EVALUACION DE LA ENSEÑANZA

(FAVOR DE NO PONER SU NOMBRE)

CURSO: FUNDAMENTOS DE ANALISIS MEDIANTE EL METODO DE ELEMENTOS FINITOS

FECHA: MARZO 15 - 19, 1976

(HOJA 1)

PROFESOR Y/O TEMA	Dominio del tema	Eficiencia en el uso de ayudas audiovisuales	Mantenimiento del interés (amenidad, facilidad de expresión, comunicación con los asistentes).	Puntualidad
INTRODUCCION: PORFIRIO BALLESTEROS B. ANTECEDENTES Y EDO. ACTUAL DEL CONOCIMIENTO DEL METODO DE ANALISIS POR ELEMENTOS FINITOS				
FUNDAMENTOS DE ALGEBRA MATRICIAL PORFIRIO BALLESTEROS B.				
PROPIEDADES DE RIGIDEZ DEL ELEMENTO PORFIRIO BALLESTEROS B.				
METODO DIRECTO DE LAS RIGIDECES PORFIRIO BALLESTEROS B.				
APLICACION TRIDIMENSIONAL DEL ELEMENTO VIGA PORFIRIO BALLESTEROS B.				
FUNDAMENTOS DE TEORIA DE ELASTICIDAD PORFIRIO BALLESTEROS B.				
METODO DIRECTO EN LA FORMULACION DE LA RIGIDEZ DEL ELEMENTO RICHARD H. GALLAGHER				
PRINCIPIO DEL TRABAJO VIRTUAL RICHARD H. GALLAGHER				
TRABAJO VIRTUAL Y ENERGIA POTENCIAL RICHARD H. GALLAGHER				
SESION DE APLICACION. SOLUCION DE PROBLEMAS BIDIMENSIONALES PORFIRIO BALLESTEROS B.				

ESCALA DE EVALUACION DE 1 A 10

EVALUACION DE LA ENSEÑANZA

(FAVOR DE NO PONER SU NOMBRE)

CURSO: FUNDAMENTOS DE ANALISIS MEDIANTE EL METODO DE ELEMENTOS FINITOS

FECHA: MARZO 15 - 19, 1976

(HOJA 2)

PROFESOR Y/O TEMA	Dominio del tema	Eficiencia en el uso de ayudas audiovisuales	Mantenimiento del interés (amenidad, facilidad de expresión, comunicación con los asistentes).	Puntualidad
ANALISIS TRIDIMENSIONAL RICHARD H. GALLAGHER				
FLEXION DE PLACAS Y ANALISIS DE CASCARONES RICHARD H. GALLAGHER				
ANALISIS PRACTICO. EJEMPLOS SUBESTRUCTURACION Y COND. DE APOYO RICHARD H. GALLAGHER				

ESCALA DE EVALUACION DE 1 A 10

EVALUACION DEL CURSO

CONCEPTO		EVALUACION
1.	APLICACION INMEDIATA DE LOS CONCEPTOS EXPUESTOS	
2.	CLARIDAD CON QUE SE EXPUSIERON LOS TEMAS	
3.	GRADO DE ACTUALIZACION LOGRADO CON EL CURSO	
4.	CUMPLIMIENTO DE LOS OBJETIVOS DEL CURSO	
5.	CONTINUIDAD EN LOS TEMAS DEL CURSO	
6.	CALIDAD DE LAS NOTAS DEL CURSO	
7.	GRADO DE MOTIVACION LOGRADO CON EL CURSO	

ESCALA DE EVALUACION DE 1 A 10

1. ¿Qué le pareció el ambiente del Centro de Educación Continua?

Muy agradable ☐ Agradable ☐ Desagradable ☐

2. Medio de comunicación por el que se enteró del curso:

Periódico Excélsior ☐
Periódico Novedades ☐
Folleto del Curso ☐
Cartel mensual ☐
Radio Universidad ☐
Comunicación carta, teléfono, verbal, etc. ☐

3. Medio de transporte utilizado para venir al Palacio de Minería:

Automóvil particular ☐ Metro ☐ Otro medio ☐

4. ¿Qué cambios haría usted en el programa para tratar de perfeccionar el curso? ____________________

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5. ¿Recomendaría el curso a otras personas? Si ☐ No ☐

6. ¿Qué curso le gustaría que ofreciera el Centro de Educación Continua? ____________________

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7. ¿Qué servicios desearía que tuviese el CEC para los asistentes a cursos? ____________________

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8. Otras sugestiones:

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Y

MARC ANALYSIS RESEARCH CORPORATION

Curso-Seminario intensivo

"TEMAS AVANZADOS DE ANALISIS POR ELEMENTOS FINITOS"

Marzo 22-26, 1976

PROGRAMA

LUNES 22

8:00 - 8:45	Inscripciones	
8:45- 9:00	Apertura del curso	Octavio Rascón Chávez Timothy J. Dwyer
9:00 - 10:30	Estudio y categorización de los métodos computacionales de análisis de ingeniería.	O. C. Zienkiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Formulaciones alternativas en mecánica estructural.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	H. Gallag
14:30 - 16:00	Formulaciones mixtas o híbridas del método de elementos finitos.	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Métodos de incremento de tiempo	O. C. Zienkiewicz

MARTES 23

9:00 - 10:30	Flujo viscoso.	O. C. Zienckiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Problemas de ingeniería ambiental.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Ecuaciones constitutivas inelásticas	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Alogaritmos de análisis por el método de elementos finitos en medios inelásticos.	R. H. Gallagher

MIERCOLES 24

9:00 - 10:30	Análisis de mecánica de propagación de grietas.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis de cascarón por el método de elementos finitos.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Visco-plasticidad.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Mecánica de suelos y rocas.	O. C. Zienkiewicz

JUEVES 25

9:00 - 10:30	Análisis por medio de elementos finitos en problemas de pandeo con desplazamientos grandes.	O. C. Zienkiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis [illegible]	arcal
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Revisión y crítica del programa MARC	T. J. Dwyer
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Caso aplicación: análisis de los componentes de reactor nuclear.	P. V. Marcal

VIERNES 26

9:00 - 10:30	Ecuaciones constitutivas del concreto.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Aplicaciones de elementos finitos en problemas de concreto.	P. V. Marcal
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Procedimientos de solución de valores en la frontera por el método de elementos finitos.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 17:00	El método de elementos finitos en el análisis de presas.	O. C. Zienkiewicz
17:00 - 18:00	Discusión final y preguntas.	O. C. Zienkiewicz P. V. Marcal T. J. Dwyer P. Ballesteros
18:00	Clausura	Octavio Rascón Chávez Pedro Martínez Pereda

ADVANCED TOPICS SEMINAR
MEXICO CITY
MARCH 22-26, 1976.

MONDAY, MARCH 22, 1976.

1) An Overview and Categorization of Computational Methods in Engineering Analysis	Zienkiewicz
2) Alternative Formulations is Structural Mechanics	Gallagher
3) Mixed and Hybrid F.E.M. Formulations	Gallagher
4) Time-Stopping Methods	Zienkiewicz

TUESDAY, MARCH 23, 1976.

5) Viscous Flows	Zienkiewicz
6) Environmental Problems	Gallagher or Zienkiewicz
7) Constitutive Equations for Inelasticity	Gallagher
8) F.E. Analysis Algorithms for Inelastic Analysis	Gallagher

WEDNESDAY, MARCH 24, 1976.

9) Shell Analysis by F.E.M.	Gallagher or Ballesteros
10) Fracture Mechanics Analysis	Gallagher
11) Viscoplasticity	Zienkiewicz
12) Soil and Rock Mechanics	Zienkiewicz

THURSDAY, MARCH 25, 1976.

13) F.E.M. Analysis for Buckling and Large Displacement	Marcal
14) Analysis for Combined Nonlinear and Dynamic Behavior	Marcal
15) MARC Review and Critique	Dwyer
16) Case Study: Nuclear Reactor Component Analysis	Marcal

FRIDAY, MARCH 26, 1976.

17) Constitutive equations of Concrete and Reinforced Concrete.	Ballesteros
18) Boundary Solution Procedures and the F.E.M.	Zienkiewicz
19) F.E.M. in Dam Analysis	Zienkiewicz
20) Final discussion and questions.	Zienkiewicz Gallagher Marcal Dwyer Ballesteros.

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WEDNESDAY, MARCH 24, 1976.

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MONDAY, MARCH 22, 1976.

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2) Alternative Formulations is Structural Mechanics	Gallagher
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TUESDAY, MARCH 23, 1976.

5) Viscous Flows	Zienkiewicz
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WEDNESDAY, MARCH 24, 1976.

9) Shell Analysis by F.E.M.	Gallagher or Ballesteros
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11) Viscoplasticity	Zienkiewicz
12) Soil and Rock Mechanics	Zienkiewicz

THURSDAY, MARCH 25, 1976.

13) F.E.M. Analysis for Buckling and Large Displacement	Marcal
14) Analysis for Combined Nonlinear and Dynamic Behavior	Marcal
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16) Case Study: Nuclear Reactor Component Analysis	Marcal

FRIDAY, MARCH 26, 1976.

17) Constitutive equations of Concrete and Reinforced Concrete.	Ballesteros
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Marzo 22-26, 1976

Palacio de Minería
Tacuba 5, primer piso. México 1, D. F.
Tels: 521-40-23 521-73-35 5123-123

PROGRAMA

LUNES 22

8:00 - 8:45	Inscripciones	
8:45- 9:00	Apertura del curso	Octavio Rascón Chávez Timothy J. Dwyer
9:00 - 10:30	Estudio y categorización de los métodos computacionales de análisis de ingeniería.	O. C. Zienkiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Formulaciones alternativas en mecánica estructural.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Formulaciones mixtas o híbridas del método de elementos finitos.	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Métodos de incremento de tiempo	O. C. Zienkiewicz

MARTES 23

9:00 - 10:30	Flujo viscoso.	O. C. Zienckiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Problemas de ingeniería ambiental.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Ecuaciones constitutivas inelásticas	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Alogaritmos de análisis por el método de elementos finitos en medios inelásticos.	R. H. Gallagher

MIERCOLES 24

9:00 - 10:30	Análisis de mecánica de propagación de grietas.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis de cascarón por el método de elementos finitos.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Visco-plasticidad.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Mecánica de suelos y rocas.	O. C. Zienkiewicz

JUEVES 25

9:00 - 10:30	Análisis por medio de elementos finitos en problemas de pandeo con desplazamientos grandes.	P. V. Marcal
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis combinado de no-linealidad y comportamiento dinámico.	P. V. Marcal
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14:30 - 16:00	Revisión y crítica del programa MARC	T. J. Dwyer
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Caso aplicación: análisis de los componentes de reactor nuclear.	P. V. Marcal

VIERNES 26

9:00 - 10:30	Ecuaciones constitutivas del concreto.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Aplicaciones de elementos finitos en problemas de concreto.	P. V. Marcal
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Procedimientos de solución de valores en la frontera por el método de elementos finitos.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 17:00	El método de elementos finitos en el análisis de presas.	O. C. Zienkiewicz
17:00 - 18:00	Discusión final y preguntas.	O. C. Zienkiewicz P. V. Marcal T. J. Dwyer P. Ballesteros
18:00	Clausura	Octavio Rascón Chávez Pedro Martínez Pereda

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TUESDAY, MARCH 23, 1976.

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WEDNESDAY, MARCH 24, 1976.

9) Shell Analysis by F.E.M.	Gallagher or Ballesteros
10) Fracture Mechanics Analysis	Gallagher
11) Viscoplasticity	Zienkiewicz
12) Soil and Rock Mechanics	Zienkiewicz

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centro de educación continua
división de estudios superiores
facultad de ingeniería, unam

Y

MARC ANALYSIS RESEARCH CORPORATION

Curso-Seminario Intensivo

"TEMAS AVANZADOS DE ANALISIS POR ELEMENTOS FINITOS"

Marzo 22-26, 1976

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FRIDAY, MARCH 26, 1976.

