LOS CALCULOS BASICOS SE MUESTRAN EN LA TABLA SIGUIENTE:

t SEGS	p TONS	$\ddot{Y}$ CM SEG^{-2}	$\dot{Y}$ CM SEG^{-1}	Y CMS	Q TONS
0.0	50.00	25.000	0.00	0.00	0.00
0.10	50.00	20.000	2.2500	0.1167	3.7330
		23.134	2.4070	0.1219	3.9000
		23.050	2.4025	0.12175	3.3960
		23.052	2.4026	0.12175	3.8960
0.20	50.00	20.000	4.5552	0.4722	15.110
		17.445	4.4270	0.46793	14.970
		17.513	4.4310	0.46804	14.977
		17.511	4.43075	0.46204	14.977
0.30	50.00	10.000	5.8060	0.98610	30.8750
		9.560	5.7840	0.98540	30.8620
		9.569	5.7848	0.98543	30.8630
0.40	50.00	0.00	6.2630	1.5958	41.849
		4.0750	6.4670	1.6026	41.972
		4.0141	6.4640	1.6025	41.970
		4.0150	6.4640	1.60250	41.970
0.50^-	50.00	0.00	6.6650	2.2623	53.846
		-1.9230			
		-1.9000	6.56975	2.2591	53.789
		-1.8944			
		-1.8946	6.5700	2.25912	53.789
0.50+	5.00	-24.3946	6.5700	2.25912	53.789
0.60	5.00	-30.000	3.8503	2.7848	63.251
		-29.126	3.8940	2.78626	63.278
		-29.136	3.89347	2.78624	63.277
		-29.138	3.89347	2.78624	63.277
0.70	5.00	-32.000	0.83657	3.025127	67.577
		-31.289			
		-31.320	0.87057	3.02626	67.598
		-31.299			
		-31.301	0.87147	3.02641	67.600
0.7278	5.00	-31.620	-0.00313	3.03850	67.818
		-31.409			
		-31.420	-0.000352	3.03853	67.818
		-31.4093	-0.000205	3.03853	67.818

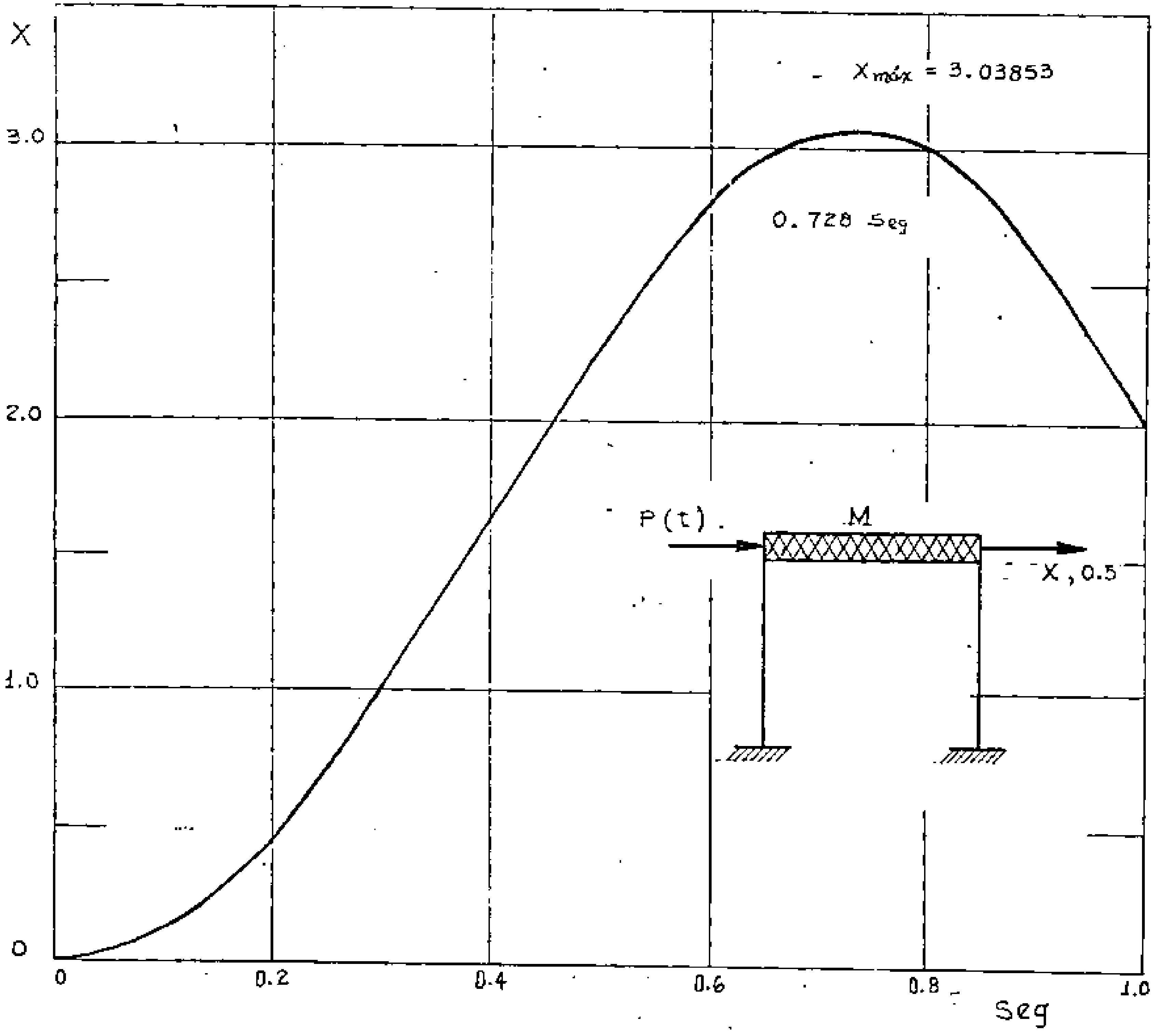
En t=0.5 + SEG, $\Delta\ddot{y} = -45/2 = -22.5 \therefore -22,5 - 1,8946 = -24.3946$

CONTINUACION DEL CUADRO ANTERIOR

t	P	$\ddot{Y}$	$\dot{Y}$	Y	Q
0.80	5.0	-28,000	-2,1449	2.959611	65.293
		-30,146			
		-30,000	-2.21708	2,957874	65.237
		-30,118			
		-30,117	-2.22127	2,95777	65.234
0,90	5,0	-27,00	-5,07712	2.59025	53.479
		-24,236			
		-25,00	-4,97712	2,59358	53,580
		-24,290			
		-24,294	-4,94182	2,59476	53.617
		-24,308	-4,94242	2.59474	53.617
1.00	5.0	-14,00	-6,85782	1.99614	34.461
		-14.7305			
		-14.7200	-6.89382	1,99494	34.423
		-14.7120	-6,89342	1,99495	34.423

EN ESTOS CALCULOS SE INTRODUJO $t = 0.50^-$ Y 0.50^+ PORQUE PARA ESTE INSTANTE SE PRODUCE UN CAMBIO BRUSCO EN LA CARGA P(t) DE 50.00 TONS A 5.00 TONS, CON LO CUAL SE PRODUCE UN CAMBIO BRUSCO EN LA ACELERACION DEL SISTEMA $\ddot{Y}$. EN ESTE INSTANTE NO SE PRODUCEN CAMBIOS EN Y Y Y. EL TIEMPO t = 0,7273 SEG. SE INTRODUJO POR LA NECESIDAD DE CALCULAR LOS VALORES DE Y Y DE Q, PUES A PARTIR DE DICHO INSTANTE SE INICIA LA DESCARGA DEL SISTEMA. ESTA CONDICION SE ENCONTRO SOBRE LA BASE DE APROXIMAR $\dot{Y}$ A CERO, OBTENIENDOSE Y_{MAX}=3.03853 CMS Y Q_{MAX} = 67.818 TON.

EN EL CUADRO SIGUIENTE SE PRESENTA UN RESUMEN DE LOS RESULTADOS.


X
Xmáx = 3.03853
0.728 Seg
3.0
2.0
1.0
0
0
0.2
0.4
0.6
0.8
1.0
Seg
P(t)
M
X, 0.5

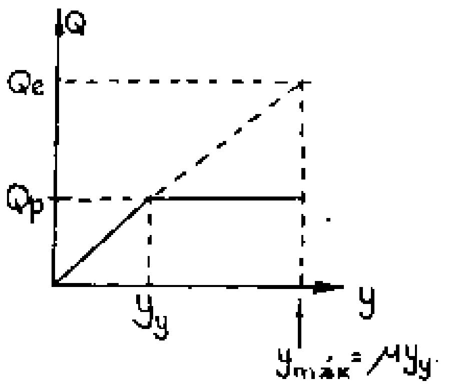
t Seg.	$\ddot{Y}$(supuesta) Cm Seg^{-2}	P Ton	Y Cm.	Q Ton	$\ddot{Y}$(calculado) Cm Seg^{-2}	$\dot{Y}$ Cm Seg^{-1}	N O T A S
0.0	- -	50.00	0.00	0.00	25.00	0.00	
0.10	23.0520	50.00	0.12175	3.896	23.0520	2.40260	
0.20	17.5110	50.00	0.46804	14.977	17.5110	4.43075	
0.30	9.5690	50.00	0.98543	30.863	9.5690	5.78480	CAMBIO DE RIGIDEZ
0.40	4.0150	50.00	1.60250	41.970	4.0150	6.4640	
0.50^{-}	-1.8946	50.00	2.25912	53.789	-1.8946	6.5700	
0.50+	- -	5.00	2.25912	53.789	-24.3945	6.5700	CAMBIO DE CARGA
0.60	-29.1380	5.00	2.78624	63.277	-29.1380	3.89347	
0.70	-31.3010	5.00	3.02641	67.600	-31.3010	0.87147	
0.7278	-31.4093	5.00	3.03853	67.818	-31.4093	-0.000205	Qmáx, Ymáx.
0.800	-30.1170	5.00	2.95777	65.234	-30.1170	-2.22127	
0.90	-24.3080	5.00	2.59474	53.617	-24.3080	-4.94242	
1.00	-14.7120	5.00	1.99495	34.423	-14.7120	-6.89342	

RESPUESTA MAXIMA

$$\begin{cases} Y\ \text{máx} = 3.03853\ \text{cms} \\ Q\ \text{máx} = 67.818\ \text{tons} \end{cases}$$

CRITERIOS PARA TRAZAR ESPECTROS DE DISEÑO ELASTOPLASTICOS A PARTIR DEL ELASTICO

1. CRITERIO DE IGUAL DESPLAZAMIENTO MAXIMO DEL SISTEMA ELASTICO Y EL ELASTOPLASTICO DE IGUAL PERIODO:

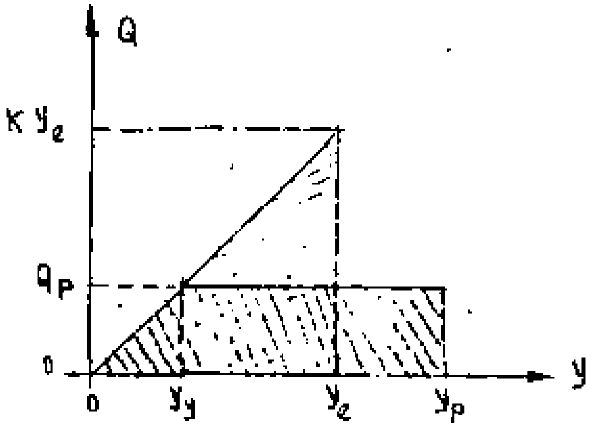


$$Q_p = Ky_y = \frac{Ky_{MAX}}{\mu} = \frac{Q_e}{\mu}$$

$$y_{MAX} = D_e = \mu y_y = \mu D_p$$

$$D_p = \frac{De}{\mu}$$

2. CRITERIO DE IGUAL ENERGIA ABSORVIDA POR LA ESTRUCTURA:



$$\frac{Ky_e y_e}{2} = \frac{Ky_y y_y}{2} + Ky_y(y_p - y_y)$$

$$\frac{1}{2} y_e^2 = \frac{1}{2} y_y^2 + y_y y_p - y_y^2 = y_y y_p - \frac{y_y^2}{2}$$

$$\frac{1}{2}\left(\frac{y_e}{y_y}\right)^2 = \frac{y_p}{y_y} - \frac{1}{2} = \mu - \frac{1}{2}$$

$$\frac{y_e}{y_y} = \sqrt{2\mu - 1}$$

$$y_y = \frac{y_e}{\sqrt{2\mu - 1}}$$

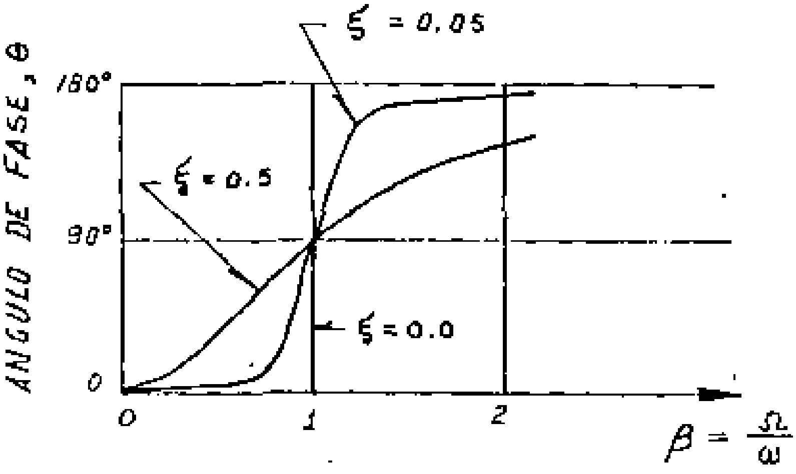
$$\left. y_y \right]_{MAX} = D_p = \frac{\left. y_e \right]_{MAX}}{\sqrt{2\mu - 1}} = \frac{D_e}{\sqrt{2 - 1}}$$

POR LO TANTO

$$D_p = D_e / \sqrt{2\mu - 1} \qquad Y \qquad Q_p = Q_e / \sqrt{2\mu - 1}$$

AMORTIGUAMIENTO HISTERETICO

SI SE CUENTA CON EQUIPO PARA MEDIR EL ANGULO DE FASE ENTRE LA FUERZA DE EXCITACION Y EL DESPLAZAMIENTO RESULTANTE, SE PUEDE EVALUAR EXPERIMENTALMENTE EL AMORTIGUAMIENTO DEL SISTEMA CON UNA SOLA PRUEBA DE VIBRACION ARMONICA EN RESONANCIA. ESTA SE LOGRA CUANDO SE AJUSTA LA FRECUENCIA DEL EXCITADOR DE TAL MANERA QUE EL ANGULO DE FASE SEA 90°, YA QUE:



EN ESTAS CONDICIONES LA FUERZA DE EXCITACION QUEDA EN FASE CON LA VELOCIDAD DE LA MASA YA QUE

$$y = A\ \mathrm{sen}(\omega t - \theta) = -A\cos\omega t, \quad \mathrm{SI}\ \theta = 90°$$

Y $\dot{y} = A\omega\mathrm{sen}\omega t \quad ; \quad \ddot{y} = A\omega^2 \cos\omega t$

Y DE LA ECUACION DIFERENCIAL DE EQUILIBRIO:

$$MA\omega^2\cos\omega t + CA\omega\mathrm{sen}\omega t + K(-A\cos\omega t) = p_o\mathrm{sen}\omega t$$

SE VE QUE SE DEBE CUMPLIR QUE:

$$CA\omega = p_o \quad , \text{ DE DONDE } \boxed{C = \frac{p_o}{A\omega}} \qquad (I)$$

DE LAS ECUACIONES ANTERIORES SE DEDUCE QUE:

$$y^2 = A^2 . \cos^2\omega^2 \quad ; \quad \frac{y^2}{A^2} = \cos^2\omega t$$

Y

$$p^2 = p_o^2\ \mathrm{sen}^2\omega t \quad ; \quad \frac{p^2}{p_o^2} = \mathrm{sen}^2\omega t$$

SUMANDO: $\dfrac{y^2}{A^2} + \dfrac{p^2}{p_o^2} = \mathrm{sen}^2\omega t + \cos^2\omega t$

QUE ES LA ECUACION DE UNA ELIPSE CON LOS EJES COORDENADOS y Y p, ASI (fig 1):

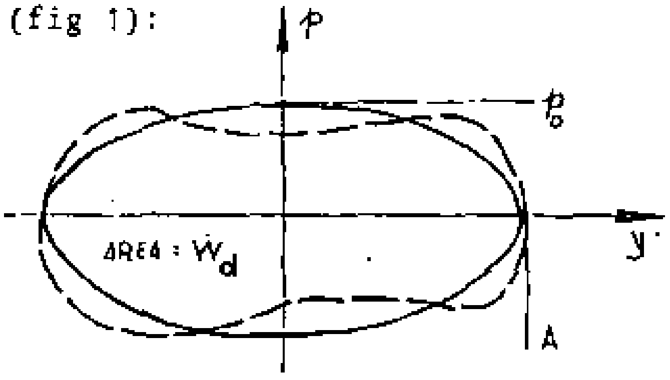


FIG 1

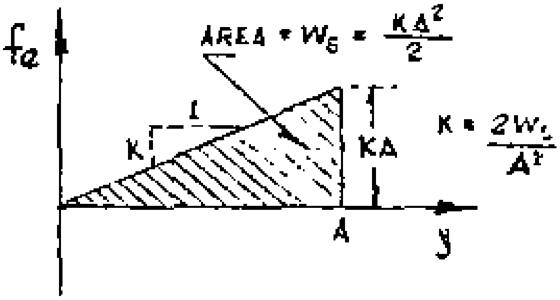


FIG 2

SI EL AMORTIGUAMIENTO NO ES EXACTAMENTE VISCOSO, LA GRAFICA QUE SE OBTENDRIA DE p CONTRA y NO SERIA EXACTAMENTE ELIPTICA, SINO ALGO COMO LA LINEA PUNTEADA AHI MOSTRADA. EN ESTE CASO SE PUEDE UTILIZAR UN AMORTIGUAMIENTO *VISCOSO EQUIVALENTE*, DE TAL MANERA QUE EL AREA W_d, DE ESTA CURVA SEA IGUAL A LA DE LA ELIPSE EQUIVALENTE, $W_{eq} = \pi A p_o$, ES DECIR

$$W_d = \pi A p_o \text{ , DE DONDE } p_o = \frac{W_d}{\pi A}$$

POR LO QUE, DE LA EC. (I)

$$\boxed{C_{eq} = \frac{W_d}{\pi \omega A^2}} \qquad \text{(II)}$$

ADEMAS, $C_{cr} = 2\sqrt{KM} = 2K/\omega$; DE FIG. 2 : $C_{cr} = 2(\frac{2W_S}{A^2})/\omega$, DE DONDE

$$\zeta_{eq} = \frac{C_{eq}}{C_{cr}}$$

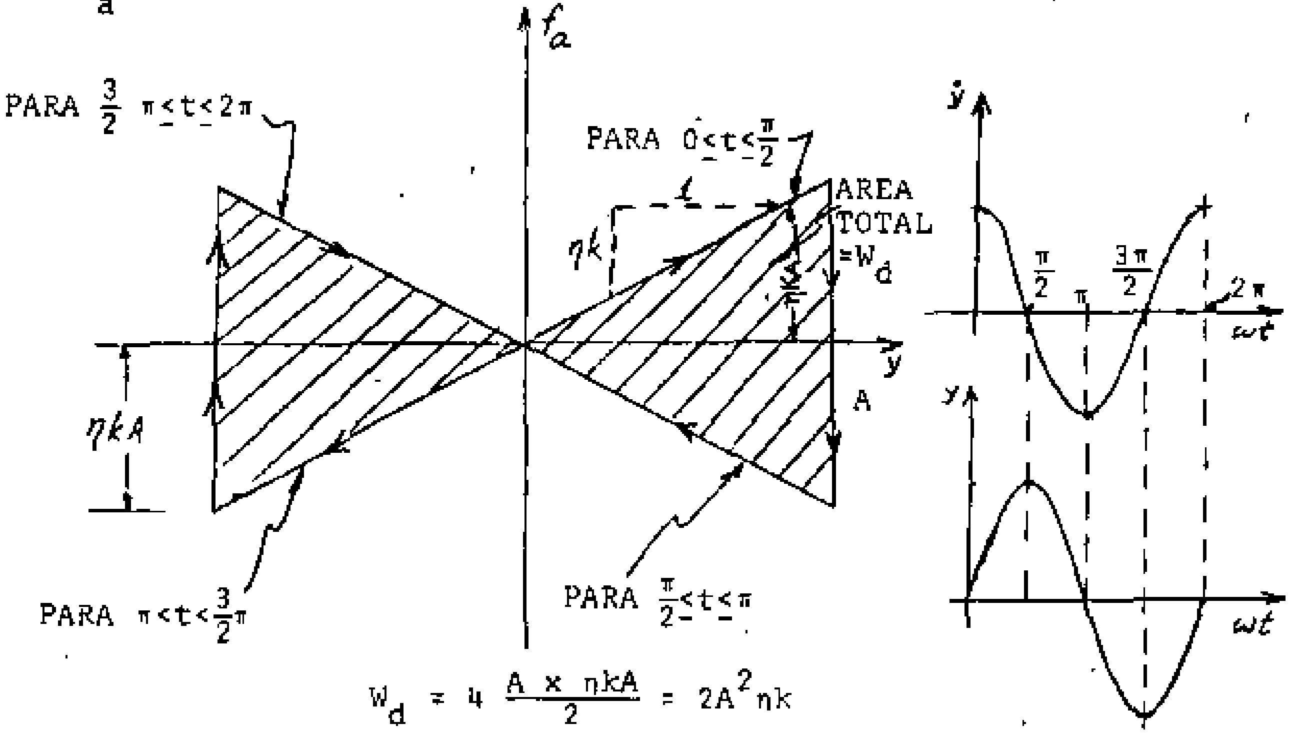
$$\boxed{\zeta_{eq} = W_d/(4\pi W_S)} \qquad \text{(II')}$$

DE LAS ECS. (I) Y (II) SE CONCLUYE QUE EL FACTOR DE AMORTIGUAMIENTO VISCOSO ES FUNCION DE LA FRECUENCIA, ω.

EXISTE OTRO TIPO DE AMORTIGUAMIENTO QUE ES INDEPENDIENTE DE LA FRECUENCIA, QUE SE CONOCE COMO AMORTIGUAMIENTO HISTERETICO, EL CUAL PRODUCE UNA FUERZA EN FASE CON LA VELOCIDAD RELATIVA DE LA MASA, PERO PROPORCIONAL AL DESPLAZAMIENTO, ES DECIR

$$f_a = \eta k |y(t)| \frac{\dot{y}(t)}{|\dot{y}(t)|} \qquad \text{(III)}$$

DONDE η ES EL COEFICIENTE DE AMORTIGUAMIENTO HISTERETICO. EL DIAGRAMA DE f_a DURANTE UN CICLO ES



$$W_d = 4 \, \frac{A \times \eta k A}{2} = 2A^2 \eta k$$

SI SE CONSIDERA QUE LA ENERGIA PERDIDA POR HISTERESIS SE PUEDE REPRESENTAR MEDIANTE UN AMORTIGUADOR VISCOSO, ENTONCES, DE LA EC. (II') Y FIG 2:

$$\zeta = \frac{2A^2 \eta k}{4\pi \frac{KA^2}{2}} = \frac{\eta}{\pi} \qquad \text{O} \qquad \boxed{\eta = \pi\zeta} \qquad \text{(IV)}$$

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V CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

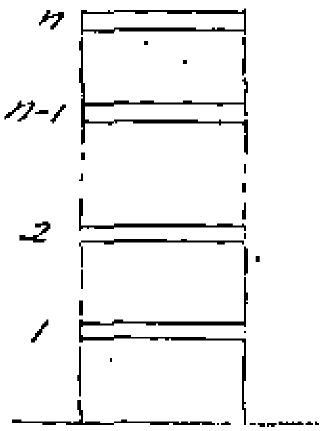
VIBRACION DE SISTEMAS DISCRETOS DE VARIOS GRADOS DE LIBERTAD

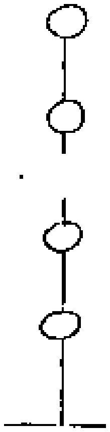
JULIO 1979

JORGE PRINCE

VIBRACION DE SISTEMAS DISCRETOS DE VARIOS GRADOS DE LIBERTAD

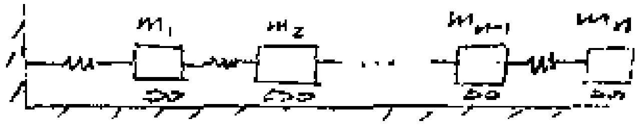
Ejemplos de sistemas de n GL



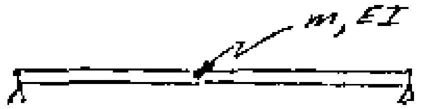


Características:

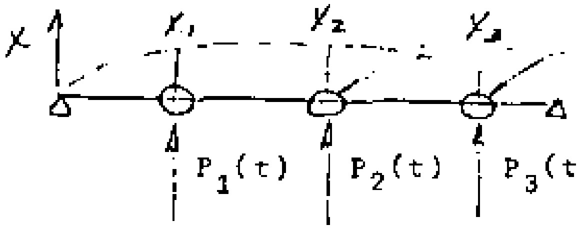
- masas concentradas rígidas
- columnas solo se deforman lateralmente
- con una coordenada por masa queda definida la configuración del sistema
- equivale a:



Además, la consideramos elástica, lineal



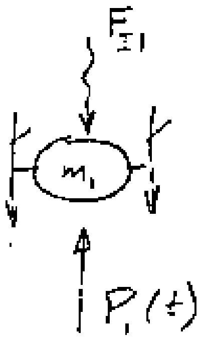
Supongamos:



masas concentradas

fuerzas concentradas exteriores

aislemos una masa:



$F_{r1} = \sum$ fuerzas resistencia elástica a la deformación

Las ecuaciones condensadas de movimiento serán:

$$F_{I1} + F_{r1} = P_1(t)$$

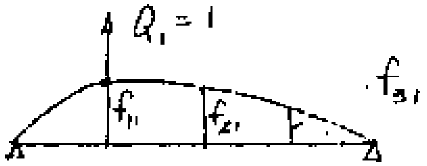
$$F_{I2} + F_{r2} = P_2(t)$$

$$F_{I3} + \underbrace{F_{r3}}_{\text{Fuerzas asociadas al desplazamiento, NO al movimiento}} = P_3(t)$$

∴ la determinación de estas fuerzas es un problema estático.

Coeficientes de influencia

1. De flexibilidad



f_{ij} = despl. de la coord. i debido a una carga unitaria en coord. j (desplazamiento y fuerza en = dirección)

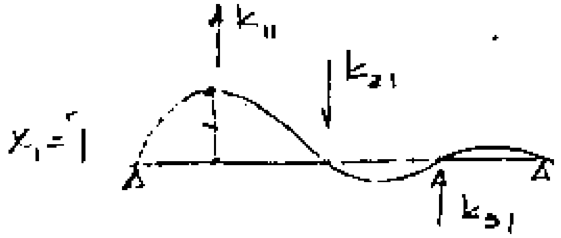
Por superposición

$$X_1 = f_{11}\ Q_1 + f_{12}\ Q_2 + f_{13}\ Q_3$$

$$X_2 = f_{21}\ Q_1 + f_{22}\ Q_2 + f_{23}\ Q_3 \qquad \text{inv. (1)}$$

$$X_3 = f_{31}\ Q_1 + f_{32}\ Q_2 + f_{33}\ Q_3$$

2. De rigidez:



K_{ij} = fuerza en coordenada i por un desplazamiento unitario en coordenada j.

Por superposición

$$
\begin{aligned}
Q_1 &= K_{11} X_1 + K_{12} X_2 + K_{13} X_3 \\
Q_2 &= K_{21} X_1 + K_{22} X_2 + K_{23} X_3 \\
Q_3 &= K_{31} X_1 + K_{32} X_2 + K_{33} X_3
\end{aligned}
\qquad (2)
$$

Desde luego $K_{ij} = K_{ji}$ (y $f_{ij} = f_{ji}$) (Maxwell-Mohr)

La ecuación 2 también puede escribirse:

$$Q_i = \sum_{j=1}^{3} K_{ij} X_j$$

o bien, en notación matricial

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}}_{\text{matriz de rigideces}} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$$

Ponemos:

$$\{Q\} = [K] \{X\}$$

Claro que $[K]^{-1} = [F] = [f_{ij}]$

Sustituyendo (2) o (3) en ecuaciones de movimiento:

$$
\begin{aligned}
m_1 \ddot{X}_1 + K_{11} X_1 + K_{12} X_2 + K_{13} X_3 &= P_1(t) \\
m_2 \ddot{X}_2 + K_{21} X_1 + K_{22} X_2 + K_{23} X_3 &= P_2(t) \\
m_3 \ddot{X}_3 + K_{31} X_1 + K_{32} X_2 + K_{33} X_3 &= P_3(t)
\end{aligned}
$$

o bien:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \\ \ddot{X}_3 \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} P_1(t) \\ P_2(t) \\ P_3(t) \end{Bmatrix}$$

o también:

$$\begin{aligned} [M] \; \{\ddot{X}\} \; + \; [K] \; \{X\} \; &= \{P(t)\} \quad \text{(vibración forzada)} \\ &= \{0\} \quad \text{(vibración libre)} \end{aligned}$$

1. VIBRACION LIBRE

$$[M] \; \{\ddot{X}\} \; + \; [K] \; \{X\} \; = \; \{0\} \tag{1.1}$$

Supongamos la solución

$$\{X\} = \underbrace{\{r\}}_{\substack{\text{constante} \\ \text{con t}}} \underbrace{(A \text{ sen } pt + B \text{ sen } pt)}_{\text{escalar} \longrightarrow} = \{r\} \quad Y(t)$$

define:
- variación armónica
- amplitud

tenemos:

$$\begin{aligned} \{X\} &= \{r\} \; (A \text{ sen } pt + B \cos pt) = r \quad Y(t) \\ \{\dot{X}\} &= \{r\} \; (Ap \cos pt - B \, p \text{ sen } pt) \\ \{\ddot{X}\} &= \{r\} \; (-Ap^2 \text{ sen } pt - B \, p^2 \cos pt) = -p^2 \{r\} \; Y(t) \end{aligned} \tag{1.2}$$

Sustituyendo 1.2 en 1.1 y dividiendo entre Y(t) nos queda:

$$-p^2 [M] \{r\} + [K] \{r\} = \{0\}$$

o sea:

$$\Big[\underbrace{[K] - p^2 [M]}_{[F]}\Big] \{r\} = \{0\} \tag{1.3}$$

$$[K]\{r\} = p^2 [M] \{r\} \qquad [K]\{r\} = p^2 [M] \{r\}$$

pre x $[M]^{-1}$ pre x $[K]^{-1} \cdot \frac{1}{p^2}$

$$[M]^{-1} [K]\{r\} = p^2 \{r\} \qquad \frac{1}{p^2} \{r\} = [K]^{-1} [M] \{r\}$$

En las dos formas llegamos a un problema de VAC

$$[L] \{u\} = \lambda \{u\}$$

Problema de valores característicos:

- Dada una matriz cuadrada de orden (nxn) $[L]$, que representa una transformación lineal de vectores n-dimensionales, debe encontrarse un vector $\{u\}$ que transformado por $[L]$ resulte en otro vector $\lambda \{u\}$ en la misma "dirección". O sea, $[L]$ solo cambia la magnitud de $\{u\}$ sin cambiar la dirección.

  El vector es un vector característico (o eigenvector) de $[L]$. λ(escalar) representa la relación entre las "longitudes" antes y después de la transformación y para llegar a los VEC debe tomar valores de un conjunto de valores característicos (VAC) (o eigenvalores).

  El problema de encontrar frecuencias y modos naturales puede considerarse un problema de VAC. - (STD)

Tenemos

$$\left[[K] - p^2 [M] \right] \{r\} = \{0\} \qquad (1.3)$$

Si en el sistema de ecuaciones

$$[A] \; \{x\} \; = \; \{0\}$$

$[A]$ es no singular, la solución única es la trivial $\{X\} = \{0\}$, de donde nos interesa el caso en que $[A]$ es singular. En este caso la adjunta* $[\hat{A}]$ existe y puede pre X por ella, con el resultado

$$|A| \; \{X\} \; = \; \{0\}$$

porque $$[\hat{A}] \; [A] \; = \; |A| \; [I] \qquad \forall [A] \; (nxn)$$

Puesto que $|A| = 0$, $\{x\}$ no necesariamente es nulo, pero si se asigna un valor dado a uno de sus elementos los demás quedan determinados en forma única.

También notamos que si $\{X\}$ es solución de $[A] \{X\} = \{0\}$ y α es una constante, entonces $\alpha \{X\}$ es también solución. Por lo tanto, hay un número infinito de soluciones. Todos estos se considerarán juntos y hablaremos de una "solución" como un conjunto de relaciones entre los elementos de $\{x\}$.

Volvemos a $$\left[\underbrace{[K] - p^2}_{[E]} [M] \right] \{r\} \; = \; \{0\} \qquad (1.3)$$

Al desarrollar $|E| = 0$ llegamos a una ecuación de grado n en p^2, cuyas raíces son los VAC.

- Como $[K]$ y $[M]$ son simétricas y positivas definidas*,

---

*Transpuesta de la matriz de cofactores.

**$[A]$ es POS. DEF. si $\{\dot{q}\} \; [A] \; \{\dot{q}\} > 0$ para todo $\{\dot{q}\}$ no nulo puede demostrarse que las raíces de la ecuación característica son reales y positivas. Las llamamos p_1^2, p_2^2 ,..., p_n^2.

Las n frecuencias naturales son los términos positivos de las raíces y la más baja es llamada <u>frecuencia fundamental</u>.

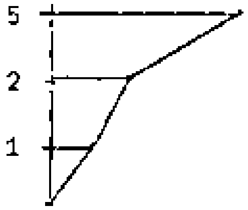
- Para la gran mayoría de los casos de interés las frecuencias son diferentes entre sí.
- Para cada frecuencia p_i existe una VEC asociado:

$$[K]\cdot\{r\}_i = p_i^2\ [M]\ \{r\}_i \qquad i = 1,\ldots, n$$

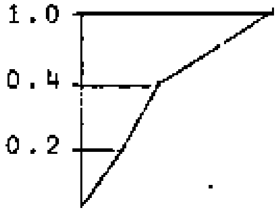
o sea para cada p_i existe una solución $\{r\}$ no trivial

- Normalización (solo conveniencia, sin significado físico)

  Varias formas:



$$\{r\}'_i\ [M]\ \{r\}_i = 1$$



(modos normales)

- Los modos y frecuencias naturales del sistema son propiedades características derivadas de las propiedades de inercia y rigidez expresadas por los elementos de $[M]$ y $[K]$.
- Llamaremos <u>matriz modal</u> $[R]$ a la que tiene los VEC, o vectores modales, como columnas.

ORTOGONALIDAD DE MODOS DE VIBRACION

Se dice que dos vectores $\{a\}$ y $\{b\}$ son ortogonales con respecto a la matriz simétrica $[J]$ si

$$\{a\}' \; [J] \; \{b\} = \{b\}' \; [J] \; \{a\} = 0$$

Demostremos que dos vectores modales $\{r\}_i$ y $\{r\}_j$, asociados a frecuencias diferentes ($p_i \neq p_j$) son ortogonales con respecto a las matrices de inercia y elástica.

- Cada uno de estos vectores satisface la ecuación 1.3

$$p^2 \, [M]\{r\} = [K]\{r\} \qquad\qquad [M]\{r\} = \frac{1}{p^2}[K]\{r\}$$

es decir:

$$p_i^{\,2} \, [M] \, \{r\}_i = [K] \, \{r\}_i \qquad [M] \, \{r\}_i = \frac{1}{p_i{}^2}[K] \, \{r\}_i$$

$$p_j \, [M] \, \{r\}_j = [K] \, \{r\}_j \qquad [M] \, \{r\}_j = \frac{1}{p_j{}^2}[K] \, \{r\}_j$$

pre X i y j por $\{r\}'_j$ y $\{r\}'_i$ respectivamente

$$\begin{array}{l} p_i^2 \, \{r\}'_j \, [M] \, \{r\}_i = \{r\}'_j [K]\{r\}_i \\ p_j^2 \, \{r\}'_i \, [M] \, \{r\}_j = \{r\}'_j [K]\{r\}_j \end{array} \left\{ \begin{array}{l} \{r\}_j [M] \{r\}_i = \dfrac{1}{p_i{}^2} \{r\}'_j [K]\{r\}_i \\ \{r\}'_i [M] \{r\}_j \quad \dfrac{1}{p_j{}^2} \{r\}'_i [K]\{r\}_j \end{array} \right. \qquad (a)$$

pero como $[M]$ y $[K]$ son simétricas:

$$\{r\}'_j \; [K] \; \{r\}_i = \{r\}'_i \, [K]\{r\}_j$$

$$\{r\}'_j \; [M] \; \{r\}_i = \{r\}'_i \, [M]\{r\}_j$$

$\therefore$, restando miembro a miembro en ecuaciones (a):

$$(p_i^2 - p_j^2)\left(\{r\}_i'\ [M]\ \{r\}_j\right) = 0 \qquad 0 = \left(\frac{1}{p_i^{\,2}} - \frac{1}{p_j^{\,2}}\right)\{r\}_i'[K]\{r\}_j$$

y como $p_i^{\,2} \neq p_j^{\,2}$

$$\{r\}_i'\,[M]\,\{r\}_j = 0 \qquad \{r\}_i'\,[K]\,\{r\}_j = 0$$

Tenemos ecuaciones de ortogonalidad:

$$\left.\begin{array}{l}\{r\}_i'\ [M]\ \{r\}_j = 0\\ \{r\}_i'\ [K]\ \{r\}_j = 0\end{array}\right. \qquad \text{si } i \neq j$$

La ec

$$[M]\ \{\ddot{x}\} + [K]\{x\} = \{0\} \qquad (a)$$

y la matriz modal $[R]$

Hagamos:

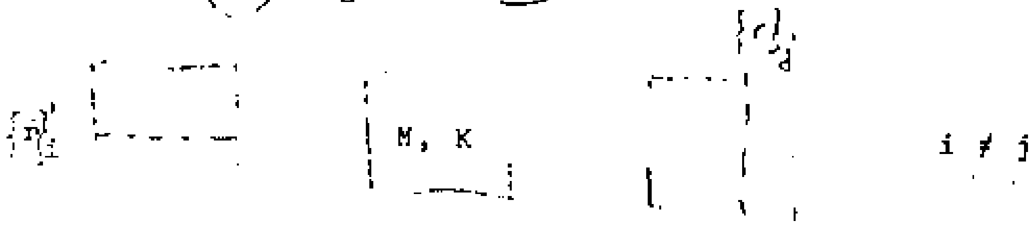
$$\{x\} = [R]\ \{y\}$$

y sustituyendo en (a):

$$[M]\ [R]\ \{\ddot{y}\} + [K]\ [R]\ \{y\} = \{0\}$$

premultiplicando por $[R]'$:

$$\underbrace{[R]'[M]\ [R]}_{}\{\ddot{y}\} + \underbrace{[R]'\ [K]\ [R]}_{\text{diagonales}}\{y\} = \{0\} \qquad (b)$$



Llamemos

$$[R]'\ [M]\ [R] = [M^*]$$

$$[R]'\ [K]\ [R] = [K^*]$$

.·. la ec (b) (p. 14) puede ponerse:

$$[M^*]\ \{\ddot{y}\} + [K^*]\ \{y\} = \{0\}$$

que equivale a:

$$m^*_{11}\cdot\ \ddot{y}_1 + k^*_{11}\ y_1 = 0$$

$$m^{**}_{22}\ \ddot{y}_2 + k^*_{22}\ y_2 = 0$$

- - - - - - - - - - - - -

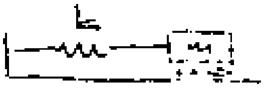
$$m^*_{nn}\ \ddot{y}_n + k_{nn}\ y_n = 0$$

de las que

$$p_1^2 = \frac{k^*_{11}}{m^*_n}, \ldots\ldots, \quad p_n^2 = \frac{k^*_{nn}}{m^*_{nn}}$$

— o — o —

Recordar que para →



$$m\ddot{x} + kx = 0$$

$$\ddot{x} + p^2x = 0 \quad y \quad p^2 = \frac{k}{m}$$

O sea, con la transformación

$$\{x\} = [R]\cdot\{y\}$$

aplicada a la ecuación

$$[M]\ \{\ddot{x}\} + [K]\ \{x\} = \{0\}$$

hemos descompuesto un sistema de nGL en <u>n sistemas de 1GL independientes</u>.

Consideremos el producto

$$[M^*]^{-1}[K^*] = \left([R]'[M][R]\right)^{-1} \quad [R]'\,[K]\,[R] = [K^*]\;[M^*]^{-1}$$

$$= [R]^{-1}[M]^{-1}\;[R]'^{-1}\;[R]' \times [K]\;[R]$$

$$= [R]^{-1}[M]^{-1}\;[K]\;[R] = [\Phi]$$

$[\Phi]$ contiene las frecuencias naturales en la diagonal principal

$\therefore$ El problema de encontrar frecuencias y modos naturales equivale al de encontrar la matriz $[R]$ que diagonalice $[M]$ y $[K]$ de acuerdo con

$$[R]'\;[M]\,[R] = [M^*]$$
$$[R]'\;[K]\,[R] = [K^*]$$

Las frecuencias naturales se obtendrán de

$$[M^*]^{-1}\;[K^*] = [K^*]\;[M^*]^{-1} = [\Phi]$$

Veámoslo en otra forma

$$[M]\cdot\{\ddot{x}\} + [K]\;\{x\} = \{P(t)\}$$

Sustituyendo $\{x\} = [R]\,\{y\}$

$$[M]\;[R]\,\{\ddot{y}\} + [K]\;[R]\;\{y\} = \{P(t)\}$$

premultiplicando por $\{r\}_j^{'}$

$$\underbrace{\{r\}_j^{'}\ [M]\ [R]}_{(a)}\ \{\ddot{y}\} + \underbrace{\{r\}_j^{'}\ [K]\ [R]}_{(b)}\ \{y\} = \underbrace{\{r\}_j^{'}\ \{P(t)\}}_{escalar}$$

En los productos (a) y (b) solo queda (por ortogonalidad):

$$\underbrace{\{r\}_j^{'}\ [M]\ \{r\}_j}_{M_j^{*}}\ \ddot{y}_j + \underbrace{\{r\}_j^{'}\ [K]\{r\}_j}_{K^{*}_j\ =\ p_j^2\ M^{*}_j}\ y_j = \underbrace{\{r\}_j^{'}\ \{P(t)\}}_{P^{*}_j\ =\ \sum_i P_i\ r_{ij}}$$

y para el modo j tenemos:

$$M_j^{*}\ \ddot{y}_j + p_j^2\ M_j^{*}\ y_j = P_j^{*}(t)$$

o bien (1.5)

$$M_j^{*}\ \ddot{y}_j + K_j^{*}\ y_j = P_j^{*}(t)$$

análoga a la ecuación de movimiento para 1 GL:

$$m\ \ddot{x} + k\ x = P(t)$$

En (1.5) tenemos:

n ecuaciones independientes para nGL

1 ecuación independiente para cada modo

Para vibración libre (1GL)

$$\ddot{x} + p^2\ x = 0 \qquad p^2 = \frac{k}{m}$$

la solución es:

$$x = A \cos pt + B \text{ sen } pt \qquad (c)$$

y para el modo j tendremos $(P_j(t) = 0)$

$$y_j = A_j \cos p_j t + B_j \text{ sen } P_j t \qquad (d)$$

Si en (c) hacemos

$$\left. X \right]_{t=0} = X_o \qquad \left. \dot{X} \right]_{t=0} = \dot{X}_o$$

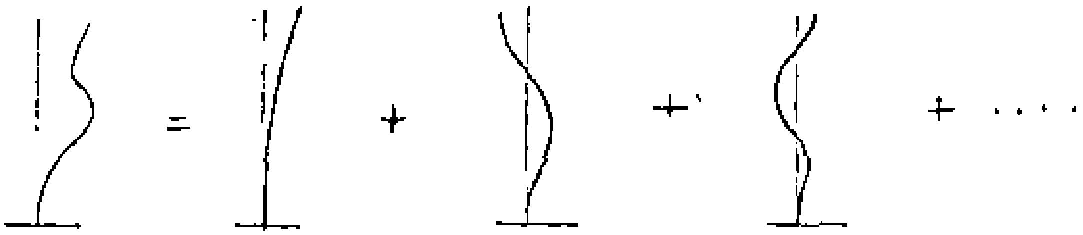
llegamos a

$$x(t) = X_o \cos pt + \frac{\dot{X}_o}{p} \text{ sen } pt$$

y ∴ en (d):

$$y_j = y_{oj} \cos p_j t + \frac{\dot{y}_{oj}}{P_j} \text{ sen } p_j t$$

Cualquier configuración del sistema puede expresarse como una suma de formas modales multiplicadas por ciertos coeficientes. Esquemáticamente:



$$\{X\} = \{r\}_1 \, Y_1 + \{r\}_2 \, Y_2 + \{r\}_3 \, Y_3 + \dots$$

estática
o
dinámica

$$(Y = Y(t))$$

$$\left(\{X\} = \{X(t)\} \right)$$

En nuestra expresión

$$\{x\} = [R]\ \{y\} \qquad 1.4$$

$\{x\}$ puede no ser función de t, por ejemplo:

$$\{1\} = [R]\ \{c\} \qquad (e)$$

donde $\{c\}$ es el vector de constantes que prex $[R]$ nos da la configuración $\{1\}$.

De la ec. (e):

$$\{c\} = [R]^{-1}\ \{1\} \qquad \left([R]\ \text{NO SING}\right)$$

En 1.4 también podríamos hacer

$$\{y\} = [R]^{-1}\ \{x\}$$

pero sigamos otro camino, premultiplicando por $\{r\}'_j\ [M]$ o por $\{r\}'_j\ [K]$

$$\{r\}'_j\ [M]\ \{x\} = \{r\}'_j\ [M][R]\ \{y\} = \{r\}'_j\ [M]\ \{r\}_1\ y_1 +$$

$$+ \{r\}'_j\ [M]\ \{r\}_2\ y_2 + \cdots$$

$$\cdots + \{r\}'_j\ [M]\ \{r\}_n\ Y_n$$

Por ortogonalidad todos estos productos son nulos excepto el término

$$\{r\}'_j\ [M]\ \{r\}_j\ Y_j$$

de donde tenemos

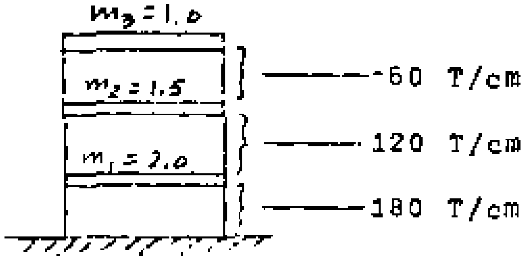
$$\{r\}_j^{'} \cdot [M] \{x\} = \{r\}_j^{'} [M] \{r\}_j \; y_j$$

de donde:

$$y_j = \frac{\{r\}_j^{'} [M] \{x\}}{\{r\}_j^{'} [M] \{r\}_j} = \frac{\{r\}_j [M] \{X\}}{M_j^*} = \frac{\{r\}_j^{'} [K] \{x\}}{K_j^*} = \frac{\{r\}_j}{p_j^2} \frac{[K] \{x\}}{M_j^*}$$

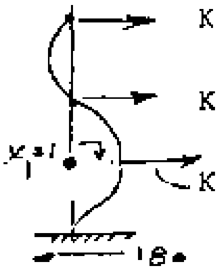
(coeficiente de participación)

Ejemplo (vigas rígidas)



$$[M] = \begin{bmatrix} 2.0 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} \frac{\text{ton seg}^2}{\text{cm}}$$

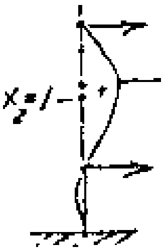
Matriz de rigideces



$K_{31} = 0$

$K_{21} = -120$

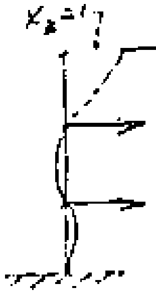
$K_{11} = 300$



$K_{32} = -60$

$K_{28} = 180$

$K_{12} = 120$



$K_{33} = 60$

$K_{23} = -60$

$K_{13} = 0$

$$\therefore [K] = \begin{bmatrix} 300 & -120 & 0 \\ -120 & 180 & -60 \\ 0 & -60 & 60 \end{bmatrix} = 60 \begin{bmatrix} 5 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (T/cm)$$

$$[E] = \left[[K] - p^2 \; [M]\right] \qquad M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 60 \begin{bmatrix} (5 - \frac{2}{60} p^2) & -2 & 0 \\ -2 & (3 - \frac{1.5}{60} p^2) & \enclose{circle}{-1} \\ 0 & \enclose{circle}{-1} & (1 - \frac{1}{60} p^2) \end{bmatrix}$$

si $d = p^2/60$:

$$[E] = 60 \begin{bmatrix} (5-2d) & -2 & 0 \\ -2 & (3-1.5\,d) & -1 \\ 0 & -1 & (1-d) \end{bmatrix}$$

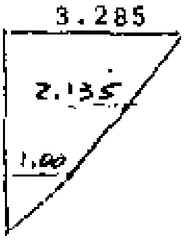
$$|E| = 0 = 60\,(d^3 - 5.5\,d^2 + 7.5\,d - 2) = 0$$

$$d_1 = 0.35$$
$$d_2 = 1.61$$
$$d_3 = 3.54$$

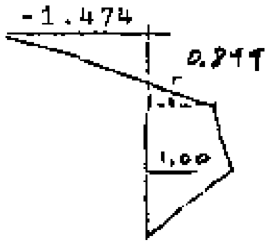
$p^2 = 60\,d$:

$$\left.\begin{matrix} p_1^{\,2} = 21.0 & p_1 = 4.58 \\ p_2^{\,2} = 96.5 & p_3 = 9.82 \\ p_3^{\,2} = 212.4 & p_3 = 14.56 \end{matrix}\right\} \text{frecuencias naturales}$$

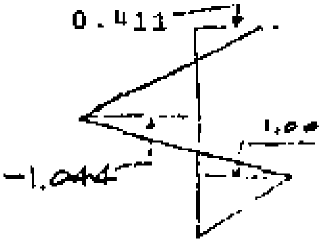
Modos:



$p_1 = 4.58$



$p_2 = 9.82$



$p_3 = 14.56$

$$[R] = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 2.135 & 0.899 & -1.044 \\ 3.285 & -1.474 & 0.411 \end{bmatrix}$$

$$\lceil M^{*} \rfloor = [R]' \, [M] \, [R] = \begin{bmatrix} 19.629 & 0.038 & 0.007 \\ 0.037 & 5.386 & -0.014 \\ 0.006 & -0.014 & 3.804 \end{bmatrix}$$

Ej:

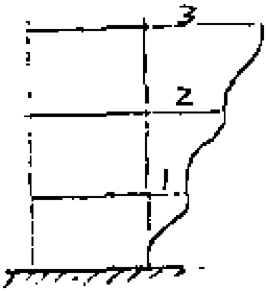
$$19.6296 = \{r\}_1' \, [M] \, \{r\}_1 = M_1^{*} = \sum_i r_{i1}^2 \, m_i$$

$$\lceil K^{*} \rfloor = [R]' \, [K] \, [R] = 60 \begin{bmatrix} 6.899 & 0.042 & 0.034 \\ 0.042 & 8.651 & -0.040 \\ 0.034 & -0.040 & 13.473 \end{bmatrix}$$

Comprobación con $\lceil K^{*} \rfloor = \lceil p^2 \, M^{*} \rfloor =$

$$= \begin{bmatrix} 412.209 & 0 & 0 \\ 0 & 519.749 & 0 \\ 0 & 0 & 807.970 \end{bmatrix} = \lceil p^2 \, M^{*} \rfloor$$

$$\lceil K^{*} \rfloor = \begin{bmatrix} 413.940 & 0\cdots & 0\cdots \\ 0\cdots & 519.060 & 0\cdots \\ 0\cdots & 0\cdots & 808.380 \end{bmatrix}$$



$X_{30} = 3$ cm

$X_{20} = 2$ cm

$X_{10} = 1$ cm

$$\{X_o\} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

$$Y_{01} = \frac{\{r\}_1' \, [M] \, \{x_o\}}{M_1^*} = \frac{2.0 + 6.405 + 9.855}{19.629} = 0.9303 \text{ cm}$$

$$Y_{02} = \frac{\{r\}_2 \, [M] \, \{x_o\}}{M_2^*} = \frac{2.0 + 2.697 - 4.422}{5.386} = 0.0511$$

$$Y_{03} = \frac{\{r\}_3 \, [M] \, \{x_o\}}{M_3^*} = \frac{2.0 - 3.132 + 1.233}{3.804} = 0.0266$$

Modo $Y_1(t)$

$P_1 = 4.58$

$P_2 = 9.62$

$P_3 = 14.56$

En p.

0.930 cm

0.051 cm

0.026 cm

son amplitudes de los modos

Para obtener los desplazamientos de las masas debemos multiplicar por las configuraciones modales:

$$x_{i1} = \{r\}_1 \quad Y_1(t) = \begin{Bmatrix}1.0\\ 2.135\\ 3.285\end{Bmatrix} \quad 0.93 \cos 4.58\ t$$

$$x_{i2} = \{r\}_2 \quad Y_2(t) = \begin{Bmatrix}1.0\\ 0.899\\ -1.474\end{Bmatrix} \quad 0.051 \cos 9.82\ t$$

$$x_{i3} = \{r\}_3 \quad Y_3(t) = \begin{Bmatrix}1.00\\ -1.044\\ 0.411\end{Bmatrix} \quad 0.0266 \cos 14.56\ t$$

<u>y sumar</u>. O sea los desplazamientos $x_i(t)$ de las masas serán

$$\{x(t)\} = [R]\{y(t)\}$$

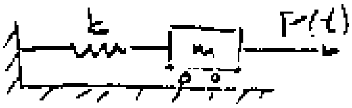
$$x_1(t) = r_{11}\ Y_1(t) + r_{12}\ Y_3(t) + r_{13}\ Y_3(t)$$

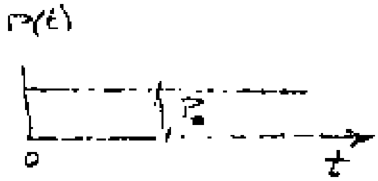
$$x_2(t) = r_{21}\ Y_1(t) + r_{22}\ Y_2(t) + r_{23}\ Y_3(t)$$

$$x_3(t) = r_{31}\ Y_1(t) + r_{32}\ Y_2(t) + r_{33}\ Y_3(t)$$

Otro ejemplo

Para 1GL teníamos





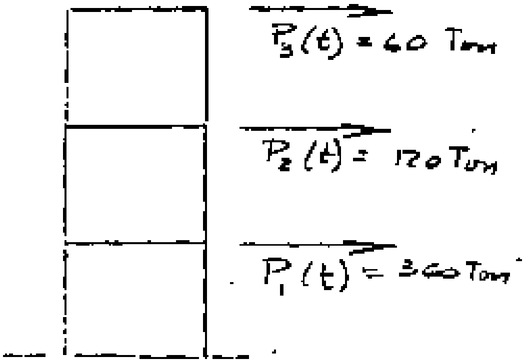
La ec:

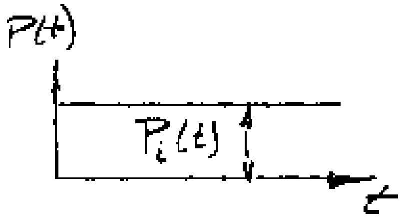
$$x + P^2\ x = \frac{P(t)}{m} = \frac{P_c}{m}$$

y para CI = 0 la solución

$$x = \frac{P_o}{K}\left(1 - \cos pt\right)$$

Tenemos ahora el problema de encontrar la respuesta de





Para el modo j:

$$\ddot{Y}_j + p^2 Y_j = \frac{P_j^*(t)}{M_j^*} = \frac{P_{jo}^*}{M_j^*} \quad \text{cuya solución es:}$$

$$Y_j = \frac{P_{jo}^*}{K_j^*}\left(1 - \cos p_j t\right) = \frac{P_{jo}^*}{p_j^2 M_j^*}\left(1 - \cos p_j t\right)$$

Cálculo de P_j^*

$$P_j^* = \{r\}_j' \{P(t)\} = \{r\}_j' \begin{Bmatrix} 360 \\ 120 \\ 60 \end{Bmatrix}$$

modo

↓

1 $P_1^* = P_1 r_{11} + P_2 r_{21} + P_3\ r_{31} = 360+256.2+197.1 = 813.3$

2 $P_2^* = P_1 r_{12} + P_2 r_{22} + P_3\ r_{32} = 360+107.88-88.4 = 379.48$

3 $P_3^* = P_1 r_{13} + P_2\ r_{23} + P_3\ r_{33} = 360-125.28+24.66 = 259.98$

Ahora bien,

$$Y_{j(st)} = \frac{P_j^*}{p_j^2\ M_j^*} = \frac{P_j^*}{K_j^*}$$

$$\therefore \quad Y_{1(st)} = \frac{813.30}{21 \times 19.629} = 1.973 \text{ cm}$$

$$Y_{2(st)} = \frac{379.48}{965 \times 5.386} = 0.730 \text{ cm}$$

$$Y_{3(st)} = \frac{259.38}{212.4 \times 3.804} = 0.321 \text{ cm}$$

de donde

$$Y_j = \frac{P_j^*}{p_j^{2} \cdot M_j^*} \left(1 - \cos p_j t\right),$$ y tenemos:

$$Y_1(t) = Y_{1(st)} \left(1 - \cos p_1 t\right)$$

$$Y_2(t) = Y_{2(st)} \left(1 - \cos p_2 t\right)$$

$$Y_3(t) = Y_{3(st)} \left(1 - \cos p_3 t\right)$$

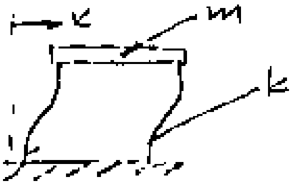
y, finalmente:

$$\{x(t)\} = \{r_1\} Y_1(t) + \{r_2\} Y_2(t) + \{r_3\} Y_3(t) = [R] \{Y\}$$

$$\begin{Bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{Bmatrix} = \begin{Bmatrix} 1.000 \\ 2.135 \\ 3.285 \end{Bmatrix} 1.973 \left(1-\cos p_1 t\right) + \ldots + \begin{Bmatrix} 1.000 \\ -1.044 \\ 0.411 \end{Bmatrix} 0.321 \left(1-\cos p_3 t\right)$$

EXCITACION SISMICA

A. Sistemas 1GL

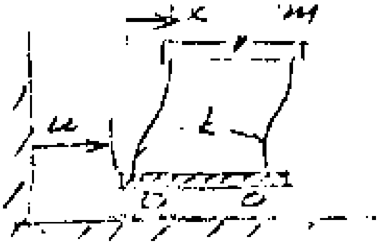


$$m\ \ddot{x} + kx + P(t) \qquad (a)$$

Para P(t) cualquiera y para CI ≠ 0 la solución de (a) es:

$$x(t) = x_o \cos pt + \frac{\dot{x}_o}{p}\ \text{sen}\ pt + \frac{1}{mp}\int_0^t P(\tau)\ \text{sen}\ p(t-\tau)d\tau$$

Para excitación sísmica:



$$m(\ddot{x} + \ddot{u}) + k\ x = 0$$

o sea,

$$m\ \ddot{x} + k\ x = -\ m\ddot{u} \qquad (b)$$

De la comparación de (a) y (b), la solución completa de ésta es:

$$x(t) = x_o\ \cos\ pt + \frac{\dot{x}_o}{p}\ \text{sen}\ pt - \frac{1}{p}\int_0^t \ddot{u}(\tau)\ \text{sen}\ p(t-\tau)\ d\tau$$

B. Sistemas de nGL:

$$[M]\ \{\ddot{x}\} + [k]\{x\} = \{P(t)\} = \begin{Bmatrix} P_1(t)\\ P_2(t)\\ \vdots\\ P_n(t)\end{Bmatrix} = \begin{Bmatrix} -m_1\ddot{u}\\ -m_2\ddot{u}\\ \vdots\\ -m_n\ddot{u}\end{Bmatrix}$$

$$= -\begin{Bmatrix} m_1\\ m_2\\ \vdots\\ m_n\end{Bmatrix}\ \ddot{u} = -\{m\}\ \ddot{u}$$

Es decir, tenemos:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{P(t)\} = -\{m\}\ddot{u}$$

sust. $$\{x\} = [R]\{y\}$$

$$[M][R]\{\ddot{y}\} + [K][R]\{y\} = \{P(t)\} = -\{m\}\ddot{u}(t)$$

pre x $\{r\}'_j$

$$\{r\}'_j[M][R]\{\ddot{y}\} + \{r\}'_j[K][R]\{y\} = \underbrace{\{r\}'_j\{P\}}_{P_j^*} = \underbrace{-\underbrace{\{r\}'_j\{m\}}_{m_j^*}\ddot{u}}_{U_j^*}$$

por ortogonalidad:

$$\{r\}'_j[M]\{r\}\ddot{y}_j + \{r\}'_j[K]\{r\}_j y_j = P_j^* = U_j^*$$

y queda:

$$M_j^*\ddot{y}_j + K_j^* y_j = P_j^* = U_j^* = -m_j^*\ddot{u}$$

∴ la solución (CI = 0) de esta ecuación es:

Para P_j^* :

$$y_j(t) = \frac{1}{p_j M_j^*}\int_0^t P_j^*(\zeta)\,\text{sen}\,p_j(t-\zeta)d\zeta$$

Para U_J^*:

$$y_j(t) = \frac{1}{p_j M_j^*}\int_0^t U_j^*(\zeta)\,\text{sen}\,p_j(t-\zeta)d\zeta$$

que puede escribirse:

$$y_j(t) = -\frac{m_j^*}{p_j M_j^*} \int_0^t \ddot{u}(Z) \text{ sen } p_j(t-Z) \, dZ$$

$$+ y_{oj} \cos p_j t + \frac{\dot{y}_{oj}}{p_j} \text{ sen } p_j t$$

términod a
para
CI ≠ 0

Una vez obtenidos los elementos de $\{y\}$ solo falta premultiplicar por $[R]$ para obtener $\{x\}$:

$$\{x(t)\} = [R] \{y(t)\}$$

GENERALIZACION DE LAS CONDICIONES DE ORTOGONALIDAD

Tenemos la ecuación:

$$\left[[K] - p^2 [M]\right] \{X\} = \{0\}$$

que convenimos en escribir en la forma:

$$(K - p^2 M)\, x = 0$$

como los vectores modales la satisfacen:

$$K\, r_j = p_j^2 \, M\, r_j \qquad \text{(a)}$$

y premultiplicando por: $r_i' \, MM^{-1}$ tenemos:

$$r_i' \, M\, M^{-1} \qquad K\, r_j = p_j^2 \, M\, M^{-1} \qquad M\, r_j = p_j^2 \, M\, M^{-1} K\, r_j = 0$$

que puede escribirse

$$r_i' \; M \; (M^{-1} \; K)^2 \; r_j = 0$$

y así podría seguirse para llegar a:

$$r_i' \; M \; (M^{-1}K)^\ell \; r_j = 0 \quad - \begin{cases} \ell \text{ entero} \\ -\infty < \ell < \infty \end{cases}$$

$$r_i' \; M \; (M^{-1} \; K)^\ell \; r_j = 0 \qquad (b)$$

en forma análoga podemos obtener

$$r_i' \; (MF)^\ell \; M \; r_j = 0 \qquad (c)$$

o

$$r_i' \; (K \; M^{-1})^\ell \; K \; r_j = 0$$

En (b):

$\ell = -2 \qquad M \; (M^{-1}K)^{-2} = M \; (M^{-1}K)^{-1} \; (M^{-1}K)^{-1}$

(en (c), con ℓ=2) $\qquad = M \; K^{-1} \; M \; K^{-1} \; M = \underline{M \; F \; M \; F \; M}$

$\ell = -1 \qquad M(M^{-1}K)^{-1} = M \; K^{-1} \; M = \underline{M \; F \; M}$

$\ell = 0 \qquad M \; (M^{-1}K)^0 = \underline{M}$

$\ell = 1 \qquad M \; (M^{-1}K)^1 = M \; M^{-1} \; K = \underline{K}$

$\ell = 2 \qquad M \; (M^{-1}K)^2 = M \; M^{-1} \; K \; M^{-1} \; K = \underline{K \; M^{-1} \; K}$

$\ell = 3 \qquad M \; (M^{-1}K)^3 = M \; M^{-1} \; K \; M^{-1} \; K \; M^{-1}K = \underline{K \; M^{-1} \; K \; M^{-1} \; K}$

VIBRACION LIBRE Y FORZADA DE SISTEMAS DE N GL CON AMORTIGUAMIENTO

Las ecuaciones de equilibrio dinámico son:

$$\{F_I\} + \{F_a\} + \{F_r\} = \{P(t)\}$$

Ya tenemos:

$$\{F_I\} = [M] \{\ddot{x}\}$$

$$\{F_r\} = [K] \{x\}$$

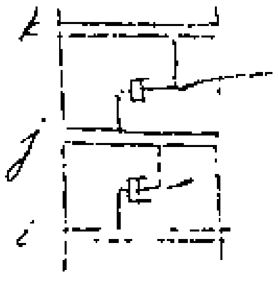
y ahora hacemos

$$\{F_a\} = [c] \{\dot{x}\}$$

donde

$$[c] = [c_{ij}]$$

y c_{ij} = fuerza de amortiguamiento en la coordenada i debido a una velocidad unitaria en la coordenada j.



$$\left.\begin{array}{l} c_{jk} = c_{kj} \\ \\ c_{ij} = c_{ji} \end{array}\right\}$$ indica acoplamiento

La ecuación de movimiento es

$$[M] \{\ddot{x}\} + [c] \{\dot{x}\} + [K] \{x\} = \{P(t)\}$$

Hagamos: $\{X\} = [R]\{y\}$ premultiplicando por $\{r\}'_j$

$$\{r\}'_j [M][R]\{\ddot{y}\} + \{r\}'_j [C][R]\{\dot{y}\} + \{r\}'_j [K][R]\{y\} = \{r\}'_j \{P(t)\}$$

Para desacoplar estas ecuaciones debemos tener

$$\left.\begin{array}{ll} \{r\}'_j [M] \{r\}_i = 0 & i \neq j \\ \{r\}'_j [K] \{r\}_i = 0 & i \neq j \end{array}\right\} \text{cierto por ortogonalidad}$$

$$\{r\}'_j [C] \{r\}_i = 0 \quad i \neq j \qquad \text{¿pero ésta?} \quad \text{(a)}$$

1° <u>admitamos</u> que se cumple:

Ya definimos

$$\{r\}'_j [M] \{r\}_j = M^*_j \qquad \{r\}'_j \{P(t)\} = P^*_j$$

$$\{r\}'_j [K] \{r\}_j = K^*_j$$

y ahora

$$\{r\}'_j [C] \{r\}_j == C^*_j = 2\beta_j \phi_j M^*_j$$

y nuestra ecuación para el modo j queda:

$$M^*_j \ddot{y}_j + 2\beta_j \phi_j M^*_j \dot{y}_j + \phi^2_j M^*_j y_j = P^*_j$$

o bien:

$$\ddot{y}_j + 2\beta_j \phi_j \dot{y}_j + \phi^2_j y_j = \frac{P^*_j}{M^*_j}$$

Como las soluciones para un sistema de 1GL (cuya ec. es $\ddot{x}+2\beta p\dot{x}+p^2x = \frac{P(t)}{m}$) ya las conocemos, solo nos falta saber cómo debe ser $[C]$ para que se cumpla

$$\{r\}_i^{t} [C] \{r\}_j = 0 \qquad i \neq j \qquad \text{(a)}$$

además, claro, de

$$\text{y} \quad \left.\begin{array}{l} \{r\}_i^{t} [M] \{r\}_j = 0 \\ \{r\}_i^{t} [K] \{r\}_j = 0 \end{array}\right\} i \neq j$$

La ec. (a) se satisface si

i) $[C]$ es proporcionala $[M]$ o a $[K]$

ii) $[C]$ es una combinacion lineal de $[M]$ y $[K]$, o sea:

$$[C] = a_0 [M] + a_1 [K]$$

esto es muy restringido.

iii) En forma más general:

$$[C] = [M] \sum_{l} a_l [M^{-1}K]^{l} = \sum_{l} [C_l] \qquad (38.1)$$

pues ya sabemos que todas las posibles formas

$[M]$ $[M^{-1}K]^{l}$ son satisfactorias y (38.1) es una C. L de matrices de este tipo.

La selección adecuada de a_l dará a $[C]$ las propiedades deseadas, o sea, podremos dar valores específicos a los elementos de $[C]$ ¿Cuáles le damos?

Asignamos un cierto valor de β a cada modo.

$$\underbrace{C^*_j = \{r\}'_j\ [C]\ \{r\}_j}_{A} = \underbrace{2\beta_j p_j M^*_j = \sum_l \{r\}'_j\ [C_l]\ \{r\}_j = \sum_l C^*_{jl}} \qquad (38.2)$$

De 38.1 y A

$$C^*_{jl} = \}r\{'_j\ [M]\cdot[M^{-1}K]^l\{r\}_j a_l \qquad (38.3)$$

Por otra parte, para vibración libre:

$$(K-p^2_jM)r_j = 0$$

$$Kr_j = p^2_jMr_j \leftrightarrow \frac{1}{p^2_j}r_j = FMr_j$$

premultiplicando por r'_jM:

$$\frac{1}{p^2_j}r'_jMr_j = r'_jMFMr_j$$

es decir

$$(p^2_j)^{-1}M^*_j = r'_jM(M^{-1}K)^{-1}r_j$$

y así podríamos llegar a que, para cualquier l:

$$(\phi_j^2)^l M_j^* = r_j' M (M^{-1}K)^l r_j = \underbrace{\frac{C_{jl}^*}{a_l}}_{\text{por 38.3}} \qquad 39.1$$

De 39.1:

$$C_{jl}^* = (\phi_j^2)^l M_j^* a_l$$

$$C_{jl}^* = (\phi_j^2)^l M_j^* a_l$$

y sumando sobre l:

$$\sum_l C_{jl}^* = \sum_l (\phi_j^2)^l M_j^* a_l$$

pero ya teníamos que

$$\sum_l C_{jl}^* = 2\beta_j \phi_j M_j^*$$

$$2\beta_j \phi_j \cancel{M_j^*} = \sum_l (P_j^2)^l \cancel{M_j^*} a_l$$

de donde:

$$\boxed{\beta_j = \frac{1}{2\phi_j} \sum_l (\phi_j^2)^l a_l}$$

Con los n valores de β_j para los n modos podemos resolver para los n valores de a_l y formar nuestra $[C]$ con la ecuación

$$[C] = [M] \sum_{1} a_1 \left[M^{-1}K\right]^{1}$$

Por ejemplo para nuestra estructura de 3GL asignemos:

$$\beta_1 = 0.10, \qquad \beta_2 = 0.05, \qquad \beta_3 = 0.02$$

$$\beta_1 = 0.10 = \frac{1}{2p_1}\left[a_1(p_1^2)^{-1}+a_0(p_1^2)^0+a_1(p_1^2)^1\right]$$

$$\beta_2 = 0.05 = \frac{1}{2p_2}\left[a_{-1}(p_2^2)^{-1}+a_0(p_2^2)^0+a_1(p_2^2)^1\right]$$

$$\beta_3 = 0.02 = \frac{1}{2p_3}\left[a_{-1}(p_3^2)^{-1}+a_0(p_3^2)^0+a_1(p_3^2)^1\right]$$

o, en forma matricial:

$$\begin{Bmatrix} 0.10 \\ 0.05 \\ 0.02 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1/p_1^3 & 1/p_1 & p_1 \\ 1/p_2^3 & 1/p_2 & p_2 \\ 1/p_3^3 & 1/p_3 & p_3 \end{bmatrix}\begin{Bmatrix} a_{-1} \\ a_0 \\ a_1 \end{Bmatrix}$$

al resolver para a_1 resulta

$$\underline{[C] = a_{-1}\,[MFM] + a_0\,[M] + a_1\,[K]}$$

En p. tenemos que para CI = 0 y β = 0, para excitación sísmica

$$y_j(t) = -\frac{m\phi_j}{p_j M_j^*}\int_0^t \ddot{u}(\tau)\,\mathrm{sen}\,p_j(t-\tau)\,d\tau$$

coeficiente de participación

$$C_j = \frac{m^*_j}{M^*_j} = \frac{\{r\}'_j \{m\}}{\{r\}'_j [M] \{r\}_j} = \frac{\sum_{i=1}^{m} m_i r_{ij}}{\sum_{i=1}^{m} m_i r^2_{ij}}$$

y ∴ podemos poner:

$$y_j(t) = C_j z_j(t)$$

en la que C_j está definida arriba y

$$z_j(t) = -\frac{1}{p_j}\int_0^t \ddot{u}(Z) \operatorname{sen} p_j (t-Z) dZ$$

(y semejante si $\beta \neq 0$)

$$y_j(t) = C_j z_j(t)$$

Además, tenemos

$$\{X\} = [R]\{y\}$$

o sea

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_i \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1j} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2j} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nj} & \cdots & r_{nn} \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{Bmatrix}$$

††
∴

$$x_i = \sum_{j=1}^{n} r_{ij} y_j = \sum_{j=1}^{n} r_{ij} C_j z_j(t)$$

De aquí (sin sumar para todos los modos)

$$\left.\begin{aligned} |X_{ij}|_{max} = r_{ij} C_j |z_j(t)|_{max} &= r_{ij} C_j S_d \\ &= r_{ij} C_j \frac{S_a}{\cancel{p}^2_j} \end{aligned}\right\} S_a = pS_v = p^2 S_d$$

De esta ec. pasamos a:

$$|X_i|_{\substack{máx \\ ABS}} = \sum_{j=1}^{n} r_{ij} C_j S_d = \sum_{j=1}^{n} r_{ij} C_j \frac{S_a}{\cancel{p}^2_j}$$

$$|x_i|_{\substack{máx \\ PROB}} = \sqrt{\Sigma(|X_{ij}|_{max})^2}$$

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V. CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

VIBRACION DE VIGAS DE CORTANTE Y DE FLEXION

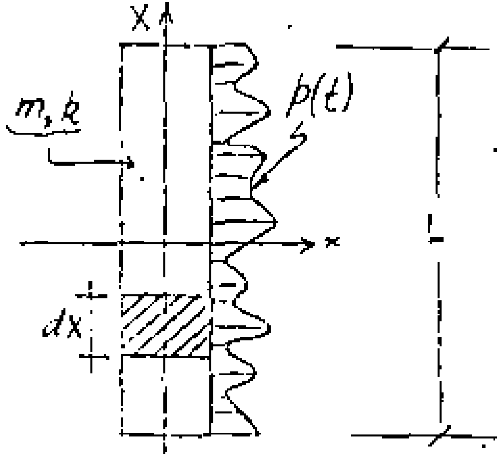
DR. OCTAVIO A. RASCON CHAVEZ

JULIO, 1979

VIGAS DE CORTANTE NO AMORTIGUADAS

SON SISTEMAS CONTINUOS CUYOS CAMBIOS DE PENDIENTE SON PROPORCIONALES AL CORTANTE QUE ACTUA EN LA SECCION.

SEAN m y p LA MASA Y FUERZA EXTERNA DISTRIBUIDAS POR UNIDAD DE LONGITUD, Y SEA k LA RIGIDEZ POR CORTANTE:



k = FAG

F = FACTOR DE FORMA

A = AREA SECCION TRANSVERSAL

G = MODULO DE ELASTICIDAD DINAMICO AL CORTANTE

$$F_I = m dX \frac{\partial^2 x}{\partial t^2}$$

POR EQUILIBRIO:

$$\frac{\partial S}{\partial x} dx + p dx - m \frac{\partial^2 x}{\partial t^2} dx = 0$$

$$m \frac{\partial^2 x}{\partial t^2} - k \frac{\partial^2 x}{\partial x^2} = p(t) \qquad (1)$$

LA EC HOMOGENEA QUEDA (CON $p=0$)

(2) $$\frac{\partial^2 x}{\partial t^2} - v^2 \frac{\partial^2 x}{\partial X^2} = 0 \; ; \quad v^2 = \frac{k}{m}$$

ESCRIBIENDO $x(t) = Z_n(X)\theta_n(t)$, LA EC (2) QUEDA

$$Z_n \ddot{\theta}_n - v^2 Z_n'' \theta_n = 0$$

$$\frac{\ddot{\theta}_n(t)}{\theta_n(t)} - v^2 \frac{Z_n''}{Z_n} = 0 \Rightarrow \frac{\ddot{\theta}_n(t)}{\theta_n(t)} = v^2 \frac{Z_n''}{Z_n} = \omega_n^2 = \text{CONSTANTE}$$

$$\Rightarrow \ddot{\theta}_n + \omega_n^2 \theta_n = 0 \quad ; \quad Z_n'' + \frac{\omega_n^2}{v^2} Z_n = 0$$

$$\theta_n = \text{sen}\,\omega_n(t-t_n), \quad Z_n = A_n \,\text{sen}\frac{\omega_n}{v}(X-a_n)$$

$$\therefore\; x_n = A_n \,\text{sen}\left[\frac{\omega_n}{v}(X-a_n)\right]\text{sen}\left[\omega_n(t-t_n)\right], \quad n=1,2\ldots$$

LAS CONSTANTES a_n Y ω_n SE DETERMINAN EN CADA PROBLEMA EN FUNCION DE LAS CONDICIONES DE FRONTERA

CONDICION DE ORTOGONALIDAD

$$\int_0^L x_n(X)x_j(X) = 0, \text{ SI } n \neq j$$

<u>EJEMPLO 1</u>:. CUERDA VIBRANTE DE LONGITUD L Y EXTREMOS FIJOS:

EN EL EXTREMO $X=0$ SE TENDRA

(3) $$x(0,t) = 0 \quad \Rightarrow \frac{\omega_n a_n}{v} = j\pi \; ; \quad j = 0,1,2,\ldots$$

EN EL EXTREMO X = L SE TENDRA

(4) $x(L,t) = 0 \Rightarrow \frac{\omega_n L}{v} = n\pi$; $n = 1,2,\ldots$

PUESTO QUE EN LA EC (3) SE TOMA $j=0$, YA QUE $j=1,2,\ldots$ DAN LA MISMA SOLUCION, LO CUAL CONDUCE A $a_n = 0$

DE LA EC (4): $\omega_n = \frac{n\pi v}{L}$; $n = 1,2,\ldots$

FRECUENCIA FUNDAMENTAL

SI $n=1$, $\omega_1 = \frac{\pi v}{L}$ $\therefore$ $\omega_n = n\,\omega_1$

Y $T_1 = \frac{2L}{v}$ $T_n = \frac{T_1}{n}$

LAS CONFIGURACIONES MODALES QUEDAN:

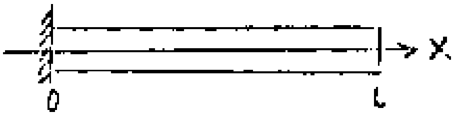
$$Z_n = A_n \text{sen}\frac{n\pi X}{L} \quad ; \quad x(t,X) = A_n \text{sen}\frac{n\pi X}{L}\text{sen}\frac{n\pi v}{L}(t-tn)$$

CONDICION DE ORTOGONALIDAD:

$$\int_0^L A_i \;\text{sen}\frac{i\pi X}{L}\, A_j \;\text{sen}\frac{i\pi X}{L}dx = 0, \quad \text{SI} \quad i \neq j$$

<u>EJEMPLO 2</u>: VIGA DE CORTANTE APOYADA EN X = 0 Y LIBRE EN X = L

DE $x(0,t) = 0 \Rightarrow a_n = 0$



DE $x'(L,t) = 0$ (PUESTO QUE EN X = L SE DEBE CUMPLIR QUE LA FUERZA CORTANTE, S, SEA NULA),

$$x'(\underline{X},t) = A_n \frac{\omega_n}{v} \cos \frac{\omega_n X}{v} \,\text{sen}\,\omega_n(t-t_n)$$

$$\therefore \quad x'(L,t) = 0 = \cos \frac{\omega_n L}{v} \Rightarrow \frac{\omega_n L}{v} = \frac{\pi}{2}(2n-1)$$

O $\quad \omega_n = \frac{v}{L} \; \frac{\pi}{2} \; (2n-1) \qquad\qquad n = 1,2,\ldots$

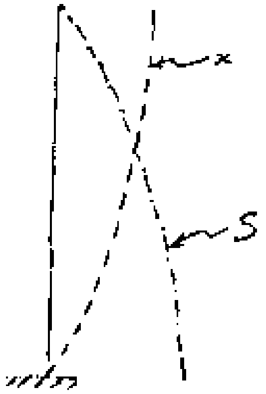
SI $\quad n=1, \; \omega_1 = \frac{\pi v}{L.2} \Rightarrow T_1 = \frac{4L}{v}$

$$\therefore \quad \omega_n = \omega_1(2n-1) \; ; \quad T_n = \frac{T_1}{2n-1}$$

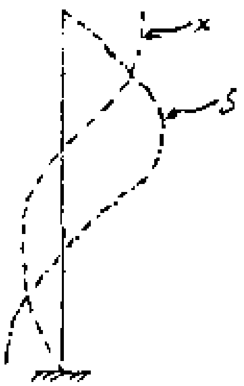
ASI: $T_2 = \frac{T_1}{3}$, $T_3 = \frac{T_1}{5}$, ETC.

DISTRIBUCION DE CORTANTES:

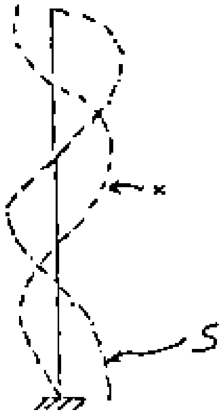
$$S_n = k \frac{\partial x}{\partial X} = A_n \; k \; \frac{\omega_n}{v} \cos \frac{\omega_n X}{v} \; \text{sen}\,\omega_n(t-t_n)$$



1er. MODO(FUNDAMENTAL)



2o. MODO



3er. MODO

VIBRACIONES FORZADAS EN VIGAS DE CORTANTE

SEA $\ddot{x}_o(t)$ LA EXCITACION DEL TERRENO. LA RESPUESTA, $x(t)$, DEL SISTEMA ES

$$(3) \quad x(t) = -\sum_{n=1}^{\infty} \frac{a_n}{\omega_n} \operatorname{sen} \frac{\omega_n}{V} X \int_0^t \ddot{x}_o(\tau) \operatorname{sen}\omega_n (t-\tau) d\tau$$

DONDE

$$(4) \quad a_n = \frac{\int_0^L n \operatorname{sen} \frac{\omega_n V}{X} dx}{\int_0^L n \operatorname{sen}^2 \frac{\omega_n V}{X} dx} = \frac{4}{(2n-1)\pi}$$

<u>TAREA</u>: DEMOSTRAR ECS (3) Y (4) Y ESTUDIAR SECCION 3.15.

<u>EJEMPLO</u>: CALCULAR EL LIMITE SUPERIOR DEL CORTANTE EN UNA VIGA DE CORTANTE A CUYA BASE SE LE SOMETE A UNA ACELERACION CONSTANTE, a .

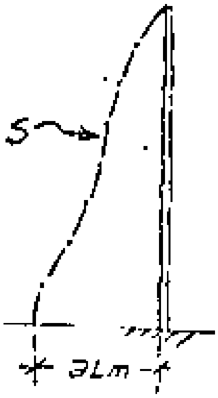
EL ESPECTRO DE ESTA EXCITACION ES $V = a/\omega$

POR LO TANTO, $S \leq k \left| \frac{\partial}{\partial X} \left(\sum_{n=1}^{\infty} \frac{a_n}{\omega_n} \operatorname{sen} \frac{\omega_n}{V} X \right) \right| V$

$$S \leq \left| \sum_{n=1}^{\infty} \frac{k a_n V}{\omega_n} \frac{\omega_n}{V} \cos \frac{\omega_n}{V} X \right| = \frac{4k}{\pi} \frac{a}{\frac{k}{m}} \left| \sum_{n=1}^{\infty} \frac{\operatorname{sen} \frac{\pi}{2L}(2n-1)X}{(2n-1)\frac{V}{L} \frac{\pi}{2}(2n-1)} \right|$$

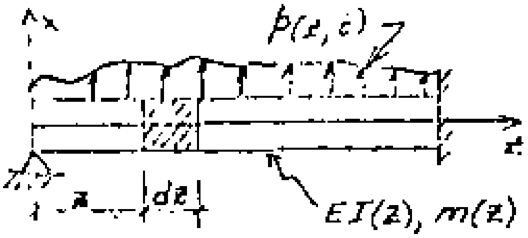
$$S \leq \frac{8aLm}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \left| \cos \frac{(2n-1)\pi X}{2L} \right| \quad ; \text{ EN } X=0$$

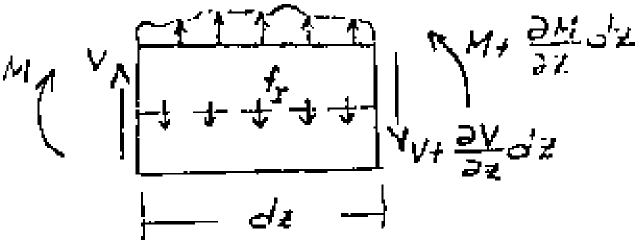
$$S \leq (8aLm)/\pi^2 \underbrace{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}}_{\pi^2/8} = aLm$$



VIBRACION DE VIGAS EN FLEXION

a. AMORTIGUAMIENTO NULO





$$V + pdz - (V + \frac{\partial V}{\partial z} dz) - f_I dz = 0 \qquad (1)$$

EN DONDE $f_I dz = mdx \ \frac{\partial^2 x}{\partial t^2}$ (2)

SUSTITUYENDO (2) EN (1) Y SIMPLIFICANDO:

$$\frac{\partial V}{\partial z} = p - m \frac{\partial^2 x}{\partial t^2} \qquad (3)$$

$$M + Vdz - (M + \frac{\partial M}{\partial z} dz) = 0 \qquad \frac{\partial M}{\partial z} = V \qquad (4)$$

(DESPRECIANDO LOS TERMINOS DE SEGUNDO ORDEN DE LOS MOMENTOS DE p Y f_I)

SUSTITUYENDO (4) EN (3) SE OBTIENE

$$\frac{\partial^2 M}{\partial z^2} + m\frac{\partial^2 x}{\partial t^2} = p \qquad (4')$$

TOMANDO EN CUENTA QUE $\frac{M}{EI} = \frac{\partial^2 x}{\partial z^2}$ SE OBTIENE FINALMENTE

$$\frac{\partial^2}{\partial z^2}(EI \ \frac{\partial^2 x}{\partial z^2}) + m\frac{\partial^2 x}{\partial t^2} = p \qquad (5)$$

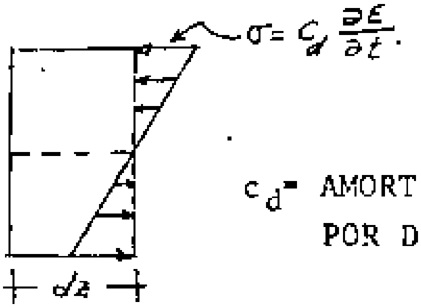
b. AMORTIGUAMIENTO VISCOSO

- FUERZA DE AMORTIGUAMIENTO POR VELOCIDAD TRANSVERSAL $= c(z)\,\frac{\partial x}{\partial t}$

$$\frac{\partial V}{\partial z} = p - m\,\frac{\partial^2 x}{\partial t^2} - c\,\frac{\partial x}{\partial t} \qquad (6)$$

- FUERZA DE AMORTIGUAMIENTO POR DEFORMACION DE LA VIGA.

ACEPTANDO LA HIPOTESIS DE NAVIER DE DEFORMACION PLANA



$$M_{amort} = \int \sigma y da = c_d I(z)\frac{\partial^3 x}{\partial z^2 \partial t}$$

c_d = AMORTIGUAMIENTO POR DEFORMACION

INCORPORANDO EL MOMENTO DEBIDO AL AMORTIGUAMIENTO EN LA EC. (5)

$$\frac{\partial^2}{\partial z^2}\left(EI\,\frac{\partial^2 x}{\partial z^2} + C_d I\,\frac{\partial^3 x}{\partial z^2 \partial t}\right) + m\frac{\partial^2 x}{\partial t^2} + c\frac{\partial x}{\partial t} = p \qquad (6)$$

SI LA EXCITACION ES POR MOVIMIENTO DE LOS APOYOS, SE PUEDE DEMOSTRAR (CLOUGH Y PENZIEN, PAG 303) QUE:

$$\frac{\partial^2}{\partial z^2}\left(EI\,\frac{\partial^2 x}{\partial z^2} + C_d I\,\frac{\partial^3 x}{\partial z^2 \partial t}\right) + m\frac{\partial^2 x}{\partial t^2} + c\frac{\partial x}{\partial t} = P_{efect.}$$

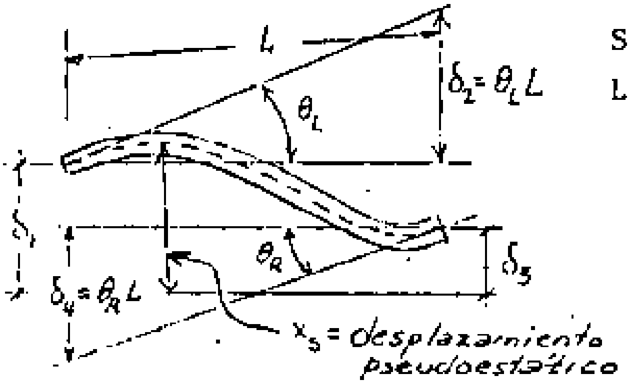
EN DONDE

$$P_{efect} = \frac{-\partial^2}{\partial z^2}\left(EI\,\frac{\partial^2 x_s}{\partial z^2} + C_d I\,\frac{\partial^3 x_s}{\partial z^2 \partial t}\right) - m\frac{\partial^2 x_s}{\partial t^2} - c\,\frac{\partial x_s}{\partial t} \qquad (7)$$

$$x_{t0t}(z,t) = x_{est}(z,t) + x(z,t)$$

x_s = DESPLAZAMIENTO PSEUDOESTATICO OCASIONADO POR EL MOV. DE LOS APOYOS DE MANERA ESTATICA

x = DESPLAZAMIENTO DINAMICO



SI SE TIENE UNA ROTACION Y UNA TRASLACION POR APOYO:

$$x_s = \sum_{i=1}^{4} \text{Ø}_i \delta_i(t) \qquad (8)$$

$\text{Ø}_i(z)$ = CONFIGURACION DE LA VIGA DEBIDA A $\delta_i = 1$

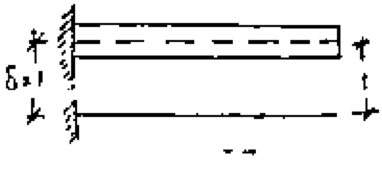
INCORPORANDO (8) EN (7):

$$P_{efect} = -\sum_{i=1}^{4} \{m\text{Ø}_i \ddot{\delta}_i(t) + c\ \text{Ø}_i \dot{\delta}_i(t) + \frac{\partial^2}{\partial z^2}[c_d I(z) \frac{\partial^2 \text{Ø}_i(z)}{\partial z^2}\ \dot{\delta}_i(t)]\} \qquad (9)$$

EN LA MAYORIA DE LOS CASOS EL AMORTIGUAMIENTO INFLUYE POCO EN LA FUERZA EFECTIVA Y LA EC.(9) SE SIMPLIFICA A

$$P_{efect} = -\sum_{i=1}^{4} m\text{Ø}_i(z)\ddot{\delta}_i(t)$$

EN EL CASO DE UN VOLADIZO



$$\text{Ø}_1(z) = 1$$

Y

$$P_{efect} = -m(z)\ \ddot{\delta}_1(t)$$

ANALISIS DE VIBRACIONES LIBRES

CONSIDEREMOS UNA VIGA DE SECCION CONSTANTE (EI= CONSTANTE ; $\bar{m}$=MASA POR UNIDAD DE LONGITUD).

DE LA EC.(5): $EI \frac{\partial^4 x}{\partial z^4} + \bar{m} \frac{\partial^2 x}{\partial t^2} = 0$

$$\frac{\partial^4 x}{\partial z^4} = \frac{\bar{m}}{EI} \frac{\partial^2 x}{\partial t^2} = 0 \qquad (10)$$

RESOLVIENDO LA EC. (10) POR SEPARACION DE VARIABLES:

$$x(z,t) = \theta(z)\, Y(t)$$

$$\theta^{IV}(z)\, Y(t) + \frac{\bar{m}}{EI}\, \theta(z)\, \ddot{Y}(t) = 0 \; ; \; \frac{\theta^{IV}(z)}{\theta(z)} + \frac{\bar{m}}{EI} \frac{\ddot{Y}(t)}{Y(t)} = 0$$

POR LO QUE

$$\frac{\theta^{IV}(z)}{\theta(z)} = -\frac{m}{EI} \frac{\ddot{Y}(t)}{Y(t)} = C = a^4 \quad (C = \text{CONSTANTE})$$

POR LO TANTO OBTENEMOS DOS ECUACIONES DIFERENCIALES ORDINARIAS:

$$\theta^{IV}(z) - a^4\, \theta(z) = 0$$

$$\ddot{Y}(t) + \omega^2 Y(t) = 0 \quad \text{DONDE} \quad \omega^2 = \frac{a^4 EI}{\bar{m}}$$

$$\text{O} \quad a^4 = \frac{\omega^2 \bar{m}}{EI}$$

LA SOLUCION DE LA SEGUNDA DE ESTAS ES:

$$Y(t) = \frac{\dot{Y}(o)}{\omega} \operatorname{sen}\omega t + Y(o) \cos\omega t \qquad (11)$$

LA SOLUCION DE LA PRIMERA ES:

$$\theta(z) = A_1 \text{ sen } az + A_2 \cos az + A_3 \text{ senhaz } + A_4 \text{ coshaz} \qquad (12)$$

EN DONDE LAS A_i SE CALCULAN EN FUNCION DE LAS CONDICIONES DE FRONTERA DE LA VIGA EN AMBOS EXTREMOS.

EJEMPLO

VIGA SIMPLEMENTE APOYADA

LAS CUATRO CONDICIONES DE FRONTERA SON:

en $z=0$: $\theta(o)=0$, $M(o)= EI\ \ddot{\theta}(o) = 0$

en $z=L$: $\theta(L)=0$, $M(L)= EI\theta''(L) = 0$

SUSTITUYENDO $\theta(o)=0$ Y $\theta''(o)=0$ EN LA EC.(12) Y SU SEGUNDA DERIVADA:

$$\left.\begin{aligned} \theta(o) &= A_2 + A_4 \cosh 0 = 0 \\ \ddot{\theta}(o) &= a^2(-A_2 + A_4 \cosh 0) = 0 \end{aligned}\right\} \Rightarrow A_2 = A_4 = 0$$

HACIENDO LO MISMO CON $\theta(L) = 0$ y $\theta''(L) = 0$:

$$\left.\begin{aligned} \theta(L) &= A_1 \text{ sen } aL + A_3 \text{ senh } aL = 0 \\ \ddot{\theta}(L) &= a^2(-A_1 \text{ sen } aL + A_3 \text{ senh } aL) = 0 \end{aligned}\right\} \rightarrow A_3 = 0$$

POR LO TANTO, $\theta(L) = A_1 \text{ sen } aL = 0$

PUESTO QUE $A_1=0$ ES LA SOLUCION TRIVIAL, SE DEBE TENER QUE A_1 SEA ARBITRARIA Y QUE

$$\text{sen } aL = 0 \rightarrow aL = n\pi; \quad n = 0, 1, 2, \ldots, \infty$$

POR LO TANTO, $a = n\pi/L$. RECORDANDO QUE

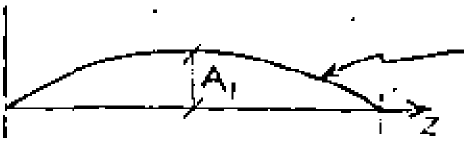
$a^4 = \omega^2\bar{m}/EI$, SE TIENE QUE

$$\omega_n^2 = (n\pi/L)^4 EI/\bar{m} \qquad \text{O} \qquad \omega_n = \frac{n^2\pi^2}{L^2}\sqrt{EI/\bar{m}}$$

SON LAS FRECUENCIAS CIRCULARES NATURALES DE VIBRACION DE LA VIGA.

LAS CONFIGURACIONES MODALES SON

$$\theta_n(z) = A_1 \text{ sen } \frac{n\pi}{L} z$$



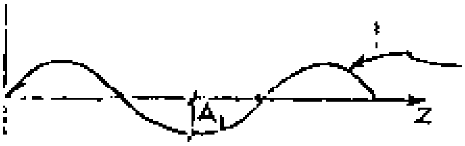
$\theta_1 = A_1 \text{sen } \frac{\pi z}{L}$; $\omega_1 = \frac{\pi^2}{L^2}\sqrt{EI/\bar{m}}$

1er. MODO

$\theta_2 = A_1 \text{sen} \frac{2\pi z}{L}$

$$\omega_2 = \frac{4\pi^2}{L^2}\sqrt{EI/\bar{m}}$$

2o. MODO



$\theta_3 = A_1 \text{sen } \frac{3\pi z}{L}$; $\omega_3 = \frac{9\pi^2}{L^2}\sqrt{EI/\bar{m}}$

3er. MODO

⋮

$$\omega_1 : \omega_2 : \omega_3 :: 1: 4: 9$$

$$\omega_i = n^2\omega_1$$

centro de educación continua
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V. CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

METODO β DE NEW MARK

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JULIO, 1979

Palacio de Minería Calle de Tacuba 5 primer piso. México 1, D. F.