17) Constitutive equations of Concrete and Reinforced Concrete. Aplicaciones en prob. de Concreto	Ballesteros P.V. Marçal
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FUNDAMENTOS DE ANALISIS MEDIANTE EL METODO DE ELEMENTOS FINITOS

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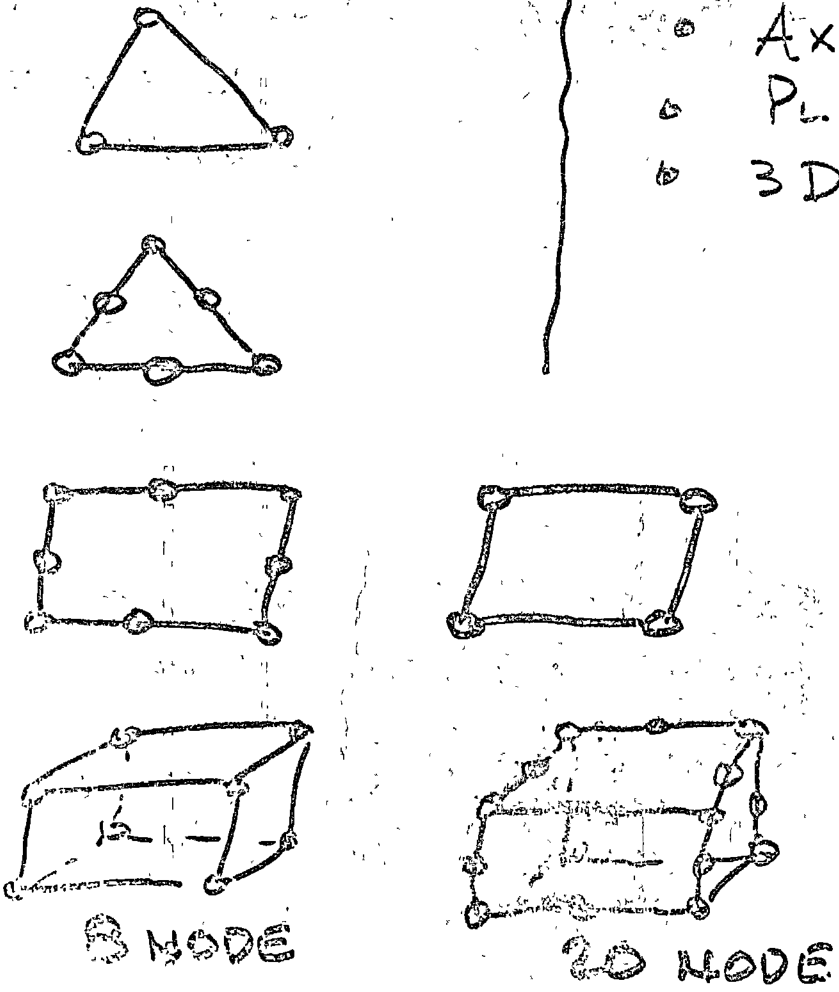
Tels.: 521-40-23 521-73-35 512-31-23

STATE OF THE ART
OF
F.E.M.
FOR
HEAT TRANSFER

I FINITE ELEMENTS

GEOMETRIES
- AXISYMMETRIC
- PLANER
- 3D SOLID



BOUNDARY CONDITIONS

- LINEAR
- NON LINEAR
  - TEMPERATURE DEPENDENT FLUENCES (FLUXES)
  - TEMPERATURE DEPENDENT FILM COEFFICIENTS
  - RADIATION

HEAT TRANSFER MAY BE
LINEAR STEADY STATE

HEAT TRANSFER MAY BE
LINEAR TRANSIENT

HEAT TRANSFER MAY BE
NON LINEAR TRANSIENT

TRANSIENT ANALYSIS IS
PERFORMED IN A STEP
BY STEP PROCEDURE.

MARC METHOD USED
IS A

CRANK-NICKELSON FINITE
[redacted] DIFFERENCE
SCHEME

MARC-HEAT Input (Version E, G.)

Card No.	Contents	Remarks
1	Title	R
2	Control Variables.	R
3	Control Variables	R.
4	Connectivity.	(MESHR) O
5	Material Properties.	R, (N1)
6 }	(@ Ref Temp.)	R, → (N1)
7 }		R,
8	Coordinates.	(MESHR) O
9	Temp. Dependent Functions.	R.
10 }		} (NOYPT) O
11 }	Functions.	O
12 }		(NOWKH) O
13	Time Interval.	R.
14	Time Sequence	R. (NTBC)

SAMPLE PROBLEM (VERSION H).

```
[illegible]
$CHANGE,********.
JOB,CM45000,CL130000,P4,T100.
ANULIM(100)
RFL,55000.
FTN.
RFL,6000.
ATTACH(NGO,MARCCDCNRP,MR=1)
REWIND(LGO)
COPYL(NGO,LGO,TAPE29)
ATTACH(JCL,MAROJCL,MR=1)
ATTACH(FUDD,MARCFUDD,MR=1)
ATTACH(PRE,MARCPHEH,MR=1)
ATTACH(HANLOAD,MARCLOADEH,MR=1)
MAP(OFF),
RFL,20000.
CCLINK(JCL)
        SUBROUTINE   CREDE (DTDL,M,NSTRES,NEQST,NSTATS)
        DIMENSION DTDL(NSTATS,NEQST,NSTRES)
        RETURN
        END
        SUBROUTINE   FLUX (F,X1,X2,NN,N,TIME)
        RETURN
        END
        SUBROUTINE   FILM (H,TINF,TS,N,TIME)
        RETURN
        END
        SUBROUTINE   ANKOND (COND,CISO,N,NN,ID)
        DIMENSION COND(ID,ID)
        RETURN
        END
TITLE      HEAT TRANSFER TEST PROB (INTERNAL HEAT GENERATION)
SIZING            6000    5   12    4    5   39
HEAT          1
FLUXES        1
END
CONNECTIVITY
    5
    1    39    1    3    4    2
    2    39    3    5    6    4
    3    39    5    7    8    6
    4    39    7    9   10    8
    5    39    9   11   12   10
COORDINATES
    2    12
    10.0        0.0
    20.0        0.2
    30.2        0.0
    40.2        0.2
    50.4        0.0
    60.4        0.2
    70.6        0.0
    80.6        0.2
    90.8        0.0
   100.8        0.2
   111.0        0.0
   121.0        0.2
PROPERTY
    1
1.          1.          1.
    1    5
BOUNDARY CONDITIONS
```

H

```
                         NODAL POINT DATA

 TOTAL TEMPERATURES

      1   100.00
      2   100.00
      3   179.99
      4   179.99
      5   219.98
      6   219.98
      7   219.98
      8   219.98
      9   179.99
     10   179.99
     11   100.00
     12   100.00

                                   END OF INCREMENT 1
```

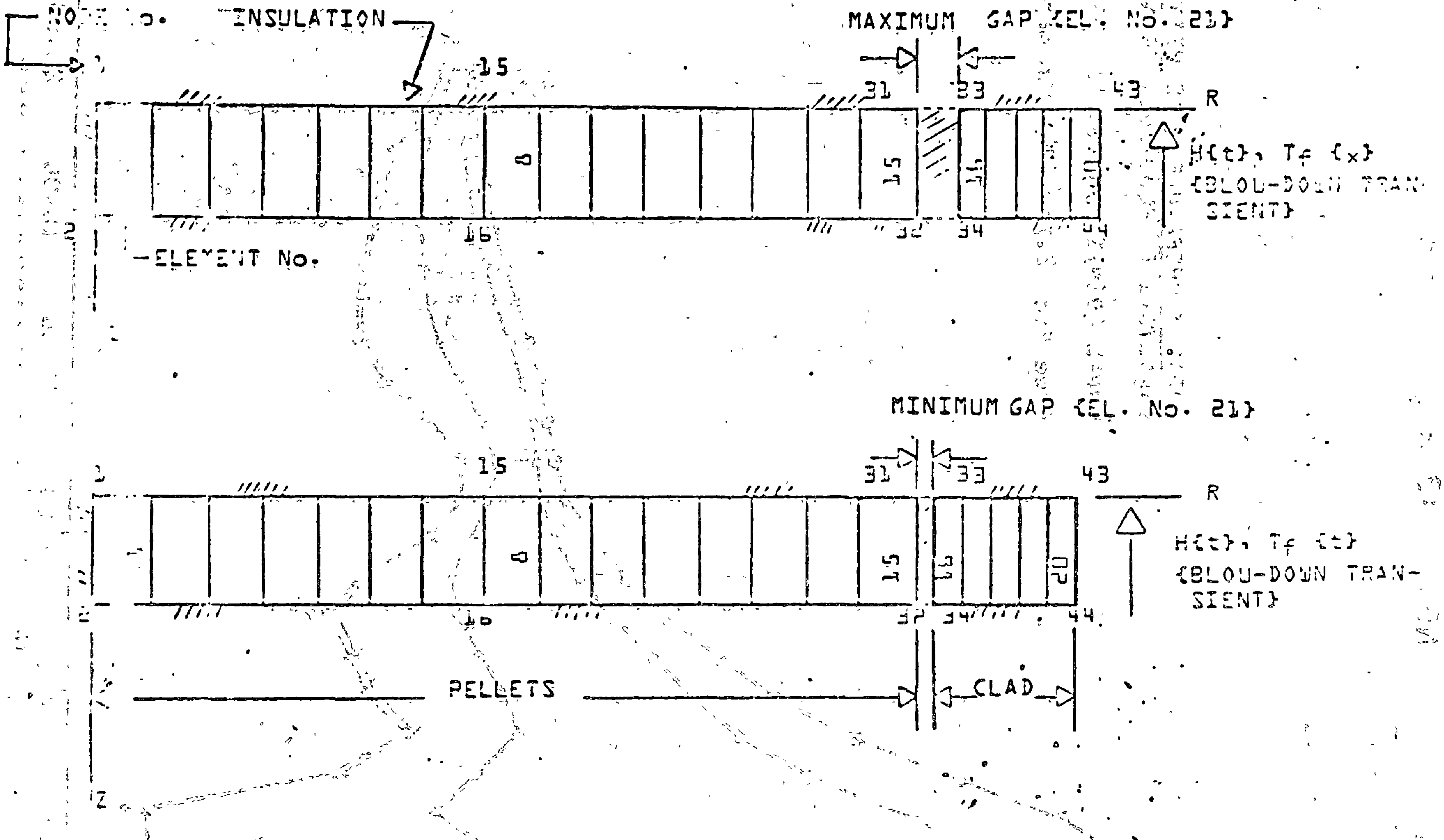
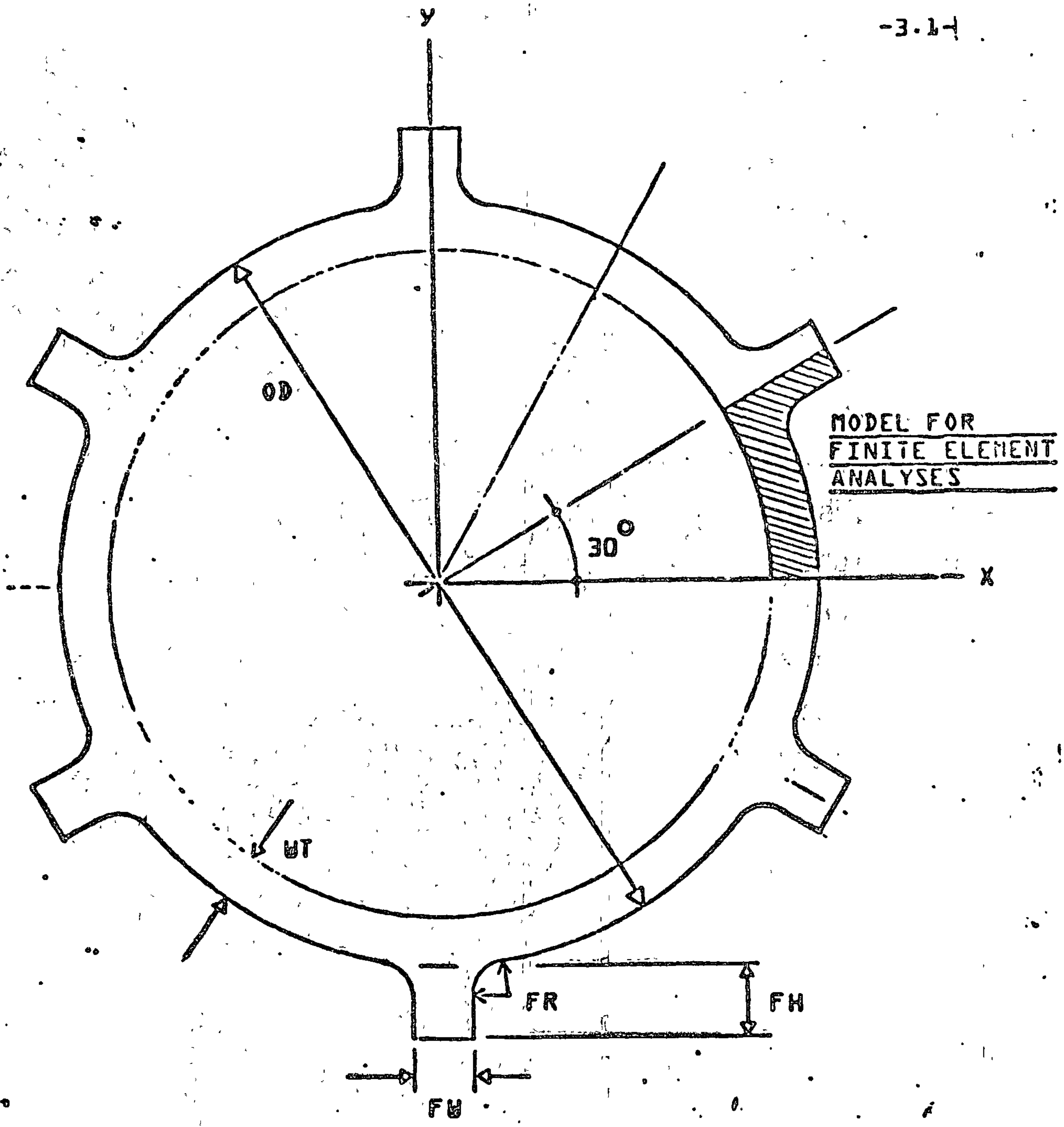