METODO β DE NEWMARK PARA SISTEMAS LINEALES DE VARIOS GRADOS DE LIBERTAD

EJEMPLO

$$\underline{K} = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} \qquad \underline{M} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad ; \quad C_i = 0 \text{ para todo } i$$

$$\left.\begin{aligned} \dot{X}_{i+1} &= \dot{X}_i + (\ddot{X}_i + \ddot{X}_{i+1}) \frac{\Delta t}{2} \\ X_{i+1} &= X_i + \dot{X}_i \Delta t + (\tfrac{1}{2} - \beta) \ddot{X}_i (\Delta t)^2 + \beta \ddot{X}_{i+1} (\Delta t)^2 \end{aligned}\right\} \text{PARA CADA MASA O GRADO DE LIBERTAD}$$

$$\Delta t = 0.2 \; ; \; \beta = 1/6$$

MOVIMIENTO DEL SUELO: $X_o = 1.2t$ (X_o EN CM Y t EN SEGUNDOS)

SI $0 \leq t \leq 2$ SEG, Y $X_o = 4.8 - 1.2\, t$ SI $0 \leq 2 \leq 4$ SEG

Y $X_o = 0$ SI $t < 0$ O $t > 4$ SEG

SI $Y_1 = X_1 - X_o$ Y $Y_2 = X_2 - X_o$

$$\underline{M}\, \ddot{\underline{Y}} + \underline{K}\, \underline{Y} = \underline{0} \rightarrow \underline{M}\, \ddot{\underline{Y}} + \underline{Q} = \underline{0} \; ; \quad \underline{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

(PUESTO QUE $\ddot{X}_o = 0$)

$$m_1 \ddot{Y}_1 + Q_1 = 0 \rightarrow \ddot{Y}_1 = -Q_1/m_1$$

$$m_2 \ddot{Y}_2 + Q_2 = 0 \rightarrow \ddot{Y}_2 = -Q_2/m_2$$

EN t=0, $Y_i = 0$, $\dot{Y}_i = 0$, $\ddot{Y}_i = 0$

EN t=0.2, SUPONGAMOS $\ddot{X}_1 = 1.35$ Y $\ddot{X}_2 = 1.50 \ \frac{CM}{SEG^2}$

$$X_o = 1.2 \times 0.2 = 0.24$$

1er CICLO

PARA LA MASA 1:

$$\dot{X}_1 = 0 + (1.35 + 0)\ \frac{0.2}{2} = 0.135$$

$$X_1 = 0 + 0 + (\frac{1}{2} - \frac{1}{6})(0.2)^2 \times 0 + \frac{1}{6} \times 1.35\ (0.2)^2 = 0.009 \text{ CM}$$

$$Y_1 = 0.009 - 0.24 = -\ 0.231$$

PARA LA MASA 2:

$$\dot{X}_2 = 0 + (1.5 + 0)\ \frac{0.2}{2} = 0.15$$

$$X_2 = 0 + 0 + 0 + \frac{1}{6} \times 1.5(0.2)^2 = 0.01$$

$$Y_2 = 0.01 - 0.24 = -0.23$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \underline{Q} = \begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -0.231 \\ -0.23 \end{bmatrix} = \begin{bmatrix} -2.54 \\ -1.381 \end{bmatrix} \rightarrow \begin{array}{l} \ddot{X}_1 = 2.54/2 = 1.27 = \ddot{Y}_1 \\ \ddot{X}_2 = 1.381/1 = 1.381 = \ddot{Y}_2 \end{array} \rightarrow \text{ETC.}$$

TABLA 2.1, EJEMPLO ~~2.1~~

t	Q_1	$\ddot{x}_1$ m/seg²	$\dot{x}_1$	x_1	x_1-x	Q	$\ddot{x}_2$ m/seg²	$\dot{x}_2$	x_2	x_2-x_0	x_0
0	0	0	0	0	0	0	0	0	0	0	0
0.2	2.540	1.35[illegible]	0.135	0.0[illegible]	-0.2[illegible]	1.390	1.500	0.150	0.0100	-0.2[illegible]	[illegible]
0.2	2.546	1.273	0.127	0.0085	0.2515	1.386	1.386	0.138	0.0[illegible]	0.2[illegible]	[illegible]
0.2	2.546	1.273	0.127	0.0085	0.2515	1.386	1.386	0.138	0.0094	0.2[illegible]	0.24
0.4	4.548	+2.300	0.484	+0.0662	-0.4138	2.468	2.100	0.486	+0.0695	0.4[illegible]	0.48
0.4	4.548	2.274	0.481	0.0660	-0.4140	2.455	2.468	0.523	0.0718	+0.408[illegible]	0.[illegible]
0.4	4.548	2.274	0.481	0.0660	-0.4140	2.45[illegible]	2.455	0.522	0.0717	-0.408[illegible]	[illegible]
0.4	4.548	2.274	0.481	0.0660	-0.4140	2.455	2.455	0.522	0.0717	0.408[illegible]	0.[illegible]
0.6	5.565	2.700	0.978	0.2105	-0.5095	2.960	3.200	1.088	0.2301	-0.48[illegible]	0.72
0.6	5.581	2.793	0.987	0.2111	+0.5089	2.967	2.960	1.064	0.2285	-0.4915	0.72
0.6	5.580	2.790	0.987	0.2111	-0.5089	2.966	2.967	1.065	0.2286	-0.4914	0.72
0.6	5.580	2.790	0.987	0.2111	-0.5089	2.966	2.966	1.065	0.2286	-0.4914	0.7[illegible]
0.8	5.409	2.900	1.556	0.4650	-0.4950	2.790	2.980	1.660	0.5010	-0.4590	[illegible]
0.8	5.423	2.704	1.536	0.4637	-0.4963	2.798	2.790	1.641	0.4997	-0.4603	0.[illegible]
0.8	5.422	2.711	1.537	0.4638	-0.4962	2.797	2.798	1.642	0.4998	-0.4602	[illegible]
0.8	5.422	2.711	1.537	0.4638	-0.4962	2.797	2.797	1.642	0.4998	-0.4602	[illegible]
1.0	4.104	2.150	2.023	0.8216	-0.3784	1.977	2.200	2.142	0.8802	-0.3198	1.2[illegible]
1.0	4.111	2.052	2.013	0.8210	-0.3790	1.985	1.977	2.120	0.8787	-0.3213	1.20
1.0	4.111	2.055	2.014	0.8210	-0.3790	1.985	1.985	2.121	0.8787	-0.3213	[illegible]
1.0	4.111	2.055	2.014	0.8210	-0.3790	1.985	1.985	2.121	0.8787	+0.3213	1.2[illegible]
1.2	1.931	0.950	2.315	1.2575	-0.1825	0.712	0.700	2.390	1.3341	-0.1059	1.44
1.2	1.930	0.965	2.316	1.2576	-0.1824	0.712	0.712	2.391	1.3341	-0.1059	1.44
1.2	1.930	0.965	2.316	1.2576	-0.1824	0.712	0.712	2.391	1.3341	-0.1059	1.44
1.4	-0.653	-0.320	2.381	1.7316	0.0516	-0.735	-0.800	2.387	1.8165	0.1365	1.68
1.4	-0.652	-0.326	2.380	1.7315	0.0515	-0.735	-0.735	2.388	1.8165	0.1365	1.68
1.4	-0.65[illegible]	-0.326	2.380	1.7315	0.0515	-0.735	-0.735	2.388	1.8169	0.1269	1.68
1.6	-3.06[illegible]	-1.50[illegible]	2.197	2.1932	0.2732	-2.026	-2.100	2.104	2.2707	0.3507	1.92
1.6	-3.06[illegible]	-1.541	2.193	2.1929	0.2729	-2.029	-2.026	2.111	2.2712	0.3512	1.92
1.6	-3.050	-1.540	2.193	2.1925	0.2729	-2.029	-2.029	2.111	2.2712	0.3512	1.92
1.8	-4.835	-2.500	1.789	2.5943	0.4343	-2.869	-2.900	1.618	2.6471	0.4871	2.16
1.8	-4.836	-2.415	1.797	2.5940	0.4349	-2.871	-2.869	1.621	2.6473	0.4872	[illegible]
1.8	-4.836	-2.418	1.797	2.5949	0.4349	-2.871	-2.871	1.621	2.6473	0.4873	2.[illegible]
2.0	-5.54[illegible]	-2.800	1.275	2.9034	0.5034	-3.069	-3.000	1.034	2.9132	0.51[illegible]	[illegible]
2.0	-5.549	-2.77[illegible]	1.278	2.9036	0.5036	-3.068	-3.069	1.027	2.9127	0.5127	[illegible]
2.0	-5.549	-2.774	1.278	2.9036	0.5036	3.068	-3.068	1.027	2.9127	0.5127	[illegible]

TABLE 2.1, EXEMPLO 2.4

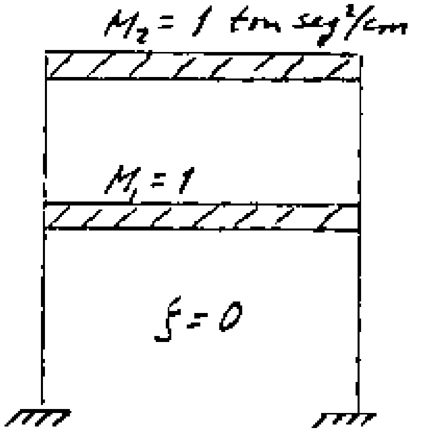
[illegible]

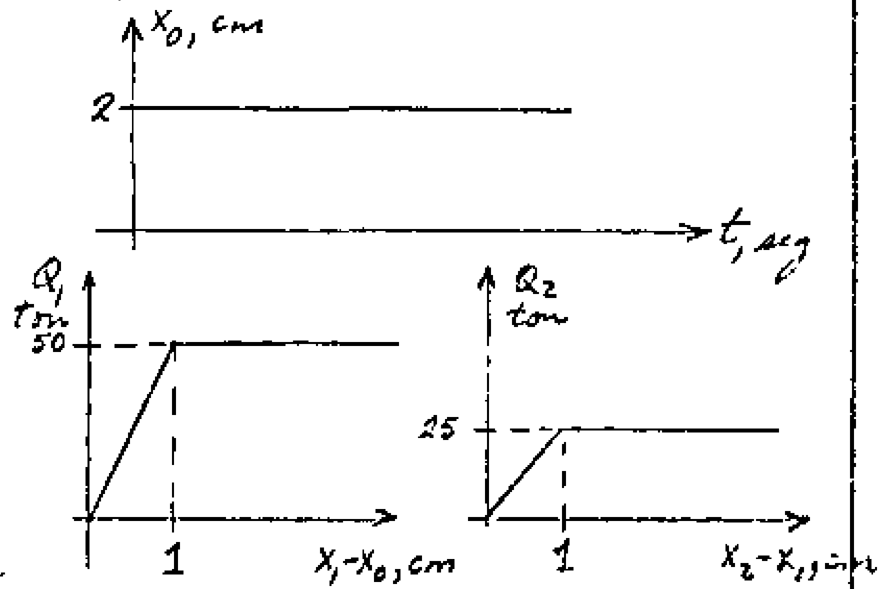
[illegible]	Q_1 ton	$\ddot{x}$ cm/sec²	$\dot{x}$ cm/sec	x_1 cm	$x_1 - x_0$ cm	Q_2 ton	$\ddot{x}$ cm/sec²	$\dot{x}$ cm/sec	x_2 cm	$x_2 - x_0$ cm	x_G cm
[illegible]	[illegible]	-5.200	0.481	3.0875	0.9275	5.332	-5.460	0.174	3.0406	[illegible]	2.16
[illegible]	[illegible]	[illegible]	0.491	3.0885	0.9283	5.337	-5.332	0.187	3.0417	[illegible]	2.16
[illegible]	[illegible]	-5.063	0.493	3.0883	0.9283	-5.337	-5.337	0.186	3.0417	0.8817	2.16
2.4	[illegible]	-6.900	-0.705	3.0731	1.1531	6.386	-6.200	-0.968	2.9665	1.0465	1.92
[illegible]	[illegible]	-6.289	-0.644	3.0772	1.1572	6.383	-6.386	-0.967	2.9652	1.0452	1.92
[illegible]	12.615	-6.299	-0.646	3.0770	1.1570	-6.383	-6.383	-0.986	2.9652	1.0452	1.92
[illegible]	12.616	-6.304	-0.646	3.0770	1.1570	-6.383	-6.383	-0.986	2.9652	1.0452	1.92
[illegible]	-12.388	-6.200	-1.897	2.8225	1.1425	-5.958	-6.000	-2.224	2.6429	0.9629	1.68
2.6	-12.388	-6.194	-1.896	2.8225	1.1425	-5.959	-5.958	-2.220	2.6432	0.9632	1.68
[illegible]	12.389	-6.194	-1.896	2.8225	1.1425	-5.959	-5.959	-2.220	2.6432	0.9632	1.65
[illegible]	-9.573	-4.300	-2.945	2.3320	0.8920	-4.155	-4.100	-3.206	+2.0925	0.6525	1.44
[illegible]	9.540	-4.787	2.994	2.3288	0.8868	-4.150	-4.155	-3.212	2.0921	0.6521	1.44
[illegible]	9.541	4.770	-2.992	2.3289	0.8889	4.150	-4.150	-3.211	2.0921	0.6521	1.44
[illegible]	[illegible]	-4.770	2.992	2.3289	0.8989	-4.150	-4.150	-3.211	2.0921	0.6521	1.44
	-4.667	-2.500	-3.719	1.6502	0.4502	-1.376	-1.400	-3.766	1.3853	0.1853	1.20
[illegible]	-4.699	-2.343	-3.703	1.6512	0.4513	-1.378	-1.376	-3.764	1.3854	0.1854	1.20
3.0	-4.698	-2.349	-3.704	1.6513	0.4513	-1.378	-1.378	-3.764	1.3854	0.1854	1.20
3.2	1.090	0.800	-3.859	0.8845	-0.0755	1.748	1.700	-3.732	0.6255	-0.3345	0.96
3.2	1.106	0.545	-3.884	0.8828	-0.0772	1.748	1.748	-3.727	0.6259	-0.3341	0.96
3.2	1.105	0.553	-3.883	0.8829	-0.0771	1.748	1.748	-3.727	0.6259	-0.3341	0.96
3.2	1.105	0.553	-3.883	0.8829	-0.0771	1.748	1.748	-3.727	0.6259	-0.3341	0.96
3.4	6.608	3.600	-3.468	0.1377	-0.5823	4.506	4.700	-3.082	-0.0649	-0.7849	0.72
3.4	6.629	3.304	-3.438	0.1357	-0.5843	4.515	4.506	-3.101	-0.0662	-0.7862	0.72
3.4	6.628	3.314	-3.439	0.1356	-0.5842	4.515	4.515	-3.100	-0.0661	-0.7861	0.72
3.4	[illegible]	3.314	-3.439	0.1358	-0.5842	4.515	4.515	-3.100	-0.0661	-0.7861	0.72
3.6	10.578	5.400	-2.568	-0.4718	-0.9518	6.251	6.900	-1.958	-0.5799	-1.0599	0.48
[illegible]	10.569	5.289	-2.579	-0.4725	-0.9525	6.277	6.251	-2.023	-0.5842	-1.0642	0.48
[illegible]	10.565	5.299	-2.577	-0.4725	-0.9525	6.277	6.277	-2.020	-0.5841	-1.0641	0.49
3.6	[illegible]	5.299	-2.577	-0.4725	-0.9525	6.277	6.277	-2.020	-0.5841	-1.0641	0.48
3.8	2.257	[illegible]	-1.427	-0.8760	-1.1160	6.612	6.800	-0.712	-0.659	-1.0991	0.24
38	[illegible]	[illegible]	-1.434	-0.8764	-1.1164	6.618	6.612	-0.731	[illegible]	-1.1003	0.24
38	[illegible]	[illegible]	-1.434	-0.8764	-1.1164	6.618	6.618	-0.730	[illegible]	[illegible]	0.24
4.0	[illegible]	[illegible]	-0.260	-1.0441	-1.0441	5.454	5.400	0.472	-0.8821	-0.8821	0
40	[illegible]	[illegible]	-0.255	-1.0437	-1.0437	5.453	5.454	0.477	[illegible]	[illegible]	0
[illegible]	11.41	[illegible]	0.255	-1.0437	[illegible]	5.453	-5.453	0.477	-0.8617	[illegible]	0
4.2	10.70	[illegible]	0.846	[illegible]	-0.9836	5.330	5.300	1.549	[illegible]	[illegible]	0
42	[illegible]	[illegible]	0.846	[illegible]	[illegible]	[illegible]	[illegible]	1.552	[illegible]	[illegible]	0

SISTEMAS DE VARIOS GRADOS DE LIBERTAD CON COMPORTAMIENTO INELASTICO

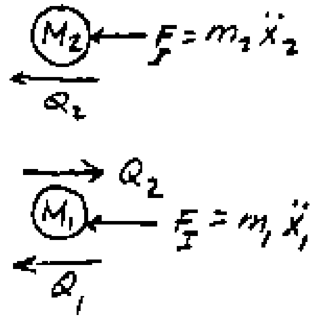
Método β de Newmark. Ejemplo

Calcular mediante el método β de Newmark los desplazamientos máximos absolutos del 1º y 2º niveles de la estructura mostrada abajo, cuando es excitada por un desplazamiento súbito de 2 cm en su base.





$\beta = \frac{1}{6}$, $\Delta t = 0.1$ seg



$\underline{M}\,\underline{\ddot{X}} + \underline{Q} = 0$

$$M_1\ddot{X}_1 + Q_1 - Q_2 = 0$$

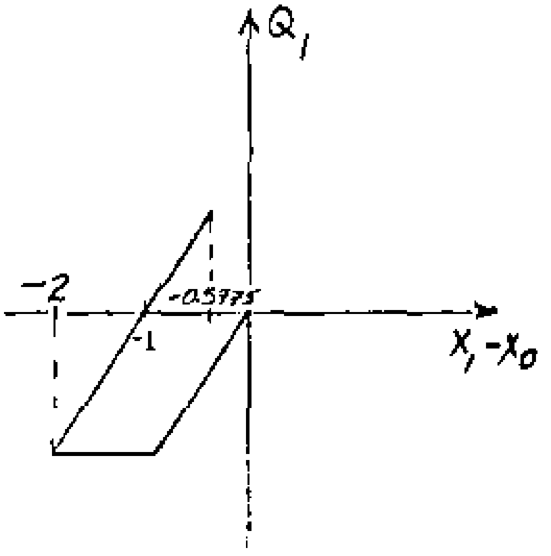
$$M_2\ddot{X}_2 + Q_2 = 0$$

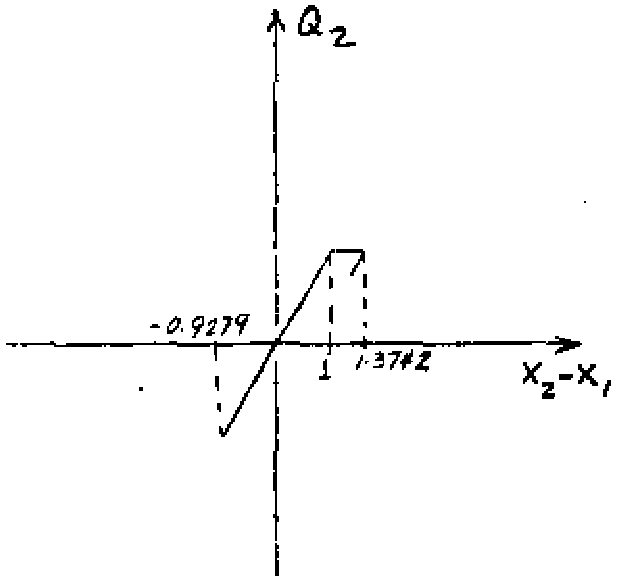
$$\Downarrow$$

$$\left.\begin{aligned} \ddot{X}_1 &= \frac{Q_2 - Q_1}{M_1} \\ \ddot{X}_2 &= \frac{Q_2}{M_2} \end{aligned}\right\} \quad (1)$$

¡En vez de suponer $\ddot{x}_1$ y $\ddot{x}_2$ al inicio de cada ciclo, supondremos Q_1 y Q_2 y calcularemos ambas aceleraciones con base en ellas mediante la ec. (1). Las ecuaciones para la velocidad y el desplazamiento son:

$$\left.\begin{aligned} \dot{x}_{i+1} &= \dot{x}_i + 0.05(\ddot{x}_i + \ddot{x}_{i+1}) \\ x_i &= x_i + 0.1\,\dot{x}_i + \frac{1}{600}(2\ddot{x}_i + \ddot{x}_{i+1}) \end{aligned}\right\} \quad (2)$$





t seg	x_o cm	Q_1 ton	Q_2 ton	$\ddot{x}_1$ cm/seg^2	$\ddot{x}_2$ cm/seg^2	$\dot{x}_1$ cm/seg	x_1 cm	$\dot{x}_2$ cm/seg	x_2 cm	x_1-x_0 cm	x_2-x_1 cm	$\dot{x}_1-\dot{x}_0$ cm/seg	$\dot{x}_2-\dot{x}_1$ cm/seg	OBSERVACIONES
0.0	2.0	-50.00	0.00	50.00	0.00	0.00	0.00	0.00	0.00	-2.00	0.00	0.00	0.00	
0.1	2.0	-37.500	-6.25	37.5	6.250	4.375	0.229	0.3125	0.0104	-1.771	-0.2185	4.375	-4.063	
		-38.550	-5.463	32.3	5.463	4.115	0.2205	0.2731	0.0091	-1.7795	-0.2113	4.115	-3.842	
		-38.975	-5.283	33.5125	5.283	4.1756	0.2225	0.2641	0.0088	-1.7774	-0.2136	4.1756	-3.9114	
		-38.875	-5.34	33.593	5.340	4.1796	0.2226	0.2670	0.0089	-1.7773	-0.2137	4.1796	-3.9126	
		-38.870	-5.343	33.530	5.3425	4.1765	0.2225	0.2671	0.0089	-1.7774	-0.2135	4.1765	-3.9093	
0.2	0.2	-20.00	-10.00	10.00	10.00	6.353	0.7685	1.0342	0.070	-1.2314	-0.6984	6.353	-5.3188	
		-11.575	-17.46	1.575	17.46	5.9317	0.7545	1.4072	0.083	-1.2455	-0.6919	5.9317	-4.5244	
		-12.275	-16.7975	-5.185	16.7975	5.5937	0.7432	1.3741	0.0814		-0.6617			
		-12.84	-16.5446	-3.9575	16.5446	5.6551	0.7453	1.3614	0.0809		-0.6643			
		-12.735	-16.6076	-3.8096	16.6076	5.6625	0.7455	1.3646	0.0810		-0.6644			
		-12.7216	-16.6100	-3.886	16.6100	5.6587	0.7454	1.3647	0.0811	-1.2546	-0.6642	5.6587	-4.294	
0.3	2.0	10.000	-25.00	-35.00	25.00	3.7144	1.2399	3.4452	0.3146		-0.9252			
		11.995	-23.1324	-36.995	23.132	3.6146	1.2366	3.3518	0.3114		-0.9251			
		11.83	-23.1277	-34.963	23.127	3.7162	1.2400	3.3510	0.3114		-0.9285			
		12.00	-23.2129	-35.13	23.213	3.7080	1.2397	3.3558	0.3116		-0.9280			
		11.9885	-23.2018	-35.202	23.202	3.7043	1.2396	3.3552	0.3116	-0.7604	-0.9279	3.7043	-0.3491	
0.4	2.0	25.000	-15.00	-40.000	15.000	-0.1193	1.4219	5.3067	0.7520		-0.6698			
		21.095	-16.745	-36.095	16.7457	0.0563	1.4272	5.3853	0.7544		-0.6727			
		21.360	-16.8193	-38.1057	16.8193	-0.0341	1.4245	5.3886	0.7545		-0.6699			
		21.225	-16.7493	-38.0443	16.7493	-0.0313	1.4245	5.3854	0.7544	-0.5755	-0.6700	-0.0313	5.4067	*
0.5	2.0	15.00												
		12.64	-10.00	-25.0	10.00	-3.1835	1.2528	6.7228	1.3654		0.1126			
		12.87	2.8159	-22.64	-2.8159	-3.0655	1.2568	6.0818	1.3440		0.0872			
		13.89	2.1819	-10.024	-2.1819	-2.4347	1.2778	6.1137	1.3451		0.0673			
		13.75	1.6833	-11.7081	-1.683	-2.5189	1.2750	6.1387	1.3459		0.0709			
		13.72	1.7853	-11.9459	-1.7853	-2.5308	1.2746	6.1336	1.3457	-0.7254	0.0741	-2.5308	8.6644	
0.6	2.0	0.000	15.000	15.000	-15.000	-1.6915	1.0524	5.2943	1.8745		0.8221			
		2.62	20.550	12.38	-20.55	-2.5090	1.0023	5.0168	1.9188		0.9165			
		0.115	22.9139	20.435	-22.9189	-2.1063	1.0157	4.8986	1.9149		0.8992			
		0.785	22.4804	22.1289	-22.4804	-1.9866	1.0185	4.9203	1.9156		0.8971			
		0.925	22.4285	21.5554	-22.4285	-2.050	1.0176	4.9229	1.9157		0.8987			
		0.88	22.4532	21.5485	-22.4532	-2.050	1.0176	4.9216	1.9156	-0.9824	0.8990	-2.050	6.9716	
0.615	2.00	-3.00	25.000	28.000	-25.000	-1.6905	0.9894	4.5657	1.9867		0.9973			
		-0.53	24.9334	25.53	-24.9334	-1.6967	0.9894	4.5659	1.9867	-1.0106	0.9973	-1.6967	6.2626	**

* $x_1-x_0-x_1$ SE HACE CASI CERO, CAMBIO DE POSITIVO A NEGATIVO. $|x_1|$máx = 1.4245 cm

** CAMBIO DE RIGIDEZ EN EL 2o. PISO x_2-x_1 = 1.00, Q_2 = 25

t seg	x_o cm	Q_1 ton	Q_2 ton	$\ddot{x}_1$ cm/seg^2	$\ddot{x}_2$ cm/seg^2	$\dot{x}_1$ cm/seg	x_1 cm	x_2 cm/seg	x_2 cm	x_1-x_0 cm	x_2-x_1 cm	$\dot{x}_1-\dot{x}_0$ cm/seg	$\dot{x}_2-\dot{x}_1$ cm/seg	OBSERVACIONES
0.70	2.00	-10.000	25.000	35.000	-25.000	0.8696	0.9484	2.441	2.285		1.3366			
		-2.580	25.000	27.580	-25.000	0.5728	0.9405	2.441	2.285		1,3445			
		-2.975	25.000	27.975	-25.000	0.5886	0.9409	2.441	2.285		1.3441			
		-2.955	25.000	27.955	-25,000	0.5878	0.9409	2.441	2.285	-1.0591	1,3441	0.5878	1.8532	
0.735	2.00	-1.000	25.000	26.000	-25.000	1.5368	0.98065	1.565	2.3549					
		-0.9675	25.000	25.9675	-25.000	1.5367	0.98065	1,566	2.3549					
		-0.9671	25.000	25.9671	-25.000	1.5367	0.98065	1.566	2.3549	-1.1093	1.3742	1.5367	0.0293	
0.80	2.00	5.000	20.000	15.000	-20.000	2.8823	1.1282	0.09128	2.4068		1.2756			
		6.365	22.575	16.245	-22.595	2.9196	1.0449	0.01343	2.4053		1.3604			
		6.570	22.4619	18.085	-22.469	2.9748	1.1300	0.0173	2.4053		1.2753			
		6.435	22.5613	16.0948	-22.5613	2.9151	1.1288	0.0144	2.4053	-0.8712	1,2765	2.9151	-2.9007	**

** x_2 SE HACE CASI CERO

$|x_2|$ máx = 2.4053 cm

V. CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

EJEMPLO

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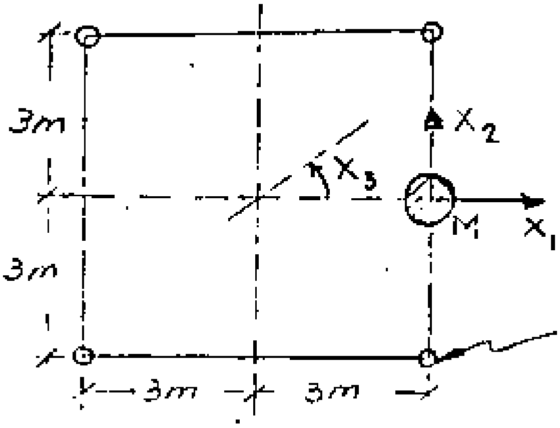
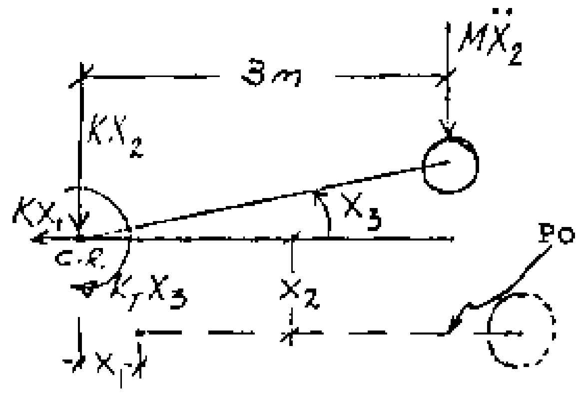
JULIO, 1979

Ejemplo: Calcular las frecuencias circulares y los modos de vibración de la siguiente estructura:



Columnas de rigidez k en cualquier dirección

$K = 4k$



Posición de equilibrio estático

$$X_2 = x_2 + 3X_3$$

$$\therefore \quad x_2 = X_2 - 3X_3$$

En la dirección de X_1: $M\ddot{X}_1 + KX_1 = 0 \Rightarrow \omega_1^2 = \frac{K}{M}$

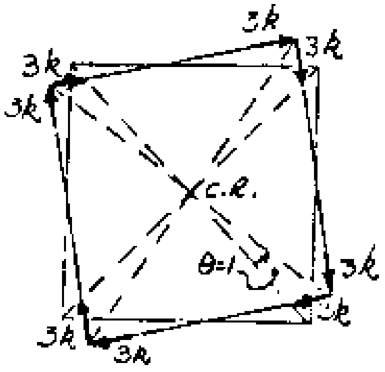
(movimiento desacoplado con X_2 y X_3)

$$\underline{z}_1^T = [1,0,0]$$

En la dirección X_2: $M\ddot{X}_2 + Kx_2 = 0$

$$M\ddot{X}_2 + KX_2 - 3KX_3 = 0 \qquad (1)$$

En la dirección X_3: $M\ddot{X}_2 \times 3 + K_T X_3 = 0 \qquad (2)$



Momento respecto a C.R.= 8 x 3k x 3 = 72k

= 18K

Sustituyendo K_T en la ec. (2):

$$3M\ddot{X}_2 + 18K\,X_3 = 0 \qquad (3)$$

$$M\ddot{X}_2 + 6K\,X_3 = 0 \qquad (4)$$

Restando la ec. (4) a la ec. (1) se obtiene:

$$KX_2 - 9KX_3 = 0 \quad \therefore \quad X_2 = 9X_3 \;\rightarrow\; \ddot{X}_2 = 9\ddot{X}_3$$

Sustituyendo esto último en la ec. (4): $\ddot{X}_3 + \frac{6K}{9M}\,X_3 = 0$

$$\therefore \; \omega_2^2 = \frac{2}{3}\,\frac{K}{M} \quad ; \; \underline{Z}_2^T = [0,9,1]$$

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V CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

SEISMICITY

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JULIO, 1979

Chapter 6

SEISMICITY

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6.1 ON SEISMICITY MODELS

Rational formulation of engineering decisions in seismic areas requires quantitative descriptions of seismicity. These descriptions should conform with their intended applications: in some instances, simultaneous intensities during each earthquake have to be predicted at several locations, while in others it suffices to make independent evaluations of the probable effects of earthquakes at each of those locations.

The second model is adequate for the selection of design parameters of individual components of a regional system (the structures in a region or country) when no significant interaction exists between response or damage of several such individual components, or between any of them and the system as a whole. In other words, it applies when the damage — or negative utility — inflicted upon the system by an earthquake can be taken simply as the addition of the losses in the individual components.

The linearity between monetary values and utilities implied in the second model is not always applicable. Such is the case, for instance, when a significant portion of the national wealth or of the production system is concentrated in a relatively narrow area, or when failure of life-line components may disrupt emergency and relief actions just after an earthquake. Evaluation of risk for the whole regional system has then to be based on seismicity models of the first type, that is, models that predict simultaneous intensities at several locations during each event; for the purpose of decision making, nonlinearity between monetary values and utilities can be accounted for by means of adequate scale transformations. These models are also of interest to insurance companies, when the probability distribution of the maximum loss in a given region during a given time interval is to be estimated.

Whatever the category to which a seismic risk problem belongs, it requires the prediction of probability distributions of certain ground motion characteristics (such as peak ground acceleration or velocity, spectral density, response or Fourier spectra, duration) at a given site during a single shock or of maximum values of some of those characteristics in earthquakes occurring during given time intervals. When the reference interval tends to infinity, the probability distribution of the maximum value of a given characteristic approaches that of its maximum *possible* value. Because different systems or subsystems are sensitive to different ground motion characteristics, the term *intensity characteristic* will be used throughout this chapter to mean a particular parameter or set of parameters of an earthquake motion, in terms of which the response is to be predicted. Thus, when dealing with the failure probability of a structure, intensity can be alternatively measured — with different degrees of correlation with structural response — by the ordinate of the response spectrum for the corresponding period and damping, the peak ground acceleration, or the peak ground velocity.

In general, local instrumental information does not suffice for estimating the probability distributions of maximum intensity characteristics, and use has to be made of data on subjective measures of intensities of past earthquakes, of models of *local seismicity*, and of expressions relating characteristics with magnitude and site-to-source distance. Models of local seismicity consist, at least, of expressions relating magnitudes of earthquakes generated in given volumes of the earth's crust with their return periods. More often than not, a more detailed description of local seismicity is required, including estimates of the maximum magnitude that can be generated in these volumes, as well as probabilistic (stochastic process) models of the possible histories of seismic events (defined by magnitudes and coordinates).

This chapter deals with the various steps to be followed in the evaluation of seismic risk at sites where information other than direct instrumental records of intensities has to be used: identifying potential sources of activity near the site, formulating mathematical models of local seismicity for each source, obtaining the contribution of each source to seismic risk at the site and adding up contributions of the various sources and combining information obtained from local seismicity of sources near the site with data on instrumental or subjective intensities observed at the site.

The foregoing steps consider use of information stemming from sources of different nature. Quantitative values derived therefrom are ordinarily tied to wide uncertainty margins. Hence they demand probabilistic evaluation, even though they cannot always be interpreted in terms of relative frequencies of outcomes of given experiments. Thus, geologists talk of the maximum magnitude that can be generated in a given area, assessed by looking at the dimensions of the geological accidents and by extrapolating the observations of other regions which available evidence allows to brand as similar to the one of interest; the estimates produced are obviously uncertain, and the degree of uncertainty should be expressed together with the most probable value. Following nearly parallel lines, some geophysicists estimate the energy that can be liberated by a single shock in a given area by making quantitative assumptions about source dimensions, dislocation amplitude and stress drop, consistent with tectonic models of the region and, again, with comparisons with areas of similar tectonic characteristics.

Uncertainties attached to estimates of the type just described are in general extremely large: some studies relating fault rupture area, stress drop, and magnitude (Brune, 1963) show that, considering not unusually high stress drops, it does not take very large source dimensions to get magnitudes 8.0 and greater, and those studies are practically restricted to the simplest types of fault displacement. It is not clear, therefore, that realistic bounds can always be assigned to potential magnitudes in given areas or that, when this is feasible, those bounds are sufficiently low, so that designing structures to withstand the corresponding intensities is economically sound, particularly when occurrence of those intensities is not very likely in the near future. Because uncertainties in maximum feasible magnitudes and in other parameters defining magnitude-recurrence laws can be as significant as their mean values when trying to make rational seismic design decisions, those uncertainties have to be explicitly recognized and accounted for by means of adequate probabilistic criteria. A corollary is that geophysically based estimates of seismicity parameters should be accompanied with corresponding uncertainty measures.

Seismic risk estimates are often based only on statistical information (observed magnitudes and hypocentral coordinates). When this is done, a wealth of relevant geophysical information is neglected, while the probabilistic prediction of the future is made to rely on a sample that is often small and of little value, particularly if the sampling period is short as compared with the desirable return period of the events capable of severely damaging a given system.

The criterion advocated here intends to unify the foregoing approaches and rationally to assimilate the corresponding pieces of information. Its philosophy consists in using the geological, geophysical, and all other available non-statistical evidence for producing a set of alternate assumptions concerning a mathematical (stochastic process) model of seismicity in a given source area. An initial probability distribution is assigned to the set of hypotheses, and the statistical information is then used to improve that probability assignment. The criterion is based on application of *Bayes theorem*, also called the *theorem of the probabilities of hypotheses*. Since estimates of risk depend largely on conceptual models of the geophysical processes involved, and these are known with different degrees of uncertainty in different zones of the earth's crust, those estimates will be derived from stochastic process models with uncertain forms or parameters. The degree to which these uncertainties can be reduced depends on the limitations of the state of the art of geophysical sciences and on the effort that can be put into compilation and interpretation of geophysical and statistical information. This is an economical problem that should be handled, formally or informally, by the criteria of decision making under uncertainty.

6.2 INTENSITY ATTENUATION

Available criteria for the evaluation of the contribution of potential seismic sources to the risk at a site make use of *intensity attenuation* expressions that relate intensity characteristics with magnitude and distance from site to source. Depending on the application envisaged, the intensity characteristic to be predicted can be expressed in a number of manners, ranging from a subjective index, such as the *Modified Mercalli intensity*, to a combination of one or more quantitative measures of ground shaking (see Chapter 1).

A number of expressions for attenuation of various intensity characteristics with distance have been developed, but there is little agreement among most of them (Ambraseys, 1973). This is due in part to discrepancies in the definitions of some parameters, in the ranges of values analyzed, in the actual wave propagation properties of the geological formations lying between source and site, in the dominating shock mechanisms, and in the forms of the analytical expressions adopted a priori.

Most intensity-attenuation studies concern the prediction of earthquake characteristics on rock or firm ground, and assume that these characteristics, properly modified in terms of frequency-dependent soil amplification factors, should constitute the basis for estimating their counterparts on soft ground. Observations about the influence of soil properties on earthquake damage support the assumption of a strong correlation between type of local ground and intensity in a given shock. Attempts to analytically predict the characteristics of motions on soil given those on firm ground or on bedrock have not been too successful, however (Crouse, 1973; Hudson and Udwadia, 1973; Salt, 1974), with the exception of some peculiar cases, like Mexico City (Herrera et al., 1965), where local conditions favor the fulfillment of the assumptions implied by usual analytical models. The following paragraphs concentrate on prediction on intensities on firm ground; the influence of local soil is discussed in Chapter 4.

6.2.1 *Intensity attenuation on firm ground*

When isoseismals (lines joining sites showing equal intensity) of a given shock are based only on intensities observed on homogeneous ground conditions, such as *firm ground* (compact soils) or bedrock, they are roughly elliptical and the orientations of the corresponding axes are often correlated with local or regional geological trends (Figs. 6.1—6.3). In some regions — for instance near major faults in the western United States — those trends are well defined and the correlations are clear enough as to permit prediction of intensity in the near and far fields in terms of magnitude and distance to the generating fault or to the centroid of the energy liberating volume. In other regions, such as the eastern United States and most of Mexico, isoseismals seem to elongate systematically in a direction that is a function of the epi-

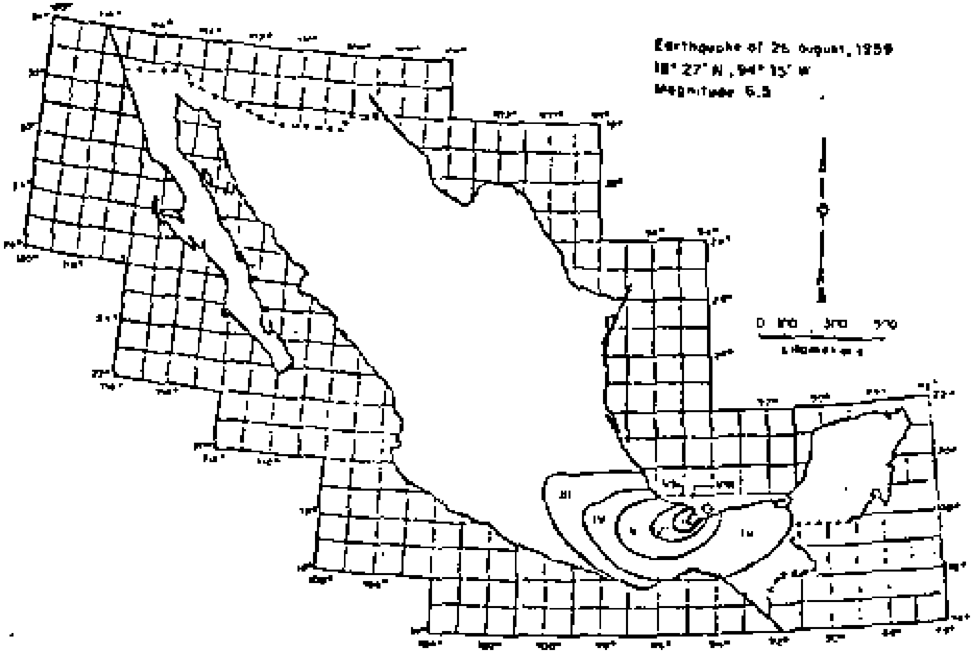


Fig. 6.1. Isoseismals of an earthquake in Mexico. (After Figueroa, 1963.)

central coordinates (Bollinger, 1973; Figueroa, 1963). In that case, intensity should be expressed as a function of magnitude and coordinates of source and site. For most areas in the world, intensity has to be predicted in terms of simple — and cruder — expressions that depend only on magnitude and distance from site to instrumental hypocenter. This stems from inadequate knowledge of geotectonic conditions and from limited information concerning the volume where energy is liberated in each shock.

A comparison of the rates of attenuation of intensities on firm ground for shocks on western and eastern North America has disclosed systematic differences between those rates (Milne and Davenport, 1969). This is the source of a basic, but often unavoidable, weakness of most intensity-attenuation expressions, because they are based on heterogeneous data, recorded in different zones, and the very nature of their applications implies that the less is known about possible systematic deviations in a given zone, as a consequence of the meagerness of local information, the greater weight is given to predictions with respect to observations.

6.2.1.1 Modified Mercalli intensities

An analysis of the Modified Mercalli intensities on firm ground reported for earthquakes occurring in Mexico in the last few decades leads to the following expression relating magnitude M, hypocentral distance R (in kilometers) and intensity I (Esteva, 1968):

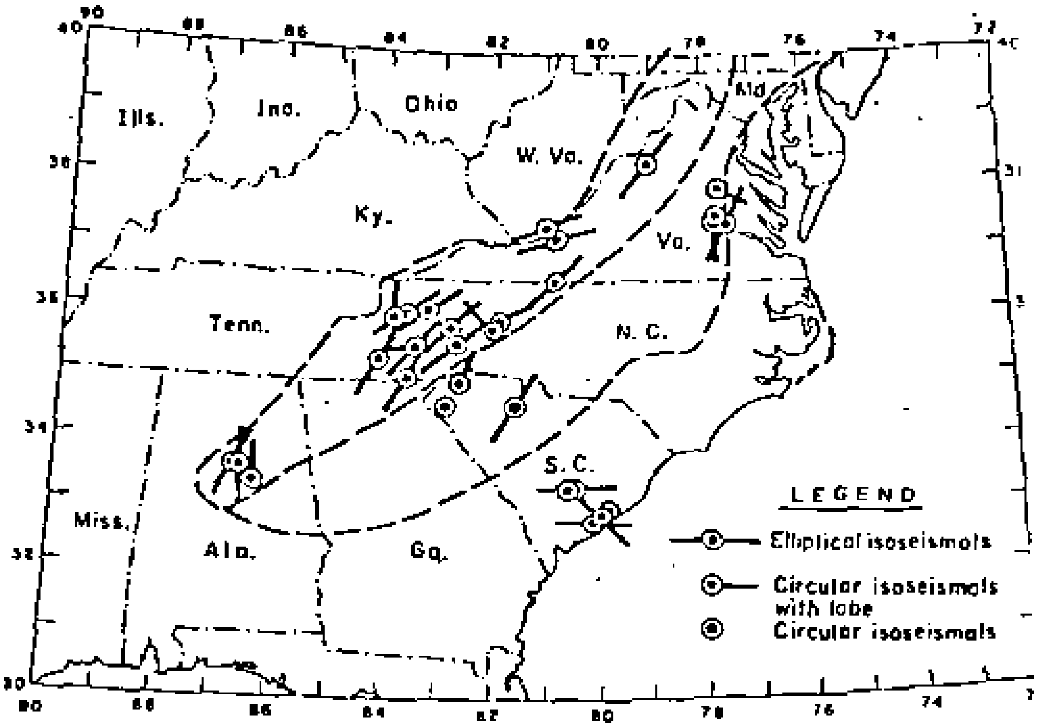


Fig. 6.2. Elongation of isoseismals in the southeastern United States. (After Bollinger, 1973.)

$$I = 1.45 M - 5.7 \log_{10} R + 7.9 \qquad (6.1)$$

The prediction error, defined as the difference between observed and computed intensity, is roughly normally distributed, with a standard deviation of 2.04, which means that there is a probability of 60% that an observed intensity is more than one degree greater or smaller than its predicted value.

6.2.1.2 Peak ground accelerations and velocities

A few of the available expressions will be described. Their comparison will show how cautiously a designer intending to use them should proceed.

Housner studied the attenuation of peak ground accelerations in several regions of the United States and presented his results graphically (1969) in terms of fault length (in turn a function of magnitude), shapes of isoseismals and areas experiencing intensities greater than given values (Fig. 6.4 and 6.5).

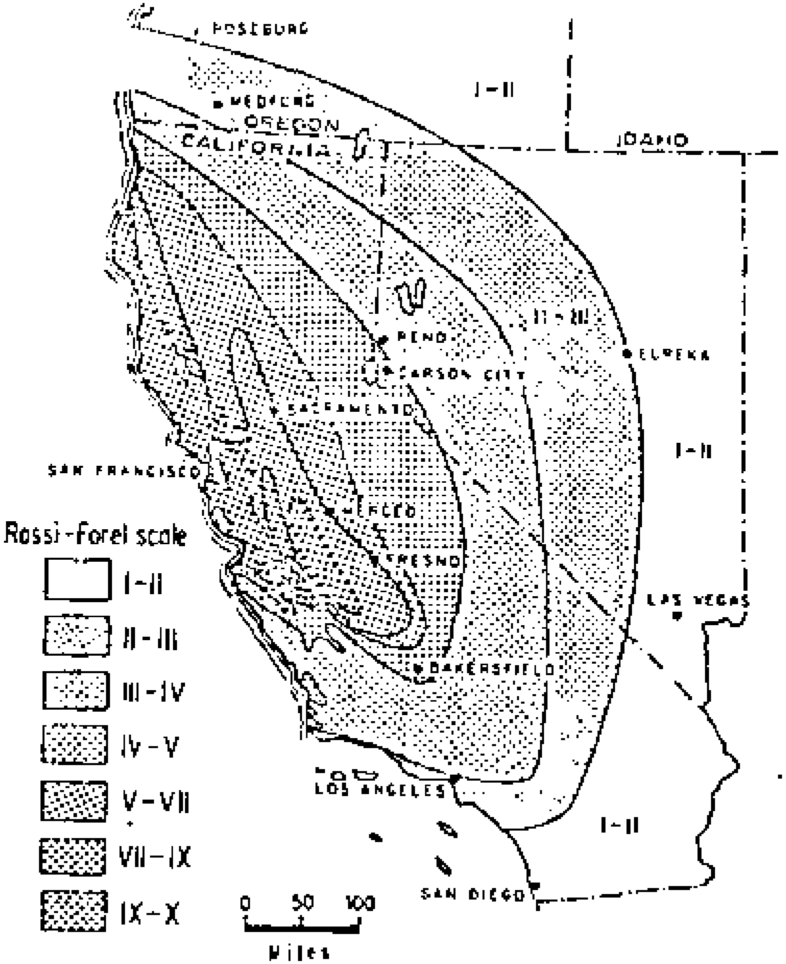
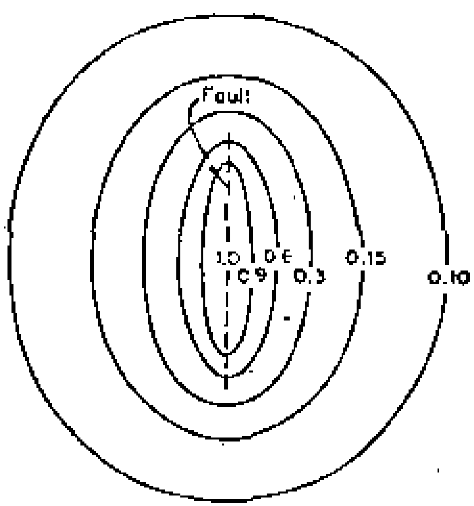


Fig. 6.3. Isoseismals in California. (After Bolt, 1970.)

He showed that intensities attenuate faster with distance on the west coast than in the rest of the country. This comparison is in agreement with Milne and Davenport (1969), who performed a similar analysis for Canada. From observations of strong earthquakes in California and in British Columbia, they developed the following expression for a, the peak ground acceleration, as a fraction of gravity:

$$a/g = 0.0069\, e^{1.6M}/(1.1\, e^{1.1M} + R^2) \tag{6.2}$$

Here, R is epicentral distance in kilometers. The acceleration varies roughly as $e^{1.64M}R^{-2}$ for large R, and as $e^{0.54M}$ where R approaches zero. This reflects to some extent the fact that energy is released not at a single point but from finite volume. A later study by Davenport (1972) led him



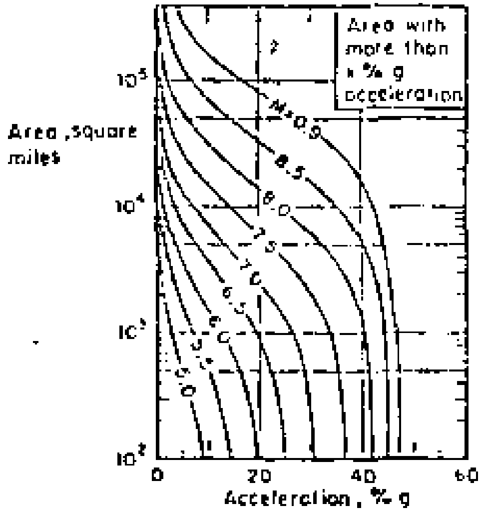


Fig. 6.4. Idealized contour lines of intensity of ground shaking. (After Housner, 1969.)

Fig. 6.5. Area in square miles experiencing shaking of x %g or greater for shocks of different magnitudes. (After Housner, 1969.)

to propose the expression:

$$a/g = 0.279\, e^{0.8M}/R^{1.64} \tag{6.3}$$

The statistical error of this equation was studied by fitting a lognormal probability distribution to the ratios of observed to computed accelerations. A standard deviation of 0.74 was found in the natural logarithms of those ratios.

Esteva and Villaverde (1973), on the basis of accelerations reported by Hudson (1971, 1972a,b), derived expressions for peak ground accelerations and velocities, as follows:

$$a/g = 5.7\, e^{0.8M}/(R + 40)^2 \tag{6.4}$$

$$v = 32\, e^{M}/(R + 25)^{1.7} \tag{6.5}$$

Here v is peak ground velocity in cm/sec and the other symbols mean the same as above. The standard deviation of the natural logarithm of the ratio of observed to predicted intensity is 0.64 for accelerations and 0.74 for velocities. If judged by this parameter, eqs. 6.2 and 6.4 seem equally reliable. However, as shown by Fig. 6.6, their mean values differ significantly in some ranges.

With the exception of eq. 6.2, all the foregoing attenuation expressions are products of a function of R and a function of M. This form, which is acceptable when the dimensions of the energy-liberating sourc re small com-

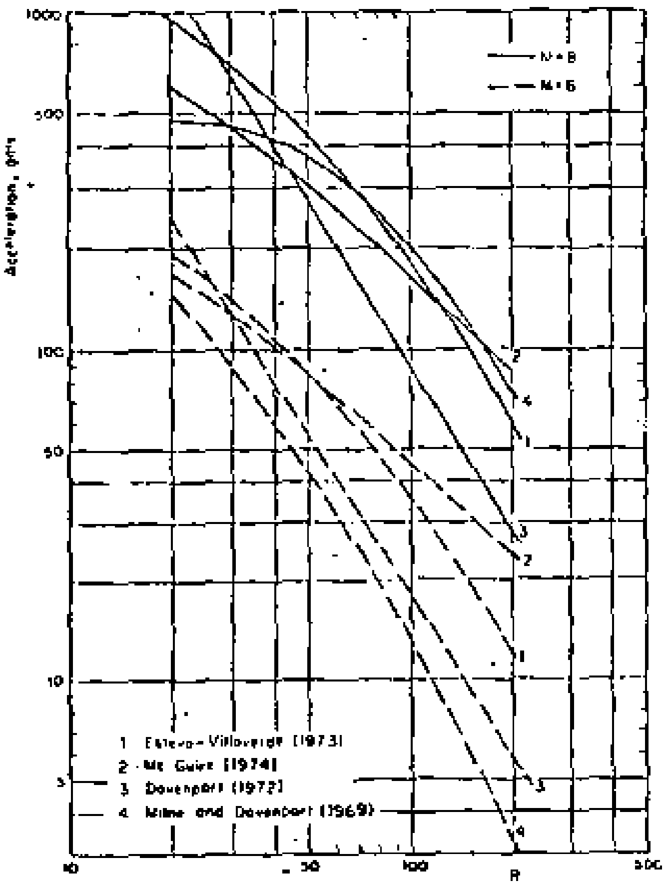


Fig. 6.6. Comparison of several attenuation expressions.

pared with R, is inadequate when dealing with earthquake sources whose dimensions are of the order of moderate hypocentral distances, and often greater than them. Although equation errors (probability distributions of the ratio of observed to predicted intensities) have been evaluated by Davenport (1972) and Esteva and Villaverde (1973), their dependence on M and R has not been analyzed. Because seismic risk estimates are very sensitive to the attenuation expressions in the range of large magnitudes and short distances, more detailed studies should be undertaken, aiming at improving those expressions in the mentioned range, and at evaluating the influence of M and R on equation error. Information on strong-motion records will probably be scanty for those studies, and hence they will have to be largely based on analytical or physical models of the generation and propagation of seismic waves. Although significant progress has been lately attained in this direction (Trifunac, 1973) the results from such models have hardly influenced the

practice of seismic risk estimation because they have remained either unknown to or imperfectly appreciated by engineers in charge of the corresponding decisions.

6.2.1.3 Response spectra

Peak ground acceleration and displacement are fairly good indicators of the response of structures possessing respectively very high and very small natural frequencies. Peak velocity is correlated with the response of intermediate-period systems, but the correlation is less precise than that tying the former parameters; hence, it is natural to formulate seismic risk evaluation and engineering design criteria in terms of spectral ordinates.

Response spectrum prediction for given magnitude and hypocentral or site-to-fault distance usually entails a two-step process, according to which peak ground acceleration, velocity and displacement are initially estimated and then used as reference values for prediction of the ordinates of the response spectrum. Let the second step in the process be represented by the operation $y_s = \alpha y_g$, where y_s is an ordinate of the response spectrum for a given natural period and damping ratio, and y_g is a parameter (such as peak ground acceleration or velocity) that can be directly obtained from the time-history record of a given shock regardless of the dynamic properties of the systems whose response is to be predicted. For given M and R, y_g is random and so is $y_s/y_g = \alpha$; the mean and standard deviation of y_s depend on those of y_g and α and on the coefficient of correlation of the latter variables. As shown above, y_g can only be predicted within wide uncertainty limits, often wider than those tied to y_s (Esteva and Villaverde, 1973). The coefficient of variation of y_s given M and R can be smaller than that of y_g only if α and y_g are negatively correlated, which is often the case: the greater the deviation of an observed value of y_g with respect to its expectation for given M and R, the lower is likely to be α. In other words, it seems that in the intermediate range of natural periods the expected values of spectral ordinates for given damping ratios can be predicted directly in terms of magnitude and focal distance with narrower (or at most equal) margins of uncertainty than those tied to predicted peak ground velocities. For the ranges of very short or very long natural periods, peak amplitudes of ground motion and spectral ordinates approach each other and their standard errors are therefore nearly equal.

McGuire (1974) has derived attenuation expressions for the conditional values (given M and R) of the mean and of various percentiles of the probability distributions of the ordinates of the response spectra for given natural periods and damping ratios. Those expressions have the same form as eqs. 6.4 and 6.5, but their parameters show that the rates of attenuation of spectral ordinates differ significantly from those of peak ground accelerations or velocities. For instance, McGuire finds that peak ground velocity attenuates in proportion to $(R + 25)^{-1.70}$, while the mean of the pseudovelocity for a

TABLE 6.1

McGuire's attenuation expressions $y = b_1 10^{b_2 M}(R + 25)^{-b_3}$

y	b_1	b_2	b_3	$V(y)$ = coeff. of var. of y
a gals	472.3	0.278	1.301	0.548
v cm/sec	5.64	0.401	1.202	0.696
d cm	0.393	0.434	0.885	0.663
Undamped spectral pseudovelocities				
T = 0.1 sec	11.0	0.278	1.346	0.941
0.5	3.05	0.391	1.001	0.636
1.0	0.631	0.378	0.549	0.768
2.0	0.0766	0.469	0.419	0.989
5.0	0.0834	0.564	0.897	1.344
5% damped spectral pseudovelocities				
T = 0.1 sec	10.09	0.233	1.341	0.651
0.5	5.74	0.356	1.197	0.591
1.0	0.432	0.399	0.704	0.703
2.0	0.322	0.466	0.675	0.941
5.0	0.0706	0.567	0.935	1.193

natural period of 1 sec and a damping ratio of 2% attenuates in proportion to $(R + 25)^{-0.59}$. These results stem from the way that frequency content changes with R and lead to the conclusion that the ratio of spectral velocity should be taken as a function of M and R.

Table 6.1 summarizes McGuire's attenuation expressions and their coefficients of variation for ordinates of the pseudovelocity spectra and for peak ground acceleration, velocity and displacement. Similar expressions were derived by Esteva and Villaverde (1973), but they are intended to predict only the maxima of the expected acceleration and velocity spectra, regardless of the periods associated with those maxima. No analysis has been performed of the relative validity of McGuire's and Esteva and Villaverde's expressions for various ranges of M and R.

6.3 LOCAL SEISMICITY

The term *local seismicity* will be used here to designate the degree of seismic activity in a given volume of the earth's crust; it can be quantitatively described according to various criteria, each providing a different amount of information. Most usual criteria are based on upper bounds to the magnitudes of earthquakes that can originate in a given seismic source, on the

amount of energy liberated by shocks per unit volume and per unit time or on more detailed statistical descriptions of the process.

6.3.1 Magnitude-recurrence expressions

Gutenberg and Richter (1954) obtained expressions relating earthquake magnitudes with their rates of occurrence for several zones of the earth. Their results can be put in the form:

$$\lambda = \alpha e^{-\beta M} \tag{6.6}$$

where λ is the mean number of earthquakes per unit volume and per unit time having magnitude greater than M and α and β are zone-dependent constants; α varies widely from point to point, as evidenced by the map of epicenters shown in Fig. 6.7, while β remains within a relatively narrow range, as shown in Fig. 6.8. Equation 6.6 implies a distribution of the energy liberated per shock which is very similar to that observed in the process of microfracturing of laboratory specimens of several types of rock subjected to gradually increasing compressive or bending strain (Mogi, 1962; Scholz, 1968). The values of β determined in the laboratory are of the same order as those obtained from seismic events, and have been shown to depend on the heterogeneity of the specimens and on their ability to yield locally. Thus, in heterogeneous specimens made of brittle materials many small shocks precede a major fracture, while in homogeneous or plastic materials the number of small shocks is relatively small. These cases correspond to large and small β-values, respectively. No general relationship is known to the writer between β and geotectonic features of seismic provinces: complexity of crustal structure and of stress gradients precludes extrapolation of laboratory results; and statistical records for relatively small zones of the earth are not, as a rule, adequate for establishing local values of β. Figure 6.8 shows that for very high magnitudes the observed frequency of events is lower than predicted by eq. 6.6. In addition, Rosenblueth (1969) has shown that β cannot be smaller than 3.46, since that would imply an infinite amount of energy liberated per unit time. However, Fig. 6.8 shows that the values of β which result from fitting expressions of the form 6.6 to observed data are smaller than 3.46; hence, for very high values of M (above 7, approximately) the curve should bend down, in accordance with statistical evidence.

Expressions alternative to eq. 6.6 have been proposed, attempting to represent more adequately the observed magnitude-recurrence data (Rosenblueth, 1964; Merz and Cornell, 1973). Most of these expressions also fail to recognize the existence of an upper bound to the magnitude that can be generated in a given source. Although no precise estimates of this upper bound can yet be obtained, recognition of its existence and of its dependence on the geotectonic characteristics of the source is inescapable. Indeed, the prac-

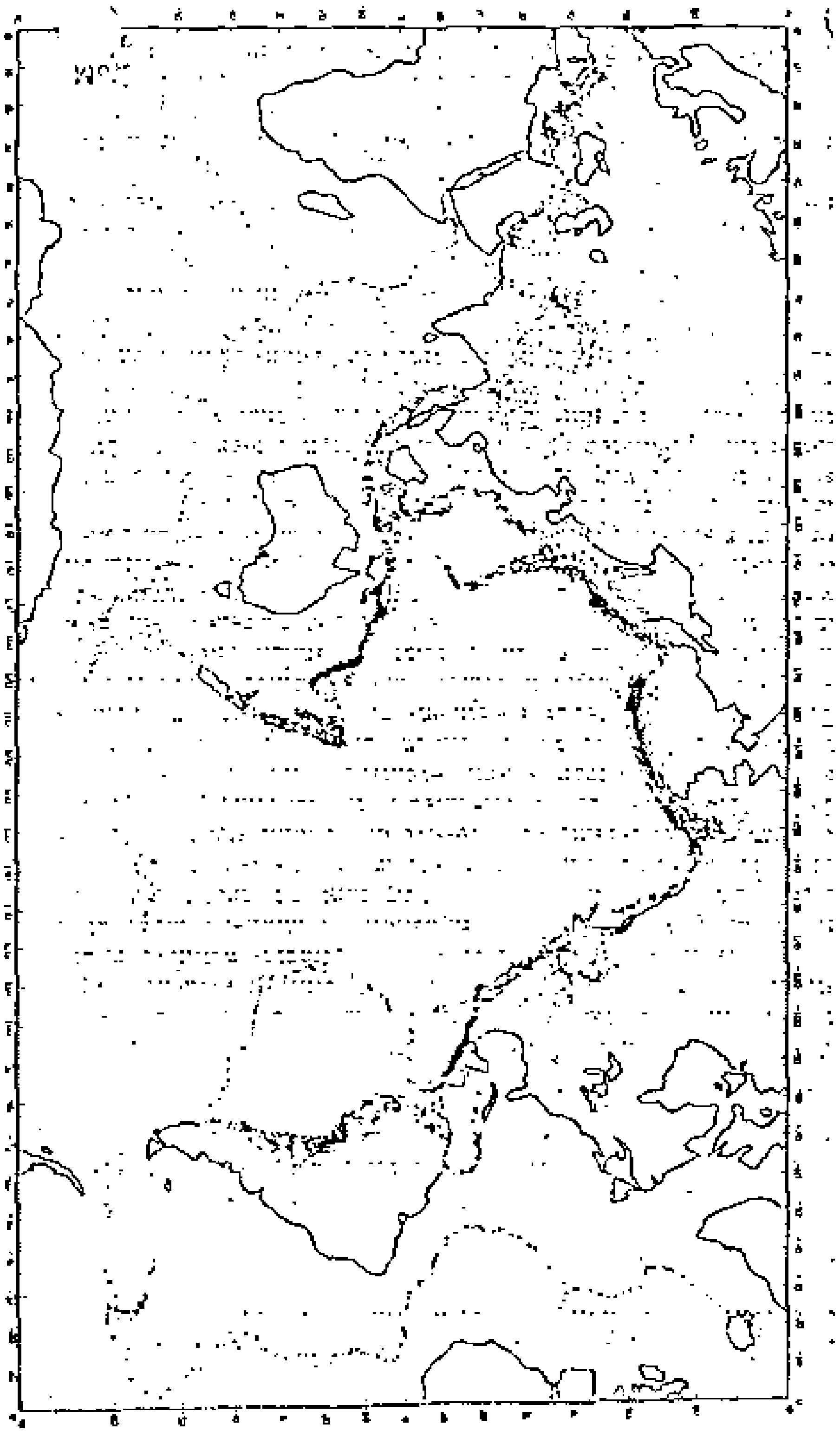


Fig. 6.7. Map showing epicenters for the interval 1961–1967. (After Newmark and Rosenblueth, 1971.)

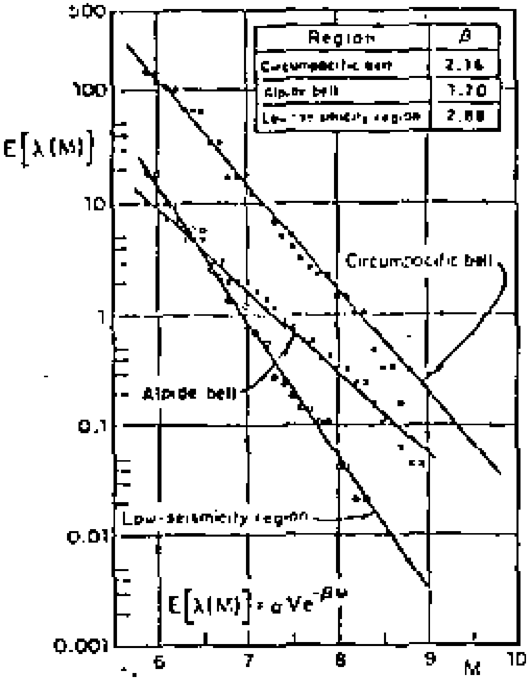


Fig. 6.8. Seismicity of macrozones. (After Esteva, 1968.)

tice of seismic zoning in the Soviet Union has been based on this concept (Gzovsky, 1962; Ananiin et al., 1968) and in many countries design spectra for very important structures, such as nuclear reactors or large dams, are usually derived from the assumption of a maximum credible intensity at a site; that intensity is ordinarily obtained by taking the maximum of the intensities that result at the site when at each of the potential sources an earthquake with magnitude equal to the maximum feasible value for that source is generated at the most unfavourable location within the same source. When this criterion is applied no attention is usually paid to the uncertainty in the maximum feasible magnitude nor to the probability that an earthquake with that magnitude will occur during a given time period. The need to formulate seismic-risk-related decisions that account both for upper bounds to magnitudes and for their probabilities of occurrence suggests adoption of magnitude recurrence expressions of the form:

$$\begin{aligned} \lambda &= \lambda_L G^*(M) && \text{for } M_L \leqslant M \leqslant M_U \\ &= \lambda_L && \text{for } M < M_L \\ &= 0 && \text{for } M > M_U \end{aligned} \qquad (6.7)$$

where M_L = lowest magnitude whose contribution to risk is significant, M_U

= maximum feasible magnitude, and $G^*(M)$ = complementary cumulative probability distribution of magnitudes every time that an event ($M \geqslant M_L$) occurs. A particular form of $G^*(M)$ that lends itself to analytical derivations is:

$$G^*(M) = A_0 + A_1 \exp(-\beta M) - A_2 \exp[-(\beta - \beta_1)M] \tag{6.8}$$

where:

$$A_0 = A\beta_1 \exp[-\beta(M_U - M_L)]$$

$$A_1 = A(\beta - \beta_1) \exp(\beta M_L)$$

$$A_2 = A_\beta \exp(-\beta_1 M_U + \beta M_L)$$

$$A = [\beta\{1 - \exp[-\beta_1(M_U - M_L)]\} - \beta_1 \{1 - \exp[-\beta(M_U - M_L)]\}]^{-1}$$

As M tends to M_L from above, eq. 6.7 approaches eq. 6.6. Adoption of adequate values of M_U and β_1 permits satisfying two additional conditions: the maximum feasible magnitude and the rate of variation of λ in its vicinity. When $\beta_1 \to \infty$, eq. 6.8 tends to an expression proposed by Cornell and Vanmarcke (1969).

Yegulalp and Kuo (1974) have applied the theory of extreme values to estimating the probabilities that given magnitudes are exceeded in given time intervals. They assume those probabilities to fit an extreme type-III distribution given by:

$$F_{M_{\max}}(M|t) = \exp[-C(M_U - M)^k t] \quad \text{for } M \leqslant M_U$$

$$= 0 \quad \text{for } M > M_U \tag{6.9}$$

Here $F_{M_{\max}}(M|t)$ indicates the probability that the maximum magnitude observed in t years is smaller than M, M_U has the same meaning as above, and C and K are zone-dependent parameters. This distribution is consistent with the assumption that earthquakes with magnitudes greater than M take place in accordance with a Poisson process with mean rate λ equal to $C(M_U - M)^k$. Equation 6.9 produces magnitude recurrence curves that fit closely the statistical data on which they are based for magnitudes above 5.2 and return periods from 1 to 50 years, even though the values of M_U that result from pure statistical analysis are not reliable measures of the upper bound to magnitudes, since in many cases they turn out inadmissibly high.

For low magnitudes, only a fraction of the number of shocks that take place is detected. As a consequence, λ-values based on statistical information lie below those computed according to eqs. 6.6 and 6.8 for M smaller than about 5.5. In addition, Fig. 6.9, taken from Yegulalp and Kuo (1974), shows that the numbers of detected shocks fit the extreme type III in eq. 6.9 better than the extreme type-I distribution implied by eq. 6.6., coupled with the assumption of Poisson distribution of the number of events. It is not clear what portion of the deviation from the extreme type-I distribution is due to the low values of the detectability levels and what portion comes from differences between the actual form of variation of λ with M and that given by eq. 6.6. The problem deserves attention because estimates of expected losses due to nonstructural damage may be sensitive to the values of λ for small magnitudes (say below 5.5) and because the evaluation of the level of seismic activity in a region is often made to depend on the recorded numbers of small magnitude shocks and on assumed detectability levels, i.e. of ratios of numbers of detected and occurred earthquakes (Kaila and Narain 1971; Kaila et al., 1972, 1974).

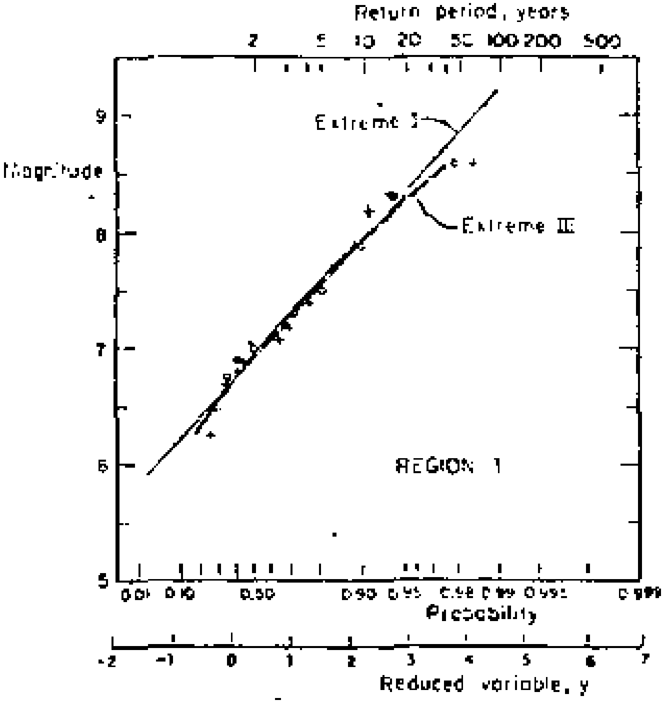


Fig. 6.9. Magnitude statistics in the Aleutian Islands region. (After Yegulalp and Kuo 1974.)

None of the expressions for λ presented in this chapter possess the desirable property that its applicability over a number of non-overlapping regions of the earth's crust implies the validity of an expression of the same form over the addition of those regions, unless some restrictions are imposed on the parameters of each λ. For instance, the addition of expressions like 6.6 gives place to an expression of the same form only if β is the same for all terms in the sum. Similar objections can be made to eq. 6.8. In what follows these forms will be preserved, however, as their accuracy is consistent with the amount of available information and their adoption offers significant advantages in the evaluation of regional seismicity, as shown later.

6.3.2 Variation with depth

Depth of prevailing seismic activity in a region depends on its tectonic structure. For instance, most of the activity in the western coast of the United States and Canada consists of shocks with hypocentral depths in the range of 20—30 km. In other areas, such as the southern coast of Mexico, seismic events can be grouped into two ensembles: one of small shallow shocks and one of earthquakes with magnitudes comprised in a wide range, and with depths whose mean value increases with distance from the shoreline (Fig. 6.10). Figure 6.11 shows the depth distribution of earthquakes with magnitude above 5.9 for the whole circum-Pacific belt.

6.3.3 Stochastic models of earthquake occurrence

Mean exceedance rates of given magnitudes are expected averages during long time intervals. For decision-making purposes the times of earthquake occurrence are also significant. At present those times can only be predicted within a probabilistic context.

Let t_i (i = 1, ..., n) be the unknown times of occurrence of earthquakes generated in a given volume of the earth's crust during a given time interval, and let M_i be the corresponding magnitudes. For the moment it will be assumed that the risk is uniformly distributed throughout the given volume, and hence no attention will be paid to the focal coordinates of each shock.

Classical methods of time-series analysis have been applied by different researchers attempting to devise analytical models for random earthquake sequences. The following approaches are often found in the literature:

(a) Plotting of histograms of waiting times between shocks (Knopoff, 1964; Aki, 1963).

(b) Evaluation of Poisson's index of dispersion, that is of the ratio of the sample variance of the number of shocks to its expected value (Vere-Jones, 1970; Shlien and Toksöz, 1970). This index equals unity for Poisson processes, is smaller for nearly periodic sequences, and is greater than one when events tend to cluster.

(c) Determination of autocovariance functions, that is, of functions representing the covariance of the numbers of events observed in given time intervals, expressed in terms of the time elapsed between those intervals (Vere-Jones, 1970; Shlien and Toksöz, 1970). The autocovariance function of a Poisson process is a Dirac delta function. This feature is characteristic for the Poisson model since it does not hold for any other stochastic process.

(d) The hazard function $h(t)$, defined so that $h(t)\,dt$ is the conditional probability that an event will take place in the interval $(t, t + dt)$ given that no events have occurred before t. If $F(t)$ is the cumulative probability distribution of the time between events:

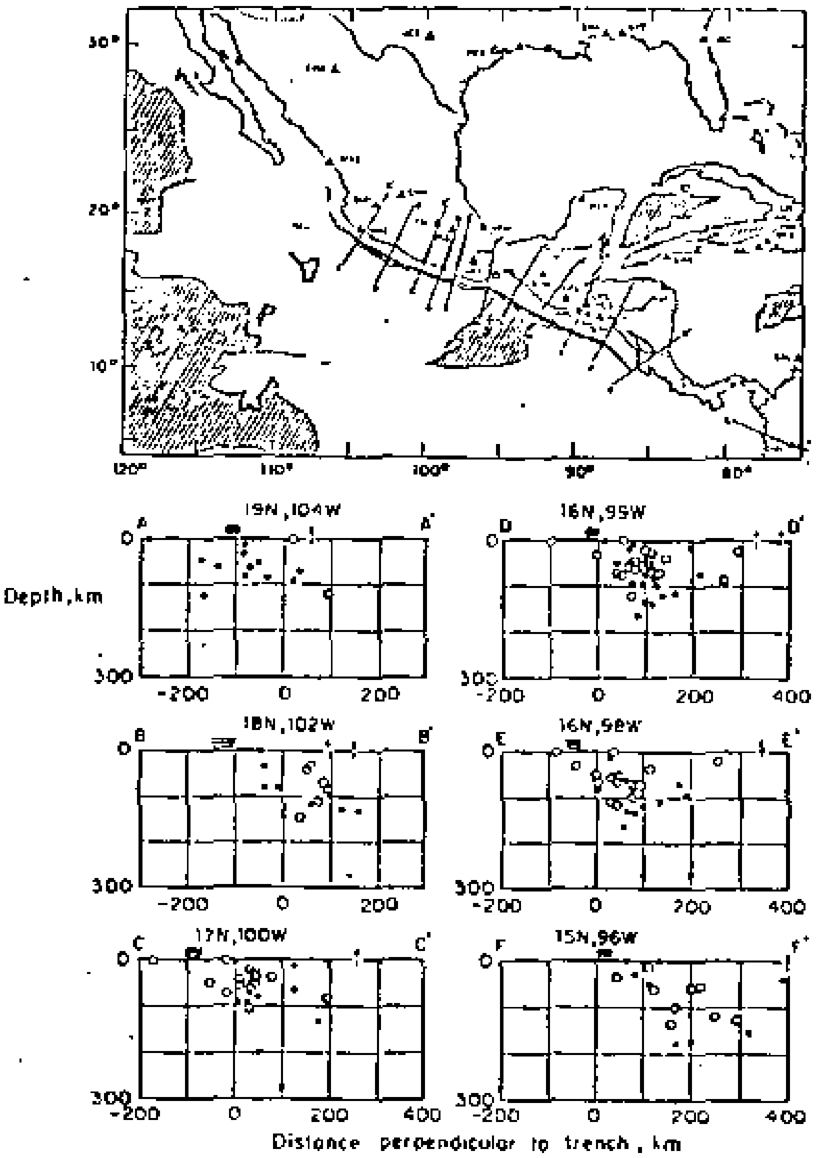


Fig. 6.10. Earthquake hypocenters projected onto a series of vertical sections through Mexico (After Molnar and Sykes, 1969.)

$$h(t) = f(t)/[1 - F(t)] \tag{6.10}$$

where $f(t) = \partial F(t)/\partial t$.

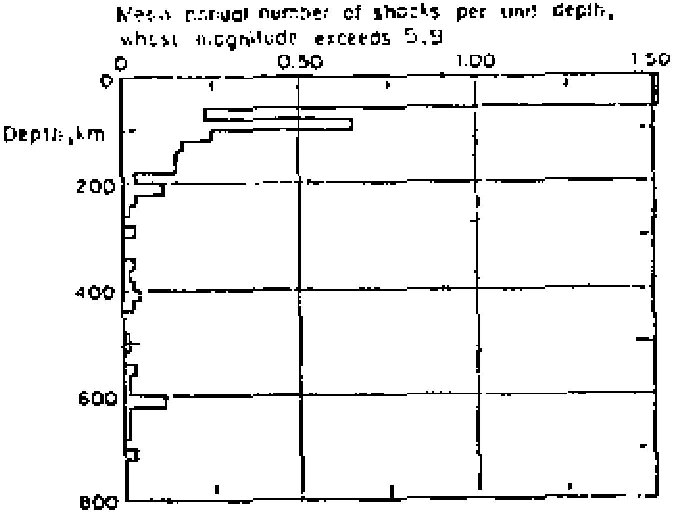


Fig. 6.11. Variation of seismicity with depth, Circum-Pacific Belt. (After Newmark and Rosenblueth, 1971.)

For the Poisson model, $h(t)$ is a constant equal to the mean rate of the process.

6.3.3.1 *Poisson model*

Most commonly applied stochastic models of seismicity assume that the events of earthquake occurrence constitute a Poisson process and that the M_i's are independent and identically distributed. This assumption implies that the probability of having N earthquakes with magnitude exceeding M during time interval $(0, t)$ equals:

$$p_N = [\exp(-\nu_M t)(\nu_M t)^N]/N! \tag{6.11}$$

where ν_M is the mean rate of exceedance of magnitude M in the given volume. If N is taken equal to zero in eq. 6.11, one obtains that the probability distribution of the maximum magnitude during time interval t is equal to $\exp(-\nu_M t)$. If ν_M is given by eq. 6.6, the extreme type-I distribution is obtained.

Some weaknesses of this model become evident in the light of statistical information and of an analysis of the physical processes involved: the Poisson assumption implies that the distribution of the waiting time to the next event is not modified by the knowledge of the time elapsed since the last one, while physical models of gradually accumulated and suddenly released energy call for a more general renewal process such that, unlike what happens in the Poisson process, the expected time to the next event decreases as time goes on [illegible]va, 1974). Statistical data show that the Poisson assumption may be acceptable when dealing with large shocks throughout the world (Ben-Menahem, 1960), implying lack of correlation between seismicities of different regions; however, when considering small volumes of the earth, of the order of those that can significantly contribute to seismic risk at a site, data often contradict Poisson's model, usually because of clustering of earthquakes in time: the observed numbers of short intervals between events are significantly higher than predicted by the exponential distribution, and values of Poisson's index of dispersion are well above unity (Figs. 6.12 and 6.13). In some instances, however, deviations in the opposite direction have been observed: waiting times tend to be more nearly periodic, Poisson's index of dispersion is smaller than one, and the process can be represented by a renewal model. This condition has been reported, for instance, in the southern coast of Mexico (Esteva, 1974), and in the Kamchatka and Pamir—Hindu Kush regions (Gaisky, 1966 and 1967). The models under discussion also fail to account for clustering in space (Tsuboi, 1958; Gajardo and Lomnitz, 1960), for the evolution of seismicity with time, and for the systematic shifting of active sources along geologic accidents (Allen, Chapter 3 of this book). On account of its simplicity, however, the Poisson process model provides a valuable tool for the formulation of some seismic-risk-related decisions, particularly of those that are sensitive only to magnitudes of events having very long return periods.

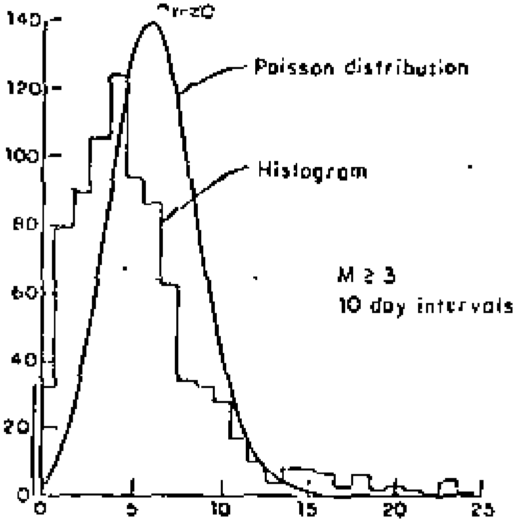
6.3.3.2 *Trigger models*

Statistical analysis of waiting times between earthquakes does not favor the adoption of the Poisson model or of other forms of renewal processes, such as those that assume that waiting times are mutually independent with lognormal or gamma distributions (Shlien and Toksöz, 1970). Alternative models have been developed, most of them of the 'trigger type' (Vere-Jones, 1970), i.e. the overall process of earthquake generation is considered as the superposition of a number of time series, each having a different origin, where the origin times are the events of a Poisson process. In general, let N be the number of events that take place during time interval $(0, t)$, τ_m = origin time of the mth series, $W_m(t, \tau_m)$ the corresponding number of events up to instant t_1 and n_t the random number of time series initiated in the interval $(0, t)$. The total number of events that occur before instant t is then:

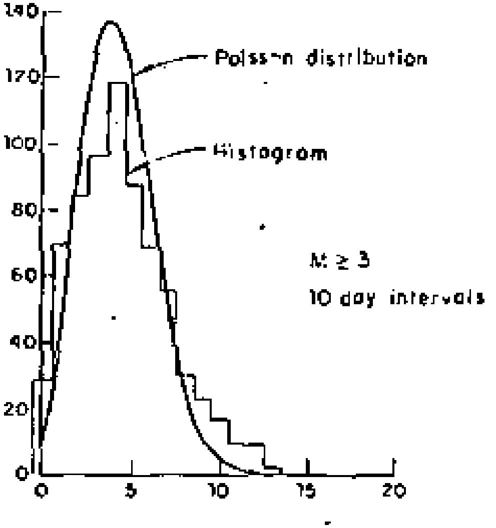
$$N = \sum_m^{n_t} W_m(t, \tau_m) \tag{6.12}$$

If origin times are distributed according to a homogeneous Poisson process with mean rate ν, and all W_m's are identically distributed stochastic processes with respect to $(t - \tau_m)$, it can be shown (Parzen, 1962) that the mean and variance of N can be obtained from:

$$(N) = \nu \int_0^t E[W(t, \tau)]\mathrm{d}\tau \tag{6.13}$$



a) Including swarms



b) Eliminating swarms

Fig. 6.12. Evaluation of Poisson process assumption. (After Knopoff, 1964.)

$$\mathrm{var}\,(N) = \nu \int_0^t E[W^2(t,\tau)]\mathrm{d}\tau \qquad (6.14)$$

Parzen (1962) gives also an expression for the probability generating function $\psi_N(Z;t)$ of the distribution of N in terms of $\psi_W(Z;t,\tau)$, the generating function of each of the component processes:

$$\psi_N(Z;t) = \exp[-\nu t + \nu \int_0^t \psi_W(Z;t,\tau)\mathrm{d}\tau] \qquad (6.15)$$

where:

$$\psi_W(Z;t,\tau) = \sum_{n=0}^{\infty} Z^n\, P\{W(t,\tau) = n\} \qquad (6.16)$$

and the probability mass function of N can be obtained from $\psi_N(Z;t)$ by recalling that:

$$\psi_N(Z;t) = \sum_{n=0}^{\infty} Z^n P\{N = n\}$$

expanding ψ_N in power series of Z, and taking $P\{N = n\}$ equal to the coefficient of Z^n in that expansion. For instance, if it is of interest to compute $P\{N = 0\}$, expansion of $\psi_N(Z;t)$ in a Taylor's series with respect to $Z = 0$ leads to:

$$\psi_N(Z;t) = \psi_N(0;t) + Z\psi'_N(0;t) + \frac{Z^2}{2!}\psi''_N(0;t) + \ldots \qquad (6.17)$$

where the prime signifies derivative with respect to Z. From the definition of ψ_N, $P\{N = 0\} = \psi_N(0;t)$.

Because the component processes of 'trigger'-type time series appear overlapped in sample histories, their analytical representation usually entails study of a number of alternative models, estimation of their parameters, and comparison of model and sample properties — often second-order properties (Cox and Lewis, 1966).

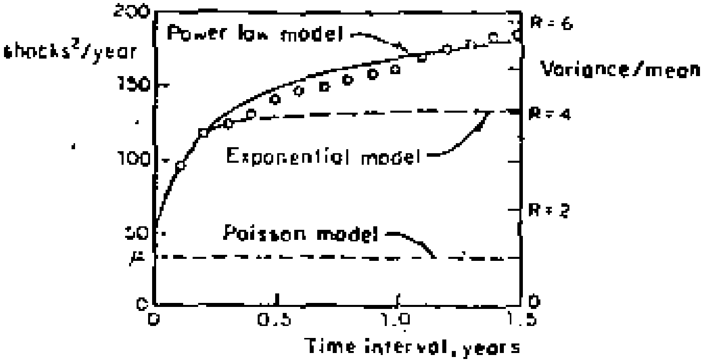


Fig. 6.13. Variance–time curve for New Zealand shallow shocks. (After Vere-Jones, 1966.)

Vere-Jones model. Applicability of some general 'trigger' models to represent loca seismicity processes was discussed in a comprehensive paper by Vere-Jones (1970), who calibrated them mainly against records of seismic activity in New Zealand. In addition to simple and compound Poisson processes (Parzen, 1962), he considered Neyman-Scott and Bartlett-Lewis models, both of which assume that earthquakes occur in clusters and that the number of events in each cluster is stocastically independent of its origin time. In the Neyman-Scott model, the process of clusters is assumed stationary and Poisson, and each cluster is defined by p_N, the probability mass function of its number of events, and $\Lambda(t)$, the cumulative distribution function of the time of an event corresponding to a given cluster, measured from the cluster origin. The Bartlett-Lewis model is a special case of the former, where each cluster is a renewal process that ends after a finite number of renewals. In these models the conditional probability of an event taking place during the interval $(t, t + dt)$, given that the cluster consists of N shocks, is equal to $N\lambda(t)dt$, where $\lambda(t) = \partial\Lambda(t)/\partial t$.

Because clusters overlap in time they cannot easily be identified and separated. Estimation of process parameters is accomplished by assuming different sets of those parameters and evaluating the corresponding goodness of fit with observed data.

Various alternative forms of Neyman-Scott's model were compared by Vere-Jones with observed data on the basis of first- and second-order statistics: hazard functions, interval distributions (in the form of power spectra) and variance time curves. The statistical record comprises about one thousand New Zealand earthquakes with magnitudes greater than 4.5, recorded from 1942 to 1961. Figures 6.13–6.15 show results of the analysis for shallow New Zealand shocks as well as the comparison of observed data with several alternative models. The process of cluster origins is Poisson in all case but the distributions of cluster sizes (N) and of times of events within clu ters differ among the various instances: in the Poisson model no clusteri takes place (the distribution of N is a Dirac delta function centered at $N =$ while in the exponential and in the power-law models the distribution of N extremely skewed towards $N = 1$, and $\Lambda(t)$ is taken respectively as $1 - e$

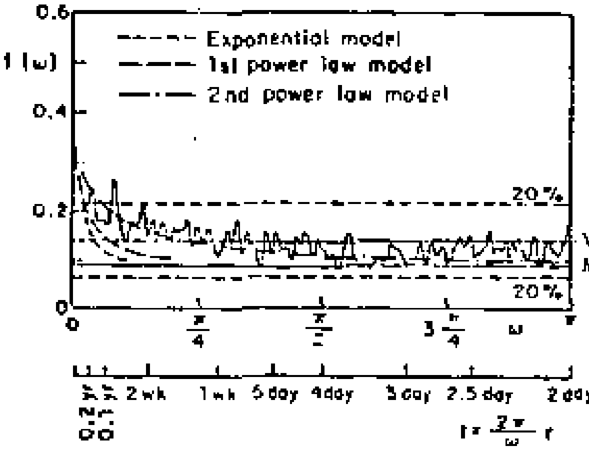


Fig. 6.14. Smoothed periodogram for New Zealand shallow shocks. (After Vere-Jones, 1966.)

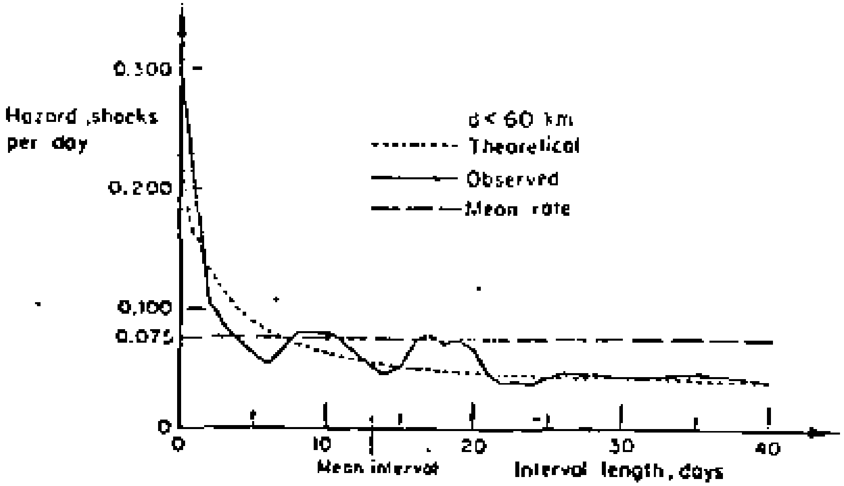


Fig. 6.15. Hazard function for New Zealand shallow shocks. (After Vere-Jones, 1970.)

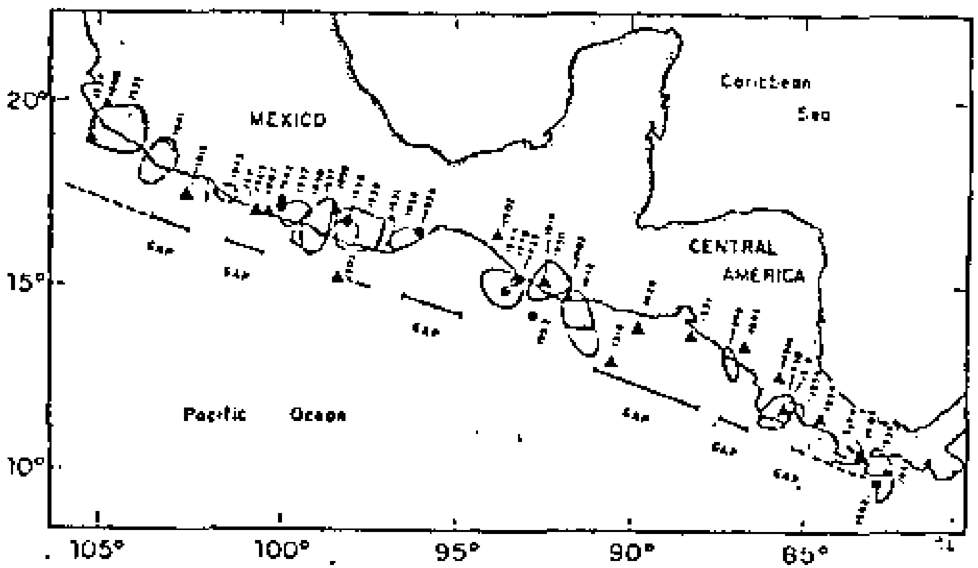


Fig. 6.16. Rupture zones and epicenters of large shallow Middle American earthquakes o this century. (After Kelleher et al., 1973.)

and $1 - [c/(c + t)]^{\delta}$ for $t > 0$, and as zero for $t < 0$, where λ, c, and δ are positive parameters. In Figs. 6.13–6.15, $\delta = 0.25$, $c = 2.3$ days, and $\lambda = 0.061$ shocks/day. The significance of clustering is evidenced by the high value of Poisson's dispersion index in Fig. 6.13, while no significant periodicity can be inferred from Fig. 6.14. Both figures show that the power-law model provides the best fit to the statistics of the samples. A similar analysis for New Zealand's deep shocks shows much less clustering: Poisson's dispersion index equals 2, and the hazard function is nearly constant with time.

Still, data reported by Gaisky (1967) have hazard functions that suggest models where the cluster origins as well as the clusters themselves may be represented by renewal processes. Mean return periods are of the order of several months, and hence these processes do not correspond, at least in the time scale, to the process of alternate periods of activity and quiescense of some geological structures cited by Kelleher et al. (1973), which have led to the concept of 'temporal seismic gaps', discussed below.

Simplified trigger models. Shlien and Toksöz (1970) proposed a simple particular case of the Neyman-Scott process; they lumped together all earthquakes taking place during non-overlapping time intervals of a given length and defined them as clusters for which $\lambda(t)$ was a Dirac delta function. Working with one-day intervals, they assumed the number of events per cluster to be distributed in accordance with the discrete Pareto law and applied a maximum-likelihood criterion to the information consisting of 35 000 earthquakes reported by the USCGS from January 1971 to August 1968. The model proposed represents reasonably well both the distribution of the number of earthquakes in one-day intervals and the dispersion index. However, owing to the assumption that no cluster lasts more than one day, the model fails to represent the autocorrelation function of the daily numbers of shocks for small time lags. The degree of clustering is shown to be a regional function, and to diminish with the magnitude threshold value and with the focal depth.

Aftershock sequences. The trigger processes described have been branded as reasonable representations of regional seismic activity, even when aftershock sequences and earthquake swarms are suppressed from statistical records, however arbitrary that suppression may be. The most significant instances of clustering are related, however, to aftershock sequences which often follow shallow shocks and only rarely intermediate and deep events. Persistence of large numbers of aftershocks for a few days or weeks has propitiated the detailed statistical analysis of those sequences since last century. Omori (1894) pointed out the decay in the mean rate of aftershock occurrence with t, the time elapsed since the main shock; he expressed that rate as inversely proportional to $t + q$, where q is an empirical constant. Utsu (1961) proposed a more general expression, proportional to $(t + c)^{-\zeta}$, where ζ is a constant; Utsu's proposal is consistent with the power-law expression for $\Lambda(t)$ presented above.

Lomnitz and Hax (1966) proposed a clustering model to represent aftershock sequences; it is a modified version of Neyman and Scott's model, where the process of cluster origins is non-homogeneous Poisson with mean rate decaying in accordance with Omori's law, the number of events in each cluster has a Poisson distribution, and $\Lambda(t)$ is exponential. All the results and methods of analysis described by Vere-Jones (1970) for the stationary process of cluster origins can be applied to the nonstationary case through a transformation of the time scale. Fitting of parameters to four aftershock sequences was accomplished through use of the second-order information of the sample defined on a transformed time scale. By applying this criterion to earthquake sets having magnitudes above different threshold values it was noticed that the degree of clustering decreases as the threshold value increases.

The magnitude of the main shock influences the number of aftershocks and the distribution of their magnitudes and, although the rate of activity decreases with time, the distribution of magnitudes remains stable throughout each sequence (Lomnitz, 1966; Utsu, 1962; Drakopoulos, 1971). Equation 6.6 represents fairly well the distribution of magnitudes observed in most aftershock sequences. Values of β range from 0.9 to 3.9 and decrease as the depth increases. Since values of β for regular (main) earthquakes are usually estimated from relatively small numbers of shocks generated throughout crust volumes much wider than those active during aftershock sequences, no relation has been established among β-values for series of both types of events. The parameters of Utsu's expression for the decay of aftershock activity with time have been estimated for several sequences, for instance those following the Aleutian earthquake of March 9, 1957, the Central Alaska earthquake of April 7, 1958, and the Southeastern Alaska earthquake of July 10, 1958 (Utsu, 1962), with magnitudes equal to 8.3, 7.3, and 7.9, respectively; c (in days) was 0.37, 0.40, and 0.01, while ζ was 1.05, 1.05 and 1.13, respectively. The relationship of the total number of aftershocks whose magnitude exceeds a given value with the magnitude of the main shock was studied by Drakopoulos (1971) for 140 aftershock sequences in Greece from 1912 to 1968. His results can be expressed by $N(M) = A \exp(-\beta M)$, where $N(M)$ is the total number of aftershocks with magnitude greater than M, and A is a function of M_0, the magnitude of the main shock:

$$A = \exp(3.62\,\beta + 1.1 M_0 - 3.46) \qquad (6.18)$$

Formulation of stochastic process models for given earthquake sequences is feasible once this relationship and the activity decay law are available for the source of interest. For seismic-risk estimation at a given site the spatial distribution of aftershocks may be as significant as the distribution of magnitudes and the time variation of activity, particularly for sources of relatively large dimensions.

6.3.3 *Renewal process models*

The trigger models described are based on information about earthquakes with magnitudes above relatively low thresholds recorded during time intervals of at most ten years. The degrees of clustering observed and the distributions of times between clusters cannot be extrapolated to higher magnitude thresholds and longer time intervals without further study.

Available information shows beyond doubt that significant clustering is the rule, at least when dealing with shallow shocks. However, there is considerable ground for discussion on the nature of the process of cluster origins during intervals of the order of one century or longer. While lack of statistical data hinders the formulation of seismicity models valid over long time intervals, qualitative consideration of the physical processes of earthquake generation may point to models which at least are consistent with the state of knowledge of geophysical sciences. Thus, if strain energy stored in a region grows in a more or less systematic manner, the hazard function should grow with the time elapsed since the last event, and not remain constant as the Poisson assumption implies. The concept of a growing hazard function is consistent with the conclusions of Kelleher et al. (1973) concerning the theory of periodic activation of *seismic gaps*. This theory is partially supported by results of nearly qualitative analysis of the migration of seismic activity along a number of geological structures. An instance is provided by the southern coast of Mexico, one of the most active regions in the world. Large shallow shocks are generated probably by the interaction of the continental mass and the subductive oceanic Cocos plate that underthrusts it and by compressive or flexural failure of the latter (Chapter 2). Seismological data show significant gaps of activity along the coast during the present century and not much is known about previous history (Fig. 6.16). Along these gaps, seismic-risk estimates based solely on observed intensities are quite low, although no significant difference is evident in the geological structure of these regions with respect to the rest of the coast, save some transverse faults which divide the continental formation into several blocks. Without looking at the statistical records a geophysicist would assign equal risk throughout the area. On the basis of seismicity data, Kelleher et al. have concluded that activity migrates along the region, in such a manner that large earthquakes tend to occur at seismic gaps, thus implying that the hazard function grows with time since the last earthquake. Similar phenomena have been observed in other regions; of particular interest is the North Anatolian fault where activity has shifted systematically along it from east to west during the last forty years (Allen, 1969).

Conclusions relative to activation of seismic gaps are controversial because the observation periods have not exceeded one cycle of each process. Nevertheless, those conclusions point to the formulation of stochastic models of seismicity that reflect plausible features of the geophysical processes.

These considerations suggest the use of renewal-process models to represent sequences of individual shocks or of clusters. Such models are characterized because times between events are independent and identically distributed. The Poisson process is a particular renewal model for which the distribution of the waiting time is exponential. Wider generality is achieved, without much loss of mathematical tractability, if inter-event times are supposed to be distributed in accordance with a gamma function:

$$f_T(t) = \frac{\nu}{(k-1)!}(\nu t)^{k-1}\, e^{-\nu t} \tag{6.19}$$

which becomes the exponential distribution when $k = 1$. If $k < 1$, short intervals are more frequent and the coefficient of variation is greater than in the Poisson model; if $k > 1$, the reverse is true. Shlien and Toksöz (1970) found that gamma models were unable to represent the sequences of individual shocks they analyzed; but these authors handled time intervals at least an order of magnitude shorter than those referred to in this section.

On the basis of hazard function estimated from sequences of small shocks in the Hindu-Kush, Vere-Jones (1970) deduces the validity of 'branching renewal process' models, in which the intervals between cluster centers, as well as those between cluster members, constitute renewal processes.

Owing to the scarcity of statistical information, reliable comparisons between alternate models will have to rest partially on simulation of the process of storage and liberation of strain energy (Burridge and Knopoff, 1967; Veneziano and Cornell, 1973).

6.3.4 *Influence of the seismicity model on seismic risk*

Nominal values of investments made at a given instant increase with time when placing them at compound interest rates, i.e. when capitalizing them. Their real value — and not only the nominal one — will also grow, provided the interest rate overshadows inflation. Conversely, for the purpose of making design decisions, nominal values of expected utilities and costs inflicted upon in the future have to be converted into present or actualized values, which can be directly compared with initial expenditures. Descriptions of seismic risk at a site are insufficient for that purpose unless the probability distributions of the times of occurrence of different intensities — or magnitudes at neighbouring sources — are stipulated; this entails more than simple magnitude-recurrence graphs or even than maximum feasible magnitude estimates.

Immediately after the occurrence of a large earthquake, seismic risk is abnormally high due to aftershock activity and to the probability that damage inflicted by the main shock may have weakened natural or man-made structures if emergency measures are not taken in time. When aftershock activity has ceased and damaged systems have been repaired, a normal risk level is attained, which depends on the probability-density functions of the waiting times to the ensuing damaging earthquakes.

For the purpose of illustration, let it be assumed that a fixed and deterministically known damage D_0 occurs whenever a magnitude above a given value is generated at a given source. If $f(t)$ is the probability-density function of the waiting time to the occurrence of the damaging event, and if the risk level is sufficiently low that only the first failure is of concern, the expected value of the actualized cost of damage is (see Chapter 9):

$$\bar{D} = D_0 \int_0^{\infty} e^{-\gamma t} f(t) dt \qquad (6.20)$$

where γ is the discount (or compound interest) coefficient and the overbar denotes expectation. If the process is Poisson with mean rate ν, then $f(t)$ is exponential and $\bar{D} \cong D_0 \ \nu/\gamma$; however, if damaging events take place in clusters and most of the damage produced by each cluster corresponds to its first event, the computation of $\bar{D}$ should make use of the mean rate ν corresponding to the clusters, instead of that applicable to individual events. Table 6.II shows a comparison of seismic risk determined under the alternative assumptions of a Poisson and a gamma model ($k = 2$), both with the same mean return period, k/ν (Esteva, 1974). Three descriptions of risk are presented as functions of the time t_0 elapsed since the last damaging event: $\bar{T}_1$, the expected time to the next event, measured from instant t_0; the expected value of the present cost of failure computed from eq. 6.20, and the hazard function (or mean failure rate). Since clustering is neglected, risk of aftershock occurrence must be either included in D_0 or superimposed on that displayed in the table.

This table shows very significant differences among risk levels for both processes. At small values of t_0, risk is lower for the gamma process, but it grows with time, until it outrides that for the Poisson process, which remains constant. The differences shown clearly affect engineering decisions.

TABLE 6.II

Comparison of Poisson and gamma processes

$t_0\,\nu/k$	$\bar{T}_1\,\nu/k$	Poisson process, $k = 1$		hk/ν	$T_1\,\nu/k$	Gamma process, $k = 2$		hk/ν
		D/D_0				D/D_0		
		$\gamma k/\nu = 10$	$\gamma k/\nu = 100$			$\gamma k/\nu = 10$	$bk/\nu = 100$	
0					1.0	0.0278	0.0004	0
0.1					0.92	0.0511	0.0036	0.367
0.2					0.86	0.0675	0.0059	0.667
0.5					0.75	0.0973	0.0100	1.333
1	1.0	0.0909	0.0099	1.0	0.67	0.120	0.0132	2 000
2					0.60	0.139	0.0158	2.667
5					0.54	0.154	0.0179	3.333
10					0.52	0.160	0.0187	3.633
					0.50	0.167	0.0196	4.000

6.4 ASSESSMENT OF LOCAL SEISMICITY

Only exceptionally can magnitude-recurrence relations for small volumes of the earth's crust and statistical correlation functions of the process of earthquake generation be derived exclusively from statistical analysis of recorded shocks. In most cases this information is too limited for that purpose and it does not always reflect geological evidence. Since the latter, as well as its connection with seismicity, is beset with wide uncertainty margins, information of different nature has to be evaluated, its uncertainty analyzed, and conclusions reached consistent with all pieces of information. A probabilistic criterion that accomplishes this is presented here: on the basis of geotectonic data and of conceptual models of the physical processes involved, a set of alternate assumptions can be made concerning the functions in question (magnitude recurrence, time, and space correlation) and an initial probability distribution assigned thereto; statistical information is used to judge the likelihood of each assumption, and a posterior probability distribution is obtained. How statistical information contibutes to the posterior probabilities of the alternate assumptions depends on the extent of that information and on the degree of uncertainty implied by the initial probabilities. Thus, if geological evidence supports confidence in a particular assumption or range of assumptions, statistical information should not greatly modify the initial probabilities. If, on the other hand, a long and reliable statistical record is available, it practically determines the form and parameters of the mathematical model selected to represent local seismicity.

6.4.1 Bayesian estimation of seismicity

Bayesian statistics provide a framework for probabilistic inference that accounts for prior probabilities assigned to a set of alternate hypothetical models of a given phenomenon as well as for statistical samples of events related to that phenomenon. Unlike conventional methods of statistical inference, Bayesian methods give weight to probability measures obtained from samples or from other sources; numbers, coordinates and magnitudes of earthquakes observed in given time intervals serve to ascertain the probable validity of each of the alternative models of local seismicity that can be postulated on the grounds of geological evidence. Any criterion intended to weigh information of different nature and different degrees of uncertainty should lead to probabilistic conclusions consistent with the degree of confidence attached to each source of information. This is accomplished by Bayesian methods.

Let H_i ($i = 1, \ldots, n$) be a comprehensive set of mutually exclusive assumptions concerning a given, imperfectly known phenomenon and let A be the observed outcome of such a phenomenon. Before observing outcome A we assign an initial probability $P(H_i)$ to each hypothesis. If $P(A|H_i)$ is the probability of A in case hypothesis H_i is true, then Bayes' theorem (Raiffa and Schlaifer, 1968) states that:

$$P(H_i|A) = P(H_i)\frac{P(A|H_i)}{\sum_j P(H_j)P(A|H_j)} \qquad (6.21)$$

The first member in this equation is the (posterior) probability that assumption H_i is true, given the observed outcome A.

In the evaluation of seismic risk, Bayes' theorem can be used to improve initial estimates of $\lambda(M)$ and its variation with depth in a given area as well as those of the parameters that define the shape of $\lambda(M)$ or, equivalently, the conditional distribution of magnitudes given the occurrence of an earthquake. For that purpose, take $\lambda(M)$ as the product of a rate function $\lambda_L = \lambda(M_L)$ by a shape function $G^*(M,B)$, equal to the conditional complementary distribution of magnitudes given the occurrence of an earthquake with $M \geqslant M_L$, where M_L is the magnitude threshold of the set of statistical data used in the estimation, and B is the vector of (uncertain) parameters $B_1, \ldots, B_r$ that define the shape of $\lambda(M)$. For instance, if $\lambda(M)$ is taken as given by eq. 6.8, B is a vector of three elements equal respectively to β, β_1, and M_U; if eq. 6.9 is adopted, B is defined by h and M_U.

The initial distribution of seismicity is in this case expressed by the initial joint probability density function of λ_L and B: $f'(\lambda_L,B)$. The observed outcome A can be expressed by the magnitudes of all earthquakes generated in a given source during a given time interval. For instance, suppose that N earthquakes were observed during time interval t and that their magnitudes were $m_1, m_2, \ldots, m_N$. Bayes' expression takes the form:

$$f''(\lambda_L, B|m_1, \ldots, m_N; t) = f'(\lambda_L, B)\frac{P[m_1, m_2, \ldots, m_N; t|\lambda_L, B]}{\int\int P[m_1, m_2, \ldots, m_N; t|l, b]f'(l,b)\mathrm{d}l\mathrm{d}b} \qquad (6.22)$$

where $f''(.)$ is the posterior probability density function, and l and b are dummy variables that stand for all values that may be taken by λ_L and B, respectively. Estimation of λ_L can usually be formulated independently of that of the other parameters. The observed fact is then expressed by N_L, the number of earthquakes with magnitude above M_L during time t, and the following expression is obtained, as a first step in the estimation of $\lambda(M)$:

$$f''(\lambda_L|N_L; t) = f'(\lambda_L)\frac{P(N_L; t|\lambda_L)}{\int P(N_L; t|l)f'(l)\mathrm{d}l} \qquad (6.23)$$

6.4.1.1 Initial probabilities of hypothetical models

Where statistical information is scarce, seismicity estimates will be very

sensitive to initial probabilities assigned to alternative hypothetical models; the opinions of geologists and geophysicists about probable models, about the parameters of these models, and the corresponding margins of uncertainty should be adequately interpreted and expressed in terms of a function f', as required by equations similar to 6.22 and 6.23. Ideally, these opinions should be based on the formulation of potential earthquake sources and on their comparison with possibly similar geotectonic structures. This is usually done by geologists, more qualitatively than quantitatively, when they estimate M_U. Initial estimates of λ_L are seldom made, despite the significance of this parameter for the design of moderately important structures (see Chapter 9).

Analysis of geological information must consider local details as well as general structure and evolution. In some areas it is clear that all potential earthquake sources can be identified by surface faults, and their displacements in recent geological times measured. When mean displacements per unit time can be estimated, the order of magnitude of creep and of energy liberated by shocks and hence of the recurrence intervals of given magnitudes can be established (Wallace, 1970; Davies and Brune, 1971), the corresponding uncertainty evaluated, and an initial probability distribution assigned. The fact that magnitude-recurrence relations are only weakly correlated with the size of recent displacements is reflected in large uncertainties (Petrushevsky, 1966).

Application of the criterion described in the foregoing paragraph can be unfeasible or inadequate in many problems, as in areas where the abundance of faults of different sizes, ages, and activity, and the insufficient accuracy with which focal coordinates are determined preclude a differentiation of all sources. Regional seismicity may then be evaluated under the assumption that at least part of the seismic activity is distributed in a given volume rather than concentrated in faults of different importance. The same situation would be faced when dealing with active zones where there is no surface evidence of motions. Hence, consideration of the overall behavior of complex geological structures is often more significant than the study of local details.

Not much work has been done in the analysis of the overall behavior of large geological structures with respect to the energy that can be expected to be liberated per unit volume and per unit time in given portions of those structures. Important research and applications should be expected, however, since, as a result of the contribution of plate-tectonics theory to the understanding of large-scale tectonic processes, the numerical values of some of the variables correlated with energy liberation are being determined, and can be used at least to obtain orders of magnitude of expected activity along plate boundaries. Far less well understood are the occurrence of shocks in apparently inactive regions of continental shields and the behavior of complex continental blocks or regions of intense folding, but even there some progress is expected in the study of accumulation of stresses in the crust.

Knowledge of the geological structure can serve to formulate initial probability distributions of seismicity even when quantitative use of geophysical information seems beyond reach. Initial probability distributions of local seismicity parameters λ_L, B in the small volumes of the earth's crust that contribute significantly to seismic risk at a site, can be assigned by comparison with the average seismicity observed in wider areas of similar tectonic characteristics, or where the extent and completeness of statistical information warrant reliable estimates of magnitude-recurrence curves (Esteva, 1969). In this manner we can, for instance, use the information about the average distribution of the depths of earthquakes of different magnitudes throughout a seismic province to estimate the corresponding distribution in an area of that province, where activity has been low during the observation interval, even though there might be no apparent geophysical reason to account for the difference. Similarly, the expected value and coefficient of variation of λ_L in a given area of moderate or low seismicity (as a continental shield) can be obtained from the statistics of the motions originated at all the supposedly stable or aseismic regions in the world.

The significance of initial probabilities in seismic risk estimates, against the weight given to purely statistical information, becomes evident in the example of Fig. 6.16: if Kelleher's theory about activation of seismic gaps is true, risk is greater at the gaps than anywhere else along the coast; if Poisson models are deemed representative of the process of energy liberation, the extent of statistical information is enough to substantiate the hypothesis of reduced risk at gaps. Because both models are still controversial, and represent at most two extreme positions concerning the properties of the actual process, risk estimates will necessarily reflect subjective opinions.

6.4.1.2 Significance of statistical information

Estimation of λ_L. Application of eq. 6.23 to estimate λ_L independently of other parameters will be first discussed, because it is a relatively simple problem and because λ_L is usually more uncertain than M_0 and much more so than β.

A model as defined by eq. 6.19 will be assumed to apply. If the possible assumptions concerning the values of λ_L constitute a continuous interval, the initial probabilities of the alternative hypotheses can be expressed in terms of a probability-density function of λ_L. If, in addition, a certain assumption is made concerning the form of this probability-density function, only the initial values of $E(\lambda_L)$ and $V(\lambda_L)$ have to be assumed. It is advantageous to assign to $\nu = k/E(T)$ a gamma distribution. Then, if ρ and μ are the parameters of this initial distribution of ν, if k is assumed to be known, and if the observed outcome is expressed as the time t_n elapsed during $n + 1$ consecutive events (earthquakes with magnitude $\geq M_1$), application of eq. 6.22 leads to the conclusion that the posterior probability function of ν is also gamma, now with parameters $\rho + nk$ and $\mu + t_n$. The initial and the posterior expected values of ν are respectively equal to ρ/μ, and to $(\rho + nk)/(\mu + t_n)$. When initial uncertainty about ν is small, ρ and μ will be large and the initial and the posterior expected values of ν will not differ greatly. On the other hand, if only statistical information were deemed significant, ρ and μ should be given very small values in the initial distribution, and $E(\nu)$, and hence λ_L, will be practically defined by n, k, and t_n. This means that the initial estimates of geologists should not only include expected or most probable values of the different parameters, but also statements about ranges of possible values and degrees of confidence attached to each.

In the case studied above only a portion of the statistical information was used. In most cases, especially if seismic activity has been low during the observation interval, significant information is provided by the durations of the intervals elapsed from the initiation of observations to the first of the n + 1 events considered, and from the last of these events until the end of the observation period. Here, application of eq. 6.23 leads to expressions slightly more complicated than those obtained when only information about t_n is used.

The particular case when the statistical record reports no events during *at least* an interval $(0, t_0)$ comes up frequently in practical problems. The probability-density function of the time T_1 from t_0 to the occurrence of the first event must account for the corresponding shifting of the time axis. Furthermore, if the time of occurrence of the last event before the origin is unknown, the distribution of the waiting time from $t = 0$ to the first event coincides with that of the *excess life* in a renewal process at an arbitrary value of t that approaches infinity (Parzen, 1962). For the particular case when the waiting times constitute a gamma process, T_1 is measured from $t = 0$, T is the waiting time between consecutive events, and it is known that $T_1 \geqslant t_0$, the conditional density function of $\tau_1 = (T_1 - t_0)/E(T)$ is given by eq. 6.24 (Esteva, 1974), where $u_0 = t_0/E(T)$:

$$f_{\tau_1}(u|T_1 \geqslant t_0) = \frac{\sum\limits_{m=1}^{k} \frac{k}{(m-1)!}\,[k(u+u_0)]^{m-1}}{\sum\limits_{m=1}^{k}\sum\limits_{n=1}^{m} \frac{1}{(n-1)!}\,(ku_0)^{n-1}}\,e^{-ku} \tag{6.24}$$

Consider now the implications of Bayesian analysis when applied to one of the seismic gaps in Fig. 6.16, under the conditions implicit in eq. 6.24. An initial set of assumptions and corresponding probabilities was adopted as described in the following. From previous studies referring to all the southern coast of Mexico, local seismicity in the gap area (measured in terms of λ for $M \geqslant 6.5$) was represented by a gamma process with $k = 2$. An initial probability density function for ν was adopted such that the expected value of $\lambda(6.5)$ for the region coincided with its average throughout the complete seismic province. Two values of ρ were considered: 2 and 10, which correspond to coefficients of variation of 0.71 and 0.32, respectively. Values in Table 6.III were obtained for the ratio of the final to the initial expected values of ν, in terms of u_0.

The last two columns in the table contain the ratios of the computed values of $E''(T_1)$ and $E'(T)$ when ν is taken as equal respectively to its initial or to its posterior expected value. This table shows that, for $\rho = 10$, that is, when uncertainty attached to the geologically based assumptions is low, the expected value of the time to the next event keeps decreasing, in accordance with the conclusions of Kelleher et al. (1973). However, as time goes on and no events occur, the statistical evidence leads to a reduction in the estimated risk, which shows in the increased conditional expected values of T_1. For $\rho = 2$, the geological evidence is less significant and risk estimates decrease at a faster rate

6.4.1.3 Bayesian estimation of jointly distributed parameters

In the general case, estimation of B will consist in the determination of the posterior Bayesian joint probability function of its components, taking as statistical evidence the relative frequencies of observed magnitudes. Thus, if event A is described as the occurrence of N shocks, with magnitudes $m_1, ..., m_N$, and b_i $(i = 1, ..., r)$ are values that may be adopted by the components of vector B being estimated, eq. 6.21 becomes:

$$f''_B(b_1, ..., b_r|A) = \frac{f_B(b_1, ..., b_r)P(A|b_1, ..., b_r)}{\int ... \int f_B(u_1, ..., u_r)P(A|u_1, ..., u_r)\mathrm{d}u_1, ..., \mathrm{d}u_r} \tag{6.25}$$

where $P(A|u_1, ..., u_r)$ is proportional to:

$$\prod_{i=1}^{N} g(m_i|u_1, ..., u_r)$$

and $g(m) = -\partial G^*(m)/\partial m$.

Closed-form solutions for f'' as given by eq. 6.25 are not feasible in general. For the purpose of evaluating risk, however, estimates of the posterior first and second moments of f'' can be obtained from eq. 6.25, making use of available first-order approximations (Benjamin and Cornell, 1970; Rosenblueth, 1975). Thus, the posterior expected value of B_i is given by $\int f''_{B_i}(u)\, u\, \mathrm{d}u$, where $f''_{B_i}(u_i) = \int ... \int f''_B(u_1, ..., u_r)\, \mathrm{d}u_1, ..., \mathrm{d}u_n$ and the multiple integral is of order $r - 1$, because it is not extended to the dominion of B_i. Hence:

$$E''(B_i) = \frac{E'_B[B_iP(A|B_1, ..., B_r)]}{E'_B[P(A|B_1, ..., B_r)]} \tag{6.26}$$

TABLE 6.III

Bayesian estimates of seismicity in one seismic gap

$u_0 = t_0/E'(T)$	$E''(\nu)/E'(\nu)$		$E''(T_1 \mid T_1 > t_0)/E'(T)$	
	$\rho = 2$	$\rho = 10$	$\rho = 2$	$\rho = 10$
0	1,0	1.0	0,75	0.75
0.1	0,95	0.99	0,76	0.74
0.5	0,75	0.94	0.91	0.71
1	0,58	0.87	1.14	0.73
5	0,20	0.54	3,11	1.05
10	0,11	0.36	5,47	1.55
20	0,06	0.22	10,50	2.48

where E' and E'' stand for initial and posterior expectation, and subscript B means that expectation is taken with respect to all the components of B. Likewise, the following *posterior moments* can be obtained:

Covariance of B_i and B_j

$$\mathrm{Cov}''(B_i, B_j) = \frac{E'_B[B_iB_jP(A|B_1, ..., B_r)]}{E'_B[P(A|B_1, ..., B_r)]} - E''(B_i)E''(B_j) \tag{6.27}$$

Expected value of $\lambda(M)$

$$E''[\lambda(M)] = E''(\lambda_1)E''[G^*(M; B)]$$

$$= E''(\lambda_1)\frac{E'_B[G^*(M; B)P(A|B_1, ..., B_r)]}{E'_B[P(A|B_1, ..., B_r)]} \tag{6.28}$$

Marginal distributions. The posterior expectation of $\lambda(M)$ is in some cases all that is required to describe seismicity for decision-making purposes. Often, however, uncertainty in $\lambda(M)$ must also be acounted for. For instance, the probability of exceedance of a given magnitude during a given time interval has to be obtained as the expectation of the corresponding probabilities over all alternative hypotheses concerning $\lambda(M)$. In this manner it can be shown that, if the occurrence of earthquakes is a Poisson process and the Bayesian distribution of λ_L is gamma with mean $\bar{\lambda}_L$ and coefficient of variation V_L, the marginal distribution of the number of earthquakes is negative binomial with mean $\bar{\lambda}_L$. In particular, the marginal probability of zero events during time interval t — equivalently, the complementary distribution function of the waiting time between events — is equal to $(1 + t/t'')^{-r''}$, where $r'' = V_L^{-2}$ and $t'' = r''/\bar{\lambda}_L$. The marginal probability-density function of the waiting time, that should be substituted in eq. 6.20, is $\bar{\lambda}_L(1 + t/t'')^{-r''-1}$, which tends to the exponential probability function as r'' and t'' tend to infinity (and $V_L \to 0$) while their ratio remains equal to $\bar{\lambda}_L$.

Bayesian uncertainty tied to the joint distribution of all seismicity parameters ($\lambda_L, B_1, \ldots, B_r$) can be included in the computation of the probability of occurrence of a given event Z by taking the expectation of that probability with respect to all parameters:

$$P(Z) = E_{\lambda_L, B}[P(Z); \lambda_L, B_1, \ldots, B_r)] \tag{6.29}$$

When the joint distribution of λ_L, B stems from Bayesian analysis of an initial distribution and an observed event, A, this equation adopts the form:

$$P''(Z) = \frac{E'_{\lambda_L, B}[P(Z|\lambda_L, B)P(A|\lambda_L, B)]}{E'_{\lambda_L, B}[P(A|\lambda_L, B)]} \tag{6.30}$$

where ' and '' stand for initial and posterior, respectively.

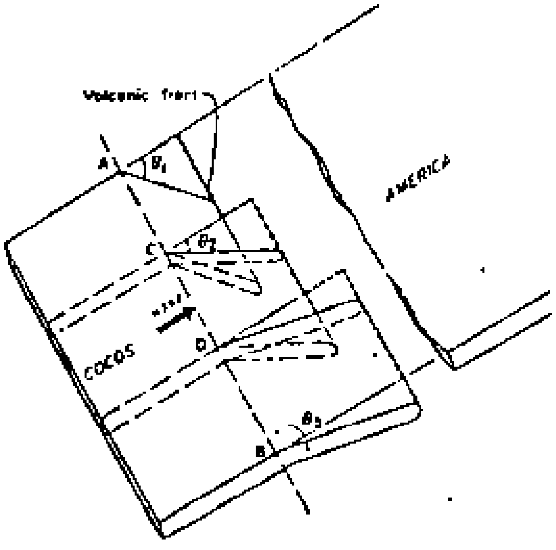


Fig. 6.18. Schematic drawing of the segmenting of Cocos plate as it subducts below American plate. (After Singh, 1974.)

Spatial variability. Figure 6.17 shows a map of geotectonic provinces of Mexico, according to F. Mooser. Each province is characterized by the large-scale features of its tectonic structure, but significant local perturbations to the overall patterns can be identified. Take for instance zone 1, whose seismotectonic features were described above, and are schematically shown in Fig. 6.18 (Singh, 1975): the Pacific plate underthrusts the continental block and is thought to break into several blocks, separated by faults transverse to the coast, that dip at different angles. The continental mass is also made up of several large blocks. Seismic activity at the underthrusting plate or at its interface with the continental mass is characterized by magnitudes that may reach very high values and by the increase of mean hypocentral depth with distance from the coast; small and moderate shallow shocks are generated at the blocks themselves. Variability of statistical data along the whole tectonic system was discussed above and is apparent in Fig. 6.10. Bayesian estimation of local seismicity averaged throughout the system is a matter of applying eq. 6.21 or any of its special forms (eqs. 6.22 and 6.23), taking as statistical evidence the information corresponding to the whole system. However, seismic risk estimates are sensitive to values of local seismicity averaged over much smaller volumes of the earth's crust; hence the need to develop criteria for probabilistic inference of possible patterns of space variability of seismicity along tectonically homogeneous zones.

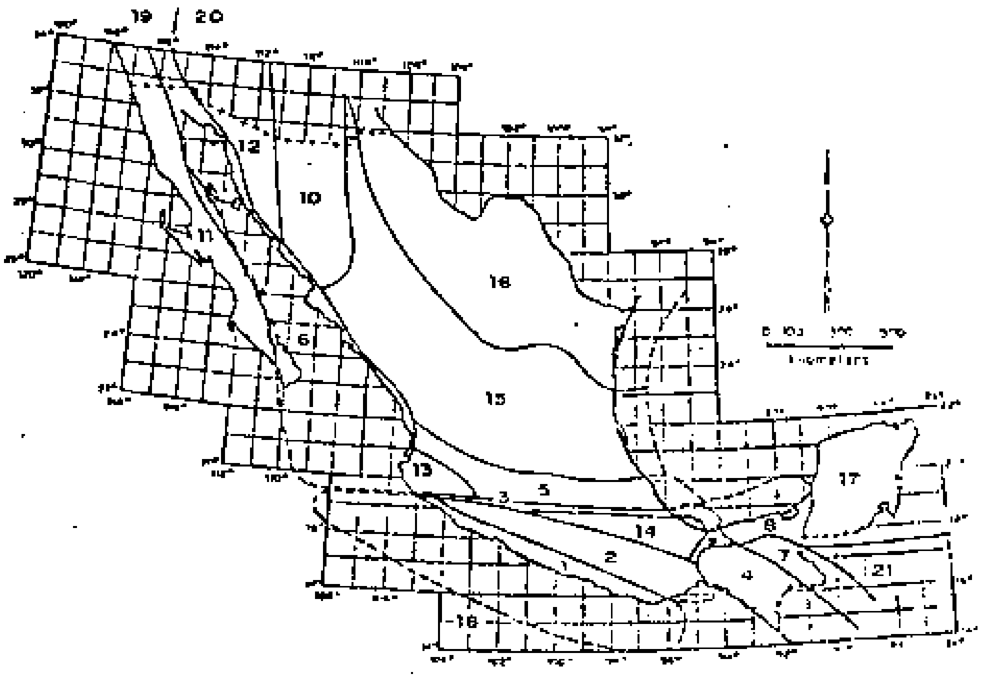


Fig. 6.17. Seismotectonic provinces of Mexico. (After F. Mooser.)

On the basis of seismotectonic information, the system under consideration can first be subdivided into the underthrusting plate and the subsystem of shallow sources; each subsystem can then be separately analyzed. Take for instance the underthrusting plate and subdivide it into s sufficiently small equal-volume subzones. Let ν_L be the rate of exceedance of magnitude M_L throughout the main system, ν_{L_i} the corresponding rate at each subzone, and define p_i as ν_{L_i}/ν_L, with p_i independent of ν_L (p_i is equal to the probability that an earthquake known to have been generated in the overall system originated at subzone i). Initial information about possible space variability of can [illegible] in terms of an initial probability distribution of p_i and of the correlation among p_i and p_j for any i and j. Because $\Sigma \nu_{L_i} = \nu_L$, one obtains $\Sigma p_i = 1$. This imposes two restrictions on the initial joint probability distribution of the p_i's: $E'(p_i) = 1$, var' $\Sigma p_i = 0$. If all p_i's are assigned equal expectations and all pairs p_i, p_j, $i \neq j$ are assumed to possess the same correlation coefficient $\rho_{ij} = \rho'$, the restrictions mentioned lead to $E'(p_i) = 1/s$ and $\rho' = -1/(s-1)$. Posterior values of $E(p_i)$ and ρ_{ij} are obtained according to the same principles that led to eqs. 6.25—6.28. Statistical evidence is in this case described by N, the total number of earthquakes generated in the system, and n_i ($i = 1, \dots, s$) the corresponding numbers for the subzones. Given the p_i's, the probability of this event is the multinomial distribution:

$$P[A|p_1, \dots, p_s] = \frac{N!}{n_1!, \dots, n_s!} p_1^{n_1} \dots p_s^{n_s} \tag{6.31}$$

If the correlation coefficients among seismicities of the various subzones can be neglected, each p_i can be separately estimated. Because p_i has to be comprised between 0 and 1, it is natural to assign it a beta initial probability distribution, defined by its parameters n_i' and N_i', such that $E'(p_i) = n_i'/N_i'$ and $\text{var}'(p_i) = n_i'(N_i' - n_i')/[N_i'^2(N_i' + 1)]$ (Raiffa and Schlaifer, 1968). The parameters of the posterior distribution will be:

$$n_i'' = n_i' + n_i, N_i'' = N_i' + N$$

Take for instance a zone whose prior distribution of λ_L is assumed gamma with expected value λ_L' and coefficient of variation V_L'. Suppose that, on the basis of geological evidence and of the dimensions involved, it is decided to subdivide the zone into four subzones of equal dimensions; a-priori considerations lead to the assignment of expected values and coefficients of variation of p_i for those subzones, say $E'(p_i) = 0.25$, $V'(p_i) = 0.25$ ($i = 1, \dots, 4$). From previous considerations for $s = 4$ take $\rho_{ij}' = -1/3$ for $i \neq j$. Suppose now that, during a given time interval t, ten earthquakes were observed in the zone, of which 0, 1, 3, and 6 occurred respectively in each subzone. If the Poisson process model is adopted, λ_L' and V_L' can be expressed in terms of a fictitious number of events $n' = V_L'^{-2}$ occurred during a fictitious time interval $t' = n'/\bar{\lambda}_L'$; after observing n earthquakes during an interval t, the Bayesian mean and coefficient of variation of λ_L will be $\bar{\lambda}_L'' = (n' + n)/(t' + t)$, $V_L'' = (n' + n)^{-1/2}$ (Esteva, 1968). Hence:

$$\lambda_L'' = (V_L'^{-2} + 10)/(V_L'^{-2} \bar{\lambda}_L'^{-1} + t), \qquad V_L'' = (V_L'^{-2} + 10)^{-1/2}$$

Local deviations of seismicity in each subzone with respect to the average λ_L can be analyzed in terms of p_i ($i = 1, \dots, 4$); Bayesian analysis of the proportion in which the ten earthquakes were distributed among the subzones proceeds according to:

$$E''(p_i|A) = \frac{E'[p_i P(A|p_1, \dots, p_4)]}{E'[P(A|p_1, \dots, p_4)]} \tag{6.32}$$

The expectations that appear in this equation have to be computed with respect to the initial joint distribution of the p_i's. In practice, adequate approximations are required. For instance, Benjamin and Cornell's (1970) first-order approximation leads to $E''(p_1) = 0.226$, $E''(p_4) = 0.294$.

If correlation among subzone seismicities is neglected, and statistical information of each subzone is independently analyzed, when the p_i's are assigned beta probability-density functions with means and coefficients of variation as defined above, one obtains $E''(p_1) = 0.206$, $E''(p_4) = 0.311$, which are not very different from those formerly obtained; however, when $E'(p_i) = 0.25$ and $V'(p_i) = 0.5$, the first criterion leads to $E''(p_i) = 0.206$, $E''(p_4) = 0.314$, while the second produces 0.131 and 0.416, respectively. Part of the difference may be due to neglect of ρ_{ij}', but probably a significant part stems from inaccuracies of the first-order approximation to the expectations that appear in eq. 6.32; alternate approximations are therefore desirable.

Incomplete data. Statistical information is known to be fairly reliable only for magnitudes above threshold values that depend on the region considered, its level of activity, and the quality of local and nearby seismic instrumentation. Even incomplete statistical records may be significant when evaluating some seismicity parameters; their use has to be accompanied by estimates of detectability values, that is, of ratios of the numbers of events recorded to total numbers of events in given ranges (Esteva, 1970; Kaila and Narain, 1971).

6.5 REGIONAL SEISMICITY

The final goal of local seismicity assessment is the estimation of regional seismicity, that is, of probability distributions of intensities at given sites, and of probabilistic correlations among them. These functions are obtained by integrating the contributions of local seismicities of nearby sources, and hence their estimates reflect Bayesian uncertainties tied to those seismicities. In the following, regional seismicity will be expressed in terms of mean rates of exceedance of given intensities; more detailed probabilistic descriptions would entail adoption of specific hypotheses concerning space and time correlations of earthquake generation.

6.5.1 Intensity-recurrence curves

The case when uncertainty in seismicity parameters is neglected will be discussed first. Consider an elementary seismic source with volume dV and local seismicity $\lambda(M)$ per unit volume, distant R from a site S, where intensity-recurrence functions are to be estimated. Every time that a magnitude M shock is generated at that source, the intensity at S equals:

$$Y = \epsilon Y_p = \epsilon b_1 \exp(b_2 M) g(R) \tag{6.33}$$

(see eqs. 6.4 and 6.5), where ϵ is a random factor and Y and Y_p stand for actual and predicted intensities, b_1 and b_2 are given constants, and $g(R)$ is a function of hypocentral distance. The probability that an earthquake originating at the source will have an intensity greater than y is equal to the probability that $\epsilon Y_p > y$. If Y_p is expressed in terms of M and randomness in ϵ is accounted for, one obtains:

$$\nu(y) = \int_{\alpha_u}^{\alpha_L} \nu_p(y/u) f_\epsilon(u) \mathrm{d}u \tag{6.34}$$

where ν and ν_p are respectively mean rates at which actual and predicted intensities exceed given values, $\alpha_U = y/y_U$, $\alpha_L = y/y_L$, y_U, and y_L are the predicted intensities that correspond to M_U and M_L, and f_ϵ the probability-density function of ϵ. If eq. 6.33 is assumed to hold:

$$\nu_p(y) = K_0 + K_1 y^{-r_1} - K_2 y^{-r_2} \tag{6.35}$$

where:

$$K_i = [b_1 g(R)]^{r_i} A_i \lambda_L \mathrm{d}V \quad (i = 0, 1, 2) \tag{6.36}$$

$$r_0 = 0, \quad r_1 = \beta/b_2, \quad r_2 = (\beta - \beta_1)/b_2 \tag{6.37}$$

Substitution of eq. 6.35 into 6.34, coupled with the assumption that $\ln \epsilon$ is normally distributed with mean m and standard deviation σ leads to:

$$\nu(y) = c_0 K_0 + c_1 K_1 y^{-r_1} - c_2 K_2 y^{-r_2} \tag{6.38}$$

where:

$$c_i = \exp(Q_i) \left[\phi\left(\frac{\ln \alpha_L - u_i}{\sigma}\right) - \phi\left(\frac{\ln \alpha_U - u_i}{\sigma}\right) \right] \tag{6.39}$$

ϕ is the standard normal cumulative distribution function, $Q_i = 1/2\, \sigma^2 r_i^2 + m r_i$, and $u_i = m + \sigma^2 r_i$. Similar expressions have been presented by Merz and Cornell (1973) for the special case of eq. 6.8 when $\beta_1 \to \infty$ and for a quadratic form of the relation between magnitude and logarithm of exceedance rate. Closed-form solutions in terms of incomplete gamma functions are obtained when magnitudes are assumed to possess extreme type-III distributions (eq. 6.9).

Intensity-recurrence curves at given sites are obtained by integration of the contributions of all significant sources. Uncertainties in local seismicities can be handled by describing regional seismicity in terms of means and variances of $\nu(y)$ and estimating these moments from eq. 6.34 and suitable first- and second-moment approximations. Influence of these uncertainties in design decisions has been discussed by Rosenblueth (in preparation).

6.5.2 Seismic probability maps

When intensity-recurrence functions are determined for a number of sites with uniform local ground conditions the results are conveniently represented by sets of seismic probability maps, each map showing contours of intensities that correspond to a given return period. For instance, Figs. 6.19 and 6.20 show peak ground velocities and accelerations that correspond to 100 years return period on firm ground in Mexico. These maps form part of a set that was obtained through application of the criteria described in this chapter. Because the ratio of peak ground accelerations and velocities does not remain constant throughout a region, the corresponding design spectra will not only vary in scale but also in shape (frequency content), in other words, seismic risk will usually have to be expressed in terms of at least the values of two parameters (for instance, as in this case, peak ground accelerations and velocities that correspond to various risk levels (return periods)).

6.5.3 Microzoning

Implicit in the above criteria for evaluation of regional seismicity is the adoption of intensity attenuation expressions valid on firm ground. Scatter of actual intensities with respect to predicted values was ascribed to differences in source mechanisms, propagation paths, and local site conditions; at least the latter group of variables can introduce systematic deviations in the

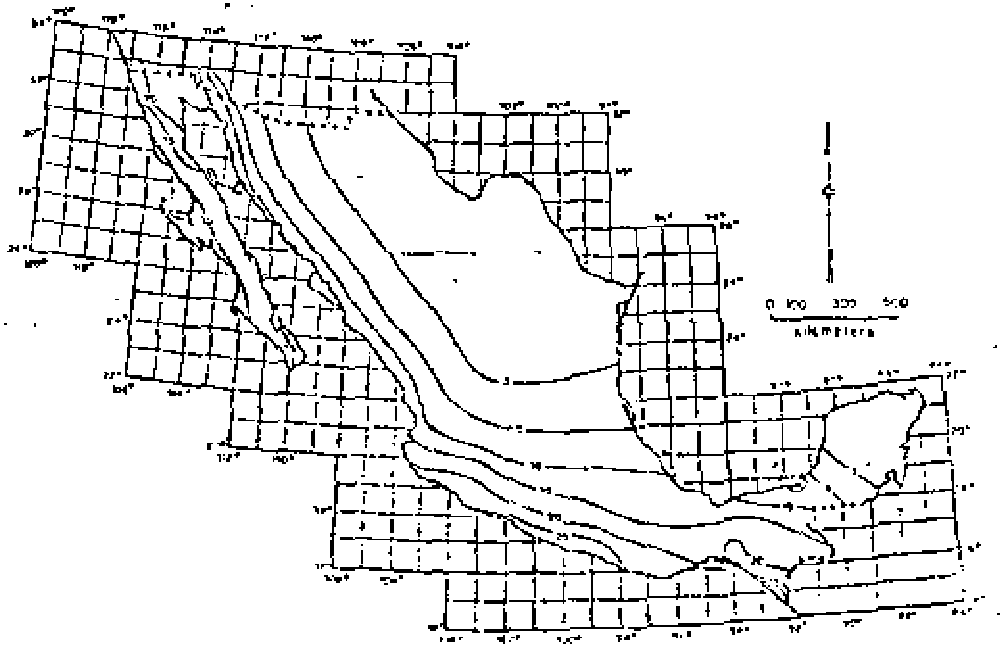


Fig. 6.19. Peak ground velocities with return period of 100 years (cm/sec).

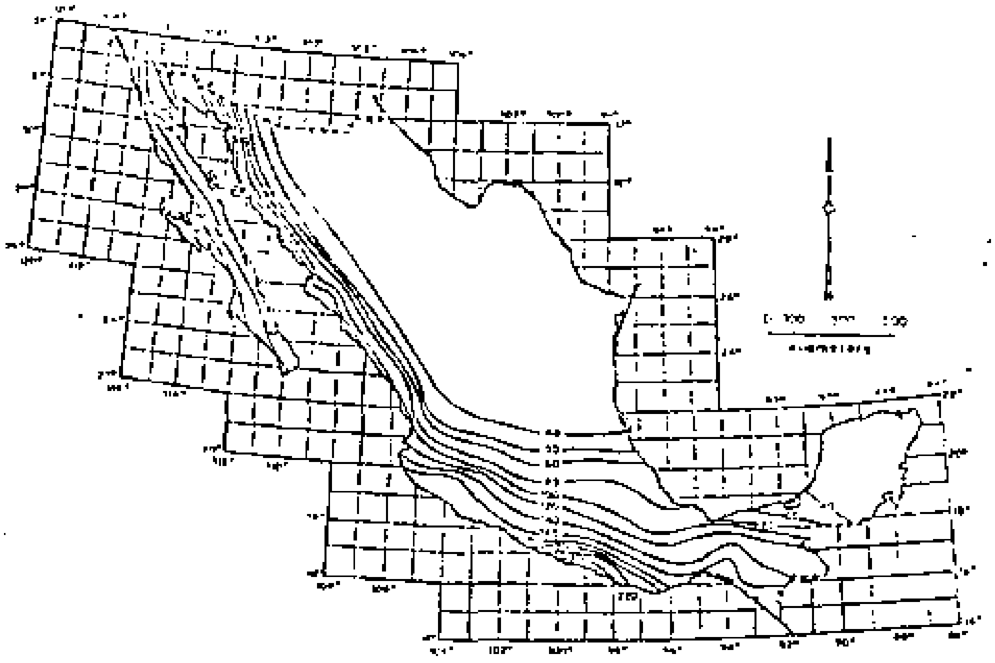


Fig. 6.20. Peak ground accelerations with return period of 100 years (cm/sec²).

ratio of actual to predicted intensities; and geological details may significantly alter local seismicity in a small region, as well as energy radiation patterns, and hence regional seismicity in the neighbourhood. These systematic deviations are the matter of microzoning, that is, of local modification of risk maps similar to Figs. 6.19 and 6.20.

Most of the effort invested in microzoning has been devoted to study of the influence of local soil stratigraphy on the intensity and frequency content of earthquakes (see Chapter 4). Analytical models have been practically limited to response analysis of stratified formations of linear or nonlinear soils to vertically traveling shear waves. The results of comparing observed and predicted behavior have ranged from satisfactory (Herrera et al., 1965) to poor (Hudson and Udwadia, 1972). Topographic irregularities, as hills or slopes of firm ground formations underlying sediments, may introduce significant systematic perturbations in the surface motion, as a consequence of wave focusing or dynamic amplification. The latter effect was probably responsible for the exceptionally high accelerations recorded at the abutment of Pacoima dam during the 1971 San Fernando earthquake.

Present practice of microzoning determines seismic intensities or design parameters in two steps. First the values of those parameters on firm ground are estimated by means of suitable attenuation expressions and then they are amplified according to the properties of local soil; but this implies an arbitrary decision to which seismic risk is very sensitive: selecting the boundary between soil and firm ground. A specially difficult problem stems when

trying to fix that boundary for the purpose of predicting the motion at the top of a hill or the slope stability of a high cliff (Rukos, 1974).

It can be concluded that rational formulation of microzoning for seismic risk is still in its infancy and that new criteria will appear that will probably require intensity attenuation models which include the influence of local systematic perturbations. Whether these models are available or the two-step process described above is acceptable, intensity-recurrence expressions can be obtained as for the unperturbated case, after multiplying the second member of eq. 6.34 by an adequate intensity-dependent corrective factor.

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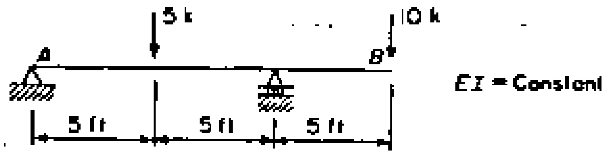

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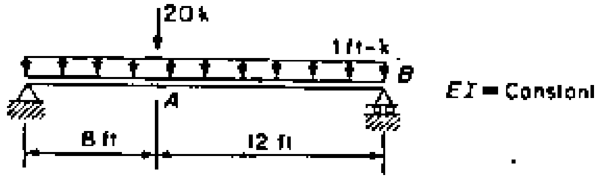

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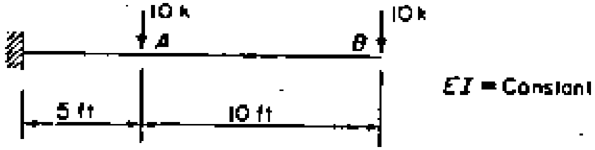

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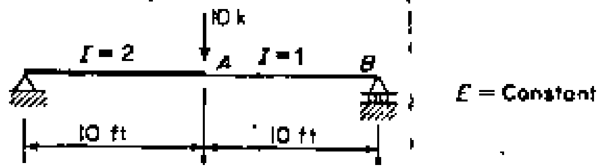

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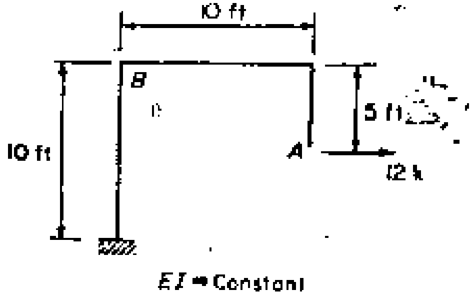

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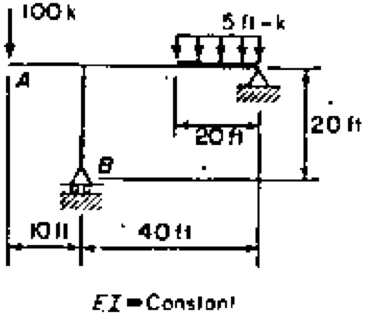

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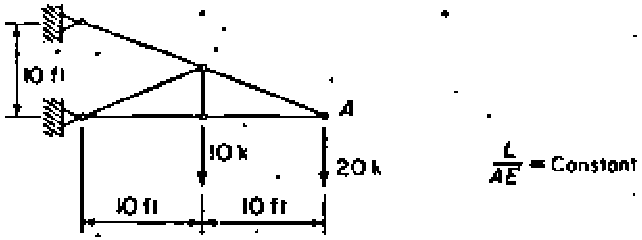

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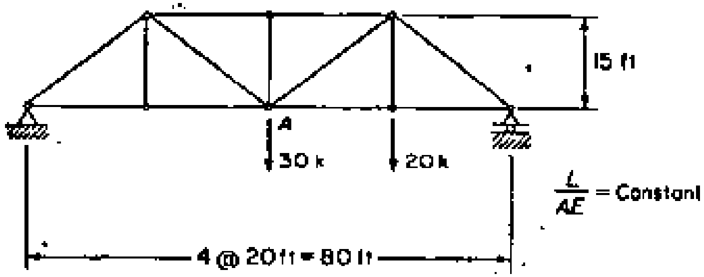
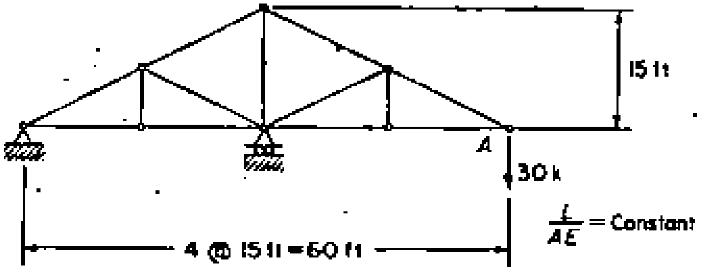
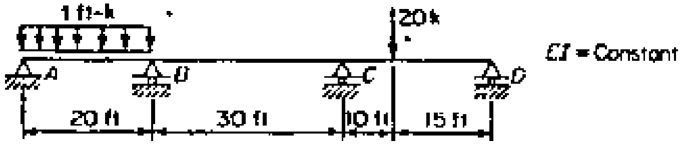


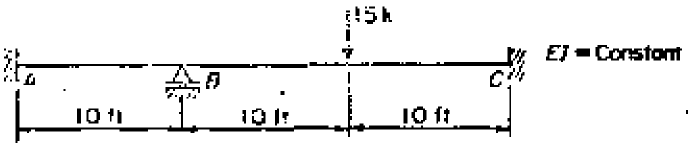
Prob. 3-8



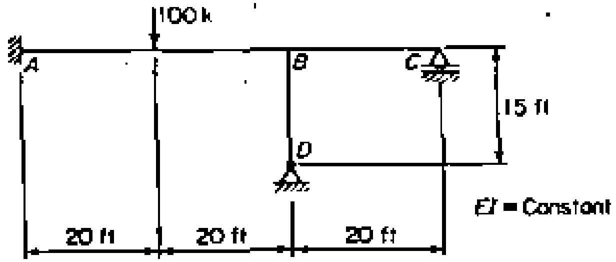
Prob. 3-9



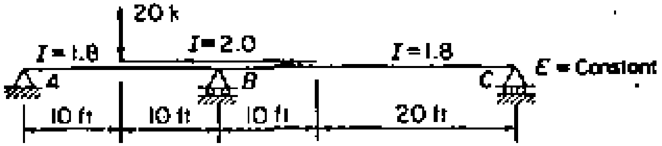
Prob. 3-10



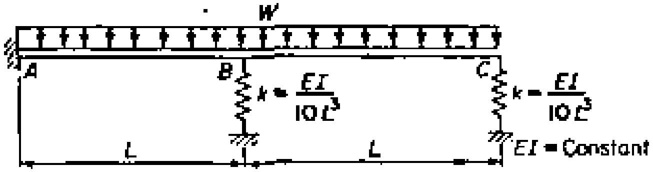
Prob. 3-17



Prob. 3-18



Prob. 3-19



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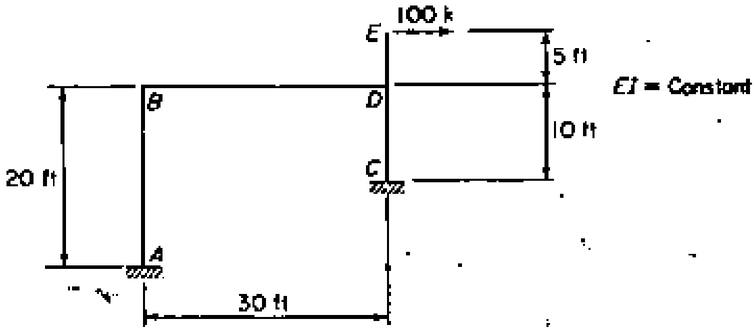
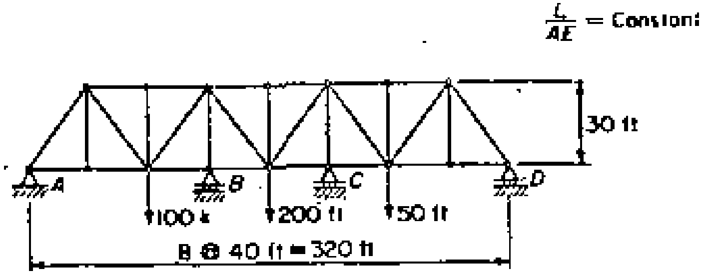


Prob. 3-21



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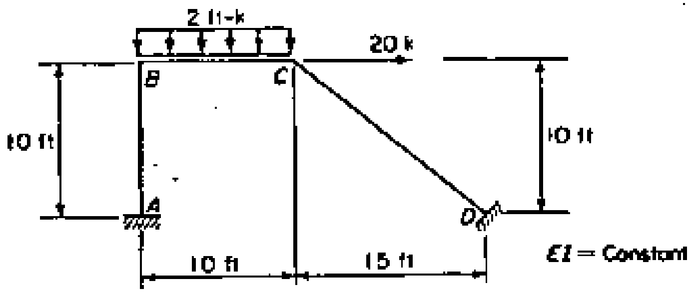
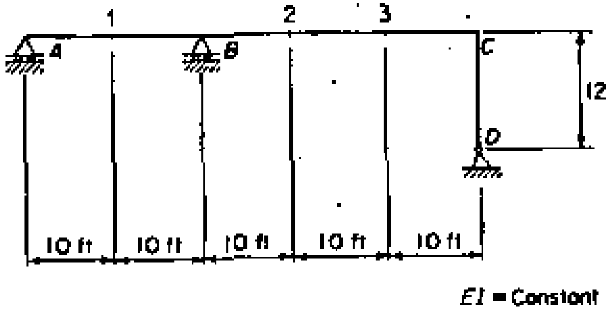


Prob. 3-23



Load case 1 — a single vertical 20 k load at point 1
Load case 2 — a single vertical 20 k load at point 2
Load case 3 — a single vertical 20 k load at point 3

Prob. 3-24

SELECTED REFERENCES

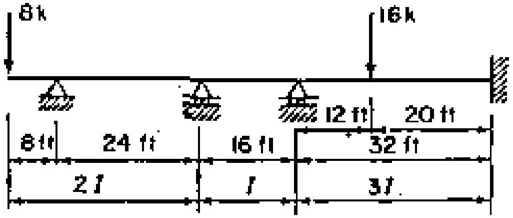
3-1 Rogers, G. L. and M. L. Causey, *Mechanics of Engineering Structures*. New York: John Wiley & Sons, Inc., 1962.

3-2 Carpenter, Samuel, *Structural Mechanics*. New York: John Wiley & Sons, Inc., 1960.

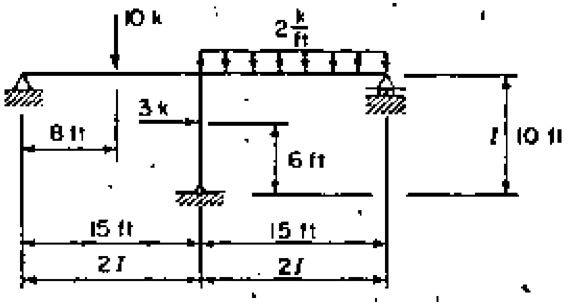
4-7

Problems

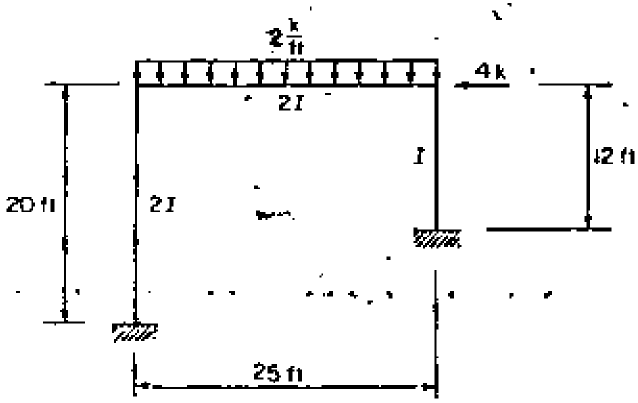
4-1 to 4-6. Analyze the structural system for the indicated loading, and draw the shear and moment diagrams for each member; E = constant and the relative value of moment of inertia is indicated for each member.



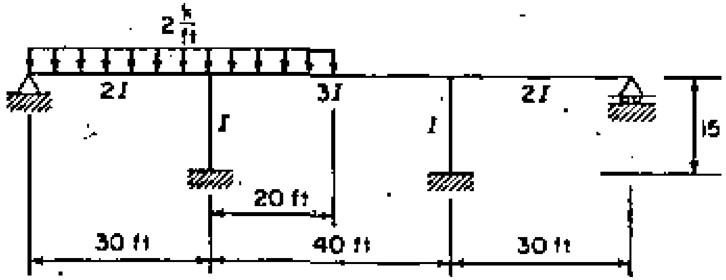
Prob. 4-1



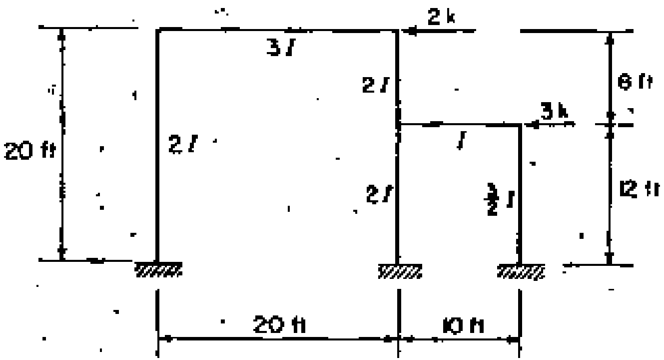
Prob. 4-2



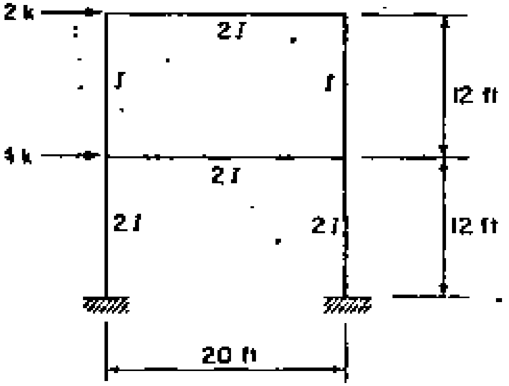
Prob. 4-3



Prob. 4-4



Prob. 4-5



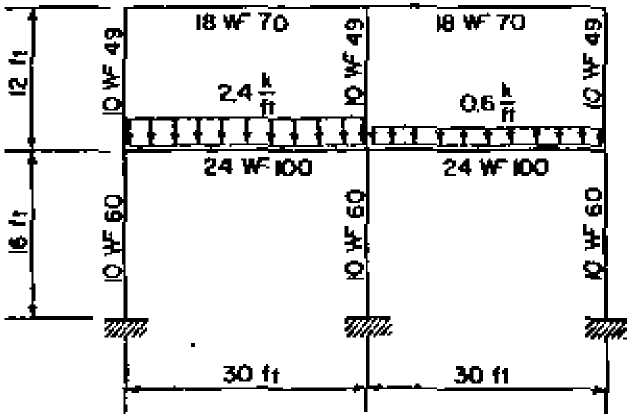
Prob. 4-6

4-7. Analyze the rigid frame of Prob. 4-2 for a settlement of the center support of 0.6 in.; $I = 500$ in^4, $E = 30{,}000$ ksi.

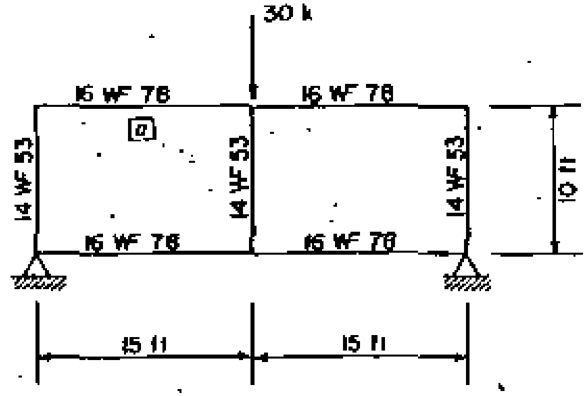
4-8. Determine the final member end actions and the support reactions for the frame of Prob. 4-3 caused by a clockwise rotation of the left support of 2°; $I = 5000$ in^4, $E = 3000$ ksi.

4-9. Analyze the planar orthogonal frame of Prob. 4-4 for a settlement of the left column support of 0.6 in., and of the right column support of 0.8 in.; $I = 6000$ in^4, $E = 3000$ ksi.

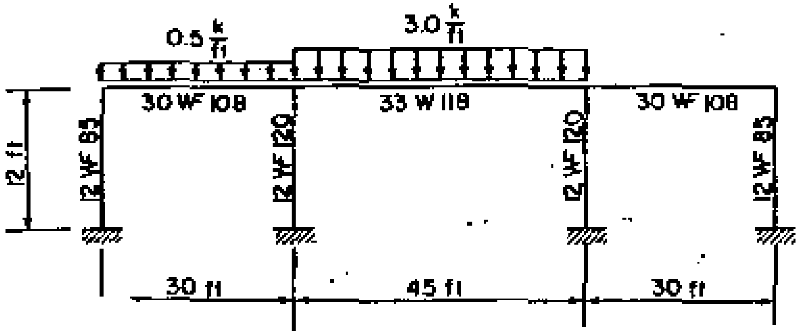
4-10 to 4-12. Calculate the final member end actions and the support reactions for the indicated loading of the structure, and draw the shear and moment diagrams for each member; $E = 30{,}000$ ksi.



Prob. 4-10



Prob. 4-11

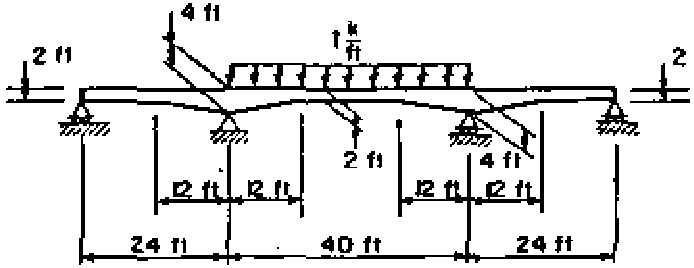


Prob. 4-12

4-13. Determine the final member end actions and the support reactions for the frame of Prob. 4-11 if member *a* is fabricated with a 6° bend (rotating the right end of the member counterclockwise) at a point 5 ft from the left end of the member.

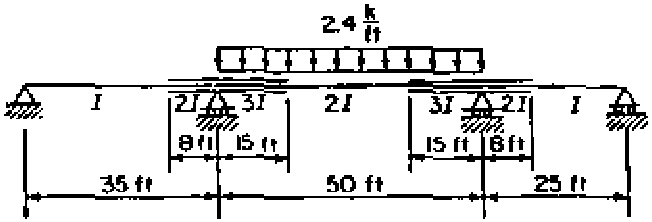
4-14. Analyze the frame of Prob. 4-12 for a fabrication error of 0.75 in. which resulted in the 33 W 118 being too short.

4-15. Analyze the continuous beam for the given loading. The beams are 1.5-ft wide and have straight haunches; $E = 3000$ ksi.



Prob. 4-15

4-16. Analyze the continuous beam for the indicated loading condition. Draw the shear and moment diagrams for each beam. The relative values of moments of inertia for each member are indicated; E = constant.



Prob. 4-16

4-17. Write a computer program using the stiffness method to analyze a continuous beam for a uniform vertical load applied to any span and acting over the entire span. Assume that the moment of inertia is constant over the span of each beam and is different for each beam; E is constant for the structure.

4-18. Develop a computer program to analyze a general planar orthogonal frame by the stiffness method for the following load cases: (1) a uniform normal load over the span of a member; (2) a normal concentrated load applied at point within the span of a member; (3) a vertical or horizontal concentrated load applied at a joint; and (4) a moment applied at a joint. Assume that the beam elements are prismatic and that E is constant for the structure.

4-19. Write a computer program to develop the member stiffness matrix and to compute the fixed end actions for a uniform normal load acting over the entire span for a nonprismatic beam element.

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4-1 Pei, Ming L., "Stiffness Method of Rigid Frame Analysis," ASCE, *Second Conference on Electronic Computation*, Pittsburgh, Pa.: September 8 and 9, 1960.

4-2 Kinney, J. Sterling, *Indeterminate Structural Analysis*. Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1957.

4-3 Gere, James M., and William Weaver, Jr., *Analysis of Framed Structures*. Princeton, N.J.: D. Van Norstrand Co., Inc., 1965.

4-4 Wang, Chu-Kia, *Statically Indeterminate Structures*. New York: McGraw-Hill Book Company, 1953.

4-5 Morice, P. B., *Linear Structural Analysis*. London: Thames and Hudson, 1959.

4-6 Hall, A. S., and R. W. Woodhead, *Frame Analysis*. New York: John Wiley & Sons, Inc., 1961.

4-7 Rubinstein, Moshe F., *Matrix Computer Analysis of Structures*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1966.

4-8 Willems, Nicholas, and William M. Lucas, Jr., *Matrix Analysis for Structural Engineers*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1968.

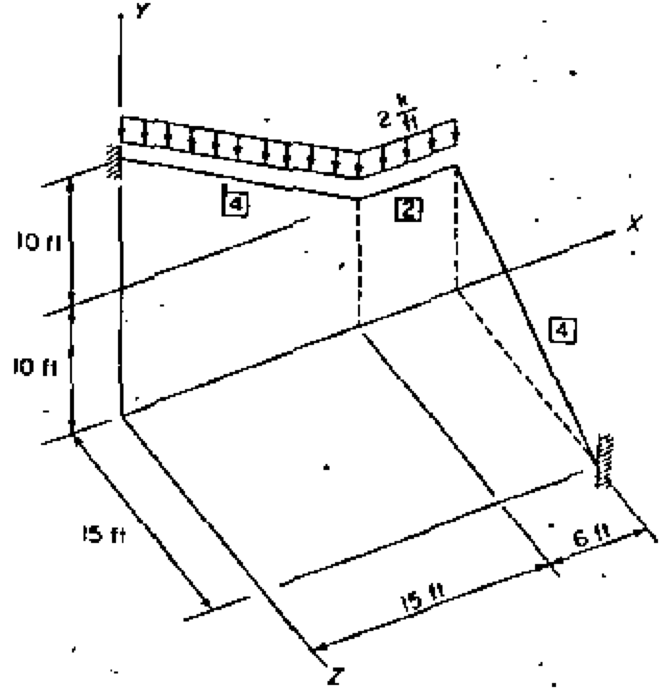
4-9 Livesley, R. K., *Matrix Methods of Structural Analysis*. New York: The Macmillan Company, Inc., 1964.

The stiffness method is a very powerful tool when coupled with the electronic digital computer for analyzing complex as well as simple structures. The procedure for carrying out the analysis is a very orderly, systematic procedure that is not restricted to a particular type of system. Only those matrices that are required to describe the behavior of particular structural elements are different. Thus, the problem of analyzing a given structure becomes one of developing the proper matrices to describe the response of the elements which make up the system.

6–9

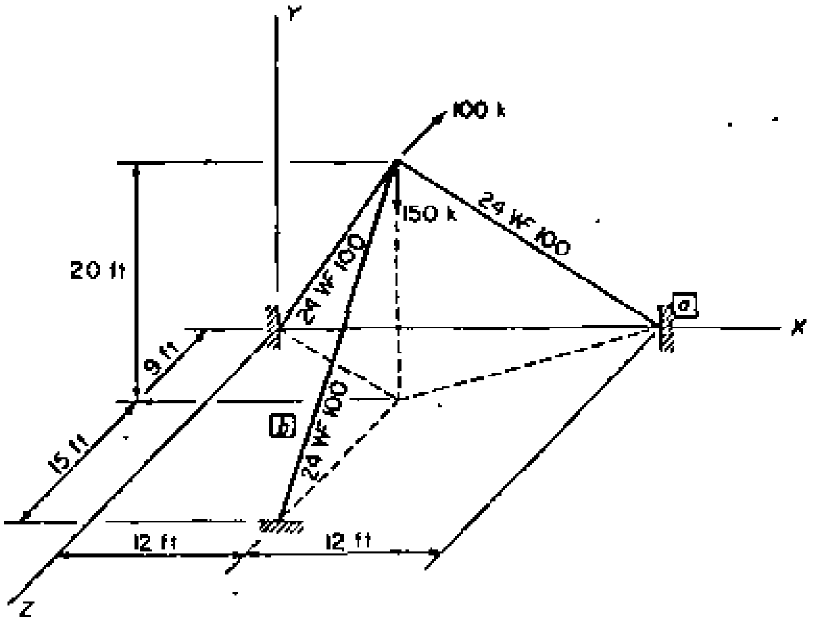
Problems

6-1. Determine the final end actions developed at the end of each member and the support reactions of the rigid frame caused by the indicated loading. For each member, $I_x = 3I/2$, $I_y = 2I$, $I_z = 4I$, and $A_x = I/4$; E = constant and $G = E/2$. The relative value of I for each member is given in the box adjacent to the member. The y_m–x_m plane of each member is perpendicular to the X–Z reference.



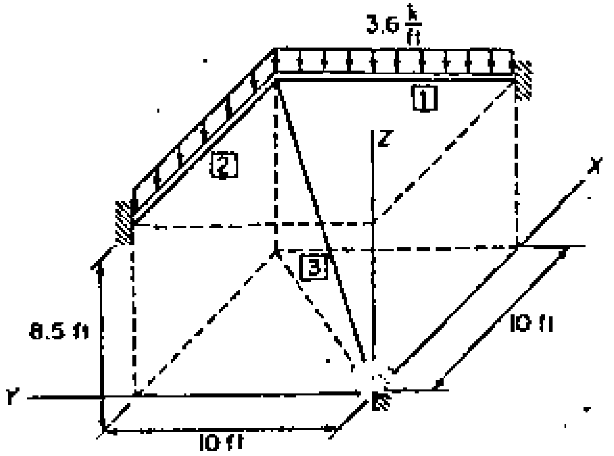
Prob. 6-1 *Note:* The y_m–x_m plane of each member is perpendicular to the X–Z plane.

6-2. Analyze the space frame for the imposed loading condition. The members are prismatic 24 W 100 steel beams; $E = 30{,}000$ ksi and $G = 12{,}000$ ksi. The members are oriented such that the y_m–x_m plane of each beam, where the y_m axis defines the minor axis of the cross-section, is perpendicular to the X–Z reference plane.



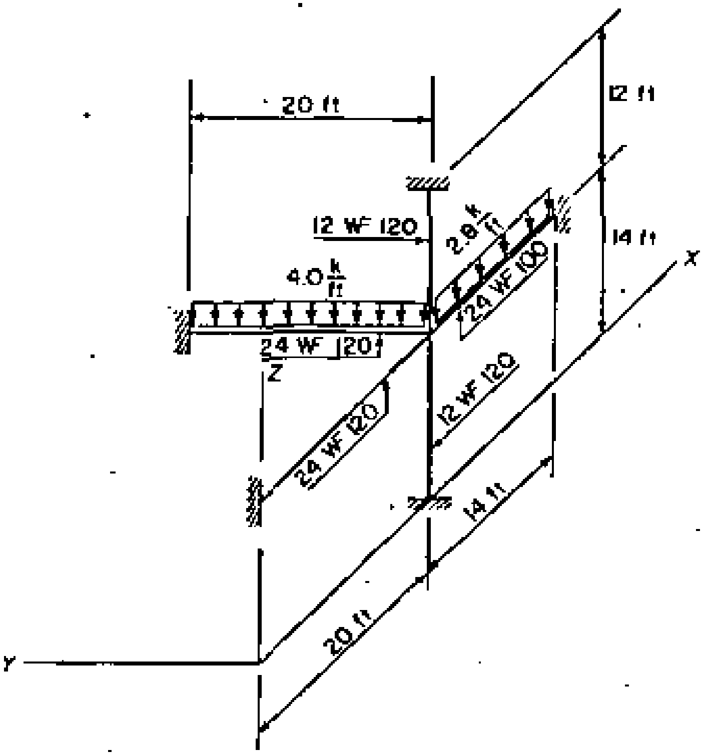
Prob. 6-2 *Note:* The y_m–x_m plane of each member is perpendicular to the X–Z plane.

6-3. Compute the support reactions and final end actions. For members 1 and 2, $I_x = 2I/3$, $I_y = I$, $I_z = 3I$, and $A_x = I/5$; for member 3, $I_x = I$, $I_y = 2I$, $I_z = 5I$, and $A_x = I/4$; E = constant and $G = E/2$. The y_m–x_m plane of each beam is perpendicular to the X–Y reference axis.



6-3 *Note:* The y_m–x_m plane of each member is perpendicular to the X–Y plane.

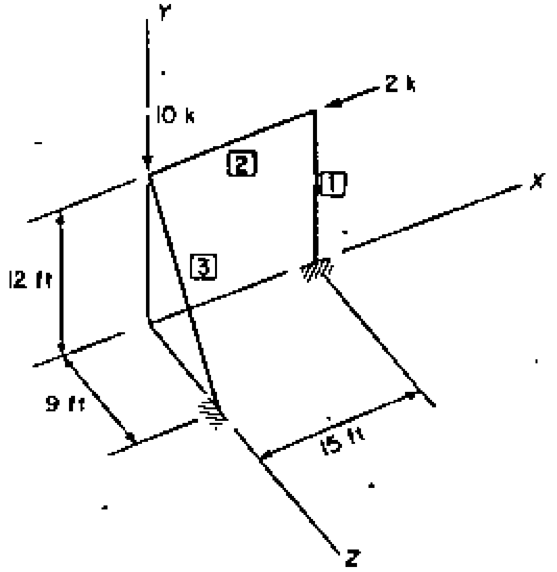
6-4. The space frame is to be analyzed for the indicated loading condition. With the y_m axis defining the minor principal axis of the cross-section, the y_m–x_m plane of each beam is perpendicular to the X–Y reference plane and the y_m–x_m plane of each column is perpendicular to the X–Z reference plane; $E = 30{,}000$ ksi and $G = 12{,}000$ ksi.



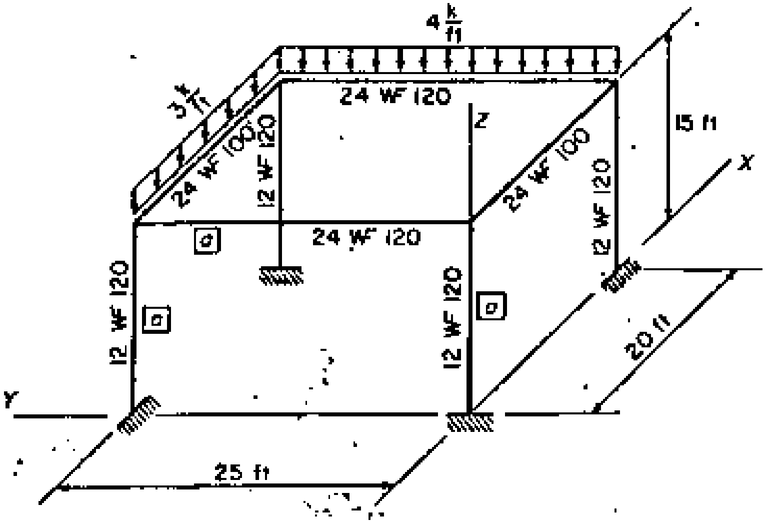
Prob. 6-4 *Note:* The y_m–x_m planes of the beams are perpendicular to the X–Y plane, and the y_m–x_m planes of the columns are perpendicular to the X–Z plane.

6-5. Determine the final end actions for each member and the support reactions for the structure caused by the applied loading. For members 1 and 2, $I_x = I/2$, $I_y = 6I$, $I_z = 8I$, and $A_x = I/5$; for member 3, $I_x = I$, $I_y = 10I$, $I_z = 10I$, and $A_x = I/4$; E = constant and $G = E/2$. Letting the y_m axis define the minor principal axis of a member's cross-section, the y_m–x_m plane of members 2 and 3 are perpendicular to the X–Z reference plane and the y_m–x_m plane of member 1 coincides with the X–Y reference plane.

6-6. Develop the complete structure stiffness matrix for the rigid space frame described in the figure and set up the complete joint load matrix for the indicated loading condition. Letting the y_m axis define the minor principal axis of a cross-section, the y_m–x_m plane of each column is parallel to the Y–Z reference plane and for each beam is perpendicular to the X–Y reference plane; $E = 30{,}000$ ksi and $G = 12{,}000$ ksi.



Prob. 6-5 *Note:* The y_m-x_m plane of member 1 coincides with the X-Y reference plane. The y_m-x_m plane of members 2 and 3 is perpendicular to the X-Z plane in both cases.



Prob. 6-6

6-7. Analyze the frame of Prob. 6-2 for a vertical settlement of 0.75 in. of the support *a*.

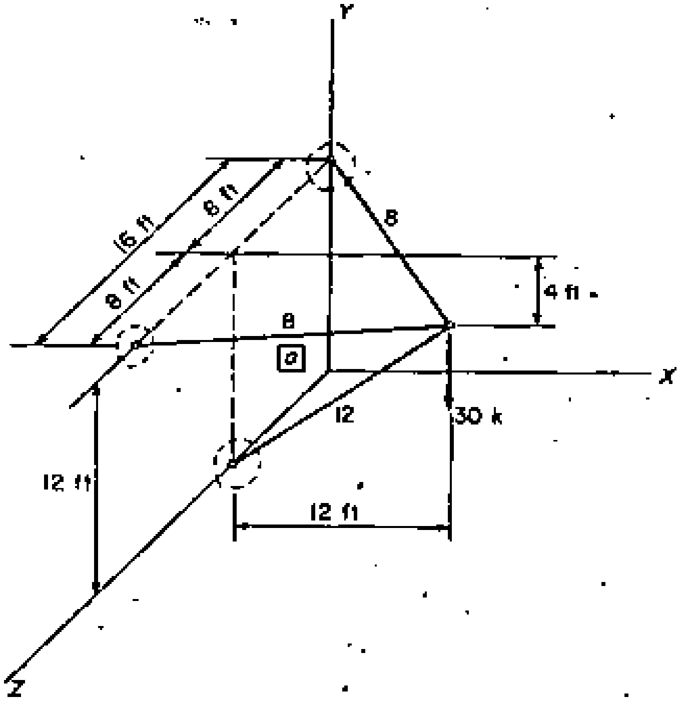
6-8. Determine the final end actions and support reactions developed by the rigid frame of Prob. 6-2 if member *b* is fabricated 1 in. too short.

6-9. Analyze the space frame of Prob. 6-5 for a vertical settlement of 0.5 in. of the support of member 1. Let $I = 1000$ in^4, $E = 3000$ ksi, and $G = 1000$ ksi.

6-10. Analyze the rigid space frame of Prob. 6-6 for the indicated loading.

6-11. Analyze the rigid frame structure of Prob. 6-6 for an increase in temperature of 40° of members *a* over the other members of the structure.

6-12 to 6-15. Determine the bar forces developed in the space truss. The orientation of the local axes for each member may be selected for convenience of computation; $E = 30{,}000$ ksi. The cross-sectional area of each member (in terms of sq in.) is indicated adjacent to the member.



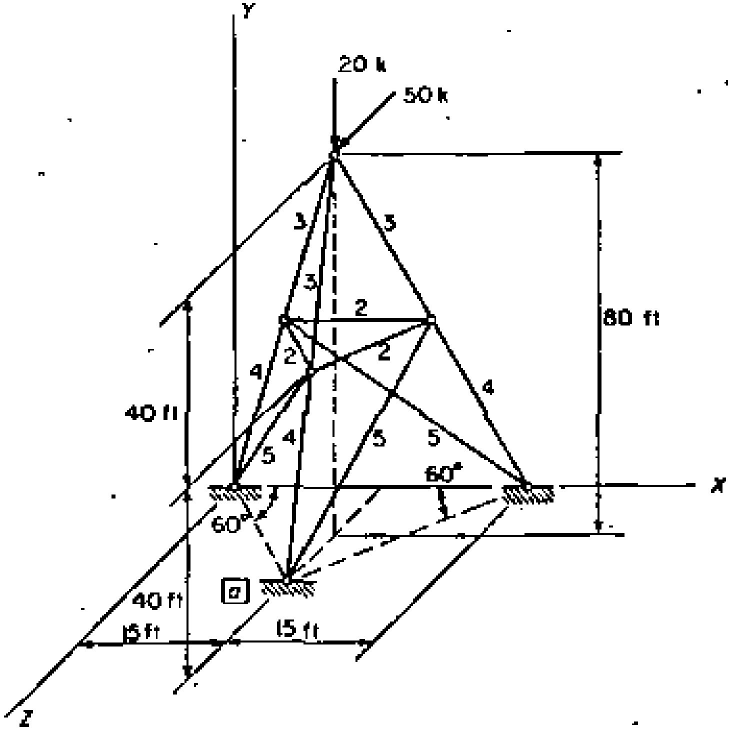
Prob. 6-12

6-16. Compute the bar forces developed in the space truss of Prob. 6-12 if member *a* is fabricated 0.5 in. too short.

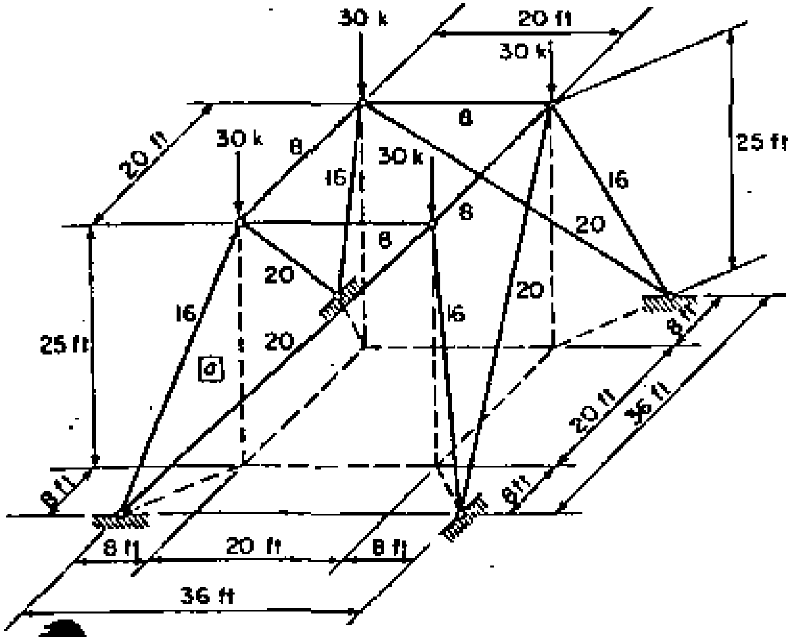
6-17. Analyze the space truss of Prob. 6-13 for a settlement of support *a* of 0.75 in.

6-18. Determine the bar forces in each member and the support reactions for the structure if member *a* of the truss of Prob 6-14 is fabricated 0.25 in. too long.

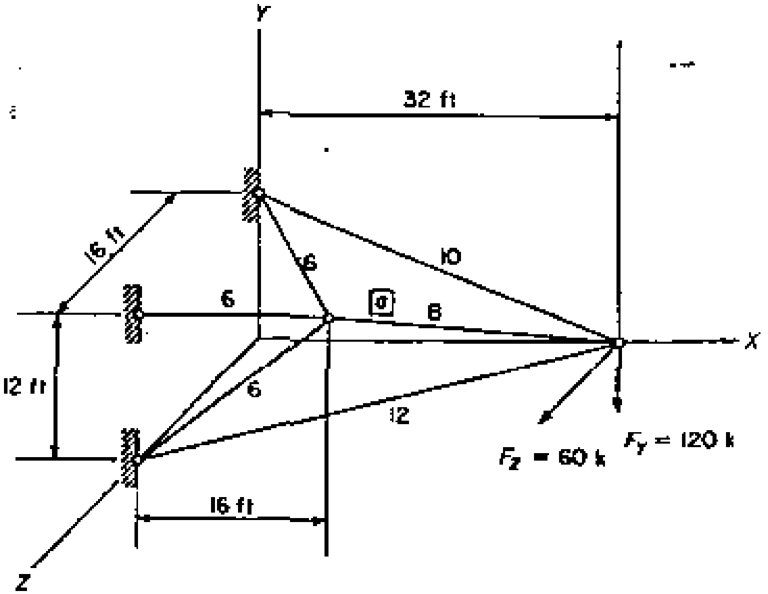
6-19. Analyze the structure of Prob. 6-15 for a fabrication error of 0.3 in. shortening the length of member *a*.



Prob. 6-13

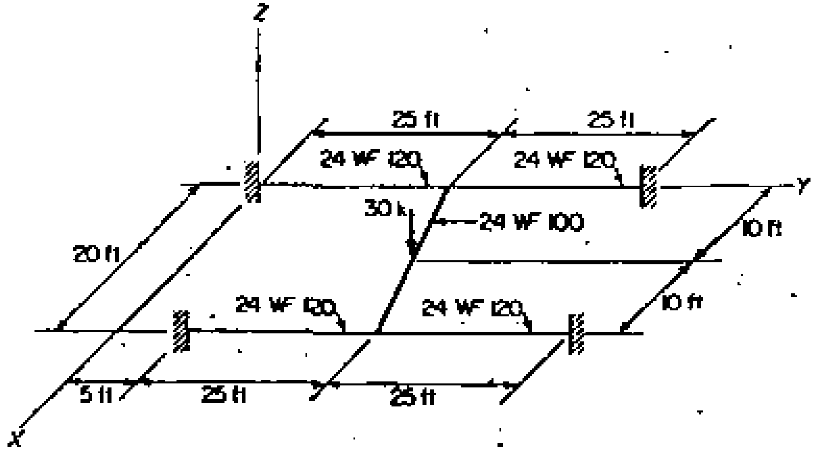


Prob. 6-14

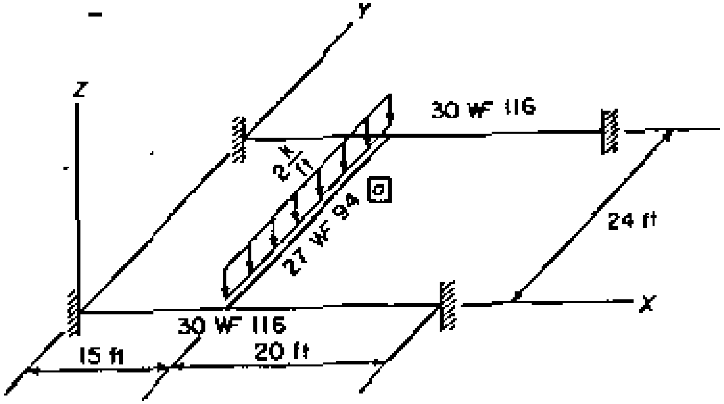


Prob. 6-16

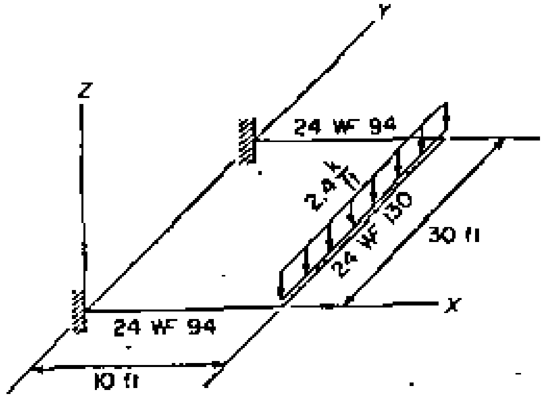
6-20 to 6-23. Analyze the planar grid structures for the indicated loading. Each member is positioned in the X–Y reference plane so that the major principal axis (y_m) of each cross-section lies in the plane; $E = 30{,}000$ ksi and $G = 12{,}000$ ksi.



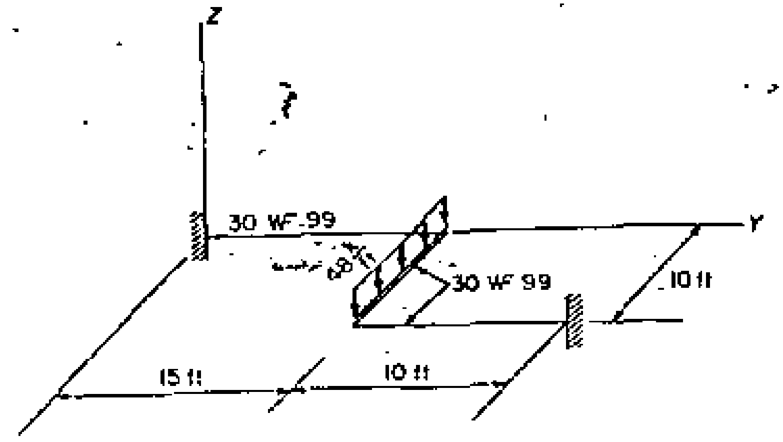
Prob. 6-20 *Note:* Major principal axis of the cross sectio each member lies in the X–Y reference plane.



Prob. 6-21 *Note:* Major principal axis of the cross section of each member lies in the *X–Y* reference plane.

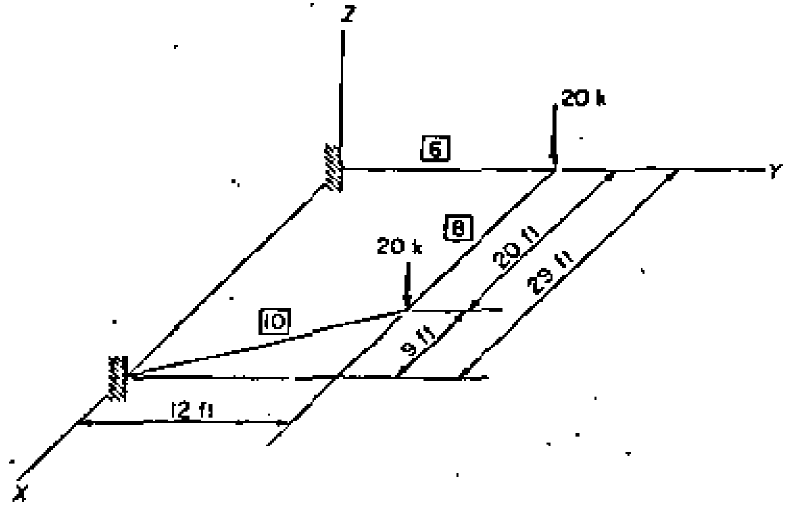


Prob. 6-22 *Note:* Major principal axis of the cross section of each member lies in the *X–Y* reference plane.



Prob. 6-23 *Note:* Major principal axis of the cross section of each member lies in the *X–Y* reference plane.

6-24. Analyze the planar grid structure shown in the figure for the indicated loading. For each member, $I_x = 3I/2$, $I_y = 2I$, $I_z = I$, and $A_x = I/6$. The major principal axis (y_m) of the cross-section of each member lies in the *X–Y* reference plane. The relative value of *I* for each member is given in the box adjacent to the member; E = constant and $G = E/3$.



Prob. 6-24 *Note:* Major principal axis of the cross section of each member lies in the *X–Y* reference plane.

6-25. Determine the final end actions and the support reactions for the structure of Prob. 6-21 if member *a* is fabricated with a bend of 5° (rotating right end in a counterclockwise direction) at midspan.

6-26. (a) Establish all of the matrices for a beam element with either a variable or constant cross-section over its span length, arbitrarily oriented in a three-dimensional space, with both ends of the member restrained against translation in the x_m, y_m, and z_m directions, both ends restrained against rotation about the x_m and y_m axes, and both ends free to rotate about the z_m axis so that this type of member could be handled in a stiffness analysis.

(b) Evaluate the member stiffness matrix for this beam element if it were a prismatic member.

6-27. Develop the grid member stiffness matrix $[\mathrm{K}_G]_i$ for a prismatic grid member with a pin at the *j*-end of the member so that it is free to rotate about the major principal axis y_m. The member is assumed to be restrained against all other possible components of end displacement. Also, establish the transformation matrix $[\mathrm{T}_G]_i$ and the transformed grid member stiffness matrix $[\bar{\mathrm{K}}_G]_i$ for this member.

6-28. Establish the member stiffness matrix $[\mathrm{K}]_i$ for a prismatic 12 in. [25# beam. Note that for this member the shear center and the centroid of the cross-section do not coincide. The x_m axis will define the centroidal axis of the beam and the z_m principal axis will contain both the centroid and the shear center of the channel section.

6-29. Develop a computer program to analyze by the stiffness method a planar grid frame for any possible loading condition. *Hint:* Let fixed end actions be input data.

6-30. Write a computer program to analyze by the stiffness method a space truss system for loads applied only at the joints.

6-31. Write a computer program to carry out the analysis of a rigid space frame by the stiffness method for any possible loading condition. *Hint:* Use fixed end actions as input.

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6-1 Willems, Nicholas, and William M. Lucas, Jr., *Matrix Analysis for Structural Engineers*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1968.

6-2 Gere, James M., and William Weaver, Jr., *Analysis of Framed Structures*. Princeton, N.J.: D. Van Nostrand Company, Inc., 1965.

6-3 Seely, Fred B., and James O. Smith, *Advanced Mechanics of Materials*, 2nd ed. New York: John Wiley & Sons, Inc., 1952.

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1 METODO DE LAS RIGIDECES PARA ANALIZAR ESTRUCTURAS ORTOGONALES PLANAS.

1.1 Convención de signos.

La siguiente convención de signos será utilizada en el desarrollo del método de las rigideces y sus aplicaciones en marcos ortogonales planos.

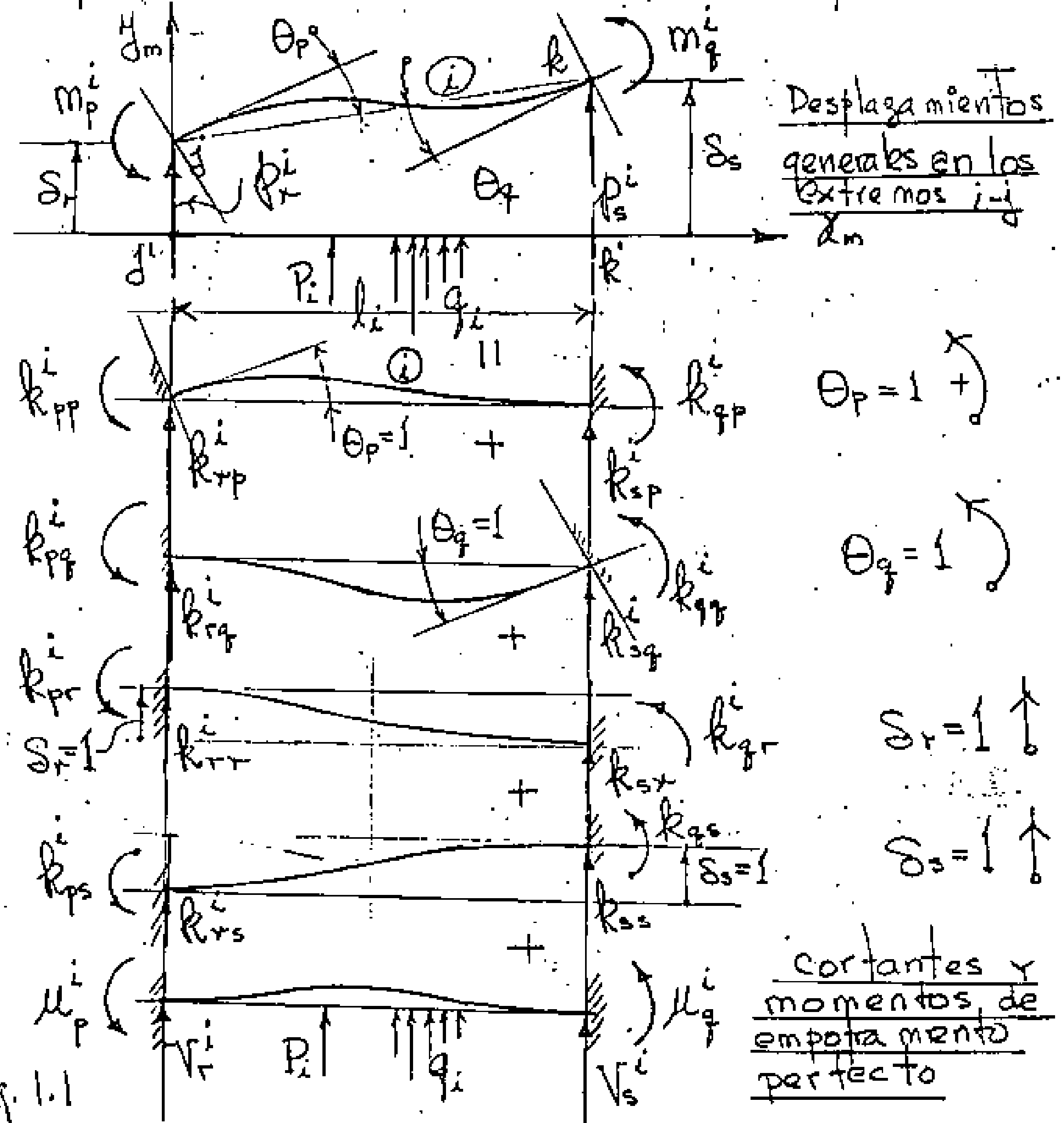


Fig. 1.1

De la Fig. 11 aceptando el principio de superposición se tiene:

$$\begin{aligned} m_p^i &= k_{pp}^i \theta_p + k_{pq}^i \theta_q + k_{pr}^i \delta_r + k_{ps}^i \delta_s + \mu_p^i \\ m_q^i &= k_{qp}^i \theta_p + k_{qq}^i \theta_q + k_{qr}^i \delta_r + k_{qs}^i \delta_s + \mu_q^i \\ p_r^i &= k_{rp}^i \theta_p + k_{rq}^i \theta_q + k_{rr}^i \delta_r + k_{rs}^i \delta_s + V_r^i \\ p_s^i &= k_{sp}^i \theta_p + k_{sq}^i \theta_q + k_{sr}^i \delta_r + k_{ss}^i \delta_s + V_s^i \end{aligned} \qquad (1.1)$$

en (1.1) se desprecia el efecto de la carga normal expresando (1.1) matricialmente se tiene

$$\{m\}_i = [k]_i \{\delta\}_i + \{\mu\}_i \qquad (1.2)$$

donde:

$$\{m\}_i = \begin{Bmatrix} m_p \\ m_q \\ p_r \\ p_s \end{Bmatrix}_i \; ; \; \{\delta\}_i = \begin{Bmatrix} \theta_p \\ \theta_q \\ \delta_r \\ \delta_s \end{Bmatrix}_i \; ; \; \{\mu\}_i = \begin{Bmatrix} \mu_p \\ \mu_q \\ V_r \\ V_s \end{Bmatrix}_i \qquad (1.3)$$

$\{m\}_i$; componentes de acciones sobre barra para mantener equil.

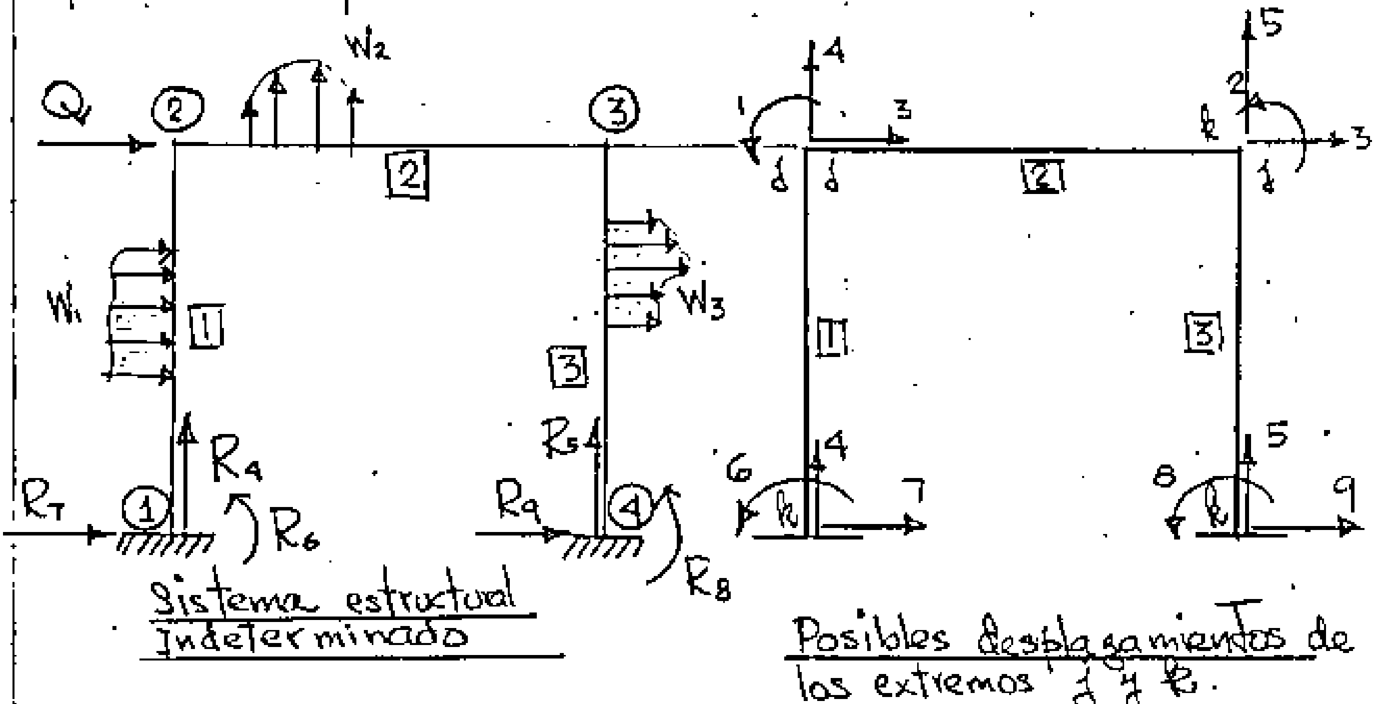
$\{\delta\}_i$; Desplazamientos en los extremos del miembro ⓘ

$\{\mu\}_i$; Momentos y cortantes de empotramiento perfecto en ⓘ

$[k]_i$; Matriz de rigidez del miembro ⓘ, la cual despreciando el efecto de cortante y carga normal, para un miembro de sección constante es:

$$[k]_i = \begin{array}{c} \left[\begin{array}{cccc} \frac{4EI}{l} & \frac{2EI}{l} & \frac{6EI}{l^2} & -\frac{6EI}{l^2} \\ \frac{2EI}{l} & \frac{4EI}{l} & \frac{6EI}{l^2} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{12EI}{l^3} \\ -\frac{6EI}{l^2} & -\frac{6EI}{l^2} & -\frac{12EI}{l^3} & -\frac{12EI}{l^3} \end{array}\right] \begin{array}{c} k \\ j \\ r \\ s \end{array} \\ \begin{array}{cccc} k & j & r & s \end{array} \end{array} \qquad (1.4)$$

La filosofía básica del método de las rigideces ha sido presentada, antes de aplicarlo a diversos sistemas estructurales su procedimiento conviene organizarlo en un programa sistemático y las ecuaciones básicas del analisis presentarlas en términos generales. Como ejemplo consideraremos el el marco siguiente.



Sistema estructural Indeterminado

Posibles desplazamientos de los extremos j y k.

Fig. 1.2

El pórtico de la Fig. 1.2 es indeterminado de tercer grado con θ_1, θ_2 y δ_3, porque las condiciones de apoyo anulan a δ_4, δ_5, θ_6, δ_7, θ_8, δ_9. Como primera etapa consideramos la estructura con los nudos fijos determinando la suma de momentos y cortantes correspondientes S_{mo}.

Aplicando las ecuaciones (1.1) al marco de la Fig. 1.2

$$\text{Miembro 1}\begin{cases} m_1^1 = k^1_{11}\theta_1 + k^1_{16}(0) + k^1_{13}\delta_3 + k^1_{17}(0) + \mu_1^1 \\ m_6^1 = k^1_{61}\theta_1 + k^1_{66}(0) + k^1_{63}\delta_3 + k^1_{67}(0) + \mu_6^1 \\ p_3^1 = k^1_{31}\theta_1 + k^1_{36}(0) + k^1_{33}\delta_3 + k^1_{37}(0) + V_3^1 \\ p_7^1 = k^1_{71}\theta_1 + k^1_{76}(0) + k^1_{73}\delta_3 + k^1_{77}(0) + V_7^1 \end{cases} \qquad (1.5)$$

$$\text{Miembro 2}\begin{cases} m_1^2 = k^2_{11}\theta_1 + k^2_{12}\theta_2 + k^2_{14}(0) + k^2_{15}(0) + \mu_1^2 \\ m_2^2 = k^2_{21}\theta_1 + k^2_{22}\theta_2 + k^2_{24}(0) + k^2_{25}(0) + \mu_2^2 \\ p_4^2 = k^2_{41}\theta_1 + k^2_{42}\theta_2 + k^2_{44}(0) + k^2_{45}(0) + V_4^2 \\ p_5^2 = k^2_{51}\theta_1 + k^2_{52}\theta_2 + k^2_{54}(0) + k^2_{55}(0) + V_5^2 \end{cases} \qquad (1.6)$$

$$\text{Miembro 3}\begin{cases} m_2^3 = k^3_{22}\theta_2 + k^3_{28}(0) + k^3_{23}\delta_3 + k^3_{29}(0) + \mu_2^3 \\ m_8^3 = k^3_{82}\theta_2 + k^3_{88}(0) + k^3_{83}\delta_3 + k^3_{8}(0) + \mu_8^3 \\ p_3^3 = k^3_{32}\theta_2 + k^3_{38}(0) + k^3_{33}\delta_3 + k^3_{39}(0) + V_3^3 \\ p_9^3 = k^3_{92}\theta_2 + k^3_{98}(0) + k^3_{93}\delta_3 + k^3_{99}(0) + V_9^3 \end{cases} \qquad (1.7)$$

Como se demostró previamente el análisis de la estructura indeterminada de la Fig. 1.2 puede ser evaluado de

$$[S_{ij}]\{\delta_i\} = \{Q_i\} \qquad (1.8)$$

en el caso de la Fig 1.2, (1.8) es igual a

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{41} & S_{51} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} M^1_{21} + M^2_{23} \\ M^2_{32} + M^3_{34} \\ V^1_{21} + V^3_{21} - Q \end{Bmatrix} \qquad (1.9)$$

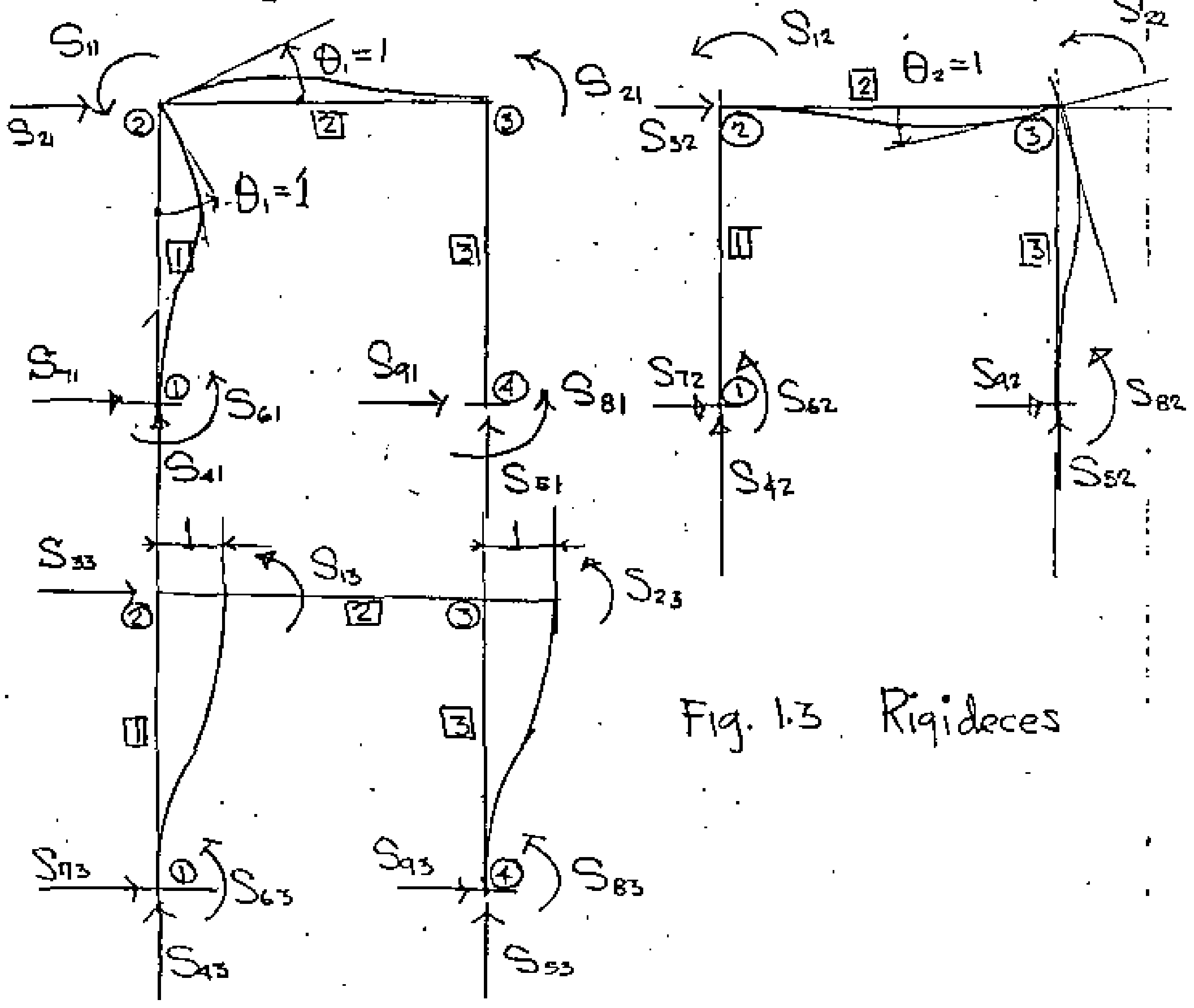


Fig. 1.3 Rigideces

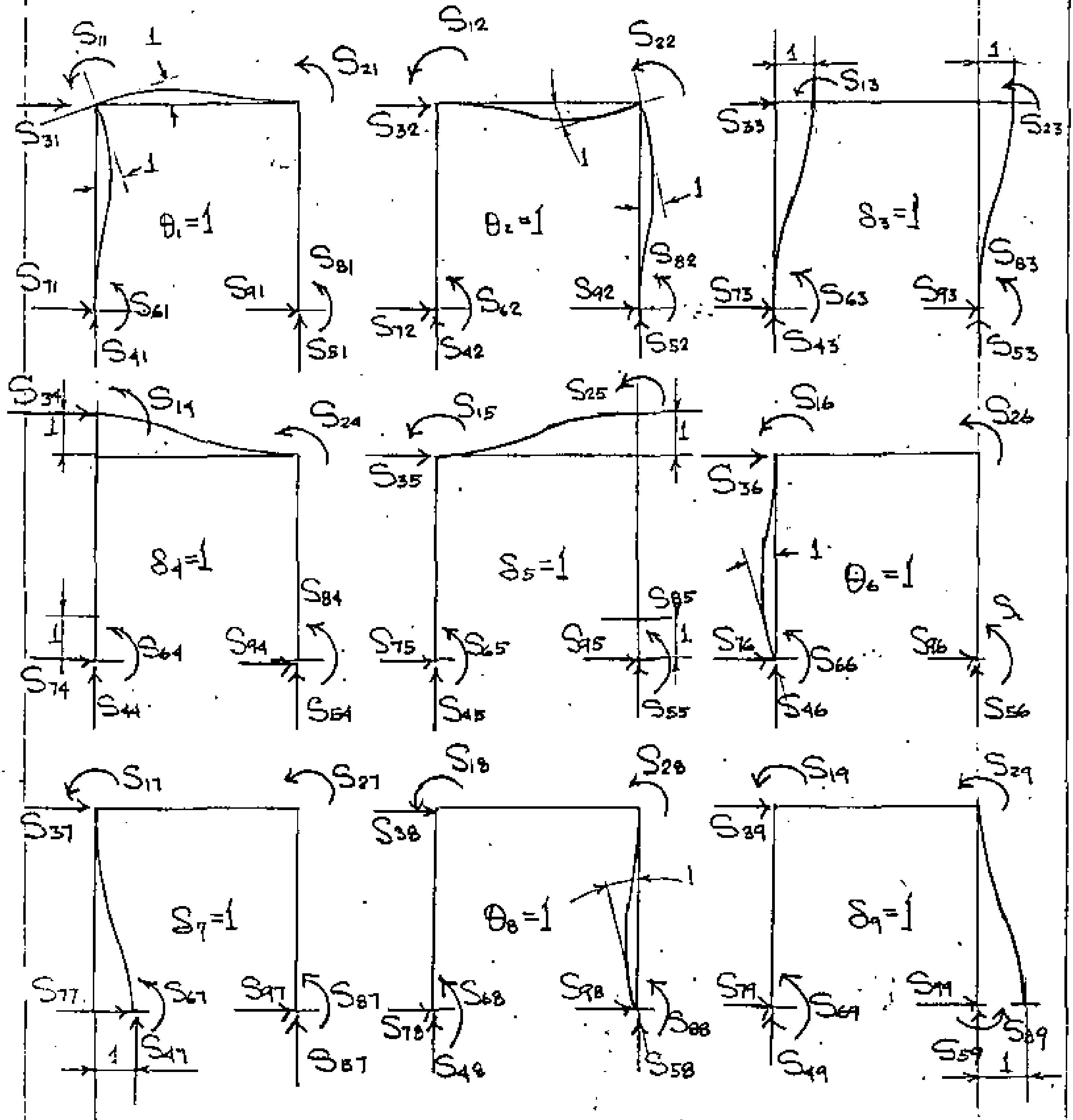


Fig. 1.4 Rigideces considerando todos los posibles grados de libertad despreciando deformaciones axiales (se suponen direcciones positivas)

De la Fig. 1.4 el desarrollo completo de las ecuaciones de superposición incluyendo reacciones es

$$
\begin{aligned}
&S_{11}\theta_1+S_{12}\theta_2+S_{13}\delta_3+S_{14}\delta_4+S_{15}\delta_5+S_{16}\theta_6+S_{17}\delta_7+S_{18}\theta_8\\
&\qquad +S_{19}\delta_9+\mu_{21}^{1}+\mu_{23}^{2}=0\\
&S_{21}\theta_1+S_{22}\theta_2+S_{23}\delta_3+S_{24}\delta_4+S_{25}\delta_5+S_{26}\theta_6+S_{27}\delta_7+S_{28}\theta_8\\
&\qquad +S_{29}\delta_9+\mu_{32}^{2}+\mu_{24}^{3}=0\\
&S_{31}\theta_1+S_{32}\theta_2+S_{33}\delta_3+S_{34}\delta_4+S_{35}\delta_5+S_{36}\theta_6+S_{37}\delta_7+S_{38}\theta_8\\
&\qquad +S_{39}\delta_9+V_{21}^{1}+V_{21}^{3}=Q\\
&S_{41}\theta_1+S_{42}\theta_2+S_{43}\delta_3+S_{44}\delta_4+S_{45}\delta_5+S_{46}\theta_6+S_{47}\delta_7+S_{48}\theta_8\\
&\qquad +S_{49}\delta_9+V_{23}^{2}=R_4\\
&S_{51}\theta_1+S_{52}\theta_2+S_{53}\delta_3+S_{54}\delta_4+S_{55}\delta_5+S_{56}\theta_6+S_{57}\delta_7+S_{58}\theta_8\\
&\qquad +S_{59}\delta_9+V_{32}^{2}=R_5\\
&S_{61}\theta_1+S_{62}\theta_2+S_{63}\delta_3+S_{64}\delta_4+S_{65}\delta_5+S_{66}\theta_6+S_{67}\delta_7+S_{68}\theta_8\\
&\qquad +S_{69}\delta_9+\mu_{12}^{1}=R_6\\
&S_{71}\theta_1+S_{72}\theta_2+S_{73}\delta_3+S_{74}\delta_4+S_{75}\delta_5+S_{76}\theta_6+S_{77}\delta_7+S_{78}\theta_8\\
&\qquad +S_{79}\delta_9+V_{12}^{1}=R_7\\
&S_{81}\theta_1+S_{82}\theta_2+S_{83}\delta_3+S_{84}\delta_4+S_{85}\delta_5+S_{86}\theta_6+S_{87}\delta_7+S_{88}\theta_8\\
&\qquad +S_{89}\delta_9+\mu_{43}^{3}=R_8\\
&S_{91}\theta_1+S_{92}\theta_2+S_{93}\delta_3+S_{94}\delta_4+S_{95}\delta_5+S_{96}\theta_6+S_{97}\delta_7+S_{98}\theta_8\\
&\qquad +S_{99}\delta_9+V_{43}^{3}=R_9
\end{aligned}
\tag{1.10}
$$

expresando (1.10) matricialmente se obtiene:

$$\underbrace{\left[\begin{array}{ccc|cccccc} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} & S_{17} & S_{18} & S_{19} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} & S_{27} & S_{28} & S_{29} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} & S_{37} & S_{38} & S_{39} \\ \hline S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} & S_{47} & S_{48} & S_{49} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} & S_{57} & S_{58} & S_{59} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} & S_{67} & S_{68} & S_{69} \\ S_{71} & S_{72} & S_{73} & S_{74} & S_{75} & S_{76} & S_{77} & S_{78} & S_{79} \\ S_{81} & S_{82} & S_{83} & S_{84} & S_{85} & S_{86} & S_{87} & S_{88} & S_{89} \\ S_{91} & S_{92} & S_{93} & S_{94} & S_{95} & S_{96} & S_{97} & S_{98} & S_{99} \end{array}\right]}_{[S_{kl}]} \underbrace{\left\{\begin{array}{c} \theta_1 \\ \theta_2 \\ \delta_3 \\ \hline \delta_4 \\ \delta_5 \\ \theta_6 \\ \delta_7 \\ \theta_8 \\ \delta_9 \end{array}\right\}}_{\{\delta_j\}} + \underbrace{\left\{\begin{array}{c} \mu^1_{21}+\mu^2_{23} \\ \mu^2_{32}+\mu^3_{34} \\ V^1_{21}+V^3_{21} \\ \hline V^2_{23} \\ V^2_{32} \\ \mu^1_{12} \\ V^1_{12} \\ \mu^3_{43} \\ V^3_{43} \end{array}\right\}}_{\{\mu\}} = \underbrace{\left\{\begin{array}{c} 0 \\ 0 \\ Q \\ \hline R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{array}\right\}}_{\{R\}} \quad (1.11)$$

Expresando (1.11) matricialmente con la notación indicada

$$[S_{kl}]\{\delta_i\} + \{\underline{\mu}\}_l = \{R\} \qquad (1.12)$$

El analisis por el método de las rigideces se reduce a evaluar de (1.8) $\{\delta_i\}$ o sea

$$\{\delta_i\} = [S_{ij}]^{-1}\{Q_i\} \qquad (1.13)$$

y substituyendo (1.13) en (1.2) se obtiene para cada barra

$$\{M_i\} = [k]_i\,[S_{ij}]^{-1}\{Q_i\} + \{\mu\}_i \qquad (1.14)$$

y las reacciones se obtienen substituyendo (1.13) en (1.12)

$$\{R\} = [S_{kl}][S_{ij}]^{-1}\{Q_i\} + \{\mu\}_k \qquad (1.15)$$

2 METODO DE LAS RIGIDECES DE ANALISIS DE ESTRUCTURAS TRIDIMENCIONALES

2.1 ELEMENTO VIGA.

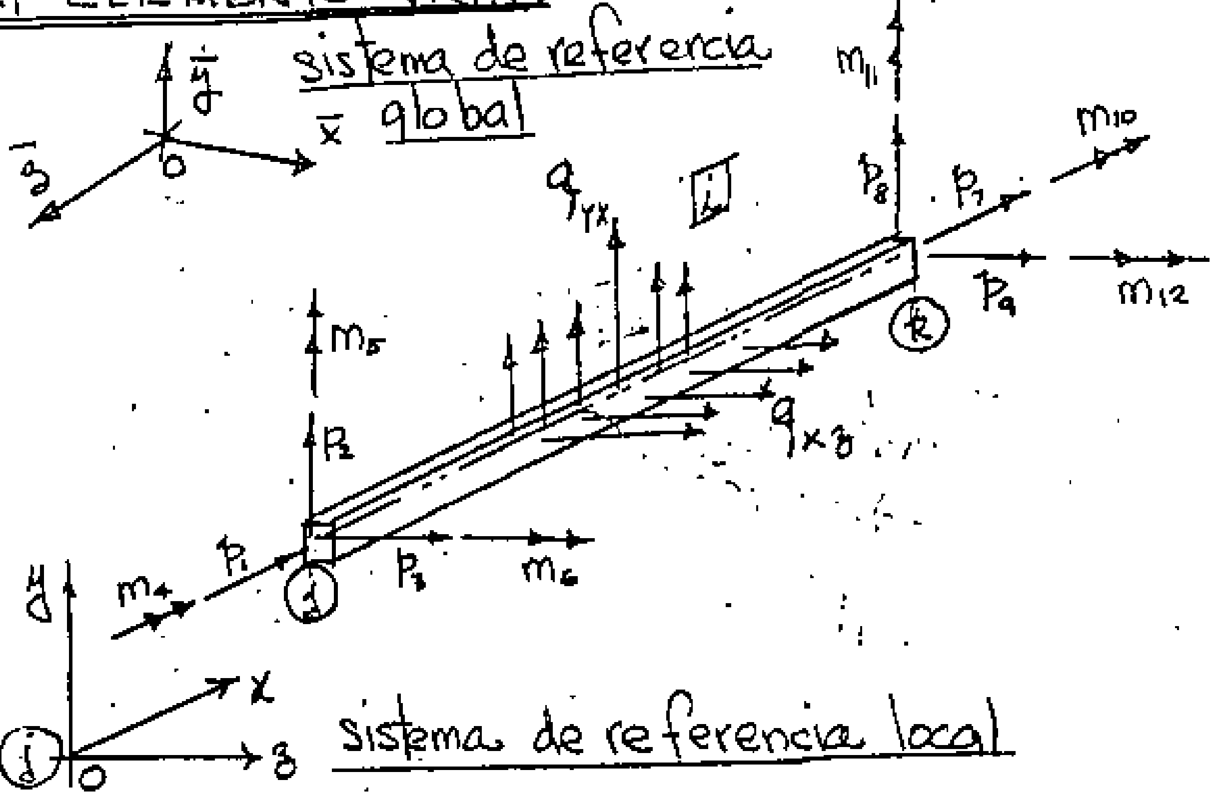


Fig. 2.1 Elemento viga; ejes y, z son centroidales y principales ($Q_y = Q_z = I_{yz} = 0$)

El elemento estructural $j \cdot k$, se supone una barra capaz de resistir fuerzas axiales, momentos flectores respecto a dos ejes principales en el plano de la sección transversal, y momentos de torsión respecto a su eje centroidal. Las siguientes fuerzas actuan en la viga jk: Fuerzas axiales p_1 y p_7; Fuerzas cortantes p_2, p_3, p_8 y p_9; Momentos flectores m_5, m_6, m_{11} y m_{12}; y Momentos de torsión m_4 y m_{10}. la localización y dirección positiva se muestra en Fig. 2.1

Los desplazamientos correspondientes serán $u_1, u_2, u_3, \ldots, u_{12}$ serán positivos en la dirección positiva de las fuerzas. La posisión del elemento viga jk será especificado por las coordenadas del extremo j y los cosenos directores del eje x (dirección jk) y del eje y con respecto al sistema global $(\bar{x}, \bar{y}, \bar{z})$.