FIG. 3 FINITE ELEMENT MODEL {THERMAL ANALYSIS}



OD: OUTER DIAMETER
UT: WALL THICKNESS
FH: FIN HEIGHT
FW: FIN WIDTH
FR: FILLET RADIUS

FIG. [illegible] AXIALLY FINNED FUEL CLAD CROSS SECTION

Marc Analysis Research Corporation

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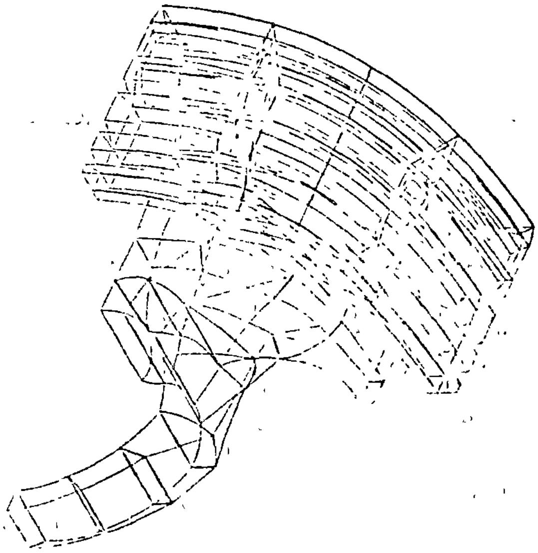

Figure 1

A linear elastic analysis indicated that the most highly stressed areas were at the wrist pin-pin bore interface and at the oil cooling channel surface, just inside the ring land area at the top of the piston.

The MARC system was used to generate the model mesh, the thermal data and the stress analysis results. One hundred and twenty-eight isoparametric twenty node brick elements were used to model the piston and the piston pin. Special modeling considerations included use of an elastic foundation stiffness in place of the crank rod and tying constraints for the interaction of the pin and the piston. The final model resulted in 1002 node points with a total of 2673 reduced degrees of freedom. The maximum nodal half-bandwidth of the optimized mesh was 175. Figure 2 is an isotherm plot of the upper piston surface.

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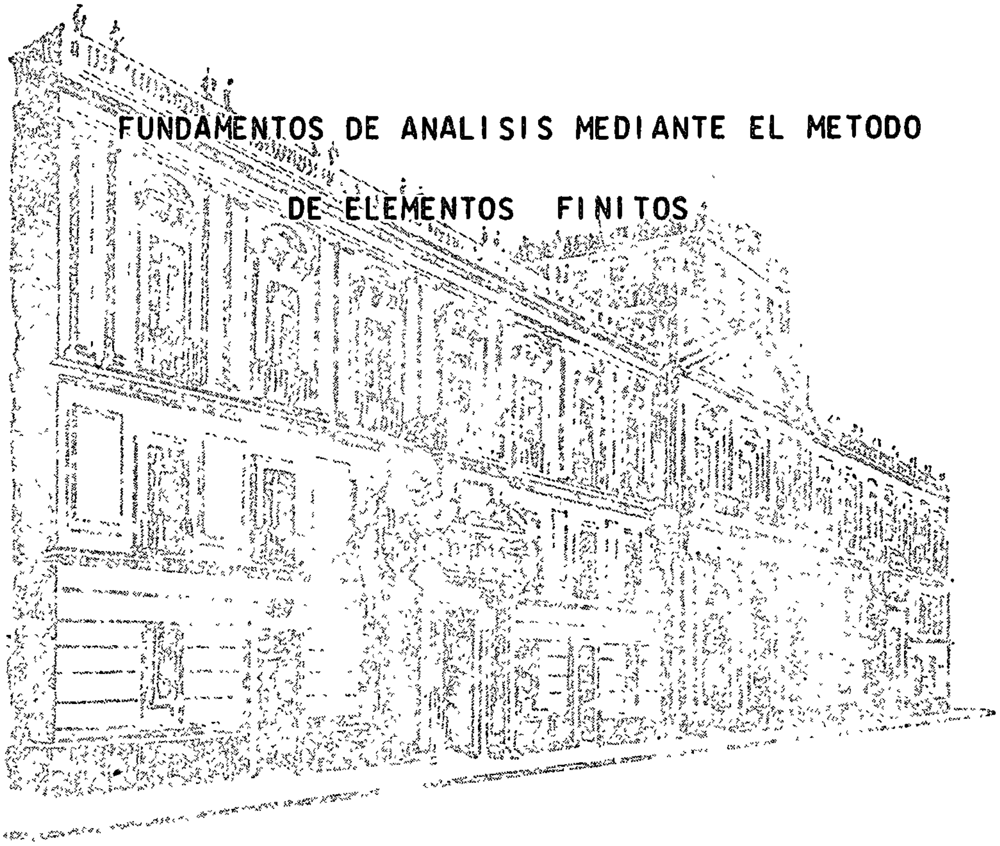
FUNDAMENTOS DE ANALISIS MEDIANTE EL METODO DE ELEMENTOS FINITOS



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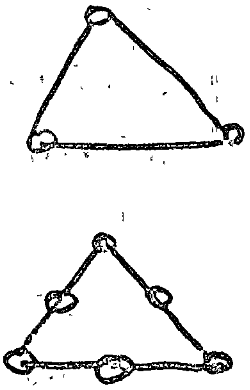
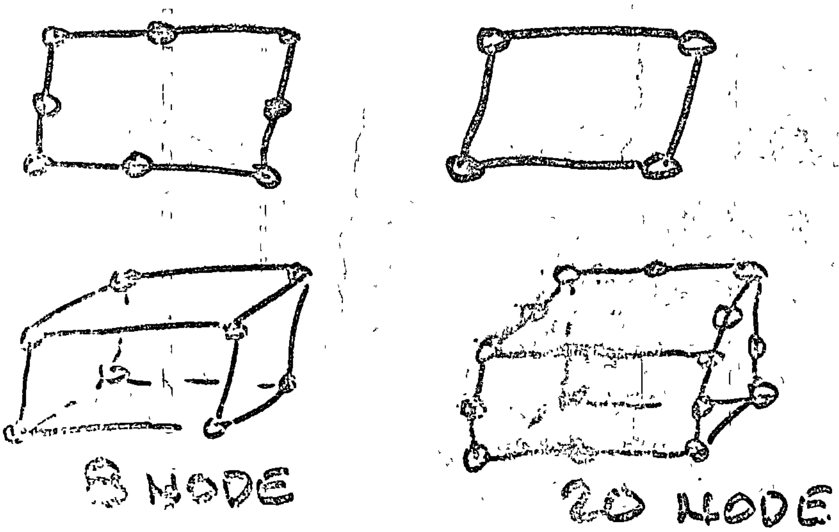
STATE OF THE ART
OF
F.E.M.
FOR
HEAT TRANSFER

FINITE ELEMENTS

GEOMETRIES

- AXISYMMETRIC
- PLANER
- 3D SOLID





BOUNDARY CONDITIONS

- LINEAR
- NON LINEAR
  - TEMPERATURE DEPENDENT FLUENCES (FLUXES)
  - TEMPERATURE DEPENDENT FILM COEFFICIENTS
  - RADIATION

HEAT TRANSFER MAY BE LINEAR STEADY STATE

HEAT TRANSFER MAY BE LINEAR TRANSIENT

HEAT TRANSFER MAY BE NON LINEAR TRANSIENT

TRANSIENT ANALYSIS IS PERFORMED IN A STEP BY STEP PROCEDURE.

MARC METHOD USED IS A

CRANK-NICKELSON FINITE [redacted] DIFFERENCE SCHEME

MARC-HEAT INPUT (VERSION E, G.)

CARD No.	CONTENTS	REMARKS
1	TITLE	R
2	CONTROL VARIABLES	R
3	CONTROL VARIABLES	R
4	CONDUCTIVITY	(MESHR) O
5	MATERIAL PROPERTIES	R, (N1)
6	(@ REF TEMP.)	R, (N1)
7		R,
8	COORDINATES	(MESHR) O
9	TEMP. DEPENDENT FUNCTIONS	R
10		(NOYPT) O
11	FUNCTIONS	O
12		(NOWKH) O
13	TIME INTERVAL	R
14	TIME SEQUENCE	R (NTBC)

SAMPLE PROBLEM (VERSION H)3

```
/ROUTE,LE,P,E.
$CHARGE,******,HAH.
JOB,CM45000,CL130000,P4,T100.
ANULIM(100)
RFL,55000.
FTN.
RFL,6000.
ATTACH(NGO,MARCCDCNAP,MR=1)
REWIND(LGO)
COPYL(NGO,LGO,TAPE29)
ATTACH(JCL,MARCJCL,MR=1)
ATTACH(FUDD,MARCFUDQ,MR=1)
ATTACH(PRE,MARCPREH,MR=1)
ATTACH(RANLDAD,MARCLDADE4,MR=1)
MAP(OFF).
RFL,20000.
ECLINK(JCL)
      SUBROUTINE   CREDE (DTOL,M,NSTRES,NEQST,NSTATS)
      DIMENSION OTOL(NSTATS,NEQST,NSTRES)
      RETURN
      END
      SUBROUTINE   FLUX (F,X1,X2,NN,N,TIME)
      RETURN
      END
      SUBROUTINE   FILM (H,TINF,TS,N,TIME)
      RETURN
      END
      SUBROUTINE   ANKOND (COND,CISO,N,NN,ID)
      DIMENSION COND(ID,ID)
      RETURN
      END
TITLE       HEAT TRANSFER TEST PROB (INTERNAL HEAT GENERATION)
SIZING            6000     5    12     4     5    39
HEAT               1
FLUXES             1
END
CONNECTIVITY
    5
    1   39     1     3     4     2
    2   39     3     5     6     4
    3   39     5     7     8     6
    4   39     7     9    10     8
    5   39     9    11    12    10
COORDINATES
    2   12
   10.0       0.0
   20.0       0.2
   30.2       0.0
   40.2       0.2
   50.4       0.0
   60.4       0.2
   70.6       0.0
   80.6       0.2
   90.8       0.0
  100.8       0.2
  111.0       0.0
  121.0       0.2
PROPERTY
    1
1.        1.        1.                                                    1
    1     5
BOUNDARY CONDITIONS
```

```
          H                 NODAL POINT DATA

  TOTAL TEMPERATURES

        1   100.00
        2   100.00
        3   179.99
        4   179.99
        5   219.98
        6   219.98
        7   219.98
        8   219.98
        9   179.99
       10   179.99
       11   100.00
       12   100.00


                                END OF INCREMENT 1
```