La matriz de rigidez del elemento viga será de 12×12 pero siempre es posible integrarla con submatrices de 2×2 y 4×4. De la teoría de flexión y torsión de vigas las fuerzas P_1 y P_7 dependen solo de sus desplazamientos correspondientes; lo mismo es cierto para los momentos torsionantes M_4 y M_{10}. Sin embargo, para una selección arbitraria de los planos de flexión, los momentos flectores y fuerzaz de corte en el plano xy dependerán no solo de sus dezplazamiento correspondientes pero también en los desplazamiento correspondientes a las fuerzas en los planos xy. Solamente si los xy y xz coinciden con los ejes principales de la sección transversal puede considerarse la flexión y corte sobre dichos planos independiente una de la otra.

$$\underbrace{\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ m_4 \\ m_5 \\ m_6 \\ p_7 \\ p_8 \\ p_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{Bmatrix}}_{\{p\}_i} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} & k_{19} & k_{1,10} & k_{1,11} & k_{1,12} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} & k_{29} & k_{2,10} & k_{2,11} & k_{2,12} \\ k_{31} & \cdot & k_{33} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{3,12} \\ k_{41} & \cdot & \cdot & k_{44} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{4,12} \\ k_{51} & \cdot & \cdot & \cdot & k_{56} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{5,12} \\ k_{61} & \cdot & \cdot & \cdot & \cdot & k_{64} & \cdot & \cdot & \cdot & \cdot & \cdot & k_{6,12} \\ k_{71} & \cdot & \cdot & \cdot & \cdot & \cdot & k_{77} & \cdot & \cdot & \cdot & \cdot & k_{7,12} \\ k_{81} & \cdot & \text{(simétrica)} & \cdot & \cdot & \cdot & \cdot & k_{88} & \cdot & \cdot & \cdot & k_{8,12} \\ k_{91} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{99} & \cdot & \cdot & k_{9,12} \\ k_{10,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{10,10} & \cdot & k_{10,12} \\ k_{11,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{11,11} & k_{11,12} \\ k_{12,1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k_{12,12} \end{bmatrix}_{[k_{ij}]} \underbrace{\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \end{Bmatrix}}_{\{\delta\}_i} + \underbrace{\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ P_7 \\ P_8 \\ P_9 \\ \mu_{10} \\ \mu_{11} \\ \mu_{12} \end{Bmatrix}}_{\{\mu\}_i} \qquad (2.1)$$

Donde:

$\{p\}$; vector de cargas actuando sobre jk

$[k_{ij}]$; matriz de rigidez de la barra jk.

$\{\delta\}$; vector de desplazamientos nodales

$\{\mu\}$; vector de reacciones de empotramiento perfecto

2.2 Elementos de la matriz de rigidez $[k_{ij}]$.

En el cálculo de las rigideces k_{ij} se utilizan los principios energéticos expuestos considerándose la energía elástica de deformación por flexión, corte y carga normal.

2.2.1 Fuerzas axiales p_1 y p_7.

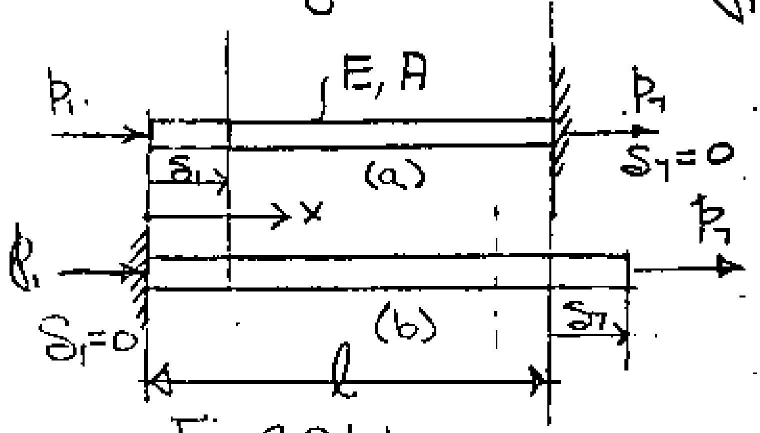


Fig. 2.2.1.1

De la ley de Hooke y la Fig. 2.2.1.2 se obtiene

$$k_{11} = \frac{p_1}{\delta_1} = \frac{EA}{l} \quad ; \quad k_{71} = -\frac{EA}{l} \qquad (a)$$

$$k_{77} = \frac{p_7}{\delta_7} = \frac{EA}{l} \quad ; \quad k_{17} = -\frac{EA}{l} \qquad (b)$$

2.2.2 Momentos de torsión m_4 y m_{10}.

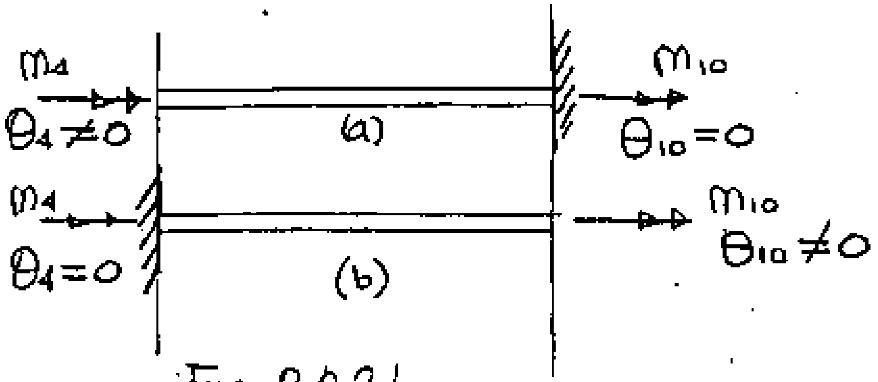


Fig. 2.2.2.1

De la teoría de torsión de barras y la: fig. 2.2.2.1 se obtiene

$$k_{44} = \frac{m_4}{\theta_4} = \frac{GJ}{l} \quad ; \quad k_{10,4} = -\frac{GJ}{l} \qquad (a)$$

$$k_{10,10} = \frac{m_{10}}{\theta_{10}} = \frac{GJ}{l} \quad ; \quad k_{4,10} = -\frac{GJ}{l} \qquad (b)$$

2.2.3 Fuerzas de corte p_2 y p_8.

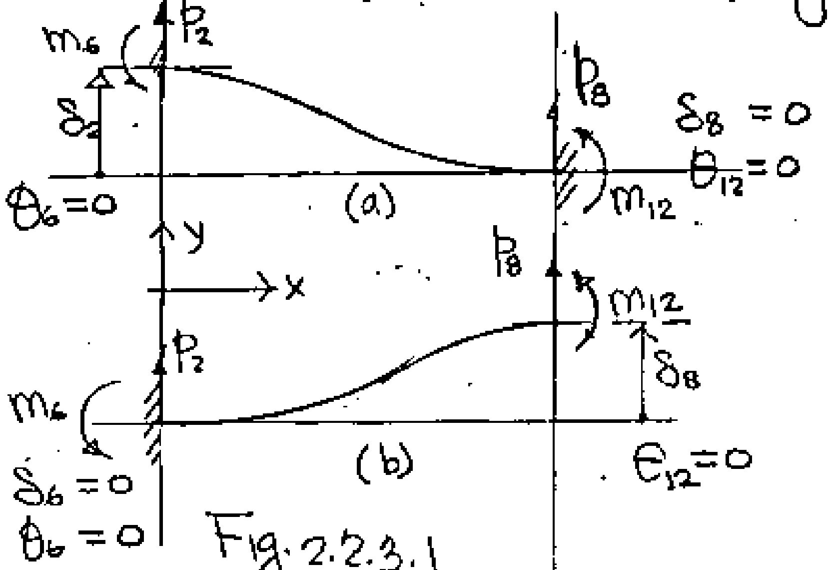


Fig. 2.2.3.1

De la Fig. 2.2.3.1 y los principios energéticos previamente expuestos, conideándo la energía de deformación por flexión y cortante se obtiene

$$k_{22} = \frac{p_2}{\delta_2} = \frac{12EI_z}{(1+\phi_Y)l^3} \qquad a$$

$$k_{62} = \frac{m_6}{\delta_2} = \frac{6EI_z}{(1+\phi_Y)l^2} \therefore k_{26} = \frac{p_2}{\theta_6} = \frac{6EI_z}{(1+\phi_Y)l^2} \qquad b$$

$$k_{82} = \frac{p_8}{\delta_2} = \frac{-12EI_z}{(1+\phi_Y)l^3} \; ; \; k_{28} = \frac{p_2}{\delta_8} = \frac{-12EI_z}{(1+\phi_Y)l^3} \qquad c$$

$$k_{12,2} = \frac{m_{12}}{\delta_2} = \frac{6EI_z}{(1+\phi_Y)l^2} \; ; \; k_{2,12} = \frac{p_2}{\theta_{12}} = \frac{6EI_z}{(1+\phi_Y)l^2} \qquad d$$

$$k_{88} = \frac{p_8}{\delta_8} = \frac{p_2}{\delta_2} = \frac{12EI_z}{(1+\phi_Y)l^3} \quad \text{(si EI es constante)} \qquad e$$

$$k_{12,8} = \frac{m_{12}}{\delta_8} = \frac{-6EI_z}{(1+\phi_Y)l^2} = -\frac{p_2}{\theta_6} = -k_{62} \therefore \qquad (f)$$

$$k_{8,12} = \frac{p_8}{\theta_{12}} = \frac{-6EI}{(1+\phi_Y)l^2} \qquad (g)$$

2.2.4 Momentos Flectores

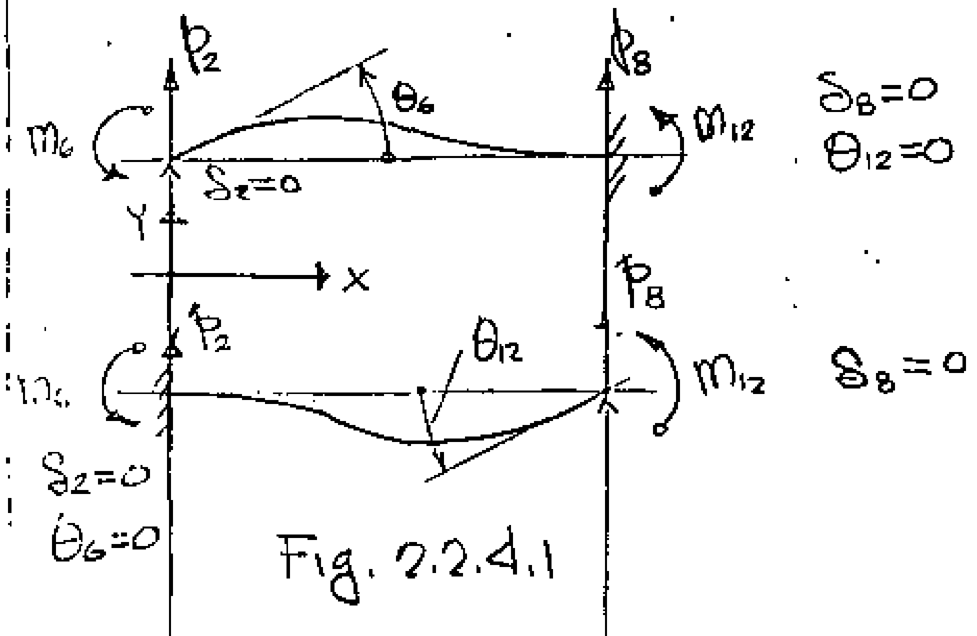


Fig. 2.2.4.1

De la Fig. 2.2.4.1 y los principios energéticos previamente expuestos, considerando la energía de deformación por flexión y corte se obtiene

$$k_{66} = \frac{M_6}{\theta_6} = \frac{(4+\phi_y)EI_z}{(1+\phi_y)l} \qquad a$$

$$k_{86} = \frac{P_8}{\theta_6} = -\frac{6EI_z}{(1+\phi_y)l^2} \; ; \; k_{68} = \frac{M_6}{\delta_8} = -\frac{6EI_z}{(1+\phi_y)l^2} \qquad b$$

$$k_{12,6} = \frac{M_{12}}{\theta_6} = \frac{(2-\phi_y)EI_z}{(1+\phi_y)l} \; ; \; k_{6,12} = \frac{M_6}{\theta_{12}} = \frac{(2-\phi_y)EI_z}{(1+\phi_y)l} \qquad c$$

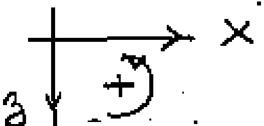
$$k_{12,12} = \frac{M_{12}}{\theta_{12}} = \frac{(4+\phi_y)EI_z}{(1+\phi_y)\;l} \qquad d$$

$$k_{8,12} = \frac{P_8}{\theta_{12}} = -\frac{6EI_z}{(1+\phi_y)l^2} \; ; \; k_{12,8} = \frac{M_{12}}{\delta_8} = k_{8,12} \qquad e$$

$$k_{6,12} = \frac{M_6}{\theta_{12}} = \frac{(2-\phi_y)EI_z}{(1+\phi_y)\;l} \; ; \; k_{12,6} = \frac{M_{12}}{\theta_6} = k_{6,12} \qquad f$$

2.2.5 Fuerzas de corte P_3 y P_9.

Los coeficientes de rigidez relacionados con los desplazamientos δ_3 y δ_9 se obtienen de los resultados previos. Debe observarse, que con la convención de signos adoptada en la Fig 2.1 las direcciones de los momentos flectores positivos en el plano x y son diferentes al plano x z :



, la convención

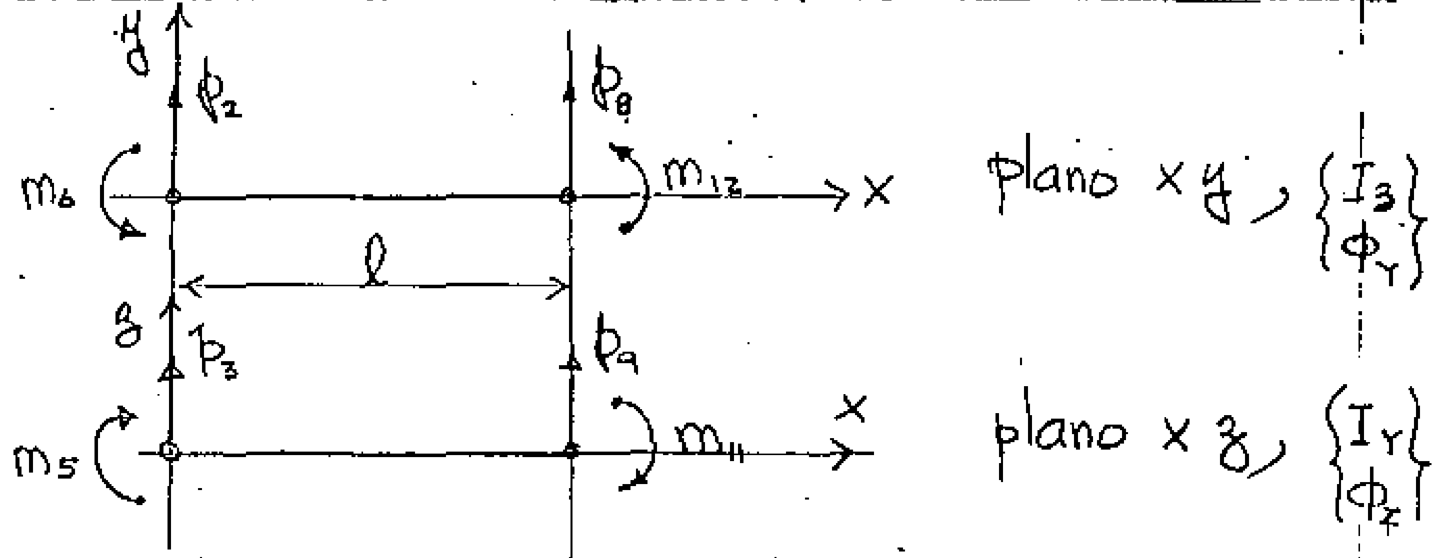


Fig. 2.2.5 Convención de signos para fuerzas de corte y momentos flectores;

de signos se muestra en la Fig. 2.2.5, basado en lo anterior es evidente que

$$k_{33} = \frac{p_3}{\delta_3} \equiv -k_{22} = -\frac{p_2}{\delta_2} \quad a$$

$$k_{53} = \frac{m_5}{\delta_3} \equiv -k_{62} = -\frac{m_6}{\delta_2} \quad b$$

$$k_{93} = \frac{p_9}{\delta_3} \equiv -k_{82} = -\frac{p_8}{\delta_2} \quad c$$

$$k_{11,3} = \frac{m_{11}}{\delta_3} \equiv -k_{12,2} = -\frac{m_{12}}{\delta_2} \quad d$$

$$k_{99} = \frac{p_9}{\delta_9} \equiv -k_{88} = -\frac{p_8}{\delta_8} \quad e$$

$$k_{11,9} = \frac{m_{11}}{\delta_9} \equiv -k_{12,8} = -\frac{m_{12}}{\delta_8} \quad f$$

Debe considerarse en el plano xz a I_Y y ϕ_z como momento de inercia y parámetro de cortante.

2.2.6 Momentos Flectores m_5 y m_{11}

Aplicando las mismas observaciones de la sección anterior, se obtiene

$$k_{55} = \frac{m_5}{\theta_5} \equiv k_{66} = \frac{m_6}{\theta_6} = \frac{(4+\phi_z)}{1+\phi_z}\frac{EI_y}{l} \qquad a$$

$$k_{95} = \frac{p_9}{\theta_5} \equiv -k_{86} = -\frac{p_8}{\theta_6} = +\frac{6EI_y}{(1+\phi_z)l} = k_{59} \qquad b$$

$$k_{11,5} = \frac{m_{11}}{\theta_5} \equiv k_{12,6} = \frac{m_{12}}{\theta_6} = \frac{(2-\phi_z)EI_y}{(1+\phi_z)\,l} = k_{5,11} \qquad c$$

substituyendo los valores k_{ij} obtenidos en las subsecciones anteriores se obtiene la matriz de rigidez de la barra jk de la Fig. 2.1 ecuación 2.5. en donde

$$\phi_y = \frac{12EJ_z}{GA_{sy}l^2} = 24(1+\nu)\frac{A}{A_{sy}}\left(\frac{r_z}{l}\right)^2 = \frac{12f_yEI_z}{GAl^2} \qquad (2.3)$$

$$\phi_z = \frac{12EI_y}{GA_{sz}l^2} = 24(1+\nu)\frac{A}{A_{sz}}\left(\frac{r_y}{l}\right)^2 = \frac{12f_zEI_y}{GAl^2}$$

ν = relación de Poison, A=area total de la sección, A_{sy} y A_{sz} = areas efectivas en cortante en direcciones y y z resp. r_y y r_z = radios de giro respecto a y y resp. a x. ϕ_y y ϕ_z = Parámetros de deformación de corte. Si r_z/l y r_y/l son pequeños comparados con la unidad como son en elementos flexibles, ambos ϕ_y y ϕ_z se pueden considerar cero. Los factores de forma son

$$f_y = \frac{A}{I_z^2}\int_A\left(\frac{Q_z}{b_z}\right)^2 dA \;,\quad f_z = \frac{A}{I_y^2}\int_A\left(\frac{Q_y}{b_y}\right)^2 dA \qquad (2.4)$$

(n.5)

$$[k_{ij}] = \begin{array}{cccccccccccc|c}
\delta_1 & \delta_2 & \delta_3 & \theta_4 & \theta_5 & \theta_6 & \delta_7 & \delta_8 & \delta_9 & \theta_{10} & \theta_{11} & \theta_{12} & \\
\hline
\frac{EA}{l} & & & & & & & & & & & & \delta_1 \\
0 & \frac{12EI_z}{l^3(1+\Phi_Y)} & & & & & & & & & & & \delta_2 \\
0 & 0 & \frac{12EI_Y}{l^3(1+\Phi_z)} & & & & & & & & & & \delta_3 \\
0 & 0 & 0 & \frac{GJ}{l} & & & & & \text{(Simétrica)} & & & & \theta_4 \\
0 & 0 & \frac{-6EI_Y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_Y}{l(1+\Phi_z)} & & & & & & & & \theta_5 \\
0 & \frac{6EI_z}{l^2(1+\Phi_Y)} & 0 & 0 & 0 & \frac{(4+\Phi_Y)EI_z}{l(1+\Phi_Y)} & & & & & & & \theta_6 \\
\frac{-EA}{l} & 0 & 0 & 0 & 0 & 0 & \frac{AE}{l} & & & & & & \delta_7 \\
0 & \frac{-12EI_z}{l^3(1+\Phi_Y)} & 0 & 0 & 0 & \frac{-6EI_z}{l^2(1+\Phi_Y)} & 0 & \frac{12EI_z}{l^3(1+\Phi_Y)} & & & & & \delta_8 \\
0 & 0 & \frac{-12EI_Y}{l^3(1+\Phi_z)} & 0 & \frac{6EI_Y}{l^2(1+\Phi_z)} & 0 & 0 & 0 & \frac{12EI_Y}{l^3(1+\Phi_z)} & & & & \delta_9 \\
0 & 0 & 0 & \frac{-GJ}{l} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{l} & & & \theta_{10} \\
0 & 0 & \frac{-6EI_Y}{l^2(1+\Phi_z)} & 0 & \frac{(2-\Phi_z)EI_Y}{l(1+\Phi_z)} & 0 & 0 & 0 & \frac{6EI_Y}{l^2(1+\Phi_z)} & 0 & \frac{(4+\Phi_z)EI_Y}{l(1+\Phi_z)} & & \theta_{11} \\
0 & \frac{6EI_z}{l^2(1+\Phi_Y)} & 0 & 0 & 0 & \frac{(2-\Phi_Y)EI_z}{l(1+\Phi_Y)} & 0 & \frac{-6EI_z}{l^2(1+\Phi_Y)} & 0 & 0 & 0 & \frac{(4+\Phi_Y)EI_z}{l(1+\Phi_Y)} & \theta_{12}
\end{array}$$

Para problemas Bi-dimensionales, el elemento viga j-k se reduce a seis fuerzas y momentos nodales y seis desplazamientos y rotaciones nodales. Utilizando

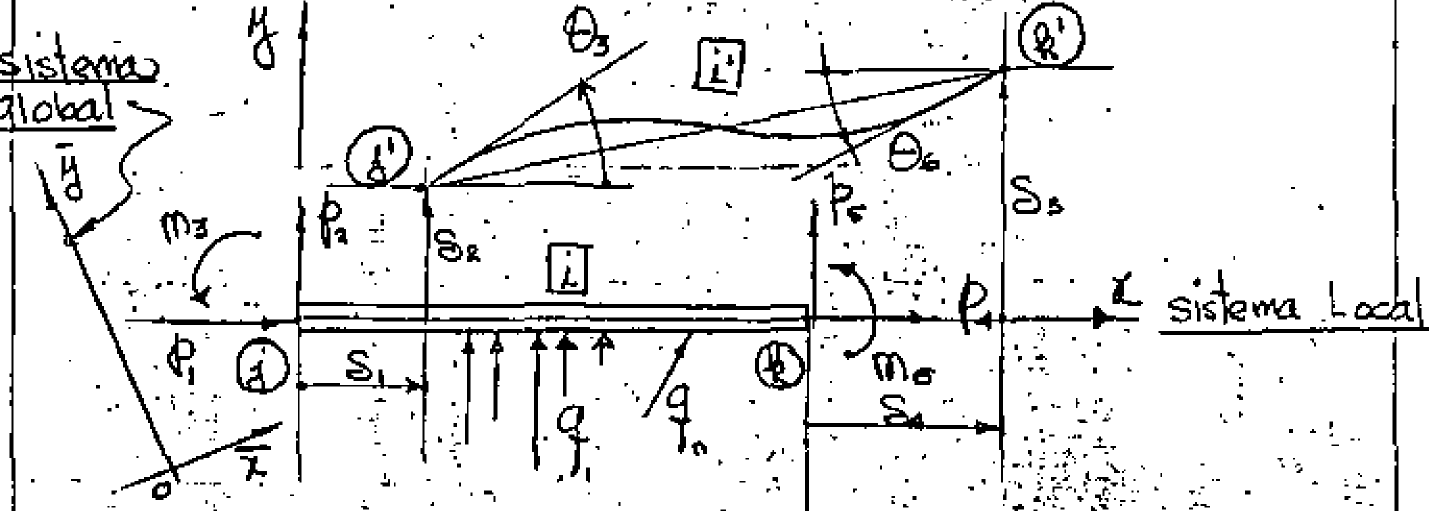


Fig. 2.2 Elemento viga para estructuras bidimensionales

la nomenclatura de la Fig. 2.2 (2.1) queda en

$$\begin{Bmatrix} P_1\\ P_2\\ M_3\\ P_4\\ P_5\\ M_6 \end{Bmatrix}_i = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16}\\ k_{21} & & & & & k_{26}\\ & & & & & \\ & & & & & \\ & & & & & \\ k_{61} & & & & & k_{66} \end{bmatrix}_i \begin{Bmatrix} \delta_1\\ \delta_2\\ \theta_3\\ \delta_4\\ \delta_5\\ \theta_6 \end{Bmatrix}_i + \begin{Bmatrix} P_1\\ P_2\\ \mu_3\\ P_4\\ P_5\\ \mu_6 \end{Bmatrix}_i \qquad (2.6)$$

o sea:

$$\{P\}_i = [k_{ij}]_i \{\delta\}_i + \{\mu\}_i \qquad (2.7)$$

De los resultados discutidos previamente la matriz de rigidez de la barra i figura 2.2 queda

$$
[k_{ij}] = \begin{array}{cccccc|c}
\delta_1 & \delta_2 & \theta_3 & \delta_4 & \delta_5 & \theta_6 & \\
\frac{EA}{l} & & & & & & \delta_1 \\
0 & \frac{12EI_z}{l^3(1+\phi_y)} & & & & & \delta_2 \\
0 & \frac{6EI_z}{l^2(1+\phi_y)} & \frac{(4+\phi_z)EI_z}{l(1+\phi_y)} & & & & \theta_3 \\
-\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & & & \delta_4 \\
0 & \frac{-12EI_z}{l^3(1+\phi_y)} & \frac{-6EI_z}{l^2(1+\phi_y)} & 0 & \frac{12EI_z}{l^3(1+\phi_y)} & & \delta_5 \\
0 & \frac{6EI_z}{l^2(1+\phi_y)} & \frac{(2-\phi_y)EI_z}{l(1+\phi_y)} & 0 & \frac{-6EI_z}{l^2(1+\phi_y)} & \frac{(4+\phi_y)EI_z}{l(1+\phi_y)} & \theta_6
\end{array} \qquad (2.8)
$$

Si las deformaciones por cortante son despreciables esto es, $\phi_y=0$, la matriz de rigidez (2.8) se simplifica a

$$
[k_{ij}] = \frac{EI_z}{l^3} \begin{bmatrix}
\frac{Al^2}{I_z} & & & & & \\
0 & 12 & & & & \\
0 & 6l & 4l^2 & & & \\
-\frac{Al^2}{I_z} & 0 & 0 & \frac{Al^2}{I_z} & & \\
0 & -12 & -6l & 0 & 12 & \\
0 & 6l & 2l^2 & 0 & -6l & 4l^2
\end{bmatrix} \qquad (2.9)
$$

La ecuación matricial relacionando los desplazamientos entre el sistema coordenado local y el global. Puede facilmente demostrarse para el elemento viga mostrado en Fig.2.1 es de la forma

$$\underbrace{\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \end{Bmatrix}}_{\{\delta\}} = \underbrace{\begin{bmatrix} \bar\lambda_{ox} & & & \\ \bar\lambda_{oY} & 0 & 0 & 0 \\ \bar\lambda_{oz} & & & \\ & \bar\lambda_{ox} & & \\ 0 & \bar\lambda_{oY} & 0 & 0 \\ & \lambda_{oz} & & \\ & & \bar\lambda_{ox} & \\ 0 & 0 & \lambda_{oY} & 0 \\ & & \lambda_{oz} & \\ & & & \bar\lambda_{ox} \\ 0 & 0 & 0 & \bar\lambda_{oY} \\ & & & \bar\lambda_{oz} \end{bmatrix}}_{[\lambda]} \underbrace{\begin{Bmatrix} \bar\delta_1 \\ \bar\delta_2 \\ \bar\delta_3 \\ \bar\theta_4 \\ \bar\theta_5 \\ \bar\theta_6 \\ \bar\delta_7 \\ \bar\delta_8 \\ \bar\delta_9 \\ \bar\theta_{10} \\ \bar\theta_{11} \\ \bar\theta_{12} \end{Bmatrix}}_{\{\bar\delta\}} \qquad (2.10)$$

o sea $\{\delta\} = [\lambda]\{\bar\delta\}$ (2.11)

donde

$$\bar\lambda_{ox} = [l_{ox} \;\; m_{ox} \;\; n_{ox}]$$

$$\bar\lambda_{oY} = [l_{oY} \;\; m_{oY} \;\; n_{oY}] \qquad (2.12)$$

$$\bar\lambda_{oz} = [l_{oz} \;\; m_{oz} \;\; n_{oz}]$$

representa las matrices de los cosenos directores para las direcciones ox, oy y oz, respectivamente, referidas al sistema global $\bar{x}$, $\bar{y}$ y $\bar{z}$, y $\{\bar{\delta}\}$ representa los desplazamientos de la barra $\boxed{i}$ respecto al sistema global.

Para problemas bidimensionales la matriz de transformacion $[\lambda]$ se reduce a

$$[\lambda] = \begin{bmatrix} l_{ox} & m_{ox} & 0 & 0 & 0 & 0 \\ l_{oy} & m_{oy} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ox} & m_{ox} & 0 \\ 0 & 0 & 0 & l_{oy} & m_{oy} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.13)$$

El analisis de marcos tridimensionales se puede describir por las mismas ecuaciones básicas usadas en la descripción del analisis de estructuras planas. Considerando el sistema total, el equilibrio estático nodal es definido por la ecuación matricial

$$[S_c]\{\delta_c\} + \{\mu_c\} = \{R_c\} \qquad (2.14)$$

donde:

$[S_c]$ = Matriz de rigidez completa de la estructura

$[\delta_c]$ = vector de desplazamientos nodales completo.

$\{\mu_c\}$ = Vector de cargas nodales completo.

$\{R\}$ vector de reacciones de la estructura

y de (2.14) se obtiene la ecuación

$$[S_{uu}]\{\delta_u\}+\{\mu_u\}=0 \qquad (2.15)$$

de donde se obtiene $\{\delta_i\}$ y $\{\delta_c\}$, el que substituyéndolo en (2.14) y (2.1) se obtiene $\{R_c\}$ y $\{P\}_i$ como

$$\{R_c\}=-[S_c][S_{uu}]^{-1}\{\mu_u\} \qquad (2.16)$$

$$\{P\}_i=[k_{ij}]_i[S_{uu}]^{-1}\{\mu_u\}+\{\mu\}_i \quad (i=1,2,\ldots,n) \qquad (2.17)$$

Ejemplo: En el sistema estructual de la Fig. 2.3, determine las reacciones nodales $\{P\}_i$ en los extremos de cada miembro y las reacciones originadas por las cargas indicadas.

La estructura tiene miembros prismáticos con las siguientes propiedades:

$$EI_Y = EI_z = EI$$
$$GI_x = \frac{EI}{4} \qquad (2.18)$$
$$EA_x = \frac{EI}{4}$$

la estructura es flexible y se puede considerar la $(\Phi_Y = \Phi_z)$ deformación por cortante despreciable

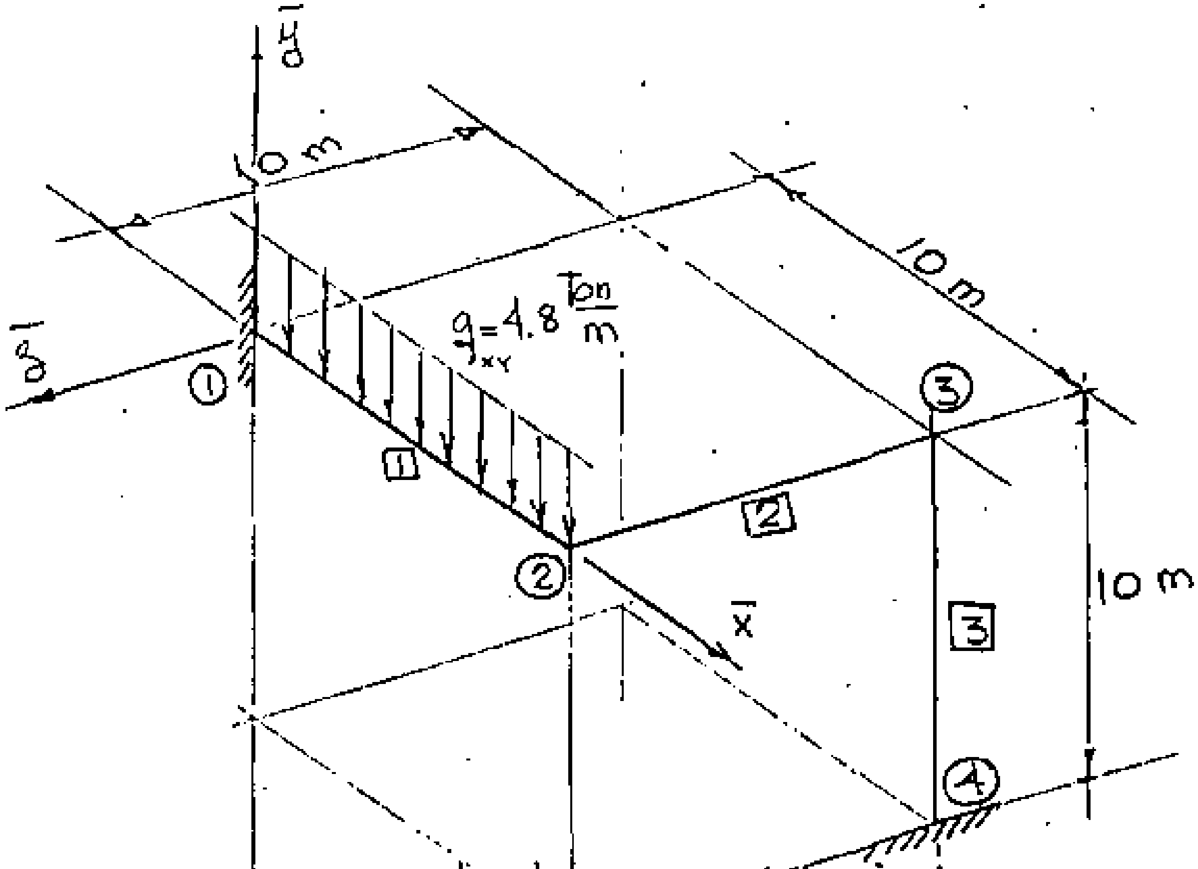


Fig 2.3 Estructura espacial rigida

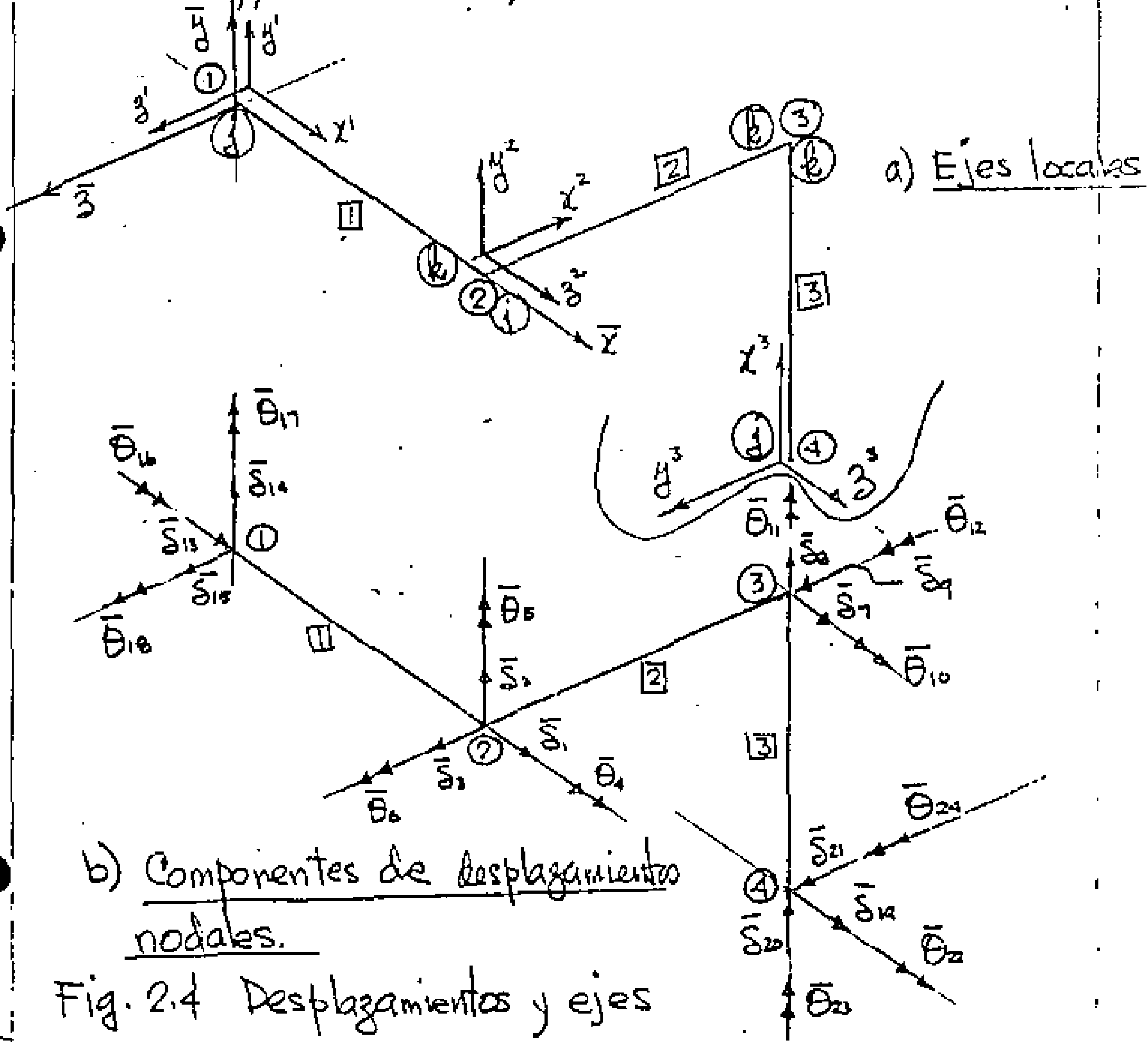
Las tablas 2.1 y 2.2 dan la información requerida para cada nodo y miembro.

Nodo	$\bar{x}$	$\bar{y}$	$\bar{z}$
1	0	0	0
2	10.0	0	0
3	10.0	0	-10.00
4	10.0	-10.00	-10.00

Tabla 2.1 coordenadas nodales en metros.

Barra	Longitud (m)	Nodo		Cosenos directores			TIPO DE TRANSFORMACION	Angulo φ
		j	k	l_{ok}	m_{ok}	n_{ok}		
1	10.0	1	2	+1	0	0	y-z-x	0
2	10.0	2	3	0	0	-1	y-z-x	0
3	10.0	4	3	0	+1	0	z-y-x	90°

Tabla 2.2. longitudes, cosenos directores y Tipos de Transformación.



a) Ejes locales

b) Componentes de desplazamientos nodales.

Fig. 2.4 Desplazamientos y ejes

vector columna de desplazamientos nodales $\{\delta_c\}$

$$\{\delta_c\} = \left\{ \begin{array}{c} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \bar{\delta}_3 \\ \bar{\theta}_4 \\ \bar{\theta}_5 \\ \bar{\theta}_6 \\ \bar{\delta}_7 \\ \bar{\delta}_8 \\ \bar{\delta}_9 \\ \bar{\theta}_{10} \\ \bar{\theta}_{11} \\ \bar{\theta}_{12} \\ \hline \bar{\delta}_{13} \\ \bar{\delta}_{14} \\ \bar{\delta}_{15} \\ \bar{\theta}_{16} \\ \bar{\theta}_{17} \\ \bar{\theta}_{18} \\ \bar{\delta}_{19} \\ \bar{\delta}_{20} \\ \bar{\delta}_{21} \\ \bar{\theta}_{22} \\ \bar{\theta}_{23} \\ \bar{\theta}_{24} \end{array} \right\} = \left\{ \begin{array}{c} \{\delta_u\} \\ \hline \{\delta_r\} \end{array} \right\} \qquad (2.19)$$

Matriz de rigidez de cada miembro

Para cada elemento viga, la matriz de rigidez se establece por medio de (2.1) con respecto a los ejes locales; la matriz de transformación se puede establecer por medio de la expresión (2.10); y la matriz de rigidez de miembro transformada, $[\bar{k}_{ij}]_i$ respecto al sistema global se obtiene de

$$[\bar{k}_{ij}]_i = [\lambda]_i^T\,[k_{ij}]_i\,[\lambda]_i \qquad (2.20)$$

Miembro $\boxed{1}$

$$[\lambda]_1 = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} = [I]\ ;\ [\bar{k}_{ij}]_1 = [I]^T[k_{ij}]_1[I] = [k_{ij}]_1 \qquad (2.21)$$

$$]_1 \quad EI \begin{array}{cccccccccccc|c} 13&14&15&16&17&18&1&2&3&4&5&6& \\ \hline .025&0&0&0&0&0&-.025&0&0&0&0&0&13\\ 0&.012&0&0&0&.060&0&-.012&0&0&0&.060&14\\ 0&0&.012&0&-.060&0&0&0&-.012&0&-.060&0&15\\ 0&0&0&.025&0&0&0&0&0&-.025&0&0&16\\ 0&0&-.06&0&0.4&0&0&0&.06&0&0.2&0&17\\ 0&.06&0&0&0&0.4&0&-.06&0&0&0&0.2&18\\ -.025&0&0&0&0&0&.025&0&0&0&0&0&1\\ 0&-.012&0&0&0&-.06&0&.012&0&0&0&-.06&2\\ 0&0&-.012&0&.06&0&0&0&.012&0&.06&0&3\\ 0&0&0&-.025&0&0&0&0&0&.025&0&0&4\\ 0&0&-.06&0&0.2&0&0&0&.06&0&.4&0&5\\ 0&.06&0&0&0&0.2&0&-.06&0&0&0&.4&6 \end{array} \qquad (2.22)$$

Miembro $\boxed{2}$ De (2.5) se obtiene:

$$[k_{ij}]_2 = EI \begin{bmatrix} .025 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 \\ 0 & .012 & 0 & 0 & 0 & .06 & 0 & -.012 & 0 & 0 & 0 & .06 \\ 0 & 0 & .012 & 0 & -.06 & 0 & 0 & 0 & -.012 & 0 & -.06 & 0 \\ 0 & 0 & 0 & .025 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 \\ 0 & 0 & -.06 & 0 & 0.4 & 0 & 0 & 0 & .06 & 0 & 0.2 & 0 \\ 0 & .06 & 0 & 0 & 0 & 0.4 & 0 & -.06 & 0 & 0 & 0 & 0.2 \\ -.025 & 0 & 0 & 0 & 0 & 0 & .025 & 0 & 0 & 0 & 0 & 0 \\ 0 & -.012 & 0 & 0 & 0 & -.06 & 0 & .012 & 0 & 0 & 0 & -.06 \\ 0 & 0 & -.012 & 0 & .06 & 0 & 0 & 0 & .012 & 0 & .06 & 0 \\ 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .025 & 0 & 0 \\ 0 & 0 & -.06 & 0 & 0.2 & 0 & 0 & 0 & .06 & 0 & .4 & 0 \\ 0 & .06 & 0 & 0 & 0 & 0.2 & 0 & -.06 & 0 & 0 & 0 & .4 \end{bmatrix} \qquad (2.23)$$

De (2.12); $\bar{\lambda}_{ox_2} = [0\ 0\ -1]_2$, $\bar{\lambda}_{oy_2} = [0\ 1\ 0]_2$, $\bar{\lambda}_{oz_2} = [1\ 0\ 0]_2$ (2.12)a

Subst. (2.12)$_a$ en (2.10) se obtiene

$$[\lambda]_2 = \begin{bmatrix} \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} & & \\ & & \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} & \\ 0 & & & \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \end{bmatrix} \qquad (2.24)$$

Subst (2.24) y (2.23) en (2.20) se obtiene

$$[\bar{k}_{ij}]_2 = EI \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} .012 & 0 & 0 & 0 & -.06 & 0 & -.012 & 0 & 0 & 0 & -.06 & 0 \\ 0 & .012 & 0 & .06 & 0 & 0 & 0 & -.012 & 0 & .06 & 0 & 0 \\ 0 & 0 & .025 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 \\ 0 & .06 & 0 & .4 & 0 & 0 & 0 & -.06 & 0 & 0.2 & 0 & 0 \\ -.06 & 0 & 0 & 0 & 0.4 & 0 & .06 & 0 & 0 & 0 & .2 & 0 \\ 0 & 0 & 0 & 0 & 0 & .025 & 0 & 0 & 0 & 0 & 0 & -.025 \\ -.012 & 0 & 0 & 0 & .06 & 0 & .012 & 0 & 0 & 0 & .06 & 0 \\ 0 & -.012 & 0 & -.06 & 0 & 0 & 0 & .012 & 0 & -.06 & 0 & 0 \\ 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .025 & 0 & 0 & 0 \\ 0 & .06 & 0 & 0.2 & 0 & 0 & 0 & -.06 & 0 & .4 & 0 & 0 \\ -.06 & 0 & 0 & 0 & 0.2 & 0 & .06 & 0 & 0 & 0 & .4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .025 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array} \qquad (2.25)$$

Miembro $\boxed{3}$, De (2.5) se obtiene la matriz de rigidez la cual resulta igual a la de los miembro $\boxed{1}$ y $\boxed{2}$

$$[k_{ij}]_3 = [k_{ij}]_2 = [k_{ij}]_1 \qquad (2.26)$$

De (2.12) se obtiene

$$\bar{\lambda}_{ox_3} = [0\ 1\ 0]_3,\ \bar{\lambda}_{oy_3} = [0\ 0\ 1]_3,\ \bar{\lambda}_{oz_3} = [1\ 0\ 0]_3 \qquad (2.27)$$

De (2.27) y (2.10) se obtiene

$$[\lambda]_3 = \begin{bmatrix} \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & & & 0 \\ & \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & & \\ & & \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & \\ 0 & & & \begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} \end{bmatrix} \qquad (2.28)$$

De (2.20) (2.26) y (2.78) se obtiene

$$[k_{ij}]_3 = EI \begin{array}{cccccccccccc|c} 19 & 20 & 21 & 22 & 23 & 24 & 7 & 8 & 9 & 10 & 11 & 12 & \\ \hline .012 & 0 & 0 & 0 & 0 & -.06 & -.012 & 0 & 0 & 0 & 0 & -.06 & 19 \\ 0 & .025 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & .012 & .06 & 0 & 0 & 0 & 0 & -.012 & .06 & 0 & 0 & 21 \\ 0 & 0 & .06 & .4 & 0 & 0 & 0 & 0 & -.06 & .2 & 0 & 0 & 22 \\ 0 & 0 & 0 & 0 & .025 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 23 \\ -.06 & 0 & 0 & 0 & 0 & .4 & .06 & 0 & 0 & 0 & 0 & .2 & 24 \\ -.012 & 0 & 0 & 0 & 0 & .06 & .012 & 0 & 0 & 0 & 0 & .06 & 7 \\ 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .025 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & -.012 & -.06 & 0 & 0 & 0 & 0 & .012 & -.06 & 0 & 0 & 9 \\ 0 & 0 & .06 & .2 & 0 & 0 & 0 & 0 & -.06 & .4 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .025 & 0 & 11 \\ -.06 & 0 & 0 & 0 & 0 & .2 & .06 & 0 & 0 & 0 & 0 & .4 & 12 \end{array} \qquad (2.29)$$

Matriz de rigidez de la estructura.

La matriz completa de la estructura $[S_c]$ se obtiene sumando los coeficientes de rigidez de miembro dados en las expresiones (2.22), (2.25) y (2.29) con respecto a la identificación de subíndices de los elementos se obtiene

$$[S_c] = EI \left[\begin{array}{cccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\
.037 & 0 & 0 & 0 & -.06 & 0 & -.012 & 0 & 0 & 0 & -.06 & 0 & 1 \\
0 & .024 & 0 & .06 & 0 & -.06 & 0 & -.012 & 0 & .06 & 0 & 0 & 2 \\
0 & 0 & .037 & 0 & .06 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 3 \\
0 & .06 & 0 & .425 & 0 & 0 & 0 & -.06 & 0 & 0.2 & 0 & 0 & 4 \\
-.06 & 0 & .06 & 0 & 0.8 & 0 & .06 & 0 & 0 & 0 & 0.2 & 0 & 5 \\
0 & -.06 & 0 & 0 & 0 & .425 & 0 & 0 & 0 & 0 & 0 & -.025 & 6 \\
-.012 & 0 & 0 & 0 & .06 & 0 & .024 & 0 & 0 & 0 & .06 & .06 & 7 \\
0 & -.012 & 0 & -.06 & 0 & 0 & 0 & .037 & 0 & -.06 & 0 & 0 & 8 \\
0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & .037 & -.06 & 0 & 0 & 9 \\
0 & .06 & 0 & 0.2 & 0 & 0 & 0 & -.06 & -.06 & .8 & 0 & 0 & 10 \\
-.06 & 0 & 0 & 0 & 0.2 & 0 & .06 & 0 & 0 & 0 & .425 & 0 & 11 \\
0 & 0 & 0 & 0 & 0 & -.025 & .06 & 0 & 0 & 0 & 0 & .425 & 12 \\
\hline
-.025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 \\
0 & -.012 & 0 & 0 & 0 & .06 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\
0 & 0 & -.012 & 0 & -.06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15 \\
0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \\
0 & 0 & .06 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 17 \\
0 & -.06 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & -.012 & 0 & 0 & 0 & 0 & -.06 & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 0 & 0 & 0 & 20 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.012 & .06 & .0 & 0 & 21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.06 & .2 & 0 & 0 & 22 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -.025 & 0 & 23 \\
0 & 0 & 0 & 0 & 0 & 0 & .06 & 0 & 0 & 0 & 0 & .2 & 24
\end{array}\right] = \left[\begin{array}{c} [S_{uu}] \\ \hline [S_{ru}] \end{array}\right] \qquad (2.30)$$

De (2.30) se obtiene $[S_{uu}]^{-1}$

$$
[S_{ij}]^{-1} = \frac{1}{EI}
\left[
\begin{array}{cccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \\
\hline
38.396 & 1.266 & -6.236 & 0.001 & 1.750 & 0.085 & 11.279 & -0.403 & -5.028 & -0.503 & 3.005 & -1.578 & 1 \\
1.266 & 210.745 & -43.160 & -21.908 & 5.487 & 30.182 & -39.151 & 11.279 & -50.707 & -13.286 & 3.124 & 7.303 & 2 \\
-6.236 & -43.160 & 102.028 & 2.421 & -11.235 & -6.537 & 50.707 & 5.028 & 84.038 & 9.312 & -2.752 & -7.543 & 3 \\
0.001 & -21.908 & 2.421 & 5.546 & -0.346 & -3.130 & 3.124 & 3.005 & 2.752 & 0.688 & -0.278 & -0.625 & 4 \\
1.750 & 5.487 & -11.235 & -0.346 & 3.048 & 0.888 & -13.286 & -0.503 & -9.312 & -1.061 & 0.688 & 1.928 & 5 \\
0.085 & 30.182 & -6.537 & -3.130 & 0.888 & 6.698 & -7.303 & 1.587 & -7.543 & -1.928 & 0.625 & 1.425 & 6 \\
11.279 & -39.151 & 50.707 & 3.124 & -13.286 & -7.303 & 210.745 & 1.266 & 43.160 & 5.487 & -21.908 & -30.182 & 7 \\
-0.403 & 11.279 & 5.028 & 3.005 & -0.503 & 1.587 & 1.266 & 38.396 & 6.236 & 1.757 & 0.001 & -0.085 & 8 \\
-5.028 & -50.707 & 84.038 & 2.752 & -9.312 & -7.543 & 43.160 & 6.236 & 102.028 & 11.235 & -2.421 & -6.537 & 9 \\
-0.503 & -13.286 & 9.312 & 0.688 & -1.061 & -1.928 & 5.487 & 1.750 & 11.235 & 3.048 & -0.346 & -0.888 & 10 \\
3.005 & 3.124 & -2.752 & -0.278 & 0.688 & 0.625 & -21.908 & 0.001 & -2.421 & -0.346 & 5.546 & 3.130 & 11 \\
-1.587 & 7.303 & -7.543 & -0.625 & 1.928 & 1.425 & -30.182 & -0.085 & -6.537 & -0.888 & 3.130 & 6.698 & 12
\end{array}
\right]
$$

Vector de momentos y reacciones fijas miembro $\boxed{1}$

$$\{\mu\}_1 = \begin{Bmatrix} P_{13} \\ P_{14} \\ P_{15} \\ \mu_{16} \\ \mu_{17} \\ \mu_{18} \\ P_1 \\ P_2 \\ P_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ 40 \\ 0 \\ 24 \\ 0 \\ 0 \\ 0 \\ -40 \end{Bmatrix}_1 = \{\bar{\mu}\}_1 \qquad (2.32)$$

$$\{\bar{\mu}\}_1 = [\lambda]_1^T \{\mu\}_1 \qquad (2.33)$$

$$\{\mu\}_2 = 0 \quad ; \{\bar{\mu}\}_2 = 0$$

$$\{\mu\}_3 = 0 \quad ; \{\bar{\mu}\}_3 = 0$$

Habiendo definido las cargas nodales en terminos de las acciones fijas en los extremos con respecto a los ejes de referencia, se deduce el vector de cargas nodales completo $\{\mu_c\}$, como.

$$\{\mu_c\} = \left\{ \begin{array}{c} \{\mu_{ij}\} \\ \hdashline \{\mu_r\} \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -24 \\ 0 \\ 0 \\ 0 \\ 40.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ -24 \\ 0 \\ 0 \\ 0 \\ -40.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \end{array} \qquad (2.34)$$

Etiqueta de grados de libertad

Substituyendo (2.31) y (2.34) en (2.15) se obtiene

$$\{\bar{\delta}_u\} = [S_{uu}]^{-1}\{\mathcal{U}_u\} \qquad (2.35)$$

$$\{\bar{\delta}_u\} = \begin{Bmatrix} \bar{\delta}_1 \\ \bar{\delta}_2 \\ \bar{\delta}_3 \\ \bar{\theta}_4 \\ \bar{\theta}_5 \\ \bar{\theta}_6 \\ \bar{\delta}_7 \\ \bar{\delta}_8 \\ \bar{\delta}_9 \\ \bar{\theta}_{10} \\ \bar{\theta}_{11} \\ \bar{\theta}_{12} \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} -26.984 \\ -3850.6 \\ 774.36 \\ 400.592 \\ -96.168 \\ -456.448 \\ 647.504 \\ -207.216 \\ 915.248 \\ 241.744 \\ -49.976 \\ -118.272 \end{Bmatrix} \qquad (2.36)$$

Los valores de los desplazamientos dados por (2.36) con respecto al sistema global son valores relativos, para obtener los valores se substituye E en ton/m² e I en m⁴ en (2.36) y se obtiene δ_i en metros y θ en radianes.

Acciones Finales en los extremos.

Habiendo evaluado las componentes de los desplazamiento nodales con respecto al sistema global de referencia por medio de (2.10) se evaluan con respecto a las coordenadas locales de cada barra y las acciones finales para cada miembro de la estructura se calculan de (2.1)

$$\{p\}_i = [k_{ij}][\lambda]_i\{\bar{S}\}_i + \{\mu\}_i \tag{2.37}$$

De la Fig. 2.4 se tiene para el ...... miembro III

$$\{\bar{\delta}\}_1 = \begin{Bmatrix} \bar{\delta}_{13} \\ \bar{\delta}_{14} \\ \bar{\delta}_{15} \\ \bar{\theta}_{16} \\ \bar{\theta}_{17} \\ \bar{\theta}_{18} \\ \bar{\delta}_{1} \\ \bar{\delta}_{2} \\ \bar{\delta}_{3} \\ \bar{\theta}_{4} \\ \bar{\theta}_{5} \\ \bar{\theta}_{6} \end{Bmatrix}_1 = \frac{1}{EI}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -26.984 \\ -3850.6 \\ 774.36 \\ 400.592 \\ -96.168 \\ -456.448 \end{Bmatrix}_1 \tag{2.38}$$

De (2.21), (2.38), (2.1) y (2.5) se obtiene

$$\{p\}_1 = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ m_4 \\ m_5 \\ m_6 \\ p_7 \\ p_8 \\ p_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{Bmatrix} = \begin{Bmatrix} 0.7 \text{ Ton} \\ 42.8 \text{ Ton} \\ -3.5 \text{ Ton} \\ -10.0 \text{ Ton-m} \\ 27.2 \text{ Ton-m} \\ 179.7 \text{ Ton-m} \\ -0.7 \text{ Ton} \\ 5.2 \text{ Ton} \\ 3.5 \text{ Ton} \\ 10.0 \text{ Ton-m} \\ 8.0 \text{ Ton-m} \\ 8.5 \text{ Ton-m} \end{Bmatrix} \qquad (2.39)$$

(índices según convención Fig. 2.4)

Miembro $\boxed{2}$ $\{\bar{\delta}\}_2 = \{\delta_u\} = [\lambda]_2 \{\bar{\delta}_u\}$ y $\{\mu\}_2 = \{0\}$

De [illegible], (2.24) (2.25), (2.1) y (2.5) se obtiene

$$\{p\}_2 = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ m_4 \\ m_5 \\ m_6 \\ p_7 \\ p_8 \\ p_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{Bmatrix} = \begin{Bmatrix} 3.5 \text{ Ton} \\ -5.2 \text{ ''} \\ 0.7 \text{ ''} \\ 8.5 \text{ Ton-m} \\ -8.0 \text{ ''} \\ -10.0 \text{ ''} \\ -3.5 \text{ Ton} \\ 5.2 \text{ ''} \\ -0.7 \text{ ''} \\ -8.5 \text{ Ton-m} \\ 1.2 \text{ ''} \\ -41.8 \text{ ''} \end{Bmatrix} \qquad (2.40)$$

(índices según convención Fg. 2.4)

Miembro $\boxed{3}$

$$\left\{\bar{\delta}\right\}_3 = \left\{\begin{matrix} \bar{\delta}_{19} \\ \bar{\delta}_{20} \\ \bar{\delta}_{21} \\ \bar{\theta}_{22} \\ \bar{\theta}_{23} \\ \bar{\theta}_{24} \\ \bar{\delta}_{7} \\ \bar{\delta}_{8} \\ \bar{\delta}_{9} \\ \bar{\theta}_{10} \\ \bar{\theta}_{11} \\ \bar{\theta}_{12} \end{matrix}\right\}_3 = \frac{1}{EI}\left\{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 647.504 \\ -207.216 \\ 915.248 \\ 241.744 \\ -49.976 \\ -118.272 \end{matrix}\right\}_3 \tag{2.41}$$

en $\boxed{3}$ También $\{u\}_3 = 0$, De (2.28)(2.29),(2.1) y (2.5) se obtiene

$$\{p\}_3 = \left\{\begin{matrix} P_1 \\ P_2 \\ P_3 \\ m_4 \\ m_5 \\ m_6 \\ P_7 \\ P_8 \\ P_9 \\ m_{10} \\ m_{11} \\ m_{12} \end{matrix}\right\}_3 = \left\{\begin{matrix} 5.2\ \text{Ton} \\ 3.5\ '' \\ -0.7\ - \\ 1.2\ \text{Ton-m} \\ 15.2\ '' \\ -6.6\ '' \\ -5.2\ \text{Ton} \\ -3.5\ '' \\ 0.7\ '' \\ -1.2\ \text{Ton-m} \\ -8.5\ '' \\ 41.8\ '' \end{matrix}\right\}_3 \tag{2.42}$$

Reacciones.

Substituyendo las matrices apropiadas en

$$\{R\} = [S_{ru}]\{\delta_u\} - \{\mu_r\}$$

se obtiene

$$\{R\} = \left\{\begin{matrix} R_{13} \\ R_{14} \\ R_{15} \\ R_{16} \\ R_{17} \\ R_{18} \\ R_{19} \\ R_{20} \\ R_{21} \\ R_{22} \\ R_{23} \\ R_{24} \end{matrix}\right\} = \left\{\begin{matrix} 0.7 \text{ Ton} \\ 42.8 \text{ ''} \\ -3.5 \text{ ''} \\ -10.0 \text{ Ton-m} \\ 27.2 \text{ Ton-m} \\ 179.7 \text{ ''} \\ -0.7 \text{ Ton} \\ 5.2 \text{ ''} \\ 3.5 \text{ ''} \\ -6.6 \text{ Ton-m} \\ 1.2 \text{ ''} \\ 15.2 \text{ ''} \end{matrix}\right\} \qquad 2.43$$

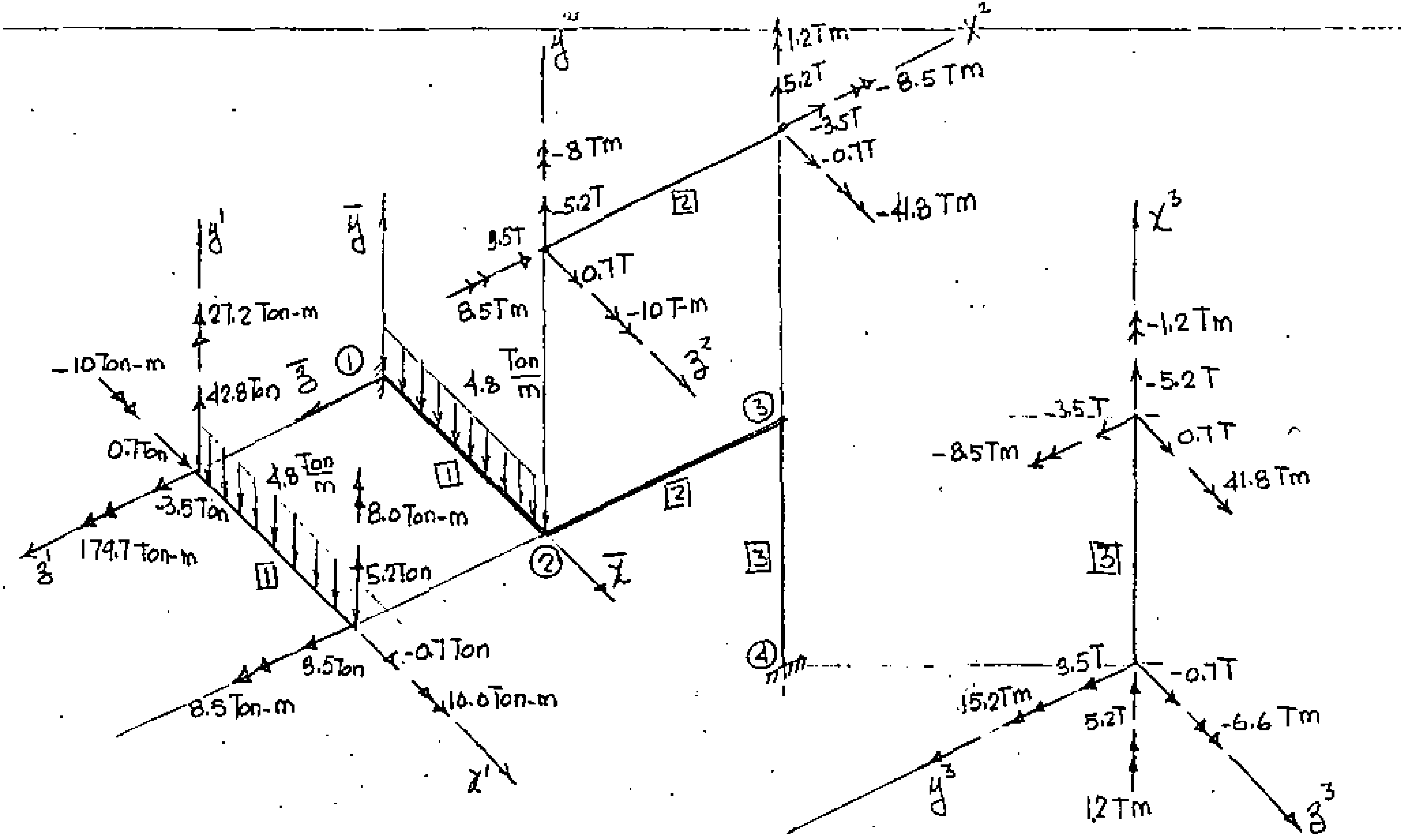


Fig. 2.5 Componentes de acciones finales $\{P_c\}$ en los extremos j k

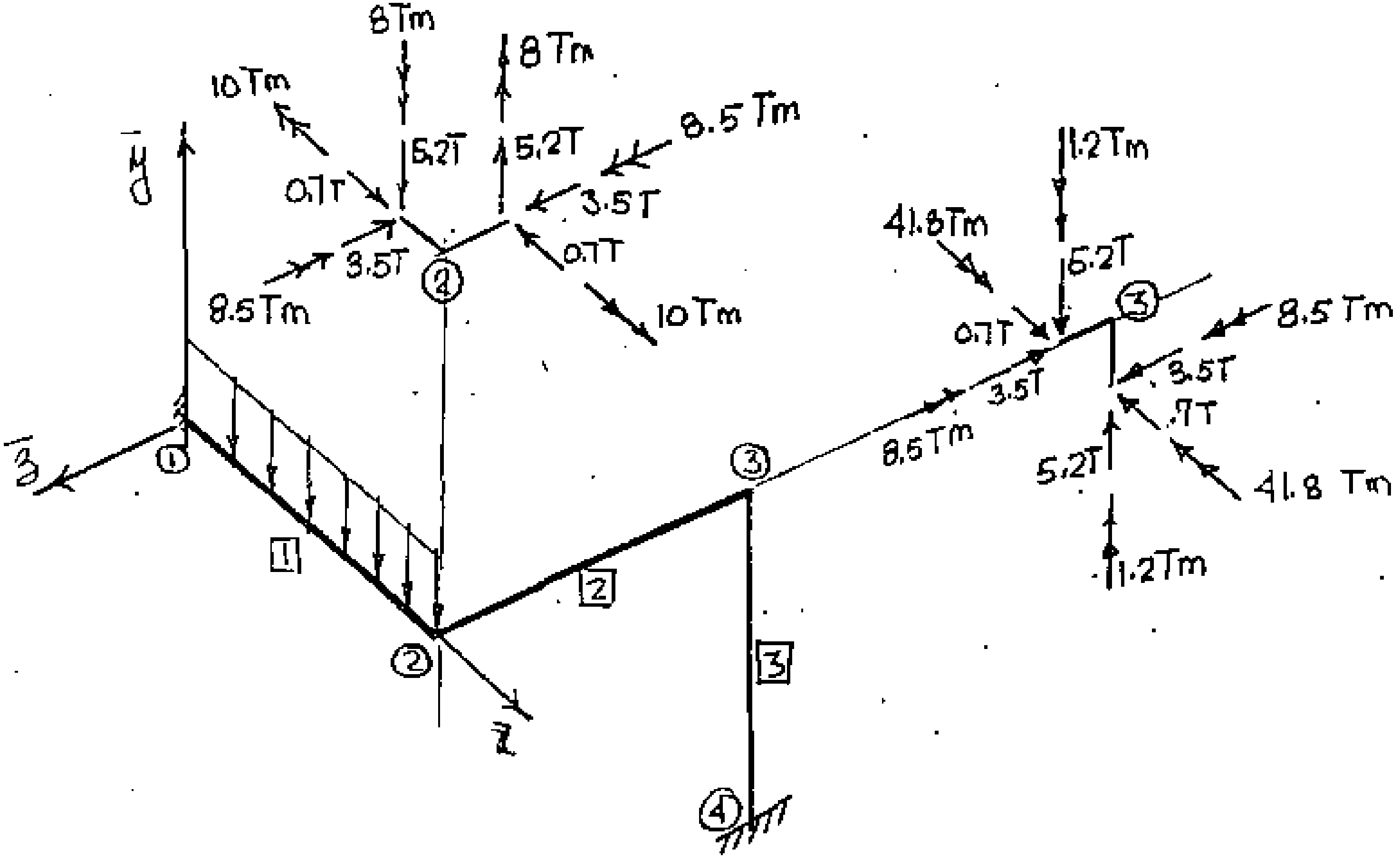


Fig.2.6 Diagrama de cuerpo libre de los nodos ② y ⑤

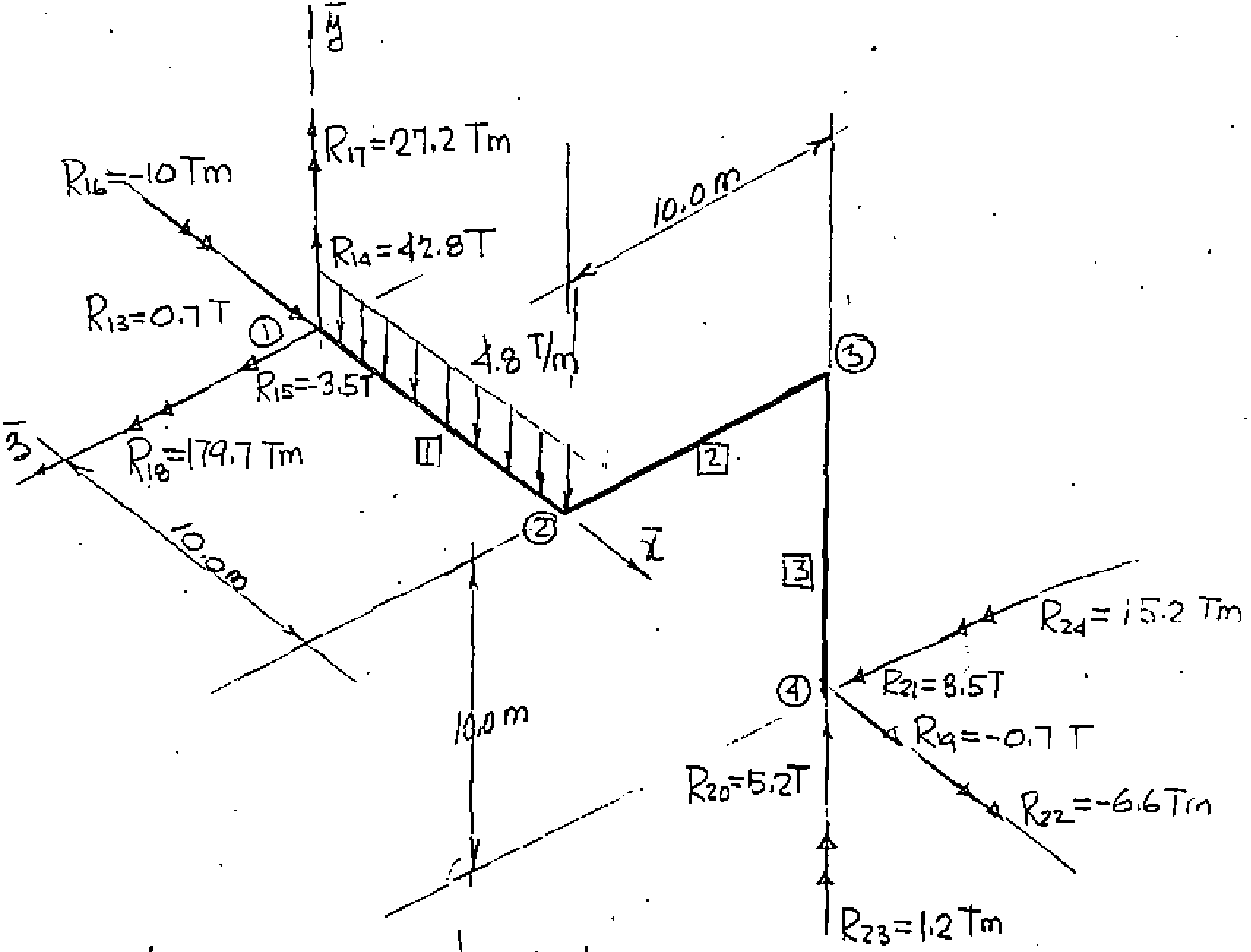


Fig. 2.7 Componentes de las reacciones en los apoyos ① y ④

METODO DE ANALISIS POR ELEMENTOS FINITOS.

INTRODUCCION.

El ingeniero en la busca de los valores numéricos adecuados para describir su proceso de diseño, se encontraba generalmente con formulaciones matemáticas difíciles. Por ejemplo, considerando el simple caso de teoría de flexión de placas, bajo las hipótesis de pequeñas deformaciones y que las secciones planas permanecen planas después de la deformación, la ecuación diferencial que gobierna el análisis para un material elástico lineal homogeneo e isotrópico es

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (1)$$

donde W es la deflexión en el punto (x, y), q es la intensidad de la carga en el punto (x, y), y $D = \frac{E h^3}{12(1-\nu^2)}$ es la rigidez flexionante de la placa la cual depende del modulo de elasticidad E, el espesor de la placa h y la relación de Poisson ν En la Fig. 1 se presenta un elemento diferencial de la placa y las acciones y reacciones sobre él. Combinando la flexión simple en dos direcciones se obtiene para los momentos y cortantes por unidad de longitud de placa lo siguiente:

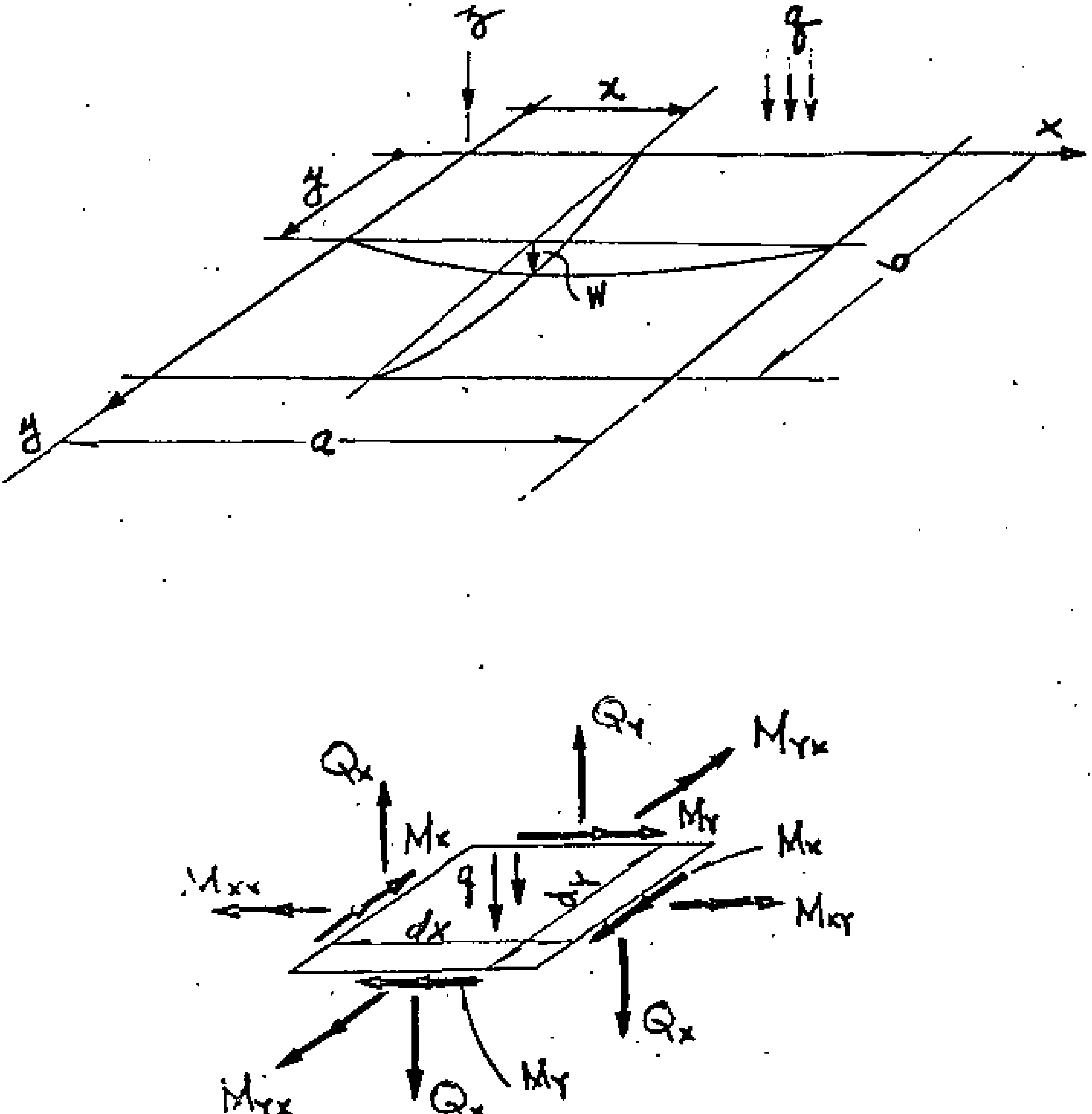


Fig. 1 Superficie media de una placa, y un elemento diferencial dx, dy.

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_Y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{XY} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \qquad (2)$$

$$Q_x = -D\frac{\partial}{\partial x}\nabla^2 w$$

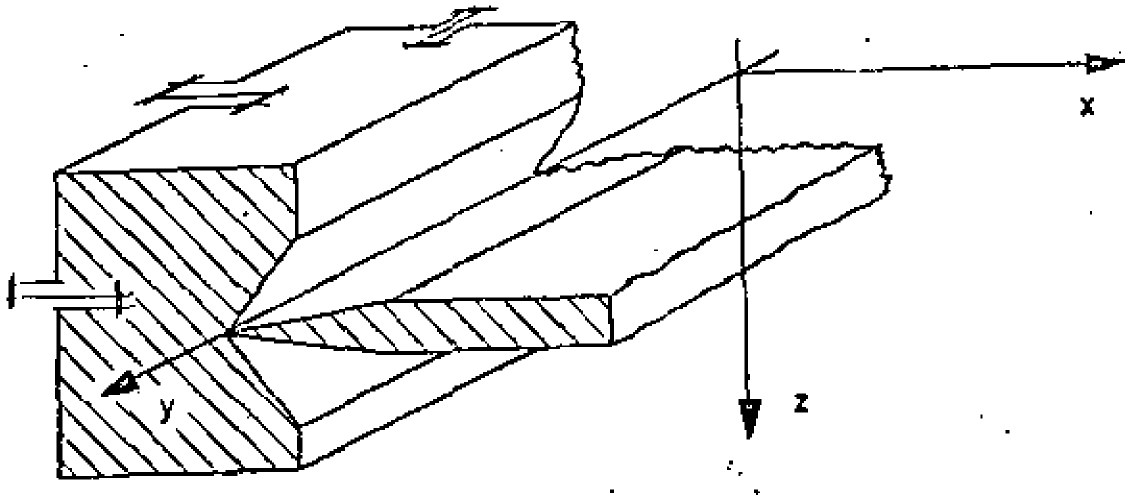
$$Q_Y = -D\frac{\partial}{\partial y}\nabla^2 w$$

donde $$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

Para el caso particular de la placa libremente apoyada, y rectangular, cuyas condiciones en la frontera (Fig. 2) son:

$$W(0,y) = 0$$

$$W_{xx}(0,y) + \nu W_{YY}(0,Y) = 0 \qquad (3)$$



Navier en 1820 presentó a la Academia Francesa de Ciencias, la solución representando la carga q (x, y), por medio de una serie trigonométrica doble

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\,\text{sen}\,\frac{m\pi}{a}x\,\text{sen}\,\frac{n\pi}{b}y \tag{4}$$

substituye (4) en (1) y considerando las propiedades de ortogonalidad de las series trigonométricas obtiene la solución de la ecuación diferencial bi-armónica (1) como

$$W=\frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\,\text{sen}\,\frac{m\pi}{a}x\,\text{sen}\,\frac{n\pi}{b}y \tag{5}$$

en donde el coeficiente Amn viene expresado por

$$a_{mn}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\,\text{sen}\,\frac{m\pi}{a}x\,\text{sen}\,\frac{n\pi}{b}y\,dx\,dy \tag{6}$$

El procedimiento de Navier consiste en lo siguiente: Conocida la función de carga q (x,y), se substituye en (6) y se obtiene el coeficiente Amn el cual nuevamente se substituye en (5) y se obtiene la deflexión W (x,y), y por medio las ecuaciones (2) se obtienen los momentos y cortantes $\{M\}$ y $\{Q\}$.
Es importante observar que las limitaciones de Navier se refieren a una placa rectangular libremente apoyada y con una función de carga q (x,y) impar con respecto a x, y con respecto a Y, es decir, $f(x)=-f(-x)$ y

Si la función fuese par, la representación de q (x,y) sería mediante una serie de cosenos, y si q (x, y) fuese una función cual_ quiera, se representaría mediante una serie trigonométrica doble completa de senos y cosenos, y se tendrían problemas en satisfacer las condiciones en la frontera. Generalmente la convergencia de la serie (5) es lenta, y en algunos casos es necesario considerar más de 500 términos para asegurar la solución correcta.

Posteriormente en 1900 M. Levy cambia de posición los ejes coordenados (Fig. 3) e utiliza una serie trigonométrica simple

$$W = \sum_{m=1}^{\infty} f_m(y) \operatorname{sen} \frac{m\pi}{a} x \qquad (7)$$

El procedimiento de Levy consiste en substituir (7) en (1) obteniendo una ecuación diferencial lineal de cuarto orden en fm(y) con coeficientes constantes no homogenea con la cual ya es posible satisfacer diferentes condiciones en la frontera $y = \pm \frac{b}{2}$, pero continua limitado a una placa rectangular libremente apoyada en las fronteras x = o y x = a.

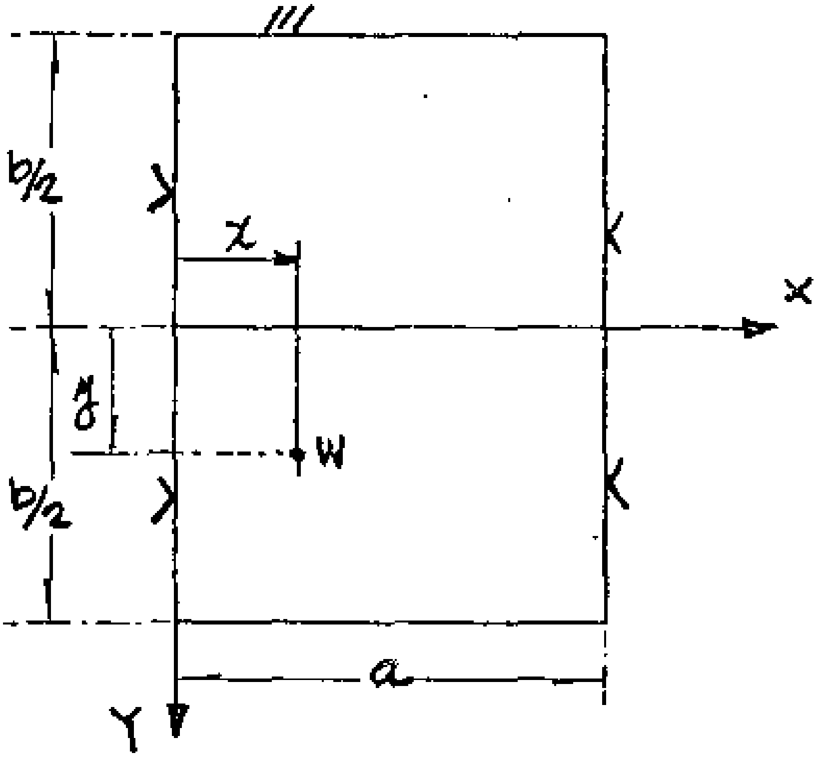


Fig. 3 Posición de ejes en solución de M. Levy.

Las limitaciones de análisis tan restringidas, como los ejemplos anteriores, aparecían en innumerables problemas de ingeniería, lo cual originó el principio de los métodos numéricos, el cual presenta dos etapas de desarrollo. Antes de la época de las computadoras, donde representa un importante papel el Prof. Southwell del Colegio Imperial de Inglaterra, desarrollando y aplicando los métodos numéricos de relajación y diferencias finitas, superando las limitaciones restringidas de los métodos analíticos de solución.

Durante la era de las computadoras digitales, el método de análisis por elementos finitos ha obtenido gran popularidad, puesto que en este procedimiento como resultado de la discretización del medio por analizar, se obtienen sistemas grandes de ecuaciones algebraicas lineales simultáneas, lo cual actualmente su solución no representa ningún problema. Por ejemplo, en el caso de análisis elástico lineal de placas, podemos tener cualquier condición de apoyo, de geometría y de cargas, prácticamente se eliminan la mayoría de las restricciones de las soluciones analíticas mencionadas, el problema más importante es verifiar adecuadamente su convergencia.

El primer trabajo referente al método se debe a Hrenikoff Ref. 1 publicado en 1941, y el segundo a McHenry publicado en 1943 en ambos trabajos (Fig. 4) se verifican soluciones de problemas de elasticidad bidemensional en estado plano de esfuerzos, discretizando el medio y buscando la analogía con la olución estructural.

Posteriormente en 1949 Newmark, en su libro de Métodos Numéricos - - Ref. 3 , presenta los métodos de Hrenikoff y McHenry. Sin embargo, el

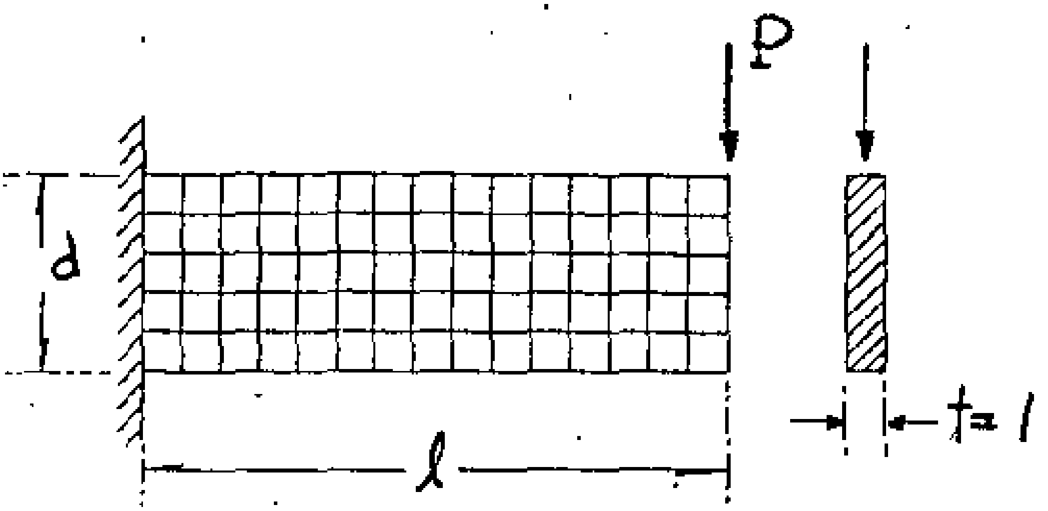


Fig. 4 Primera solución presentada por Hrenikoff en 1941.

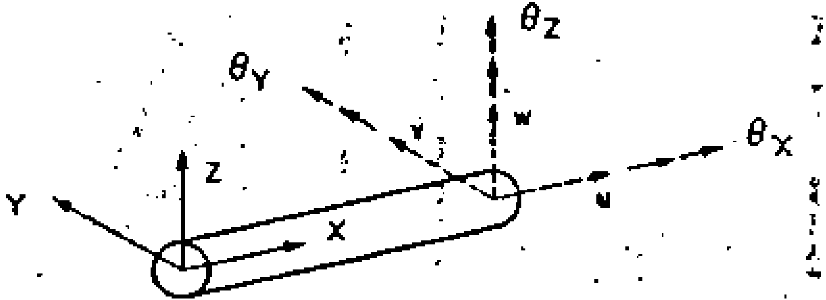
crédito de aplicarlo a medios continuos es de Turner, Clough, Martin y Topp Ref. 5 , y no es, sino hasta 1960 con Clough, Ref. 6 nace por primera vez el nombre mágico de "Elemento Finito" , derivando más correctamente las propiedades básicas del elemento triangular y el rectangular, y el hecho de que en el mismo tiempo la computadora comienza a ser una herramienta muy efectiva, conduce rápidamente a la solución numérica de problemas elástico lineales complejos, en los cuales una solución analítica no era posible.

Se inician la derivación de las propiedades de rigidez de los elementos finitos, el campo de desplazamientos en el medio se expresa en función de los desplazamientos nodales del elemento, satisfaciendo continuidad, las fuerzas internas se definen aplicando el principio del trabajo virtual, la identidad de este proceso con el de minimizar la energía potencial total, o sea, el proceso de Rayleigh-Ritz Ref. 7 es obvia. El desarrollo anterior se acentúa en el campo de la Mecánica de Sólidos y posteriormente Zienkiewicz Ref: 13 y Wilson Ref. 14 lo aplican en Mecánica de fluídos y en problemas de análisis de conducción de calor.

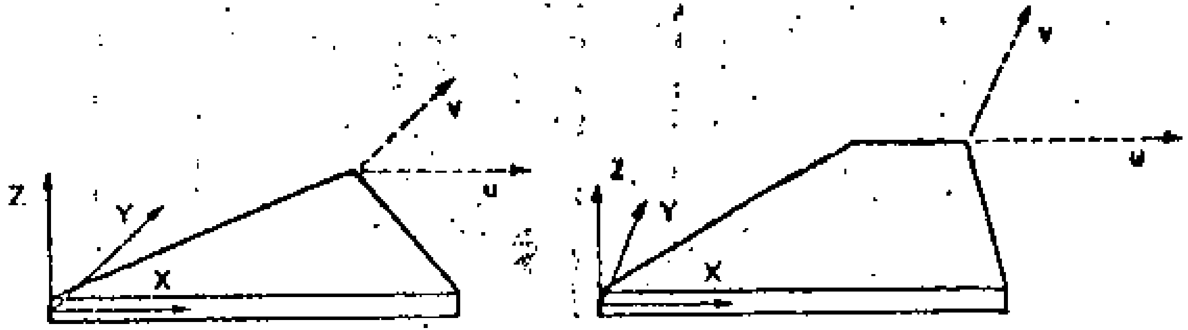
Se presenta al final una lista de referencias de importancia del método del elemento finito.

Al iniciar la determinación de esfuerzos y desplazamientos en cierto problema de diseño, las ecuaciones que gobiernan el problema en cualquier forma deben satisfacer equilibrio y continuidad.

El Método del Elemento Finito es un procedimiento analítico, y cuando se aplica a un medio continuo, éste se modela analíticamente subdividiéndolo en sub-regiones (los elementos finitos) en los que el comportamiento de cada uno es definido por grupos separados de funciones que supuestamente definen esfuerzos y desplazamientos en esa región, las funciones se seleccionan en forma tal que se satisfaga la condición de continuidad a través de todo el medio, por lo tanto, el método del elemento finito en común con las soluciones por series y diferencias finitas representa una aproximación a la solución del problema

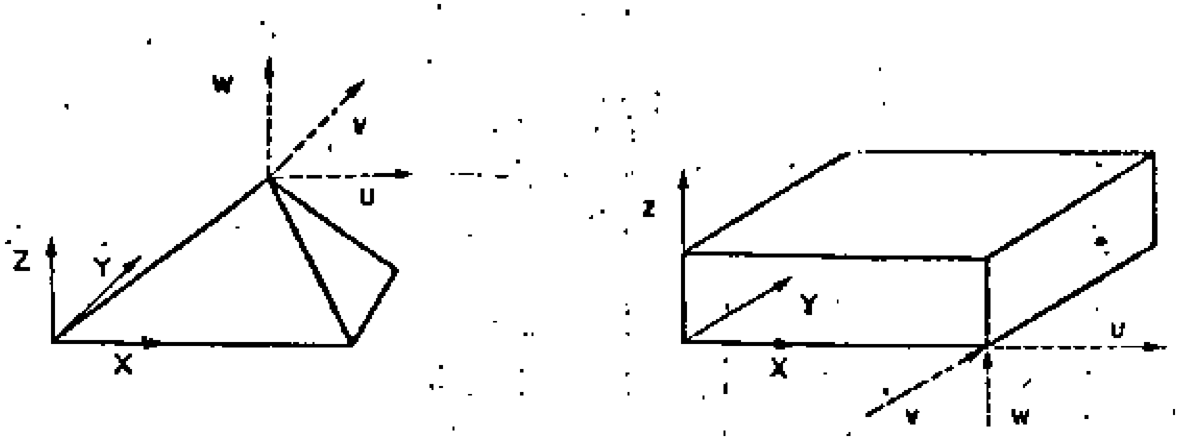


a) Elemento estructural

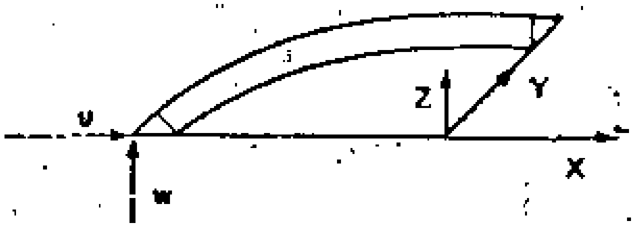


$$\gamma_{XY} = \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}$$

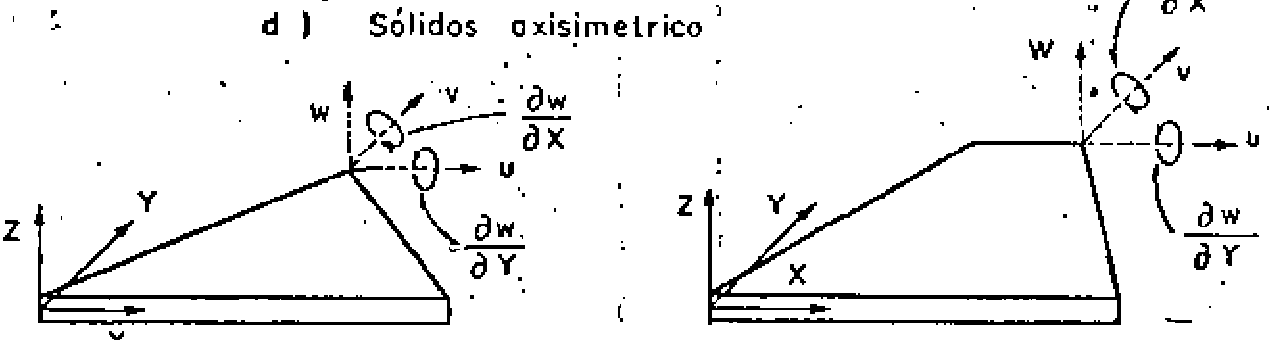
b) Esfuerzos planos



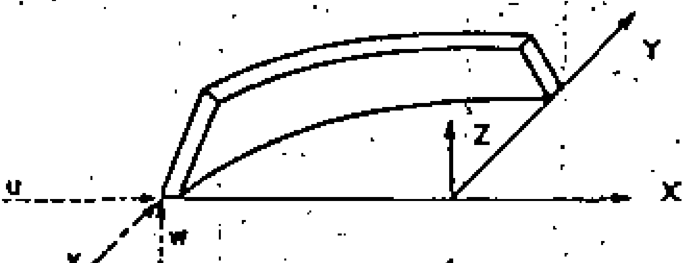
c) Elementos sólidos



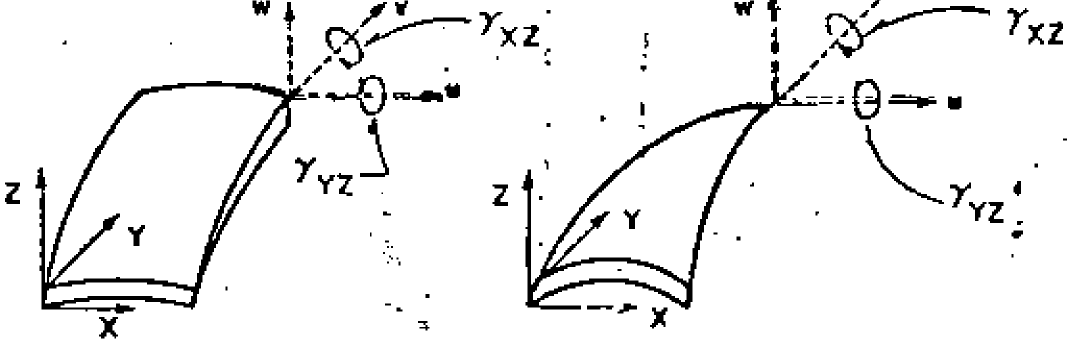
d) Sólidos axisimetrico



e) Flexión de placas



f) Cascarón delgado axisimétrico



g) Cascarones delgados curvos

Fig 5 Tipos de elementos finitos

TIPOS DE ELEMENTOS.

Elementos que son usados comunmente en la práctica son ilustrados en la Fig. 5.

El elemento estructural simple, Fig. 5 (a), es un miembro de la familia total de elementos finitos. Cuando se usa con elementos del mismo tipo describe armaduras y estructuras espaciales. Cuando se combina con elementos de tipo diferente, especialmente con elementos de placa generalmente se describen miembros de rigidez.

Los elementos básicos en análisis por elementos finitos son placas delgadas con cargas contenidas en su plano (condición de esfuerzos planos), triangulares y cuadriláteros se ilustran en la Fib 5b. Se denominan básicos porque los primeros desarrollos concernientes con el método se refieren a ellos.

Los elementos sólidos, Fig. 5 (c), son la generalización tridimensional de los elementos de esfuerzos planos. El tetrahedro y el hexaedro son las formas más comunes y son esenciales para modelar analíticamente problemas de mecánica de suelos, rocas y estructuras nucleares. Es conveniente mencionar que la única forma práctica de resolver problemas tridimensionales prácticos, es el método de elementos finitos.

Uno de los campos más importantes de aplicación del método de elementos finitos es en el análisis de "sólidos axisimétricos", Fig. 5 (d). Una gran variedad de problemas de ingeniería caen en esta categoría, incluyendo concreto, tanques, recipientes nucleares, rotores, pistones, flechas de motores, y la cabeza de los roquets. Generalmente son medios de carga y geometría axisimétrica.

En la Fig. 5 (d) se muestra el elemento triangular, también se usan secciones cuadriláteras.

Elemento de placa plana en flexión es empleado no solo en conección con el comportamiento de placas planas, sino también en cascarones y miembros de pared delgada. Fig. 5 (e).

Estructuras de cascarón delgado axisimétricas, Fig. 5 (f), tienen el mismo rango de significado en la aplicación práctica que los sólidos axisimétricos. Sin-embargo, las relaciones gobernantes se derivan de la teoría de cascarones delgados.