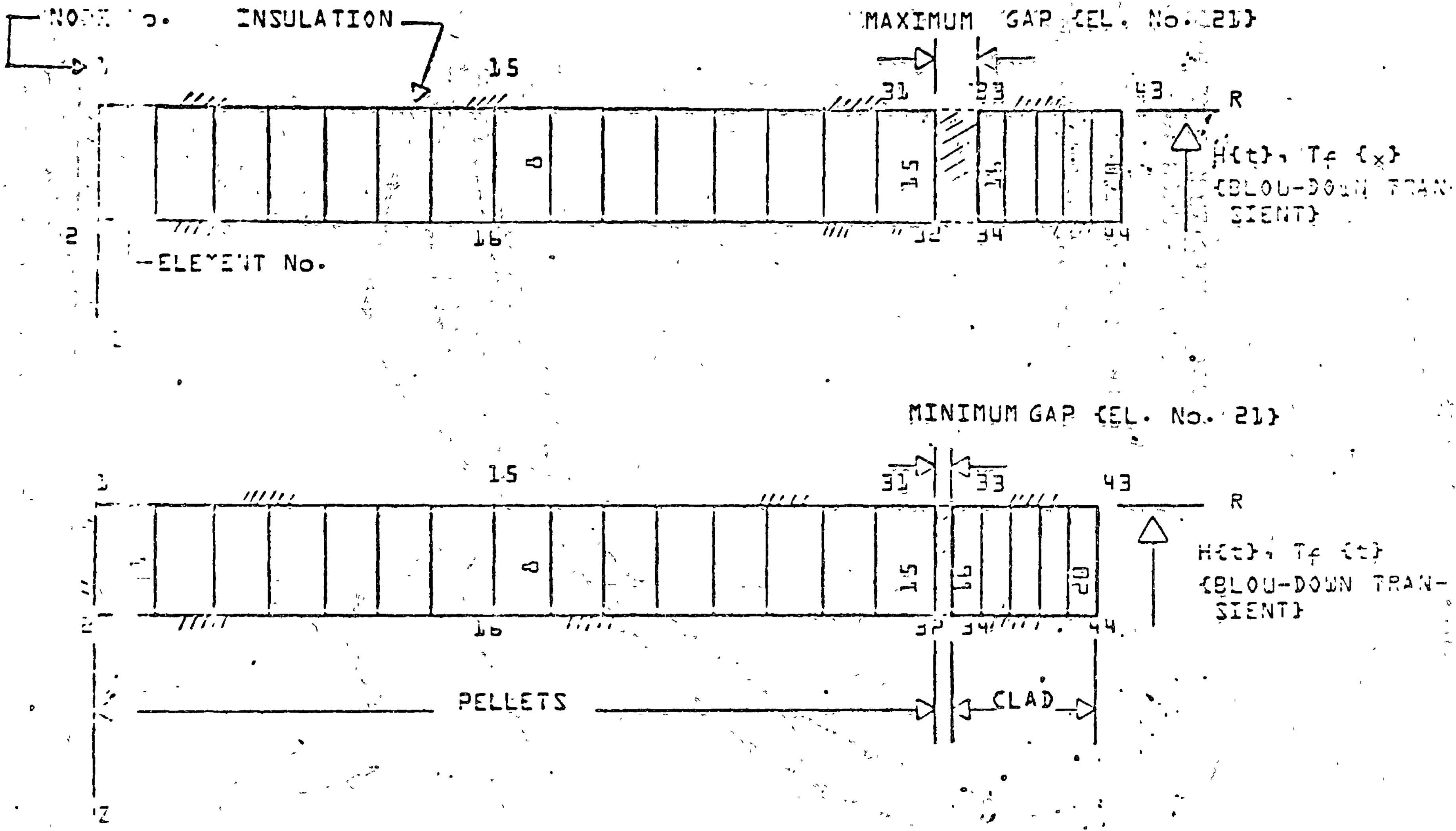
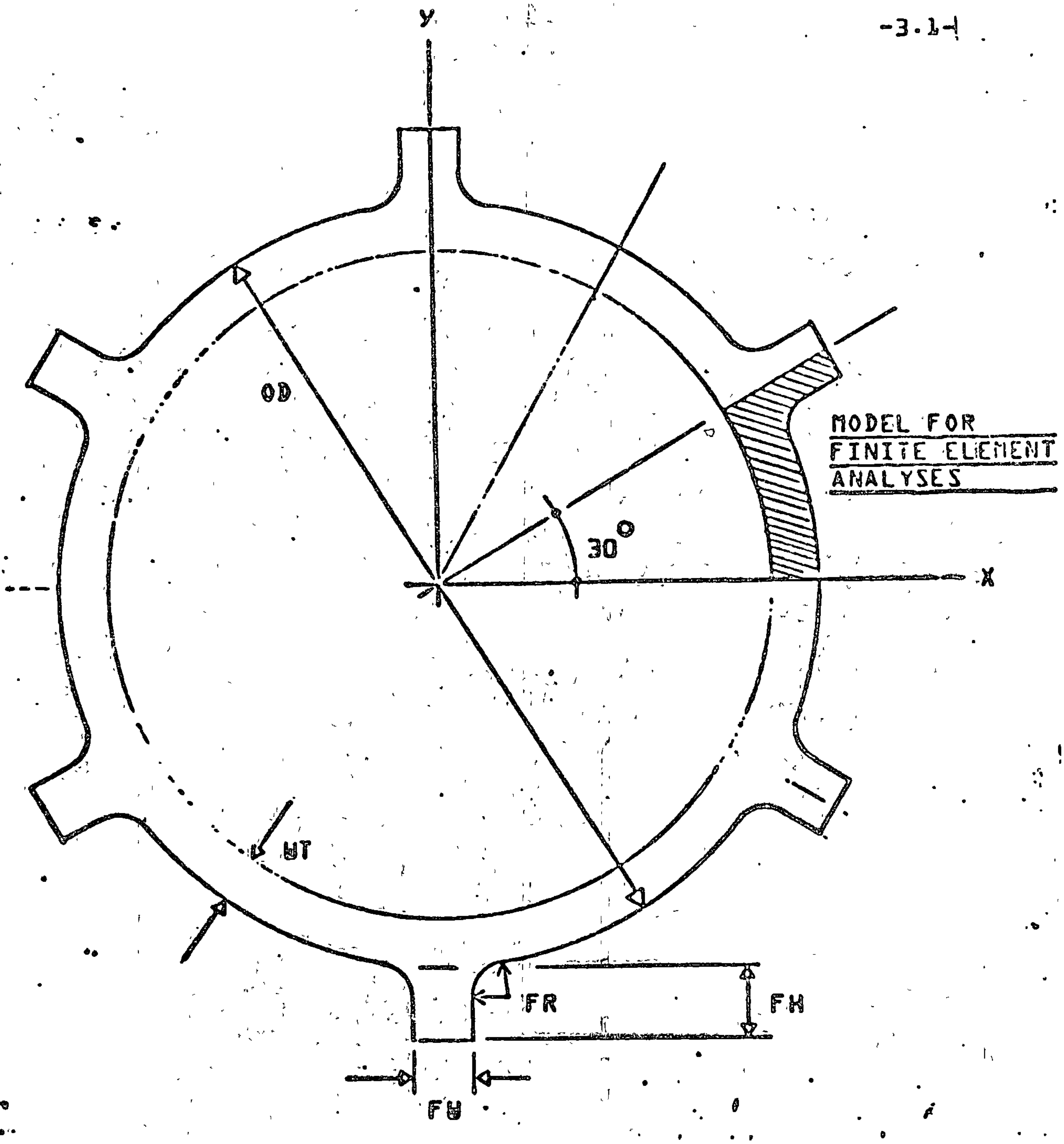


FIG. 3 1D FINITE ELEMENT MODEL (THERMAL ANALYSIS)



OD: OUTER DIAMETER
WT: WALL THICKNESS
FH: FIN HEIGHT
FW: FIN WIDTH
FR: FILLET RADIUS

FIG. [illegible] AXIALLY FINNED FUEL CLAD CROSS SECTION

Marc Analysis Research Corporation

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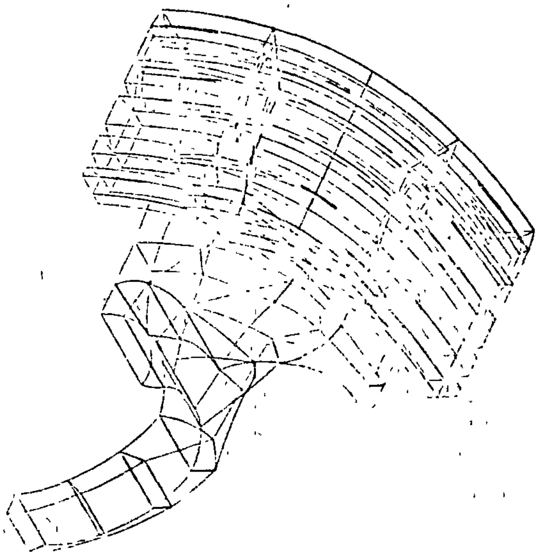

Figure 1

A linear elastic analysis indicated that the most highly stressed areas were at the wrist pin-pin bore interface and at the oil cooling channel surface, just inside the ring land area at the top of the piston.

The MARC system was used to generate the model mesh, the thermal data and the stress analysis results. One hundred and twenty-eight isoparametric twenty node brick elements were used to model the piston and the piston pin. Special modeling considerations included use of an elastic foundation stiffness in place of the crank rod and tying constraints for the interaction of the pin and the piston. The final model resulted in 1002 node points with a total of 2673 reduced degrees of freedom. The maximum nodal half-bandwidth of the optimized mesh was 175. Figure 2 is an isotherm plot of the upper piston surface.

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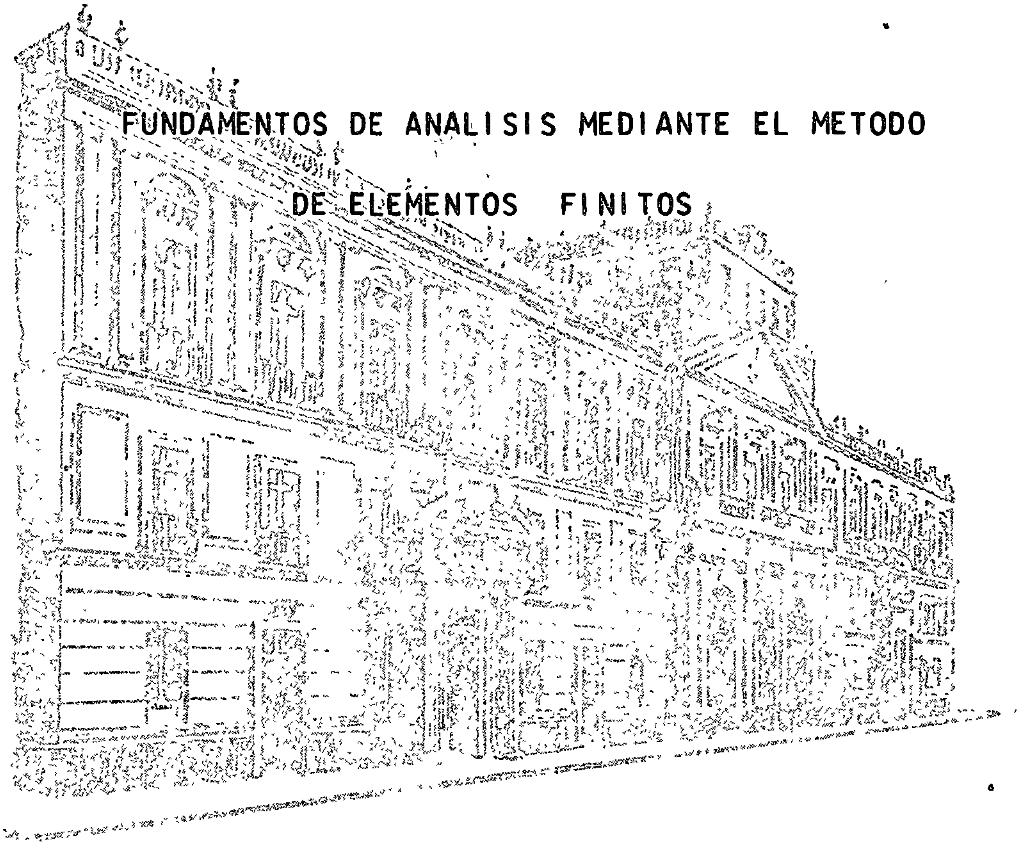