Cuando una estructura de cascarón delgado que de hecho es curva, es preferible emplear elementos de cascarón curvos delgados para el modelo analítico, tienen la ventaja de describir más aproximadamente la superficie curva del cascarón, y la apropiada representación del acoplamiento de deformación y equilibrio entre cada elemento.. Elementos típicos de cascarones de doble curvatura se muestran en Fig. 5 (g). Gran número de formulaciones para este elemento existen.

ALGUNAS APLICACIONES DE ELEMENTOS FINITOS.

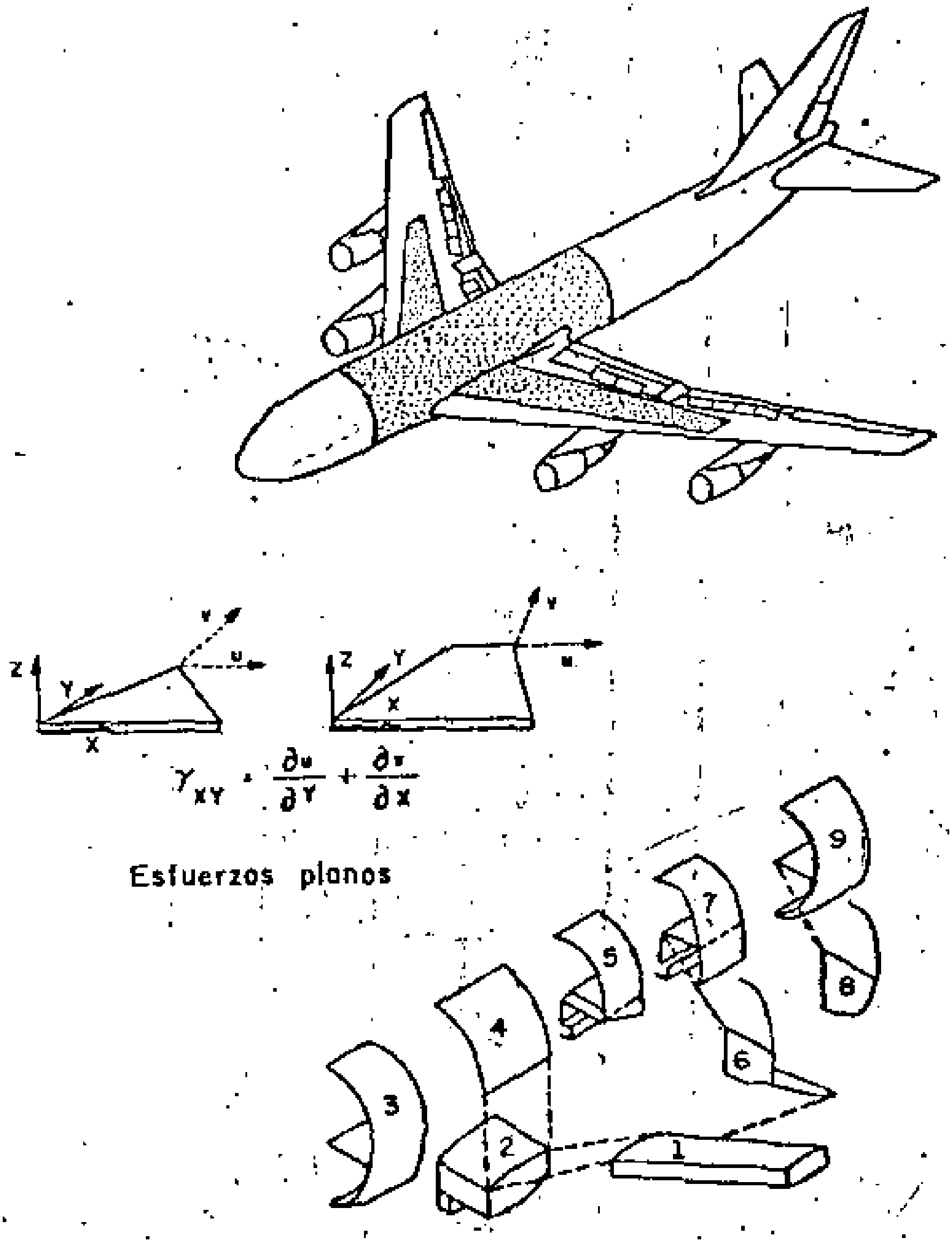
Examinaremos algunas aplicaciones delmétodo de elementos finitos en diseño estructural con el objeto de ilustrar la forma en la cual se usan los elementos de la Fig. 5, y la escala y complejidad de los problemas.

El desarrollo del método del elemento finito se debe a los investigadores relacionados con la industria aeronáutica. La Figura 6 muestra la forma en que se aplicó el análisis por elementos finitos de una porción del avión Boeing 747. La estructura del fuselaje de un avión consiste de laminas de aluminio ligadas a una estructura interna formada por armaduras y atiezadores. La experiencia ha mostrado que los efectos locales de flexión en el cascarón son despreciables, por lo tanto, se supone que consiste de elementos en condición plana de esfuerzos Fig. 5(b). El análisis de elementos finitos del Boeing 747, de la parte achurada, región que conecta el cuerpo o Cascarón Monocoque con las alas, área achurada en Fig. 6, consiste de 7000 incógnitas. Por lo tanto, es común en la práctica dividir la estructura en regiones, o subestructuras, y analizar cada una por elementos finitos con el objeto de producir un superelemento. Los superelementos se ligan entre sí por medio de un procedimiento convencional que determina la fase final del análisis.

El esquema de subestructuración del Boeing 747 es mostrado en la Fig. 6 y los detalles son listados en la Tabla 1.

Sub-Estructura	Descripción	Nodos	Condición Carga	Elemento Viga	Elemento Placa	Grados libertad interacción elementos.	Grado de libertad total.
1	Ala	262	14	355	363	104	796
2	Centro ala	267	8	414	295	198	880
3	Cascarón Monocoque	291	7	502	223	91	1,026
4	Cascarón M.	213	5	377	185	145	820
5	Cascarón M	292	7	415	241	200	936
6	Caja Tren Aterrizaje	170	10	221	103	126	656
7	Cascarón M	285	6	392	249	233	909
8	Caja Tren Aterrizaje	129	10	201	93	148	503
9	Cascarón M	286	7	497	227	92	1,038
TOTAL		2,195	63	3,374	1,979	553	7,594

Tabla 1 Subestructuación del Boeing 747



$$\gamma_{XY} = \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}$$

Esfuerzos planos

Fig 6 Boeing 747

Como es usual en el diseño de aviones, se hicieron pruebas en el prototipo y los resultados se compararon con la solución por elementos finitos, coincidiendo como se muestra en la Fig. 7

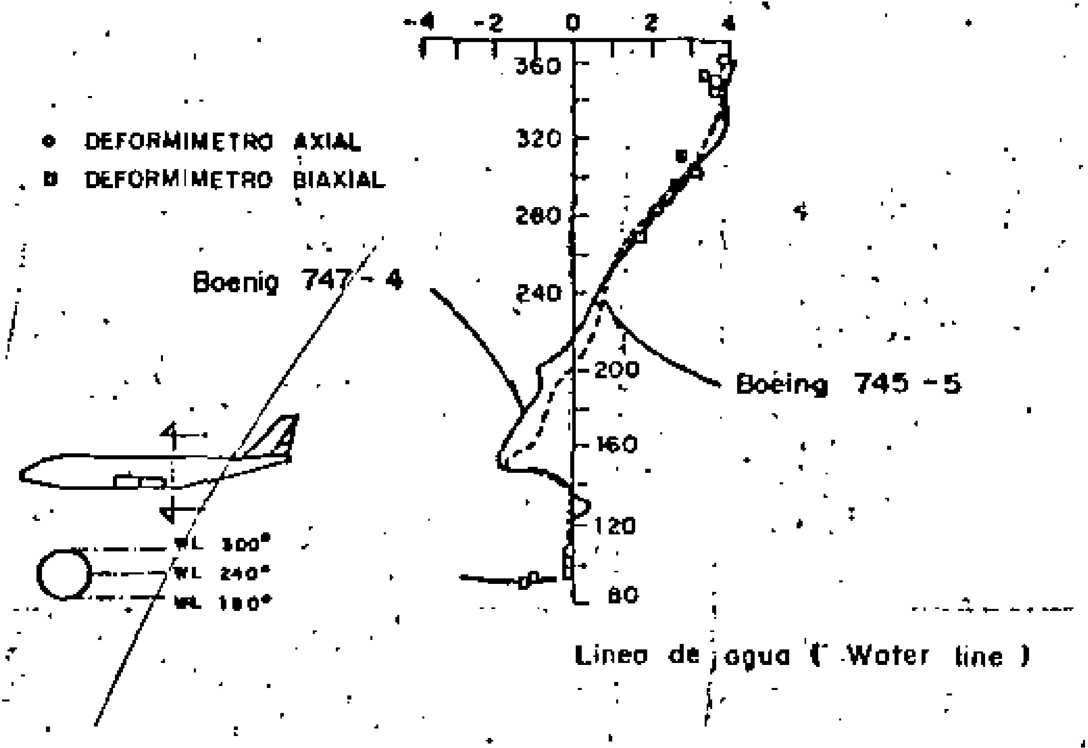


Fig. 7 Comparación entre análisis y experimentación del Boing 747

Es importante agregar que la respuesta dinámica de un avión es muy importante, así como su inestabilidad elástica es una forma importante de falla. Ninguno de estos fenómenos puede tratarse por los métodos simplificados, pero su análisis usando el método de elementos finitos ha probado ser muy aceptable.

Problemas similares se encuentran en Arquitectura Naval. Figura 8 una porción de una estructura de un transbordador. La parte plana es representada por elementos en estado plano de esfuerzos, Fig. 5 (b). Elementos estructurales, Fig. 5 (a), son empleados en la representación de la estructura interna.

El número total de incógnitas para definir las partes importantes de un barco es del orden de 50,000, y de nuevo se subdivide el problema en subestructuras obteniendo menos incógnitas.

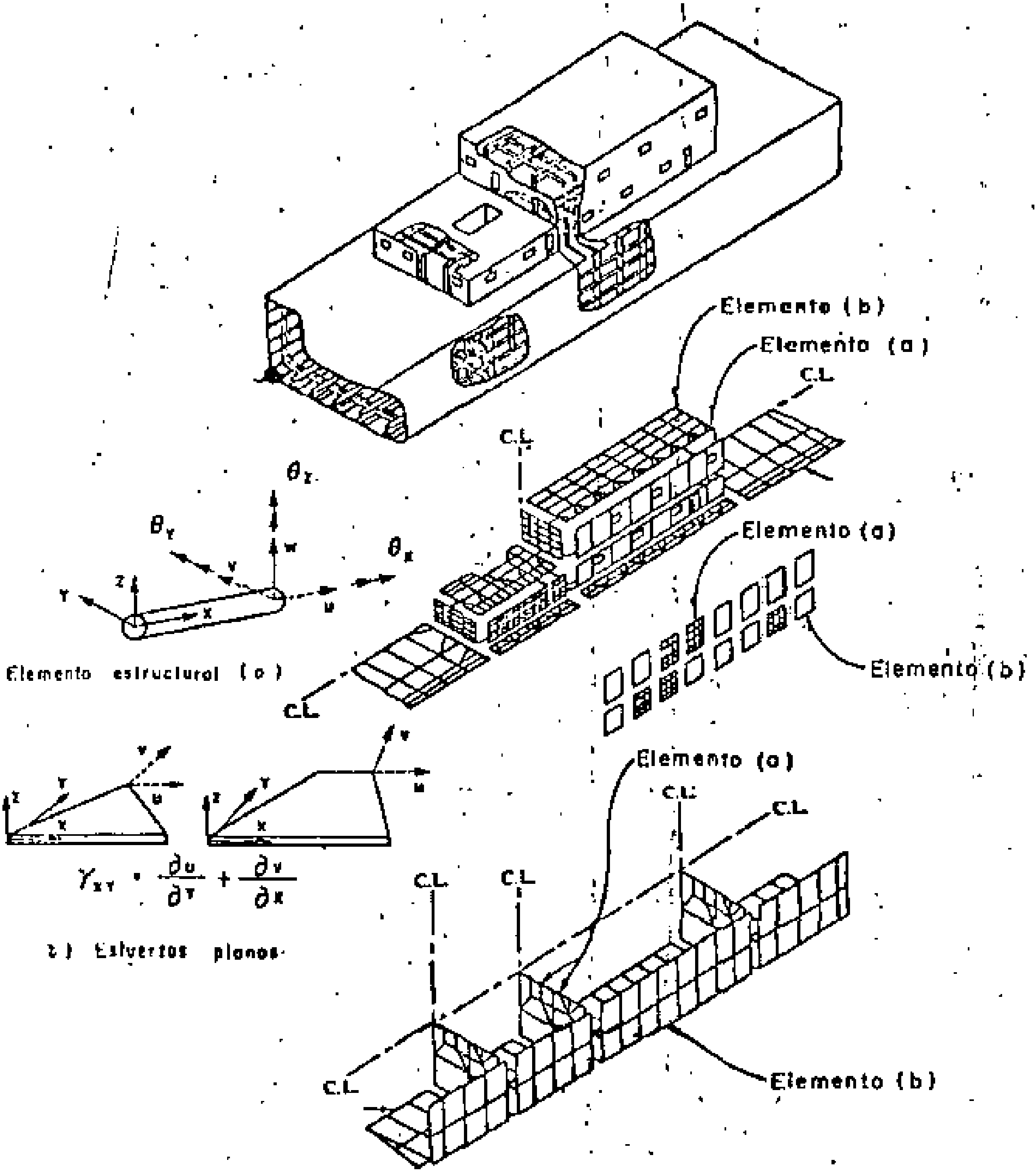


Fig. 8 Análisis por elemento finito de estructura de un transbordador

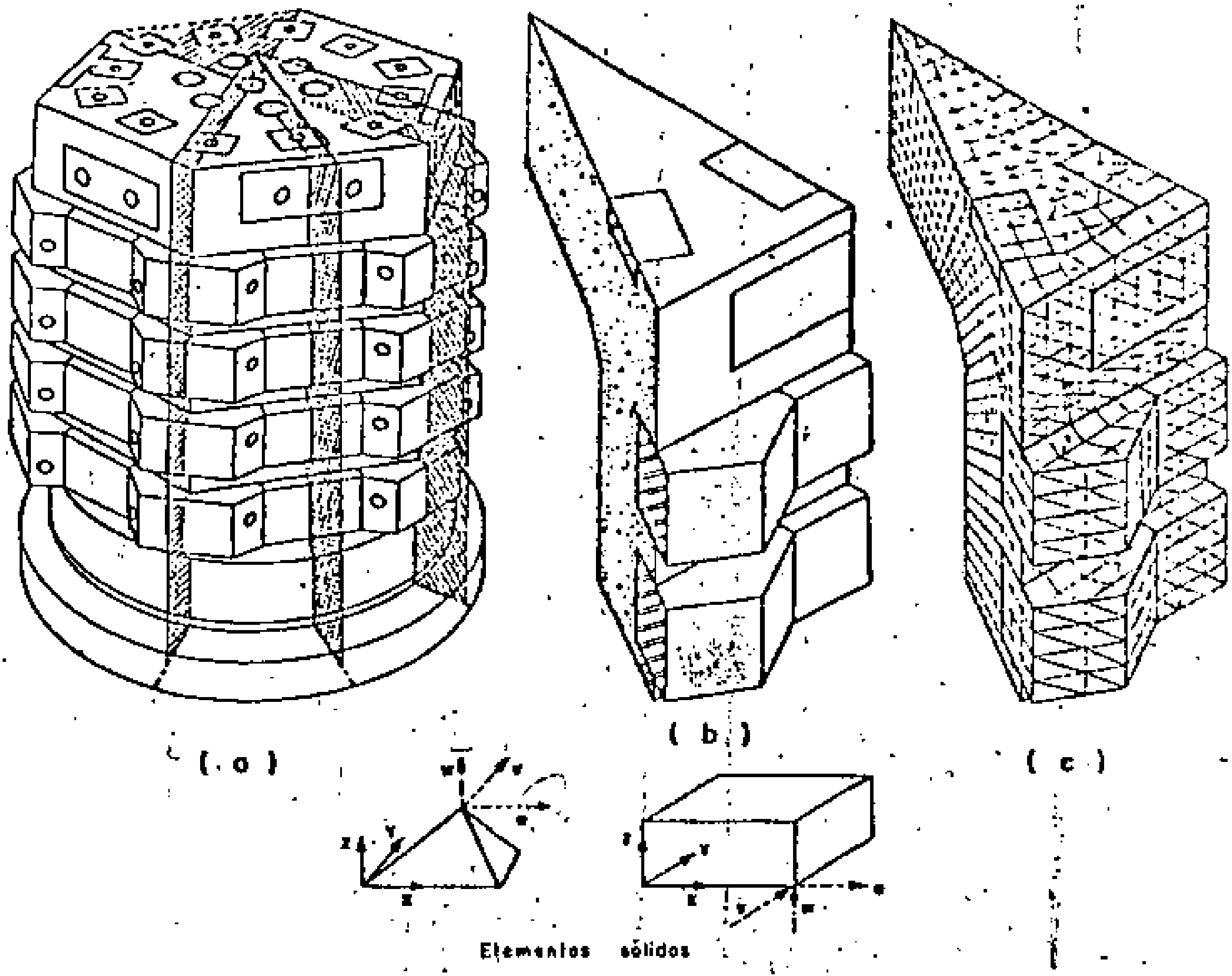


Fig 9 Analisis por elementos finitos de un recipiente reactor de concreto presforzado

Requerimientos de seguridad en el diseño estructural de los reactores nucleares han causado que la industria use ampliamente el análisis por elementos finitos. Figura 9 (a) un recipiente reactor de concreto presforzado. Debido a la simetría es posible analizar solamente un doceavo de la estructura total, - Fig. 9 (b). Su volumen se modela analíticamente en un ensamble de elementos tetaedrales y hexaedrales, Fig. 5 (c). En problemas de este tipo, el número de incógnitas es del orden de 20,000, y muy común hacer el análisis en condiciones no lineales en material y geometría.

No todos los problemas de aplicación del método de elementos finitos son de proporciones monumentales. Las figuras 10 y 11 muestran aplicaciones básicas a ciertos problemas de ingeniería civil. Una forma de incrementar la eficiencia de diseño en secciones roladas de acero estructural es cortando el alma en la forma dentada mostrada en la Fig. 10 (a), colocando una sección sobre la otra y soldándolas, Fig. 10 (b). Y se obtiene una viga más aperaltada reduciendo el acero en el alma, y por supuesto que en este problema rutinario de diseño, no es necesario el uso del método de elementos finitos.

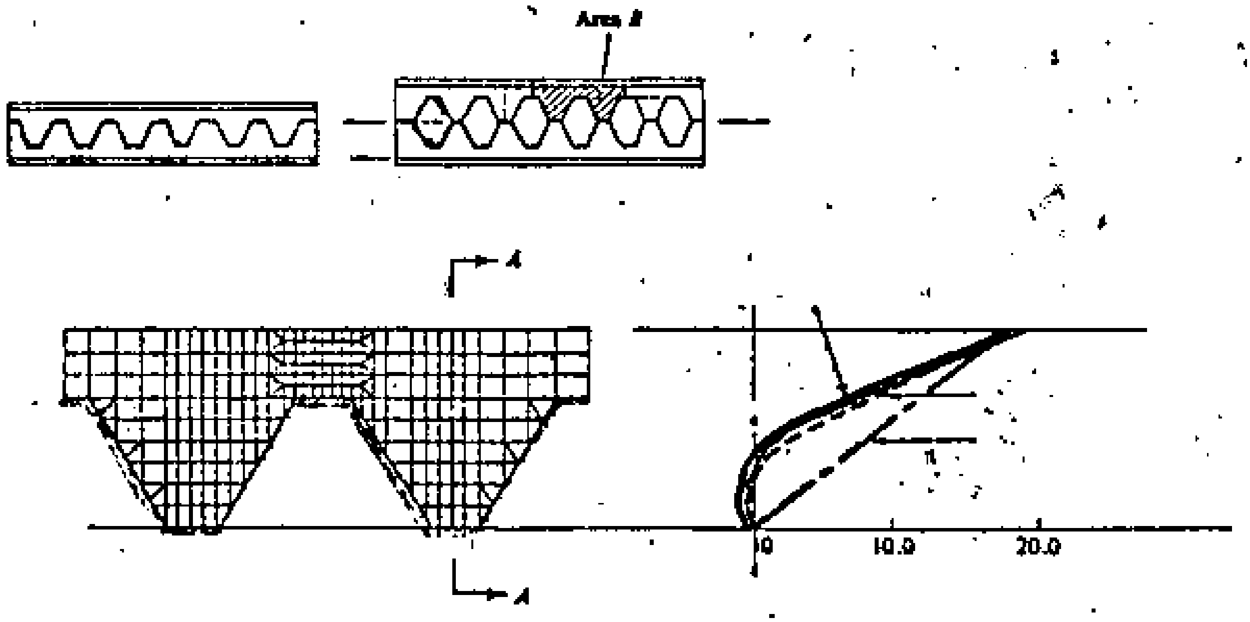


Fig. 10 Análisis de elementos finitos de una viga aperaltada en celosía.

Un problema todavía más común es el de una viga de concreto reforzado, Fig. 11, para el cual se conoce muy poco respecto a la adherencia entre el acero de refuerzo y el concreto, y la formación y crecimiento de las grietas al aumentar la carga. La Figura 11 (a) muestra el modelo analítico de elementos finitos y la descripción de las trayectorias de grietas y las gráficas de esfuerzos se muestran en la Fig. 11 (b).

Los pocos ejemplos mostrados muestran que el método de elementos finitos puede ser usado ventajosamente en cualquier situación que se requiera la predicción de esfuerzos y deformaciones internas, desplazamientos, vibraciones, inestabilidad elástica, mecánica de fluídos, transferencia de calor. Situaciones que se levantan de diversos campos que tradicionalmente han sido considerados como disciplinas ingenieriles separadas. Ejem., Ingeniería Civil, Mecánica, Aeroespacial, Arquitectura Naval. El método del elemento finito proporciona una tecnología unificada de análisis en casi todos los campos.

Es nuestro intento en este curso desarrollar los conceptos teóricos básicos y estudiar problemas específicos de carácter práctico. Un compendio de tales problemas llenaría muchos volumenes, por lo tanto es recomendable consultar las memorias de congresos y publicaciones periódicas correspondientes.

PROGRAMAS DE PROPOSITOS GENERALES.

Se ha indicado que las ecuaciones del método de elementos finitos son de una forma tal que su carácter general permite teóricamente escribir un solo programa de computadora que resuelva la mayoría de los problemas que se presentan en la Mecánica de Medio Continuos. Programas de computadora con este objetivo, aún en escala restringida, son llamados programas "de propósitos generales". La ventaja de programas de propósitos generales no es sólo su capacidad,

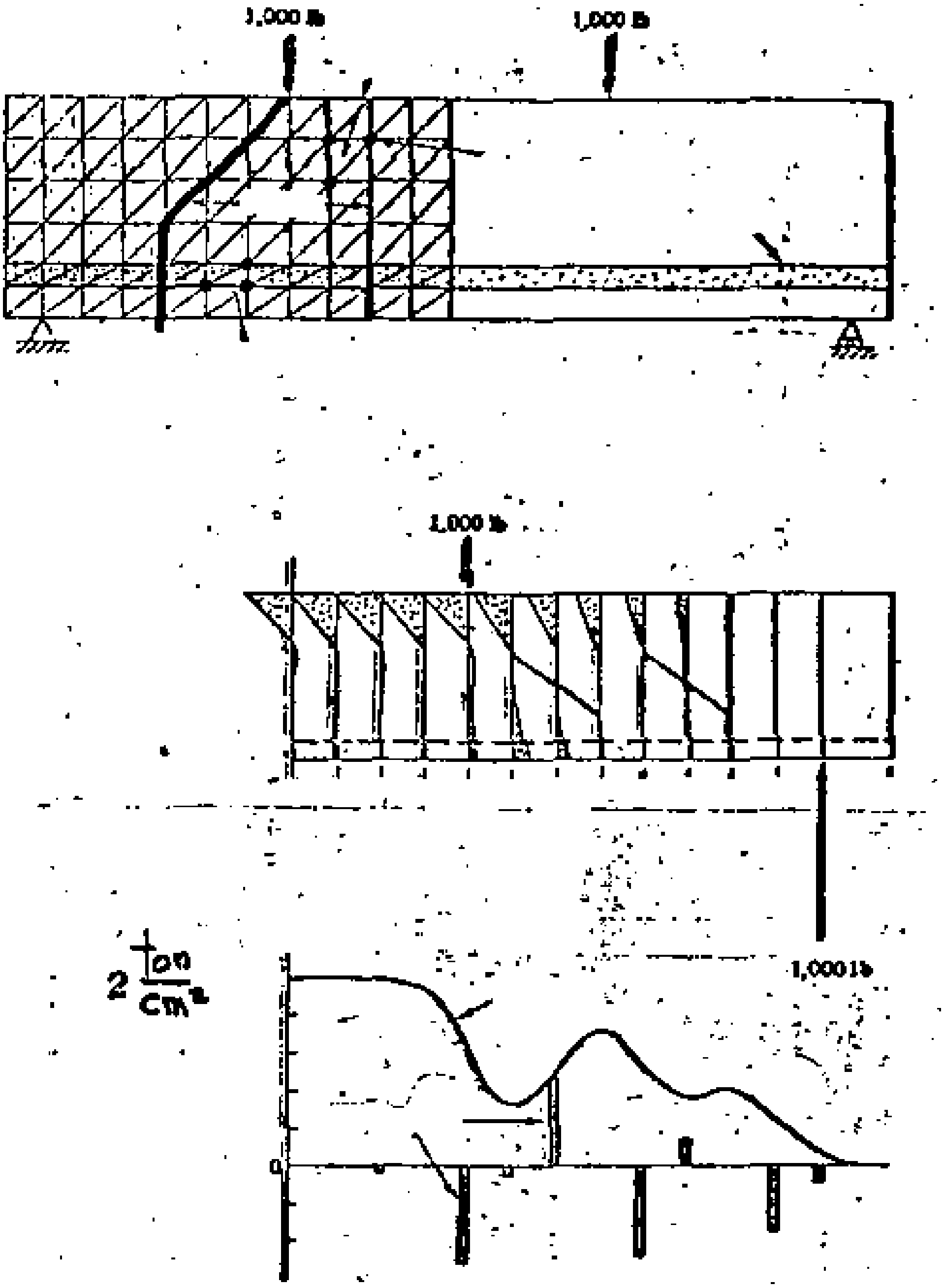


Fig. 11 Análisis por elementos finitos de una viga de concreto reforzado.

sino también en la instrucción de los probables usuarios respecto a la interpretación de la documentación, los datos y procedimientos de entrada y salida de resultados.

El costo de desarrollo de un !programa de propósitos generales es usualmente muy alto por lo que la amortización de la inversión es esencial. Ciertos programas de propósitos generales son codificados en un lenguaje computacional que permite operar el programa a muchas organizaciones diferentes localizadas en grandes separaciones geográficas. Otros programas de propósitos especiales de limitada capacidad se usan en organizaciones industriales y gubernamentales con un costo menor en su desarrollo y operación.

Las cuatro componentes mostradas en el diagrama de flujo de la Fig. 12, son comunes en el desarrollo de programas de propósitos generales, fase de datos de entrada, requiere del usuario información del medio o materal, descripción geométrica de la representación por elementos finitos y las condiciones de carga y de frontera. Los programas de propósitos generales más sofisticados facilitan el proceso de entrada como propiedades constitutivas del material, almacenados previamente, esquemas de modelar analíticamente el medio, trazar esterográficamente la idealización por elementos finitos en forma tal que los errores pueden detectarse antes de efectuar los cálculos.

La fase de biblioteca de elementos finitos es de interés primordial en el curso. En ella se tienen los procesos de codificación formulativos para los elementos individualmente. La mayoría de los programas de propósitos generales contienen todos los elementos de la Fig. 5, así como ciertas otras alternativas de formulación para un tipo dado de elemento, por ejemplo el trián-

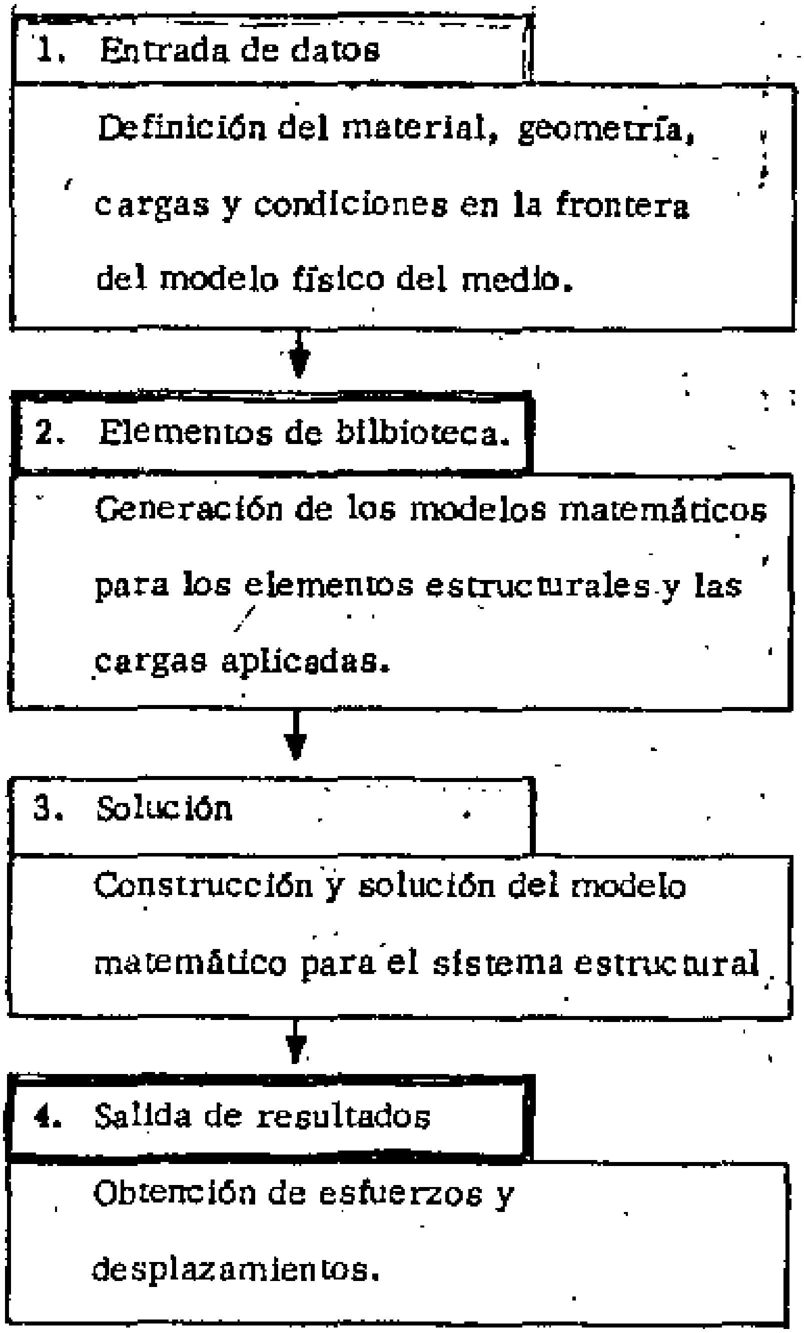


Fig. 12 Diagrama de flujo computacional en Análisis Estructural.

gulo en flexión. Teóricamente el elemento biblioteca es de extremos abiertos y capaz de acomodar cualquier nuevo elemento de cualquier grado de complejidad.

La fase elemento de biblioteca recibe los datos almacenados y establece las relaciones algebráicas del elemento por medio de la aplicación de los procesos formulativos relevantes de codificación. Esta fase del programa de propósitos generales también incluye todas las relaciones algebráicas para interconectar los elementos vecinos y la conección del proceso en sí. Las operaciones posteriores producen un conjunto de ecuaciones algebráicas lineales simultáneas para representar la estructura completa por elementos finitos.

La fase solución del programa de propósitos generales opera sobre las ecuaciones del problema formadas en la fase anterior. En el caso de un problema de análisis estructural solo significa la solución de un conjunto de ecuaciones lineales algebráicas. Soluciones para respuesta dinámica requerirán computaciones más extensas sobre la historia-tiempo de las cargas aplicadas. En algunos casos hay que operar en regiones subdivididas como en el caso del análisis del Boeing 747, o efectuar operaciones especiales en las ecuaciones construídas originalmente. Incluídas en esta fase están las operaciones necesarias de substitución para obtener todos los aspectos deseados de la solución.

La fase salida de resultados presenta el análisis con un registro de la solución sobre la cual se pueden tomar decisiones respecto al dimensionamiento estructural o diseño. El registro comunmente es presentado mediante una lista impresa de esfuerzos y desplazamientos de los respectivos elementos. Así como en la fase de entrada existe una fuerte tendencia a la representación gráfica de datos, tales como gráficas de trayectorias principales de esfuerzos o modos de pandeo y vibración.

ALGUNOS PROGRAMAS DE PROPOSITOS GENERALES.

ICES-STRUDL II, Integrated Civil Engineering System, (ICES), MIT, Maneja problemas de deformación y esfuerzos planos, cascarones rebajados, sólidos tridimensionales, flexión de placas con y sin deformación axial. Su uso en problemas muy especializados resulta caro. ASKA, Automatic System for Kinematic Analysis. Desarrollado por J. H. Argyris, H. A. Kamel y otros en la Universidad de Stuttgar. Sistema general muy potente el cual incluye una biblioteca de 42 elementos diferentes. Puede ser costoso para un usuario especializado. SAP, A General Structural Analysis Program, elaborado por E. L. Wilson de la Universidad de California. Incluye análisis lineal estático y dinámico de estructuras elásticas, estructuras tridimensionales, sólidos axisimétricos, sólidos tridimensionales, esfuerzos y deformación plana, placas y cascarones.

Zienkiewcz, O.C., programa desarrollando en la Universidad de Wales, Swansea. Incluye lo de los programas anteriores y problemas de Mecánica de Fluídos y transferencia de calor.

NASTRAN, NAsa STRuctural ANalysis. Desarrollado por U. S. National Aeronautical and Space Administration para análisis elástico de varias estructuras incluye, análisis de expansión térmica, respuesta dinámica a cargas transitorias y extitaciones random, cálculo de valores característicos reales y complejos, estabilidad dinámica. Ofrece capacidad limitada para análisis no lineal.

SAMIS, Structural Analysis and Matrix Interpretarive System. Desarrollado por Jet Propulsion Laboratory, y Manned Spacecraft Center. Contiene un elemento unidimensional general y elementos triangulares para deformaciones por flexión y membrana.

ELAS y ELAS 8, Equilibrium Problems of Linear Structures.. Desarrollado por el Jet Propulsion Laboratory. Incluye una biblioteca de elementos unidimensionales, triangulares, cuadriláteros, tetaedros, hexaedros, cónicos, sólidos axisimétricos de secciones cuadriláteros y triangulares.

MARC, elaborado por P. V. Marcal, incluye análisis lineal y no lineal de problemas de Mecánica de Medios Continuos.

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FINITE ELEMENT METHOD
THEORY AND APPLICATION

1. INTRODUCTION

1.1 HISTORICAL BACKGROUND

The finite element method (FEM) has become a powerful numerical technique for solving complex problems in science and engineering, mainly due to the advances made earlier in the numerical methods particularly in matrix methods as well as due to the rapid introduction of high speed computers in the market. However, the introduction of concepts and applications of FEM dates back to the era of mathematicians who tried to calculate the perimeter and area of a circle by idealizing it as a regular polygon. It is also interesting to note that the bound solutions which are often discussed in FEM can be traced back to the solution of the area of a circle. If the circle is modelled with an inscribed polygon, a lower bound solution is obtained whereas an upper bound solution is obtained by replacing the circle by a circumscribed polygon. Even though the basic concepts of FEM existed for over two thousand years, for all practical purposes, one can only say that these concepts were actually used for solving physical problems in 1950s by the aeronautical engineers.

In 1956, Turner et al (Ref 1) presented the stiffness analysis for the complex structures, which is the starting point in the rediscovery of FEM. Nevertheless, Clough (Ref 2) was the one who actually used the term FEM in 1960. Since then, a tremendous amount of research has been done in this field and quite a large number of papers have been published in almost all the journals related to all fields of engineering as well as some in the fields of mathematics and science. In addition, several conferences have been held all over the world and hundreds of papers have been presented in each. The theory and application of FEM have also been presented in numerous text books (Ref 3-22). In order to help the research workers in tracing the references required for their particular work several bibliographics have either been published or under preparation, among them notably Ref (23) is a good source of information.

1.2 APPLICATIONS OF FEM

The FEM is applicable to a variety of boundary value and initial value problems in engineering as well as applied science. Some of these applications are:

1. Stress Analysis of Structures, Stability of Structures, Dynamic response of structures, Thermal Stress Analysis, Torsion of prismatic members
2. Stress Analysis of Geomechanics problems, Soil-Structure Interaction, Slope Stability problems, Soil Dynamics and Earthquake Engineering, Seepage in soils and rocks, Consolidation settlement
3. Solutions in Fluid Mechanics, Harbour oscillations, Pollution Studies, Sedimentation
4. Analysis of Nuclear Reactor Structures
5. Stress Analysis and Flow Problems in Biomechanics
6. Characteristic Study of Composites in Fibre Technology
7. Wave Propagation in Geophysics
8. Field Problems in Electrical Engineering

Apart from the above mentioned areas, the FEM is also applicable to any other problem as long as the analyst makes certain that the problem is amenable to solution based on the assumptions introduced in the formulation of FEM and appropriate material properties can be provided in a realistic manner.

1.3 METHODS OF ANALYSIS

In general, there are four basic methods of analysis in FEM-displacement method, equilibrium method, mixed method and hybrid method. The field variables or unknown quantities in each of these methods are as follows.

Displacement method - displacements and their derivatives
Equilibrium method - stress components
Mixed method - some displacements and some stress components
Hybrid method - displacements or boundary forces

In the displacement method, smooth displacement distribution is assumed within an element, interelement compatibility of displacement is generally assured and minimum potential energy criterion is used in the formulation.

In the equilibrium method, the interior stress distribution is assumed to be smooth, the equilibrium of boundary tractions is maintained and the minimum complimentary energy is the basis for the formulation.

In the mixed method which is generally used for plate and shell problems, both displacements and stresses are assumed smooth in the interior, the displacement components and the equivalent stress components are considered to be continuous at the interelement boundaries and the formulation is based on Reissner's principle.

In the hybrid method, depending on whether the model is displacement type or equilibrium type, the distribution of displacements or stresses within the element is considered to be smooth and along the interelement boundary either assumed compatible displacements or assumed equilibrating boundary tractions are ensured and either modified complementary energy or modified potential energy principle is adopted for the formulation.

Among these four methods, the displacement method is the most widely used approach. However, for plate bending problems either the equilibrium or mixed method is preferred and for some field problems hybrid method is more suitable.

1.4 DESCRIPTION OF FEM

A structure, continuum or a domain is divided into a number of arbitrary shaped parts or regions known as *elements*. These elements are interconnected at joints known as *nodes*. The principal unknown is termed as the *field variable*. This field variable can be displacement, temperature, pore-pressure or stress. The distribution of the field variable within an element is approximated by the use of certain polynomial functions. Variational methods or residual methods are employed to develop the finite element equations which relate the field variables at the nodes to the corresponding action vector at the nodes of the element. This relationship is provided by the so called property matrix which is based on the material and the geometric properties of the element. Finally these finite element equations are assembled to form a system of algebraic equations for the entire domain. The unknown field variable is obtained by solving this system of algebraic equations.

1.5 BASIC STEPS IN FE ANALYSIS

The basic steps in the finite element analysis of general problems are as follows.

1. The continuum is divided into finite elements of any arbitrary shape.
2. A suitable polynomial is chosen to represent the distribution of the field variable within an element in terms of its nodal values. Thus, the field variables at the nodes become the primary unknowns.
3. Using variational methods or residual methods, the finite element equations are formulated.
4. The individual finite element equations obtained in step 3 are assembled to form a set of algebraic equations for the overall continuum.
5. The solution of the algebraic equations obtained in step 4 yields the values of the field variables at the nodes.
6. From the field variables at the nodes, the secondary variables such as stress, strain for an element can be obtained.

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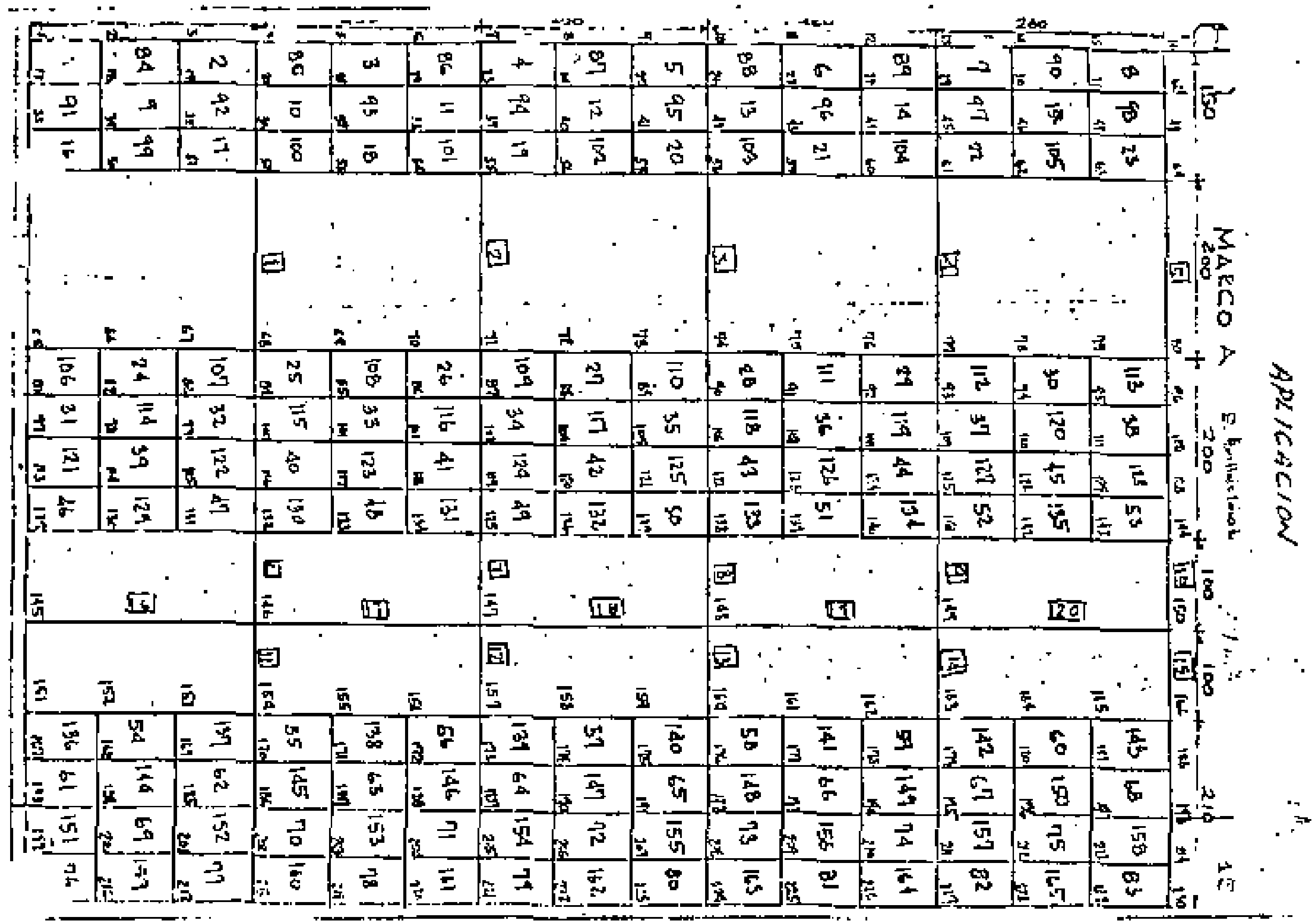

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INSTITUTO DE INGENIERIA

FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Balbuena REVISADO POR Salinas FECHA MARZO 19
PAGINA 1 DE 10

TARJETAS DE CONTROL

```
7 JOB MURØ/MARCØ ; USER XYTZ/AB ; CLASS=3 ; BEGIN ;
EXECUTE (XPAD) DESFI/SMTA/P7405 ;
PROCESS 180 ; 10 180 ;
BCL TQUE
 1
A11   TITULO   PROBLEMA   GENERAL
ANALISIS DE UNA ESTRUCTURA TIPO MURO MARCO
A12   ARCHIVOS
 1 10   15   10   25        [illegible]
A13   ESTRUCTURAS POR ANALIZAR
 1 1
A141  TITULO   PROBLEMA   PARTICULAR
ANALISIS DEL MURO MARCO   A         ([illegible])
A142  CANTIDAD DE BARRAS, CUADRADOS, MATERIAL, NUDOS Y SECCIONES
 1 20  145     1  230     2
A143  CARACTERISTICAS  DEL  MATERIAL (Identificador de material, E, ν, γ)
 1 1  1500000.0     0.15      2.4
A144  CARACTERISTICAS DE LAS SECCIONES   (Tipo Area rectangular, b, h)
  1     2       70.       15.
  2     2      100.       20.
A145  COORDENADAS PUNTOS NODALES
 1 1       0.0      0.0
 1 16      0.0     13.0    1     ([illegible])
 1 17      0.5      0.0
 1 32      0.5     13.0    1     ([illegible])
```

— IDENTIFICACION DE LA TARJETA POR PERFORAR (NO SE PERFORA)

FORMA 2-0L-71

FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO 1976 PAGINA 2 DE 10

```
   33         1.0         0.0
   48         1.0        13.0    1
   49         1.5         0.0
   64         1.5        13.0    1
   65         3.5         0.0
   80         3.5        13.0    1
   81         4.0         0.0
   96         4.0        13.0    1
   97         4.5         0.0
  112         4.5        13.0    1
  113         5.0         0.0
  128         5.0        13.0    1
  129         5.5         0.0
  144         5.5        13.0    1
  145         6.5         0.0
  150         6.5        13.0    1
  151         7.5         0.0
  166         7.5        13.0    1
  167       8.025         0.0
  182       8.025        13.0    1
  183       8.550         0.0
  198       8.550        13.0    1
  199       9.075         0.0
  214       9.075        13.0    1
  215       9.600         0.0
```

FORMA F-01-71

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO 1 PAGINA 3 DE 10

```
  230       9.600        13.0    1      (5 líneas agregadas de 215 a 230
A147 LOCALIZACION DE BARRAS Y CARACTERISTICAS
    1    52    68        1                Barra nodo i, nodo j, tipo de sección
    5    64    80        1         1   "    "    "    "    [illegible]
    6   132   146        1
    7   136   147        1
    8   138   148        1
    9   141   149        1
   10   144   150        1
   11   146   154        1
   12   147   157        1
   13   148   160        1
   14   149   168        1
   15   156   166        1
   16   145   146        2
   20   147   150        2         1                 [illegible]
A139 CALCULO FUERZA DE GRAVEDAD Y ESPESOR DEL MURO
   2200.0        0.20
A140 TIPO DE LOS CUADRADOS  [illegible]
11111111111111111111111111111111111111111111111111111111111111111111111111111111
00000000000000000000000000000000000000000000000000000000000000000000000000000000
  100  [illegible]
A141 LOCALIZACION DE CUADRADOS  [illegible]
    1     2     1    17    18             [illegible]
    8    16    15     [illegible]    32             [illegible]
```

FORMA F-01-71

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR [illegible] REVISADO POR Salinas FECHA MARZO 1976
PAGINA 5 DE 10

```
  105   47   46   62   63                   1
  106   66   65   81   82                        1
  113   80   79   95   96                   1
  114   83   82   98   99
  120   95   94  110  111                   1
  121   98   97  113  114                        1
  128  112  111  127  128                   1
  129  115  114  130  131
  135  127  126  142  143                   1
  136  152  151  167  168                        1
  143  166  165  181  182                   1
  144  169  168  184  185
  150  181  180  196  197                   1
  151  184  183  199  200                        1
  158  198  197  213  214                   1
  159  201  200  216  217
  165  213  212  228  229                   1
AI412   CANTIDAD DE NUDOS RESTRINGIDOS, CONDICIONES DE CARGA Y RIGIDEZ REQUERIDA
   15    1    0
AI413    TIPO DE RESTRICCION DE NUDOS   [illegible]
    1111    17111    33111    49111    65111    81111    97111   113111   129111   145111
  151111   167111   183111   199111   215111
AI4141  CANTIDAD DE NIVELES Y NUDOS POR NIVEL
    5   15
AI4142   INDICACION [illegible] EL CALCULO DE RIGIDEZ DE ENTREPISO
```

FORMA P-01-71

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR [illegible] REVISADO POR Salinas FECHA MARZO 1976
PAGINA 4 DE 10

```
    9   19   18   34   35
   15   31   30   46   47                   1
   16   34   33   49   50                        1
   23   48   47   63   64                   1
   24   67   66   82   83
   30   79   78   94   95                   1
   31   82   81   97   98                        1
   38   96   95  111  112                   1
   39   99   98  114  115
   45  111  110  126  127                   1
   46  114  113  129  130                        1
   53  128  127  143  144                   1
   54  153  152  168  169
   60  165  164  180  181                   1
   61  168  167  183  184                        1
   68  182  181  197  198                   1
   69  185  184  200  201
   75  197  196  212  213                   1
   76  200  199  215  216                        1
   83  214  213  229  230                   1
   84    3    2   18   19
   90   15   14   30   31                   1
   91   18   17   33   34                        1
   98   32   31   47   48                   1
   99   35   34   5[illegible]   51
```

FORMA P-01-71

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO 1976

PAGINA 6 DE 10

```
CALCULO DE LAS RIGIDECES DE ENTREPISO
A1A143    CANTIDAD DE NUDOS EN CADA NIVEL
   15   15   15   15   15
A1A144    NUMERACION DE LOS NUDOS POR NIVEL
    4   20   36   52   68   84  100  116  132  146  154  170  186  202  218
    7   23   39   55   71   87  103  119  135  147  157  173  189  205  221
   10   26   42   58   74   90  106  122  138  148  160  176  192  208  224
   13   29   45   61   77   93  109  125  141  149  163  179  195  211  227
   16   32   48   64   80   96  112  128  144  150  166  182  198  214  230
A1A145    ALTURAS DE ENTREPISO
      2.60      2.60      2.60      2.60      2.60                  [illegible]
A1A146    PESOS POR NIVEL
     33.23     33.23     33.23     33.23     33.23                  [illegible]
A1A147    COEFICIENTE SISMICO
      0.08                                                          [illegible]
A1A15     INDICACION DEL TIPO DE CONDICION DE CARGA
ANALISIS DEL MURO MARCO A CONSIDERANDO CARGA ESTATICA Y EFECTO SISMICO
A1A16     CANTIDAD DE BARRAS Y NUDOS CARGADOS
   20    90
A1A171    NUMERO DE BARRAS CON INDICE DE GRAFICACION
    1    1    2    1    3    1    4    1    5    1    6    1    7    1    8    1
    9    1   10    1   11    1   12    1   13    1   14    1   15    1   16    1
   17    1   18    1   19    1   20    1
A1A172    CARGAS EN LAS BARRAS
    1    4        [illegible] intermedias [illegible] con la misma carga
```

FORMA P-01-72

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO

PAGINA 7 DE 10

```
    1        -0.50        las cargas verticales -05 ton/m
    1    9                [illegible] cargas horizontales [illegible] con carga igual
    1        -0.25        [illegible] vertical [illegible]
    1    4                [illegible]
    4         0.00        [illegible] carga [illegible]
A1A18     CARGAS EN LOS NUDOS
    1          0.0     -1.491
   17          0.0     -2.982
   33          0.0     -2.982
   49          0.0     -2.982
   65          0.0     -2.982
   81          0.0     -2.982
   97          0.0     -2.982
  113          0.0     -2.982
  129          0.0     -2.982
  145          0.0     -2.982
  161          0.0     -2.982
  177          0.0     -2.982
  193          0.0     -2.982
  209          0.0     -2.982
  225          0.0     -1.491
    4        0.056     -1.491
   20        0.056     -2.982
   36        0.056     -2.982
   52        0.056     -2.982
```

FORMA P-01-72

(9)

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO 1976
PAGINA 8 DE 10

68	0.056	-2.982
84	0.056	-2.982
100	0.056	-2.982
116	0.056	-2.982
132	0.056	-2.982
146	0.056	-2.982
154	0.056	-2.982
170	0.056	-2.982
186	0.056	-2.982
202	0.056	-2.982
218	0.056	-1.491
7	0.112	-1.491
23	0.112	-2.982
39	0.112	-2.982
55	0.112	-2.982
71	0.112	-2.982
87	0.112	-2.982
103	0.112	-2.982
119	0.112	-2.982
135	0.112	-2.982
147	0.112	-2.982
157	0.112	-2.982
173	0.112	-2.982
189	0.112	-2.982
205	0.112	- .982

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FORMA PARA CODIFICACION

PROGRAMA MURO-MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA MARZO 1976
PAGINA 9 DE 10

221	0.112	-1.491
10	0.168	-1.491
26	0.168	-2.982
42	0.168	-2.982
58	0.168	-2.982
74	0.168	-2.982
90	0.168	-2.982
106	0.168	-2.982
122	0.168	-2.982
138	0.168	-2.982
148	0.168	-2.982
160	0.168	-2.982
176	0.168	-2.982
192	0.168	-2.982
208	0.168	-2.982
224	0.168	-1.491
13	0.220	-1.491
29	0.220	-2.982
45	0.220	-2.982
61	0.220	-2.982
77	0.220	-2.982
93	0.220	-2.982
109	0.220	-2.982
125	0.220	-2.982
141	0.220	[illegible].982

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FORMA PARA CODIFICACION

PROGRAMA MURO - MARCO CODIFICADO POR Ballesteros REVISADO POR Salinas FECHA [illegible] 20 1976

PAGINA 10 DE 10

149	0.220	-2.982
163	0.220	-2.982
179	0.220	-2.982
195	0.220	-2.982
211	0.220	-2.982
227	0.220	-1.491
16	0.280	-1.491
32	0.280	-2.982
48	0.280	-2.982
64	0.280	-2.982
80	0.280	-2.982
96	0.280	-2.982
112	0.280	-2.982
128	0.280	-2.982
144	0.280	-2.982
150	0.280	-2.982
166	0.280	-2.982
182	0.280	-2.982
198	0.280	-2.982
214	0.280	-2.982
230	0.280	-1.491

END 600

FORMA 3-01-01

LISIS DE UNA ESTRUCTURA TIPO MARCO

```
       20  NO. DE ELEMENTOS
      690  NO. DE ECUACIONES
        1  NO. DE TIPOS DE MATERIAL
      235  NO. DE PUNTOS DE LA ESTRUCTURA
      165  NO. DE CUADRILATEROS
        2  NO. DE TIPOS DE SECCION
       2[illegible]  NO. DEL PRIMER PUNTO FRONTERA


  CONSTANTES ELASTICAS DE LOS MATERIALES
**MODULO DE ELASTICIDAD**COEFICIENTE DE POISSON**PESO VOLUMETRICO
        (TON/M**2)                                  (TON/M**3)

      1500000.00                  0.15              2.400


 P A R A M E T R O S   Q U E   D E F I N E N   L A S   S E C C I O N E S

TIPO*        *SECCION*            *PARAMETROS**

  0        ESPECIAL               (A,IZ,FY)
  1        RECTANGULAR            (B,H)
  2        T                      (B,H,V,T)
  3        I                      (B,H,V,T)
  4        CANAL                  (B,H,V,T)
  5        ANGULO                 (B,H,V,T)
  6        CIRCULAR               (D)
  7        CAJON                  (B,H,V,T)
  8        CIRCULAR HUECA         (D,TC)
  9        CRUZ                   (B,H,V,T,C)
 10        ZETA                   (BI,H,V,TI,C,P)
 11        E                      (B,H,V,T)

**NOMENCLATURA**     (VER FIGURAS DEL CATALOGO DE SECCIONES)

  B        ANCHO DE LA SECCION TIPO 1,2,3,4,5,7,9,10 Y 11
  BI       ANCHO INFERIOR DE LA SECCION TIPO 10
  D        DIAMETRO DE LAS SECCIONES TIPO 6 Y 8
  H        ALTURA DE LAS SECCIONES TIPO 1,2,3,4,5,7,9,10 Y 11
  TC       ESPESOR DE LA SECCION CIRCULAR HUECA
  V        ANCHO DEL ALMA DE LAS SECCIONES TIPO 2,3,4,7,9,10  Y 11
  T        ESPESOR DEL PATIN DE LAS SECCIONES TIPO 2,3,4,5,7,9 Y 11
  TI       ESPESOR DEL PATIN INFERIOR DE LA SECCION TIPO 10
  P        ESPESOR DEL PATIN SUPERIOR DE LA SECCION TIPO 10
  BS       ANCHO SUPERIOR DE LA SECCION TIPO 10
  C        DISTANCIA ENTRE LAS FIBRAS SUPERIORES DEL ALMA Y PATIN RESPECTIVAME
```

RESPECTIVAMENTE DE LAS SECCIONES 9 Y 10

```
  (CM)     CENTIMETROS
  (M**2)   METROS A LA SEGUNDA POTENCIA
  (M**4)   METROS A LA CUARTA POTENCIA
  A        AREA
  IZ       MOMENTO DE INERCIA RESPECTO AL EJE Z
  FY       FACTOR DE FORMA PARA LA DIRECCION Y
```

```
SECCION NO.--- TIPO ------- B=DI=D=A --------- -- H=IZ=TC ------- V=FY -------
                               (CM)                    (CM)                 (CM)

      1            1           70.000                 15.000               0.000
      2            1          100.000                 20.000               0.000

SECCION NO.      TIPO ------ A ---------------- IZ ------------- FY
                          (M**2)             (M**4)

      1            1      0.105000000      0.000196875      1.200000000
      2            1      0.200000000      0.000666667      1.200000000

NUDO----ABSCISA---ORDENADA
NO.       (M)        (M)

  1       0.000      0.000
  2       0.000      0.867
  3       0.000      1.733
  4       0.000      2.600
  5       0.000      3.467
  6       0.000      4.333
  7       0.000      5.200
  8       0.000      6.067
  9       0.000      6.933
 10       0.000      7.800
 11       0.000      8.667
 12       0.000      9.533
 13       0.000     10.400
 14       0.000     11.267
 15       0.000     12.133
 16       0.000     13.000
 17       0.500      0.000
 18       0.500      0.867
 19       0.500      1.733
 20       0.500      2.600
 21       0.500      3.467
 22       0.500      4.333
 23       0.500      5.200
 24       0.500      6.067
 25       0.500      6.933
 26       0.500      7.800
 27       0.500      8.667
 28       0.500      9.533
 29       0.500     10.400
 30       0.500     11.267
 31       0.500     12.133
 32       0.500     13.000
 33       1.000      0.000
 34       1.000      0.867
 35       1.000      1.733
 36       1.000      2.600
 37       1.000      3.467
 38       1.000      4.333
 39       1.000      5.200
 40       1.000      6.067
 41       1.000      6.933
```

sigue igual ↓

3

```
     9.600        6.067
     9.600        6.933
     9.600        7.800
     9.600        8.667
     9.600        9.533
     9.600       10.400
     9.600       11.267
     9.600       12.133
     9.600       13.000

NE.-----NUDO I----NUDO J----MAT.NO----SEC.NO----APOY I----APOY J----LONGITUD

   1         52        68           1            1            0          0          2.000
   2         55        71           1            1            0          0          2.000
   3         58        74           1            1            0          0          2.000
   4         61        77           1            1            0          0          2.000
   5         64        80           1            1            0          0          2.000
   6        132       145           1            1            0          0          1.000
   7        135       147           1            1            0          0          1.000
   8        138       148           1            1            0          0          1.000
   9        141       149           1            1            0          0          1.000
  10        144       150           1            1            0          0          1.000
  11        146       154           1            1            0          0          1.000
  12        147       157           1            1            0          0          1.000
  13        148       160           1            1            0          0          1.000
  14        149       163           1            1            0          0          1.000
            150       166           1            1            0          0          1.000
            145       146           1            2            0          0          2.600
            146       147           1            2            0          0          2.600
  18        147       148           1            2            0          0          2.600
  19        148       149           1            2            0          0          2.600
  20        149       150           1            2            0          0          2.600

    270.00  GRADOS, ANGULO ENTRE GRAVEDAD Y EJE X GLOBAL
      0.20  METROS, ESPESOR CONSTANTE EN EL MURO

ELEM NO.      TIPO DE ELEMENTO
    1               2
    2               2
    3               2
    4               2
    5               2
    6               2
    7               2
    8               2
    9               2
   10               2
   11               2
   12               2
   13               2
                    2
                    2
   16               2
   17               2
   18               2
   19               2
   20               2
   21               2
```

{ sigue igual

```
       142                 1
       143                 1
       144                 1
       145                 1
       146                 1
       147                 1
       148                 1
       149                 1
       150                 1
       151                 1
       152                 1
       153                 1
       154                 1
       155                 1
       156                 1
       157                 1
       158                 1
       159                 1
       160                 1
       161                 1
       162                 1
       163                 1
       164                 1
       165                 1

  ELEM.NO.-NODO I-NODO J-NODO K-NODO L-MAT.NO.-ESPESOR(H) INGEN

       1        2       1      17      18       1          0.20    0
       2        4       3      19      20       1          0.20    1
       3        6       5      21      22       1          0.20    1
       4        8       7      23      24       1          0.20    1
       5       10       9      25      26       1          0.20    1
       6       12      11      27      28       1          0.20    1
       7       14      13      29      30       1          0.20    1
       8       16      15      31      32       1          0.20    1
       9       19      18      34      35       1          0.20    0
      10       21      20      36      37       1          0.20    1
      11       23      22      38      39       1          0.20    1
      12       25      24      40      41       1          0.20    1
      13       27      26      42      43       1          0.20    1
      14       29      28      44      45       1          0.20    1
      15       31      30      46      47       1          0.20    1
      16       34      33      49      50       1          0.20    0
      17       36      35      51      52       1          0.20    1
      18       38      37      53      54       1          0.20    1
      19       40      39      55      56       1          0.20    1
      20       42      41      57      58       1          0.20    1
      21       44      43      59      60       1          0.20    1
      22       46      45      61      62       1          0.20    1
      23       48      47      63      64       1          0.20    1
      24       67      66      82      83       1          0.20    0
      25       69      68      84      85       1          0.20    1
      26       71      70      86      87       1          0.20    1
      27       73      72      88      89       1          0.20    1
      28       75      74      90      91       1          0.20    1
      29       77      76      92      93       1          0.20    1
      30       79      78      94      95       1          0.20    1
      31       82      81      97      98       1          0.20    0
```

sigue igual

```
 152    186    185    201    202        1          0.20    1
 153    188    187    203    204        1          0.20    1
 154    190    189    205    206        1          0.20    1
 155    192    191    207    208        1          0.20    1
 156    194    193    209    210        1          0.20    1
 157    196    195    211    212        1          0.20    1
 158    198    197    213    214        1          0.20    1
 159    201    200    216    217        1          0.20    0
 160    203    202    218    219        1          0.20    1
 161    205    204    220    221        1          0.20    1
 162    207    206    222    223        1          0.20    1
 163    209    208    224    225        1          0.20    1
 164    211    210    226    227        1          0.20    1
 165    213    212    228    229        1          0.20    1


        54  ANCHO DE SEMIBANDA DE LA MATRIZ DE RIGIDECES


        15  NO. DE NUDOS FRONTERA
         1  NO. DE CONDICIONES DE CARGA
         0  INDICADOR  DE RIGIDECES DE ENTREPISO


  NUDO        RESTRICCION TIPO 0=LIBRE, 1=FIJO
RESTRINGIDO   D-X  D-Y  R-Z

      1          1    1    1
     17          1    1    1
     33          1    1    1
     49          1    1    1
     65          1    1    1
     81          1    1    1
     97          1    1    1
    113          1    1    1
    129          1    1    1
    145          1    1    1
    151          1    1    1
    167          1    1    1
    183          1    1    1
    199          1    1    1
    215          1    1    1


         NO. DE RESTRICCIONES DE LA ESTRUCTURA    45
 NO. RESTRICCION    G.L.RESTRINGIDO

          1                  1
          2                  2
          3                  3
          4                 49
          5                 50
          6                 51
          7                 97
          8                 98
          9                 99
         10                145
         11                146
```

```
      12              147
      13              193
      14              194
      15              195
      16              241
      17              242
      18              243
      19              289
      20              290
      21              291
      22              337
      23              338
      24              339
      25              385
      26              386
      27              387
      28              433
      29              434
      30              435
      31              451
      32              452
      33              453
      34              499
      35              500
      36              501
      37              547
      38              548
      39              549
      40              595
      41              596
      42              597
      43              643
      44              644
      45              645


              CALCULO  DE  LAS  RIGIDECES  DE  ENTREPISO


      PUNTOS NODALES EN CADA NIVEL
 NIVEL    NO.DE NUDOS
    1        15
    2        15
    3        15
    4        15
    5        15


      NUMERACION DE LOS NUDOS CONTENIDOS EN EL NIVEL NO.      1
  4   20   36   52   60   84  100  116  132  146  154  170  186  202  218

      NUMERACION DE LOS NUDOS CONTENIDOS EN EL NIVEL NO.      2
  7   23   39   55   71   87  103  119  135  147  157  173  189  205  221

      NUMERACION DE LOS NUDOS CONTENIDOS EN EL NIVEL NO.      3
 10   26   42   58   74   90  106  122  138  148  160  176  192  208  224
```

```
      NUMERACION DE LOS NUDOS CONTENIDOS EN EL NIVEL NO.    4
  13  29  45  61  77  93 109 125 141 147 163 179 195 211 227

      NUMERACION DE LOS NUDOS CONTENIDOS EN EL NIVEL NO.    5
  16  32  40  64  80  96 112 128 144 150 166 102 198 214 236

      ALTURAS DE LOS ENTREPISOS
ENTREPISO NO.  ALTURA (M)
      1            2.60
      2            2.60
      3            2.60
      4            2.60
      5            2.60

      PESOS DE LOS NIVELES
NIVEL NO.   PESO (TON)
    1         33.230
    2         33.230
    3         33.230
    4         33.230
    5         33.230

      COEFICIENTE SISMICO=   0.080

NIVEL,ALTURA(M),FZA.RIGIDEZ(TON)
     1    2.60          0.886
     2    5.20          1.772
     3    7.80          2.658
     4   10.40          3.545
     5   13.00          4.431

      RIGIDECES DE ENTREPISO,EN TON/M
      2.9806E+041.2200E+048.2321E+035.8534E+033.3549E+03
```

```
              ANALISIS DEL MURO MARCO A CONSIDERANDO CARGA ESTATICA Y EFE

           1   NO. DE CONDICION DE CARGA
          20   NO. DE BARRAS CARGADAS
           0   NO. DE NUDOS CARGADOS

       DATOS PARA EL CASO DE BARRAS CON CARGAS INTERMEDIAS DISTINTAS A PESO PRO
RRA NO TIPO.GRAFI
     1        1
     2        1
     3        1
     4        1
     5        1
     6        1
     7        1
     8        1
     9        1
    10        1
    11        1
    12        1
    13        1
    14        1
    15        1
    16        1
    17        1
    18        1
    19        1
    20        1

 BARRA     1 CARGA DIST UNIFOR CONTIN(TON/M)=    -0.5000
 BARRA     2 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     3 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     4 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     5 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     6 CARGA DIST UNIFOR CONTIN(TON/M)=     0.2500
 BARRA     7 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     8 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA     9 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA    10 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA    11 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA    12 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
 BARRA    13 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
```

```
BARRA   14 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
BARRA   15 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
BARRA   16 CARGA DIST UNIFOR CONTIN(TON/M)=    0.0000
BARRA   17 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
BARRA   18 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
BARRA   19 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR
BARRA   20 CARGA DIST UNIFOR CONTIN(TON/M)=A LA BARRA ANTERIOR

CIONES CONCENTRADAS EN LOS NUDOS (EN TON Y TON-M)
UDO NO.FZA.HORIZONTAL FZA. VERTICAL      MOMENTO

      1            0.000          -1.491         0.00000
     17            0.000          -2.982         0.00000
     33            0.000          -2.982         0.00000
     49            0.000          -2.982         0.00000
     65            0.000          -2.982         0.00000
     81            0.000          -2.982         0.00000
     97            0.000          -2.982         0.00000
    113            0.000          -2.982         0.00000
    129            0.000          -2.982         0.00000
    145            0.000          -2.982         0.00000
     51            0.000          -2.982         0.00000
    167            0.000          -2.982         0.00000
    183            0.000          -2.982         0.00000
    199            0.000          -2.982         0.00000
    215            0.000          -1.491         0.00000
      4            0.056          -1.491         0.00000
     20            0.056          -2.982         0.00000
     36            0.056          -2.982         0.00000
     52            0.056          -2.982         0.00000
     68            0.056          -2.982         0.00000
     84            0.056          -2.982         0.00000
    100            0.056          -2.982         0.00000
    116            0.056          -2.982         0.00000
    132            0.056          -2.982         0.00000
    146            0.056          -2.982         0.00000
    154            0.056          -2.982         0.00000
    170            0.056          -2.982         0.00000
    166            0.056          -2.982         0.00000
    202            0.056          -2.982         0.00000
    218            0.056          -1.491         0.00000
      7            0.112          -1.491         0.00000
     23            0.112          -2.982         0.00000
     39            0.112          -2.982         0.00000
     55            0.112          -2.982         0.00000
     71            0.112          -2.982         0.00000
     87            0.112          -2.982         0.00000
    103            0.112          -2.982         0.00000
    119            0.112          -2.982         0.00000
    135            0.112          -2.982         0.00000
    147            0.112          -2.982         0.00000
    157            0.112          -2.982         0.00000
```

10

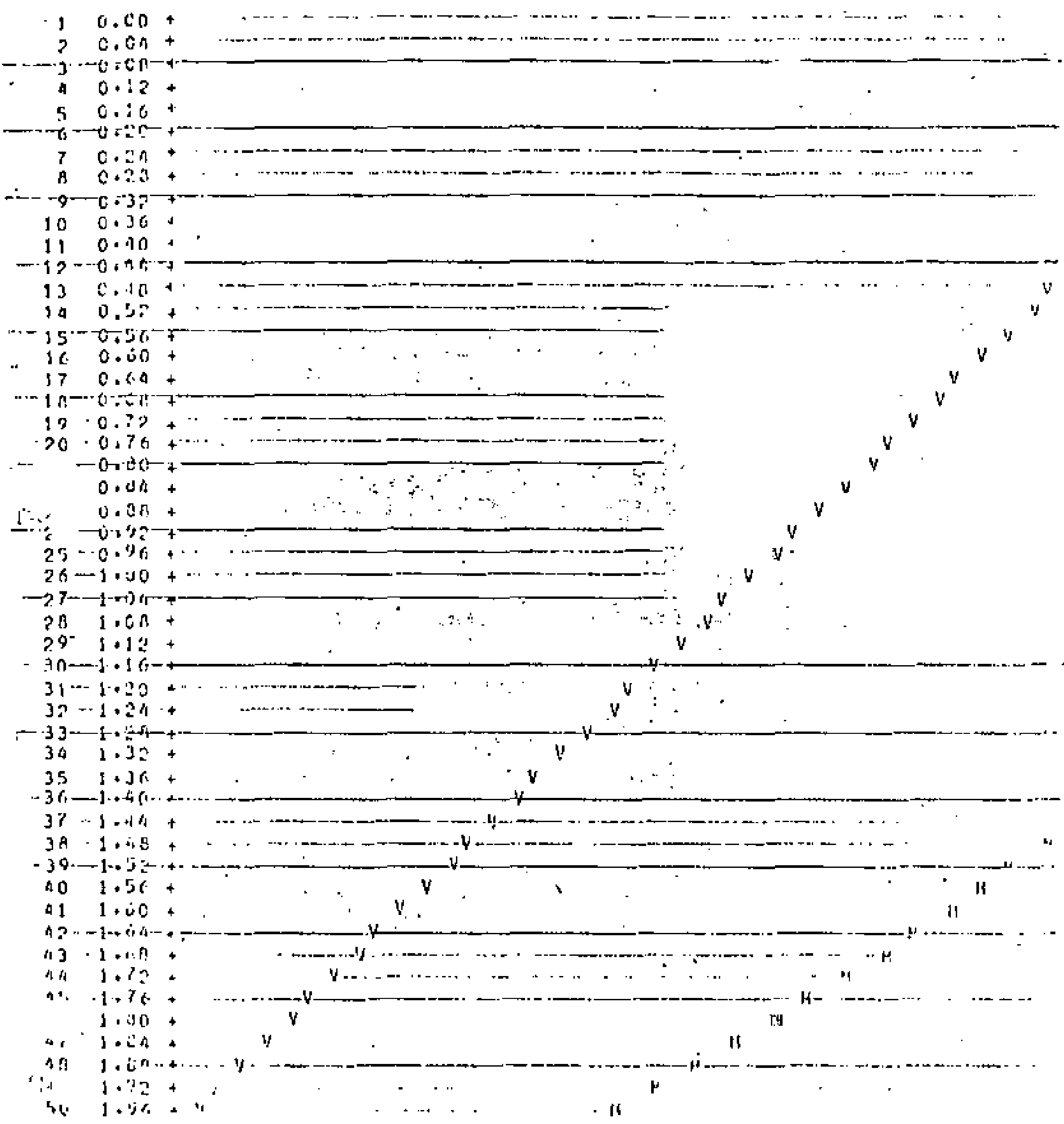
```
  173          0.112        -2.982        0.00000
  189          0.112        -2.982        0.00000
  205          0.112        -2.982        0.00000
  221          0.112        -1.491        0.00000
   10          0.168        -1.491        0.00000
   26          0.168        -2.982        0.00000
   42          0.168        -2.982        0.00000
   58          0.168        -2.982        0.00000
   74          0.168        -2.982        0.00000
   90          0.168        -2.982        0.00000
  106          0.168        -2.982        0.00000
  122          0.168        -2.982        0.00000
  138          0.168        -2.982        0.00000
  144          0.168        -2.982        0.00000
  160          0.168        -2.982        0.00000
  176          0.168        -2.982        0.00000
  192          0.168        -2.982        0.00000
  208          0.168        -2.982        0.00000
  224          0.168        -1.491        0.00000
   13          0.220        -1.491        0.00000
   29          0.220        -2.982        0.00000
   45          0.220        -2.982        0.00000
   61          0.220        -2.982        0.00000
   77          0.220        -2.982        0.00000
   93          0.220        -2.982        0.00000
  109          0.220        -2.982        0.00000
  125          0.220        -2.982        0.00000
  141          0.220        -2.982        0.00000
  147          0.220        -2.982        0.00000
  163          0.220        -2.982        0.00000
  179          0.220        -2.982        0.00000
  195          0.220        -2.982        0.00000
  211          0.220        -2.982        0.00000
  227          0.220        -1.491        0.00000
   16          0.280        -1.491        0.00000
   32          0.280        -2.982        0.00000
   48          0.280        -2.982        0.00000
   64          0.280        -2.982        0.00000
   80          0.280        -2.982        0.00000
   96          0.280        -2.982        0.00000
  112          0.280        -2.982        0.00000
  128          0.280        -2.982        0.00000
  144          0.280        -2.982        0.00000
  150          0.280        -2.982        0.00000
  166          0.280        -2.982        0.00000
  182          0.280        -2.982        0.00000
  198          0.280        -2.982        0.00000
  214          0.280        -2.982        0.00000
  230          0.280        -1.491        0.00000

        ----------DESPLAZAMIENTOS NODALES DE LA ESTRUCTURA (M)----------
PTO.   D. H O R I Z O N T A L      V E R T I C A L      G I R O S   (RAD)
   1        0.                       0.                    0.
   2        3.4578361E-05           -4.3734830E-05        -9.6972576E-05
   3        1.6178531E-04           -6.8345851E-05        -1.9723313E-04
   4        3.6482475E-04           -1.4563466E-04        -2.7239484E-04
   5        6.2696840E-04           -1.9244611E-04        -3.3305638E-04
   6        9.3653634E-04           -2.4714974E-04        -3.8158451E-04
            1.2833962E-03           -3.1083084E-04        -4.0108536E-04
```

Sigue igual

```
ESCALA DE LA GRAFICA=  1.16380E-02 UNIDADES/COLUMNA
ORDENADA MENOR       = -9.02614E-01
ORDENADA MAYOR       =  2.61189E-01

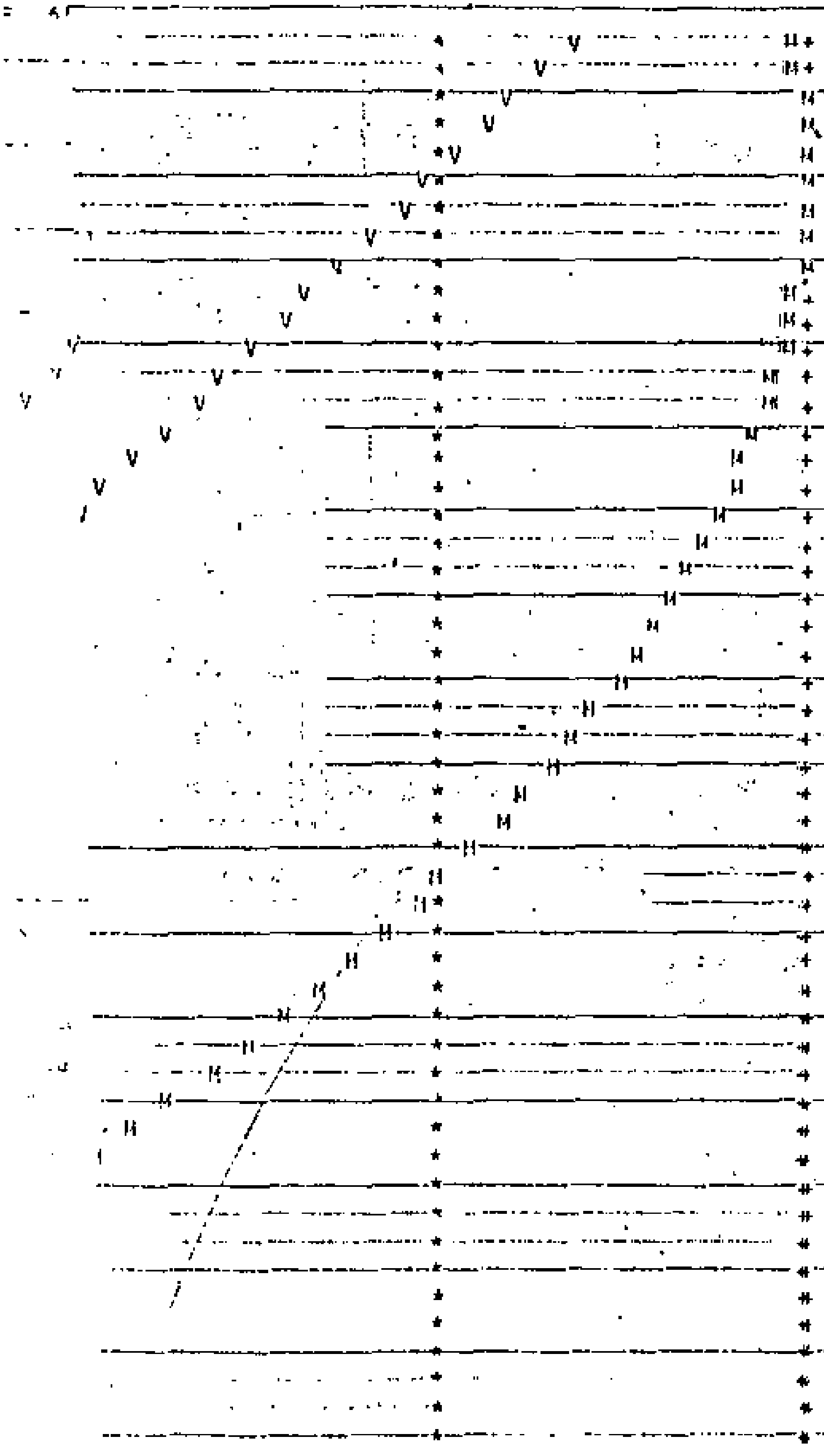
                    BARRA NO.   1,NUDO INICIAL=  52,NUDO FINAL
```



CORTANTES EN TON. Y MOMENTOS EN TON-M

HOJA 12

```
NUDO FINAL=  68                                   V            M
```



```
 9.74E-02   2.52E-01
 7.74E-02   2.55E-01
 5.74E-02   2.58E-01
 3.74E-02   2.60E-01
 1.74E-02   2.61E-01
-2.61E-03   2.61E-01
-2.26E-02   2.61E-01
-4.26E-02   2.59E-01
-6.26E-02   2.57E-01
-8.26E-02   2.54E-01
-1.03E-01   2.51E-01
-1.23E-01   2.46E-01
-1.43E-01   2.41E-01
-1.63E-01   2.35E-01
-1.83E-01   2.28E-01
-2.03E-01   2.20E-01
-2.23E-01   2.12E-01
-2.43E-01   2.02E-01
-2.63E-01   1.92E-01
-2.83E-01   1.81E-01
-3.03E-01   1.70E-01
-3.23E-01   1.57E-01
-3.43E-01   1.44E-01
-3.63E-01   1.30E-01
-3.83E-01   1.15E-01
-4.03E-01   9.91E-02
-4.23E-01   8.26E-02
-4.43E-01   6.53E-02
-4.63E-01   4.72E-02
-4.83E-01   2.83E-02
-5.03E-01   8.58E-03
-5.23E-01  -1.19E-02
-5.43E-01  -3.32E-02
-5.63E-01  -5.53E-02
-5.83E-01  -7.82E-02
-6.03E-01  -1.02E-01
-6.23E-01  -1.26E-01
-6.43E-01  -1.52E-01
-6.63E-01  -1.78E-01
-6.83E-01  -2.05E-01
-7.03E-01  -2.32E-01
-7.23E-01  -2.61E-01
-7.43E-01  -2.90E-01
-7.63E-01  -3.20E-01
-7.83E-01  -3.51E-01
-8.03E-01  -3.83E-01
-8.23E-01  -4.15E-01
-8.43E-01  -4.49E-01
-8.63E-01  -4.83E-01
-8.83E-01  -5.18E-01
-9.03E-01  -5.54E-01
```

13

```
                    ESCALA DE LA GRAFICA=  7.12272E-03 UNIDADES/COLUMNA
                    ORDENADA MENOR       = -3.59968E-01
                    ORDENADA MAYOR       =  3.52308E-01

                                    BARRA NO.  16,NUDO INICIAL= 145,NUDO FINAL=

    1  0.00 M                                                 *
    2  0.05 + M                                               *
----3--0.10-+---M---------------------------------------------*---------------
    4  0.16 +     M                                           *
    5  0.21 +       M                                         *
----6--0.26-+---------M---------------------------------------*---------------
    7  0.31 +           M                                     *
    8  0.36 +             M                                   *
----9--0.42-+---------------M---------------------------------*---------------
   10  0.47 +                 M                               *
   11  0.52 +                   M                             *
---12--0.57-+---------------------M---------------------------*---------------
   13  0.62 +                       M                         *
   14  0.68 +                         M                       *
---15--0.73-+---------------------------M---------------------*---------------
   16  0.78 +                             M                   *
   17  0.83 +                               M                 *
---18--0.88-+---------------------------------M---------------*---------------
   19  0.94 +                                   M             *
   20  0.99 +                                     M           *
---21--1.04-+---------------------------------------M---------*---------------
       1.09 +                                         M       *
       1.14 +                                           M     *
----2--1.20-+---------------------------------------------M---*---------------
       1.25 +                                               M *
    2  1.30 +                                                M*
---27--1.35-+-------------------------------------------------*M--------------
   28  1.40 +                                                 *  M
   29  1.46 +                                                 *    M
---30--1.51-+-------------------------------------------------*------M--------
   31  1.56 +                                                 *        M
   32  1.61 +                                                 *          M
---33--1.66-+-------------------------------------------------*------------M.
   34  1.72 +                                                 *
   35  1.77 +                                                 *
---36--1.82-+-------------------------------------------------*---------------
   37  1.87 +                                                 *
   38  1.92 +                                                 *
---39--1.98-+-------------------------------------------------*---------------
   40  2.03 +                                                 *
   41  2.08 +                                                 *
---42--2.13-+-------------------------------------------------*---------------
   43  2.18 +                                                 *
   44  2.24 +                                                 *
---45--2.29-+-------------------------------------------------*---------------
       2.34 +                                                 *
       2.39 +                                                 *
---48--2.44-+-------------------------------------------------*---------------
   49  2.50 +                                                 *
   50  2.55 +                                                 *
   51  2.60-+-------------------------------------------------*---------------
          CORTANTES EN TON. Y MOMENTOS EN TON-M
```

```
     UAGESZCCISUA         14

               -----------------------------------------------------------------
     AL= 145.NUDO FINAL= 146                                     V

I= 14   ------------------------------------------------------------------------
        *                                     V          +  2.74E-01 -3.60E-01
        *                                     V          +  2.74E-01 -3.46E-01
     ---*-------------------------------------V----------+--2.74E-01 -3.31E-01
        *                                     V          +  2.74E-01 -3.17E-01
        *                                     V          +  2.74E-01 -3.03E-01
     ---*-------------------------------------V----------+--2.74E-01 -2.89E-01
        *                                     V          +  2.74E-01 -2.74E-01
        *                                     V          +  2.74E-01 -2.60E-01
     ---*-------------------------------------V----------+--2.74E-01 -2.46E-01
        *                                     V          +  2.74E-01 -2.32E-01
        *                                     V          +  2.74E-01 -2.18E-01
     ---*-------------------------------------V----------+--2.74E-01 -2.03E-01
        *                                     V          +  2.74E-01 -1.89E-01
        *                                     V          +  2.74E-01 -1.75E-01
     ---*-------------------------------------V----------+--2.74E-01 -1.61E-01
        *                                     V          +  2.74E-01 -1.46E-01
        *                                     V          +  2.74E-01 -1.32E-01
     ---*-------------------------------------V----------+--2.74E-01 -1.18E-01
        *                                     V          +  2.74E-01 -1.04E-01
        *                                     V          +  2.74E-01 -8.93E-02
     ---*-------------------------------------V----------+--2.74E-01 -7.51   2
        *                                     V          +  2.74E-01 -6.0    2
        *                                     V          +  2.74E-01 -4.66.  2
     ---*-------------------------------------V----------+--2.74E-01 -3.23E-02
     I  *                                     V          +  2.74E-01 -1.81E-02
       H*                                     V          +  2.74E-01 -3.83E-03
     ---*H------------------------------------V----------+--2.74E-01  1.04E-02
        *  H                                  V          +  2.74E-01  2.47E-02
        *    H                                V          +  2.74E-01  3.89E-02
     ---*------H------------------------------V----------+--2.74E-01  5.32E-02
        *        H                            V          +  2.74E-01  6.74E-02
        *          H                          V          +  2.74E-01  8.16E-02
     ---*------------H------------------------V----------+--2.74E-01  9.59E-02
        *              H                      V          +  2.74E-01  1.10E-01
        *                H                    V          +  2.74E-01  1.24E-01
     ---*------------------H------------------V----------+--2.74E-01  1.39E-01
        *                    H                V          +  2.74E-01  1.53E-01
        *                      H              V          +  2.74E-01  1.67E-01
     ---*------------------------H------------V----------+--2.74E-01  1.81E-01
        *                          H          V          +  2.74E-01  1.96E-01
        *                            H        V          +  2.74E-01  2.10E-01
     ---*------------------------------H------V----------+--2.74E-01  2.24E-01
        *                                H    V          +  2.74E-01  2.38E-01
        *                                  H  V          +  2.74E-01  2.53E-01
     ---*------------------------------------HV----------+--2.74E-01  2.67E-01
        *                                     VH         +  2.74E-01  2.81E
        *                                     V  H       +  2.74E-01  2.95E
     ---*-------------------------------------V----H-----+--2.74E-01  3.10E-01
        *                                     V      H   +  2.74E-01  3.24E-01
        *                                     V        H +  2.74E-01  3.38E-01
     ---*-------------------------------------V----------H--2.74E-01  3.52E-01
```

```
BARRA  E X T R E M O                               EXTREMO INICIAL (TON Y TON-M)          ***
 NO. INICIAL FINAL      N O R M A L        C O R T A N T E        FLEXIONANTE


ELEM.****COORDENADAS (M)*****ESFUERZOS REFERIDOS A XY DE (TON/M**2)*****ESFUERZOS P
 NO.        X            Y        TXX O TSS       TYY O TTT        TXY  O TST          T11

   1      0.250        0.433    -2.7130E+00     -1.1632E+02     -3.2908E+00     -2.7617E+
   2      0.250        2.167    -1.6729E+00     -1.2962E+02      5.1504E+00     -1.4649E+
   3      0.250        3.900     1.1331E-01     -1.1664E+02      2.7319E+00      1.7720E-
   4      0.250        5.633     1.3506E+00     -1.0003E+02      1.3674E+00      1.3850E+
   5      0.250        7.367    -1.8471E+00     -1.0671E+02      3.3696E+00     -1.7390E+
   6      0.250        9.100     9.4658E-02     -7.8666E+01      3.7715E-01      9.6667E-
   7      0.250       10.833     1.1013E+00     -4.5407E+01      1.7925E+00      1.2507E+
   8      0.250       12.567    -3.4803E+00     -3.0996E+01      2.1333E+00     -3.3159E+
   9      0.750        1.300     2.2708E+00     -2.0299E+02      6.8115E+00      2.5245E+
  10      0.750        3.033     1.9747E+00     -1.6211E+02      4.5920E+00      2.7011E+
  11      0.750        4.767    -8.6727E-01     -1.5610E+02      6.2326E+00     -6.1734E-
  12      0.750        6.500     2.2665E-01     -1.1867E+02      6.3487E+00      3.8539E-
  13      0.750        8.233     9.9559E-01     -8.1042E+01     -5.8073E-01      9.9970E-
  14      0.750        9.967    -5.4520E-01     -7.5519E+01      1.2940E+00     -5.2590E-
  15      0.750       11.700     4.9256E-01     -3.8914E+01     -1.5702E+00      5.5535E-
  16      1.250        0.433    -3.1459E+01     -2.8913E+02      7.2605E+00     -3.1255E+
  17      1.250        2.167    -1.9015E+00     -2.6637E+02      1.9460E+00     -1.8972E+
  18      1.250        3.900    -3.7152E-01     -2.0097E+02      3.0142E+00     -3.2617E-
  19      1.250        5.633    -4.4169E+00     -1.8299E+02     -1.2770E+00     -4.0603E+
  20      1.250        7.367     1.9265E+00     -1.2741E+02      2.6868E+00      1.0923E+
          1.250        9.100     3.0246E-01     -7.7599E+01      2.2454E-01      3.0311E-
          1.250       10.833    -5.3077E+00     -3.6086E+01     -8.4242E+00     -4.6646E+
  2       1.250       12.567     1.7134E+00     -4.2566E+01     -2.1240E+00      1.8151E+
  24      3.750        1.300     2.2063E+00     -8.4175E+01      7.7766E+00      2.9065E+
  25      3.750        3.033     9.3962E+00     -7.6809E+01      1.3074E+01      1.1321E+
  26      3.750        4.767    -1.0298E+01     -1.1618E+02      1.2253E+01     -1.6787E-
  27      3.750        6.500     1.0979E+00     -9.2124E+01      6.5565E+00      1.3201E+
  28      3.750        8.233     1.2549E+01     -6.8625E+01      1.1860E+01      1.9246E+
  29      3.750        9.967    -1.9053E+01     -9.1638E+01      9.9401E+00     -1.9562E+
  30      3.750       11.700     5.3633E+00     -4.8286E+01      1.3925E+01      5.3094E+
  31      4.250        0.433    -1.6448E+01     -1.5290E+02     -7.9152E+00     -1.5990E+
  32      4.250        2.167    -5.8705E+00     -1.6255E+02      1.5659E+01     -4.3295E+
  33      4.250        3.900     4.0725E-01     -1.3557E+02      1.9716E+01      3.2925E+
  34      4.250        5.633     6.4613E+00     -1.0595E+02     -2.1942E+00      6.9197E+
  35      4.250        7.367    -6.1507E+00     -1.0897E+02      1.9155E+01     -5.1574E+
  36      4.250        9.100     2.1116E-01     -7.7235E+01      1.5605E+01      3.2372E+
  37      4.250       10.833     6.3101E+00     -4.8527E+01      6.9079E-01      6.3150E+
  38      4.250       12.567    -8.9754E+00     -3.3925E+01      2.8655E+00     -8.6505E+
  39      4.750        1.300     1.1622E-01     -2.3719E+02      1.8896E+01      1.6114E+
  40      4.750        3.033    -1.4096E+00     -1.8885E+02      1.1051E+01     -8.1702E-
  41      4.750        4.767    -4.9239E+00     -1.6749E+02      1.0149E+01      5.5193E+
  42      4.750        6.500     4.3703E-01     -1.2396E+02      1.0857E+01      3.2327E+
  43      4.750        8.233    -4.1312E+00     -5.1678E+01      4.2561E+00     -3.8083E+
  44      4.750        9.967     7.1906E+00     -7.2909E+01      4.1159E+00      7.8033E+
  45      4.750       11.700    -6.0367E-02     -3.7660E+01      1.4248E+01      4.7200E+
  4       5.250        0.433    -4.1364E+01     -3.3618E+02     -1.4760E+01     -4.0633E+
  47      5.250        2.167     5.6458E+00     -2.6609E+02      9.2066E+00      5.9353E+
  4       5.250        3.900    -2.0970E-01     -2.1472E+02      5.4979E+00     -1.6951E+
  49      5.250        5.633    -2.3122E+01     -1.6247E+02     -6.4186E+00     -2.2927E+
  50      5.250        7.367     1.5496E+01     -1.2390E+02      1.5407E+01      1.7170E+
  51      5.250        9.100     1.2946E+00     -7.5610E+01      2.5903E+00      1.3823E+
```

10

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     ****                                     EXTREMO FINAL  (TON Y TON-M)
MTE         N O R M A L        C O R T A N T E      FLEXIONANTE

ESFUERZOS PRINCIPALES REFXY (TON/M**2) ***DIR.PPAL*
   T11                  T22              TAUMAX      (GRADOS) T

  -2.6178E+00     -1.1642E+02       5.6900E+01     -14.578
  -1.4460E+00     -1.2982E+02       6.4179E+01      23.015
   1.7720E-01     -1.1670E+02       5.8440E+01      13.397
   1.3650E+00     -1.0007E+02       5.0726E+01      10.548
  -1.7390E+00     -1.0681E+02       5.2538E+01      18.386
   9.6467E-02     -7.8670E+01       3.9383E+01       2.746
   1.2507E+00     -4.5476E+01       2.3363E+01     -22.087
  -3.3159E+00     -3.1161E+01       1.3922E+01      44.070
   2.5245E+00     -2.0322E+02       1.0287E+02      18.983
   2.1031E+00     -1.4224E+02       7.2169E+01      16.018
  -6.1744E-01     -1.5635E+02       7.7866E+01      22.954
   3.8549E-01     -1.1843E+02       5.9408E+01      20.919
   9.9970E-01     -8.1046E+01       4.1023E+01      -4.056
  -5.2590E-01     -7.5538E+01       3.7506E+01       9.191
   5.5535E-01     -3.8976E+01       1.9766E+01     -22.841
  -3.1255E+01     -2.8039E+02       1.2908E+02      16.127
  -1.8872E+00     -2.6639E+02       1.3225E+02       4.216
  -3.2617E-01     -2.0101E+02       1.0034E+02       8.613
  -4.8043E+00     -1.4290E+02       6.8997E+01      -5.202
   1.9823E+00     -1.2746E+02       6.4722E+01      11.896
   3.0311E-01     -7.7599E+01       3.8951E+01       1.651
  -4.6884E+00     -3.6710E+01       1.6013E+01     -30.106
   1.0151E+00     -4.2658E+01       2.2201E+01     -27.800
   2.9009E+00     -8.4869E+01       4.3885E+01      51.035  Sigue
   1.1321E+01     -7.8733E+01       4.5027E+01     -84.063  igual
  -1.6787E+01     -1.1769E+02       5.0450E+01      70.281
   1.3201E+00     -9.2346E+01       4.6833E+01      22.917
   1.4246E+01     -7.0323E+01       4.2285E+01      81.005
  -1.8502E+01     -9.2989E+01       3.7243E+01     -77.398
   5.3098E+00     -4.8322E+01       2.6816E+01     -14.859
  -1.5090E+01     -1.5336E+02       6.8684E+01     -33.087
  -4.3205E+00     -1.6410E+02       7.9891E+01      56.516
   3.2925E+00     -1.3837E+02       7.0831E+01      80.805
   6.9197E+00     -1.0645E+02       5.6684E+01     -36.058
  -5.1574E+00     -1.0996E+02       5.2402E+01      55.867
   3.2372E+00     -8.0261E+01       4.1749E+01     109.743
   6.3150E+00     -4.2532E+01       2.4423E+01      -5.757
  -8.6505E+00     -3.4250E+01       1.2800E+01      60.683
   1.6114E+00     -2.3860E+02       1.2015E+02      85.242
  -8.1209E-01     -1.8551E+02       9.2349E+01     -34.363
   5.5193E+00     -1.6809E+02       8.6803E+01      33.573
   3.2327E+00     -1.2675E+02       6.4993E+01      80.332
  -3.8063E+00     -8.1911E+01       3.9006E+01     -31.221
   7.4033E+00     -7.2222E+01       3.9813E+01      29.670
   4.7208E+00     -4.2457E+01       2.3589E+01     185.789
  -4.0633E+01     -3.3691E+02       1.4814E+02     -28.875
   5.9353E+00     -2.8718E+02       1.4656E+02      18.008
  -1.4891E-01     -2.1486E+02       1.0736E+02      14.678
  -2.2827E+01     -1.6277E+02       6.9970E+01     -24.317
   1.7170E+01     -1.2567E+02       7.1420E+01      62.287
   1.3823E+00     -7.5706E+01       3.8544E+01      19.326
```

```
FUERZAS EN PUNTOS NODALES DE LOS SIGUIENTES ELEMENTOS
                                                                    17
ELEMENTO        NUDO      FX                  FY                M
-------------------------------------------------------------------------
       1          2       0.06401            -3.01552          0.00000
       1          1       0.65426             7.38313          0.00000
-------------------------------------------------------------------------


-------------------------------------------------------------------------

       1         17      -0.71632             4.28767          0.00000
       1         18      -0.20195            -8.65229          0.00000
      16         34       2.78918            -9.61457          0.00000
      16         33       2.02773            10.02470          2.00000
      16         49      -4.46482            10.93294          0.00000
      16         50      -0.35209           -19.34308          2.00000
      31         82       1.77110            -5.20479          0.00000
      31         81       0.96367             8.96433          1.00000
      31         97      -2.70536             6.30227          0.00000
      31         98      -0.02942           -10.96185          1.00000
      46        114       3.98498           -11.91785          0.00000
      46        113       2.21670            20.62108          2.00000
      46        129      -5.91376            13.06553         -1.00000
      46        130      -0.28993           -21.76875          2.00000
      61        168       1.92696            -5.42451          0.00000
      61        167       0.90419             9.31794         -1.00000
      61        153      -2.95614             6.92760          0.00000
      61        154       0.12563           -10.52102          1.00000
      76        200       4.30672           -13.14723          0.00000
      76        199       2.24360            22.61114          3.00000
      76        215      -6.35623            14.42406          1.00000
      76        216      -0.27409           -23.48797          3.00000
      91         18       0.93066           -12.98592         -1.00000
      91         17       2.83964             5.90690          0.00000
      91         33      -1.70014            10.11524         -1.00000
      91         34      -2.07016            -7.03622          0.00000
     106         66       0.32843            -3.99076          0.00000
     106         65       0.98254             1.42865          0.00000
     106         81      -0.63325             5.15606          0.00000
     106         82      -0.67772            -2.59401          0.00000
     121         98       1.89749           -16.60581         -2.00000
     121         97       2.47282             6.44249          0.00000
     121        113      -2.30616            17.43689         -1.00000
     121        114      -2.06425            -7.27354          0.00000
     136        152       0.34806            -3.76478          0.00000
     136        151       0.77570             1.12420          0.00000
     136        167      -0.61772             5.07866         -0.00000
     136        168      -0.50624            -2.43869          0.00000
     151        184       2.14604           -18.15582         -2.00000
     151        183       2.30852             6.84765          0.00000
     151        199      -2.40606            19.04229         -2.00000
     151        200      -1.96850            -7.73411          0.00000
-------------------------------------------------------------------------

TIEMPO DE EJECUCION         =   118.5833       SEG
TIEMPO DE ENTRADA Y SALIDA  =    27.2667       SEG


TIEMPO DE EJECUCION         =   119.0167       SEG
TIEMPO DE ENTRADA Y SALIDA  =    27.6333       SEG
```

In the above the surface integral is only taken on external boundaries where $\partial u/\partial n$ or $\partial v/\partial n$ is specified. If u and v are given there the equations are not formed in boundary points.

Although a standard form of a finite element relationship has been established the element matrix is not symmetric. Such non-symmetric matrices often arise in flow problems[14] but the reader will observe that here a simple change of sign of Eq. (3.36) re-establishes symmetry after integration by parts. Galerkin process is thus not unique.

An alternative approach to the above problem could be pursued by introducing a stream function concept. If we define

$$u = -\frac{\partial \theta}{\partial y}, \qquad v = \frac{\partial \theta}{\partial x} \tag{3.46}$$

then Eq. (3.36) is identically satisfied and we are left with two governing equations:

$$\begin{aligned} X-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(-\frac{\partial \theta}{\partial y}\right) &= 0, \\ Y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial \theta}{\partial x}\right) &= 0. \end{aligned} \tag{3.47}$$

Differentiating the first with respect to y and second with respect to x and subtracting, p is eliminated and only one equation is left.

$$\mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}\right)+\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y} = 0. \tag{3.48}$$

A similar process of approximate formulation as before can be adopted and the reader can perform this as an exercise. He will find that now symmetric element matrices arise and indeed the formulation will be very similar to that discussed in the chapter on plate bending. The shape function now, however, will have to satisfy continuity of first derivatives between elements as second order differentials occur in the various integrals. Such problems have been dealt with in an axi-symmetric context by Atkinson *et al.*[14] from the basis of a variational form given in Chapter 15, p. 317.

These examples have been introduced to illustrate the general applicability of the method. The particular problem discussed here, however, is of some considerable engineering interest and much work in the solution of the Navier–Stokes

equation is currently in progress. In the illustration, to linearize the equations, the dynamic terms

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y},\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$$

have been omitted from the two Eqs. (3.35) respectively. Their retention is obviously possible but then it will be found that the resulting equations of the general form (3.43) are non-linear, $[K]$ being dependent itself on the velocities. The derivation is too complex to be discussed in detail here but the reader could consider extension of the non-linear techniques of Chapter 18 to be applicable here

3.7 Concluding Remarks

In addition to generalizing the finite element concept to that of approximately solving a variational problem, the alternative of proceeding directly by approximating to the differential expression was presented. Both procedures open up many, as yet unexplored, fields of application. Some general ideas in similar context are given by Oden.[15] Other uses of finite element process, such as minimization of the root mean square value of errors, can easily be envisaged.

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4. Plane Stress and Plane Strain

4.1 Introduction

Two-dimensional elastic problems were the first successful examples of the application of the finite element method.[1,2] Indeed, we have already used this situation to illustrate the basis of the finite element formulation in Chapter 2 where the general relationships were derived. These basic relationships are given in Eqs. (2.1), (2.2), (2.3), (2.9), (2.10), and (2.16) and for quick reference are summarized in Appendix II.

In this chapter the particular relationships for the problem in hand will be derived in more detail, and illustrated by suitable practical examples, a procedure that will be followed throughout the remainder of the book.

Only the simplest, triangular, element will be discussed in detail but the basic approach is general. More elaborate elements to be discussed in later chapters would be introduced to the same problem in an identical manner.

The reader not familiar with the applicable basic definitions of elasticity is referred to elementary texts on the subject, in particular to the text by Timoshenko and Goodier,[3] whose notation will be widely used here.

In both problems of plane stress and plane strain the displacement field is uniquely given by the u and v displacements in directions of the cartesian, orthogonal x and y axes.

Again, in both, the only strains and stresses that have to be considered are the three components in the x–y plane. In the case of *plane stress*, by definition, all other components of stress are zero and therefore give no contribution to internal work. In *plane strain* the stress in a direction perpendicular to the x–y plane is not zero. However, by definition, the strain in that direction is zero, and therefore no contribution to internal work is made by this stress, which can in fact be explicitly evaluated from the three main stress components, if desired, at the end of all computation.

4.2 Element Characteristics

4.2.1 *Displacement functions.* Figure 4.1 shows the typical triangular element considered, with nodes i, j, m numbered in an anti-clockwise order.

The displacements of a node have two components

$$\{\delta_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{4.1}$$

and the six components of element displacements are listed as a vector

$$\{\delta\}^e = \begin{Bmatrix} \delta_i \\ \delta_j \\ \delta_m \end{Bmatrix}. \tag{4.2}$$

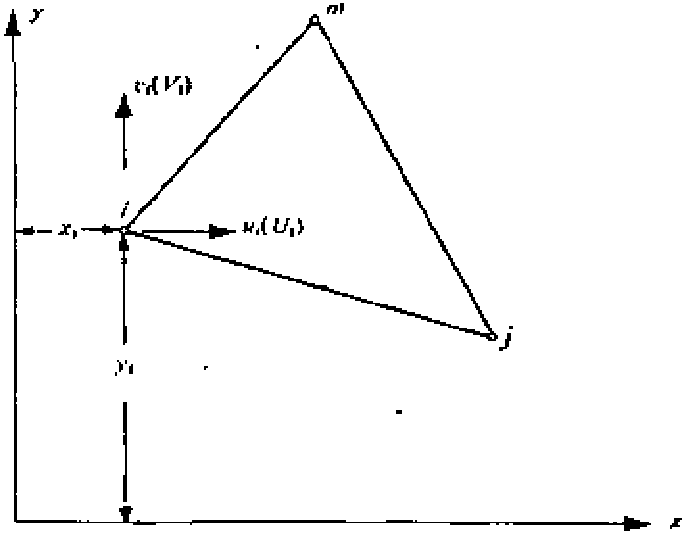


Fig. 4.1 An element of a continuum in plane stress or plane strain

The displacements within an element have to be uniquely defined by these six values. The simplest representation is clearly given by two linear polynomials

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y, \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y. \end{aligned} \tag{4.3}$$

The six constants α can be evaluated easily by solving the two sets of three simultaneous equations which will arise if the nodal co-ordinates are inserted and the displacements equated to the appropriate nodal displacements. Writing, for example,

$$\begin{aligned} u_i &= \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j &= \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_m &= \alpha_1 + \alpha_2 x_m + \alpha_3 y_m \end{aligned} \tag{4.4}$$

we can easily solve for α_1, α_2, and α_3 in terms of the nodal displacements u_i, u_j, u_m and obtain finally

$$u = \frac{1}{2\Delta}\{(a_i+b_ix+c_iy)u_i+(a_j+b_jx+c_jy)u_j+(a_m+b_mx+c_my)u_m\} \quad (4.5a)$$

in which

$$\begin{aligned} a_i &= x_jy_m-x_my_j \\ b_i &= y_j-y_m = y_{jm} \\ c_i &= x_m-x_j = x_{mj} \end{aligned} \quad (4.5b)$$

with the other coefficients obtained by a cyclic permutation of subscripts in the order, i, j, m, and where

$$2\Delta = \det\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = 2 \quad \text{(area of triangle } ijm). \quad (4.5c)$$

As the equations for the vertical displacement v are similar we also have

$$v = \frac{1}{2\Delta}\{(a_i+b_ix+c_iy)v_i+(a_j+b_jx+c_jy)v_j+(a_m+b_mx+c_my)v_m\}. \quad (4.6)$$

Though not strictly necessary at this stage we can represent the above relations Eqs. (4.5a) and (4.6) in the standard form of Eq. (2.1)

$$\{f\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{\delta\}^e = [IN'_i, IN'_j, IN'_m]\{\delta\}^e \quad (4.7)$$

with I a two by two identity matrix, and

$$N'_i = (a_i+b_ix+c_iy)/2\Delta \text{ etc.} \quad (4.8)$$

Note: if co-ordinates are taken from the centroid of the element then
$x_i+x_m+x_j = y_i+y_j+y_m = 0$ and $a_i = 2\Delta/3 = a_j = a_m$.

The chosen displacement function automatically guarantees continuity of displacements with adjacent elements because the displacements vary linearly along any side of the triangle and, with identical displacement imposed at the nodes, the same displacement will clearly exist all along an interface.

4.2.2 *Strain (total).* The total strain at any point within the element can be defined by its three components which contribute to internal work.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x} \end{Bmatrix}. \quad (4.9)$$

Using Eqs. (4.7) or (4.5a) *and* (4.6) we have

$$\{\varepsilon\} = \begin{bmatrix} \dfrac{\partial N'_i}{\partial x} & 0 & \dfrac{\partial N'_j}{\partial x} & 0 & \dfrac{\partial N'_m}{\partial x} & 0 \\ 0 & \dfrac{\partial N'_i}{\partial y} & 0 & \dfrac{\partial N'_j}{\partial y} & 0 & \dfrac{\partial N'_m}{\partial y} \\ \dfrac{\partial N'_i}{\partial y} & \dfrac{\partial N'_i}{\partial x} & \dfrac{\partial N'_j}{\partial y} & \dfrac{\partial N'_j}{\partial x} & \dfrac{\partial N'_m}{\partial y} & \dfrac{\partial N'_m}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$= \frac{1}{2\Delta}\begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix}\{\delta\}^e \quad (4.10)$$

which defines the matrix $[B]$ of Eq. (2.2) explicitly.

It will be noted that in this case the $[B]$ matrix is independent of the position within the element, and hence the strains are constant throughout it. Obviously, the criterion of constant strain mentioned in Chapter 2 is satisfied by the shape functions.

4.2.3 *Initial strain (thermal strain).* 'Initial' strains, that is strains which are independent of stress, may be due to many causes. Shrinkage, crystal growth or, most frequently, temperature changes will, in general, result in an initial strain vector.

$$\{\varepsilon_0\} = \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix}. \quad (4.11)$$

Although this initial strain may, in general, depend on the position within the element, it will usually be defined by average, constant, values. This is consistent with the constant strain conditions imposed by the prescribed displacement function.

Thus, for the case of *plane stress* in an isotropic material in an element subject to a temperature rise θ^e with a coefficient of thermal expansion α, we will have, for instance,

$$\{\varepsilon_0\} = \begin{Bmatrix} \alpha\theta^e \\ \alpha\theta^e \\ 0 \end{Bmatrix} \tag{4.12}$$

as no shear strains are caused by a thermal dilatation.

In *plane strain* the situation is more complex. The presumption of plane strain implies that stresses perpendicular to the x–y plane will develop due to thermal expansion even without the three main stress components, and hence the initial strain will be affected by the elastic constants.

It will be shown that in such a case

$$\{\varepsilon_0\} = (1+\nu)\begin{Bmatrix} \alpha\theta^e \\ \alpha\theta^e \\ 0 \end{Bmatrix} \tag{4.13}$$

where ν is the Poisson's ratio.

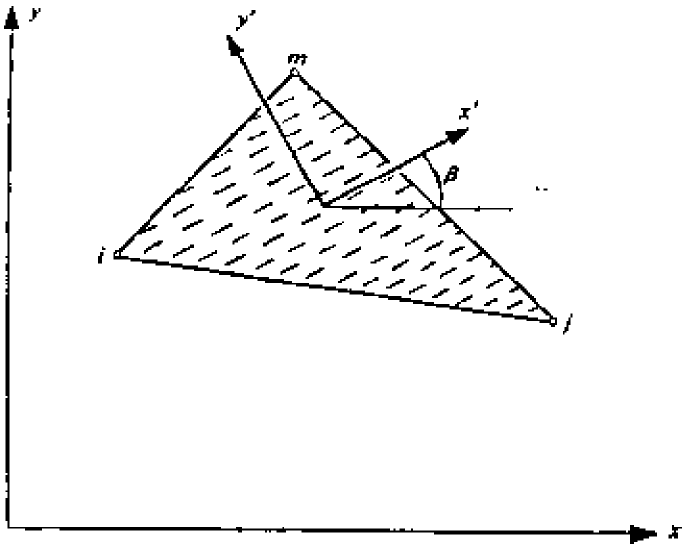


Fig. 4.2 An element of a stratified (transversely-isotropic) material

Anisotropic materials present special problems, since the coefficients of thermal expansion may vary with direction. Let x' and y' in Fig. 4.2 show the principal directions of the material. The initial strain due to thermal expansion becomes, with reference to these co-ordinates for plane stress

$$\{\varepsilon_0\}' = \begin{Bmatrix} \varepsilon_{x'0} \\ \varepsilon_{y'0} \\ \gamma_{x'y'0} \end{Bmatrix} = \begin{Bmatrix} \alpha_1\theta^e \\ \alpha_2\theta^e \\ 0 \end{Bmatrix} \tag{4.14}$$

where α_1 and α_2 are the expansion coefficients referred to the x' and y' axes respectively.

To obtain the strain components in the x, y system it is necessary to use an appropriate strain transformation matrix $[T]$ giving

$$\{\varepsilon_{0'}\} = [T]^T\{\varepsilon_0\}. \tag{4.15}$$

With the β as defined in Fig. 4.2 it is easily verified that

$$[T] = \begin{bmatrix} \cos^2\beta & \sin^2\beta & -2\sin\beta\cos\beta \\ \sin^2\beta & \cos^2\beta & 2\sin\beta\cos\beta \\ \sin\beta\cos\beta & -\sin\beta\cos\beta & \cos^2\beta-\sin^2\beta \end{bmatrix}.$$

Thus, $\{\varepsilon_0\}$ can be simply evaluated. It will be noted that no longer is the shear component of strain equal to zero in the x–y co-ordinates.

4.2.4 *Elasticity matrix.* The matrix $[D]$ of the relation Eq. (2.3)

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \{\varepsilon_0\}\right) \tag{4.16}$$

can be explicitly stated for any material (excluding here $\{\sigma_0\}$ which is simply additive).

Plane stress—isotropic material. For plane stress in an isotropic material we have, by definition,

$$\begin{aligned} \varepsilon_x &= \sigma_x/E-\nu\sigma_y/E+\varepsilon_{x0} \\ \varepsilon_y &= -\nu\sigma_x/E+\sigma_y/E+\varepsilon_{y0} \\ \gamma_{xy} &= 2(1+\nu)\tau_{xy}/E+\varepsilon_{xy0}. \end{aligned} \tag{4.17}$$

Solving the above for the stresses, we obtain matrix $[D]$ as

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{4.18}$$

in which E is the elastic modulus and ν is the Poisson's ratio.

Plane strain—isotropic material. In this case a normal stress σ_z exists in addition to the three other stress components. For the special case of isotropic thermal expansion we have

$$\begin{aligned} \varepsilon_x &= \sigma_x/E-\nu\sigma_y/E-\nu\sigma_z/E+\alpha\theta^e \\ \varepsilon_y &= -\nu\sigma_x/E+\sigma_y/E-\nu\sigma_z/E+\alpha\theta^e \\ \gamma_{xy} &= 2(1+\nu)\tau_{xy}/E. \end{aligned} \tag{4.19}$$

but in addition

$$\varepsilon_z = 0 = -\nu\sigma_x/E - \nu\sigma_y/E + \sigma_z/E + \alpha\theta^e.$$

On eliminating σ_z and solving for the three remaining stresses we obtain the previously quoted expression for the initial strain Eq. (4.13), and by comparison with Eq. (4.16), the matrix $[D]$

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & 0 \\ \nu/(1-\nu) & 1 & 0 \\ 0 & 0 & (1-2\nu)/2(1-\nu) \end{bmatrix}. \qquad (4.20)$$

Anisotropic materials. For a completely anisotropic material, 21 independent elastic constants are necessary to define completely the three-dimensional stress-strain relationship.[4,5]

If two-dimensional analysis is to be applicable a symmetry of properties must exist, implying at most six independent constants in the $[D]$ matrix. Thus, it is always possible to write

$$[D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ & d_{22} & d_{23} \\ (\text{sym}) & & d_{33} \end{bmatrix} \qquad (4.21)$$

to describe the most general two-dimensional behaviour. (The necessary symmetry of the $[D]$ matrix follows from the general equivalent of the Maxwell-Betti reciprocal theorem and is a consequence of invariant energy irrespective of the path taken to reach a given strain state.)

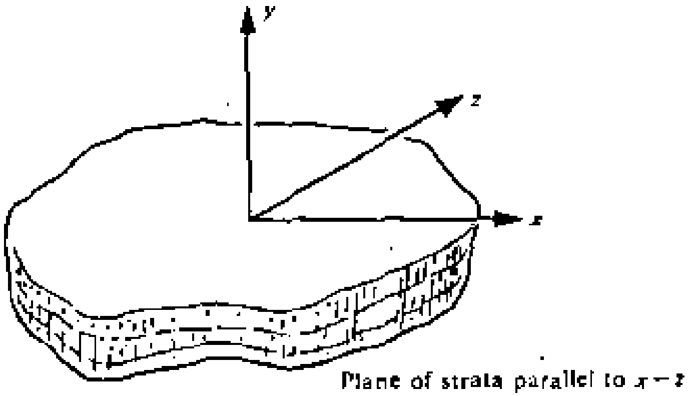


Fig. 4.3 A stratified (transversely-isotropic) material

A case of particular interest in practice is that of a 'stratified' or transversely-isotropic material in which a rotational symmetry of properties exists within the plane of the strata. Such a material possesses only five independent elastic constants.

The general stress-strain relations give in this case, following the notation of Lekhnitskii,[4] and taking now the y axis as perpendicular to the strata (neglecting initial strain), Fig. 4.3.

$$\begin{aligned} \varepsilon_x &= \sigma_x/E_1 - \nu_2\sigma_y/E_2 - \nu_1\sigma_z/E_1 \\ \varepsilon_y &= -\nu_2\sigma_x/E_2 + \sigma_y/E_2 - \nu_2\sigma_z/E_2 \\ \varepsilon_z &= -\nu_1\sigma_x/E_1 - \nu_2\sigma_y/E_2 + \sigma_z/E_1 \\ \gamma_{xz} &= \{2(1+\nu_1)/E_1\}\tau_{xz} \\ \gamma_{xy} &= \frac{1}{G_2}\tau_{xy} \\ \gamma_{yz} &= \frac{1}{G_2}\tau_{yz}. \end{aligned} \qquad (4.22)$$

in which the constants E_1, ν_1 (G_1 is dependent) are associated with the behaviour in plane of the strata and E_2, G_2, ν_2 with a direction normal to these.

The $[D]$ matrix in two-dimensions becomes now, taking

$$\frac{E_1}{E_2} = n \quad \text{and} \quad \frac{G_2}{E_2} = m$$

$$[D] = \frac{E_2}{(1-n\nu_2^2)}\begin{bmatrix} n & n\nu_2 & 0 \\ n\nu_2 & 1 & 0 \\ 0 & 0 & m(1-n\nu_2^2) \end{bmatrix} \qquad (4.23)$$

for plane stress, or

$$D = \frac{E_2}{(1+\nu_1)(1-\nu_1-2n\nu_2^2)}$$

$$\begin{bmatrix} n(1-n\nu_2^2) & n\nu_2(1+\nu_1) & 0 \\ n\nu_2(1+\nu_1) & (1-\nu_1^2) & 0 \\ 0 & 0 & m(1+\nu_1)(1-\nu_1-2n\nu_1^2) \end{bmatrix} \qquad (4.24)$$

for plane strain.

When, as in Fig. 4.2, the direction of strata is inclined to the x-axis then to obtain the $[D]$ matrices in the universal co-ordinates a transformation is necessary. Taking $[D']$ as relating the stresses and strains in the inclined co-ordinate system (x', y') it is easy to show that

$$[D] = [T][D'][T]^T \tag{4.25}$$

where $[T]$ is the same as given in Eq. (4.15)

If the stress systems $\{\sigma'\}$ and $\{\sigma\}$ correspond to $\{\varepsilon'\}$ and $\{\varepsilon\}$ respectively then by equality of work

$$\{\sigma'\}^T\{\varepsilon'\} = \{\sigma\}^T\{\varepsilon\}$$

or

$$\{\varepsilon'\}^T[D']\{\varepsilon'\} = \{\varepsilon\}^T[D]\{\varepsilon\}$$

from which Eq. (4.25) follows on substitution of Eq. (4.15). (*See also* Chapter 1.)

4.2.5 *The stiffness matrix.* The stiffness matrix of the element *ijm* is defined from the general relationship Eq. (2.10) as

$$[k] = \int [B]^T[D][B]t\,dx\,dy \tag{4.26}$$

where t is the thickness of the element and the integration is taken over the area of the triangle. If the thickness of the element is assumed to be constant, an assumption convergent to the truth as size of elements decreases, then, as neither of the matrices contains x or y we have, simply

$$[k] = [B]^T[D][B]t\Delta \tag{4.27}$$

where Δ is the area of the triangle (defined already by Eq. (3.5)). This form is now sufficiently explicit for computation with the actual matrix operations being left to the computer.

The matrix $[B]$ defined by Eq. (4.10) can be written as

$$[B] = [B_i, B_j, B_m] \quad \text{with} \quad [B_i] = \begin{Bmatrix} b_i & 0 \\ 0 & c_i \\ c_i & b_i \end{Bmatrix} \Big/ 2\Delta, \text{ etc.} \tag{4.28}$$

Now the stiffness matrix can be written in a partitioned form as

$$[k] = \begin{bmatrix} k_{ii} & k_{ij} & k_{im} \\ k_{ji} & k_{jj} & k_{jm} \\ k_{mi} & k_{mj} & k_{mm} \end{bmatrix} \tag{4.29}$$

in which the 2 by 2 submatrices are built up as

$$[k_{rs}] = [B_r]^T[D][B_s]t\Delta. \tag{4.30}$$

This form is often convenient for computation.

4.2.6 *Nodal forces due to initial strain.* These are given directly by the expression Eq. (2.12) which, on performing the integration, becomes

$$\{F\}^e_{\varepsilon_0} = -[B]^T[D][\varepsilon_0]t\Delta, \text{ etc.} \tag{4.31}$$

Partitioning, one can write alternatively

$$\{F_i\}^e_{\varepsilon_0} = -[B_i]^T[D][\varepsilon_0]t\Delta, \text{ etc.} \tag{4.32}$$

These 'initial strain' forces are contributed to the nodes of an element in an unequal manner and require precise evaluation. Similar expressions are derived for initial stress forces.

4.2.7 *Distributed body forces.* In the general case of plane stress or strain each element of unit area in the x–y plane is subject to forces

$$\{p\} = \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

in the direction of the appropriate axes.

Again, by Eq. (2.11), the contribution of such forces to these at each node is given by

$$\{F\}^e_p = -\int [N]^T \begin{Bmatrix} X \\ Y \end{Bmatrix} dx\,dy,$$

or by Eq. (4.7)

$$\{F_i\}_p = -\begin{Bmatrix} X \\ Y \end{Bmatrix} \int N_i\,dx\,dy, \text{ etc.} \tag{4.33}$$

if the body forces X and Y are constant. As N_i is no longer constant the integration has to be carried out explicitly. Some general integration formulae for a triangle are given in Appendix III.

In this special case the calculation will be simplified if the origin of co-ordinates is taken at the centroid of the element. Now

$$\int x\,dx\,dy = \int y\,dx\,dy = 0$$

and on using Eq. (3.8)

$$\{F_i\}_p = -\begin{Bmatrix} X \\ Y \end{Bmatrix} \int a_i\,dx\,dy/2\Delta = -\begin{Bmatrix} X \\ Y \end{Bmatrix} a_i/2$$

or

$$\{F_i\}_p = -\begin{Bmatrix} X \\ Y \end{Bmatrix} \Delta/3 = \{F_j\}_p = \{F_m\}_p \tag{4.34}$$

by relations noted on p. 50.

Explicitly, for the whole element

$$\{F\}^e_p = -\begin{Bmatrix} X \\ Y \\ X \\ Y \\ X \\ Y \end{Bmatrix} \Delta/3 \tag{4.35}$$

which means simply that the total forces acting in x and y direction due to the body forces are distributed to the nodes in three equal parts. This fact corresponds with physical intuition, and was often assumed implicitly.

4.2.8 *Body force potential.* In many cases the body forces are defined in terms of a body force potential ϕ as

$$X = -\frac{\partial \phi}{\partial x}, \qquad Y = -\frac{\partial \phi}{\partial y} \tag{4.36}$$

and this potential, rather than the values of X and Y, is known throughout the region and is specified at nodal points. If $\{\phi\}^e$ lists the three values of the potential associated with the nodes of the element, i.e.,

$$\{\phi\}^e = \begin{Bmatrix} \phi_i \\ \phi_j \\ \phi_m \end{Bmatrix} \tag{4.37}$$

and has to correspond with constant values of X and Y, ϕ must vary linearly within the element. The 'shape function' of its variation will obviously be given by a procedure identical to that used in deriving Eqs. (4.4) to (4.6), and yields

$$\phi = [N'_i, N'_j, N'_m]\{\phi\}^e. \tag{4.38}$$

Thus,

$$X = -\frac{\partial \phi}{\partial x} = -[b_i, b_j, b_m]\{\phi\}^e/2\Delta$$

and

$$Y = -\frac{\partial \phi}{\partial y} = -[c_i, c_j, c_m]\{\phi\}^e/2\Delta. \tag{4.39}$$

The vector of nodal forces due to the body force potential will now replace Eq. (4.35) by

$$\{F\}^e_p = \frac{1}{6}\begin{bmatrix} b_i & b_j & b_m \\ c_i & c_j & c_m \\ b_i & b_j & b_m \\ c_i & c_j & c_m \\ b_i & b_j & b_m \\ c_i & c_j & c_m \end{bmatrix}\{\phi^e\} \tag{4.40}$$

4.2.9 *Evaluation of stresses.* The formulae derived enable the full stiffness matrix of the structure to be assembled, and a solution for displacements to be obtained.

The stress matrix given in general terms in Eq. (2.15) is obtained by the appropriate substitutions for each element.

The stresses are, by the basic assumption, constant within the element. It is usual to assign these to the centroid of the element, and in most of the examples in this chapter this procedure is followed. An alternative consists of obtaining stress values at the nodes by averaging the values in the adjacent elements. Some 'weighting' procedures have been used in this context on an empirical basis but their advantage appears small.

It is usual to arrange for the computer to calculate the principal stresses and their directions of every element.

4.3 Examples—An Assessment of Accuracy

There is no doubt that the solution to plane elasticity problems as formulated in Section 4.2 is, in the limit of subdivision, an exact solution. Indeed at any stage of a finite subdivision it is an approximate solution as, say, a Fourier series solution with a limited number of terms.

As already explained in Chapter 2 the total strain energy obtained during any stage of approximation will be below the true strain energy of the exact solution. In practice it will mean that the displacements, and hence also the stresses, will be underestimated by the approximation in its *general picture*. However, it must be emphasized that this is not necessarily true at every point of the continuum individually; hence the value of such a bound in practice is not great.

What is important for the engineer to know is the order of accuracy achievable in typical problems with a certain fineness of element subdivision. In any particular case the error can be assessed by comparison with known, exact, solutions or by a study of the convergence, using two or more stages of subdivision.

With the development of experience the engineer can assess *a priori* the order of approximation that will be involved in a specific problem tackled with a given element subdivision. Some of this experience will perhaps be conveyed by the examples considered in this book.

In the first place attention will be focused on some simple problems for which exact solutions are available.

Uniform stress field. If the exact solution is in fact that of a uniform stress field then, whatever the element subdivision, the finite element solution will coincide exactly with the exact one. This is an obvious corollary of the formulation, nevertheless it is useful as a first check of written computer programs.

Linearly varying stress field. Here, obviously, the basic assumption of constancy of stress within elements means that solution will be approximate only. In Fig. 4.4 a simple example of a beam subject to constant bending moment is shown with a fairly coarse subdivision. It is readily seen that the axial (σ_y) stress given by the element 'straddles' the exact values and, in fact, if the constant stress values are associated with centroids of the elements and plotted, the best 'fit' line represents the exact stresses.

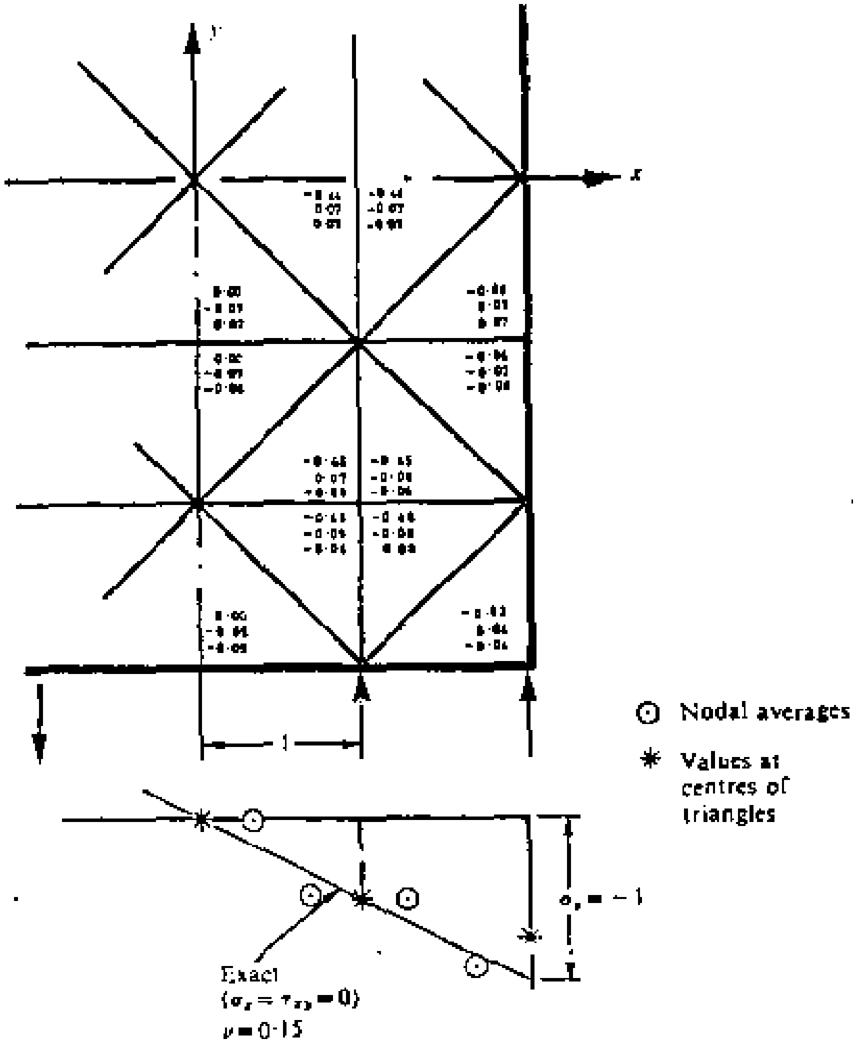


Fig 4.4 Pure bending of a beam solved by a coarse subdivision into elements of triangular shape. (Values of σ_y, σ_x, and τ_{xy} listed in that order)

The horizontal and shear stress components differ again from the exact values (which are simply zero). Again, however, it will be noted that they oscillate by equal, small amounts around the exact values.

At internal nodes, if the average of stresses of surrounding elements is taken it will be found that the exact stresses are very closely represented. The average at external faces is not, however, so good. The overall improvement in representing the stresses by nodal averages, as shown on Fig. 4.4, is often used in practice for improvement of the approximation.

A weighting of averages near the faces of the structure can further be used for refinement. Without being dogmatic on this point, it seems preferable, when accuracy demands this, simply to use a finer mesh subdivision.

Stress concentration. A more realistic test problem is shown in Figs. 4.5 and 4.6. Here the flow of stress around a circular hole in an isotropic and in an anisotropic stratified material is considered when the stress conditions are uniform.[6] A graded division into elements is used to allow a more detailed study in the region where high stress gradients are expected. The high degree of accuracy achievable can be assessed from Fig. 4.6 where some of the results are compared against exact solutions.[3,7]

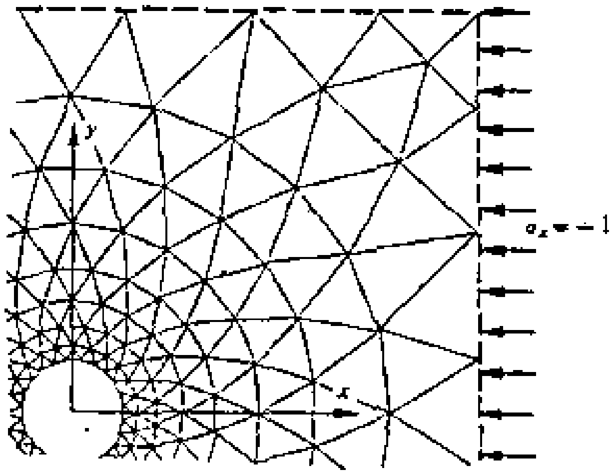
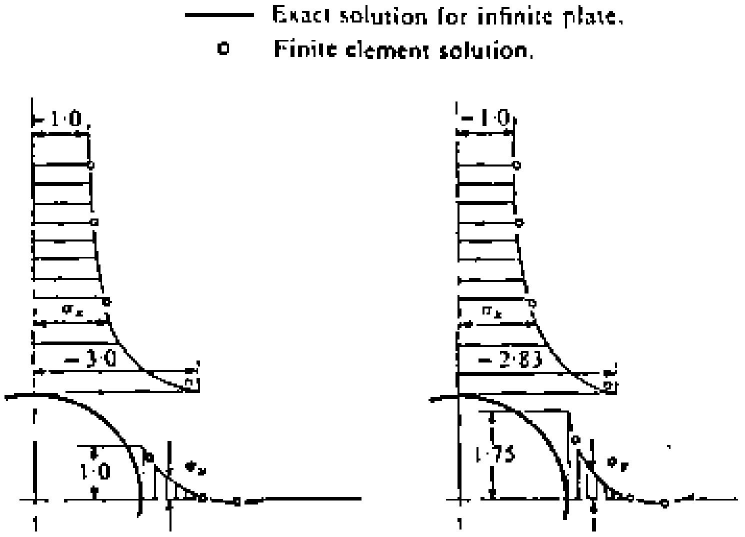


Fig. 4.5 A circular hole in a uniform stress field. (a) isotropic material; (b) stratified (orthotropic) material; $E_x = E_1 = 1$, $E_y = E_2 = 3$, $\nu_1 = 0{\cdot}1$, $\nu_2 = 0$, $G_{xy} = 0{\cdot}42$

4.4 Some Practical Applications

Obviously, the practical applications of the method are limitless, and indeed at this moment of time the use of the finite element method is superseding experimental technique for plane problems because of its high accuracy, low cost, and versatility. The ease of treatment of material anisotropy, thermal stresses, or body force problems add to its advantages.



(a) Isotropic. (b) Orthotropic.

Fig. 4.6 Comparison of theoretical and finite element results for cases (a) and (b) of Fig. 4.5

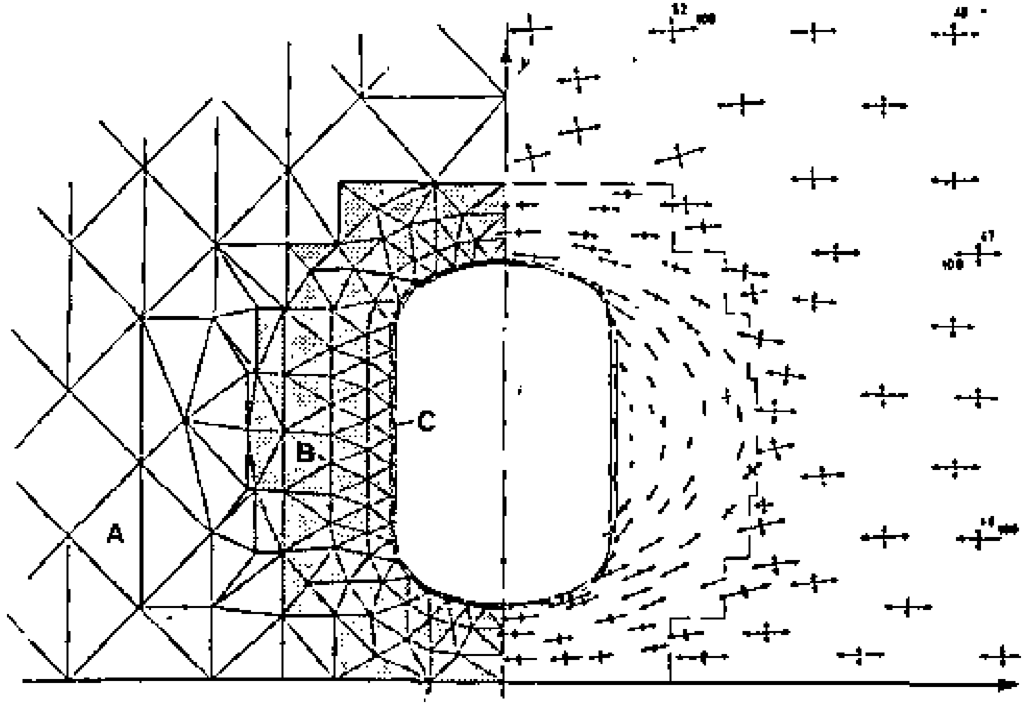
A few examples of actual applications to complex problems of engineering practice will now be given.

Stress flow around a reinforced opening (Fig. 4.7). In steel pressure vessels or aircraft structures, openings have to be introduced in the stressed skin. The penetrating duct itself provides some reinforcement round the edge and, in addition, the skin itself is increased in thickness to reduce the stresses due to the concentration effects.

Analysis of such problems treated as cases of plane stress presents no difficulties. The elements are so chosen as to follow the thickness variation, and appropriate values of this are assigned.

The narrow band of thick material near the edge can be represented either by special beam-type elements, or more easily in a standard programme by very thin triangular elements of the usual type, to which appropriate thickness is assigned. The latter procedure was used in the problem shown in Fig. 4.7 which gives some of the resulting stresses near the opening itself. The fairly large extent of the region introduced in the analysis and the grading of the mesh should be noted.

An anisotropic valley subject to tectonic stress[6] (Fig. 4.8). A symmetrical valley subject to a uniform horizontal stress is considered. The material is stratified, hence is 'transversely isotropic', and the direction of strata varies from point to point.



Restrained in *y* direction from movement.

Fig. 4.7 A reinforced opening in a plate. Uniform stress field at a distance from opening $\sigma_x = 100$, $\sigma_y = 50$. Thickness of plate regions A, B, and C is in the ratio of 1:3:23

The stress plot shows the tensile region that develops. This phenomenon is of considerable interest to geologists and engineers concerned with rock mechanics.

A dam subject to external and internal water pressures[8,9] (Fig. 4.9). A buttress dam on a somewhat complex rock foundation is here analysed. The heterogeneous foundation region is subject to plane strain conditions while the dam itself is considered as a plate (plane stress) of variable thickness.

With external and gravity loading no special problems of analysis arise, though perhaps it should be mentioned that it was found worth while to 'automatize' the computation of gravity nodal loads.

When pore pressures are considered, the situation, however, requires perhaps some explanation.

It is well known that in a porous material the water pressure is transmitted to the structure as a *body force of magnitude*

$$X = -\frac{\partial p}{\partial x}, \qquad Y = -\frac{\partial p}{\partial y}$$

and that now the external pressure need not be considered.

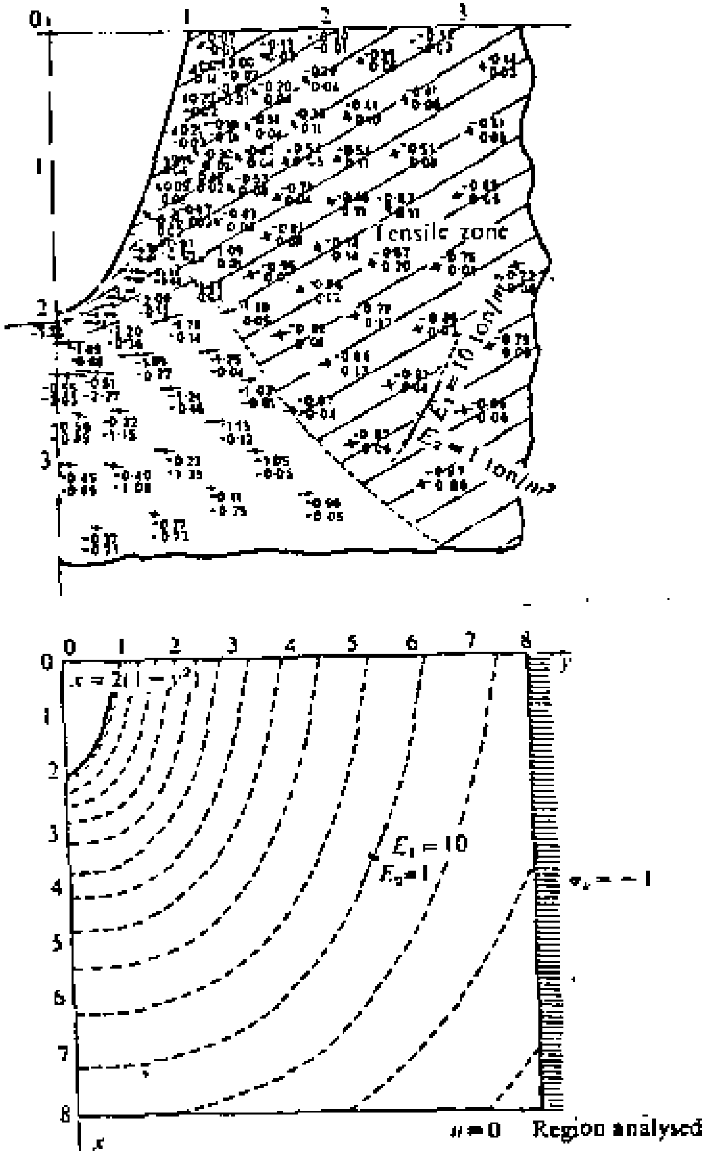
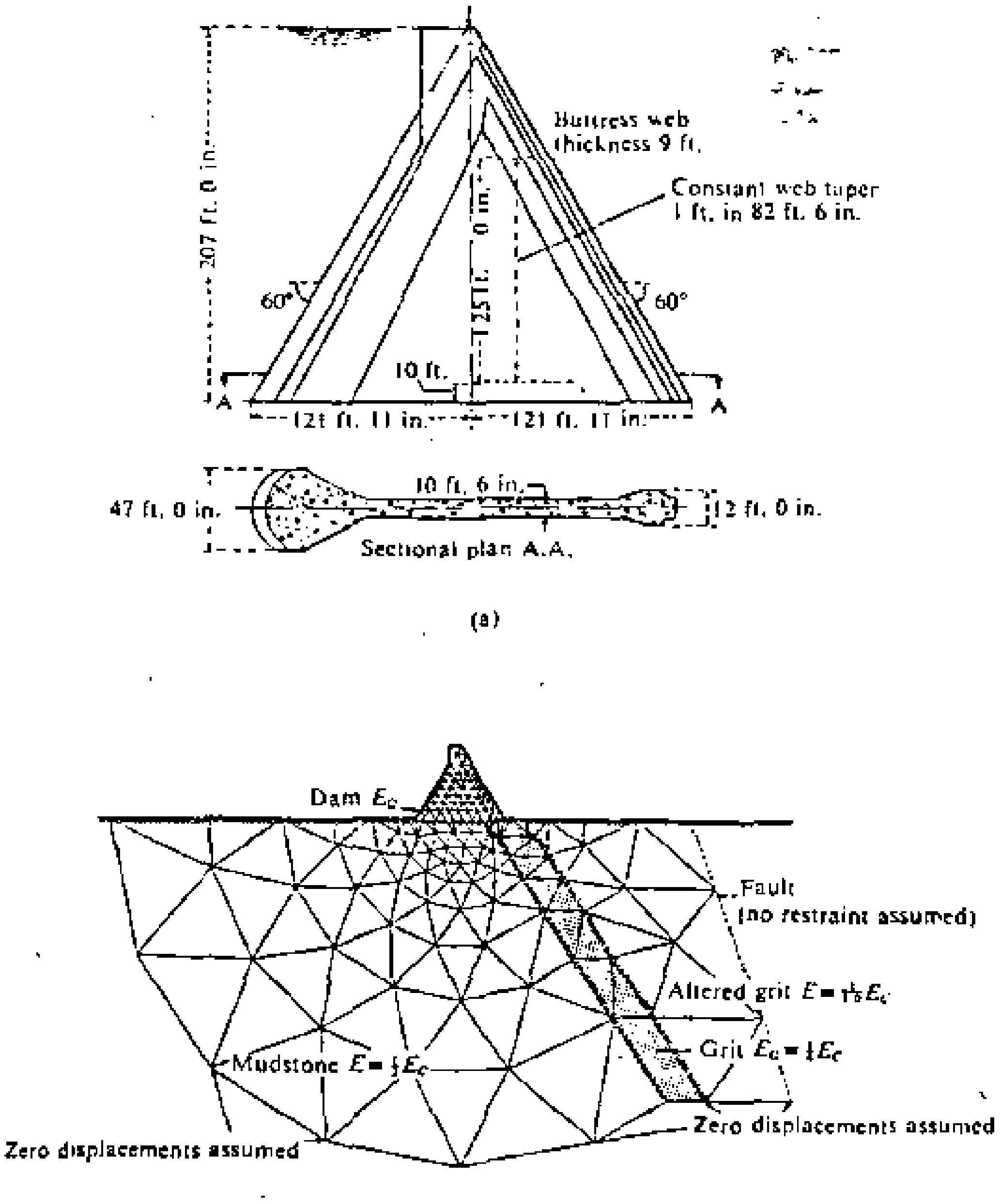


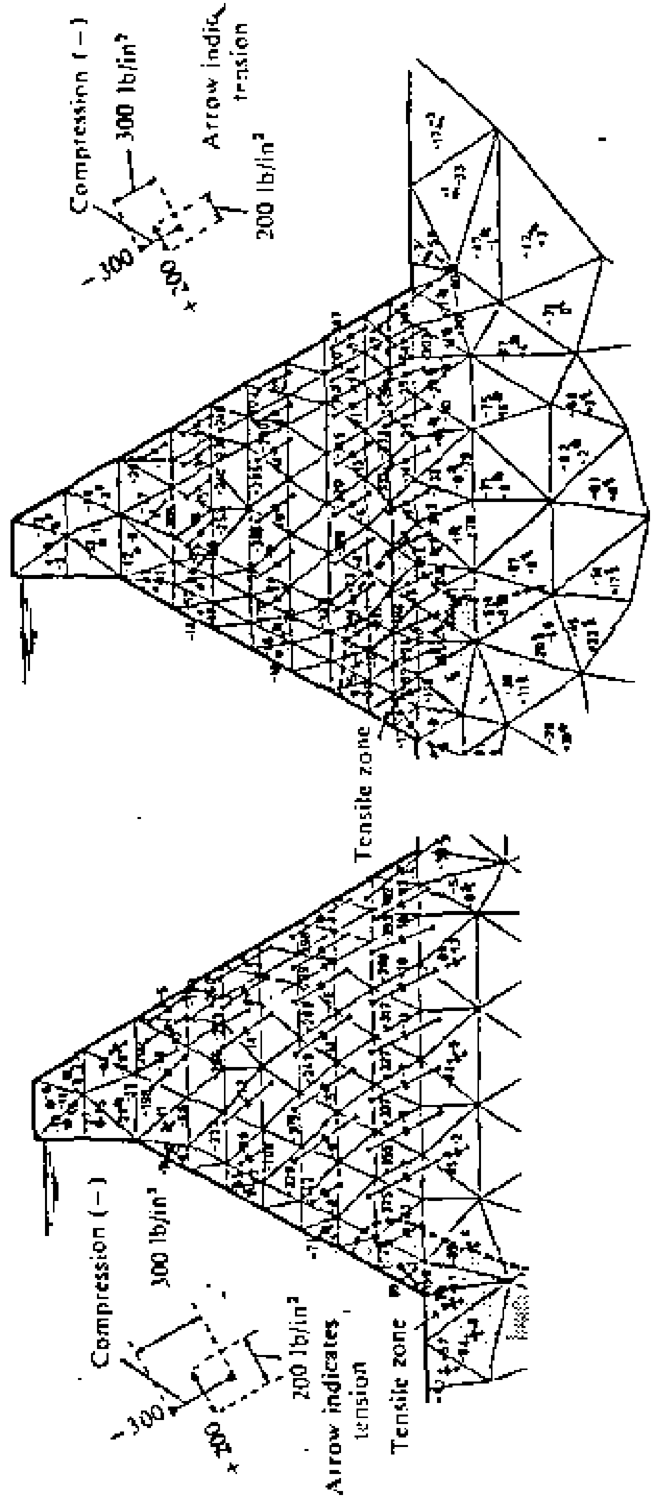
Fig. 4.8 A valley with curved strata subject to a horizontal tectonic stress (plane strain 170 nodes, 298 elements)



(a)

(b)

Fig. 4.9 Stress analysis of a buttress dam. Plane stress condition assumed in dam and plane strain in foundation. (*a*) The buttress section analysed. (*b*) Extent of foundation considered and division into finite elements



Below the foundation initial rock stresses should be superimposed

(a)

(b)

Fig. 4.10 Stress analysis of the buttress dam of Fig. 4.9. Principal stresses for gravity loads combined with water pressures, which are assumed to act (*a*) as external loads, (*b*) as body forces due to pore pressure

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The pore pressure *p* is, in fact, now a body force potential, as defined in Eq. (4.36). Figure 4.9 shows the element subdivision of the region and the outline of the dam. Figure 4.10(*a*) and (*b*) show the stresses resulting from gravity (applied to the dam only) and due to water pressure assumed to be acting as an external load or, alternatively as an internal pore pressure. Both solutions indicate large tensile regions, but the increase of stresses due to the second assumption is important.

Cracking. The tensile stresses in the previous example will doubtless cause the rock to crack. If a stable situation can develop when such a crack spreads then the dam can be considered safe.

Cracks can be introduced very simply into the analysis by assigning zero elasticity values to chosen elements. An analysis with a wide cracked wedge is shown in Fig. 4.11, where it can be seen that with the extent of the crack assumed no tension within the dam body develops.

A more elaborate procedure for following crack propagation and resulting stress redistribution can be developed and will be discussed later (*see* Chapter 18).

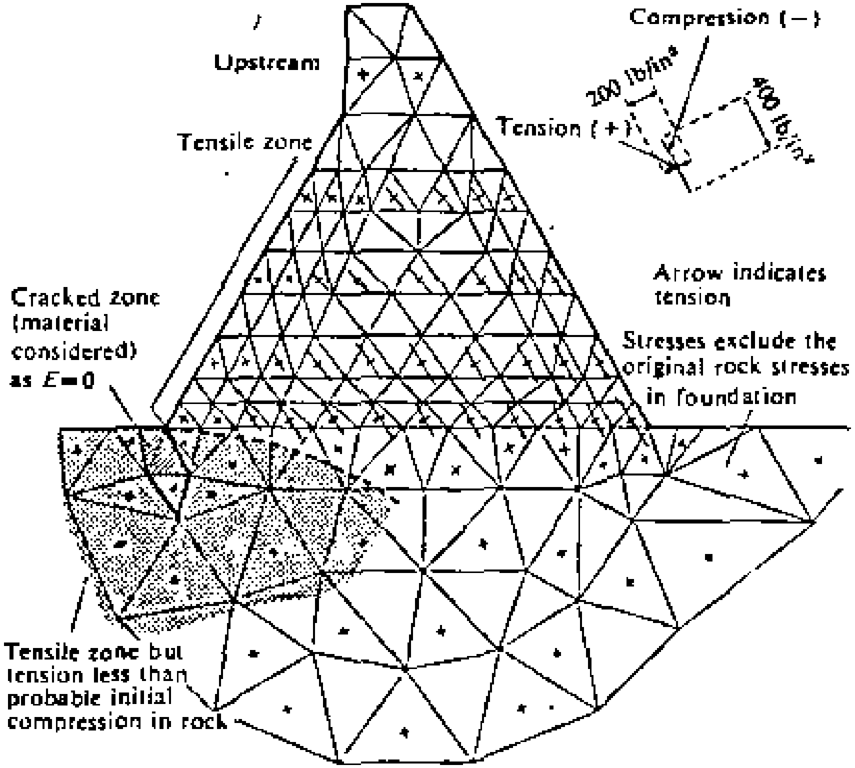

Fig. 4.11 Stresses in a buttress dam. An introduction of a 'crack' modifies stress distribution (same loading as Fig 4.10(*b*))

Thermal stresses. As an example of thermal stress computation the same dam is shown under simple temperature distribution assumptions. Results of this analysis are given in Fig. 4.12.

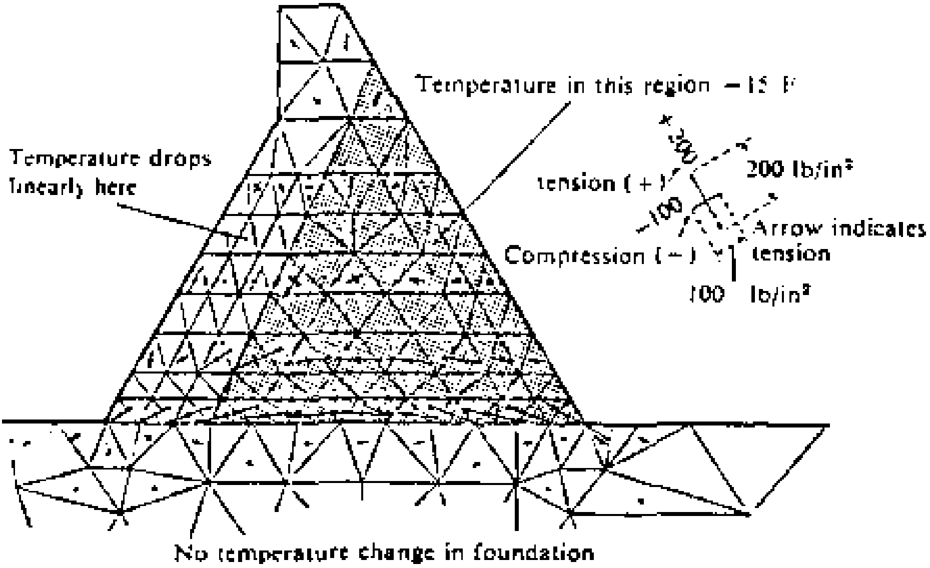

Fig. 4.12 Stress analysis of a buttress dam. Thermal stresses due to cooling of shaded area by 15°F ($E = 3 \times 10^6$ lb/in², $\alpha = 6 \times 10^{-6}$/deg F)

Gravity dams. A buttress dam is a natural example for the application of finite element methods. Other types, such as gravity dams with or without piers and so on, can also be simply treated. Figure 4.13 shows an analysis of a large dam with piers and crest gates.

In this case an approximation of assuming a two-dimensional treatment in the vicinity of the abrupt change of section, i.e., where the piers join the main body of the dam, is clearly involved, but this leads to localized errors only.

It is important to note here how, in a single solution, the grading of element size is used to study concentration of stress at the cable anchorages, the general stress flow in the dam, and the foundation behaviour. The linear ratio of size of largest to smallest elements is of the order of 30 to 1 (the largest elements occurring in the foundation are not shown in the figure).

Underground power station. This last example illustrated in Figs. 4.14 and 4.15 shows an interesting large-scale application. Here principal stresses are plotted automatically. In this analysis very many different

components of $\{\sigma_0\}$, the initial stress, were used due to uncertainty of knowledge about geological conditions. The rapid solution and plot of many results enabled the limits within which stresses vary to be found and an engineering decision arrived at.

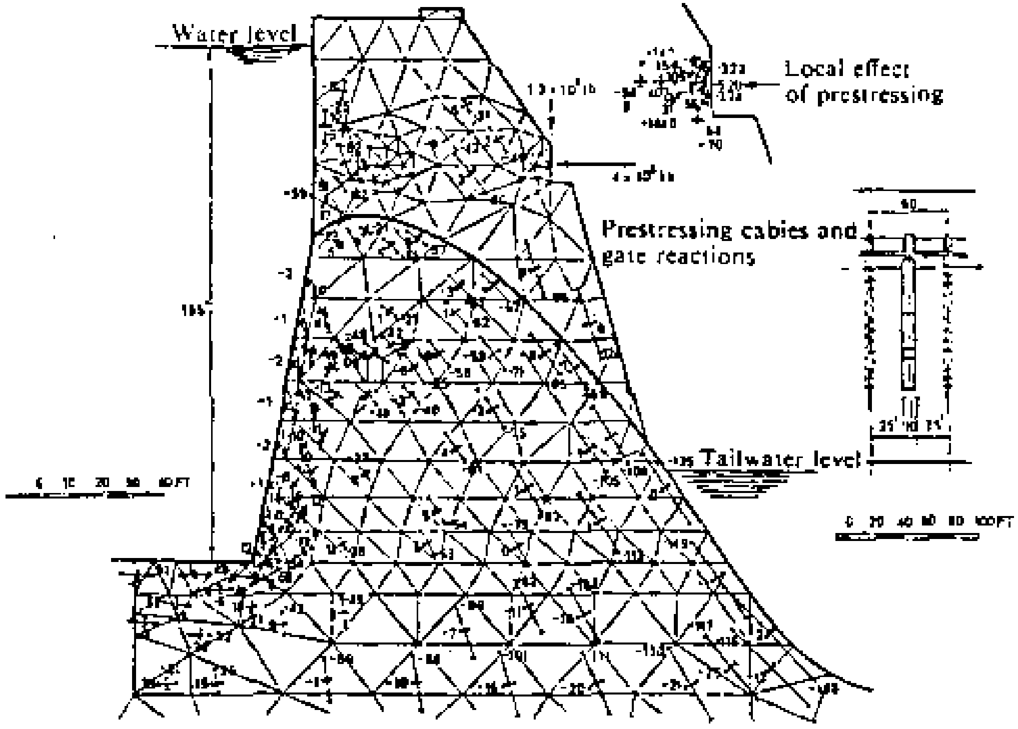

Fig. 4.13 A large barrage with piers and prestressing cables

4.5 Special Treatment of Plane Strain with an Incompressible Material

It will have been noted that the relationship Eq. (4.20) defining the elasticity $[D]$ matrix for an isotropic material breaks down when the Poisson's ratio reaches a value of 0·5 as the factor in the parentheses becomes infinite. A simple way of side-stepping the difficulty presented is to use values of Poisson's ratio approximate to 0·5 but not equal to it. Experience shows, however, that if this is done the approximation of solution deteriorates. An alternative procedure has been suggested by Herrman.[10] This involves the use of a new variational formulation, and readers are referred to his work for details

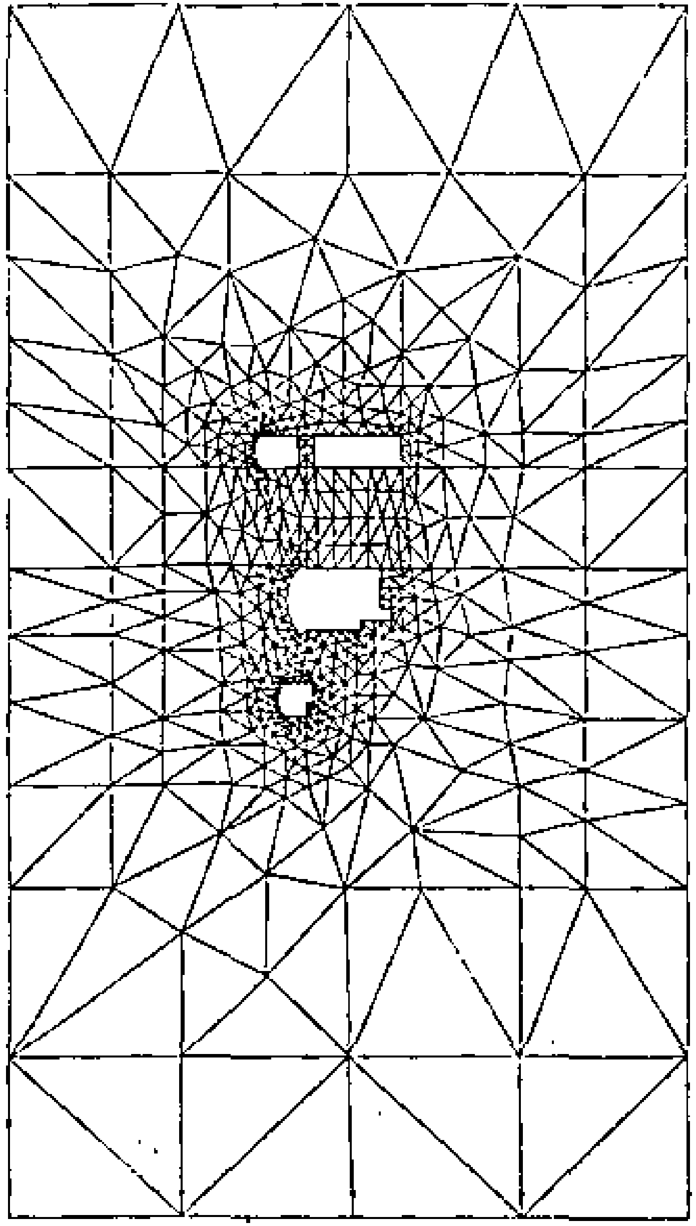

Fig. 4.14 An underground power station. Mesh used in analysis.

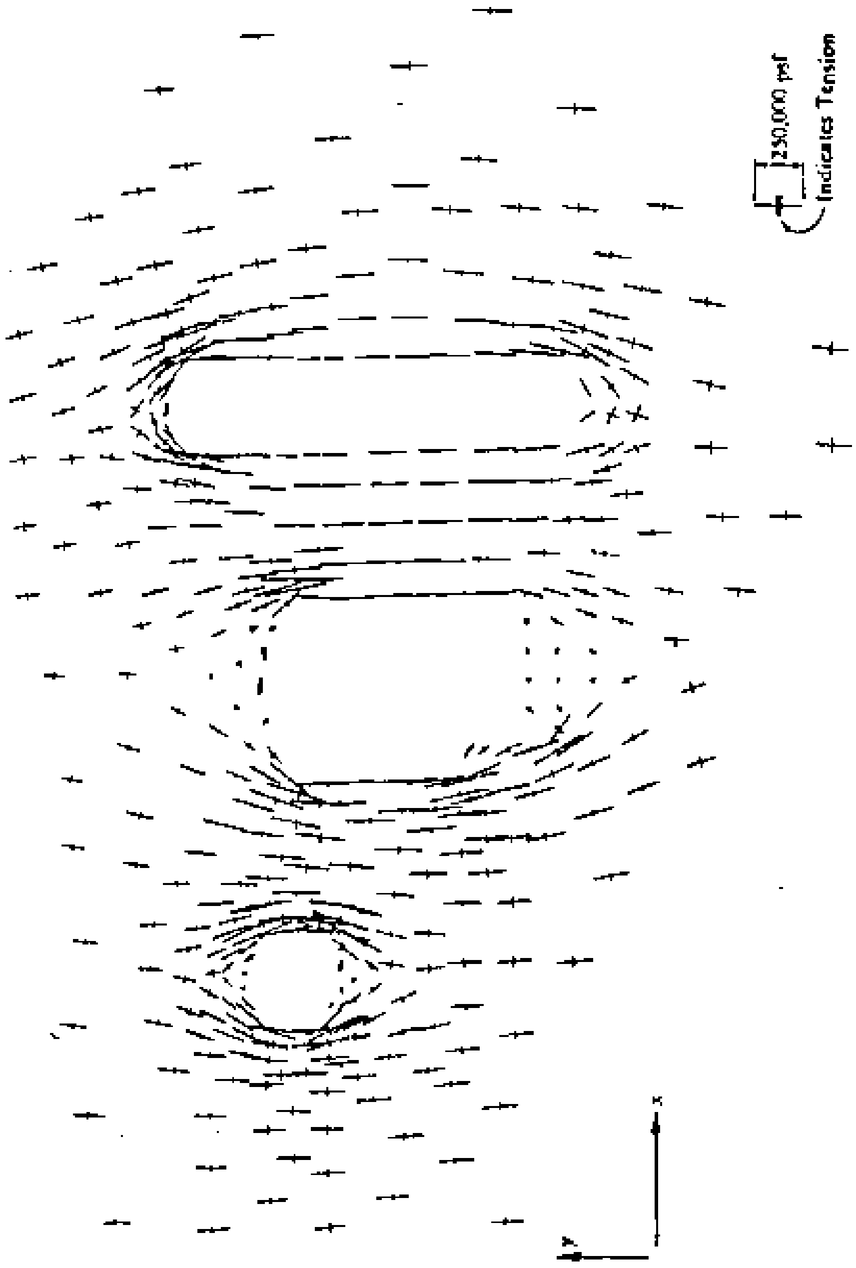

Fig. 4.15 An underground power station. Plot of principal stresses.

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TRABAJO VIRTUAL

Principio del trabajo virtual: Sí una partícula se encuentra en equilibrio, el trabajo total efectuado por todas las fuerzas actuando sobre la partícula, bajo cualquier desplazamiento virtual es cero. Sean

$\delta u, \delta v, \delta w$: Componentes del desplazamiento virtual en las direcciones x, y, z.

$\Sigma F_x, \Sigma F_y, \Sigma F_z$: Sumas de fuerzas en las direcciones x, y, z que actuan sobre la partícula o cuerpo rígido.

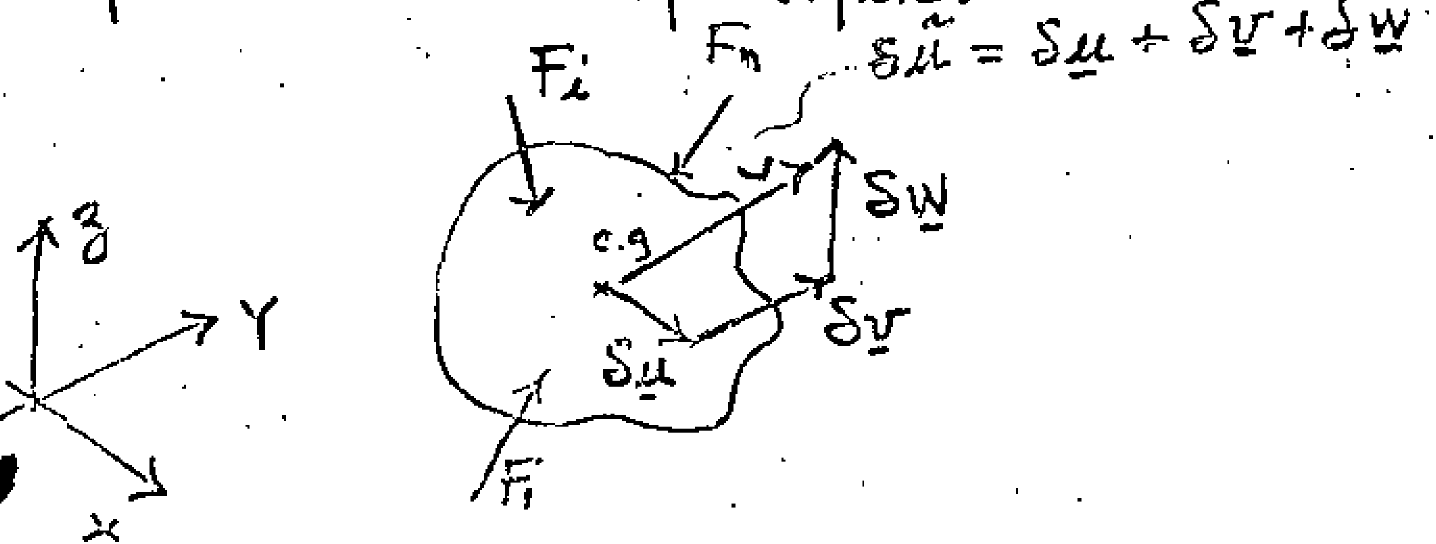


Fig. 1

El principio del desplagamiento virtual establace que

$$\delta u \sum_{i=1}^{n} F_{xi} = 0$$

$$\delta v \sum_{1}^{n} F_{yi} = 0 \qquad (1)$$

$$\delta w \sum_{1}^{n} F_{zi} = 0$$

Si el sistema esta en equilibrio y permanece en equilibrio después del desplagamiento virtual se satisface ①.

Un cuerpo elástico en réposo constituye un conjunto de partículas sobre las que en cada elemento actua un subconjunto de fuergas en equilibrio. En cualquier desplagamiento virtual, el trabajo virtual sobre cada partícula es cero, por lo tanto el Trabajo virtual total también debe ser cero. Es conveniente q[illegible] δu_i sean consistentes con las condiciones de apoyo.

Sean:

$\lfloor u\ v\ w \rfloor$ componentes de los desplazamientos debido a las cargas en x_i.

$\lfloor \delta u\ \delta v\ \delta w \rfloor$ componentes del desplazamiento virtual en x_i. (funciones arbitaria de x_i)

Para deformaciones lineales pequeñas, los desplazamientos virtuales correspondientes a las seis componentes de deformación son

$$\delta \epsilon_x = \frac{\partial}{\partial x}(\delta u), \quad \delta \gamma_{xy} = \frac{\partial}{\partial x}(\delta v) + \frac{\partial}{\partial y}(\delta u)$$

$$\delta \epsilon_y = \frac{\partial}{\partial y}(\delta v), \quad \delta \gamma_{yz} = \frac{\partial}{\partial y}(\delta w) + \frac{\partial}{\partial z}(\delta v) \qquad (2)$$

$$\delta \epsilon_z = \frac{\partial}{\partial z}(\delta w), \quad \delta \gamma_{zx} = \frac{\partial}{\partial z}(\delta u) + \frac{\partial}{\partial x}(\delta w)$$

y el trabajo virtual en un elemento dxdydz es

$$\delta U_0\, dV = \Big[\sigma_x(\delta \epsilon_x) + \sigma_y(\delta \epsilon_y) + \sigma_z(\delta \epsilon_z) + \tau_{xy}(\delta \gamma_{xy}) + \tau_{yz}(\delta \gamma_{yz}) + \tau_{zx}(\delta \gamma_{zx})\Big] dV \qquad (3)$$

Sean:

$\bar{X} dA, \bar{Y} dA, \bar{Z} dA$, Fuerzas de superficie en el elemento $dV = dx\,dy\,dz$.

$X dV, Y dV, Z dV$, Fuerzas de cuerpo en el elemento $dV = dx\,dy\,dz$.

La afirmación de que el Trabajo virtual es cero es

$$\int_A (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dA + \int_V (X\delta u + Y\delta v + Z\delta w)\,dV - \int_V \delta U_0\,dV = 0$$ ④

Puesto que las fuerzas de superficie $[\bar{X}]$, las de cuerpo $[X]$ y los esfuerzo $[\sigma]$, no varían durante un desplazamiento virtual pequeño, el símbolo variacional δ se puede sacar fuera del signo integr quedando

⑤

$$\delta\Big[\underbrace{\int_V U_o\,dV}_{(a)} - \underbrace{\int_V (Xu+Yv+Zw)\,dV}_{(b)} - \underbrace{\int_A (\bar{X}u+\bar{Y}v+\bar{Z}w)\,dA}_{(c)}\Big]=0$$

P. Ballestero

Matricialmente:

$$\boxed{\delta\Big[\int_V \lfloor\sigma\rfloor\{\varepsilon\} - \int_V \lfloor X\rfloor\{u\}\,dV - \int_A \lfloor\bar{X}\rfloor\{u\}\,dA\Big]=0} \qquad (6)$$

(a) ≡ Energía potencial de deformación

(b) ≡ " " " fuerzas de cuerpo

(c) ≡ " " " " " superficie

en (6)

$\lfloor\sigma\rfloor = [\sigma_x\ \sigma_Y\ \sigma_z\ \tau_{xY}\ \tau_{Yz}\ \tau_{zx}]$

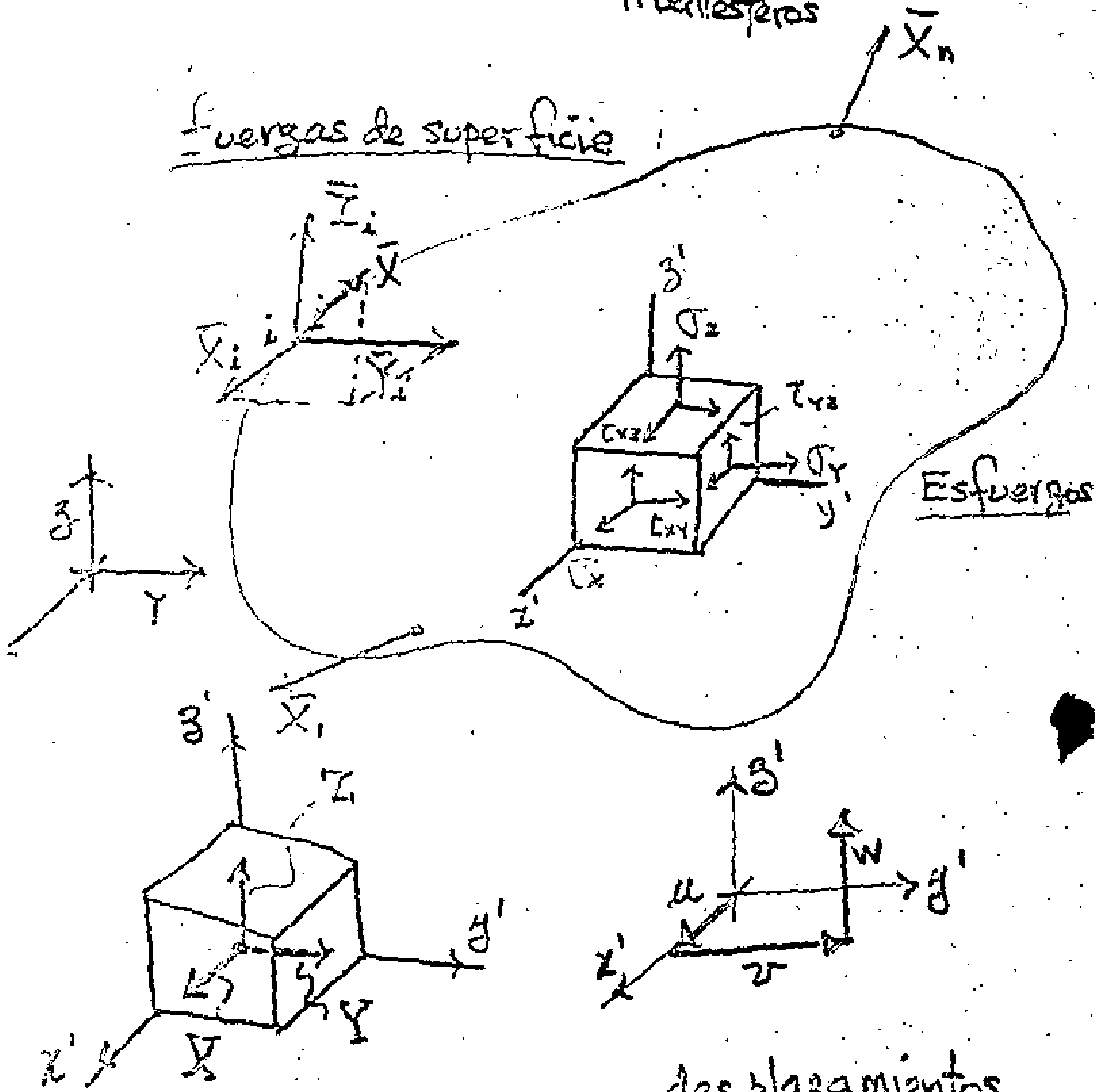
$\{\varepsilon\}^T = \{\varepsilon_x\ \varepsilon_Y\ \varepsilon_z\ \gamma_{xY}\ \gamma_{Yz}\ \gamma_{zx}\}^T$

$\lfloor X\rfloor = \lfloor X\ Y\ Z\rfloor$

$\{u\}^T = \lfloor u\ v\ w\rfloor$

$\lfloor\bar{X}\rfloor = \lfloor\bar{X}\ \bar{Y}\ \bar{Z}\rfloor$

P. Ballesteros



(a) Establece que los desplazamientos $\lfloor u\ v\ w \rfloor$ bajo ciertas fuerzas de superficie y de cuerpo dadas, son tal que la variación de primer orden de la energía potencial total ES CERO para cualquier desplazamiento virtual, ó brevemente La energía potencial total es estacionaria.

El termino desplazamiento o trabajo virtual implican multiplicadores arbitrarios $\lfloor \delta u\ \delta v\ \delta w \rfloor$ con las ecuaciones de equilibrio, es conveniente referirse a ellos como variaciones de u. v. w.

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V CURSO INTERNACIONAL DE INGENIERIA SISMICA

DINAMICA ESTRUCTURAL

ANALISIS ESTRUCTURAL II

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JULIO,1979.

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Principios Fundamentales.

a).- Continuidad

$$e = a\ d$$

b).- Ley de Hooke

$$p = k\ e$$

c).- Equilibrio

$$F = a^T\ p$$

Solución de Estructuras.

Método de los desplazamientos o de las rigideces.
(Continuidad —— Ley de Hooke —— Equilibrio)

$$F = a^T\ p = a^T\ k\ e = a^T\ k\ a\ d$$

$$F = K\ d\ ;\ K = a^T\ k\ a\ .$$

Solución:

$$d = K^{-1}\ F$$

$$e = a\ d = a\ K^{-1}\ F$$

$$p = k\ e = k\ a\ K^{-1}\ F$$

Si la estructura es isostética, a^T es no singular, por consiguiente:

$$p = (a^T)^{-1}\ F$$

$$e = (k)^{-1}\ p$$

$$d = (a)^{-1}\ e$$

Si la estructura es hiperestática estable, la matriz a^T se puede particionar.

$$\left[a^T\right] = \left[\begin{array}{c:c} a_o^T & a_1^T \end{array}\right]$$

a_o^T debe ser una matriz cuadrada no singular, que define la estructura primaria o isostática.

$$P = \left[\begin{array}{c:c} a_o^T & a_1^T \end{array}\right]\left[\begin{array}{c} p_o \\ \hdashline p_1 \end{array}\right]$$

p_o = fuerzas en las barras que forman la estructura primaria.

p_1 = fuerzas en las barras sobrantes.

Método de las fuerzas o método de las flexibilidades.

Equilibrio:

$$\left[p\right] = \left[\begin{array}{c:c} b_o & b_1 \end{array}\right]\left[\begin{array}{c} F \\ \hdashline R \end{array}\right]$$

o bien:

$$\left[\begin{array}{c} p_o \\ \hdashline p_1 \end{array}\right] = \left[\begin{array}{c:c} b_{oo} & b_{1o} \\ \hdashline 0 & b_{11} \end{array}\right]\left[\begin{array}{c} F \\ \hdashline R \end{array}\right]$$

Ley de Hooke:

$$\left[e\right] = \left[f\right]\left[p\right]$$

$$f = k^{-1}$$

Continuidad:

$$\left[\begin{array}{c} d \\ \hdashline u \end{array}\right] = \left[\begin{array}{c} b_o^T \\ \hdashline b_1^T \end{array}\right]\left[e\right]$$

Relaciones de b_{oo}, b_{1o}, a .

$$b_{oo} = (a_o^T)^{-1}$$

$$b_{1o} = - b_{oo} \, a_1^T \, b_{11}$$

b_{11} = depende de R

Las redundantes (R) se obtienen obligando a que $u = 0$:

$$u = b_1^T\ e = b_1^T\ f\ p = b_1^T\ f\ (b_o\ F + b_1\ R)$$

$$\underline{R = -\ (b_1^T\ f\ b_1\ )^{-1}\ b_1^T\ f\ b_o\ F}$$

$$p = \left[\ b_o - b_1\ (b_1^T\ f\ b_1\ )^{-1}\ b_1^T\ f\ b_o\right] F$$

$$e = f\ p = f\ b\ F$$

$$d = b_o^T\ e = b_o^T\ f\ b\ F$$

Notas:

a).- Si: $p = b\ F$

Por el principio de contragradiencia:

$$d = b^T\ e$$

$$d = b^T\ f\ b\ F$$

b).- Se demuestra que:

$$b_o^T\ f\ b = b^T\ f\ b = b^T\ f\ b_o = K^{-1}$$

$(K)^{-1}$ = matriz de flexibidades de la estructura.

<u>Demostraciones</u>

1).- $a^T\ b_o = I$; $a^T\ b_1 = 0$

Por la ecuación de equilibrio:

$$p = b_o\ F + b_1\ R$$

Premultiplicando por a^T :

$$a^T p = a^T b_o F + a^T b_1 R$$

Pero: $a^T p = F$

$$\therefore F = a^T b_o F + a^T b_1 R$$

La ecuación anterior se debe cumplir para cualquier F y cualquier R

$$a^T b_o = I$$

$$a^T b_1 = 0 \qquad \text{l.q.q.d.}$$

2).- $b_{oo} = (a_o^T)^{-1}$

Por el teorema (1)

$$\left[a^T\right]\left[b_o\right] = \left[a_o^T \mid a_1^T\right] \begin{bmatrix} b_{oo} \\ \hline 0 \end{bmatrix} = I$$

$$a_o^T b_{oo} = 1$$

La matriz a_o^T es cuadrada, por lo tanto:

$$b_{oo} = (a_o^T)^{-1} \qquad \text{l.q.q.d.}$$

3).- $b_{1o} = - b_{oo} a_1^T b_{11}$

Por el teorema (1)

$$\left[a^T\right]\left[b_1\right] = \left[a_o^T \mid a_1^T\right] \begin{bmatrix} b_{1o} \\ b_{11} \end{bmatrix} = 0$$

$$a_o^T b_{1o} + a_1^T b_{11} = 0$$

Despejando a b_{1o} :

$$b_{1o} = -(a_o^T)^{-1} a_1^T b_{11}$$

Pero por (2)

$$\underline{b_{10} = - b_{00}\, a_1^T\, b_{11} \qquad \text{l.q.q.d.}}$$

4).- $b_0^T f b = b^T f b = b^T f b_0 = K^{-1}$

Recordemos que: $b = b_0 - b_1 (b_1^T f b_1)^{-1} b_1^T f b_0$ por consiguiente:

$$b^T f b = \left[b_0 - b_1 (b_1^T f \cdot b_1)^{-1} b_1^T f b_0 \right]^T f b =$$

$$= b_0 f b - \underbrace{b_0^T f b_1 (b_1^T f b_1)^{-1} b_1^T f b_1}_{0}$$

$$b_0 f b_1 (b_1^T f b_1)^{-1} b_1^T f b_0 - b_0 f b_1 \underbrace{(b_1^T f b_1)^{-1} b_1^T f b_1}_{I}$$

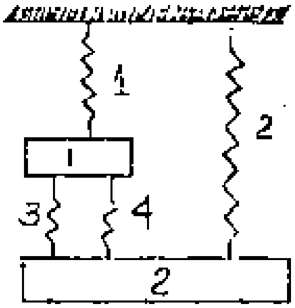
$$(b_1^T f b_1)^{-1} b_1^T f b_0$$

$$\therefore - b_0^T f b_1 (b_1^T f b_1)^{-1} b_1^T f b_0 = 0 \qquad \text{l.q.q.d.}$$

La matriz K^{-1} es simétrica, por consiguiente:

$$K^{-1} = (K^{-1})^T \quad \therefore \quad b_0^T f b = (b_0^T f b)^T = b^T f b_0$$

Ejemplos:



$$F = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

$k_j = 1$ T/cm.

$j = 1, 2, 3, 4$

a).- Método de las rigideces

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} \quad ; \; k = I$$

$$a^T k a = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} = K$$

$$K^{-1} = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}$$

$$d = K^{-1}\ F = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ cm.}$$

$$e = a\ d = \begin{bmatrix} 4 \\ 1 \\ -3 \\ -3 \end{bmatrix} \text{ cm.}$$

$$p = k\ e = \begin{bmatrix} 4 \\ 1 \\ -3 \\ -3 \end{bmatrix} \text{ Ton.}$$

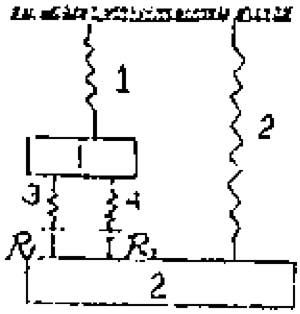
Solución para cualquier F

$$d = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} \begin{bmatrix} F \end{bmatrix}$$

$$e = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \\ -1/5 & 1/5 \\ -1/5 & 1/5 \end{bmatrix} \begin{bmatrix} F \end{bmatrix}$$

$$p = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \\ -1/5 & 1/5 \\ -1/5 & 1/5 \end{bmatrix} \begin{bmatrix} F \end{bmatrix}$$

b).- Método de las flexibilidades



$$b_0 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] \qquad b_1 = \left[\begin{array}{cc} 1 & 1 \\ -1 & -1 \\ \hdashline 1 & 0 \\ 0 & 1 \end{array}\right]$$

$$b_1^T \, f \, b_1 = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$(b_1^T \, f \, b_1)^{-1} = \begin{bmatrix} 3/5 & -2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$b_1 \, (b_1^T \, f \, b_1)^{-1} \; b_1^T \, f \, b_O = 1/5 \begin{bmatrix} 2 & -2 \\ -2 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$b = b_O - b_1 \, (b_1^T \, f \, b_1)^{-1} \; b_1^T \, f \, b_O = 1/5 \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} \quad \therefore p = b$$

$$d = b_O^T \, f \, b \, F$$

$$b_O^T \, f \, b = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} = K^{-1}$$

Resumen (2)

I.- Armaduras

1.- Vectores fuerza y desplazamiento

a).- armadura plana

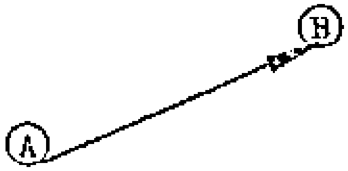
$$F_j = \begin{bmatrix} F_{jx} \\ F_{jy} \end{bmatrix} \quad ; \quad d_j = \begin{bmatrix} d_{jx} \\ d_{jy} \end{bmatrix}$$

b).- armadura en el espacio

$$F_j = \begin{bmatrix} F_{jx} \\ F_{jy} \\ F_{jz} \end{bmatrix} \quad ; \quad d_j = \begin{bmatrix} d_{jx} \\ d_{jy} \\ d_{jz} \end{bmatrix}$$

2.- Matriz de continuidad (a)

$$(a) = \begin{array}{c|ccc|} & \textcircled{A} & & \textcircled{B} \\ \hline i & -u_i & & +u_i \\ & \vdots & & \vdots \end{array}$$



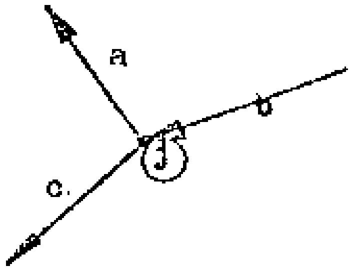
Barra i con extemos en los nudos (A) y (B)

$$u_i = \left[\text{sen } \theta \ , \ \cos \theta\right] \quad ó \quad u_i = \left[\cos\alpha , \cos\beta , \cos\gamma\right]$$

3.- Matriz de equilibrio $\left[a^T\right]$

$$\left[a^T\right] = \begin{array}{c|ccc|} & a & b & c \\ \hline j & -u_a^T & +u_b^T & -u_c^T \\ & \vdots & & \end{array}$$

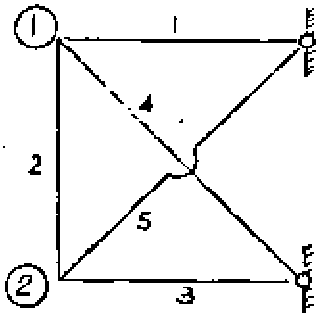
Si entra al nudo +
Si sale del nudo -

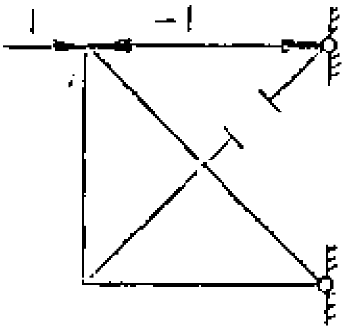


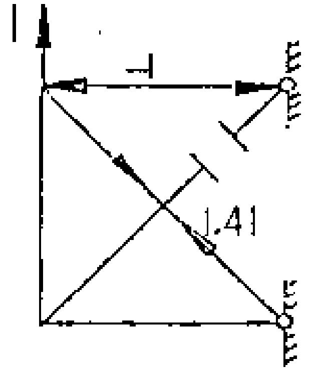
Nudo ⓙ concurren barras a, b, c, ...

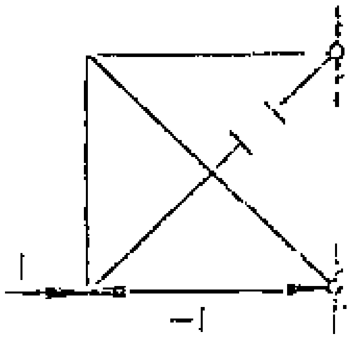
II.- Interpretación física de algunas matrices en el método de las fuerzas

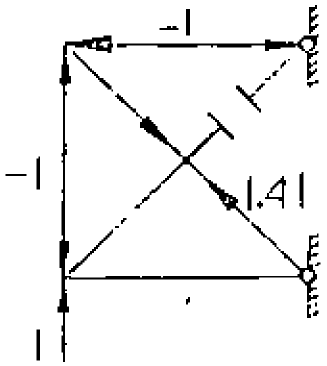
b_O = fuerzas en las barras producidas por fuerzas unitarias aplicadas en la estructura primaria





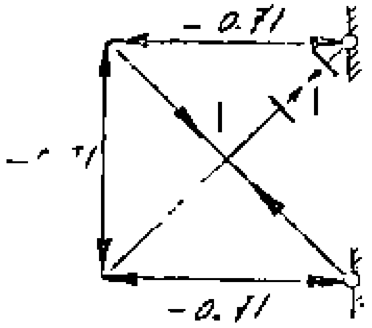






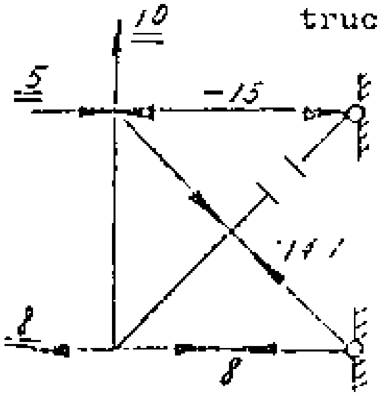
$$b_O = \begin{bmatrix} -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1.41 & 0 & 1.41 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b_1 = fuerzas en las barras producidas por redundantes unitarios aplicados en la estructura primaria



$$b_1 = \begin{bmatrix} -0.71 \\ -0.71 \\ -0.71 \\ +1 \\ +1 \end{bmatrix}$$

$b_0 F$ = fuerzas en las barras producidas por las fuerzas $\left[F\right]$ aplicadas en la estructura primaria.



$$F = \begin{bmatrix} 5 \\ 10 \\ -8 \\ 0 \end{bmatrix}; \quad b_0 F = \begin{bmatrix} -15 \\ 0 \\ +8 \\ 14.1 \\ 0 \end{bmatrix}$$

III.- Casos particulares de $\left[a_0^T\right]$

Si la estructura es una armadura, que se puede resolver la estructura primaria por el método de los nudos, es posible que la matriz sea triangular interior.

Posteriormente veremos que para algunas estructuras es posible hacer que $\left[a_0^T\right]$ sea triangular inferior ó superior. (Ver ejemplo)

IV.- Obtención directa de las reacciones y efecto de desplazamiento en los apoyos.

Sean $\left[P\right]$ las reacciones y $\left[d_A\right]$ los desplazamientos de los apoyos - (en general $d_A = 0$) si consideramos a estos como nudos, en el método de los desplazamientos, se obtiene:

$$\begin{bmatrix} F \\ \hdashline P \end{bmatrix} = \left[\begin{array}{c:c} K_{11} & K_{21} \\ \hdashline K_{12} & K_{21} \end{array}\right] \begin{bmatrix} d \\ \hdashline d_A \end{bmatrix}$$

$$(K_{11} = K_{21}^T)$$

O bien:
$$F = K_{11}\, d + K_{21}\, d_A$$

$$F - K_{21}\, d_A = K_{11}\, d \qquad d = K_{11}^{-1} F_{et}$$

$$P = K_{12}\, d + K_{22}\, d_A$$

O bien:

$$P = K_{12} K_{11}^{-1} F + \left[K_{22} - K_{12} K_{11}^{-1} K_{21} \right] d_A$$

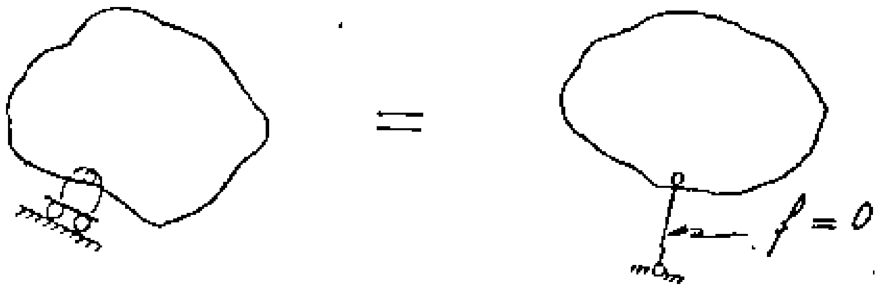
Observe que si $F = 0$

$$P = \tilde{K} \, d_A$$

donde $\tilde{K} = K_{22} - K_{12} K_{11}^{-1} K_{21}$ es la contracción de la matriz $[K]$

Cuando un apoyo no es completo (tiene algun grado de libertad) se puede substituir por un sistema de barras de rigidez infinita (flexibilidad nula) que se apoyen en apoyos completos.

Ejemplo:



Si el apoyo es completo no es necesario ni se debe hacer esta substitución.

V.- Apéndice

1.- Inversión de una matriz triangular inferior

Sea L una matriz triangular inferior y M su inversa, es muy facil demostrar el siguiente algoritmo para obtener los elementos de M.

$m_{ii} = (l_{ii})^{-1}$ (elementos diagonales)

$m_{ij} = -(l_{ii})^{-1} \sum_{r=j} l_{ir} \, m_{rj}$ $(i > j)$

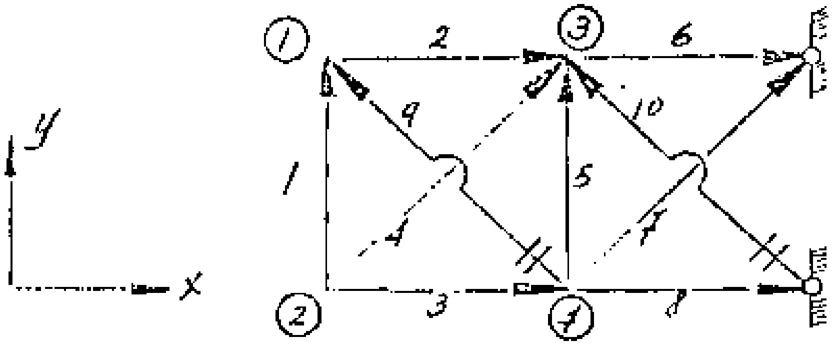
$m_{ij} = 0$ $(i < j$, M es también triangular inferior)

2.- J. Robinson ha publicado dos artículos:
"Automatic Selection of Redundancies in the Matrix Force Method: The Rank Technique"
Car. Aeron Space J; 11: 9-12 (1965)
"Dissertation on the Rank Technique and its Application" J Roy Aeron Soc., 69:280-283 (1965)

en los cuales desarrolla un método bastante ingenioso, basado en la eliminación de Jordan, para elegir las redundantes y obtener $\left[a_o^T\right]$; el método es aplicable a cualquier tipo de estructura.

VI.- Ejemplos

1.- Matriz $\left[a_o^T\right]$ triangular inferior:

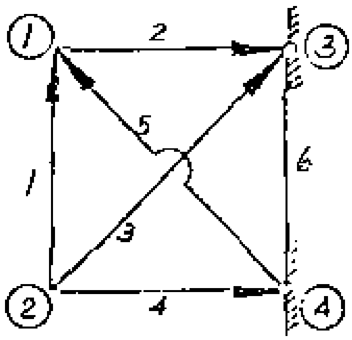


$$\left[a^T\right] = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0 \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 \\ 2 & 0 & 0 & -1 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\ & -1 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0.71 & 0 & -1 & 0 & 0 & 0 & -0.71 \\ & 0 & 0 & 0 & 0.71 & 1 & 0 & 0 & 0 & 0 & 0.71 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & -0.71 & -1 & 0.71 & 0 \\ & 0 & 0 & 0 & 0 & -1 & 0 & -0.71 & 0 & -0.71 & 0 \end{array}$$

Obtengamos $\left[a_o^T\right]$ con el algoritmo de la inversión de matrices triangular inferior trabajando con submatrices de 2 x 2.

$$\left[a_o^T\right]^{-1} = \left[b_{oo}\right] = \begin{array}{|cc|cc|cc|cc|} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.41 & 0 & -1.41 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ \hline 0 & -1.41 & 0 & -1.41 & 0 & -1.41 & 0 & -1.41 \\ 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \end{array}$$

2.- Desplazamientos en los apoyos y obtención de reacciones



$$d_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{cm.}$$

$$d_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{cm.}$$

$$k_i = 1 \text{ Ton/cm.}$$

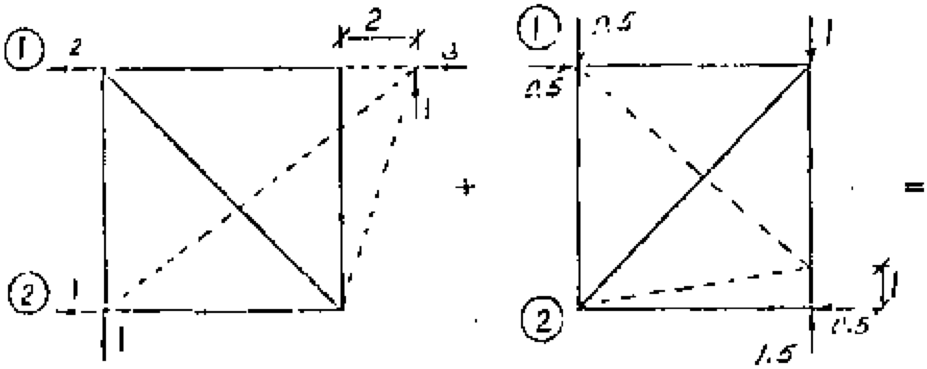
$$a = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -0.71 & -0.71 & 0.71 & 0.71 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ \hline -0.71 & 0.71 & 0 & 0 & 0 & 0 & 0.71 & -0.71 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \hline \end{array}$$

Calculemos $a^T k a$:

$$a \quad k\, a = K = \left[\begin{array}{cccc|cccc} 1.5 & -0.5 & 0 & 0 & -1 & 0 & -0.5 & 0.5 \\ -0.5 & 1.5 & 0 & -1 & 0 & 0 & 0.5 & -0.5 \\ 0 & 0 & 1.5 & 0.5 & -0.5 & -0.5 & -1 & 0 \\ 0 & -1 & 0.5 & 1.5 & -0.5 & -0.5 & 0 & 0 \\ \hline -1 & 0 & -0.5 & -1.5 & 1.5 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & -0.5 & 0.5 & 1.5 & 0 & -1 \\ -0.5 & 0.5 & -1 & 0 & 0 & 0 & 1.5 & -0.5 \\ 0.5 & -0.5 & 0 & 0 & 0 & -1 & -0.5 & 0 \end{array}\right] = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]$$

$$d_A = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad -K_{21}\, d_A = \begin{bmatrix} +1.5 \\ +0.5 \\ +1 \\ +1 \end{bmatrix}$$

Observe que:



$= +K_{21}\ d_A$

$$P_{ex} = \begin{bmatrix} -10 \\ 0 \\ 5 \\ 10 \end{bmatrix} + \begin{bmatrix} 1.5 \\ 0.5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8.5 \\ 0.5 \\ 6 \\ 11 \end{bmatrix}$$

I.- Generalización de vectores fuerza y desplazamiento.

$$\begin{bmatrix} F \end{bmatrix} \qquad \begin{bmatrix} d \end{bmatrix}$$

Armadura plana

$$\begin{bmatrix} Fx \\ Fy \end{bmatrix} \qquad \begin{bmatrix} dx \\ dy \end{bmatrix}$$

Armadura en el espacio

$$\begin{bmatrix} Fx \\ Fy \\ Fz \end{bmatrix} \qquad \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

Marco plano

$$\begin{bmatrix} Fx \\ Fy \\ Mz \end{bmatrix} \qquad \begin{bmatrix} dx \\ dy \\ \theta z \end{bmatrix}$$

Marco en el espacio

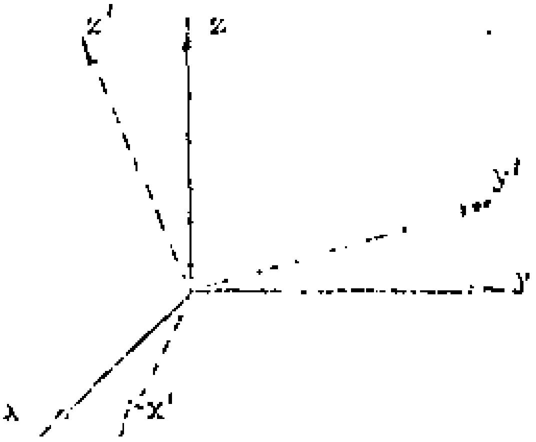
$$\begin{bmatrix} Fx \\ Fy \\ Fz \\ Mx \\ My \\ Mz \end{bmatrix} \qquad \begin{bmatrix} dx \\ dy \\ dz \\ \theta x \\ \theta y \\ \theta z \end{bmatrix}$$

Malla plana

$$\begin{bmatrix} Fz \\ Mx \\ My \end{bmatrix} \qquad \begin{bmatrix} dz \\ \theta x \\ \theta y \end{bmatrix}$$

II.- Transformación de coordenadas.

a).- Rotación
(Espacio tridimensional)



La rotación queda definida con la matriz Λ_3

$$\Lambda_3 = \begin{bmatrix} \cos\alpha_{x'} & \cos\beta_{x'} & \cos\gamma_{x'} \\ \cos\alpha_{y'} & \cos\beta_{y'} & \cos\gamma_{y'} \\ \cos\alpha_{z'} & \cos\beta_{z'} & \cos\gamma_{z'} \end{bmatrix}$$

α, β, γ = Angulos que forman los ejes X',Y',Z', con los ejes X,Y,Z.

(Λ_3 es ortogonal; $\Lambda_3^T = \Lambda_3^{-1}$)

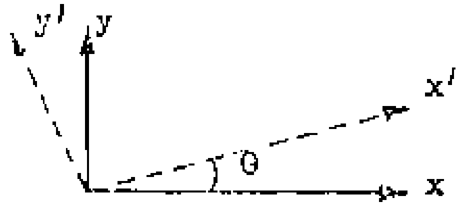
$$\left[F'\right] = \left[T\right]\left[F\right]$$

Donde: V' = Vector referido al sistema X,Y,Z (6 componentes)

V = Vector referido al sistema X,Y,Z (6 componentes)

$$T = \left|\begin{array}{c:c} \Lambda_3 & 0 \\ \hdashline 0 & \Lambda_3 \end{array}\right| = \text{matriz de transformación (6 x 6)}$$

(Plano)



$$\left[F'\right] = \left[T\right]\left[F\right] \text{, etc.}$$

Donde:

$$T = \left[\begin{array}{cc:c} \cos\theta & \operatorname{Sen}\theta & 0 \\ -\operatorname{Sen}\theta & \cos\theta & \\ \hdashline \multicolumn{2}{c:}{0} & 1 \end{array}\right]$$

O bien: $= \left[\begin{array}{c:c} \Lambda_2 & 0 \\ \hdashline 0 & 1 \end{array}\right]$

F, d, V = Vectores con tres componentes (Fx; Fy, Mz), etc.

III.- Transformación de rigideces

Si: $F' = T F$

Se sigue que: $d = T^T d'$ (por contragradiencia)

Si $[k]$ es la matriz de rigidez para $[F]$ y $[d]$

$$F = k d$$

Se puede obtener $[k']$ (matriz de rigidez para $[F']$ y $[d']$)

$$F = k d = k T^T d'$$

$$T F = T k T^T d'$$

$$F' = (T k T^T) d'$$

$$\therefore k' = T k T^T$$

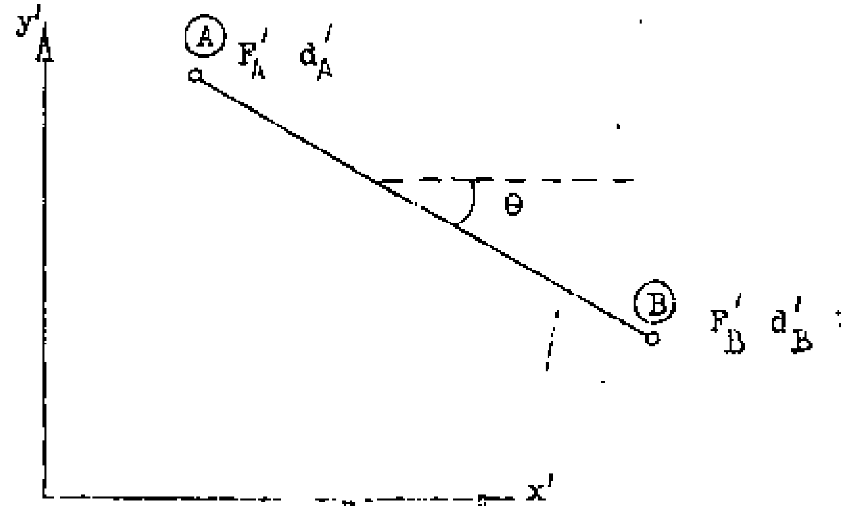
Aplicaciones:

a).- Barras de armaduras

Sistema F, d:

$$F = [N]; \quad d = [e]; \quad k = [EA/L]$$

Sistema F', d'



$$F' = \left[\begin{array}{c} F_{x'A} \\ F_{y'A} \\ \hline F_{x'B} \\ F_{y'B} \end{array}\right] = \left[\begin{array}{c} F'_A \\ \hline F'_B \end{array}\right]$$

$$d' = \left[\begin{array}{c} d_{x'A} \\ d_{y'A} \\ \hline d_{x'B} \\ d_{y'B} \end{array}\right] \quad \left[\begin{array}{c} d'_A \\ \hline d'_B \end{array}\right]$$

Matriz T: (Por estático)

$$T = \begin{bmatrix} -\text{Cos } \theta \\ +\text{Sen } \theta \\ +\text{Cos } \theta \\ -\text{Sen } \theta \end{bmatrix}$$

$$k' = T\ k\ T^T$$

$$k' = EA/L \left[\begin{array}{cc|cc} c^2 & -cs & -c^2 & cs \\ -cs & s^2 & cs & s^2 \\ \hline -c^2 & cs & c^2 & -cs \\ cs & -s^2 & -cs & s^2 \end{array}\right]$$

Si $\theta = 0^\circ$

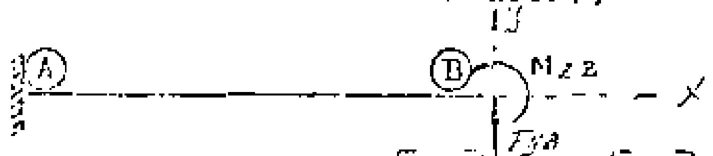
$$k'^{\circ} = EA/L \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ANALISIS

ESTRUCTURAL II

RESUMEN 6

b).- Vigas (considerando únicamente flexión),

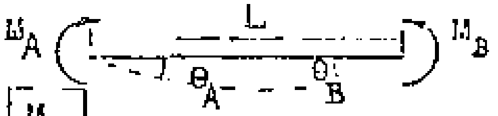


Sistema F, d:

$$F = \begin{bmatrix} F_{yB} \\ M_{zB} \end{bmatrix}; \quad d = \begin{bmatrix} d_{yB} \\ \theta_{zB} \end{bmatrix}$$

$$k = \begin{bmatrix} 12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L \end{bmatrix} \text{ (Ver apéndice)}$$

Sistema F', d':



$$F' = \begin{bmatrix} M_A \\ M_B \end{bmatrix} \text{(momentos Flex.)}; \quad d' = \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

Matriz T: (Por estática)

$$T = \begin{bmatrix} L & 1 \\ 0 & 1 \end{bmatrix}$$

$$k' = EI/L \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$$

Nota: Observese que los ángulos θ_A y θ_B se miden a partir de la linea (A)-(B), se considera que (A) y (B) no se desplazan relativamente.

IV.- Ensamble de la matriz K .

Se ha visto que la matriz K (rigidez ensamblada de la estructura), se obtiene:

$$K = a^T k a$$

donde:

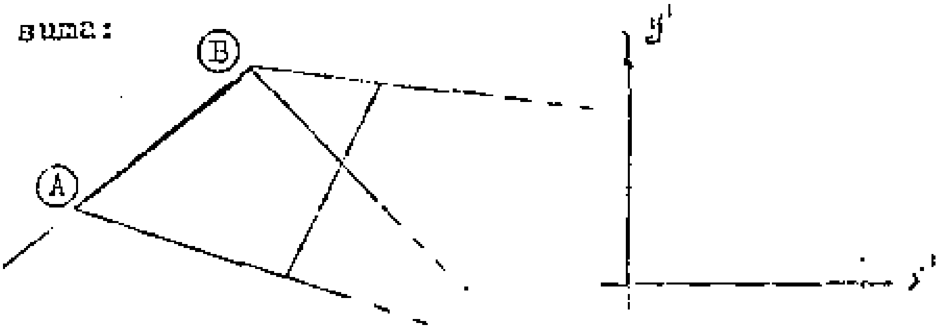
$$F = a^T p$$

$$e = a d$$

Obsérvese que $[a^T]$ es semejante a la matriz $[T]$, donde el sistema $[p]$, $[e]$ se transforma al sistema $[F]$, $[d]$.

Hay dos formas de obtener K sin afectar el producto (a^T k a) directamente.

1).- Regla de la suma:

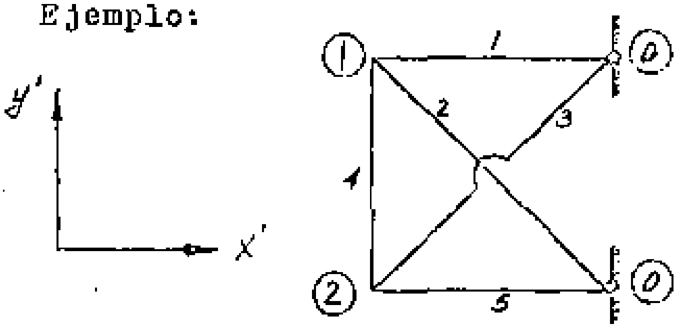


Consideremos la barra Ⓐ-Ⓑ de una estructura, sea $[k']$ su rigidez acoplada referida al sistema global x', y'

$$[k'] = \left[\begin{array}{c|c} k'_{AA} & k'_{AB} \\ \hline k'_{BA} & k'_{BB} \end{array}\right]$$

$$[K] = \sum_{j=1}^{j=b} \begin{array}{c} \\ Ⓐ \\ Ⓑ \end{array} \begin{array}{cc} Ⓐ & Ⓑ \\ \multicolumn{2}{c}{\left[\begin{array}{c|c} k_{AA_j} & k_{AB_j} \\ \hline k_{BA_j} & k_{BB_j} \end{array}\right]} \end{array}$$

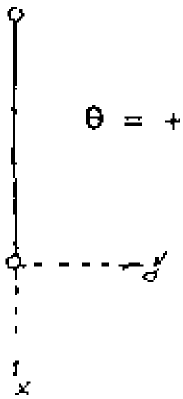
Ejemplo:



$A_1 = A_4 = A_5 = A$

$A_2 = A_3 = A/\sqrt{2}$

Barra ①-② (4)



θ = + 90°

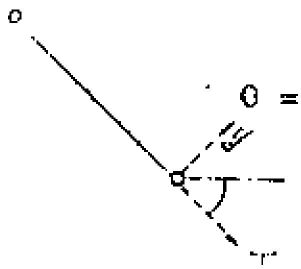
$$k'_4 = EA/L \begin{array}{c|cc|cc} & \multicolumn{2}{c|}{①} & \multicolumn{2}{c}{②} \\ \hline ① & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -1 \\ \hline ② & 0 & 0 & 0 & 0 \\ & 0 & -1 & 0 & 1 \end{array}$$

Barra ①-⓪ (1)

θ = 0°

$$k'_1 = EA/L \begin{array}{c|cc|cc} & ① & & ⓪ & \\ \hline ① & 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline ⓪ & -1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \end{array}$$

Barra ①-⓪ (2)



θ = 45°

$$k'_2 = \begin{array}{c|cc|cc} & ① & & ⓪ & \\ \hline ① & 1/2 & -1/2 & -1/2 & 1/2 \\ & -1/2 & 1/2 & +1/2 & -1/2 \\ \hline ⓪ & -1/2 & 1/2 & 1/2 & -1/2 \\ & 1/2 & -1/2 & -1/2 & 1/2 \end{array} EA/L$$

Barra ⓪-② (3)

θ = - 45°

$$k'_3 = \begin{array}{c|cc|cc} & ⓪ & & ② & \\ \hline ⓪ & 1/2 & 1/2 & -1/2 & -1/2 \\ & 1/2 & 1/2 & -1/2 & -1/2 \\ \hline ② & -1/2 & -1/2 & 1/2 & 1/2 \\ & -1/2 & -1/2 & 1/2 & 1/2 \end{array} EA/$$

Barra ②-⓪ (5)

θ = 0°

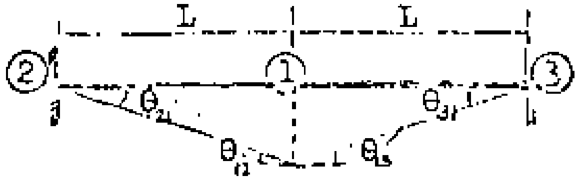
$$k'_5 = EA/L \begin{array}{c|cc|cc} & ② & & ⓪ & \\ \hline ② & 1 & 0 & -1 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline ⓪ & -1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \end{array}$$

$$K = \left[\begin{array}{c|cc|cc} & \textcircled{1} & & \textcircled{2} & \\ \textcircled{1} & 0+1+1/2 & 0+0-1/2 & 0 & 0 \\ & 0+0-1/2 & 1+0+1/2 & 0 & -1 \\ \hline \textcircled{2} & 0 & 0 & 0+1/2+1 & 0+1/2+0 \\ & 0 & -1 & 0+1/2+0 & 1+1/2+0 \end{array}\right]$$

$$K = EA/L \left[\begin{array}{cc|cc} 3/2 & -1/2 & 0 & 0 \\ -1/2 & 3/2 & 0 & -1 \\ \hline 0 & 0 & 3/2 & 1/2 \\ 0 & -1 & 1/2 & 3/2 \end{array}\right]$$

2).- Generalización de a^T

(Ejemplo)



EI = cte.

Se quiere obtener $[K']$, considerando solo el desplazamiento vertical $[\Delta]$ en ① producido por una carga $[P]$, usemos la rigidez a flexión en función de momentos.

Sistema F, d:

$$F = \begin{bmatrix} M_{21} \\ M_{12} \\ M_{13} \\ M_{31} \end{bmatrix} \quad ; \quad d = \begin{bmatrix} \theta_{21} \\ \theta_{12} \\ \theta_{13} \\ \theta_{31} \end{bmatrix}$$

$$K = EI/L \left[\begin{array}{cc|cc} 4 & -2 & & \\ -2 & 4 & & \\ \hline & & 4 & -2 \\ & & -2 & 4 \end{array}\right]$$

Sistema F', d':

$$F' = [P] \; ; \quad d' = [\Delta]$$

Matriz T (a^T generalizada)

$$T = 1/L \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$$

$K' = T K T^T$ (K' matriz ensomblada)

$$K' = EI/L^3 \begin{bmatrix} 24 \end{bmatrix}$$

V.- Apendice

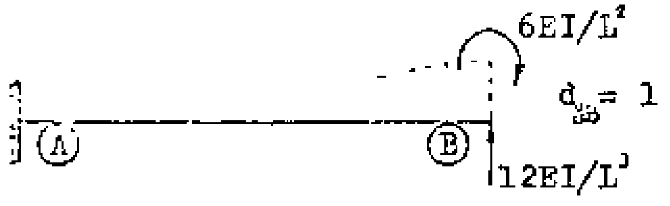
1).- Interpretación de K

$$F = K d$$

$$\text{Si: } d = I$$

$$F = K$$

Por consiguiente $[K]$ son las fuerzas que hay que aplicar para producir desplazamientos unitarios; ejemplo:

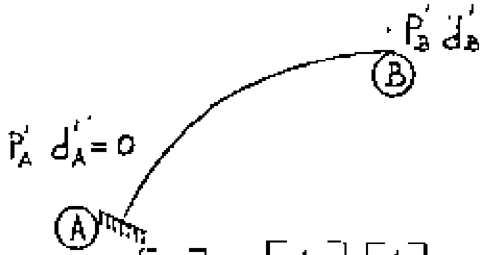


$$K = \begin{bmatrix} 12EI/L^3 & -6EI/L^2 \\ -6EI/L^2 & 4EI/L \end{bmatrix}$$

Resumen (4)

I.- Relación entre rigideces "no acopladas" y rigideces "acopladas"

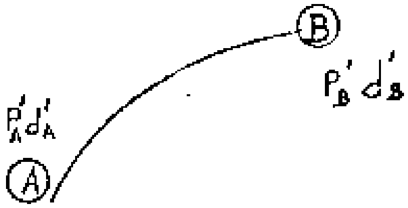
No acoplada



$$[p'_B] = [k'_{BB}]\,[d'_B]$$

$$[k'_{BB}] = [f_{BB}]^{-1}$$

Acoplada



$$\begin{bmatrix} p'_A \\ \hline p'_B \end{bmatrix} = \left[\begin{array}{c|c} k'_{AA} & k'_{AB} \\ \hline k'_{BA} & k'_{BB} \end{array}\right] \begin{bmatrix} d_A \\ \hline d_B \end{bmatrix}$$

Por estática: (Ver apéndice)

$$\begin{bmatrix} p_A \\ \hline p_B \end{bmatrix} = \begin{bmatrix} -H_{BA} \\ \hline I \end{bmatrix} [p_B]$$

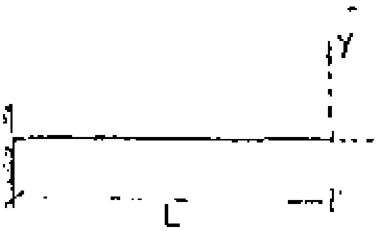
Por consiguiente:

$$\left[\begin{array}{c|c} k'_{AA} & k'_{AB} \\ \hline k'_{BA} & k'_{BB} \end{array}\right] = \begin{bmatrix} -H_{BA} \\ \hline I \end{bmatrix} [k_{BB}] \left[\begin{array}{c|c} -H^T_{BA} & I \end{array}\right]$$

$$\left[\begin{array}{c|c} k'_{AA} & k'_{AB} \\ \hline k'_{BA} & k'_{BB} \end{array}\right] = \left[\begin{array}{c|c} H_{BA}\;k_{BB}\;H^T_{BA} & -H_{BA}\;k_{BB} \\ \hline -k_{BB}\;H^T_{BA} & k_{BB} \end{array}\right]$$

Para una viga recta de sección constante, considerando los efectos de flexión, fuerza normal y cortante:

$$f_{BB} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI}(1+c) & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$



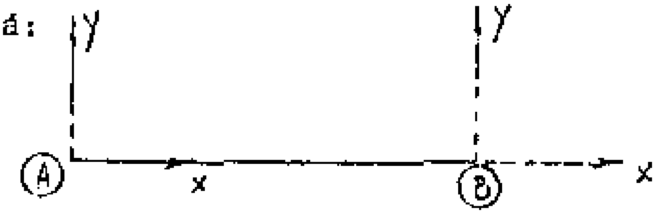
$$c = 6(1+u)\left(\rho/L\right)^2 k_{forma}$$

$$\left(\rho = \sqrt{I/A}\;;\; k_{forma} = A/A_{corte}\right)$$

Invirtiendo a f_{BB}, se obtiene a k_{BB}

$$k = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3(1+4c)} & \frac{-6EI}{L^2(1+4c)} \\ 0 & \frac{-6EI}{L^2(1+4c)} & \frac{4EI\,(1+c)}{L\,(1+4c)} \end{bmatrix}$$

La matriz acoplada será:



$$\begin{bmatrix} k_{AA} & k_{AB} \\ k_{BA} & k_{BB} \end{bmatrix} = \left[\begin{array}{ccc|ccc} \frac{EA}{L} & & & & & \\ 0 & \frac{12EI}{L^3(1+4c)} & & & \text{SIMETRICA} & \\ 0 & \frac{6EI}{L^2(1+4c)} & \frac{4EI(1+c)}{L(1+4c)} & & & \\ \hline -\frac{EA}{L} & 0 & 0 & \frac{GA}{L} & & \\ 0 & \frac{-12EI}{L^3(1+4c)} & \frac{-6EI}{L^2(1+4c)} & 0 & \frac{12EI}{L^3(1+4c)} & \\ 0 & \frac{6EI}{L^2(1+4c)} & \frac{2EI(1-2c)}{L(1+4c)} & 0 & \frac{-6EI}{L^2(1+4c)} & \frac{4EI(1+c)}{L(1+4c)} \end{array}\right]$$

Conociendo la matriz acoplada de las barras que forman una estructura, bastará aplicar la regla de la suma para obtener la matriz de rigidez de la estructura K' (matriz de rigidez ensamblada)

Nota: para pasar de coordenadas locales a globales, recuerde que:

$$k' = T\ k\ T^T$$

O bien:

$$k'_{BB} = T\ k_{BB}\ T^T$$

$$k'_{AA} = T\ k_{AA}\ T^T \quad \text{etc.}$$

II.- Matriz de continuidad para marcos planos

a).- Alternativa 1

$$[a] = ⓘ \begin{array}{c} \\ Ⓐ \\ Ⓑ \end{array} \begin{array}{cc} Ⓐ & Ⓑ \\ \left[\begin{array}{cc} T^T & 0 \\ 0 & T^T \end{array}\right] \end{array}$$

$[a]$ (6b x 3n)

$$[k] = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_b \end{bmatrix}$$

ki = rigidez acoplada de la barra i

$$[K'] = [a^T][k][a]$$

Solución:

$$d' = K^{-1}\ F'$$

$$e = a\ d'$$

$p = k\ e$ (Obteniendose p y p , en coordenadas locales, de cada barra)

b).- Alternativa 2

$$[a] = ⓘ \begin{array}{cc} Ⓐ & Ⓑ \\ \left[\begin{array}{cc} -H^T_{BA}\ T & T^T \end{array}\right] \end{array}$$

$[a]$ (3b x 3n)

$$[k] = \begin{bmatrix} k_{BB1} & & & & \\ & k_{BB2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k_{BBb} \end{bmatrix}$$

k_{BBi} = matriz de rigidez "no acopla
de la barra i.

$$K' = a^T \; k \; a$$

Solución:

$$d' = (K')^{-1} F'$$

$$e = a \, d'$$

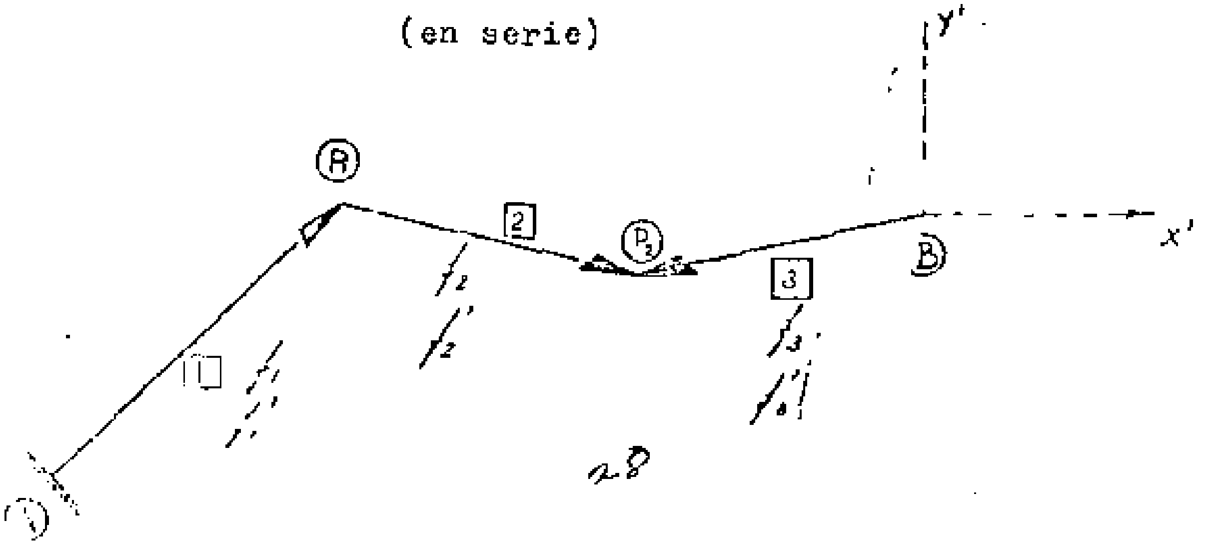
$p = k \, e$ Obteniendose sólo p_B , en coordenadas locales, de cada barra; p_A se obtiene por estática:

$$p_A = -H_{BA} \, p_B$$

$$H_{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L & 1 \end{bmatrix} \quad \text{(barra recta)}$$

(Coordenadas locales)

III.- Flexibilidad de barras encadenadas
(en serie)



Por definición:

$$\underline{\underline{d_B' = f_{BB}\ p_B'}}$$

$$f_{BB} = H_{BP_3}'^{T}\ f_3'\ H_{BP_3}' + H_{BP_2}'^{T}\ f_2'\ H_{BP_2}' + H_{BP_1}'^{T}\ f_1'\ H_{BP_1}'$$

Nota: Si se tienen dos sistemas F, d y F', d' las relaciones entre sus flexibilidades son las siguientes:

Si: $P = A\ F'$

$d' = A^T\ d$ (por contragradiencia)

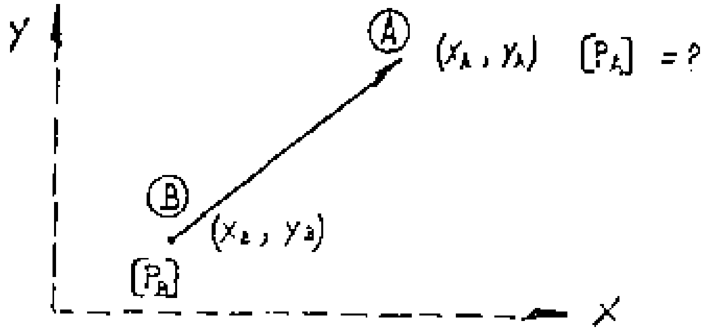
$f' = A^T\ f\ A$

Por consiguiente: $(F = T^T\ F')$

$$f' = T\ f\ T^T$$

Apendice:

I.- Matriz $[H]$ de transporte de fuerzas.



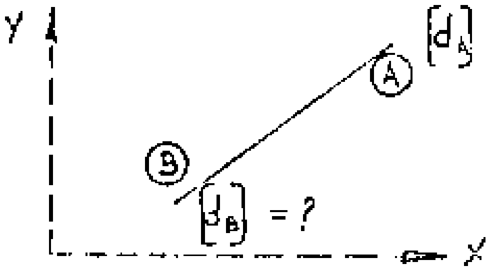
Aplicada $[P_B]$ en Ⓑ se quiere transportar a A: por estática:

$$\begin{bmatrix} P_{AX} \\ P_{AY} \\ M_{AZ} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (y_A - y_B) & -(x_A - x_B) & 1 \end{bmatrix} \begin{bmatrix} P_{BX} \\ P_{BY} \\ M_{BZ} \end{bmatrix}$$

H_{BA} = Matriz para transportar fuerzas de B a A

Nota: Observe que $H_{BA} = (H_{Ab})^{-1}$

II.- Matriz $[H']$ de transporte de desplazamientos:



Supongamos que a Ⓐ se le da un desplazamiento $[d_A]$, cuanto vale $[d_B]$ si la barra $\overline{AB}$ se considera rígida, por geometría:

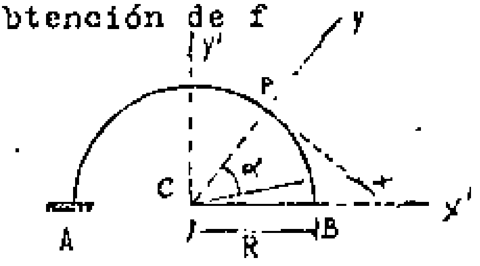
$$\begin{bmatrix} d_{BX} \\ d_{BY} \\ \theta_{BZ} \end{bmatrix} = \begin{bmatrix} 1 & 0 & (y_A - y_B) \\ 0 & 1 & -(x_A - x_B) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_{AX} \\ d_{AY} \\ \theta_{AZ} \end{bmatrix}$$

H^T_{BA} = Matriz para transportar desplazamientos de Ⓐ, Ⓑ

Nota: Este resultado se puede obtener directamente aplicando el principio de contragradiencia a la formulación anterior.

Ejemplos:

I.- Obtención de f



E = cte.

G = cte

Sección = constante

Conviene obtener f_{cc}, suponiendo que Ⓒ este unido rígidamente a Ⓑ

$$f_{BB} = H^T_{BC} \; f_{cc} \; H_{BC}$$

f_{cc} es más fácil de calcular que f_{AB}, por ser Ⓒ el centro de la circunferencia.

$$f_{cc} = \int_L H'^T_{cP} \; T \; \phi \; T^T \; H'_{cP} \; ds$$

...

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ R \text{ sen}\,\alpha & -R \cos\alpha & 1 \end{bmatrix} ; \quad T = \begin{bmatrix} \text{sen}\,\alpha & \cos\alpha & 0 \\ -\cos\alpha & \text{sen}\,\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x' = R \text{ sen}\,\alpha \qquad \theta = 90-\alpha$$

$$y' = R \cos\alpha$$

$$\phi = \begin{bmatrix} \frac{1}{EA} & & \\ & \frac{1}{GA_c} & \\ & & \frac{1}{EI} \end{bmatrix} \qquad A_c = \frac{A}{k_{forma}} \qquad G = \frac{E}{2(1+u)}$$

$$T^{T} H_{cp} = \begin{bmatrix} s & -c & 0 \\ c & s & 0 \\ Rs & -Rc & 1 \end{bmatrix} ; \quad H_{cp} T^{T} = \begin{bmatrix} s & c & Rs \\ -c & s & -Rc \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi\,(T^{T} H_{cp}) = \begin{bmatrix} \frac{s}{EA} & \frac{-c}{EA} & 0 \\ \frac{c}{GA_c} & \frac{s}{GA_c} & 0 \\ \frac{Rs}{EI} & \frac{-Rc}{EI} & \frac{1}{EI} \end{bmatrix}$$

$$(H_{cp}^{T}\,T)\,(\phi\,T^{T}H_{cp}) = \begin{bmatrix} \left[\frac{s^2}{EA} + \frac{c^2}{GA_c} + \frac{R^2 s^2}{EI}\right] & & \\ \left[\frac{-cs}{EA} + \frac{cs}{GA_c} - \frac{Rcs}{EI}\right] & \left[\frac{c^2}{EA} + \frac{s^2}{GA_c} + \frac{R^2 c^2}{EI}\right] & \\ \left[\frac{Rs}{EI}\right] & \left[\frac{-Rc}{EI}\right] & \left[\frac{1}{EY}\right] \end{bmatrix}$$

$$d\,s = R\,d\alpha$$

Obtengamos las integrales que figuren:

$$\int_0^{\pi} s^2\,d\alpha = \left[\frac{\alpha}{2} - \frac{\text{sen }2\alpha}{4}\right]_0^{\pi} = \frac{\pi}{2}$$

$$\int c^2\,d\alpha = \left[\frac{\alpha}{2} + \frac{\cos 2}{4}\right]_0^{\pi} = \frac{\pi}{2} + \frac{1}{4} - \frac{1}{4} = \frac{\pi}{2}$$

$$\int_0^{\pi} c\ s\ d\alpha = \left[\frac{\text{sen}^2\alpha}{2}\right]_0^{\pi} = 0\ ;\quad \int_0^{\pi} s\ d\alpha = 2\ ;\ \int c\ d\alpha = 0$$

$$f_{cc} = \begin{bmatrix} \left[\frac{\pi R}{2EA} + \frac{\pi R}{2GA_c} + \frac{\pi R}{2EI}\right] & \textit{Simétrica} & \\ 0 & \left[\frac{\pi R}{2EA} + \frac{\pi R}{2GA_c} + \frac{\pi R^3}{2EI}\right] & \\ \left[\frac{2R^2}{EI}\right] & 0 & \left[\frac{\pi R}{EI}\right] \end{bmatrix}$$

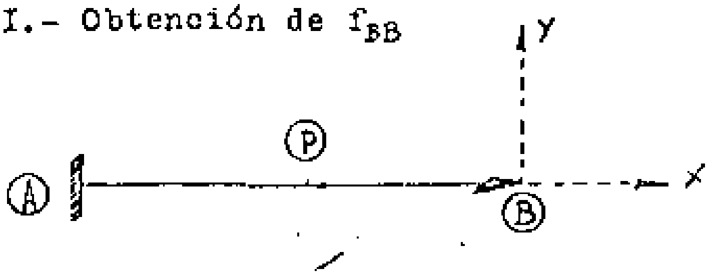
Obtengamos f_{BB} :

$$H_{BC} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & R & 1 \end{bmatrix}$$

$$f_{cc}\ H_{BC} = \begin{bmatrix} \frac{\pi R}{2}\left[\frac{1}{EA} + \frac{1}{GA_c} + \frac{R^2}{EI}\right] & \left[\frac{2R^3}{EI}\right] & \left[\frac{2R^2}{EI}\right] \\ 0 & \frac{\pi R}{2}\left[\frac{1}{EA} + \frac{1}{GA_c} + \frac{R^2}{EI}\right] & 0 \\ \left[\frac{2R^2}{EI}\right] & \left[\frac{\pi R^2}{EI}\right] & \left[\frac{\pi R}{EI}\right] \end{bmatrix}$$

$$H_{BC}^{T}\,(f_{cc}\,H_{BC}) = f_{bb} = \begin{bmatrix} \frac{\pi R}{2}\left[\frac{1}{EA} + \frac{1}{GA_c} + \frac{R^2}{EI}\right] & \left[\frac{2R^3}{EI}\right] & \left[\frac{2R^2}{EI}\right] \\ \left[\frac{2R^3}{EI}\right] & \frac{\pi R}{2}\left[\frac{1}{EA} + \frac{1}{GA_c} + \frac{3R^2}{EI}\right] & \left[\frac{\pi R^2}{EI}\right] \\ \left[\frac{2R^2}{EI}\right] & \left[\frac{\pi R^2}{EI}\right] & \left[\frac{\pi R}{EI}\right] \end{bmatrix}$$

II.- Obtención de f_{BB}



E = cte.

G = cte.

Sección constante

$$H_{BP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x & 1 \end{bmatrix} ; \quad T = I ; \quad \phi = \begin{bmatrix} \frac{1}{EA} & & \\ & \frac{1}{GA_c} & \\ & & \frac{1}{EI} \end{bmatrix}$$

$$H_{BP}^{T}\,\phi = \begin{bmatrix} \frac{1}{EA} & 0 & 0 \\ 0 & \frac{1}{GA_c} & -\frac{x}{EI} \\ 0 & 0 & \frac{1}{EI} \end{bmatrix}$$

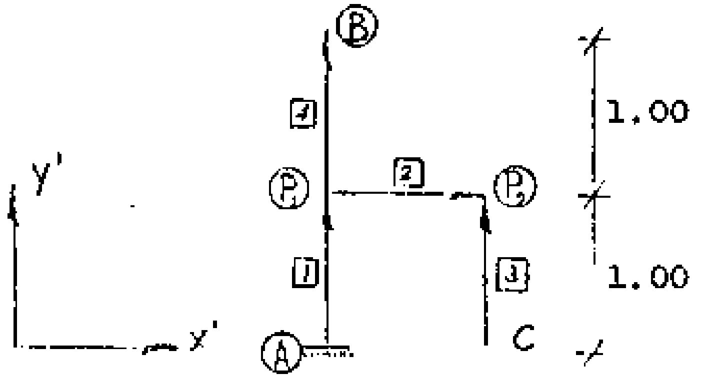
$$(H_{BP}\phi)\,H_{BP} = \begin{bmatrix} \frac{1}{EA} & 0 & 0 \\ 0 & \left[\frac{1}{GA_c} + \frac{x^2}{EI}\right] & -\frac{x}{EI} \\ 0 & -\frac{x}{EI} & \frac{1}{EI} \end{bmatrix}$$

Integrado: $\displaystyle\int_{x=-L}^{x=0}$ $\qquad ds = dx$

$$f_{BB} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \left(\frac{L^3}{3EI} + \frac{L}{GA_c} \right) & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

Nota: $\frac{L^3}{3EI} + \frac{L}{GA_c} = \frac{L^3}{3EI} \left[1 + \frac{3EI}{GA_c L^2} \right] = \frac{L^3}{3EI} \left[1 + c \right]$

III.- Obtención de f_{BB} f_{CC} f_{BC} f_{CB} :



$E = 2 \times 10^7$ T/m^2

$I = 2 \times 10^{-4}$ m^4

$A = 0.01$ m^2

Despreciemos el efecto del cortante ($c = 0$)

$$f_{BB} = f_4' + H_{BP_2}^T f_1' \; H_{BP_2}$$

$$f_{CC} = H_{CP_1}^T f_3' \; H_{CP_1} + H_{CP_1}^T f_2' \; H_{CP_1} + H_{CP_2}^T f_1' \; H_{CP_2}$$

$$f_{BC} = H_{BP_2}^T f_{P_2P_2}' H_{CP_2}$$

$$f_{CB} = f_{BC}^T$$

Obtención de f_i (flexibilidad en coordenadas locales)

$$\frac{L}{EA} = \frac{1}{2 \times 10^7 \times 0.01} = 0.5 \times 10^{-5} \text{ m/Ton.} = 0.05 \times 10^{-4} \text{ m/Ton.}$$

$$\frac{L}{3EI} = \frac{1}{3 \times 2 \times 10^7 \times 2 \times 10^{-4}} = 0.083 \times 10^{-3} \text{ m/Ton.}$$

$$\frac{L}{2EI} = \frac{1}{2 \times 2 \times 10^7 \times 2 \times 10^{-4}} = 0.125 \times 10^{-3} \text{ 1/Ton.}$$

$$\frac{L}{EI} = \frac{1}{2 \times 10^7 \times 2 \times 10^{-4}} = 0.25 \times 10^{-3} \text{ 1/Ton-m.}$$

$$f_1 = f_2 = f_3 = f_4 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.83 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix} \times 10^{-4}$$

Obtención de f'_i (flexibilidad en coordenadas globales)

a).- Barras $\boxed{1}$ $\boxed{3}$ $\boxed{4}$

$$\theta = -90^\circ \qquad T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f'_1 = f'_3 = f'_4 = \begin{bmatrix} 0.83 & 0 & -1.25 \\ 0 & 0.05 & 0 \\ -1.25 & 0 & 2.5 \end{bmatrix} \times 10^{-4}$$

b).- Barra $\boxed{2}$

$$f'_2 = f_2 = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.83 & 1.25 \\ 0. & 1.25 & 2.5 \end{bmatrix} \times 10^{-4}$$

Obtención de las matrices de transporte:

$$H_{BP_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.0 & 0 & 1 \end{bmatrix}; \quad H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +1.0 & 0 & 1 \end{bmatrix}$$

$$H_{CP_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1.00 & 1.00 & 1 \end{bmatrix}$$

Obtención de f_{BB} :

$$H_{BP_2}^{T}\ f_1' = \begin{bmatrix} 2.08 & 0 & -3.75 \\ 0 & 0.05 & 0 \\ 1.25 & 0 & 2.5 \end{bmatrix}$$

$$(H_{BP_2}^{T}\ f_1')\ H_{BP_2} = \begin{bmatrix} 5.83 & 0 & -3.75 \\ 0 & 0.05 & 0 \\ -3.75 & 0 & 2.50 \end{bmatrix}$$

$$f_{BB} = \begin{bmatrix} 6.66 & 0 & -5.00 \\ 0 & 0.10 & 0 \\ -5.00 & 0 & 5.00 \end{bmatrix} \times 10^{-4}$$

Obtención de f_{cc} :

$$H_{CP_1}^{T}\ f_3' = \begin{bmatrix} -0.42 & 0 & 1.25 \\ 0 & 0.05 & 0 \\ -1.25 & 0 & 2.50 \end{bmatrix} ; \quad (H_{CP_1}^{T}\ f_3')\ H_{CP_1} = \begin{bmatrix} 0.83 & 0 & 1. \\ 0 & 0.05 & 0 \\ 1.25 & 0 & 2.50 \end{bmatrix}$$

$$H_{CP_1}^{T}\ f_2' = \begin{bmatrix} 0.05 & 1.25 & 2.50 \\ 0 & 0.83 & 1.25 \\ 0 & 1.25 & 2.5 \end{bmatrix} ; \quad (H_{CP_1}^{T}\ f_2')\ H_{CP_1} = \begin{bmatrix} 2.55 & 1.25 & 2.50 \\ 1.25 & 0.83 & 1.25 \\ 2.50 & 1.25 & 2.50 \end{bmatrix}$$

$$H_{CP_3}^{T}\ f_1' = \begin{bmatrix} -0.42 & 0 & 1.25 \\ -1.25 & 0.05 & 2.50 \\ -1.25 & 0 & 2.5 \end{bmatrix} ; \quad (H_{CP_3}^{T}\ f_1')\ H_{CP_2} = \begin{bmatrix} 0.83 & 1.25 & 1.25 \\ 1.25 & 2.55 & 2.50 \\ 1.25 & 2.5 & 2.50 \end{bmatrix}$$

$$f_{cc} = \begin{bmatrix} 4.21 & 2.50 & 5.0 \\ 2.50 & 3.43 & 3.75 \\ 5.0 & 3.75 & 7.50 \end{bmatrix} \times 10^{-4}$$

Obtención de f_{BC}:

$$f'_{P_2P_2} = f'_1$$

$$H^T_{BP_2} f'_1 = \begin{bmatrix} 2.08 & 0 & -2.50 \\ 0 & 0.05 & 0 \\ -1.25 & 0 & 2.50 \end{bmatrix}; \quad f_{BC} = (H^T_{bP_2} f'_1)\, H_{CP_2}$$

$$f_{BC} = \begin{bmatrix} -0.42 & -2.50 & -2.50 \\ 0 & 0.05 & 0 \\ 1.25 & 2.50 & 2.50 \end{bmatrix} \times 10^{-4}$$

Recordemos que:

$$\begin{bmatrix} d'_B \\ d'_C \end{bmatrix} = \begin{bmatrix} f'_{BB} & f'_{BC} \\ f'_{CB} & f_{CC} \end{bmatrix} \begin{bmatrix} P'_B \\ P'_C \end{bmatrix} \qquad (\text{Si: } d'_A = 0)$$

O bien: $d' = [\gamma]\, P'$

Obtengamos: $[\gamma]^{-1}$

$$[\gamma] = \begin{bmatrix} 6.66 & 0 & -5.00 & -0.42 & -2.50 & -2.50 \\ 0 & 0.10 & 0 & 0 & 0.05 & 0 \\ -5.00 & 0 & 5.00 & 1.25 & 2.50 & 2.50 \\ -0.42 & 0 & 1.25 & 4.21 & 2.50 & 5.0 \\ -2.50 & 0.05 & 2.50 & 2.50 & 3.43 & 3.75 \\ -2.50 & 0 & 2.50 & 5.0 & 3.75 & 7.50 \end{bmatrix} \times 10^{-4}$$

$$[\gamma]^{-1} = \left[\begin{array}{c|c} k'_{BB} & k'_{BC} \\ \hline k'_{CB} & k'_{CC} \end{array}\right]$$

$$[\gamma]^{-1} = \begin{bmatrix} 1.217 & -0.067 & 1.055 & -1.230 & 0.133 & 0.808 \\ -0.067 & 10.224 & 0.081 & 0.133 & -0.447 & 0.086 \\ 1.055 & 0.081 & 1.253 & -0.904 & 0.162 & 0.618 \\ -1.230 & 0.133 & -0.904 & 2.460 & -0.266 & 1.615 \\ 0.133 & -0.447 & -0.162 & -0.266 & 0.895 & -0.171 \\ 0.808 & 0.086 & 0.618 & -1.615 & -0.171 & 1.360 \end{bmatrix} \times 1$$

Obtengamos la matriz de rigidez "acoplada", esto es cuando $d'_A \neq 0$; $d'_B \neq 0$; $d'_C \neq 0$

Por estática:

$$\begin{bmatrix} p'_A \\ p'_B \\ p'_C \end{bmatrix} = \begin{bmatrix} -H'_{BA} & -H'_{CA} \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p'_B \\ p'_C \end{bmatrix}$$

$$\left[\begin{array}{c|cc} k'_{AA} & k'_{AB} & k'_{AC} \\ \hline k_{BA} & k_{BB} & k_{BC} \\ \hline k_{CA} & k_{CB} & k_{CC} \end{array}\right] = \left[\begin{array}{c:c} -H'_{BA} & -H_{CA} \\ \hdashline I & 0 \\ \hdashline 0 & I \end{array}\right] \begin{bmatrix} k'_{BB} & k'_{BC} \\ k'_{CB} & k'_{CC} \end{bmatrix} \left[\begin{array}{c:c:c} -H^T_{BA} & I & 0 \\ \hdashline -H & 0 & I \end{array}\right] =$$

$$= \left[\begin{array}{c|c:c} \begin{array}{c}(H'_{BA} k'_{BB} H'^T_{BA} + H'_{CA} k'_{CC} H'^T_{CA} + \\ H'_{CA} k'_{CB} H'^T_{BA} + H'_{BA} k'_{BC} H^T_{CA})\end{array} & -H'_{BA} k'_{BB} - H'_{CA} k_{CB} & -H'_{BA} k'_{BC} - H'_{CA} k'_{CC} \\ \hline -k'_{BB} H'^T_{BA} - k'_{BC} H'^T_{CA} & k'_{BB} & k'_{BC} \\ \hdashline -k'_{CB} H'^T_{BA} - k_{CC} H'^T_{CA} & k'_{CB} & k'_{CC} \end{array}\right]$$

Generalización de la fórmula dada en la hoja 24

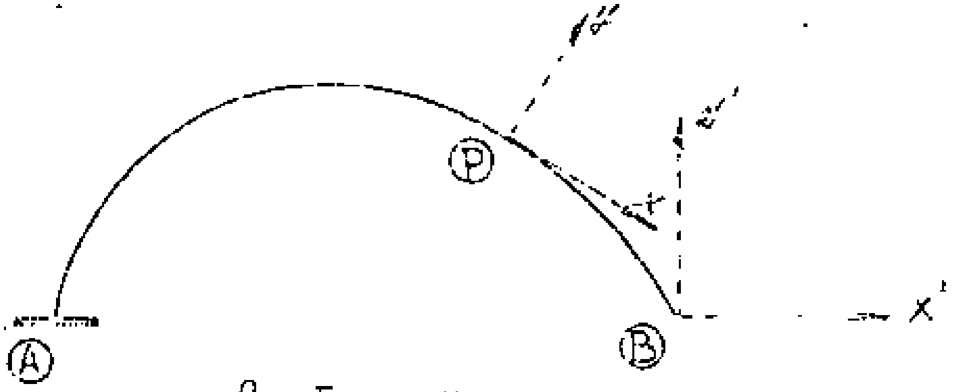
```
SOLEC        19:00      034   10/16/70