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STATE OF THE ART
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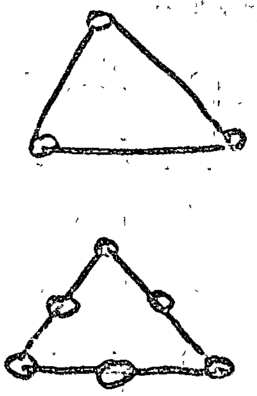
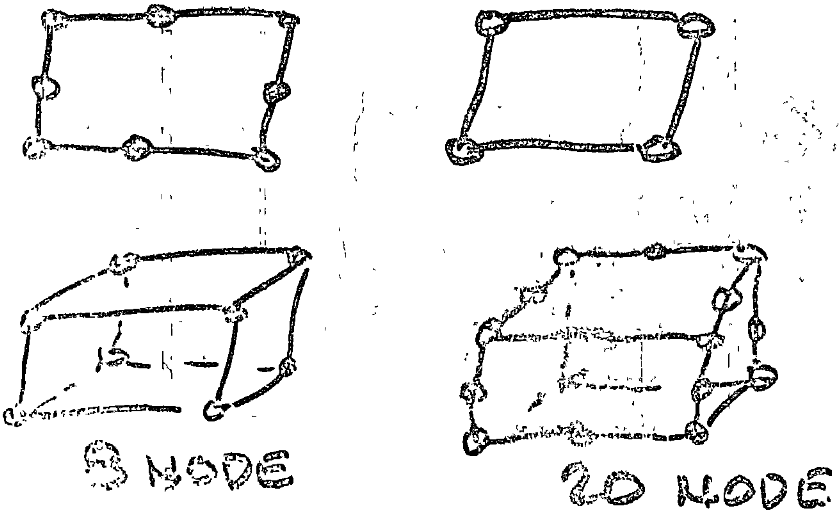
FOR
HEAT TRANSFER

FINITE ELEMENTS

GEOMETRIES
- AXISYMMETRIC
- PLANER
- 3D SOLID





BOUNDARY CONDITIONS

- LINEAR
- NON LINEAR
    - TEMPERATURE DEPENDENT FLUENCES (FLUXES)
    - TEMPERATURE DEPENDENT FILM COEFFICIENTS
    - RADIATION

HEAT TRANSFER MAY BE LINEAR STEADY STATE

HEAT TRANSFER MAY BE LINEAR TRANSIENT

HEAT TRANSFER MAY BE NON LINEAR TRANSIENT

TRANSIENT ANALYSIS IS PERFORMED IN A STEP BY STEP PROCEDURE.

MARC METHOD USED IS A

CRANK-NICKELSON FINITE [redacted] DIFFERENCE SCHEME

MARC-HEAT. INPUT. (VERSION E, G.)

CARD No.	CONTENTS.	REMARKS.
1	TITLE.	R
2	CONTROL VARIABLES.	R
3	CONTROL VARIABLES.	R.
4.	CONNECTIVITY.	(MESHR) O
5	MATERIAL PROPERTIES.	R, (N1)
6}	(@ REF TEMP.).	R, (N1)
7}		R,
8	COORDINATES.	(MESHR). O
9	TEMP. DEPENDENT FUNCTIONS.	R.
10}		} (NOYPT) O
11}	FUNCTIONS.	O.
12}		(NOWKH), O
13	TIME INTERVAL.	R,
14	TIME SEQUENCE.	R. (NTBC).

SAMPLE PROBLEM (VERSION H).

```
 DAEWUENLE,MM.
 $CHANGE,********=HMM.
 JOB,CM45000,CL130000,P4,T100.
 ANULIM(100)
 RFL,55000.
 FTN.
 RFL,6000.
 ATTACH(NGO,MARCCDCNRP,MR=1)
 REWIND(LGO)
 COPYL(NGO,LGO,TAPE29)
 ATTACH(JCL,MARCJCL,MR=1)
 ATTACH(FUDD,MARCFUDD,MR=1)
 ATTACH(PRE,MARCPREH,MR=1)
 ATTACH(RANLOAD,MARCLOADER,MR=1)
 MAP(OFF).
 RFL,20000.
 CCLINK(JCL)
       SUBROUTINE   CREDE (DTOL,M,NSTRES,NEQST,NSTATS)
       DIMENSION DTOL(NSTATS,NEQST,NSTRES)
       RETURN
       END
       SUBROUTINE   FLUX (F,X1,X2,NN,N,TIME)
       RETURN
       END
       SUBROUTINE   FILM (H,TINF,TS,N,TIME)
       RETURN
       END
       SUBROUTINE   ANKOND (COND,CISO,N,NN,ID)
       DIMENSION COND(ID,ID)
       RETURN
       END
 TITLE      HEAT TRANSFER TEST PROB (INTERNAL HEAT GENERATION)
 SIZING            6000     5   12     4     5    39
 HEAT          1
 FLUXES        1
 END
 CONNECTIVITY
    5
    1   39     1     3     4     2
    2   39     3     5     6     4
    3   39     5     7     8     6
    4   39     7     9    10     8
    5   39     9    11    12    10
 COORDINATES
    2   12
    10.0        0.0
    20.0        0.2
    30.2        0.0
    40.2        0.2
    50.4        0.0
    60.4        0.2
    70.6        0.0
    80.6        0.2
    90.8        0.0
   100.8        0.2
   111.0        0.0
   121.0        0.2
 PROPERTY
    1
 1.         1.         1.                                                  1
    1    5
 BOUNDARY CONDITIONS
```

```
        H          N O D A L   P O I N T   D A T A

   TOTAL TEMPERATURES

         1   100.00
         2   100.00
         3   179.99
         4   179.99
         5   219.98
         6   219.98
         7   219.98
         8   219.98
         9   179.99
        10   179.99
        11   100.00
        12   100.00



                                  E N D   O F   I N C R E M E N T   1
```

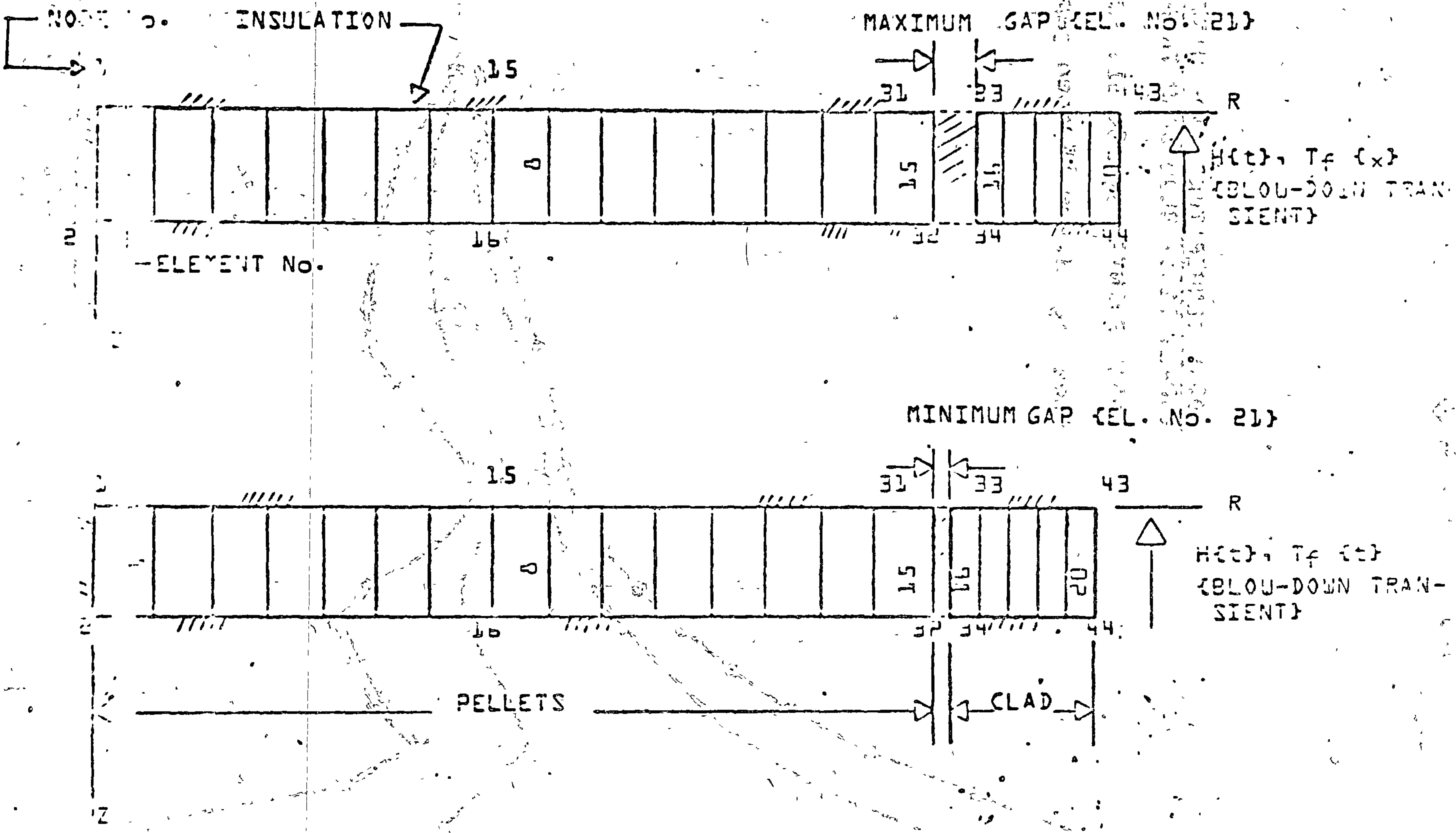
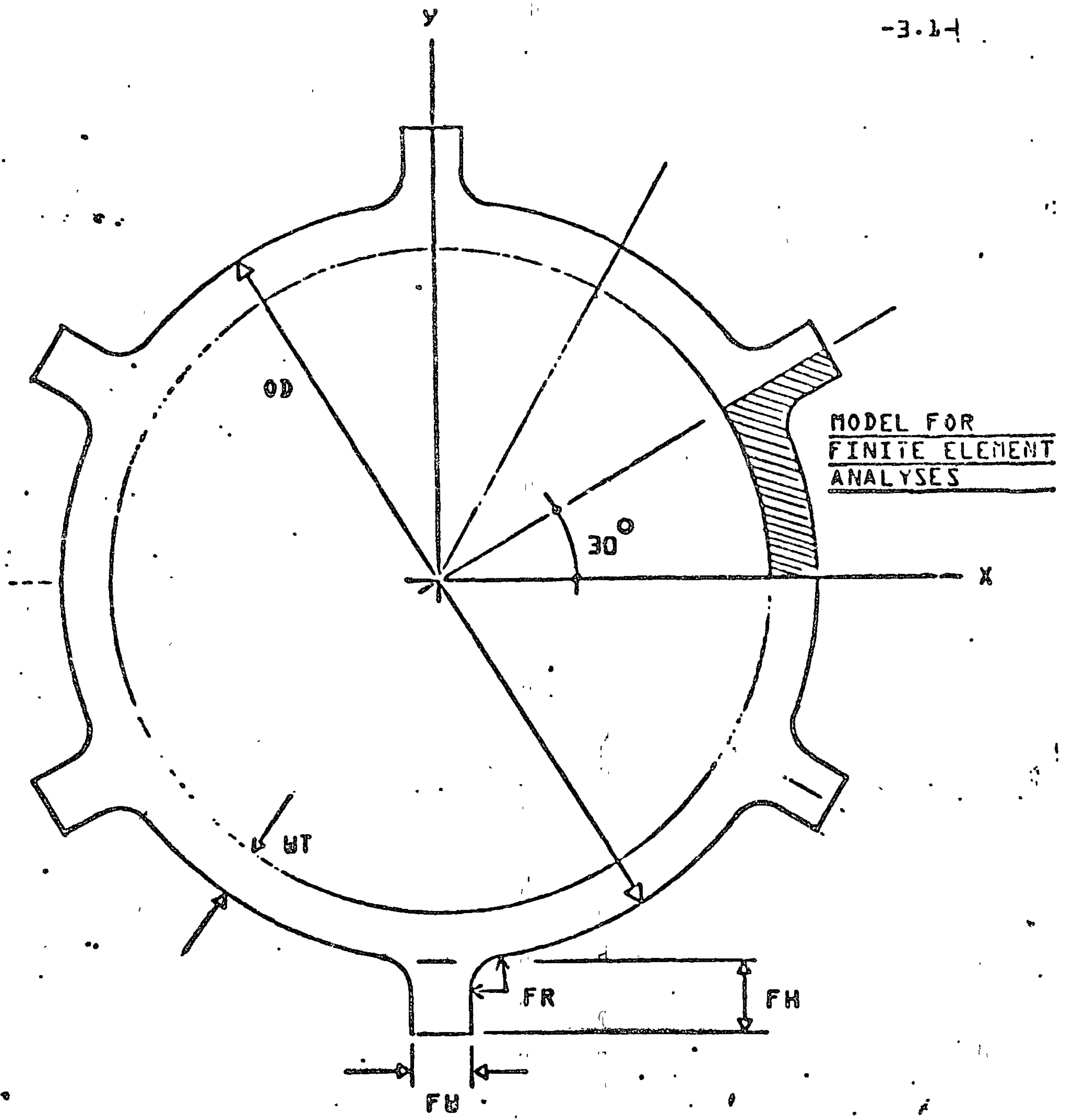