MATRIZ A
 .66600000E+01   .00000000E+00  -.50000000E+01  -.42000000E+00
-.25000000E+01  -.25000000E+01   .00000000E+00   .10000000E+00
 .00000000E+00   .00000000E+00   .50000000E-01   .00000000E+00
-.50000000E+01   .00000000E+00   .50000000E+01   .12500000E+01
 .25000000E+01   .25000000E+01  -.42000000E+00   .00000000E+00
 .12500000E+01   .42100000E+01   .25000000E+01   .50000000E+01
-.25000000E+01   .50000000E-01   .25000000E+01   .25000000E+01
 .34300000E+01   .37500000E+01  -.25000000E+01   .00000000E+00
 .25000000E+01   .50000000E+01   .37500000E+01   .75000000E+01

MATRIZ (A) (-1)
 .12174584E+01  -.66563716E-01   .10545104E+01  -.12300975E+01
 .13312743E+00   .80781727E+00  -.66563715E-01   .10223654E+02
 .80941477E-01   .13312743E+00  -.44730816E+00   .85734063E-01
 .10545104E+01   .80941476E-01   .12526551E+01  -.90420154E+00
-.16188295E+00   .61769426E+00  -.12300975E+01   .13312743E+00
-.90420153E+00   .24601949E+01  -.26625486E+00  -.16156345E+01
 .13312743E+00  -.44730816E+00  -.16188295E+00  -.26625486E+00
 .89461633E+00  -.17146813E+00   .80781726E+00   .85734063E-01
 .61769425E+00  -.16156345E+01  -.17146813E+00   .13595314E+01
```

Resumen (5)

I.- Flexibilidades de barras curvas de sección variable:



$$f'_{PB} = \int_L H'^{T}_{BP} \, T \, \phi \, T^{T} H'_{BP} \, ds \, .$$

donde:

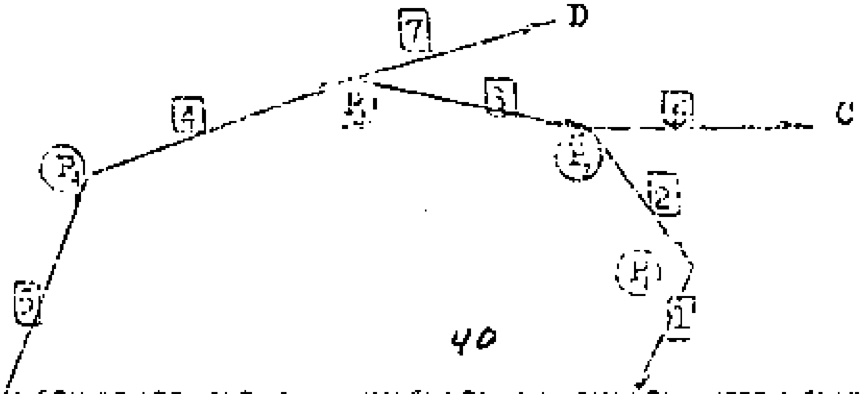
$$H'_{BP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y'_P & -x'_P & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} \text{Cos } \theta & \text{Sen } \theta & 0 \\ -\text{Sen } \theta & \text{Cos } \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\phi = \begin{bmatrix} \frac{1}{EA} & & \\ & \frac{1}{GA_c} & \\ & & \frac{1}{EI} \end{bmatrix} ; \qquad \left(A_c = \frac{A}{K_{forma}} \right)$$

II.- Flexibilidades y rigideces de árboles

Consideremos un elemento con 4 nudos (4 puntos donde se pueden aplicar fuerzas externas)



La matriz k' ("acoplada") tendrá los siguientes elementos:

$$k' = \left[\begin{array}{c|ccc} k'_{AA} & k'_{AB} & k'_{AC} & k'_{AD} \\ \hline & k'_{BB} & k'_{BC} & k'_{BD} \\ & & k'_{CC} & k'_{CD} \\ & & & k'_{DD} \end{array}\right]$$ Considerando empotrado a Ⓐ

Para obtener esta matriz habra que invertir a la matriz γ'

$$\gamma' = \begin{bmatrix} f'_{BB} & f'_{BC} & f'_{BD} \\ & f'_{CC} & f'_{CD} \\ & & f'_{DD} \end{bmatrix}$$

donde: $f'_{BB} = f'_1 + H'^T_{BP_1} f'_2 H_{BP_1} + H'^T_{BP_2} f_3 H'_{BP_2} + H'^T_{BP_3} f'_4 H'_{BP_3} + H'^T_{BP_4} f'_5 H_{BP_4}$

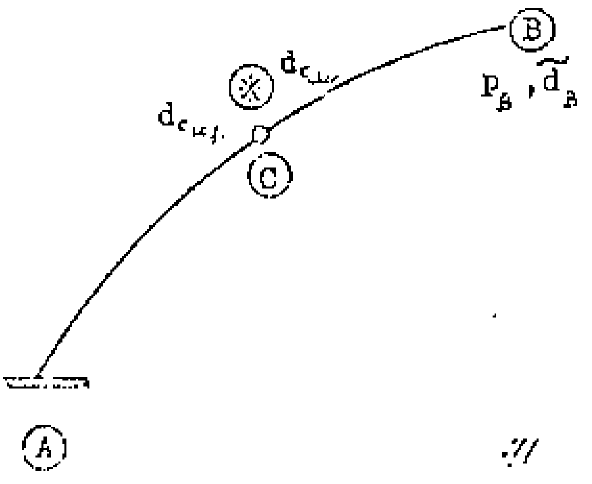
$f'_{CC} = f'_2 + H'^T_{CP_2} f'_3 H'_{CP_2} + H'^T_{CP_3} f'_4 H_{CP_3} + H'^T_{CP_4} f_5 H_{CP_4}$

$f'_{BC} = H'^T_{BP_2} f'_{BP_2} H'_{CP_2}$

donde: $f'_{BP_2} = f'_3 + H'^T_{P_2P_3} f'_4 H'_{BP_3} + H'^T_{P_2P_4} f'_5 H_{P_2P_4}$

Nota: Con elementos de n nudos las matrices de rigideces acopladas serán de dimensión (n x n) y se podrá aplicar la <u>regla de la suma</u> para ensamblar la matriz K', utilizando las rigideces acopladas en coordenadas globales.

III.- Rigideces de barras con discontinuidades (Releases)



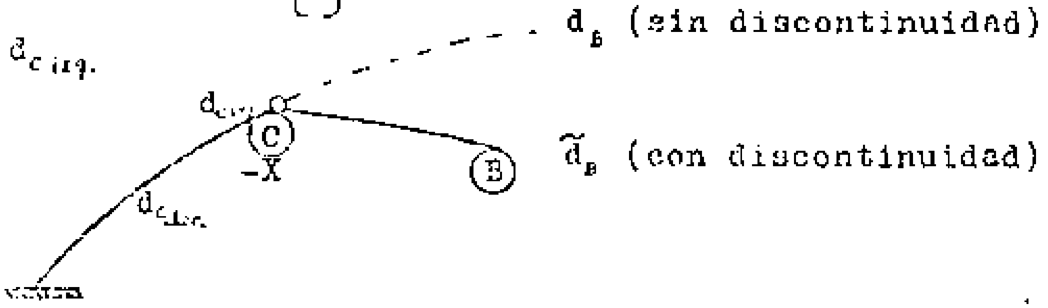
⊛ discontinuidad en algunos componentes del desplazamiento; por ejemplo: giro o si se trata de una articulación.

Se desea obtener $\widetilde{k}_{BB}$ tal que

$$p_B = \widetilde{k}_{BB}\ \widetilde{d}_B$$

Sea $\left[X\right]$ la discontinuidad de $\left[d_c\right]$

$$X = d_{c\,der.} - d_{c\,izq.}$$



$$d_B = \widetilde{d}_B + H_{BC}^{T}\ (-X)$$

$$p_B = k_{BB}\ d_B = k_{BB}\ (\widetilde{d}_B - H_{BC}^{T}\ X)$$

En la discontinuidad se tiene que:

$$\Lambda\, p_c = \Lambda\ H_{BC}\ p_B = 0$$

donde Λ es una matriz que define la discontinuidad en Ⓒ

Ejemplos: a).- Articulación —o— $\Lambda = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$p_{c\,z} = 0$ (Momento flexionante en Ⓒ = 0)

b).- Discontinuidad y —||— $\Lambda = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$p_{c\,y} = 0$ (Fuerza cortante en Ⓒ = 0)

erve que: $X = \Lambda^{T}\ x$

nde x = Parte no nula de X

(Si se trata de una articulación:)

$$X = \begin{bmatrix} 0 \\ 0 \\ x_1 \end{bmatrix}; \quad x = \left(x_1\right)$$

$$p_B = k_{BB} (\tilde{d}_B - H_{BC}^T \Lambda^T x) \ldots\ldots (5.1)$$

pero $\Lambda H_{BC} p_B = \Lambda H_{BC} k_{BB} \tilde{d}_B - \Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T x = 0$

$$x = (\Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T)^{-1} \Lambda H_{BC} k_{BB} \tilde{d}_B$$

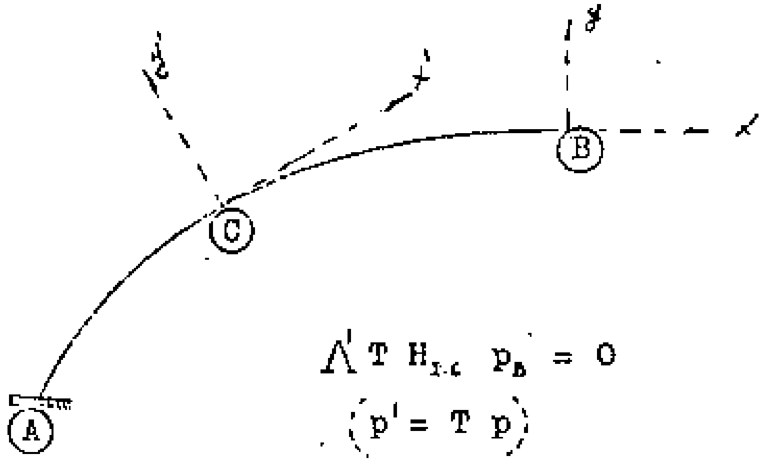
Sustituyendo en (5.1)

$$p_B = k_{BB} \underbrace{\left[I - H_{BC}^T \Lambda^T (\Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T)^{-1} \Lambda H_{BC} k_{BB} \right]}_{= \tilde{k}_{BB}} \tilde{d}_B$$

Por consiguiente:

$$\tilde{k}_{BB} = k_{BB} \left[I - A \right] \qquad A = H_{BC}^T \Lambda^T (\Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T)^{-1} \Lambda H_{BC} k_{BB}$$

Nota: Si la discontinuidad en C esta referida a un sistema x', y', se tiene:



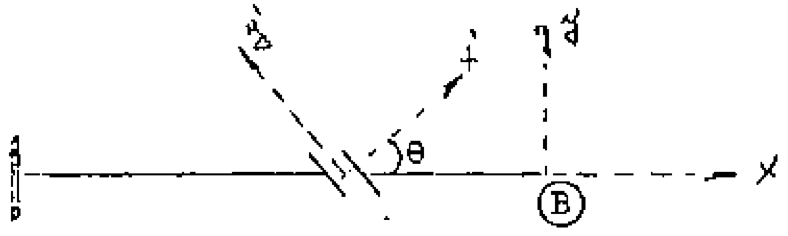
$$\Lambda' T H_{BC} p_B = 0$$

$$(p' = T p)$$

En este caso:

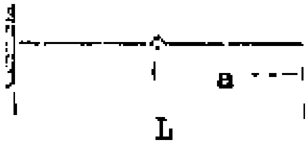
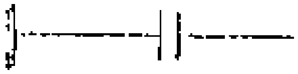
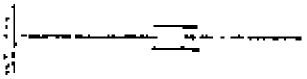
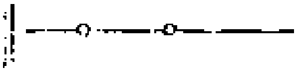
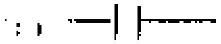
$$A = H_{BC}^T - T^T \Lambda'^T (\Lambda' T H_{BC} k_{BB} H_{BC} T^T \Lambda'^T)^{-1} \Lambda'^T H_{BC} k_{BB}$$

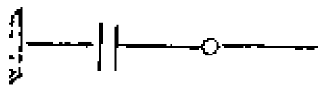
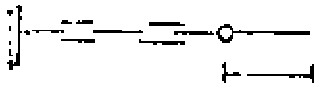
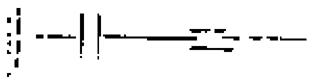
Ejemplo:



$$\Lambda' = \left[0 \ 1 \ 0 \right]$$

IV.- Rigideces de miembros rectos de sección uniforme con discontinuidad

Discontinuidad	$\tilde{k}_{SS}$
	$\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3\psi(\alpha)} & \frac{-3EI\alpha}{L^2\psi(\alpha)} \\ 0 & \frac{-3EI\alpha}{L^2\psi(\alpha)} & \frac{3EI\alpha^2}{L\psi(\alpha)} \end{bmatrix}$ $\alpha = a/L$ $\psi(\alpha) = 1+c-3\alpha+3\alpha^2$
	$\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{EI}{L} \end{bmatrix} = \tilde{k}_{AA}$
	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{12EI}{L^3(1+4c)} & \frac{-6EI}{L^2(1+4c)} \\ 0 & & \frac{4EI(1+c)}{L\ (1+4c)} \end{bmatrix}$
	$\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{k}_{AA}$
	$\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{EI}{L} \end{bmatrix} = \tilde{k}_{AA}$

Discontinuidad	k_{BA}
	IGUAL CASO (3)
	$\begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{k}_{AA}$
	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\text{CASO (1)}} \end{bmatrix}$
	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{EI}{L} \end{bmatrix} = \tilde{k}_{AA}$

APENDICE

I.- Demostración de que $\tilde{k}_{bb}$ es singular

Se sabe que: $\tilde{k}_{bb} = k_{bb}(I-A)$

Observe que:

$$A \cdot A = H_{bc}^{T} \; \Lambda \; (\; \Lambda H_{bc} k_{bb} H_{bc}^{T} \; \Lambda^{T})^{-1} \Lambda H_{bc} k_{bb} H_{bc}^{T} \Lambda (\quad)^{-1} \Lambda H_{bc} k_{bb}$$

$$A \cdot A = A$$

Premultipliquemos por A

$$A^{-1} A\ A = A^{-1} A$$

$$\therefore A = I$$

a).- Si A no es la identidad A^{-1} no existe pues su existencia contradice el hecho de que $A \neq I$, por lo tanto A es singular.

b).- A es la identidad. (no singular)

Si usamos la alternativa (a)

$$B = \left[I - A\right] \neq 0$$

pero: $B\ B = \left[I - A\right]\left[I - A\right] = I - 2A + A^2$

$$= I - 2A + A = I - A = B$$

Por consiguiente para $\left[B\right]$ se tienen las siguientes alternativas

c).- $\left[B\right]$ no es la identidad por lo tanto es singular y también $\left[k_{BB}\right]$

d).- $\left[B\right]$ es la identidad, lo que es imposible porque sería el caso de que no hubiera discontinuidad

$$k_{BB} = \tilde{k}_{BB}$$

Si usamos la alternativa (b)

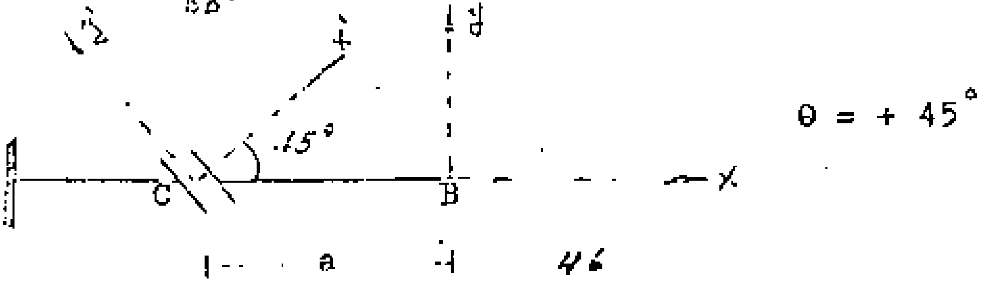
$$A = I \therefore I - A = 0 \quad \therefore \tilde{k}_{BB} = 0 \ *$$

$\tilde{k}_{BB}$ evidentemente que será singular.

Caso de que la discontinuidad sea total $\Lambda = I$.

Ejemplos:

- Obtener $\tilde{k}_{BB}$, viga de sección uniforme:



$$H_{BC} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1/\sqrt{2} & +1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda' = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad ; \quad \Lambda' \ T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$\Lambda' \ T \ H_{BC} = \begin{bmatrix} -1/\sqrt{2} & +1/\sqrt{2} & 0 \end{bmatrix}$$

$$\Lambda' \ T \ H_{BC} \ k_{BB} = \begin{bmatrix} -1/\sqrt{2} & +1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3(1+4c)} & \frac{-6EI}{L^2(1+4c)} \\ 0 & & \frac{4EI(1+c)}{L\ (1+4c)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-EA}{\sqrt{2}\,L} & \frac{+\ 12EI}{\sqrt{2}\,L^3(1+4c)} & \frac{-6EI}{\sqrt{2}\,L^2(1+4c)} \end{bmatrix}$$

$$\Lambda' \ T \ H_{BC} \ k_{BB} (\Lambda' \ T \ H_{BC})^T = \left(\frac{EA}{2L} + \frac{6EI}{L^3(1+4c)} \right)$$

$$(\qquad)^{-1} = \frac{1}{\frac{EA}{2L} + \frac{6EI}{L^3(1+4c)}}$$

$$(\qquad)^{-1} \ \Lambda' \ T \ H_{BC} \ k_{BB} = (\qquad)^{-1} \begin{bmatrix} \frac{-EA}{\sqrt{2}\,L} & \frac{+\ 12EI}{\sqrt{2}\,L^3(1+4c)} & \frac{-6EI}{\sqrt{2}\,L^2(1+4c)} \end{bmatrix}$$

$$(\Lambda' \ T \ H_{BC})^T \cdot (\qquad)^{-1}\Lambda \ T \ H_{BC} \ k_{BB} = A$$

$$A = \begin{bmatrix} \frac{+EA}{2\ L} & \frac{-6EI}{L^3(1+4c)} & \frac{+3EI}{L^2(1+4c)} \\ \frac{-EA}{2\ L} & \frac{+6EI}{L^3(1+4c)} & \frac{-3EI}{L^2(1+4c)} \\ 0 & 0 & 0 \end{bmatrix}$$

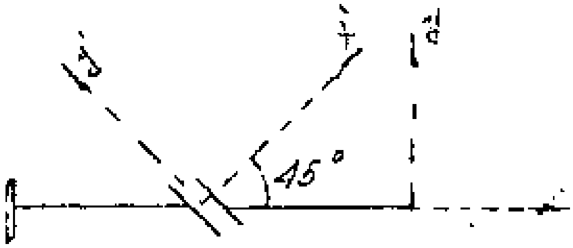
$$I-A = \begin{bmatrix} \frac{6EI}{L^3(1+4c)} & \frac{+6EI}{L^3(1+4c)} & \frac{-3EI}{L^2(1+4c)} \\ \frac{+EA}{2\ L} & \frac{EA}{2\ L} & \frac{+3EI}{L^2(1+4c)} \\ 0 & 0 & \frac{EA}{2\ L} + \frac{6EI}{L^3(1+4c)} \end{bmatrix} \frac{1}{\frac{EA}{2\ L} + \frac{6EI}{L^3(1+4c)}}$$

$$k_{EA}\left[I-A\right] = \begin{bmatrix} \frac{6EI}{L\ (1+4c)}\ \frac{EA}{L} & \frac{+6EI}{L\ (1+4c)}\ \frac{EA}{L} & & \frac{-3EI}{L\ (1+4c)}\ \frac{EA}{L} \\ & \frac{6EI}{L\ (1+4c)}\ \frac{EA}{L} & \frac{36E\ I}{L\ (1+4c)} & \frac{-6EI}{L\ (1+4c)}\ \frac{EA}{2\ L}\ \frac{+6EI}{L\ (1+4c)} \\ & \text{simétrico} & \frac{-18E\ I}{L\ (1+4c)} & \frac{+4EI(1+c)}{L(1+4c)}\ \frac{EA}{2\ L}\ \frac{+6EI}{L\ (1+4c)} \end{bmatrix} v$$

$$x\ \frac{1}{\frac{EA}{2\ L} + \frac{6EI}{L^3(1+4c)}}$$

La expresión de k para

Se puede simplificar:



$$k = \begin{bmatrix} \frac{12EI}{L^3(1+\beta)} & \frac{+12EI}{L^3(1+\beta)} & \frac{-6EI}{L^2(1+\beta)} \\ \frac{+12EI}{L^3(1+\beta)} & \frac{12EI}{L^3(1+\beta)} & \frac{-6EI}{L^2(1+\beta)} \\ \frac{-6EI}{L^2(1+\beta)} & \frac{-6EI}{L^2(1+\beta)} & \frac{EI}{L}\left\{4\ (1+c) - \frac{3\beta}{1+\beta}\right\} \end{bmatrix} \frac{1}{(1+4c)}$$

donde:

$$\beta = \frac{12}{(1+4c)}\left(\frac{\rho}{L}\right)^2 \quad ; \qquad \rho = \sqrt{\frac{I}{A}}$$

Observese que si se desprecia el efecto de fuerza normal ($A = \infty$), se tiene:

$$\rho = 0 \quad ; \quad \beta = 0$$

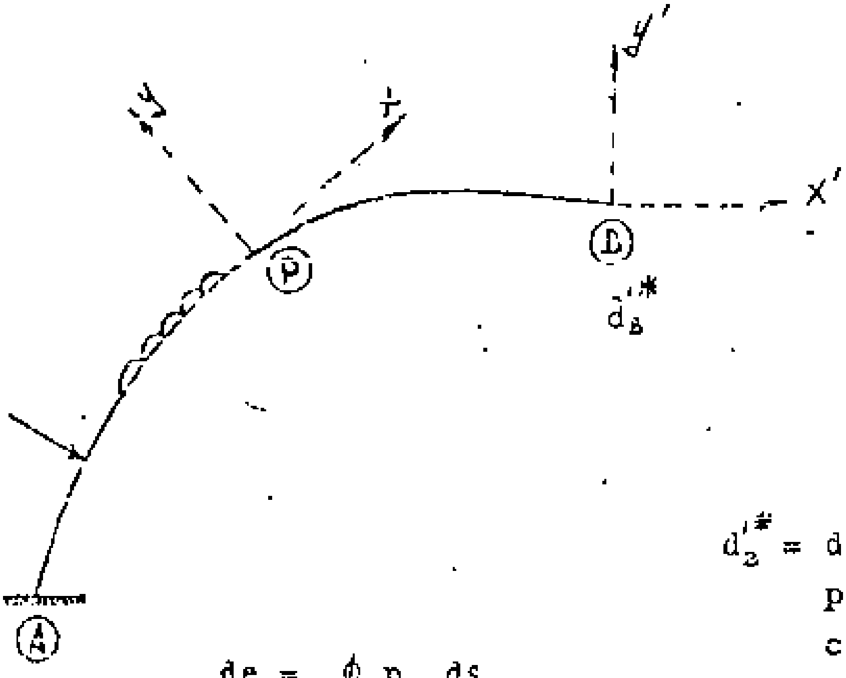
Si también se desprecia el efecto de la fuerza cortante; $c = 0$

Se obtiene:

$$k_{55} = \frac{EI}{L}\begin{bmatrix} \frac{12}{L^2} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{12}{L^2} & \frac{12}{L^2} & \frac{-6}{L} \\ \frac{-6}{L} & \frac{-6}{L} & 4 \end{bmatrix}$$

Resumen (6)

I.- Desplazamientos producidos por cargas intermedias.



d'^{*}_{B} = desplazamiento en B producido por las cargas intermedias — considerando empotrado a A

$$de = \phi \, p_p \, ds$$

$$d\, d'^{*}_{B} = H'^{T}_{BP} \, T \, \phi \, p_p \, ds \; ; \; p_p = T^{T} p'_p$$

$$\boxed{d^{*}_{B} = \int_L H'^{T}_{BP} \, T \, \phi \, T^{T} \, p_p \, ds} \cdot (6.1)$$

p'_p = elementos mecánicos en P referidos a coordenadas globales

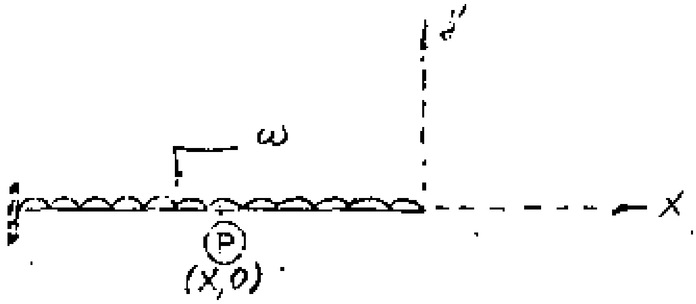
$$d'^{*}_{B} = \int_L H'^{T}_{BP} \, T \, \phi \, p_p \, ds \; \ldots\ldots\ldots (6.2)$$

p_p = elementos mecánicos en P referidos a coordenadas locales

$$\phi = \begin{bmatrix} \frac{1}{EA} & & \\ & \frac{1}{GA_c} & \\ & & \frac{1}{EI} \end{bmatrix} \; ; \quad H^{T}_{BP} = \begin{bmatrix} 1 & 0 & (y'_p - y'_B) \\ 0 & 1 & -(x'_p - x'_B) \\ 0 & 0 & 1 \end{bmatrix}$$

Ejemplos:

①



$$H'^{T}_{BP} = H^{T}_{BP} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad d^{*}_{B} = \int_{X=-L}^{X=0} H^{T}_{BP}\,\phi\,p_{p}\,ds$$

$$ds = dX$$

$T = I$

$$p_{p} = p'_{p} = \begin{bmatrix} 0 \\ \omega x \\ \frac{-\omega x^{2}}{2} \end{bmatrix}$$

$$H^{T}_{BP}\,\phi = \begin{bmatrix} \frac{1}{EA} & 0 & 0 \\ 0 & \frac{1}{GA_{c}} & \frac{-x}{EI} \\ 0 & 0 & \frac{1}{EI} \end{bmatrix} \quad ; \quad H^{T}_{BP}\,\phi\,p_{p} = \begin{bmatrix} 0 \\ \frac{\omega x}{GA_{c}} + \frac{\omega x^{3}}{2EI} \\ \frac{-\omega x^{2}}{2EI} \end{bmatrix}$$

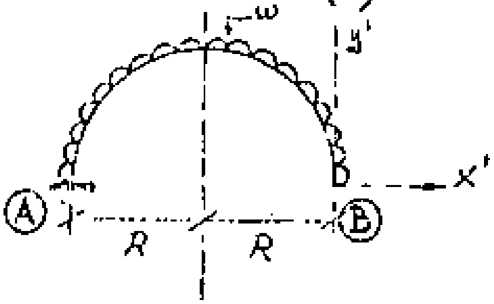
$$d^{*}_{B} = d'^{*}_{B} = \begin{bmatrix} 0 \\ \frac{-\omega L^{2}}{2GA_{c}} - \frac{\omega L^{4}}{8EI} \\ \frac{-\omega L^{3}}{6EI} \end{bmatrix}$$

O bien:

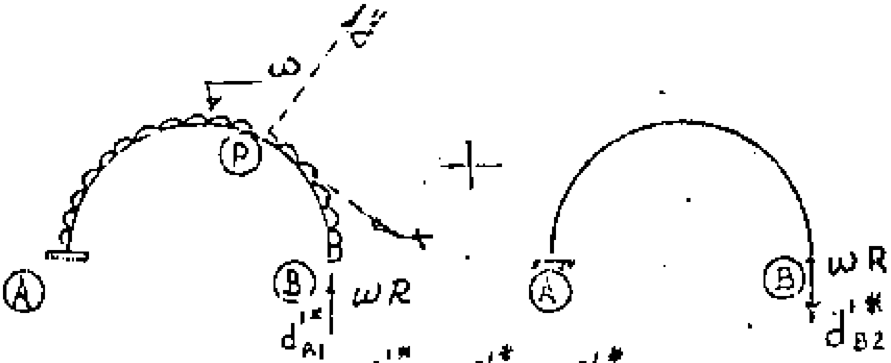
$$d^{*}_{B} = \begin{bmatrix} 0 \\ \frac{-\omega L^{4}}{8EI}\left(1 + \frac{4c}{3}\right) \\ \frac{-\omega L^{3}}{6EI} \end{bmatrix}$$

donde: $c = 6\ (1+\mu)\left(\frac{r}{L}\right)^2 k = \frac{3EI}{GA_c L^2}$

②

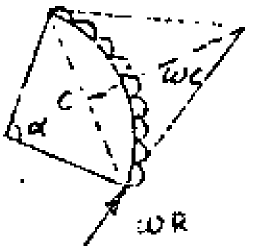


Para obtener $d_B'^*$ hagamos la siguiente superposición de fuerzas:

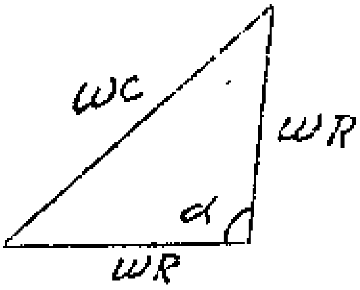


$$\underline{d_B'^* = d_{B1}'^* + d_{B2}'^*}$$

a).- Obtención de $d_{B1}'^*$



c = cuerda



$$P_p = \begin{bmatrix} -\omega R \\ 0 \\ 0 \end{bmatrix} \quad \text{(en coordenadas locales)}$$

Utilizemos la fórmula (6.2) $(ds = R\, d\alpha)$

$$T = \begin{bmatrix} \operatorname{sen}\alpha & \cos\alpha & 0 \\ -\cos\alpha & \operatorname{sen}\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad H_{BP}'^{T} = \begin{bmatrix} 1 & 0 & R\operatorname{sen}\alpha \\ 0 & 1 & R(1-\cos\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

$$H'^{T}_{BP}\ T = \begin{bmatrix} s & c & R\ \text{sen}\ \alpha \\ -c & s & R(1-\cos\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

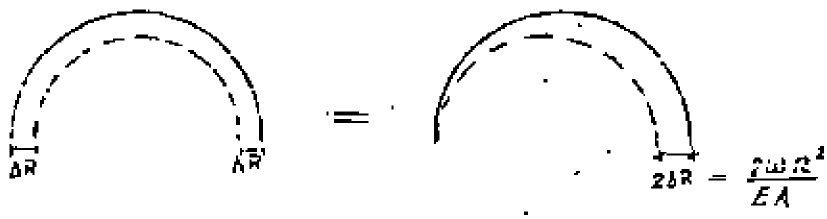
$$H'^{T}_{BP}\ T\ \phi = \begin{bmatrix} \frac{s}{EA} & \frac{c}{GA_c} & \frac{R\ s}{EI} \\ \frac{-c}{EA} & \frac{s}{GA_c} & \frac{R(1-c)}{EI} \\ 0 & 0 & \frac{1}{EI} \end{bmatrix}$$

$$H \quad T \quad p = \begin{bmatrix} \frac{-\omega R\ \text{sen}}{EA}\alpha \\ \frac{-\omega R\ \cos}{EA}\alpha \\ 0 \end{bmatrix}$$

$$d'^{*}_{B1} = \int_0^{\pi} H'^{T}_{BP}\ T\ \phi\, p_p\ ds$$

$$d'^{*}_{B1} = \begin{bmatrix} \frac{-2\omega R^2}{EA} \\ 0 \\ 0 \end{bmatrix}$$

Nota: Este resultado es obvio porque el arco esta trabajando a solo fuerza normal, por consiguiente solo sufrirá un acortamiento su radio, igual a: $\frac{\omega R}{EA} \cdot R = \frac{\omega R^2}{EA} = \Delta R$



b).- Obtención de d'^{*}_{B1}

$$d'^{*}_{B1} = \begin{bmatrix} f_{BB} \end{bmatrix} \begin{bmatrix} 0 \\ -\omega R \\ 0 \end{bmatrix}$$

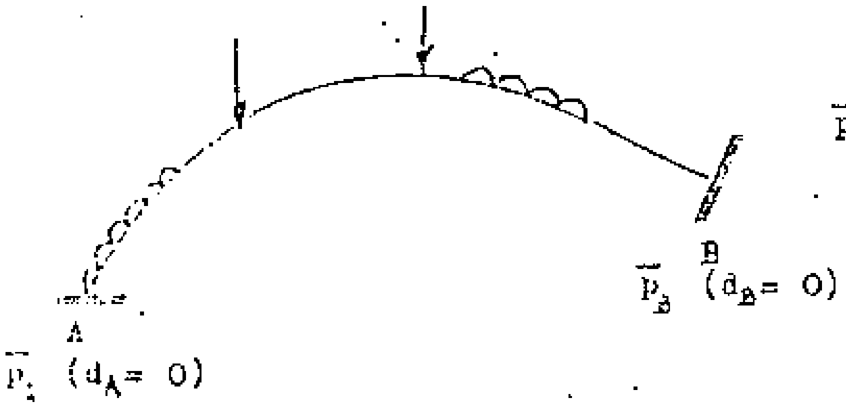
Utilizemos el valor de f_{BB}

$$d'^{*}_{B1} = \begin{bmatrix} \dfrac{-2\omega R^{4}}{EI} \\ -\dfrac{\pi \omega R^{2}}{2} \left(\dfrac{1}{EA} + \dfrac{1}{GA_{c}} + \dfrac{3\ R^{2}}{EI} \right) \\ -\dfrac{\pi \omega R^{3}}{EI} \end{bmatrix}$$

Por consiguiente:

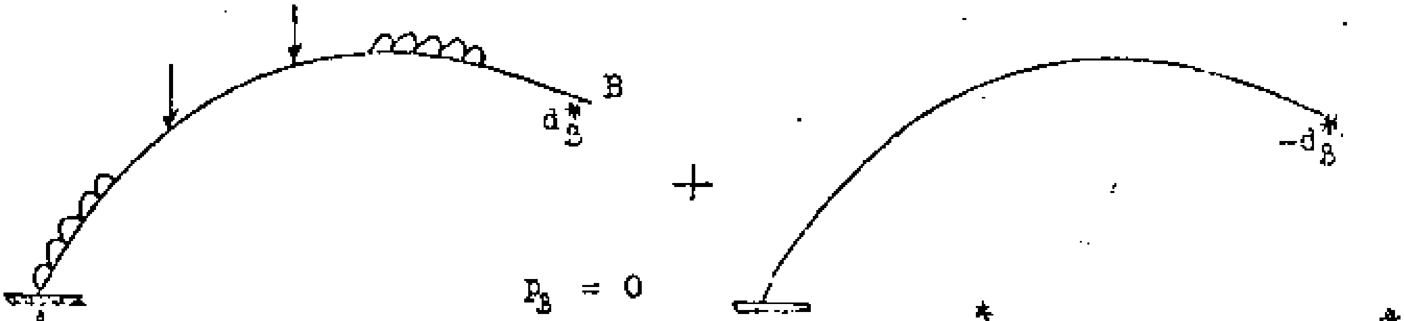
$$d^{*}_{B} = \begin{bmatrix} -\dfrac{2\omega R^{2}}{EA} - \dfrac{2\omega R^{4}}{EI} \\ -\dfrac{\pi \omega R^{2}}{2} \quad \dfrac{1}{EA} + \dfrac{1}{GA_{c}} + \dfrac{3\ R^{2}}{EI} \\ -\dfrac{\pi \omega R^{3}}{EI} \end{bmatrix}$$

II.- Fuerzas de fijación



$\overline{P}_A, \overline{P}_B$ = reacciones en Ⓐ y Ⓑ, considerando empotrados a Ⓐ y Ⓑ

Para obtener $\overline{p}_A$ y $\overline{p}_B$ usamos la siguiente superposición:



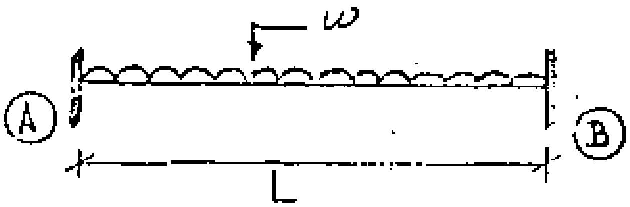
$p_A = p_A^*$ $\qquad p_B = 0$ $\qquad p_A = H_{BA}\, k_{BB}\, d_B^*$ $\qquad p_B = -k_{BB}\, d_B^*$

$$\boxed{\begin{aligned} \overline{p}_A &= p_A^* + H_{BA}\, k_{BB}\, d_B^* \\ \overline{p}_B &= -\, k_{BB}\, d_B^* \end{aligned}}$$

$p_A^* =$ Reacción en Ⓐ considerando suelto a Ⓑ (cantiliver)

Ejemplos:

①



Usando los resultados de los ejemplos anteriores

$$d_B^* = \begin{bmatrix} 0 \\ \dfrac{-\omega L^4}{8EI}\left(1+\dfrac{4}{3}c\right) \\ \dfrac{-\omega L^3}{6EI} \end{bmatrix}$$

$$k_{BB} = \begin{bmatrix} \dfrac{EA}{L} & 0 & 0 \\ 0 & \dfrac{12EI}{L^3(1+4c)} & \dfrac{-6EI}{L^2(1+4c)} \\ 0 & \dfrac{-6EI}{L^2(1+4c)} & \dfrac{4EI(1+c)}{L\,(1+4c)} \end{bmatrix}$$

$$k_{BB}\, d_B^{*} = \begin{bmatrix} 0 \\ -\frac{3}{2}\, \omega L \left(\frac{1+4c/3}{1+4c}\right) + \frac{\omega L}{(1+4c)} \\ +\frac{3}{4}\, \omega L^2 \left(\frac{1+4c/3}{1+4c}\right) - \frac{2}{3}\, L\, \frac{(1+c)}{(1+4c)} \end{bmatrix}$$

Simplificando:

$$k_{BB}\, d_B^{*} = \begin{bmatrix} 0 \\ \frac{-\omega L}{2} \\ \frac{+\omega L^2}{12} \end{bmatrix}$$

Por estática, se tiene:

$$P_A^{*} = \begin{bmatrix} 0 \\ \omega L \\ \frac{+\omega L^2}{2} \end{bmatrix}$$

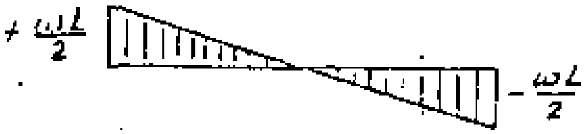
Obtengamos $H_{BA}\, k_{BB}\, d_B^{*}$

$$H_{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & L & 1 \end{bmatrix}$$

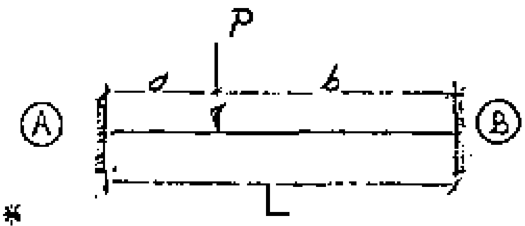
$$H_{BA}\, k_{BB}\, d_B^{*} = \begin{bmatrix} 0 \\ \frac{-\omega L}{2} \\ -\frac{5}{12}\, \omega L^2 \end{bmatrix}$$

$$\bar{P}_A = \begin{bmatrix} 0 \\ +\frac{\omega L}{2} \\ +\frac{\omega L^2}{12} \end{bmatrix} \; ; \quad \bar{p}_B = \begin{bmatrix} 0 \\ +\frac{\omega L}{2} \\ -\frac{\omega L^2}{12} \end{bmatrix}$$

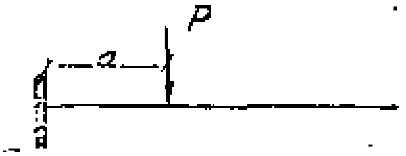
Resultado que es interesante, porque no se ve afectado por el trabajo de la fuerza cortante, lo cual es evidente porque el diagrama de T es antisimétrico y se anula al ser integrado.



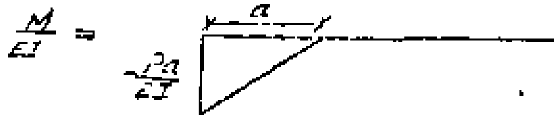
②

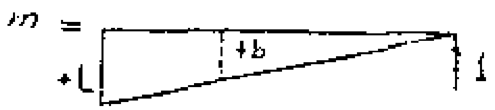


a).- Obtengamos d_B^*



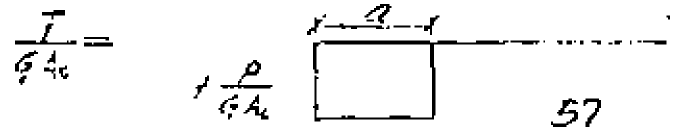
Utilizemos las ecuaciones del trabajo virtual, obtención de d_{Bj}^*

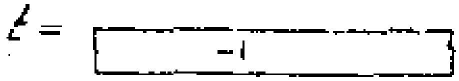




$$\int \frac{Mm}{EI}\, ds = \frac{1}{6}\; a \left(\frac{-Pa}{EI}\right)\left(+\; 2\; L + b\right)$$

$$= \frac{-Pa^2}{6EI}\; (2\; L + b)$$





$$\int \frac{Tt}{GA_c} = -\frac{Pa}{GA_c}$$

$$d^*_{By} = -\frac{Pa^2}{6EI}\,(2L+b)\,-\frac{Pa}{GA_c}$$

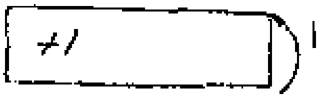
Simplificando:

$$d^*_{By} = -\frac{Pa^2}{6EI}\left[2L(1+cL/a)+b\right]$$

$$\left(C = \frac{3EI}{GA_cL^2}\right)$$

Obtención de d^*_{Bz} $(= \theta^*_{Bz})$

$m =$



$t = 0$

$$\int \frac{Mm}{EI}\,ds = -\frac{Pa^2}{2EI}$$

$$d^*_{Bz} = -\frac{Pa^2}{2EI}$$

$$d^*_B = \begin{bmatrix} 0 \\ -\frac{Pa^2}{6EI}\quad 2L(1+cL/a)+b \\ -\frac{Pa^2}{2EI} \end{bmatrix}$$

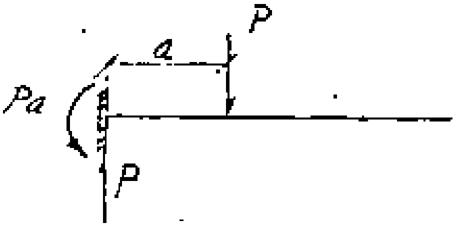
b).- Obtengamos $k_{BB}\,d^*_B$ (simplificando)

$$k_{BB}\,d^*_B = \begin{bmatrix} 0 \\ \frac{-Pa}{L(1+4c)}\left[1+4c+\frac{ab-b^2}{L^2}\right] \\ \frac{+Pa^2b}{L^2(1+4c)}\left[1+2cL/a\right] \end{bmatrix}$$

c).- Obtengamos $H_{BA} k_{BB} d_B^*$

$$H_{BA}\, k_{BB}\, d_B^* = \begin{bmatrix} 0 \\ \frac{-Pa}{L(1+4c)} \left[1+4c+\frac{ab-b^2}{L^2} \right] \\ \frac{Pab^2}{L^2(1+4c)} \left[1+2cL-\frac{L^2}{b^2}\,(1+4c) \right] \end{bmatrix}$$

d).- Por estática



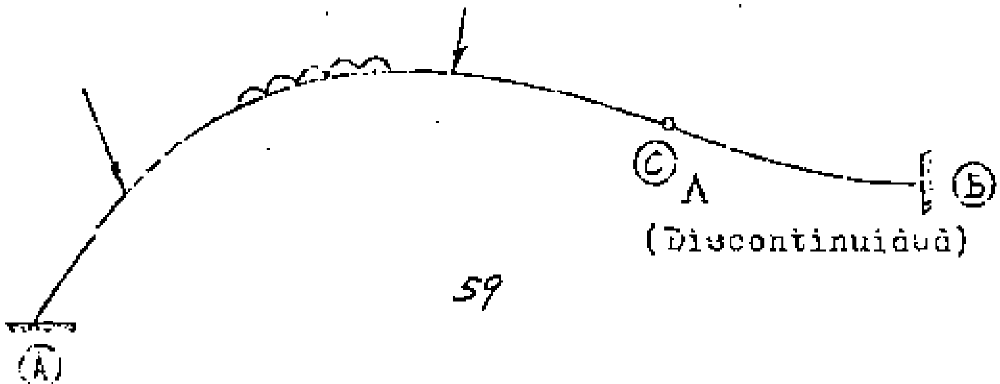
$$p_A^* = \begin{bmatrix} 0 \\ +P \\ +Pa \end{bmatrix}$$

$$\overline{p}_A = \begin{bmatrix} 0 \\ \frac{Pb}{L(1+4c)} \left[1+4c+\frac{ab-a^2}{L^2} \right] \\ +\frac{Pab^2}{L^2(1+4c)} \left[1+\frac{2cL}{b} \right] \end{bmatrix} ; \quad \overline{p}_B = \begin{bmatrix} 0 \\ \frac{Pa}{L(1+4c)} \left[1+4c+\frac{ab-b^2}{L^2} \right] \\ -\frac{Pa^2b}{L^2(1+4c)} \left[1+\frac{2cL}{a} \right] \end{bmatrix}$$

Nota: Observese que cuando $a = b = L/2$
(Diagrama de T antisimétrico)

$$\overline{p}_A = \begin{bmatrix} 0 \\ +\frac{P}{2} \\ +\frac{PL}{8} \end{bmatrix} ; \quad \overline{p}_B = \begin{bmatrix} 0 \\ +\frac{P}{2} \\ -\frac{PL}{8} \end{bmatrix}$$

III.- Fuerzas de fijación de barras con discontinuidades.

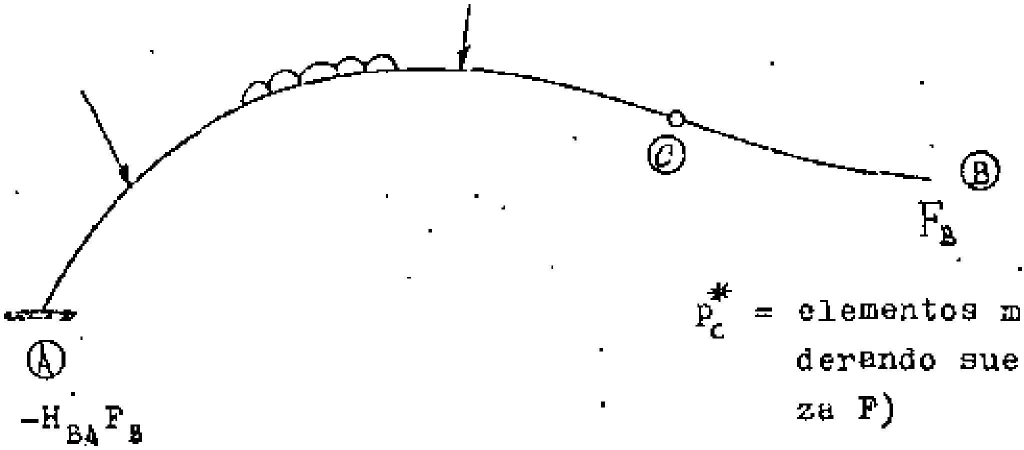


a).- Obtengamos d_B^* sin considerar la discontinuidad en C.

b).- Consideremos en B una fuerza F_B que cumpla con el equilibrio en C

($\Lambda p_c = 0$), o sea que:

$$\Lambda(p_c^* + H_{BC} F_B) = 0 \text{------} (6.3)$$



p_c^* = elementos mecánicos en C considerando suelto a B (sin la fuerza F)

La fuerza F_B no esta determinada por (6.3), hay muchos valores que la satisfacen, por ejemplo:

$$F_B = -H_{CB} \Lambda^T \Lambda \; p_c^* \text{------} (6.4)$$

En efecto, sustituyendo en (6.3)

$$\Lambda(I - H_{BC} H_{CB} \Lambda^T \Lambda) p_c^* = $$
$$= (\Lambda - \Lambda \Lambda^T \Lambda) p_c^* = 0$$

Porque: $\Lambda \Lambda^T \Lambda = \Lambda$

La fuerza F_B, produce en B un desplazamiento $f_{BB} F_B$, por lo que el desplazamiento total en B sera:

$$\tilde{d}_B^* = d_B^* + f_{BB} F_B$$

c).- Obtención de p_B :

$\widetilde{k}_{BB}$ = rigidez modificada en B.

$$\bar{p}_B = - \widetilde{k}_{BB} \widetilde{d}_B^* + F_B$$

$$p_B = -\widetilde{k}_{BB} \widetilde{d}_B^* - \widetilde{k}_{BB} f_{BB} F_B + F_B$$

$$\boxed{\bar{p}_B = - \widetilde{k}_{BB} d_B^* + (I - \widetilde{k}_{BB} f_{BB}) F_B} \cdots\cdots (6.5)$$

O bien sustituyendo a (6.4) en (6.5)

$$\bar{p}_B = - \widetilde{k}_{BB} d_B^* - (I - \widetilde{k}_{BB} f_{BB}) H_{CB} \Lambda^T \Lambda p_C^* \qquad (6.6)$$

Recordando que:

$$\widetilde{k}_{BB} = k_{BB} \left[I - A \right] = \left[I - D \right] k_{BB}$$

donde:

$$A = H_{BC}^T \Lambda^T (\Lambda H_{BC} k_{BB} H_{BC} \Lambda^T)^{-1} \Lambda H_{BC} k_{BB}$$

$$D = k_{BB} H_{BC}^T \Lambda^T (\Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T)^{-1} \Lambda H_{BC}.$$

(Ver resumen (5))

y sustituyendo en (6.6) y simplificando

$$\bar{p}_B = - k_{BB} \left[(I - A) d_B^* + H_{BC}^T \Lambda^T (\Lambda H_{BC} k_{BB} H_{BC}^T \Lambda^T)^{-1} \Lambda p_C^* \right] \qquad (6.7$$

d).- Por estática:

$$\bar{p}_A = p_A^* - H_{BA} \bar{p}_B \quad \text{..........} (6.8)$$

Sustituyendo (6.6) en (6.8)

ANALISIS ESTRUCTURAL II

RESUMEN 7

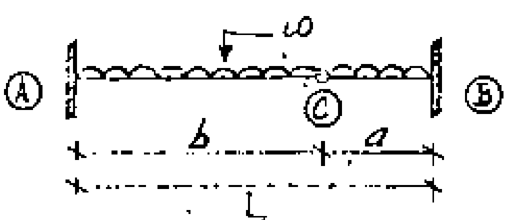
$$\bar{P}_A = p_A^* + H_{BA}\left[\tilde{k}_{BB}\, d_B^* + (I - \tilde{k}_{BB}\, f_{BB})\, H_{CB}\, \Lambda^T \Lambda\, p_C^*\right] \quad - - - \;\; - - - - - . \; (6.9)$$

O sustituyendo (6.7) en (6.8)

$$\vec{P}_A = p_A^* + H_{BA}\, k_{BB}\left[(I - A)\, d_B^* + H_{BC}^T \Lambda^T (\Lambda\, H_{BC}\, k_{BB} H_{BC} \Lambda^T)^{-1} \Lambda\, p_C^*\right] \quad . - - - . \; (6.10)$$

Ejemplos:

①



$$\Lambda = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

a).- Obtención de d_B^* (ejemplo anterior)

$$d_B^* = \begin{bmatrix} 0 \\ \frac{-wL^4}{8EI} \; (1+4\,c/3) \\ \frac{-wL^3}{6EI} \end{bmatrix}$$

b).- Obtención de $\tilde{k}_{BB}$ (Tabla en el resumen (5))

$$\tilde{k}_{BB} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 \\ 0 & \frac{3EI}{L^3 \psi(\alpha)} & \frac{-3EI\alpha}{L^2 \psi(\alpha)} \\ 0 & \frac{-3EI\alpha}{L^2 \psi(\alpha)} & \frac{3EI\alpha^2}{L\,\psi(\alpha)} \end{bmatrix}; \qquad \alpha = a/L$$

$$\psi(\alpha) = 1 + c - 3\alpha + 3\alpha^2$$

c).- Obtención de p_A^* y p_C^* (Por estática)

$$p_A^* = \begin{bmatrix} 0 \\ wL \\ \frac{+wL^2}{2} \end{bmatrix}; \qquad p_C^* = \begin{bmatrix} 0 \\ -wa \\ \frac{-wa^2}{2} \end{bmatrix}$$

Reacción en Ⓐ $\qquad$ Elementos mecánicos en Ⓒ

d).- Apliquemos la ecuación (6.6) para obtener $\overline{p}_B$:

$$-\tilde{k}_{BB}\, d_b^* = \begin{bmatrix} 0 \\ \frac{\omega L}{8\psi(\alpha)}\left[3+4c-4\frac{a}{L}\right] \\ \frac{-\omega a L}{8\psi(\alpha)}\left[3+4c-4\frac{a}{L}\right] \end{bmatrix}$$

$$f_{BB} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI}\quad(1+c) & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

$$\tilde{k}_{BB}\, f_{BB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\left(1+c-\frac{3a}{2L}\right)}{\psi(\alpha)} & \frac{3}{2L\psi(\alpha)}\left(1-\frac{2a}{L}\right) \\ 0 & \frac{-a}{\psi(\alpha)}\left(1+c-\frac{3a}{2L}\right) & \frac{-3a}{2L\psi(\alpha)}\left(1-\frac{2a}{L}\right) \end{bmatrix}$$

$$I - \tilde{k}_{BB}\, f_{BB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{3a}{2L\psi(\alpha)}\left(1-\frac{2a}{L}\right) & -\frac{3}{2L\psi(\alpha)}\left(1-\frac{2a}{L}\right) \\ 0 & \frac{a}{\psi(\alpha)}\left(1+c-\frac{3a}{2L}\right) & \frac{1}{\psi(\alpha)}\left(1+c-\frac{3a}{2L}\right) \end{bmatrix}$$

$$\Lambda^T \Lambda\, p_c^* = \begin{bmatrix} 0 \\ 0 \\ -\frac{\omega a^2}{2} \end{bmatrix} \quad ; \qquad H_{cB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -a & 1 \end{bmatrix}$$

$$-H_{cB}\, \Lambda^T \Lambda\, p_c^* = \begin{bmatrix} 0 \\ 0 \\ +\frac{\omega a^2}{2} \end{bmatrix} \quad (= F_B)$$

$$-\left(I-\widetilde{k}_{BB}f_{BB}\right)H_{CB}\,\Lambda^{T}\Lambda\;p_c^{*}=\begin{bmatrix}0\\ -\dfrac{3\omega a^2}{4L\psi(\alpha)}\quad\left(1-\dfrac{2a}{L}\right)\\ \dfrac{\omega a^2}{2\psi(\alpha)}\quad\left(1+c-\dfrac{3a}{2L}\right)\end{bmatrix}$$

$$\vec{p}_B=\begin{bmatrix}0\\ \dfrac{\omega L}{8\psi(\alpha)}\left[3+4c-4\alpha-6\alpha^2+12\alpha^3\right]\\ \dfrac{-\omega L^2\alpha}{8\psi(\alpha)}\left[+3+4c-4\alpha(2+c)+6\alpha^2\right]\end{bmatrix}$$

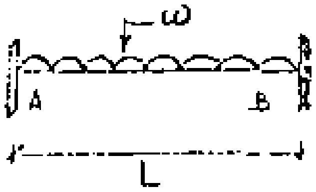
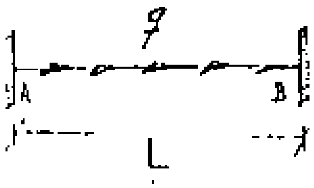
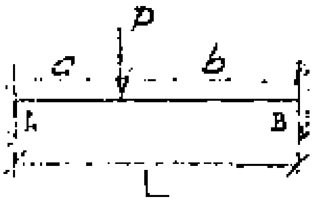
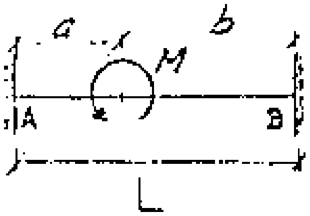
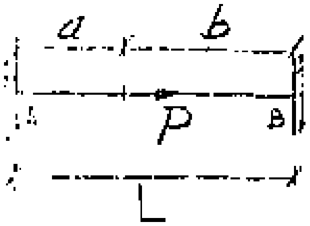
Obtengamos $\overline{p}_A$ utilizando (6.8)

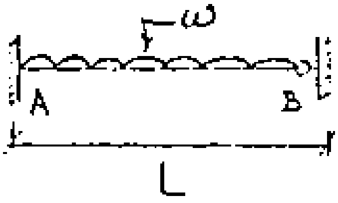
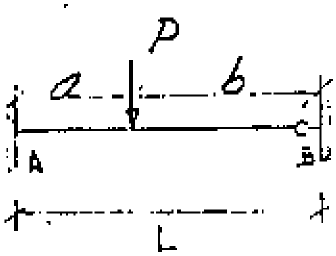
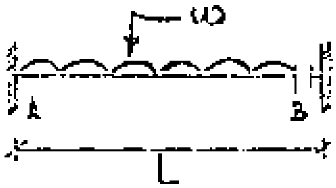
$$-H_{BA}\,\overline{p}_B=\begin{bmatrix}0\\ \dfrac{-\omega L}{8\psi(\alpha)}\quad 3+4c-4\alpha-6\alpha^2+12\alpha^3\\ \dfrac{\omega L^2}{8\psi(\alpha)}\quad 3+4c-\alpha(7+4c)+2\;(1+2c)+6\alpha^3\end{bmatrix}$$

$$\overline{p}_A=\begin{bmatrix}0\\ \dfrac{\omega L}{8\psi(\alpha)}\quad 5+4c-20\alpha+30\alpha^2-12\alpha^3\\ \dfrac{\omega L^2}{8\psi(\alpha)}\quad 1+\alpha(4c-5)+2\alpha^2(5-2c)-6\alpha^3\end{bmatrix}$$

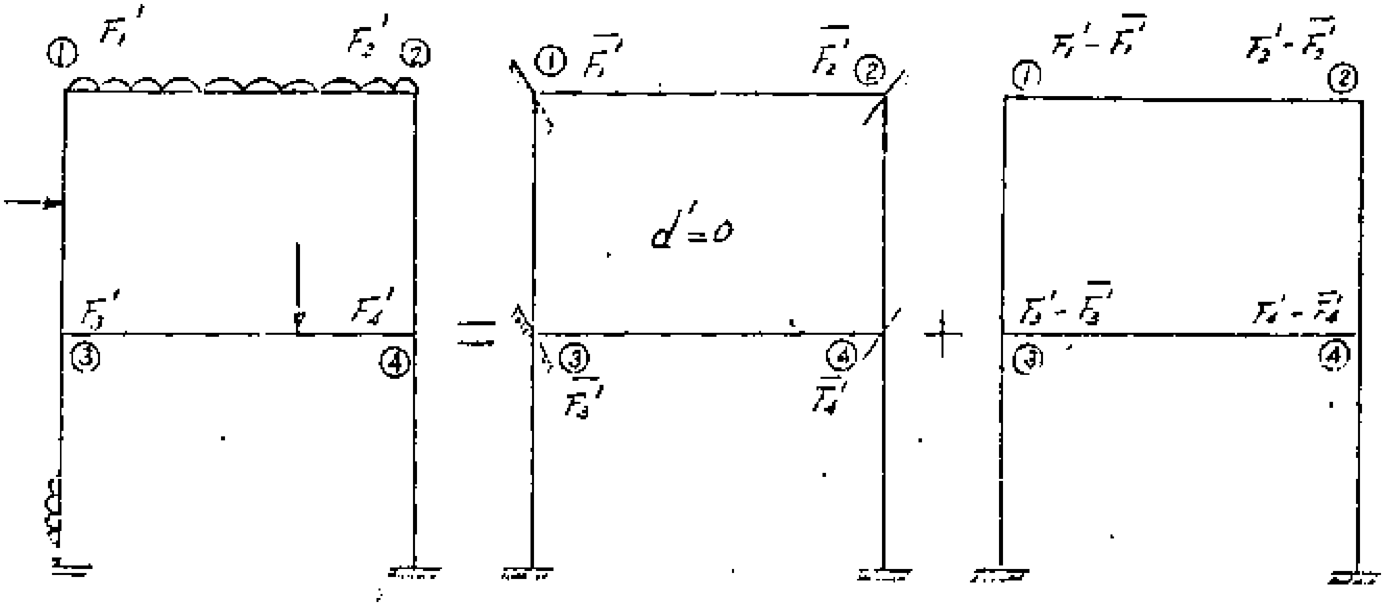
Resumen (7)

I.- Tabla de fuerzas de fijación (Barras rectas de sección cte.)

Carga	$\bar{P}_A$	$\bar{P}_B$
	$\begin{bmatrix} 0 \\ +\frac{\omega L}{2} \\ +\frac{\omega L^2}{12} \end{bmatrix}$	$\begin{bmatrix} 0 \\ +\frac{\omega L}{2} \\ -\frac{\omega L^2}{12} \end{bmatrix}$
	$\begin{bmatrix} -\frac{qL}{2} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -\frac{qL}{2} \\ 0 \\ 0 \end{bmatrix}$
	$\begin{matrix} 0 \\ \frac{Pb}{L(1+4c)} \left\{1+4c+\frac{ab-a^2}{L^2}\right\} \\ \frac{Pab^2}{L^2(1+4c)} \left\{1+\frac{2cL}{b}\right\} \end{matrix}$	$\begin{matrix} 0 \\ \frac{Pa}{L(1+4c)} \left\{1+4c+\frac{ab-b^2}{L^2}\right\} \\ \frac{-Pa^2b}{L^2(1+4c)} \left\{1+\frac{2cL}{a}\right\} \end{matrix}$
	$\begin{matrix} 0 \\ \frac{6Mab}{L^3(1+4c)} \\ \frac{-Mb}{L(1+4c)} \left\{1+4c-\frac{3a}{L}\right\} \end{matrix}$	$\begin{matrix} 0 \\ \frac{-6Mab}{L^3(1+4c)} \\ \frac{-Ma}{L(1+4c)} \left\{1+4c-\frac{3b}{L}\right\} \end{matrix}$
	$\begin{bmatrix} -\frac{Pb}{L} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -\frac{Pa}{L} \\ 0 \\ 0 \end{bmatrix}$

Carga	$\bar{p}_A$	$\bar{p}_B$
	0 $\frac{5\omega L}{8}\left(\frac{1+\frac{4c}{5}}{1+c}\right)$ $\frac{\omega L^2}{8(1+c)}$	0 $\frac{3\omega L}{8}\left(\frac{1+\frac{4c}{3}}{1+c}\right)$ 0
	0 $\frac{Pb}{L(1+c)}\left\{\frac{a(L+b)}{2L^2}+1+c\right\}$ $+\frac{Pab(L+b)}{2L^2(1+c)}$	0 $\frac{Pa}{L(1+c)}\left\{\frac{a(2L+b)}{2L^2}+c\right\}$ 0
	$\begin{bmatrix} 0 \\ +\omega L \\ +\frac{\omega L^2}{3} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ +\frac{\omega L^2}{6} \end{bmatrix}$

II.- Fuerzas efectivas producidas por fuerzas en las barras.



$\bar{F}_j'$ = Suma de fuerzas de fijación de las barras que concurren a (j) (en coordenadas globales)

F_j' = Fuerzas en el nudo (j)

Los desplazamientos finales se obtendrán:

$$F' - \bar{F}' = K' d'$$

O bien:

$$F_{ex}' = K' d' \qquad F_{ex}' = F' - \bar{F}'$$

La solución será:

$$d' = (K')^{-1} F_{ex}'$$

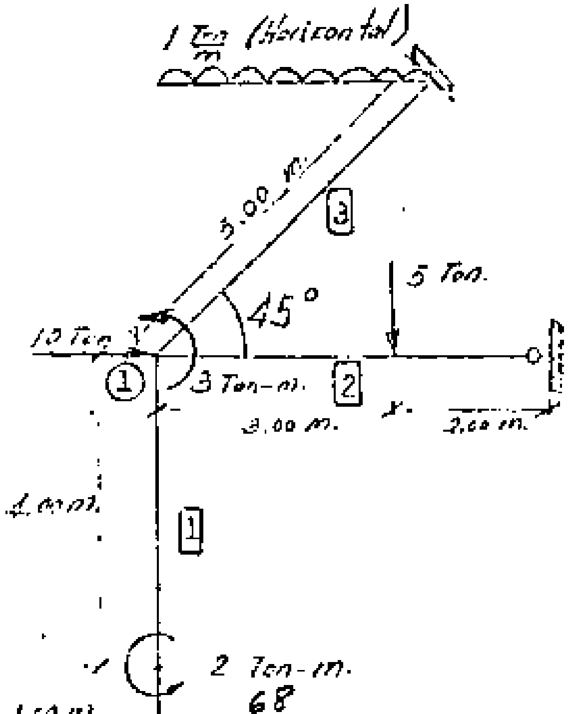
$$e = a d'$$

$$p_{b(1)} = k_{BB(1)} e_{(1)} + \bar{p}_{B(1)}$$

$$p_{A(1)} = -H_{BA(1)} k_{BB(1)} e_{(1)} + \bar{p}_{A(1)}$$

donde: $\bar{p}_{A(1)}, \bar{p}_{B(1)}$ = fuerzas de fijación en la barra ①, en coordenadas locales.

Ejemplo: (Por simplificación consideramos una estructura de un solo nudo)



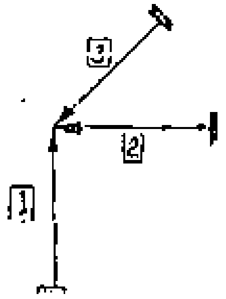
Las tres barras son de igual sección:

$$E = 2\times10^{6}\ kg/cm^{2}$$
$$I = 10^{5}\ cm^{4}$$
$$A = 100\ cm^{2}$$
$$C = 0.1$$

Por consiguiente:

$$k_{BB} = \begin{bmatrix} 4.0 & 0 & 0 \\ 0 & 0.137 & -0.343 \\ 0 & -0.343 & 1.142 \end{bmatrix} \times 10^{3} \quad (Ton,\ m)$$

Orientemos las barras en la forma siguiente:

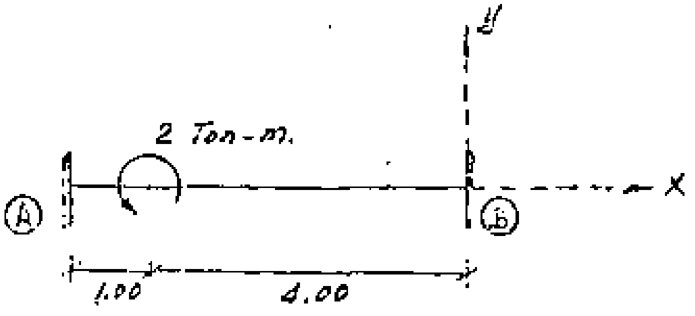


Obtengamos $\tilde{k}_{BB}$ de la barra $\boxed{2}$ (Tabla del resumen (5)

$$(\alpha = 1.0) \quad \tilde{k}_{BB_2} = \begin{bmatrix} 4.0 & 0 & 0 \\ 0 & 0.044 & -0.219 \\ 0 & -0.219 & 1.093 \end{bmatrix} \times 10^{3}$$

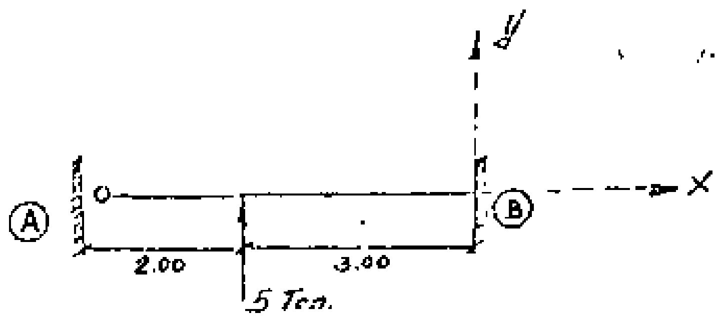
c).- Obtengamos $\vec{p}_A$, $\bar{p}_B$ de cada barra: (Tabla hoja (1))

<u>Barra 1</u>



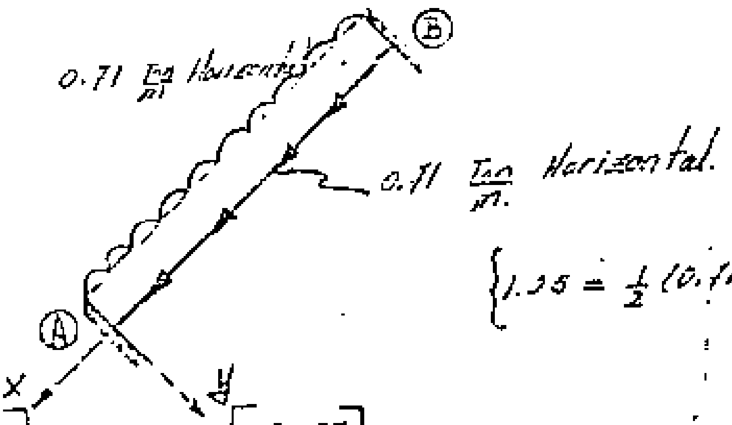
$$\overline{p}_A = \begin{bmatrix} 0 \\ 0.27 \\ -0.91 \end{bmatrix}; \qquad \overline{p}_B = \begin{bmatrix} 0 \\ -0.27 \\ +0.29 \end{bmatrix}$$

Barra 2



$$\overline{p}_A = \begin{bmatrix} 0 \\ -2.24 \\ 0 \end{bmatrix}; \qquad \overline{p}_B = \begin{bmatrix} 0 \\ -2.76 \\ +3.82 \end{bmatrix}$$

Barra 3

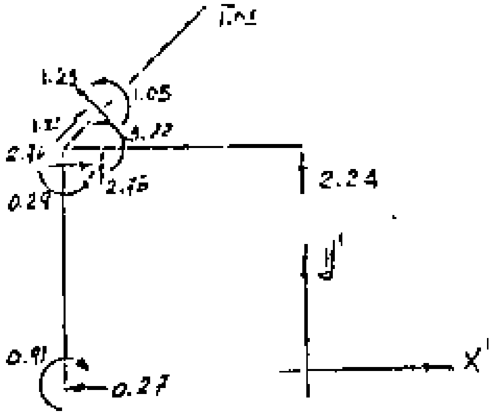


$\left\{1.25 = \frac{1}{2}(0.71)(3.55)\right\}$

$$\overline{p}_A = \begin{bmatrix} -1.250 \\ -1.25 \\ -1.05 \end{bmatrix}; \qquad \overline{p}_B = \begin{bmatrix} -1.25 \\ -1.25 \\ +1.05 \end{bmatrix}$$

Nota:
$$\begin{cases} \frac{\omega L^2}{12} = \omega L \, \frac{L}{12} = W \, \frac{L}{12} \\ \frac{L}{2} = \frac{1}{2} \, \omega L = \frac{1}{2} \, W \end{cases}$$

En resumen:



b).- Obtengamos $\overline{F}'$ (en coordenadas globales)

$$\overline{F}' = \begin{bmatrix} +0.27 \\ +4.53 \\ +5.16 \end{bmatrix}$$ (Obtenidas por estatica elemental)

c).- Obtengamos F'_{ex}

$$F' = \begin{bmatrix} +10 \\ 0 \\ +3 \end{bmatrix} \quad ; \quad -\overline{F}' = \begin{bmatrix} -0.27 \\ -4.53 \\ -5.16 \end{bmatrix}$$

$$F'_{ex} = \begin{bmatrix} +9.73 \\ -4.53 \\ -2.16 \end{bmatrix}$$

d).- Obtengamos K $(= a^T ka)$ (ó usando la regla la suma, que en este caso son equivalentes)

$$a = \begin{bmatrix} T_1^T \\ T_2^T \\ T_3^T \end{bmatrix} ; \quad T_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad T_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} -0.71 & +0.71 & 0 \\ -0.71 & -0.71 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Nota:

$$T = \begin{bmatrix} X_{x'} & Y_{x'} & 0 \\ X_{y'} & Y_{y'} & 0 \\ 0 & 0 & 1 \end{bmatrix};$$ donde X, Y son vectores unitarios paralelos a los ejes x, y.

$$k = \begin{bmatrix} k_{BB} & & \\ & \tilde{k}_{BB} & \\ & & k_{BB} \end{bmatrix}$$

(Ver hoja (67) para los valores de k_{BB} y $\tilde{k}_{BB}$)
Efectuando multiplicaciones:

$$K' = \begin{bmatrix} 6.205 & 1.932 & +0.100 \\ 1.932 & 6.112 & +0.462 \\ +0.100 & 0.462 & +3.377 \end{bmatrix} \times 10^{3}$$

e).- Obtengamos d', resolviendo el sistema $F'_{ak} = k'd'$

$$d' = \begin{bmatrix} 1.991 \\ -1.331 \\ -0.516 \end{bmatrix} \times 10^{-3}$$

f).- Obtengamos e (= ad')

$$e = \left[\begin{array}{c} -1.331 \\ -1.991 \\ -0.516 \\ \hline -1.991 \\ +1.331 \\ -0.516 \\ \hline -0.467 \\ +2.340 \\ -0.516 \end{array}\right] \times 10^{-3}$$

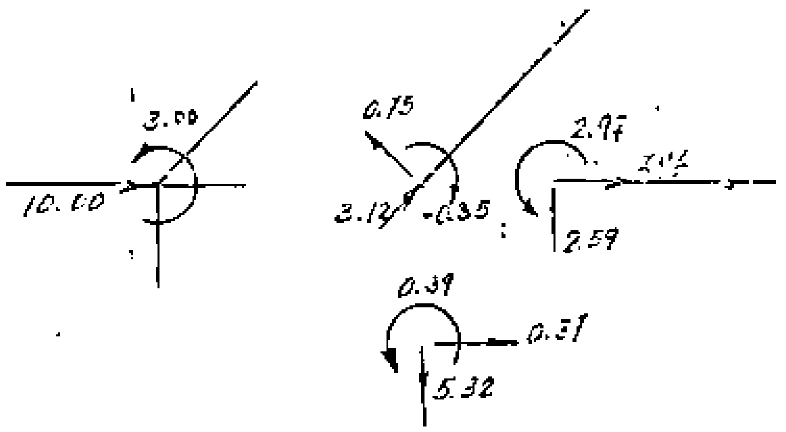
g).- Obtengamos p_B : $(p_B = ke + \bar{p}_B)$

$$p_{B_1} = \begin{bmatrix} -5.324 \\ -0.097 \\ +0.095 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.27 \\ +0.29 \end{bmatrix} = \begin{bmatrix} -5.324 \\ -0.367 \\ +0.385 \end{bmatrix}$$

$$p_{B_2} = \begin{bmatrix} -7.964 \\ +0.172 \\ -0.855 \end{bmatrix} + \begin{bmatrix} 0 \\ -2.76 \\ +3.82 \end{bmatrix} = \begin{bmatrix} -7.964 \\ -2.588 \\ +2.965 \end{bmatrix}$$

$$p_{B_3} = \begin{bmatrix} -1.868 \\ +0.497 \\ -1.398 \end{bmatrix} + \begin{bmatrix} -1.25 \\ -1.25 \\ +1.050 \end{bmatrix} = \begin{bmatrix} -3.118 \\ -0.753 \\ -0.348 \end{bmatrix}$$

Comprobación: (Equilibrio)



$\sum F_x = 10.00$

$\sum F_y = 0$

$\sum M = 3.01$

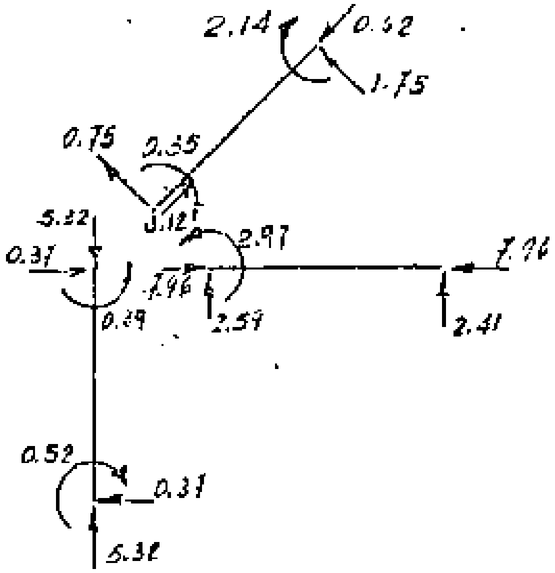
h).- Obtengamos p_A ; $(p_A = -H_{BA}\, ke + \bar{p}_A)$

$$H_{BA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$ (Para todas las barras)

$$p_{A_1} = \begin{bmatrix} +5.324 \\ +0.097 \\ +0.390 \end{bmatrix} + \begin{bmatrix} 0 \\ +0.270 \\ -0.910 \end{bmatrix} = \begin{bmatrix} +5.324 \\ +0.367 \\ -0.520 \end{bmatrix}$$

$$p_{A_2} = \begin{bmatrix} +7.964 \\ -0.172 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2.240 \\ 0 \end{bmatrix} = \begin{bmatrix} +7.964 \\ -2.412 \\ 0 \end{bmatrix}$$

$$p_{A_3} = \begin{bmatrix} +1.868 \\ -0.497 \\ -1.087 \end{bmatrix} + \begin{bmatrix} -1.250 \\ -1.250 \\ -1.050 \end{bmatrix} = \begin{bmatrix} +0.618 \\ -1.747 \\ -2.137 \end{bmatrix}$$



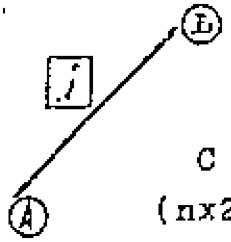
Tratamiento matricial general:

a).- $\bar{F} = C\ \bar{p}$

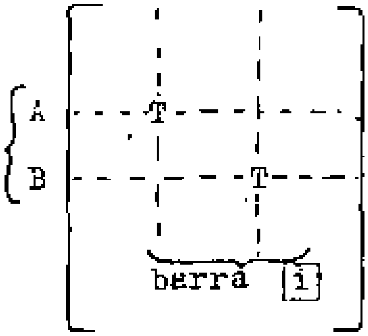
donde: $\vec{F} = \begin{bmatrix} \vec{F}_1 \\ \vec{F}_2 \\ \vec{F}_3 \\ \vdots \\ \vec{F}_n \end{bmatrix}$; $\bar{p} = \begin{bmatrix} \bar{p}_{A_1} \\ \bar{p}_{B_1} \\ \bar{p}_{A_2} \\ \bar{p}_{B_2} \\ \vdots \\ \bar{p}_{A_b} \\ \bar{p}_{B_b} \end{bmatrix}$

n = No. de nudos

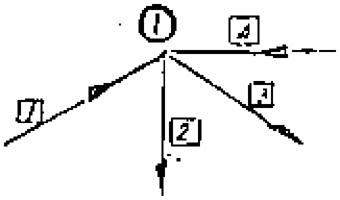
b = No. de barras



C = Nudos
(nx2b)



O bien la siguiente regla:



$\overline{F}_1 = T_1 \overline{p}_{B1} + T_2 \overline{p}_{A2} + T_3 \overline{p}_{A3} + T_4 \overline{p}$

b).- $\boxed{\tilde{p} = \overline{p} + D\ k\ e}$

donde:

$$\tilde{p} = \left[\begin{array}{c} p_{A1} \\ p_{B1} \\ \hdashline p_{A2} \\ p_{B2} \\ \hdashline \vdots \\ \hdashline p_{Ab} \\ p_{Bb} \end{array}\right] \text{(fuerzas finales en las barras)}$$

$$D = \overset{\text{barra}}{\boxed{i}} \begin{array}{c|ccccc|} & & & \boxed{i} & & \\ & & & 0 & & \\ & & & 0 & & \\ A & 0 & 0 & H_{BA} & 0 & 0 \\ B & 0 & 0 & I & 0 & 0 \\ & & & 0 & & \\ & & & 0 & & \\ & & & 0 & & \end{array}$$

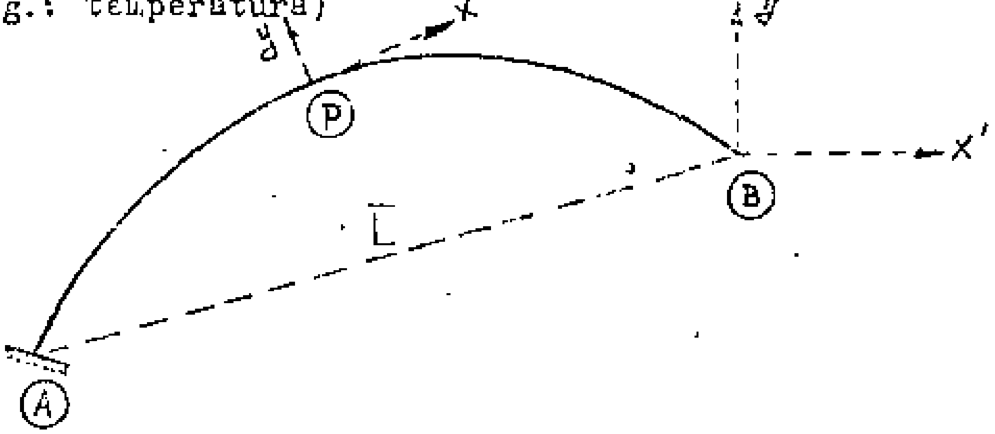
(2bxb)

Ejemplo: (El ejemplo anterior)

$$C = \textcircled{1}\begin{array}{c} \left[\begin{array}{cc:cc:cc} 0 & T_1 & 0 & T_2 & 0 & T_3 \end{array}\right] \\ \begin{array}{ccc} \boxed{1} \quad\quad & \boxed{2} \quad\quad & \boxed{3} \end{array} \end{array}$$

$$D = \begin{array}{cc} \begin{array}{c} \boxed{1} \\ \\ \boxed{2} \\ \\ \boxed{3} \end{array} & \left[\begin{array}{c:c:c} \begin{matrix} -H_{BA_1} \\ I \end{matrix} & 0 & 0 \\ \hdashline 0 & \begin{matrix} -H_{BA_2} \\ I \end{matrix} & 0 \\ \hdashline 0 & 0 & \begin{matrix} -H_{BA_3} \\ I \end{matrix} \end{array}\right] \\ & \begin{array}{ccc} \boxed{1} & \boxed{2} & \boxed{3} \end{array} \end{array}$$

III.- Desplazamientos producidos por deformaciones inducidas (v.g.: temperatura)



$_T d'_B$ = desplazamiento en Ⓑ producido por deformaciones inducidas (de)

$$_T d'_B = \int_L H'^{T}_{BP} \;\; T \; de$$

Si se trata de una dilatación producida por un cambio de temperatura θ

$$de = \alpha\theta ds \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

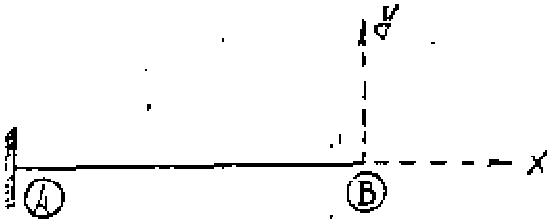
donde: α = coeficiente de dilatación, constante en la sección.
θ = incremento en temperatura, constante en la sección.

Si α y θ son constantes a lo largo de la barra

$$ {}_T d'_B = \alpha\theta \begin{bmatrix} L_{x'} \\ L_{y'} \\ 0 \end{bmatrix} $$

$L_{x'} L_{y'}$ = proyecciones del vector $\overline{L}$ (A - B), con respecto a x', y'.

Si la barra es recta:



$$ {}_T d_B = \alpha\theta L \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} $$

IV.- Fuerzas efectivas producidas por deformaciones inducidas.

a).- Deformaciones (e) en las barras $\boxed{e = a\,d' - {}_T d_B}$

b).- Fuerzas en las barras $\boxed{\begin{array}{l} p = k\,e \\ p = k\,a\,d' - k\,{}_T d_B \end{array}}$

Por equilibrio

$$ F' = a^T p = a^T k\,a\,d' - a^T k\,{}_T d_B $$

$$ \boxed{F' + a^T k\,{}_T d_B = F'_{ax}} $$

Si $F' = 0$, se tiene la siguiente solución:

$$\begin{cases} d' = (a^T k\ a)^{-1}\ a^T k\ {}_T d_B; \\ e = \left[a\ (a^T k\ a)^{-1} a^T k\ - I\right]\ {}_T d_B \\ p = k\left[a\ (a^T k\ a)^{-1}\ a^T k - I\right]\ {}_T d_B \end{cases}$$

Observe que si la estructura es isostática, la matriz $[a]$ es cuadrado y no singular, por lo tanto:

$$(a^T k\ a)^{-1} = a^{-1} k^{-1} (a^T)^{-1}$$

$$\left[a\ (a^T k\ a)^{-1} a^T k - I\right] = 0$$

O sea que las deformaciones inducidas <u>no causan</u> esfuerzos en las estructuras isostáticas, aunque <u>si producen</u> desplazamientos.

Ejemplo: (Ejemplo hoja 66).

$$\begin{array}{c} \text{Supongamos:} \\ \text{(Temperatura)} \end{array} \begin{Bmatrix} \alpha = 0.000015/°C \\ \theta = 20°C \end{Bmatrix} \text{ Para todas las barras}$$

Por lo tanto: $${}_T d_B = \begin{bmatrix} 0.0015 \\ 0 \\ 0 \end{bmatrix} (m) \quad \text{(Para todas las barras)}$$

a).- Fuerzas efectivas:

$${}_T d_B = \left[\begin{array}{c} 0.0015 \\ 0 \\ 0 \\ \hdashline 0.0015 \\ 0 \\ 0 \\ \hdashline 0.0015 \\ 0 \\ 0 \end{array}\right]; \quad k_T d_B = \left[\begin{array}{c} 6.0 \\ 0 \\ 0 \\ \hdashline 6.0 \\ 0 \\ 0 \\ \hdashline 6.0 \\ 0 \\ 0 \end{array}\right]$$

$$F'_{ex} = a^T(k_T d_3) = \begin{bmatrix} -10.25 \\ +1.75 \\ 0 \end{bmatrix}$$

b).- Obtengamos d' ; $d' = (k')^{-1} F'_{ex}$ (ó resolviendo el sistema)

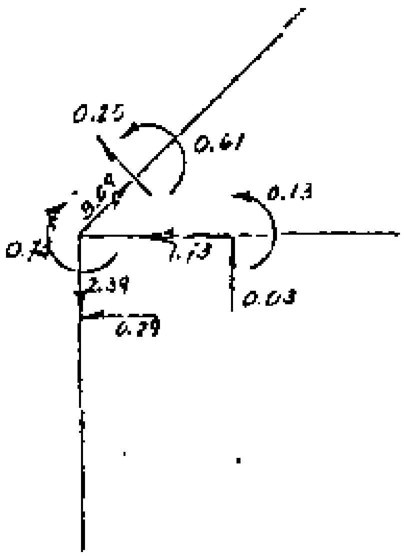
$$d' = \begin{bmatrix} -1.932 \\ +0.902 \\ -0.067 \end{bmatrix} \times 10^{-3}$$

c).- Obtengamos e (= a $d' - {}_T d_B$)

$$a\,d' = \begin{bmatrix} +0.902 \\ +1.932 \\ -0.067 \\ +1.932 \\ -0.902 \\ -0.067 \\ +0.728 \\ -2.000 \\ -0.067 \end{bmatrix} \times 10^{-3} \quad e = \begin{bmatrix} -0.598 \\ +1.932 \\ -0.067 \\ +0.432 \\ -0.902 \\ -0.067 \\ -0.772 \\ -2.000 \\ -0.067 \end{bmatrix} \times 10^{-3}$$

d).- Obtengamos p (=p_2): (p = k e)

$$p = \begin{bmatrix} -2.39 \\ +0.29 \\ -0.74 \\ +1.73 \\ -0.03 \\ +0.13 \\ -3.09 \\ -0.25 \\ +0.61 \end{bmatrix}$$