FIG. 3 2D FINITE ELEMENT MODEL {THERMAL ANALYSIS}



OD: OUTER DIAMETER
WT: WALL THICKNESS
FH: FIN HEIGHT
FW: FIN WIDTH
FR: FILLET RADIUS

FIG. [redacted] AXIALLY FINNED FUEL CLAD CROSS SECTION

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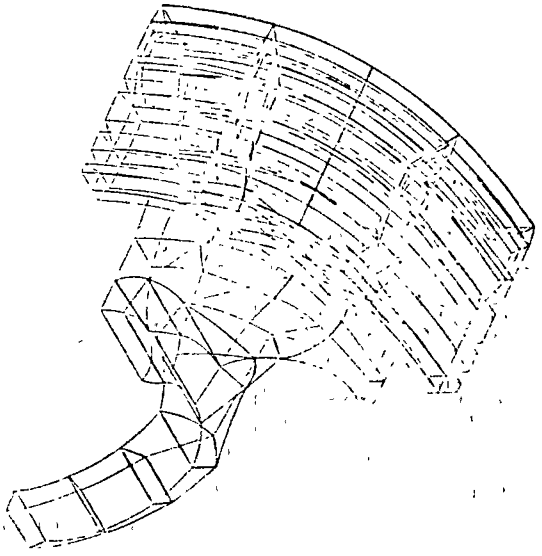

Figure 1

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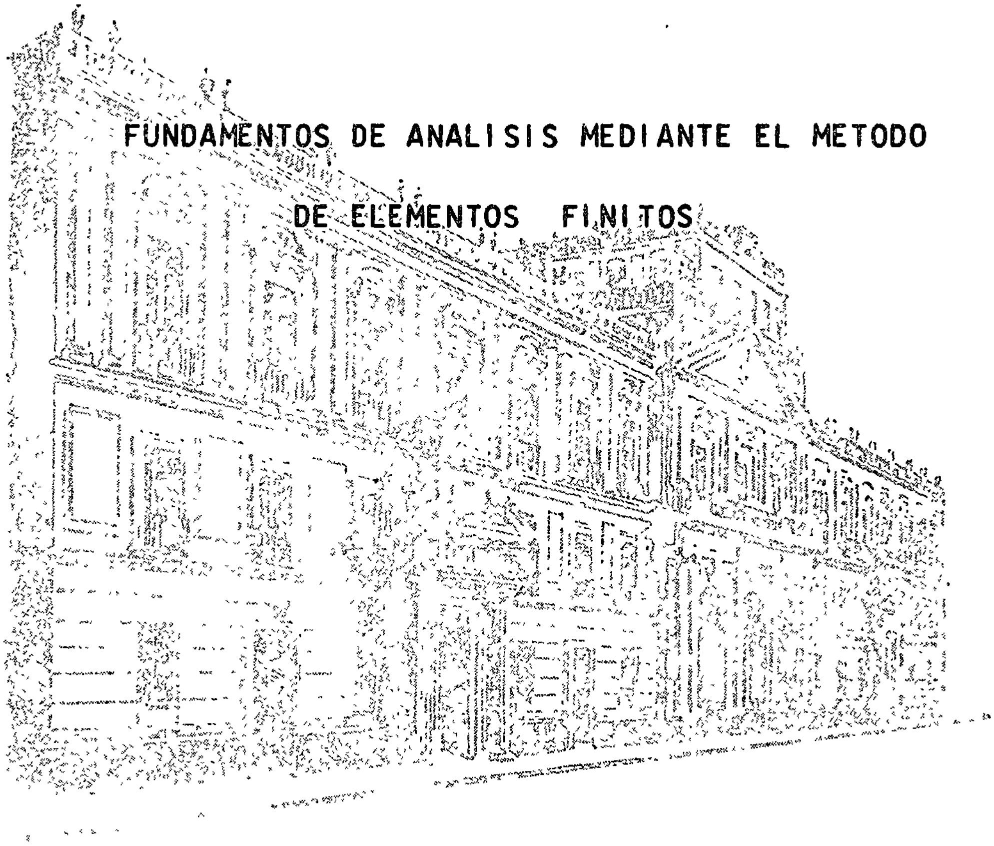

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STATE OF THE ART
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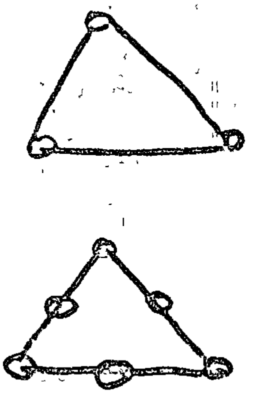
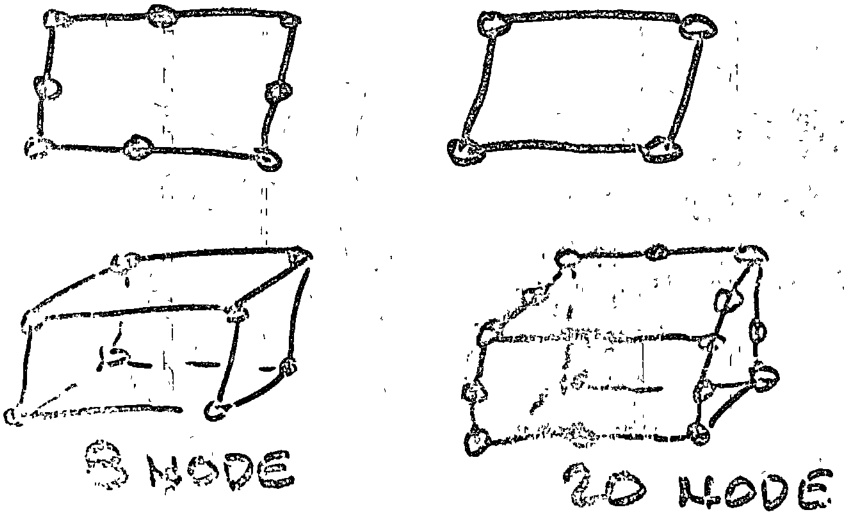
FOR
HEAT TRANSFER

FINITE ELEMENTS



GEOMETRIES

- AXISYMMETRIC
- PLANER
- 3D SOLID



BOUNDARY CONDITIONS

- LINEAR
- NON LINEAR
  - TEMPERATURE DEPENDENT FLUENCES (FLUXES)
  - TEMPERATURE DEPENDENT FILM COEFFICIENTS
  - RADIATION

HEAT TRANSFER MAY BE LINEAR STEADY STATE

HEAT TRANSFER MAY BE LINEAR TRANSIENT

HEAT TRANSFER MAY BE NON LINEAR TRANSIENT

TRANSIENT ANALYSIS IS PERFORMED IN A STEP BY STEP PROCEDURE.

MARC METHOD USED IS A

CRANK-NICKELSON FINITE [redacted] DIFFERENCE SCHEME

MARC-HEAT Input (Version E, G.)

Card No.	Contents	Remarks
1	Title	R
2	Control Variables	R
3	Control Variables	R
4	Connectivity	(MESHR) O
5	Material Properties	R, (N1)
6 }	(@ Ref. Temp.)	R, (N1)
7 }		R,
8	Coordinates	(MESHR) O
9	Temp. Dependent Functions	R
10 }		} (NOYPT) O
11 }	Functions	O
12 }		(NOWKH) O
13	Time Interval	R
14	Time Sequence	R (NTBC)

Sample Problem
(Version H)3

```
[illegible]
$CHANGE,[illegible]
JOB,CM45000,CL130000,P4,T100.
ANULIM(100)
RFL,55000.
FTN.
RFL,6000.
ATTACH(NGO,MARCCDCNRP,MR=1)
REWIND(LGO)
COPYL(NGO,LGO,TAPE29)
ATTACH(JCL,MARCJCL,MR=1)
ATTACH(FUDD,MARCFUDD,MR=1)
ATTACH(PRE,MARCPREH,MR=1)
ATTACH(HANLDAD,MARCLDADEH,MR=1)
MAP(OFF).
RFL,20000.
CCLINK(JCL)
      SUBROUTINE   CREDE (DTOL,M,NSTRES,NEQST,NSTATS)
      DIMENSION DTOL(NSTATS,NEQST,NSTRES)
      RETURN
      END
      SUBROUTINE   FLUX (F,X1,X2,NN,N,TIME)
      RETURN
      END
      SUBROUTINE   FILM (H,TINF,T3,N,TIME)
      RETURN
      END
      SUBROUTINE   ANKOND (COND,CISO,N,NN,ID)
      DIMENSION COND(ID,ID)
      RETURN
      END
TITLE      HEAT TRANSFER TEST PROB (INTERNAL HEAT GENERATION)
SIZING            6000    5   12     4     5    39
HEAT            1
FLUXES          1
END
CONNECTIVITY
    5
    1    39    1    3    4    2
    2    39    3    5    6    4
    3    39    5    7    8    6
    4    39    7    9   10    8
    5    39    9   11   12   10
COORDINATES
    2   12
    10.0         0.0
    20.0         0.2
    30.2         0.0
    40.2         0.2
    50.4         0.0
    60.4         0.2
    70.6         0.0
    80.6         0.2
    90.8         0.0
   100.8         0.2
   111.0         0.0
   121.0         0.2
PROPERTY
    1
1.         1.         1.
    1    5
BOUNDARY CONDITIONS
```

H

```
                 N O D A L   P O I N T   D A T A

TOTAL TEMPERATURES

      1   100.00
      2   100.00
      3   179.99
      4   179.99
      5   219.98
      6   219.98
      7   219.98
      8   219.98
      9   179.99
     10   179.99
     11   100.00
     12   100.00

                                 E N D   O F   I N C R E M E N T   1
```

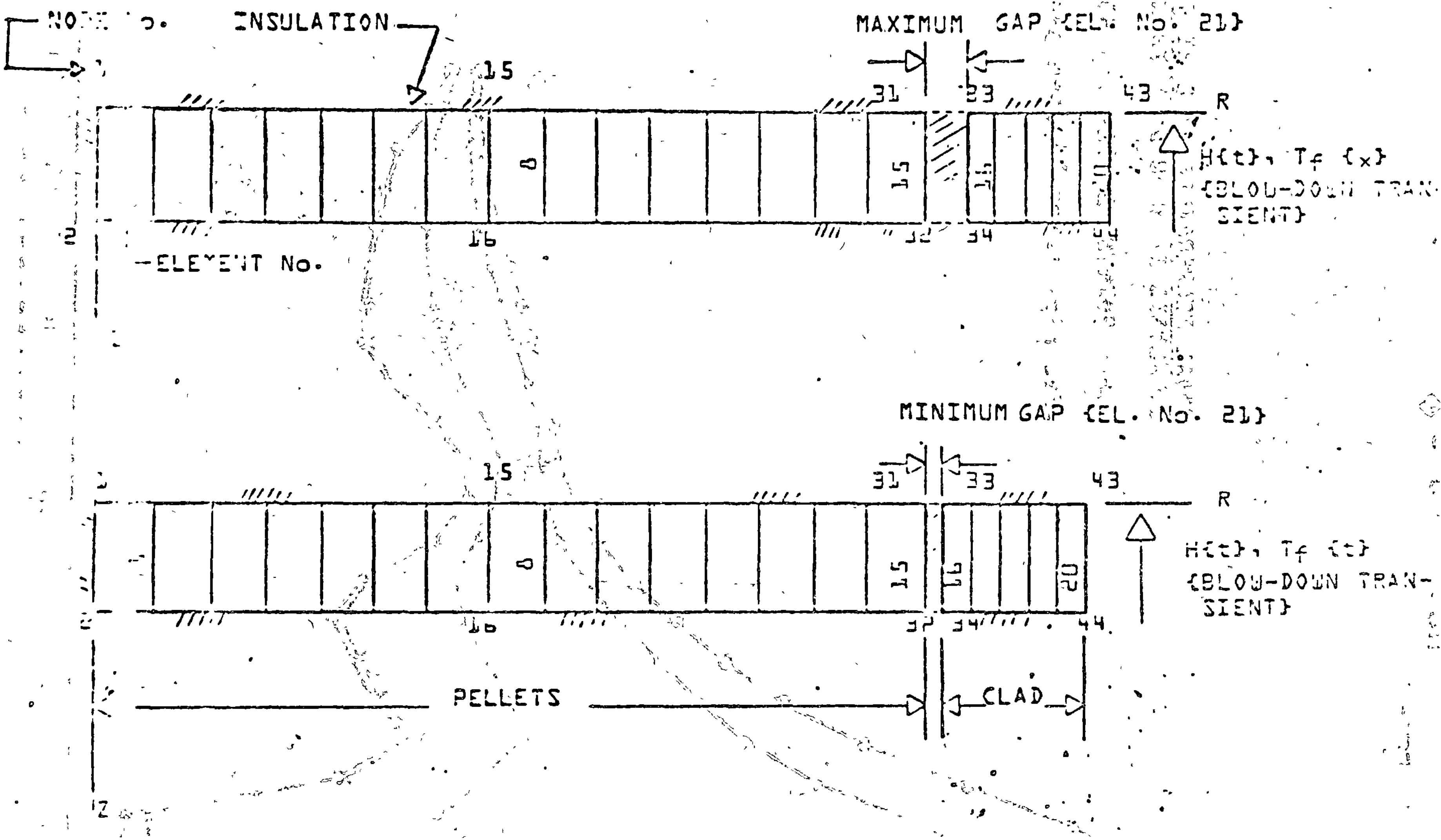
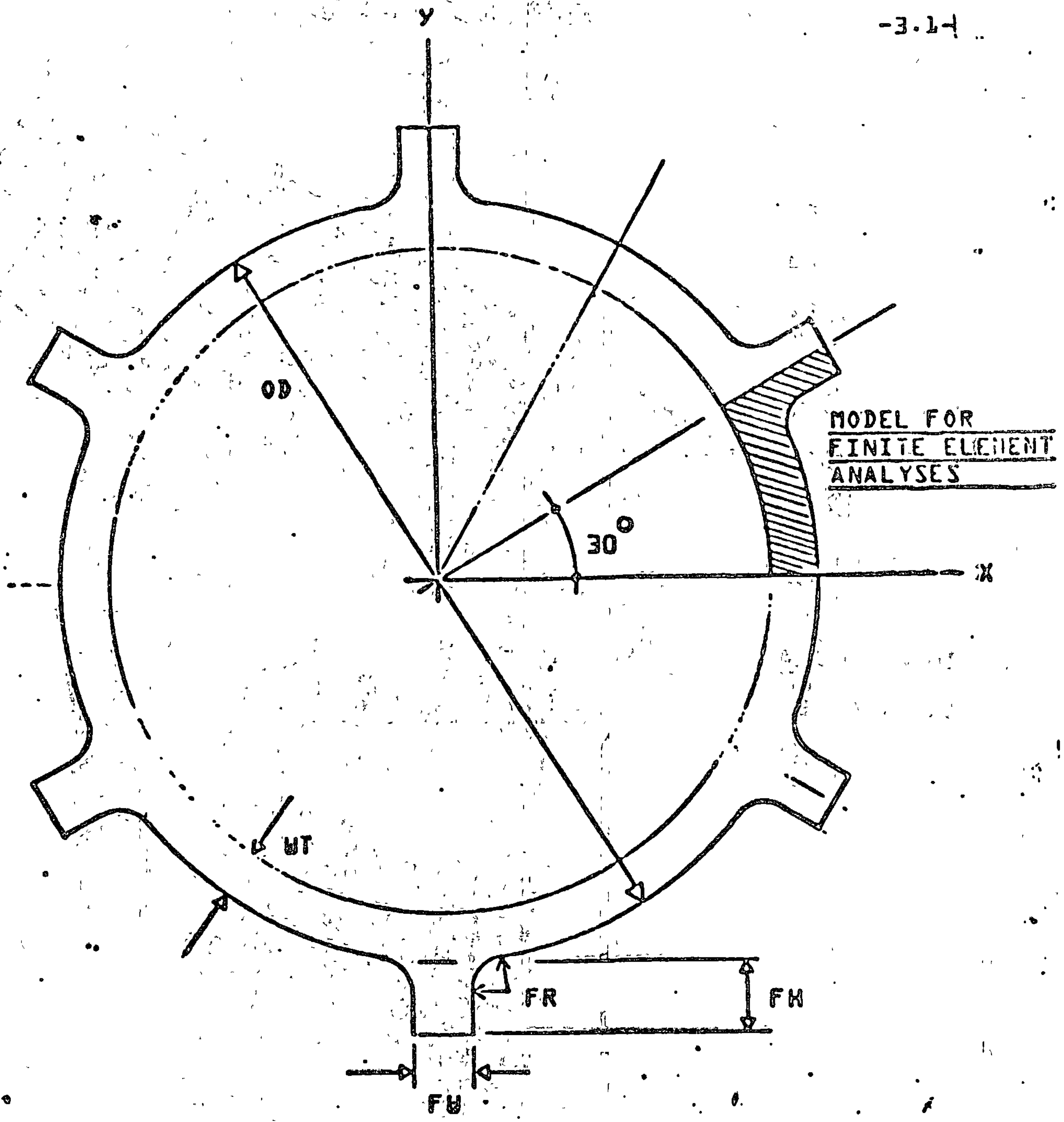


FIG. 3 DISCRETE ELEMENT MODEL {THERMAL ANALYSIS}



OD: OUTER DIAMETER
WT: WALL THICKNESS
FH: FIN HEIGHT
FW: FIN WIDTH
FR: FILLET RADIUS

FIG. [redacted] AXIALLY FINNED FUEL CLAD CROSS SECTION

Marc Analysis Research Corporation

MARC APPLICATION SUMMARY

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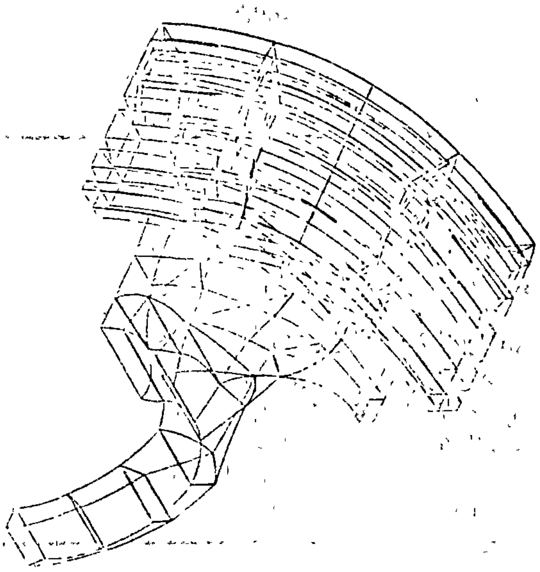

Figure 1

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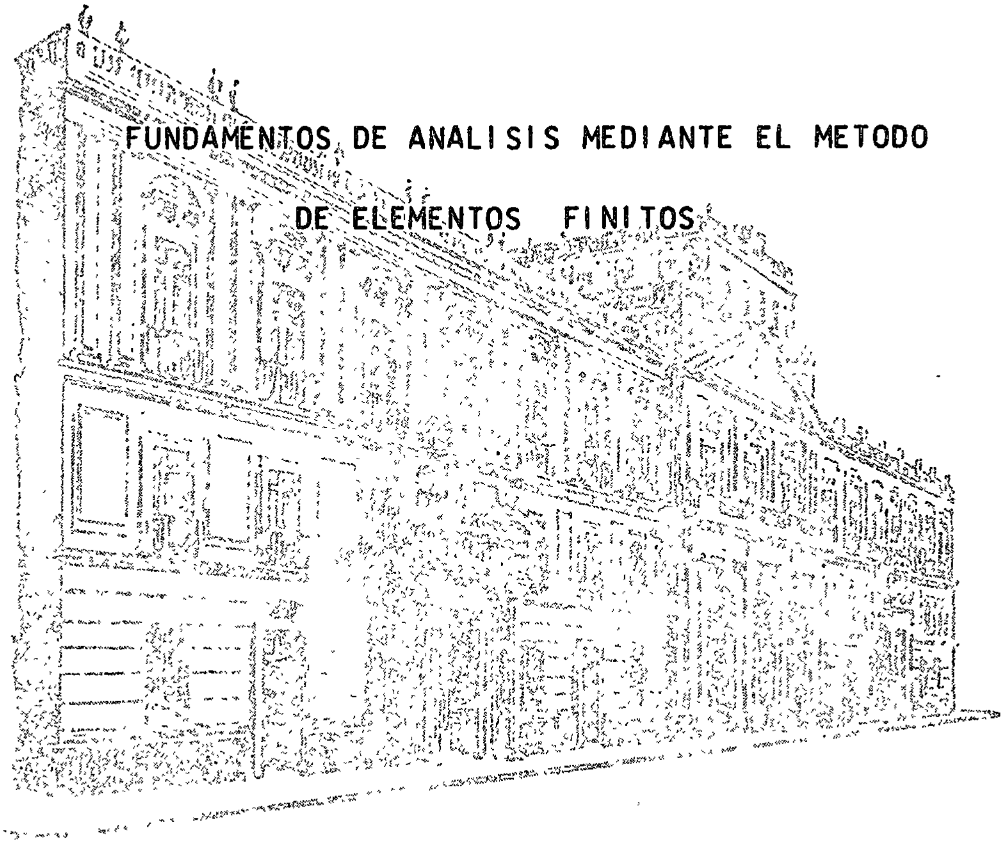
división de estudios superiores
facultad de ingeniería, unam

FUNDAMENTOS DE ANALISIS MEDIANTE EL METODO DE ELEMENTOS FINITOS



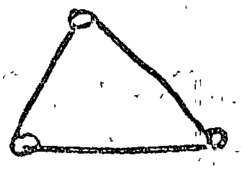
Palacio de Minería
Tacuba 5, primer piso. México 1, D. F.
Tels.: 521-40-23 521-73-35 512-31-23

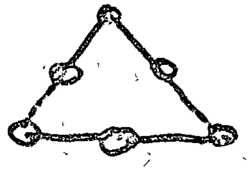
STATE OF THE ART
OF
F.E.M.
FOR
HEAT TRANSFER

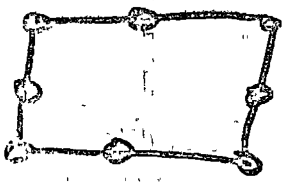
FINITE ELEMENTS

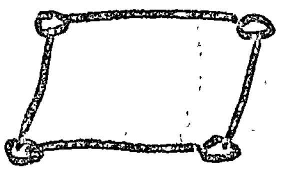
GEOMETRIES

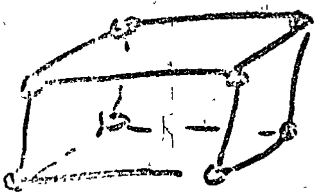
- AXISYMMETRIC
- PLANER
- 3D SOLID



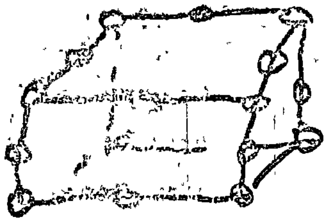








8 NODE



20 NODE

BOUNDARY CONDITIONS

- LINEAR
- NON LINEAR
  - TEMPERATURE DEPENDENT FLUENCES (FLUXES)
  - TEMPERATURE DEPENDENT FILM COEFFICIENTS
  - RADIATION

HEAT TRANSFER MAY BE LINEAR STEADY STATE

HEAT TRANSFER MAY BE LINEAR TRANSIENT

HEAT TRANSFER MAY BE NON LINEAR TRANSIENT

TRANSIENT ANALYSIS IS
PERFORMED IN A STEP
BY STEP PROCEDURE.

MARC METHOD USED
IS A

CRANK-NICKELSON FINITE
[redacted] DIFFERENCE
SCHEME

MARC-HEAT Input (Version E, G.)