Ejemplo: (Ejemplo hoja (66))

La barra $\boxed{1}$ tiene una deformación previa (por error de fábrica) igual a

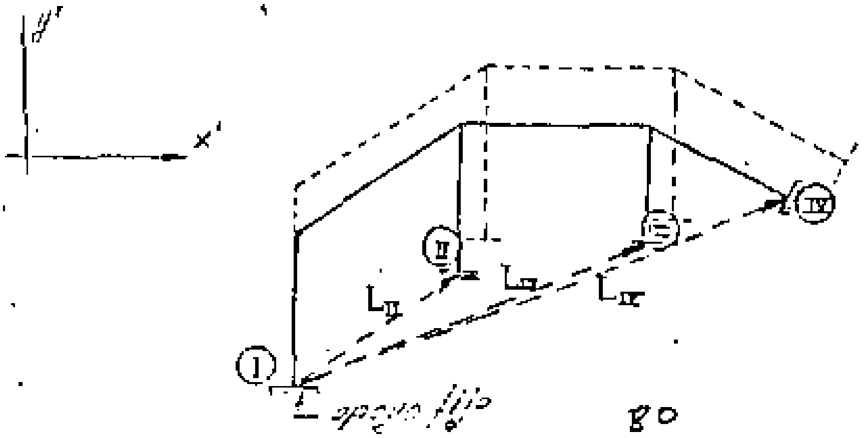
$$ {}_{T}d_{B_1} = \begin{bmatrix} 0.001 \\ 0.001 \\ 0.0005 \end{bmatrix} $$

Obtener las fuerzas efectivas, producidas por esta deformación inducida.

$$ k_{T}d_{B} = \left[\begin{array}{c} +4.000 \\ -0.034 \\ +0.228 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] \qquad F'_{ex} = \begin{bmatrix} -0.034 \\ -4.000 \\ +0.228 \end{bmatrix} $$

V.- Simplificación de los efectos por temperatura, cuando α y θ son iguales para todas las barras.

Consideramos una estructura cualquiera con α y θ iguales para todas las barras, soltemos todos los apoyos que le impidan dilatarse libremente, sea Ⓘ el apoyo fijo y (II), (III) y (IV) los apoyos que se han removido.



Estructura dilatada (homóloga a la estructura original)

Los desplazamientos de los apoyos serán:

$$ {}_T d'_I = 0; \; {}_T d'_{II} = \alpha \theta \begin{bmatrix} L_{II x'} \\ L_{II y'} \\ 0 \end{bmatrix}; \quad {}_T d_{III} = \alpha \theta \begin{bmatrix} L_{III x'} \\ L_{III y'} \\ 0 \end{bmatrix} \quad {}_T d_{IV} = \alpha \theta \begin{bmatrix} L_{IV} \\ L_{IV y'} \\ 0 \end{bmatrix} $$

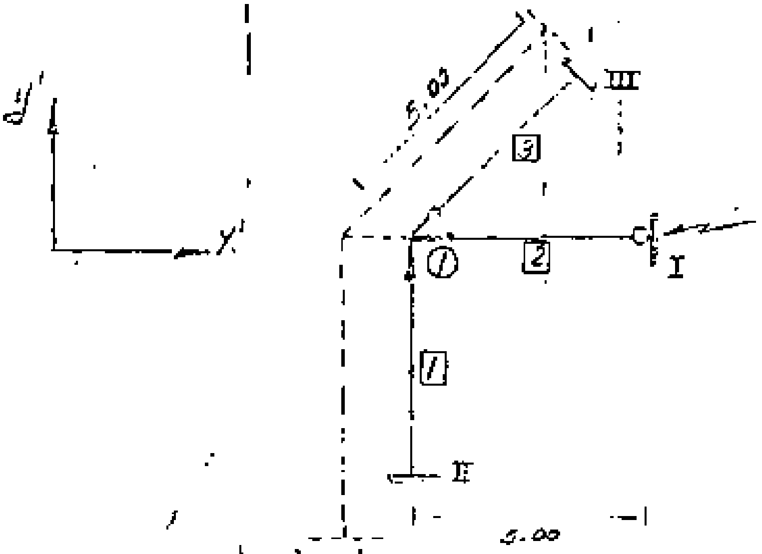
donde: $\left\{ \begin{array}{lcl} L_{i x'} & = & x'_i - x'_I \\ L_{i y'} & = & y'_i - y'_I \end{array} \right\}$

El problema es <u>equivalente</u> a una estructura cuyos apoyos han sufrido desplazamientos iguales y de signo contrario a los que se obtienen al dilatarse libremente la estructura. (Ver inciso IV del Resumen (2))

$$ \left\{ F'_{ix} \right. = - K_{12} \left. d'_{k} \right\} $$

Ejemplo: (Ejemplo hoja (78))

$$ \left\{ \begin{array}{l} \alpha = 0.000015/ °C \\ \theta = 20° C \end{array} \right. $$



Apoyo fijo

$$ \bar{L}_{II} = \begin{bmatrix} -5.00 \\ -5.00 \end{bmatrix} $$

$$ L_{III} = \begin{bmatrix} -1.465 \\ +3.535 \end{bmatrix} $$

Obtengamos ${}_T d'_I, {}_T d'_{II}, {}_T d'_{III}$

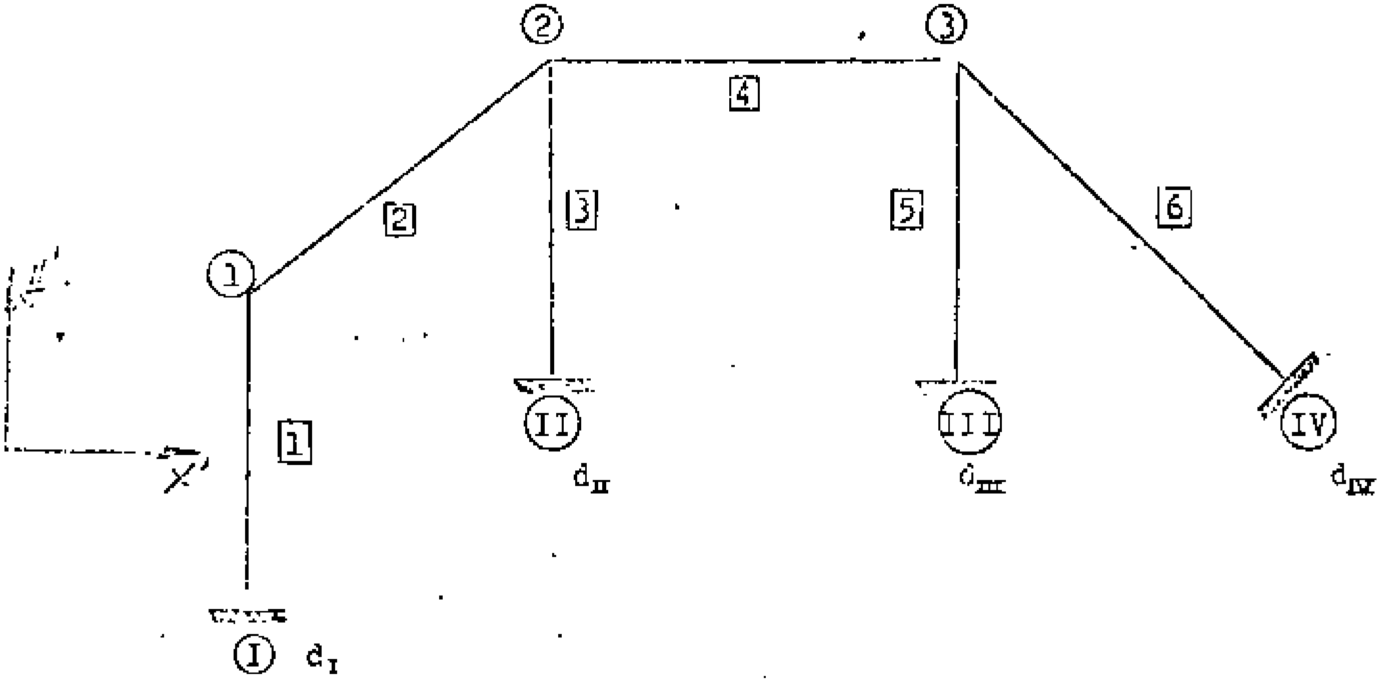
$$ {}_T d'_I = 0 \; ; \; {}_T d'_{II} = \begin{bmatrix} -0.00150 \\ -0.00150 \\ 0 \end{bmatrix} ; \qquad {}_T d'_{III} = \begin{bmatrix} -0.00044 \\ +0.00106 \\ 0 \end{bmatrix} $$

Resolvamos la estructura con los siguientes desplazamientos en los apoyos:

$$d = \left[\begin{array}{c} 0\\ 0\\ 0\\ +0.00150\\ +0.00150\\ 0\\ +0.00044\\ -0.00106\\ 0 \end{array}\right] \begin{array}{l} \\ I\\ \\ \\ II\\ \\ \\ III\\ \\ \end{array}$$

Nota:

Para la obtención de las fuerzas en los apoyos efectivas, producidas por desplazamientos, no es necesario usar el procedimiento indicado en el inciso IV del Resumen (2), que consistía en considerar a los apoyos como nudos; basta con obtener las fuerzas de fijación en las barras vecinas a los apoyos, producidas por los desplazamientos de estos, a continuación damos un ejemplo.



Las fuerzas de fijación en las barras $\boxed{1}$, $\boxed{3}$, $\boxed{5}$ y $\boxed{6}$ serán:

Barra $\boxed{1}$ $\begin{cases} \overline{p}_{A_1} = k_{AA}\ d_{I} ; \\ \overline{p}_{B_1} = k_{BA}\ d_{I} \end{cases}$ Barra $\boxed{3}$ $\begin{cases} \overline{p}_{A_3} = k_{AA}\ d_{II} \\ \overline{p}_{B_3} = k_{BA}\ d_{II} \end{cases}$

Barra $\boxed{5}$ $\begin{cases} \overline{p}_{A_5} = k_{AA}\ d_{III} ; \\ p_{B_5} = k_{BA}\ d_{III} \end{cases}$ Barra $\boxed{6}$ $\begin{cases} \overline{p}_{A_6} = k_{AA}\ d_{IV} \\ \overline{p}_{B_6} = k_{BA}\ d_{IV} \end{cases}$

d_I, d_{II}, d_{III}, d_{IV} en coordenadas locales, esto es:

$$d_I = T_1^T\ d_I' \ ; \ d_{II} = T_3^T\ d_{II}'$$
$$d_{III} = T_5^T\ d_{III}' \ ; \ d_{IV} = T_6^T\ d_{IV}'$$

Por lo tanto las fuerzas efectivas (F_{ex}') en los nudos ①, ② y ③, serán:

$$F_{ex}' = \begin{array}{c} \left[\begin{array}{cc} -T_1\ \overline{p}_{B_1} & \\ \hline -T_3\ \overline{p}_{B_3} & \\ \hline -T_5\ \overline{p}_{A_5} & -T_6\ \overline{p}_{B_6} \end{array}\right] \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$$

Apliquemos este procedimiento a nuestro ejemplo.

a).- Obtención de d_{II} d_{III}

$$d_{II} = T_1^T\ d_{II}' = \begin{bmatrix} +0.00150 \\ -0.00150 \\ 0 \end{bmatrix}$$

$$d_{III} = T_3^T\ d_{III}' = \begin{bmatrix} +0.000438 \\ +0.001060 \\ 0 \end{bmatrix}$$

b).- Fuerzas de fijación en las barras $\boxed{1}$ y $\boxed{3}$

Barras $\boxed{1}$ $\boxed{3}$

$$\left\{\begin{array}{l} k_{BA} = - k_{BB} H^{T}_{BA} = \begin{bmatrix} -4.00 & 0 & 0 \\ 0 & -0.137 & -0.343 \\ 0 & +0.343 & +0.573 \end{bmatrix} \times 10^{3} \\ \\ k_{AA} = H_{BA} k_{BB} H^{T}_{BA} = \begin{bmatrix} 4.00 & 0 & 0 \\ 0 & 0.137 & +0.343 \\ 0 & +0.343 & 1.142 \end{bmatrix} \times 10^{3} \end{array}\right.$$

Barra $\boxed{1}$ $$\left\{ \overline{p}_{A_1} = k_{AA} d_{II} = \begin{bmatrix} +6.000 \\ -0.205 \\ -0.514 \end{bmatrix} \; ; \; \overline{p}_{B_1} = k_{BA} d_{II} = \begin{bmatrix} -6.00 \\ +0.205 \\ -0.514 \end{bmatrix} \right.$$

Barra $\boxed{3}$ $$\left\{ \overline{p}_{A_3} = k_{AA} d_{III} = \begin{bmatrix} +1.752 \\ +0.145 \\ +0.365 \end{bmatrix} \; ; \; \overline{p}_{B_3} = k_{BA} d_{III} = \begin{bmatrix} -1.752 \\ -0.145 \\ +0.365 \end{bmatrix} \right.$$

c).- Obtención F'_{ex}

$$F'_{ex} = \begin{bmatrix} - T_1 \overline{P}_{B_1} & - T_3 \; \overline{P}_{B_3} \end{bmatrix}$$

$$F'_{ex} = \begin{bmatrix} -0.930 \\ +4.660 \\ +0.149 \end{bmatrix}$$

d).- Obtengamos $[d']$ $\left(d' = (K')^{-1} \; F'_{ex} \right)$

$$d' = \begin{bmatrix} -0.430 \\ +0.903 \\ -0.067 \end{bmatrix} \times 10^{-3}$$

Observese que los valores de $\left[d'\right]$ no son iguales a los obtenidos en la hoja (75) porque falta sumar el desplazamiento de ① cuando se dilató libremente, que es igual a:

$$\begin{bmatrix} -1.500 \\ 0 \\ 0 \end{bmatrix} \times 10 \qquad \text{que sumado al anterior}$$

$$\text{nos da:} \quad \begin{bmatrix} -1.930 \\ +0.903 \\ -0.067 \end{bmatrix} \qquad \text{que es el mismo valor que el obtenido en la hoja (77)}$$

e).- Obtengamos $[e]$ (= a d')

$$e = \left[\begin{array}{c} +0.903 \\ +0.430 \\ -0.067 \\ \hline +0.430 \\ -0.903 \\ -0.067 \\ \hline -0.334 \\ -0.942 \\ -0.067 \end{array}\right]$$

f).- Obtengamos k e:

$$k\ e = \left[\begin{array}{c} +3.61 \\ +0.08 \\ -0.22 \\ \hline +1.72 \\ -0.03 \\ +0.13 \\ \hline -1.34 \\ -0.11 \\ +0.25 \end{array}\right] \begin{array}{c} \\ \boxed{1} \\ \\ \\ \boxed{2} \\ \\ \\ \boxed{3} \\ \\ \end{array}$$

g).- Obtengamos p_B $(= K_B e + \overline{p}_B)$

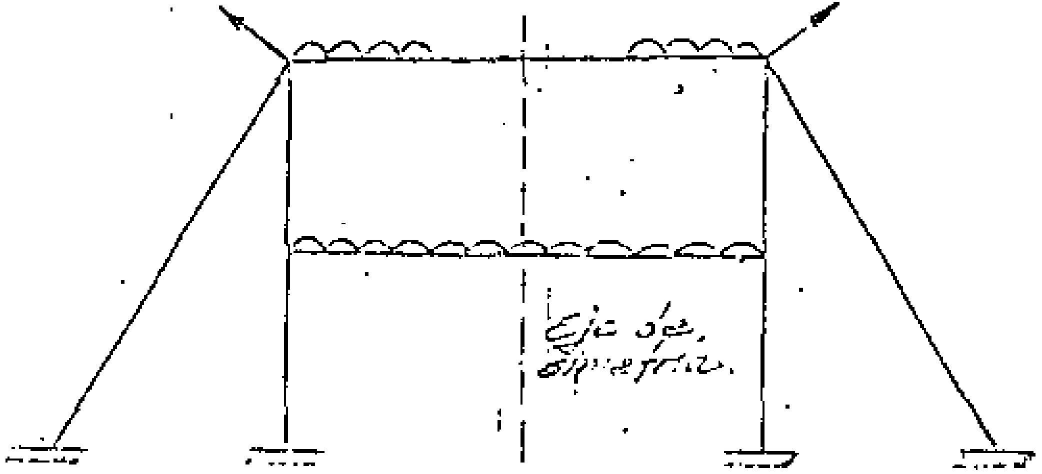
$$p_B = \begin{bmatrix} -2.\text{[illegible]} \\ +0.29 \\ -0.73 \\ +1.72 \\ -0.03 \\ +0.13 \\ -3.09 \\ -0.26 \\ +0.61 \end{bmatrix}$$

Iguales resultados a los obtenidos en la hoja (77)

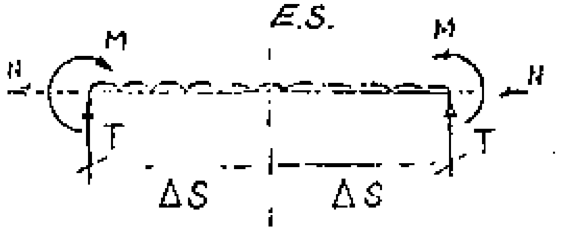
VI.- Simplificación cuando una estructura es simétrica en geometría. (Marcos planos)

1).- Carga simétrica

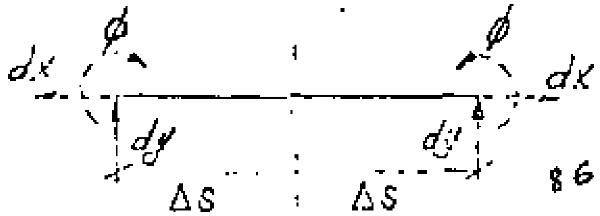
a).- No hay nudos en el eje de simetría



Por simetría: (reflexión)

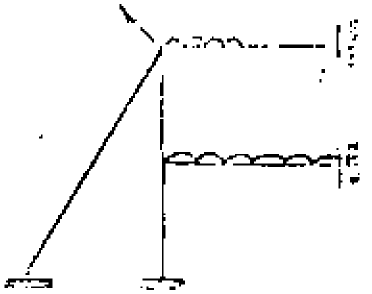


Si Δs → 0
Por equilibrio
T = 0



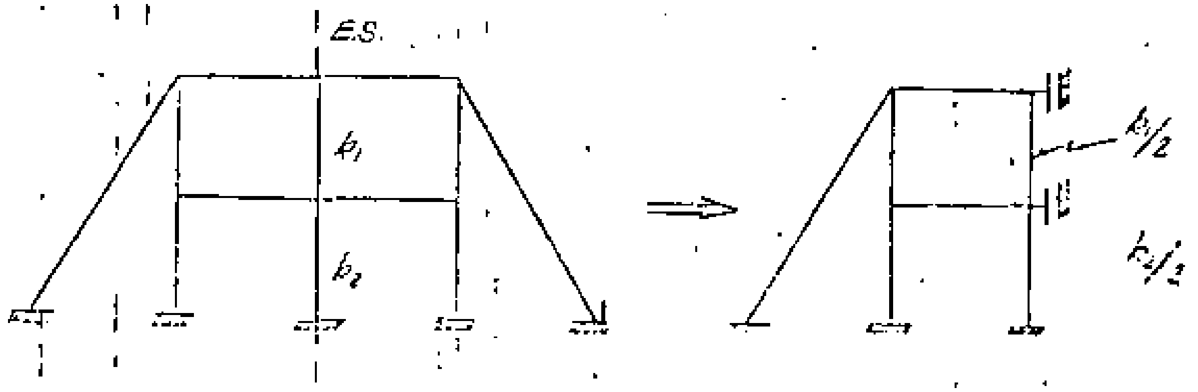
Si Δs → 0
Por continuidad
$d_x = 0$
$\phi = 0$

Por lo tanto, la estructura es equivalente a:

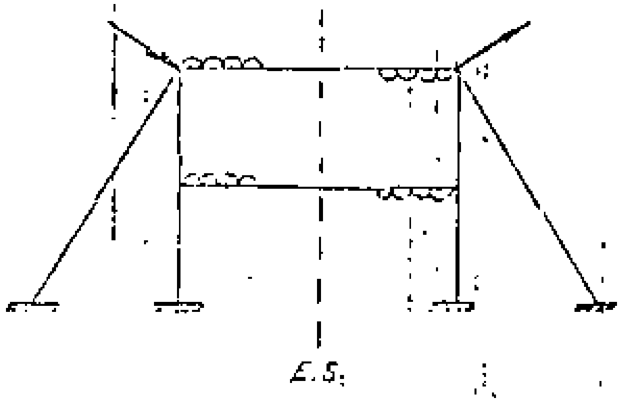


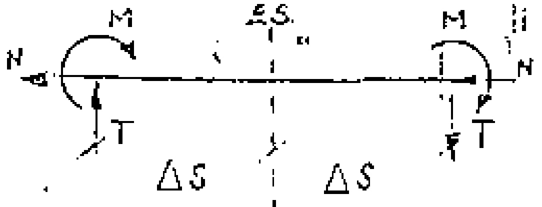
$(\phi = d_x = 0)$

b).- Hay nudos (barras) en el eje de simetría



2).- Carga antisimétrica:



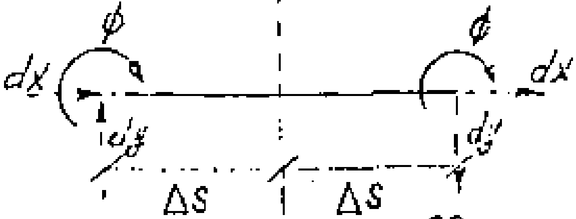


Por antisimetría (antireflexión)

Si $\Delta s \rightarrow 0$

Por equilibrio: $N = 0$

$M = 0$

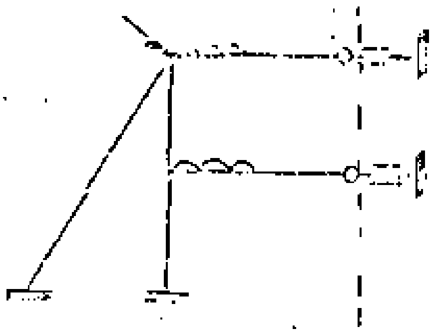


Si $\Delta s \rightarrow 0$

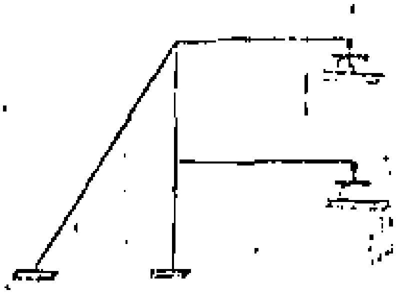
Por continuidad

$d_y = 0$

Por lo tanto la estructura es equivalente a:

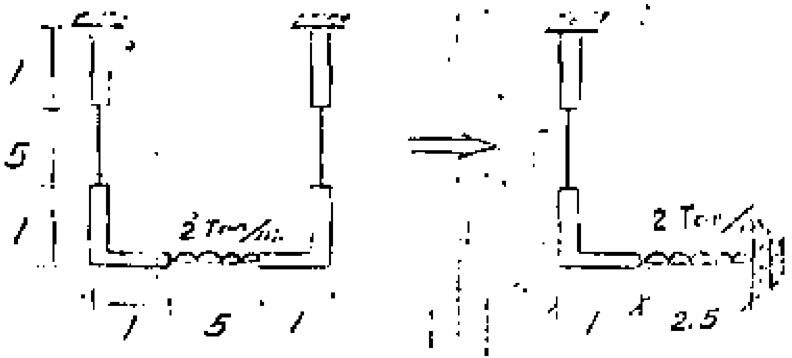

(d, = 0)

o bien:

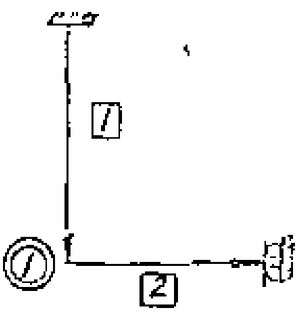


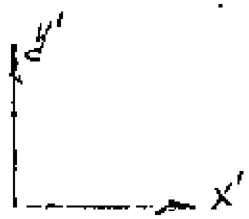
Si hay barras en el eje se toma la mitad de su rigidez, como en el caso anterior.

Ejemplo:



Orientemos las barras:





a).- Rigideces de las barras (k)

$$k_{BB_1} = \begin{bmatrix} 0.400 & 0 & 0 \\ 0 & 0.191 & -0.668 \\ 0 & -0.668 & 2.730 \end{bmatrix} \times 10^5$$

$$k_{BB_2} = \begin{bmatrix} 0.800 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.800 \end{bmatrix} \times 10^5$$

(Ver tabla hoja 43 del resumen (5))

$$k_{AA_2} = H_{BA}\, k_{BB}\, H^T_{BA} = \begin{bmatrix} 0.800 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.800 \end{bmatrix} \times 10^5$$

b).- Obtengamos K' (con la regla de la suma)

$$k'_{BB_1} = \begin{bmatrix} 0.191 & 0 & -0.668 \\ 0 & 0.400 & 0 \\ -0.668 & 0 & 2.730 \end{bmatrix} \times 10^5$$

$$k'_{AA_2} = k_{AA_2}$$

$$K' = \begin{bmatrix} 0.991 & 0 & -0.668 \\ & 0.400 & 0 \\ -0.668 & 0 & 3.530 \end{bmatrix} \times 10^5$$

c).- Las fuerzas efectivas y las de fijación son las mismas que las del nudo C en el problema original (Ver tabla al -- principio de este resumen)

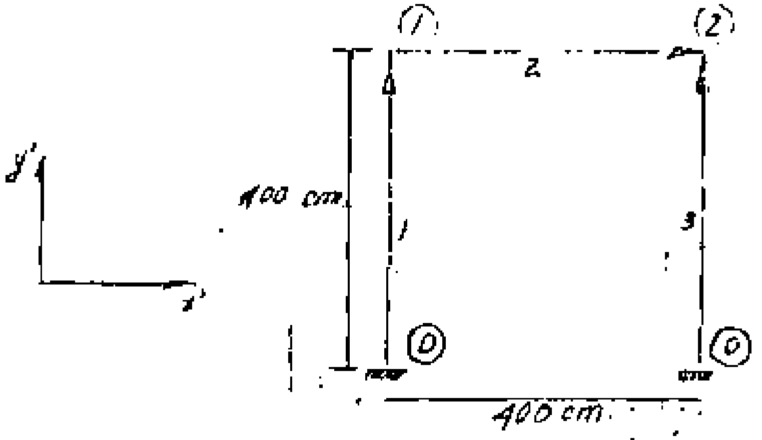
$$F'_{ex} = \begin{bmatrix} 0 \\ -5 \\ -9.167 \end{bmatrix}$$

d).- Resolviendo el sistema $K'd' = F'$ se obtiene:

$$d' = \begin{bmatrix} -\ 2.01 \\ -12.50 \\ -\ 2.98 \end{bmatrix} \times 10^{5}$$

Ejemplos:

I.- Obtener K'



E = 2.1×10^6 kg/cm²

I = 2×10^4 cm^4

A = 100 cm²

$\mu = 0.4$

$K_{forma} = 1.2$

$$\rho = \sqrt{20000/100} = 14.1 \text{ cm}$$

$$c = 6(1+\mu)\left(\frac{\rho}{L}\right)^2 K_{forma} = 6x1.4x\left(\frac{14.1}{400}\right)^2 x1.2 = 0.012$$

$$\frac{EA}{L} = \frac{2.1x10^6 x100}{400} = 5.25x10^5 \text{ kg/cm}$$

$$\frac{12EI}{L^3(1+4c)} = \frac{12x2.1x10^6 x2x10^4}{400^3 \; x \; (1.05)} = 0.075x10^5 \text{ kg/cm}$$

$$\frac{6EI}{L^2(1+4c)} = \frac{6x2.1x10^6 x2x10^4}{400^2 \; x \; (1.05)} = 15.00x10^5 \text{ kg}$$

$$\frac{4EI(1+c)}{L(1+4c)} = \frac{4x2.1x10^6 x2x10^4(1.01)}{400 \quad (1.05)} = 4040x10^5 \text{ kg-cm}$$

$$\frac{2EI(1-2c)}{L(1+4c)} = \frac{2x2.1x10^6 x2x10^4 x(0.98)}{400 \quad (1.05)} = 1960x10^5 \text{ kg-cm}$$

Regla de la suma

Barra $\boxed{1}$ (⓪ - ①) y Barra $\boxed{3}$ (⓪ - ②)

$$\theta = -90^\circ \quad ; \quad T = \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

$$k = \left[\begin{array}{ccc|ccc} 5.25 & 0 & 0 & & & \\ 0 & 0.075 & 15 & & \text{Simetrica} & \\ 0 & 15 & 4040 & & & \\ \hline -5.25 & 0 & 0 & 5.25 & 0 & 0 \\ 0 & -0.075 & -15 & 0 & 0.075 & -15 \\ 0 & 15 & 1960 & 0 & -15 & 4040 \end{array}\right] \times 10^{5}$$

$k' = T\,k\,T^{T}$ etc.

$$k = \begin{array}{c} ⓪ \\ \\ \\ ① \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 0.075 & 0 & -15 & -0.075 & 0 & -15 \\ 0 & 5.25 & 0 & 0 & -5.25 & 0 \\ -15 & 0 & 4040 & 15 & 0 & 1960 \\ \hline -0.075 & 0 & 15 & 0.075 & 0 & 15 \\ 0 & -5.25 & 0 & 0 & 5.25 & 0 \\ -15 & 0 & 1960 & 15 & 0 & 4040 \end{array}\right] \times 10^{5}$$

$$\begin{array}{cc} \qquad ⓪ \qquad & \qquad \overset{①}{②} \qquad \end{array}$$

Barra $\boxed{2}$ (① — ②):

$\theta = 0^{o}$; $T = I$

$$k = k' = \begin{array}{c} \\ ① \\ \\ \\ ② \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 5.25 & 0 & 0 & -5.25 & 0 & 0 \\ 0 & 0.075 & 15 & 0 & -0.075 & 15 \\ 0 & 15 & 4040 & 0 & -15 & 1960 \\ \hline -5.25 & 0 & 0 & 5.25 & 0 & 0 \\ 0 & -0.075 & -15 & 0 & 0.075 & -15 \\ 0 & 15 & 1960 & 0 & -15 & 4040 \end{array}\right] \times 10$$

$$\begin{array}{cc} \qquad ① \qquad & \qquad ② \qquad \end{array}$$

Aplicando la regla de la suma

$$K' = \left[\begin{array}{c|ccc|ccc}
 & 5.325 & 0 & 15 & -5.25 & 0 & 0 \\
\textcircled{1} & 0 & 5.325 & 15 & 0 & -0.075 & 15 \\
 & 15 & 15 & 8080 & 0 & -15 & 1960 \\
\hline
 & -5.25 & 0 & 0 & 5.325 & 0 & 15 \\
\textcircled{2} & 0 & -0.075 & -15 & 0 & 5.325 & -15 \\
 & 0 & 15 & 1960 & 15 & -15 & 8080 \\
\hline
 & & \textcircled{1} & & & \textcircled{2} &
\end{array}\right] \times 10^5$$

Alternativa 2

1) Matriz $[a]$

Barra $\boxed{1}$ y barra $\boxed{3}$:

$$-T^T H^T_{1,0} = -T^T H^T_{3,0} = \quad \text{(No se necesita)}$$

$$T^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Barra $\boxed{2}$

$$-T^T H^T_{2,1} = -I\ H^T_{2,1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -400 \\ 0 & 0 & -1 \end{bmatrix}$$

$$
a = \begin{array}{c}
 \\ \boxed{1} \\ \\ \\ \boxed{2} \\ \\ \\ \boxed{3} \\ \\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} & ① & & & ② & \end{array} \\
\left[\begin{array}{ccc:ccc}
0 & 1 & 0 & & & \\
-1 & 0 & 0 & & 0 & \\
0 & 0 & 1 & & & \\
\hdashline
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & -400 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
\hdashline
 & & & 0 & 1 & 0 \\
 & 0 & & -1 & 0 & 0 \\
 & & & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

2) Matriz k

$$
k = \begin{array}{c}
 \\ \boxed{1} \\ \\ \\ \boxed{2} \\ \\ \\ \boxed{3} \\ \\
\end{array}
\left[\begin{array}{ccc:ccc:ccc}
5.25 & 0 & 0 & & & & & & \\
0 & 0.075 & -15 & & & & & & \\
0 & -15 & 4040 & & & & & & \\
\hdashline
 & & & 5.25 & 0 & 0 & & & \\
 & & & 0 & 0.075 & -15 & & & \\
 & & & 0 & -15 & 4040 & & & \\
\hdashline
 & & & & & & 5.25 & 0 & 0 \\
 & & & & & & 0 & 0.075 & -15 \\
 & & & & & & 0 & -15 & 4040
\end{array}\right] \mathrm{x}10^{3}
$$

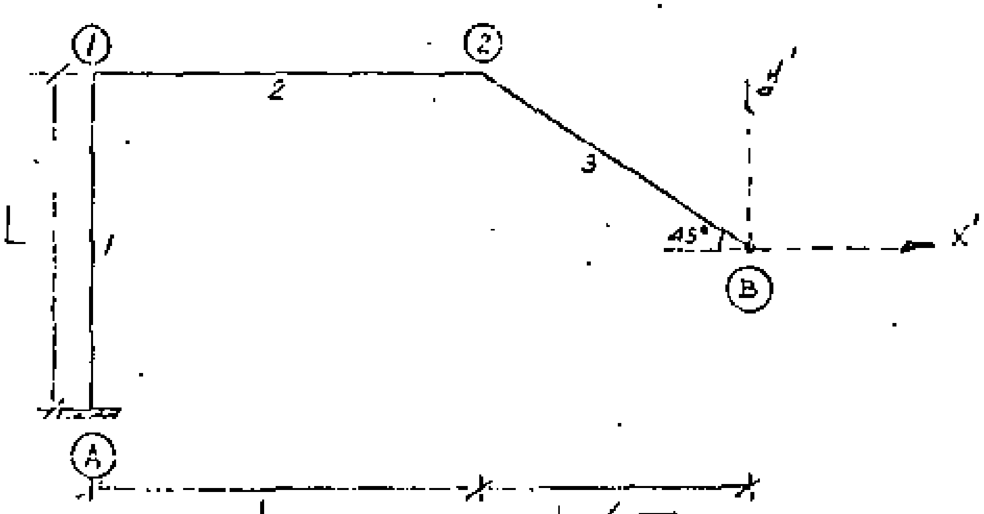
3) Obtención de $K' = a^T k a$

$$[k]\,[a] = \left[\begin{array}{ccc|ccc} 0 & 5.25 & 0 & & & \\ -0.075 & 0 & -15 & & 0 & \\ 15 & 0 & 4040 & & & \\ \hline -5.25 & 0 & 0 & 5.25 & 0 & 0 \\ 0 & -0.075 & -15 & 0 & 0.075 & -15 \\ 0 & 15 & 1960 & 0 & -15 & 4040 \\ \hline & & & 0 & 5.25 & 0 \\ & 0 & & -0.075 & 0 & -15 \\ & & & 15 & 0 & 4040 \end{array}\right] \times 10^5$$

$$a^T k a = K' = \left[\begin{array}{ccc|ccc} 5.325 & 0 & 15 & -5.25 & 0 & 0 \\ 0 & 5.325 & 15 & 0 & -0.075 & 15 \\ 15 & 15 & 8080 & 0 & -15 & 1960 \\ \hline -5.25 & 0 & 0 & 5.325 & 0 & 15 \\ 0 & -0.075 & -15 & 0 & 5.325 & -15 \\ 0 & 15 & 1960 & 15 & -15 & 8080 \end{array}\right] \times 10^5$$

Igual a la obtenida con la regla de la suma.

II) Obtener $\left[f_{LL}\right]$



EI = cte

c = 0 (despreciemos el efecto del cortante)

EA = cte

Apliquemos la fórmula:

$$f_{BB} = f'_3 + H^T_{B,2}\ f'_2\ H_{B,2} + H^T_{B,1}\ f'_1\ H_{B,1}$$

donde: $f' = T\ F\ T^T$

$$f = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L^3}{3EI} & \frac{L^2}{2EI} \\ 0 & \frac{L^2}{2EI} & \frac{L}{EI} \end{bmatrix}$$

(Flexibilidad en coordenadas locales)

Barra $\boxed{3}$: $\theta = 45^0$

$$T = \begin{bmatrix} 0.71 & 0.71 & 0 \\ -0.71 & 0.71 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f'_3 = \begin{bmatrix} \left(\frac{L}{2EA} + \frac{L^3}{6EI}\right) & \left(\frac{-L}{2EA} + \frac{L^3}{6EI}\right) & \left(\frac{L^2}{2\sqrt{2}EI}\right) \\ \left(\frac{-L}{2EA} + \frac{L^3}{6EI}\right) & \left(\frac{L}{2EA} + \frac{L^3}{6EI}\right) & \left(\frac{L^2}{2\sqrt{2}EI}\right) \\ \left(\frac{L^2}{2\sqrt{2}EI}\right) & \left(\frac{L^2}{2\sqrt{2}EI}\right) & \left(\frac{L}{EI}\right) \end{bmatrix}$$

Barra $\boxed{2}$ $\theta = 0^0$

$$f'_2 = f_2\ ;\ H_{B,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{L}{\sqrt{2}} & \frac{L}{\sqrt{2}} & 1 \end{bmatrix} \qquad \left(y_2 = \frac{L}{\sqrt{2}}\ ;\ x_2 = \frac{-L}{\sqrt{2}}\right)$$

$$H \quad f'_2 \; H_{B,2} = \begin{bmatrix} \left(\frac{L}{EA} + \frac{L^3}{2EI}\right) & \frac{L^3}{2EI}\left(1 + \frac{1}{\sqrt{2}}\right) & \frac{L^2}{\sqrt{2}EI} \\ \frac{L^3}{2EI}\left(1 + \frac{1}{\sqrt{2}}\right) & \frac{L^3}{EI}\left(\frac{5}{6} + \frac{1}{\sqrt{2}}\right) & \frac{L^2}{2EI}\left(1+\sqrt{2}\right) \\ \frac{L^2}{\sqrt{2}EI} & \frac{L^2}{2EI}\left(1+\sqrt{2}\right) & \frac{L}{EI} \end{bmatrix}$$

Barre $\boxed{1}$: $\theta = -90^\circ$

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f'_1 = \begin{bmatrix} \frac{L^3}{3EI} & 0 & -\frac{L^2}{2EI} \\ 0 & \frac{L}{EA} & 0 \\ -\frac{L^2}{2EI} & 0 & \frac{L}{EI} \end{bmatrix} \; ; \quad H_{B,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{L}{\sqrt{2}} & L\left(1 + \frac{1}{\sqrt{2}}\right) & 1 \end{bmatrix}$$

$$H^T_{B,1} \; f'_1 \; H_{B,1} = \begin{bmatrix} \frac{L^3}{EI}\left(\frac{5}{6} - \frac{1}{\sqrt{2}}\right) & \frac{L^3}{2\sqrt{2}EI} & \frac{L^2}{EI}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) \\ \frac{L^3}{2\sqrt{2}EI} & \left[\frac{L}{EA} + \frac{L^3}{EI}\left(\frac{3}{2} + \sqrt{2}\right)\right] & \frac{L^2}{EI}\left(1 + \frac{1}{\sqrt{2}}\right) \\ \frac{L^2}{EI}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) & \frac{L^2}{EI}\left(1 + \frac{1}{\sqrt{2}}\right) & \frac{L}{EI} \end{bmatrix}$$

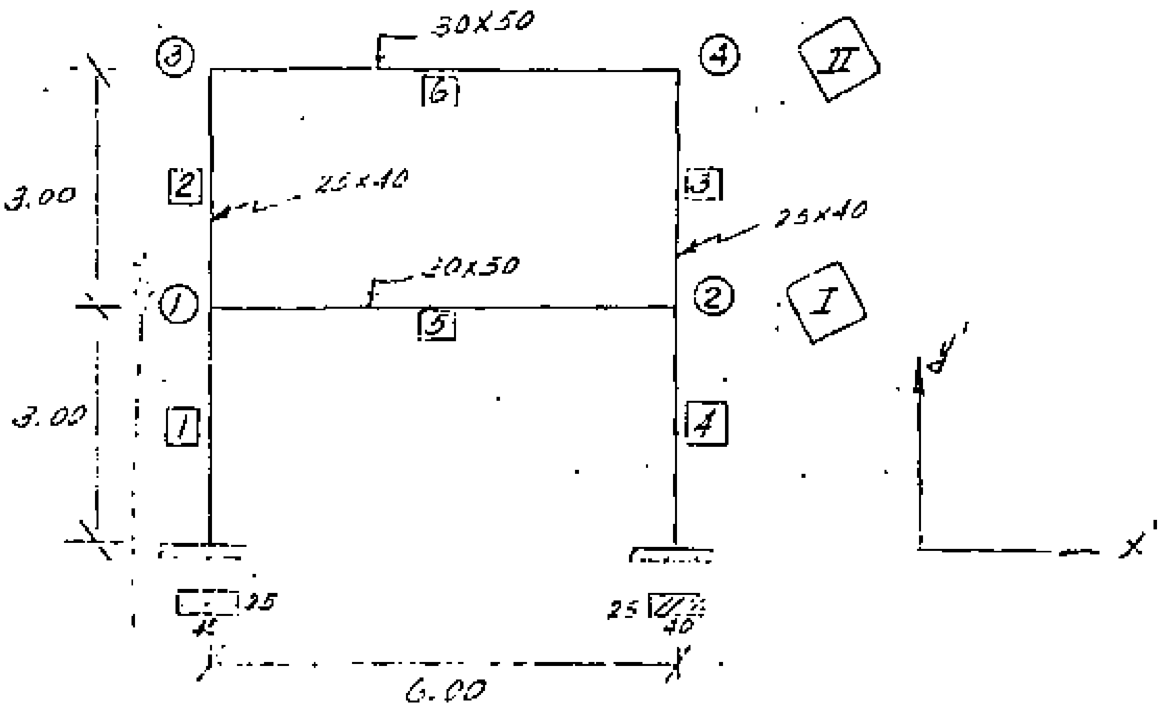
$$f_{bb} = \begin{bmatrix} \left[\frac{L}{2EA} + \frac{L^3}{EI}\left(\frac{3}{2} - \frac{1}{\sqrt{2}}\right)\right] & \left[-\frac{L}{2EA} + \frac{L^3}{EI}\left(\frac{2}{3} + \frac{1}{\sqrt{2}}\right)\right] & \frac{L^2}{2EI}\left(\frac{5}{\sqrt{2}} - 1\right) \\ \left[-\frac{L}{2EA} + \frac{L^3}{EI}\left(\frac{2}{3} + \frac{1}{\sqrt{2}}\right)\right] & \left[\frac{3L}{2EA} + \frac{L^3}{EI}\left(\frac{5}{2} + \frac{3}{\sqrt{2}}\right)\right] & \frac{L^2}{2EI}\left(\frac{5}{\sqrt{2}} + 3\right) \\ \frac{L^2}{2EI}\left(\frac{5}{\sqrt{2}} - 1\right) & \frac{L^2}{2EI}\left(\frac{5}{\sqrt{2}} + 3\right) & \frac{3L}{EI} \end{bmatrix}$$

ANALISIS ESTRUCTURAL II

EJEMPLOS

Ejemplos de análisis dinámico de estructuras

Ejemplo 1: Obtener los períodos y modos de vibrar (vibración libre)



Datos: $W_1 = W_2 = 30$ ton.

$E = 147000 \text{ kg/cm}^2$ ($f'_c = 210 \text{ kg/cm}^2$)

$\mu = 0.25$

Hagamos las siguientes alternativas.

1) Análisis dinámico completo.

a) Obtención de $\left[K \right]$ (12x12), usando la regla de la suma:

Trabes: $\boxed{5}$ y $\boxed{6}$

$$I = \frac{30\times50^3}{12} = 3.125\times10^5 \text{ cm}^4; \quad A = 30\times50 = 1500 \text{ cm}^2$$

$$I/A = 208 \text{ cm}^2; \quad c = 6\times1.25\times\frac{208}{600^2}\times1.2 = 0.0052$$

$$\frac{12EI}{L^3(1+4c)} = \frac{12\times1.47\times10^6\times3.125\times10^{-3}}{6.00^3 \times 1.0208} = 249.0 \text{ ton/m}$$

$$\frac{6EI}{L^2(1+4c)} = \frac{6\times1.47\times10^6\times3.125\times10^{-3}}{6.00^2 \times 1.0208} = 750.0 \text{ ton}$$

$$\frac{4EI(1+c)}{L(1+4c)} = \frac{4\times1.47\times10^6\times3.125\times10^{-3}\times1.0052}{6.00 \times 1.0208} = 2980.0 \text{ ton/m}$$

$$\frac{EA}{L} = \frac{1.47\times10^6\times1500\times10^{-4}}{6} = 36750.0 \text{ ton/m.}$$

$$\frac{2EI(1-2c)}{L(1+4c)} = 1470$$

$$k'_{EP_{(5)}} = k'_{FP_{(6)}} = \begin{bmatrix} 36750.0 & 0 & 0 \\ 0 & 249 & -750 \\ 0 & -750 & +2980 \end{bmatrix}$$

Columnas $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$

$$I = \frac{25\times40^3}{12} = 1.33\times10^5 \text{ cm}^4; \quad A = 1000 \text{ cm}^2$$

$$I/A = 133.3; \quad c = 6\times1.25\times\frac{133.3}{300^2}\times1.2 = 0.0133$$

$$\frac{12EI}{L^3(1+4c)} = \frac{12\times1.47\times1.333\times10^3}{3.00^3 \times 1.0532} = 824$$

$$\frac{6EI}{L^2(1+4c)} = 1236$$

$$\frac{4EI(1+c)}{L(1+4c)} = \frac{4\times1.47\times1.33\times10^3\times1.0133}{3 \times 1.0532} = 2520$$

$$\frac{EA}{L} = 49000.0$$

$$\frac{2EI(1-2c)}{L(1+4c)} = 1210.0$$

$$k'_{BB_{(1)}} = k_{bb_{(2)}} = \begin{bmatrix} 824 & 0 & 1236 \\ 0 & 49000 & 0 \\ 1236 & 0 & 2520 \end{bmatrix}$$

Aplicando la regla de la suma:

$$K = \begin{array}{cccccccccccc} & \text{①} & & & \text{②} & & & \text{③} & & & \text{④} & \\ \end{array}$$
$$\left[\begin{array}{ccc|ccc|ccc|ccc} 37398 & 0 & 0 & -36150 & 0 & 0 & -824 & 0 & -1236 & & & \\ 0 & 98249 & 150 & 0 & -249 & 150 & 0 & -49000 & 0 & & 0 & \\ & 150 & 8020 & 0 & -150 & 1410 & 1236 & 0 & 1210 & & & \\ \hline -36150 & 0 & 0 & 37398 & 0 & 0 & & & & -824 & 0 & -1236 \\ 0 & -249 & -150 & 0 & 98249 & -150 & & 0 & & 0 & -49000 & 0 \\ 0 & 150 & 1410 & 0 & -150 & 8020 & & & & 1236 & 0 & 1210 \\ \hline -824 & 0 & 1236 & & & & 37574 & 0 & 1236 & -36150 & 0 & 0 \\ 0 & -49000 & 0 & & 0 & & 0 & 49249 & 150 & 0 & -249 & 150 \\ -1236 & 0 & 1210 & & & & 1236 & 150 & 5500 & 0 & -150 & 1410 \\ \hline & & & -824 & 0 & 1236 & -36150 & 0 & 0 & 37574 & 0 & 1236 \\ & 0 & & 0 & -49000 & 0 & 0 & -249 & -150 & 0 & 49249 & -150 \\ & & & -1236 & 0 & 1210 & 0 & 150 & 1410 & 1236 & -150 & 5500 \end{array}\right]$$

b) Obtención de $\widetilde{k}$ (permutando k) tal que:

$$\begin{bmatrix} F_{x'} \\ F_{y'} \\ \hline R \end{bmatrix} = \left[\begin{array}{c|c} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \hline \widetilde{K}_{21} & \widetilde{K}_{22} \end{array}\right] \begin{bmatrix} d_{x'} \\ d_{y'} \\ \hline \phi \end{bmatrix}$$

$$K = \left[\begin{array}{cccc|cccc|cccc}
38398 & -36150 & -824 & 0 & & & & & 0 & 0 & -1236 & 0 \\
-36150 & 38398 & 0 & -824 & & & 0 & & 0 & 0 & 0 & -12.36 \\
-824 & 0 & 31314 & -36150 & & & & & 12.36 & 0 & 12.36 & 0 \\
0 & -824 & -36150 & 31314 & & & & & 0 & 1236 & 0 & 1236 \\
\hline
 & & & & 98249 & -249 & -49000 & 0 & 150 & 150 & 0 & 0 \\
 & & & & -249 & 98249 & 0 & -49000 & -150 & -150 & 0 & 0 \\
 & 0 & & & -49000 & 0 & 49249 & -249 & 0 & 0 & 150 & 150 \\
 & & & & 0 & -49000 & -249 & 49249 & 0 & 0 & -150 & -150 \\
\hline
0 & 0 & 1236 & 0 & 150 & -150 & 0 & 0 & 8020 & 1410 & 1210 & 0 \\
0 & 0 & 0 & 12.36 & 150 & -150 & 0 & 0 & 1410 & 8020 & 0 & 1210 \\
-1236 & 0 & 1236 & 0 & 0 & 0 & 150 & -150 & 1210 & 0 & 5500 & 1410 \\
0 & -12.36 & 0 & 1236 & 0 & 0 & 150 & -150 & 0 & 1210 & 1410 & 5500
\end{array}\right]$$

La ecuación característica será:

$$\left[\begin{array}{c|c} \tilde{K}_{11} & \tilde{K}_{12} \\ \hline \tilde{K}_{21} & \tilde{K}_{22} \end{array}\right] \left[\begin{array}{c} d_{x'} \\ d_{y'} \\ \hline \phi \end{array}\right] = \left[\begin{array}{c} \omega^2 m d_{x'} \\ \omega^2 m d_{y'} \\ \hline 0 \end{array}\right]$$

Por lo tanto se puede contraer (eliminando a ϕ)

$$\underbrace{\left[\tilde{K}_{11} - \tilde{K}_{12}\,\tilde{K}_{22}^{-1}\,\tilde{K}_{21}\right]}_{\tilde{\tilde{K}}} \left[\begin{array}{c} d_{x'} \\ \hline d_{y'} \end{array}\right] = \left[\begin{array}{c} \omega^2 m d_{x'} \\ \hline \omega^2 m d_{y'} \end{array}\right]$$

c) Obtengamos: $\widetilde{\widetilde{K}} = \widetilde{K}_{11} - \widetilde{K}_{12}\,\widetilde{K}_{22}^{-1}\,\widetilde{K}_{21}$

$$\widetilde{\widetilde{K}} = \left[\begin{array}{cccc|cccc}
 & d_{x'} & & & & d_{y'} & & \\
38085 & -36661 & -563 & -65.8 & -17.3 & 17.3 & 136.0 & -136. \\
-36661 & 38085 & -65.8 & -563 & -17.3 & 17.3 & 136.0 & -136. \\
-563 & -65.8 & 37158 & -36666 & -82.6 & 82.6 & -118.7 & 118. \\
-65.8 & -563 & -36666 & 37158 & -82.6 & 82.6 & -118.7 & 118. \\
\hline
-17.3 & -17.3 & -82.6 & -82.6 & 98128 & -128 & -48979 & -21. \\
17.3 & 17.3 & 82.6 & 82.6 & -128 & 98128 & -21.0 & -4897 \\
136.0 & 136.0 & -118.7 & -118.7 & -48979 & -21.0 & 49084 & -83. \\
-136.0 & -136.0 & 118.7 & 118.7 & -21.0 & -48979 & -83.9 & 4908
\end{array}\right.$$

Nota: Observe que las lineas de $\widetilde{K}_{22}^{-1}$ son los giros producidos por par
unitarios, sin desplazamientos, por lo que se puede aplicar Cro
ó Kani (modificados) para efectuar su inversión, ó simplemente
aplicar el método de Gauss - Seidel.

d) Obtengamos los períodos y modos naturales, con la ecuación:

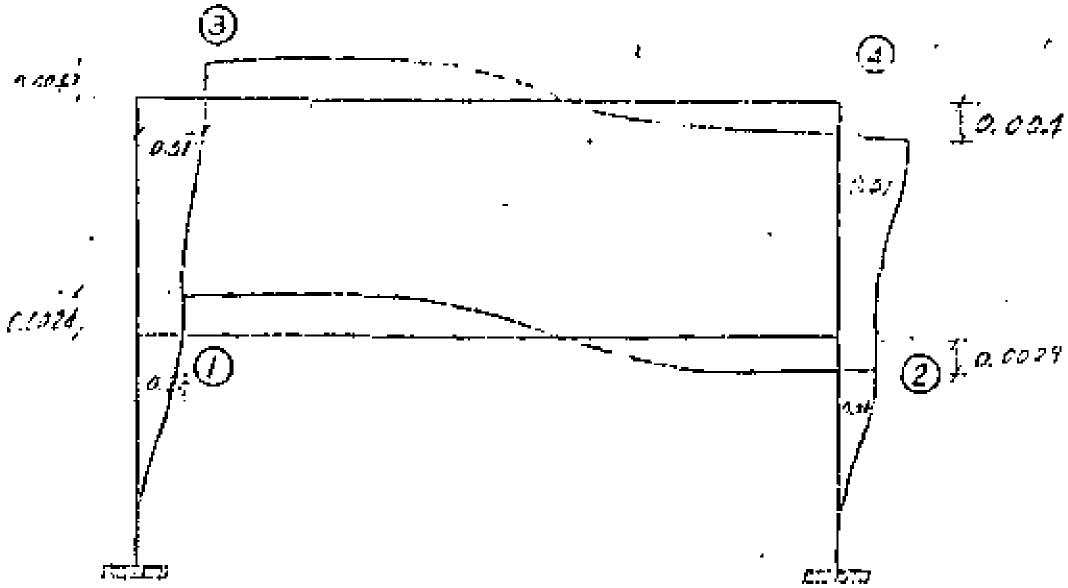
$$\left[\widetilde{\widetilde{K}}\right]\left[d'\right] = \omega^2 \left[M\right]\left[d'\right]$$

donde: $d' = \left[\begin{array}{c} d_{x'} \\ \hline d_{y'} \end{array}\right]$; $M = \left[\begin{array}{c|c} m & 0 \\ \hline 0 & m \end{array}\right]$

$$m_i = 1/2 \ \frac{W_i}{g} = 1/2\left(\frac{30}{9.8}\right) = 1.53 \ \frac{T\text{-}s^2}{m}$$

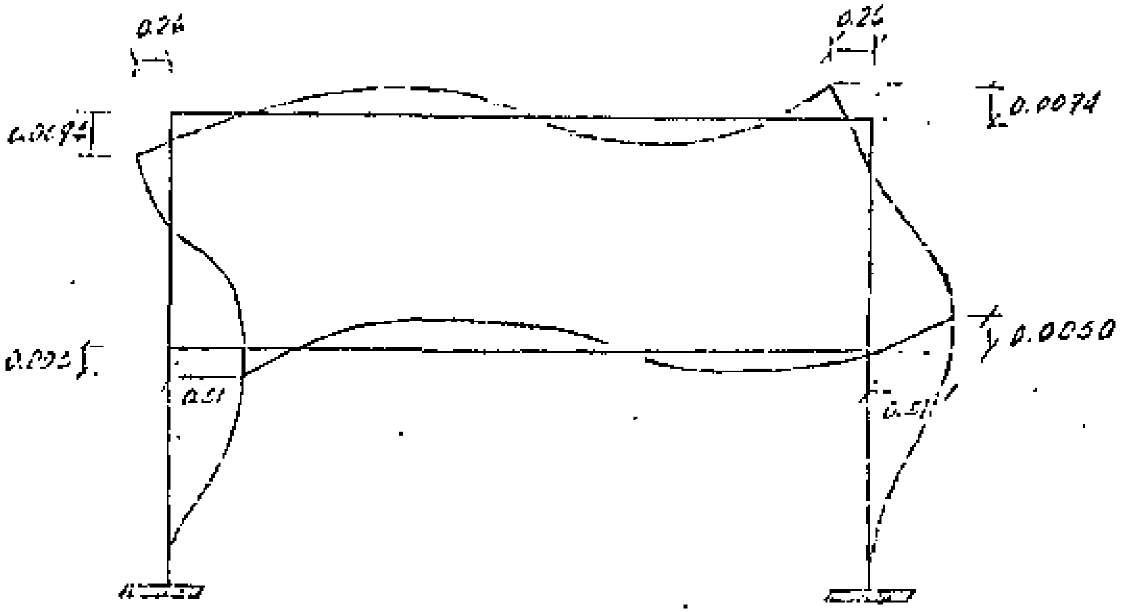
$$\left[M\right] = \left[\begin{array}{cccc|cccc}
1.53 & & & & & & & \\
 & 1.53 & & & & & & \\
 & & 1.53 & & & & & \\
 & & & 1.53 & & & & \\
\hline
 & & & & 1.53 & & & \\
 & & & & & 1.53 & & \\
 & & & & & & 1.53 & \\
 & & & & & & & 1.53
\end{array}\right]$$

1er modo



$T_1 = 0.589$ seg

2° modo



$T_2 = 0.187$ seg.

```
ADIN      11:48     034 11/24/70

PROGRAMA PARA LA OBTENCION DE PERIODOS Y MODOS NATURA-
LES DE VIBRAR. PROGRAMADO POR J. DAMY R.. MEXICO NOVIEMBRE DE 1970


 MATRIZ K
   .36085000E+05 -.36661000E+05 -.56300000E+03 -.65800000E+02
  -.17300000E+02  .17300000E+02  .13600000E+03 -.13600000E+03
   .36085000E+05 -.65800000E+02 -.56300000E+03 -.17300000E+02
   .17300000E+02  .13600000E+03 -.13600000E+03  .37158000E+05
  -.36666000E+05 -.82600000E+02  .82600000E+02 -.11870000E+03
   .11870000E+03  .37158000E+05 -.82600000E+02  .82600000E+02
  -.11870000E+03  .11870000E+03  .98128000E+05 -.12600000E+03
  -.48979000E+05 -.21000000E+02  .98126000E+05 -.21000000E+02
  -.48979000E+05  .49084000E+05 -.83900000E+02  .49084000E+05
 MASAS
   .15300000E+01  .15300000E+01  .15300000E+01  .15300000E+01
   .15300000E+01  .15300000E+01  .15300000E+01  .15300000E+01

 MODO  1
 OMEGA =  10.66051200  PERIODO =     .58938871
   .25623737E+00  .25623737E+00  .51099425E+00  .51099425E+00
   .29466487E-02 -.29466486E-02  .39979469E-02 -.39979469E-02

 MODO  2
 OMEGA =  33.67011100  PERIODO =     .18661017
   .51096589E+00  .51096589E+00 -.25612073E+00 -.25612073E+00
  -.50193496E-02  .50193496E-02 -.93926081E-02  .93926080E-02

 MODO  3
 OMEGA =  110.60265000  PERIODO =     .05680863
   .59273579E-09  .59857214E-09 -.83033499E-09 -.83031192E-09
   .30054070E+00  .30054064E+00  .48628462E+00  .48628453E+00

 MODO  4
 OMEGA =  111.30092000  PERIODO =     .05645223
  -.72833551E-02 -.72833551E-02  .91885201E-02  .91885201E-02
  -.30022755E+00  .30022759E+00 -.48633659E+00  .48633667E+00

 MODO  5
 OMEGA =  219.33793000  PERIODO =     .02864614
   .22869770E+00 -.22869771E+00  .52392246E+00 -.52392246E+00
   .21103940E-13 -.21103940E-13  .39491379E-13 -.39491376E-13

 MODO  6
 OMEGA =  221.34915000  PERIODO =     .02838586
   .52392246E+00 -.52392246E+00 -.22869770E+00  .22869771E+00
  -.11766490E-11  .11766492E-11 -.19060459E-11  .19060462E-11

 MODO  7
 OMEGA =  289.56096000  PERIODO =     .02169901
   .46862398E-09  .46862254E-09  .40180199E-10  .39996558E-10
   .48628428E+00  .48628487E+00 -.30054049E+00 -.30054086E+00

 MODO  8
 OMEGA =  289.78019000  PERIODO =     .02168259
  -.77499100E-03 -.77499100E-03 -.67084765E-04 -.67084765E-04
   .48644342E+00 -.48644283E+00 -.30028316E+00  .30028279E+00
```

Se obtuvo: (Utilizando el servicio de tiempo compartido de G E)

$\omega_1 = 10.66\ 1/s;\quad T_1 = 0.589$ seg.

$\omega_2 = 33.67 \quad T_2 = 0.187$

$\omega_3 = 110.60 \quad T_3 = 0.057$

$\omega_4 = 111.30 \quad T_4 = 0.056$

$\omega_5 = 219.34 \quad T_5 = 0.029$

$\omega_6 = 221.35 \quad T_6 = 0.028$

$\omega_7 = 289.56 \quad T_7 = 0.022$

$\omega_8 = 289.78 \quad T_8 = 0.022$

2) Análisis dinámico, sin considerar acortamiento en trabes

En este caso la matriz $\left[\tilde{\tilde{K}}\right]$ se modifica de acuerdo con lo visto en el Resumen (8) obteniéndose la matriz $\left[K^{=}\right]$

$$
\left[K^{III}\right] = \begin{array}{cccccc}
\Delta_1 & \Delta_2 & d_{y_1'} & d_{y_2'} & d_{y_3'} & d_{y_4'} \\
\hline
2848 & -1257.6 & -34.6 & 34.6 & 272 & -272 \\
-1257.6 & 984 & -34.6 & 34.6 & 272 & -272 \\
 & & 98128 & -128 & -48979 & -21 \\
 & \text{Simétrica} & & 98128 & -21 & -48979 \\
 & & & \text{Simétrica} & 49084 & -83.8 \\
 & & & & & 49084
\end{array}
$$

donde: $\Delta_1' = d_{x_1'} = d_{x_2'}$; $\Delta_2 = d_{x_3'} = d_{x_4'}$

La matriz de masas M será:

$$M = \left[\begin{array}{cc|cccc} 3.06 & & & & & \\ & 3.06 & & & & \\ \hline & & 1.53 & & & \\ & & & 1.53 & & \\ & & & & 1.53 & \\ & & & & & 1.53 \end{array}\right]$$

Obteniendose:

$\omega_1 = 10.66$ 1/s ; $T_1 = 0.589$ seg

$\omega_2 = 33.67$ $T_2 = 0.187$

$\omega_3 = 110.60$ $T_3 = 0.057$

$\omega_4 = 111.30$ $T_4 = 0.056$

$\omega_5 = 289.56$ $T_5 = 0.022$

$\omega_6 = 289.78$ $T_6 = 0.022$

Observe que los modos 1º, 2º, 3º y 4º son iguales al caso anterior y que los modos 5º y 6º son iguales al 7º y 8º del caso anterior.

3) <u>Análisis dinámico, sin considerar acortamiento en trabes y columnas.</u>

En este caso la matriz de rigidez $\left[K^{IV}\right]$ será:

$$\left[K^{IV}\right] = \begin{array}{cc} \Delta_1 & \Delta_2 \\ \left[\begin{array}{c} 2848 \\ -1257.6 \end{array}\right. & \left.\begin{array}{c} -1257.6 \\ 984 \end{array}\right] \end{array}$$

$d_{y_1}' = d_{y_2}' = d_{y_3}' = d_{y_4}' = 0$

$\Delta_1 = d_{x_1}' = d_{x_2}'$

$\Delta_2 = d_{x_3}' = d_{x_4}'$

```
MATRIZ K
   .25480000E+04 -.12576000E+04 -.34600000E+02  .34600000E+02
   .27200000E+03 -.27200000E+03  .98400000E+03 -.16520000E+03
   .16520000E+03 -.23740000E+03  .23740000E+03  .98128000E+05
  -.12800000E+03 -.48979000E+05 -.21000000E+02  .98128000E+05
  -.21000000E+02 -.48979000E+05  .49084000E+05 -.83900000E+02
   .49084000E+05
MASAS
   .30600000E+01  .30600000E+01  .15300000E+01  .15300000E+01
   .15300000E+01  .15300000E+01

MODO  1
OMEGA =  10.66050900  PERIODO =   .58938884
   .25623737E+00  .51099425E+00  .29466487E-02 -.29466486E-02
   .39979469E-02 -.39979469E-02

MODO  2
OMEGA =  33.67011000  PERIODO =   .18661018
   .51096589E+00 -.25612072E+00 -.50193496E-02  .50193496E-02
  -.93926081E-02  .93926080E-02

MODO  3
OMEGA =  110.60265000  PERIODO =   .05680863
   .60344747E-09 -.82412281E-09  .30054070E+00  .30054064E+00
   .48623462E+00  .48628453E+00

MODO  4
OMEGA =  111.30092000  PERIODO =   .05645223
  -.72833551E-02  .91865200E-02 -.30022755E+00  .30022759E+00
  -.48633659E+00  .48633667E+00

MODO  5
OMEGA =  289.56096000  PERIODO =   .02169901
   .46912016E-09  .40092534E-10  .48628428E+00  .48628487E+00
  -.30540349E+00 -.30054086E+00

MODO  6
OMEGA =  289.78019000  PERIODO =   .02168259
  -.77499100E-03 -.67084765E-04  .48644342E+00 -.48644283E+00
  -.30028316E+00  .30028279E+00
```

La matriz de masas $\left[M\right]$ será:

$$M = \begin{bmatrix} 3.06 & \\ & 3.06 \end{bmatrix}$$

Se obtiene: $\omega_1 = 10.71\ 1/s$; $T = 0.587$ seg

$\omega_2 = 33.73\ 1/s$; $T = 0.185$ seg

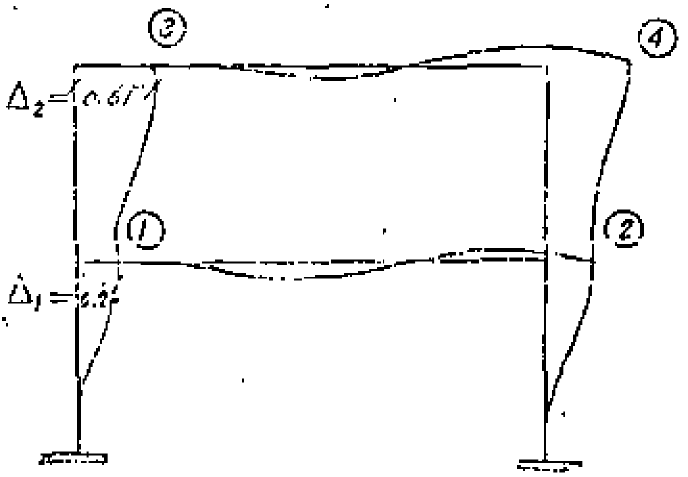
```
Matriz K
  .28 5000OE+04  -.12576000E+04  .98400000E+03
Masas
  .30600000E+01   .30600000E+01

Modo  1
Omega =  10.70541900  Periodo =      .58691633
  .25711735E+00  .51057620E+00

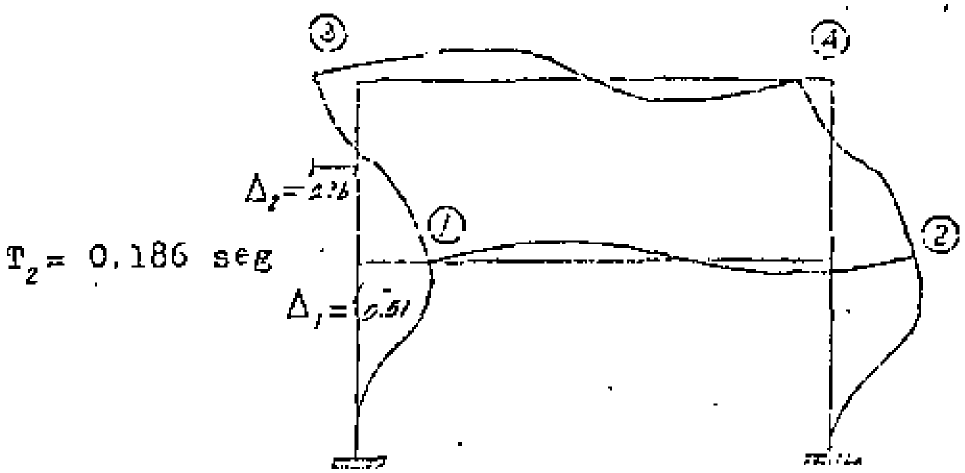
Modo  2
Omega =  33.72953600  Periodo =      .18628140
  .51057620E+00 -.25711735E+00
```

1er modo

$T_1 = 0.587$ seg



2° modo



Observese que son muy parecidos a los dos primeros modos de los casos anteriores.

Nota: En los tres casos anteriores, para obtener el valor de $\left[\phi\right]$, se usara la expresión:

$$\left[\phi\right] = \left[-\widetilde{K}_{22}^{-1}\ \widetilde{K}_{21}\right]\begin{bmatrix} d_{x'} \\ d_{y'} \end{bmatrix}$$

(Ver hojas 98, 99, 100 y 101)

donde $\widetilde{K}_{22}^{-1}$:

$$\widetilde{K}_{22}^{-1} = \begin{bmatrix} 1.3470 & -0.2694 & -0.3362 & 0.1491 \\ & 1.3470 & 0.1491 & -0.3362 \\ & & 2.0472 & -0.5800 \\ & \text{Simétrico} & & 2.0472 \end{bmatrix}$$

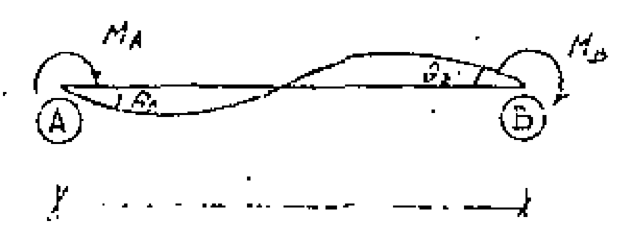
Obtención directa de la matriz $\boxed{K^M}$

La matriz K^M se puede obtener directamente contruyendo una matriz $\mathcal{K}$, que se obtiene:

$$\mathcal{K} = a^T \; k \; a$$

donde k son las rigideces de las barras sin considerar acortamiento, referidos a los momentos extremos M_A, M_B

La matriz k en función de M_A y M_B, para una barra recta de sección uniforme, es:



EI = cte.

$$\begin{bmatrix} M_A \\ \\ M_B \end{bmatrix} = \begin{bmatrix} \dfrac{4EI(1+c)}{L(1+4c)} & \dfrac{2EI(1-2c)}{L(1+4c)} \\ \\ \dfrac{2EI(1-2c)}{L(1+4c)} & \dfrac{4EI(1+c)}{L(1+4c)} \end{bmatrix} \begin{bmatrix} \theta_A \\ \\ \theta_B \end{bmatrix}$$

Para la deducción de esta matriz, vease los Resumenes (3) y (4)

Para nuestro ejemplo:

Barras $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$; $\qquad k = \begin{bmatrix} 2520 & 1210 \\ 1210 & 2520 \end{bmatrix}$

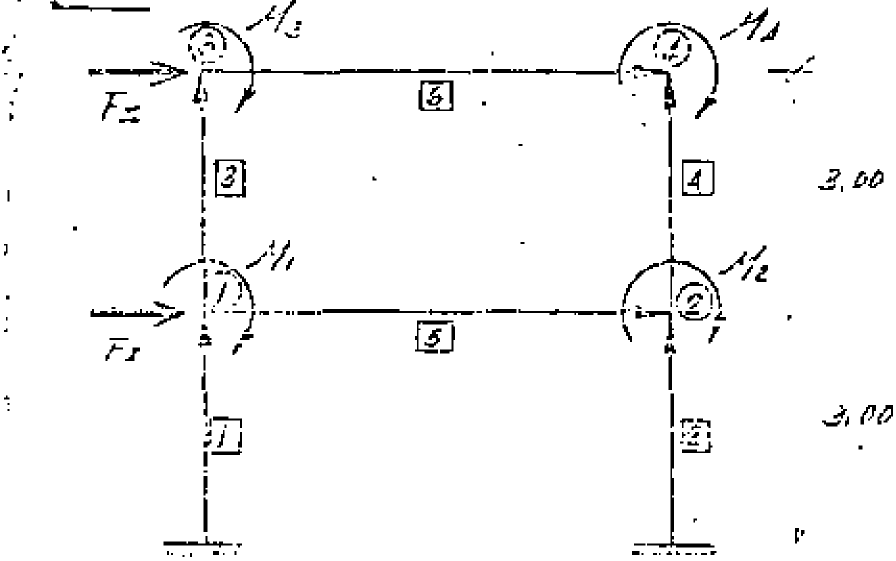
Ver hojas (96, 97, 98 y 99)

Barras $\boxed{5}$, $\boxed{6}$ $\qquad k = \begin{bmatrix} 2980 & 1470 \\ 1470 & 2980 \end{bmatrix}$

Por consiguiente:

$$[k] = \begin{bmatrix} k_1 & & & & & \\ & k_2 & & & & \\ & & k_3 & & & \\ & & & k_4 & & \\ & & & & k_5 & \\ & & & & & k_6 \end{bmatrix}$$

La matriz a^T se obtendrá planteando el equilibrio de los nudos y de los pisos:



$$\begin{bmatrix} F \\ F \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathcal{M}_3 \\ \mathcal{M}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.33 & -0.33 & -0.33 & -0.33 & 0.33 & 0.33 & 0.33 & 0.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.33 & -0.33 & -0.33 & -0.33 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{[a^T]} \begin{bmatrix} M_{h1} \\ M_{k1} \\ M_{h2} \\ M_{k2} \\ M_{h3} \\ M_{k3} \\ M_{h4} \\ M_{k4} \\ M_{h5} \\ M_{k5} \\ M_{h6} \\ M_{k6} \end{bmatrix}$$

Efectuando el producto a^T k a , se obtiene $\mathcal{K}$:

$$a^T \; k \; a = \left[\mathcal{K}\right] = \begin{array}{cc|cccc} \Delta_I & \Delta_{II} & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \left[\begin{array}{cc} 3296 & -1648 \\ -1648 & 1648 \\ \hline 0 & -1236 \\ 0 & -1236 \\ 1236 & -1236 \\ 1236 & -1236 \end{array}\right. & & \left.\begin{array}{cccc} 0 & 0 & 1236 & 1236 \\ -1236 & -1236 & -1236 & -1236 \\ \hline 8020 & 1470 & 1210 & 0 \\ 1470 & 8020 & 0 & 1210 \\ 1210 & 0 & 5500 & 1470 \\ 0 & 1210 & 1470 & 5500 \end{array}\right] \end{array}$$

Nota: Esta matriz puede obtenerse directamente de la matriz $\tilde{K}$ (hoja 98 y 99); por ejemplo:

$$3296 = 38398+38398-36750-36750$$
$$-1648 = -824-824$$
etc...

La matriz $\mathcal{K}$ se puede particionar y obtenerse la siguiente ecuación característica:

$$\left[\begin{array}{c|c} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \hline \mathcal{K}_{21} & \mathcal{K}_{22} \end{array}\right] \left[\begin{array}{c} \Delta \\ \hline \phi \end{array}\right] = \omega^2 \left[\begin{array}{c} M\Delta \\ \hline 0 \end{array}\right]$$

Por lo que se puede eliminar a ϕ, obteniéndose:

$$\left[\mathcal{K}_{11} - \mathcal{K}_{12}\,\mathcal{K}_{22}^{-1}\,\mathcal{K}_{21}\right]\Delta = \omega^2 M\Delta$$

Es obvio que:

$$\left[K^{IV}\right] = \mathcal{K}_{11} - \mathcal{K}_{12}\,\mathcal{K}_{22}^{-1}\mathcal{K}_{21}$$

Obsérvese que $\mathcal{K}_{22} = \tilde{K}_{22}$.

Cuando la matriz $\mathcal{K}_{21}$ tiene menos columnas que $\mathcal{K}_{22}$ no conviene invertir esta última para obtener K^{II} , ya que $\mathcal{K}_{22}^{-1}\,\mathcal{K}_{21}$ tiene por elementos a las raíces del siguiente sistema de ecuaciones:

$$\mathcal{K}_{22}\,X = \mathcal{K}_{21}$$

$$X = \mathcal{K}_{22}^{-1}\,\mathcal{K}_{21}$$

en nuestro ejemplo la obtención de $K_{22}^{-1} K_{21}$ se reduce a resolver el sistema

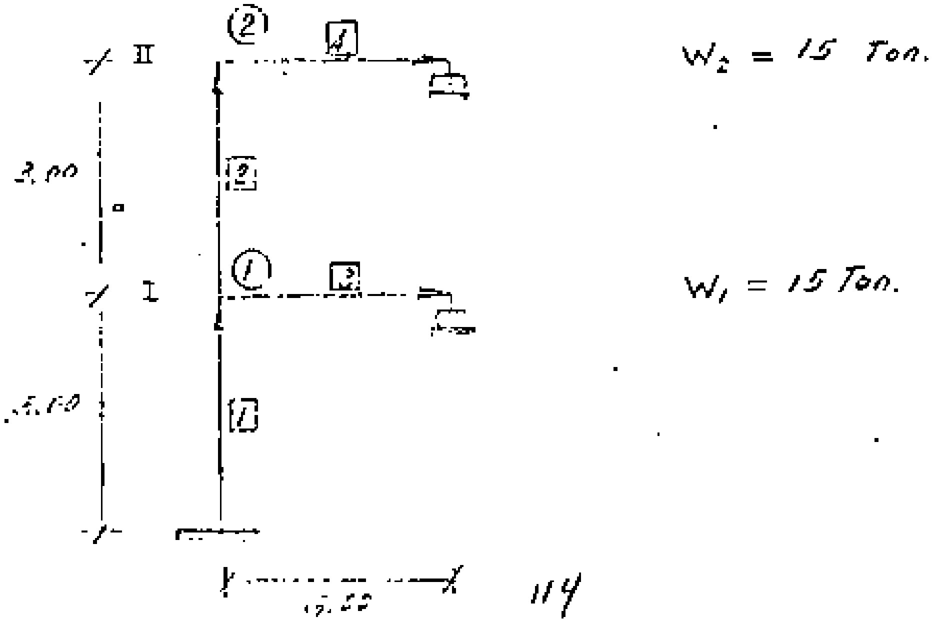
$$\begin{bmatrix} 8020 & 1470 & 1210 & 0 \\ 1470 & 8020 & 0 & 1210 \\ 1210 & 0 & 5500 & 1470 \\ 0 & 1210 & 1470 & 5500 \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0 & -1236 \\ 0 & -1236 \\ 1236 & -1236 \\ 1236 & -1236 \end{bmatrix}$$

este procedimiento simplifica bastante el problema de la contracción de K, cuando se trata de marcos con muchos nudos y pocos niveles.

Simplificación por la simetría de la estructura.

Con las alternativas (1) y (2) (exacta y sin acortamiento en trabes, respectivamente) no se puede llevar a cabo ninguna simplificación por la simetría de la estructura ya que solo los modos 1, 2, 4, 8 (en la alternativa (1)) son completamente antisimétricos, los modos 3, 5, 6, 7 son simétricos y antisimétricos en forma simultanea (Ver hoja 100 A).

Con la alternativa (3) (sin acortamiento en trabes y columnas) si se puede efectuar simplificaciones, ya que los modos corresponden siempre a condiciones antisimétricas. En nuestro ejemplo consideraríamos la siguiente estructura:



La matriz k (en función de M_A y M_B) para las barras $\boxed{3}$ y $\boxed{4}$ son:

$$M_A = \left[\frac{3EI}{L(1+c)}\right] \theta_A$$

$$k_{3,4} = \frac{3EI}{L(1+c)} \quad \ast$$

La matriz $\left[a^T\right]$ será:

$$\left[\begin{array}{c} F_I \\ F_{II} \\ \hline M_1 \\ M_2 \end{array}\right] = \underbrace{\left[\begin{array}{cc|cc|c|c} -0.33 & -0.33 & 0.33 & 0.33 & 0 & 0 \\ 0 & 0 & -0.33 & -0.33 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right]}_{\left[a^T\right]} \left[\begin{array}{c} M_{A1} \\ M_{B1} \\ \hline M_{A2} \\ M_{B2} \\ \hline M_{A3} \\ M_{B3} \end{array}\right]$$

La matriz $\left[k\right]$ será:

$$\left[k\right] = \left[\begin{array}{cc|cc|c|c} 2520 & 1210 & & & & \\ 1210 & 2520 & & & & \\ \hline & & 2520 & 1210 & & \\ & & 1210 & 2520 & & \\ \hline & & & & 4500 & \\ & & & & & 4500 \end{array}\right]$$

Efectuando $a^T k\ a$, se obtiene:

$$[\mathcal{K}] = \left[\begin{array}{cc|cc} 1648 & -824 & 0 & 1236 \\ -824 & 824 & -1236 & -1236 \\ \hline 0 & -1236 & 9540 & 1210 \\ 1236 & -1236 & 1210 & 7070 \end{array}\right]$$

Obtengamos $\mathcal{K}_{12}\,\mathcal{K}_{22}^{-1}\,\mathcal{K}_{21}$:

$$\mathcal{K}_{12}\,\mathcal{K}_{22}^{-1}\,\mathcal{K}_{21} = \begin{bmatrix} 223 & -194 \\ -194 & 330 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 1425 & -630 \\ -630 & 494 \end{bmatrix}$$

* Barras $\boxed{3}$ y $\boxed{4}$ $\quad \dfrac{3EI}{L(1+c)} = \dfrac{3x1.47x10\ x3125x10}{3.00\ x\ 1.0208} = 4500$

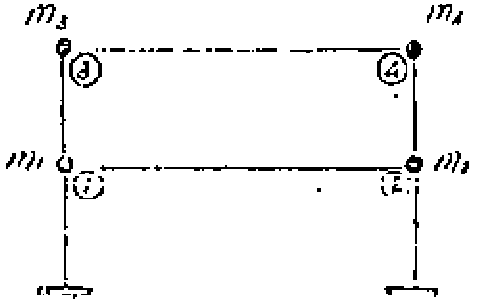
(Nota: $c = 4x0.0052 = 0.0208$)

Observese que: $[K] \approx 1/2\,[K^{II}]$ y las masas son la mitad de las masas de la alternativa (3), por lo tanto los períodos y los modos son los mismos.

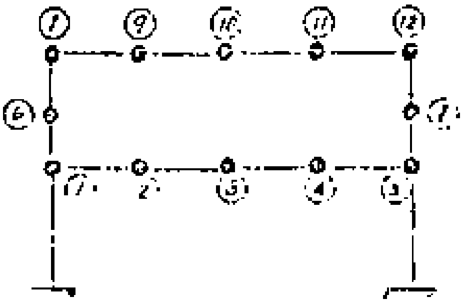
Conclusiones:

Para efectuar el análisis dinámico de una estructura ha sido necesario discretizar a las masas, considerándolas concentradas en los "nudos", porque las matrices de rigideces de las estructuras, están referidas a ellos. Si se desea una mejor aproximación al análisis dinámico de una estructura cualquiera, bastará aumentar el número de "nudos" y por consiguiente el número de masas concentradas.

1ª aproximación.



2ª aproximación.



Este tratamiento es conveniente cuando se quiere considerar la inercia rotacional de las barras, que puede ser considerable en barras largas.

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