Card No.	Contents	Remarks
1	Title	R
2	Control Variables	R
3	Control Variables	R
4	Connectivity	(MESHR) O
5	Material Properties	R, (N1)
6 }	(@ Ref Temp.)	R, (N1)
7 }		R,
8	Coordinates	(MESHR) O
9	Temp. Dependent Functions	R
10 }		} (NOYPT), O
11 }	Functions	O
12 }		(NOWKH), O
13	Time Interval	R
14	Time Sequence	R (NTBC)

SAMPLE PROBLEM
(VERSION H).

```
[illegible]
$CHANGE,[illegible].
JOB,C445000,CL130000,P4,T100.
ANULIM(100)
RFL,55000.
FTN.
RFL,6000.
ATTACH(NGO,MARCEDCNRP,MR=1)
REWIND(LGO)
COPYL(NGO,LGO,TAPE29)
ATTACH(JCL,MAROJCL,MR=1)
ATTACH(FUDO,MARCFUDO,MR=1)
ATTACH(PRE,MARCPHEH,MR=1)
ATTACH(HANLOAD,MARCLOADE4,MR=1)
MAP(OFF).
RFL,20000.
CCLINK(JCL)
      SUBROUTINE   CREDE (OTOL,M,NSTRES,NEOST,NSTATS)
      DIMENSION OTOL(NSTATS,NEOST,NSTRES)
      RETURN
      END
      SUBROUTINE   FLUX (F,X1,X2,NN,N,TIME)
      RETURN
      END
      SUBROUTINE   FILM (H,TINF,TS,N,TIME)
      RETURN
      END
      SUBROUTINE   ANKOND (COND,CISO,N,NN,ID)
      DIMENSION COND(ID,ID)
      RETURN
      END
TITLE     HEAT TRANSFER TEST PROB (INTERNAL HEAT GENERATION)
SIZING          6000     5    12     4     5    39
HEAT               1
FLUXES             1
END
CONNECTIVITY
    5
    1    39     1     3     4     2
    2    39     3     5     6     4
    3    39     5     7     8     6
    4    39     7     9    10     8
    5    39     9    11    12    10
COORDINATES
    2    12
    10.0        0.0
    20.0        0.2
    30.2        0.0
    40.2        0.2
    50.4        0.0
    60.4        0.2
    70.6        0.0
    80.6        0.2
    90.8        0.0
   100.8        0.2
   111.0        0.0
   121.0        0.2
PROPERTY
    1
1.        1.        1.                                                      1
    1    5
BOUNDARY CONDITIONS
```

```
        H               N O D A L   P O I N T   D A T A

  TOTAL TEMPERATURES

      1     100.00
      2     100.00
      3     179.99
      4     179.99
      5     219.98
      6     219.98
      7     219.98
      8     219.98
      9     179.99
     10     179.99
     11     100.00
     12     100.00

                              E N D   O F   I N C R E M E N T   1
```

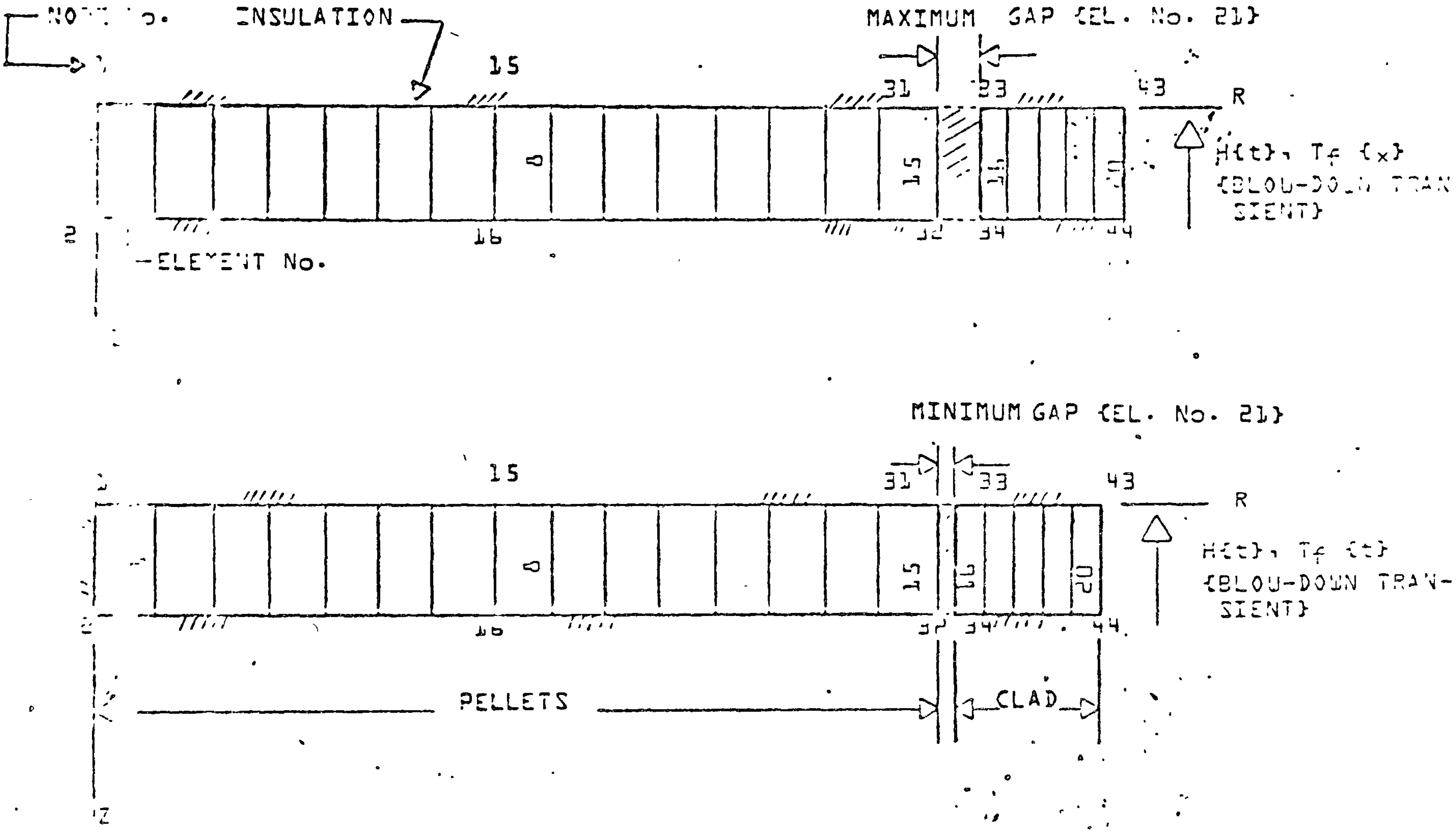
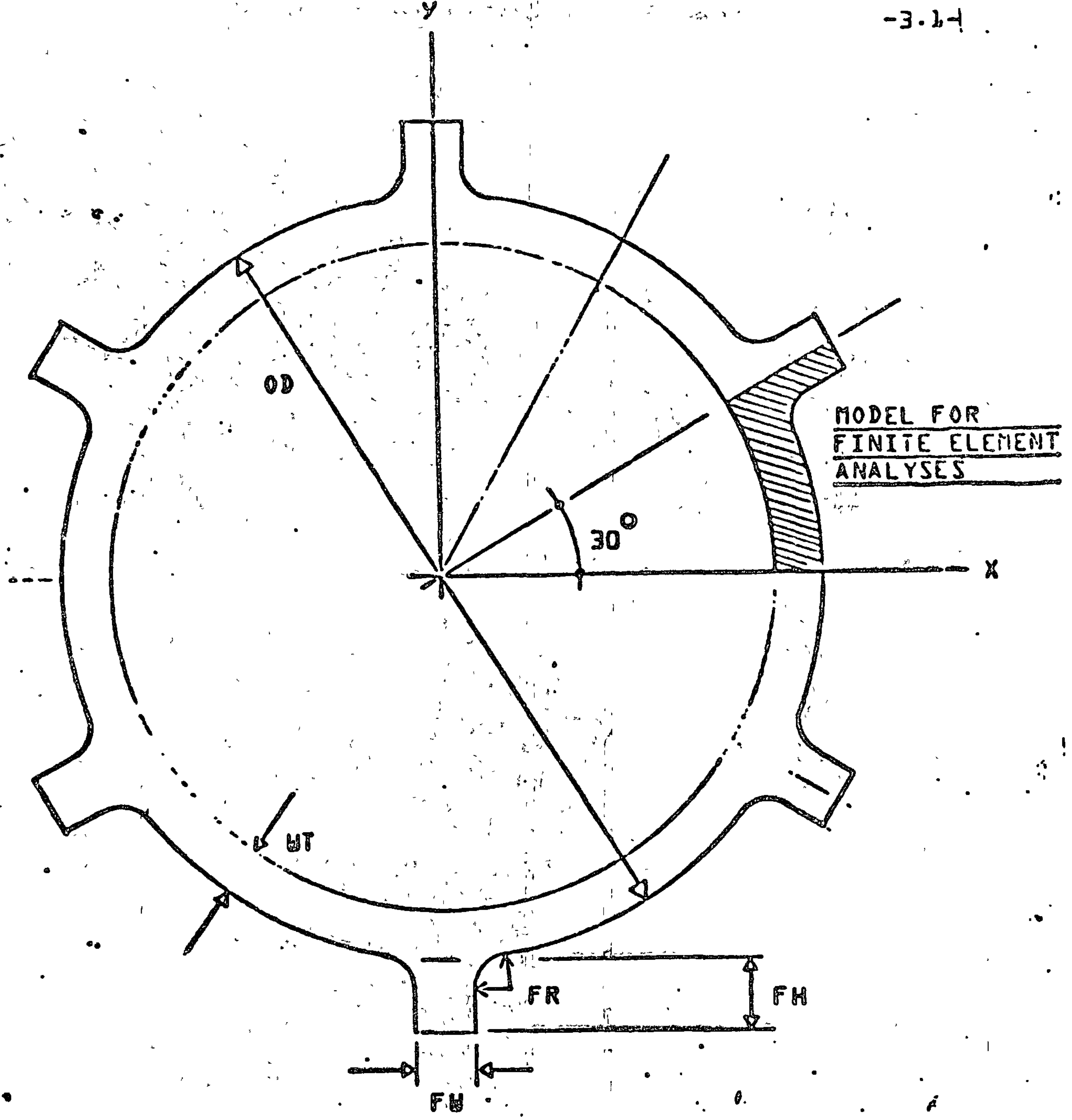


FIG. 3 2D FINITE ELEMENT MODEL (THERMAL ANALYSIS)



OD: OUTER DIAMETER
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FIG. [illegible] AXIALLY FINNED FUEL CLAD CROSS SECTION

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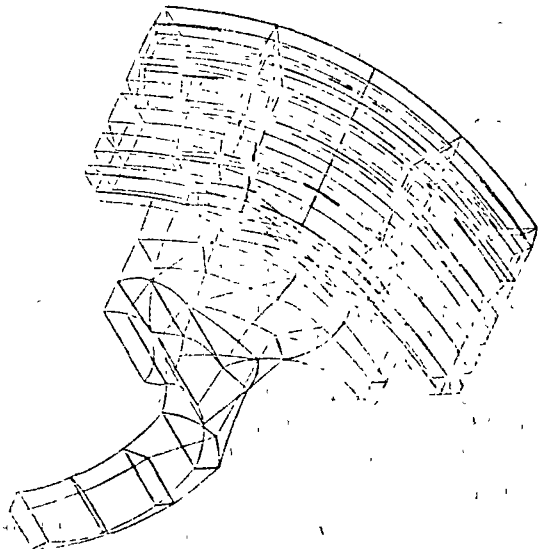

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1 Cuadrica de Esfuerzos de Cauchy, superficies de esfuerzos, Esfuerzos principales, Invariantes

las componentes del tensor de esfuerzo en notación indice e Ingeniería son

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \qquad (1)$$

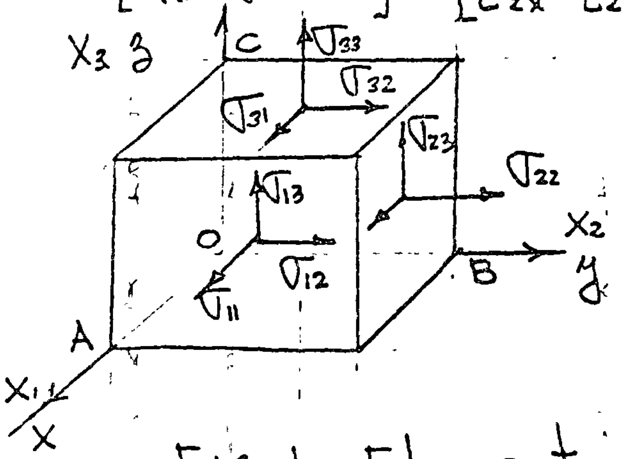


Fig. 1 Elemento diferencial, actuando los esfuerzos $[\sigma_{ij}]$.

Llevando un plano a Travez de ABC y considerando su diagrama de cuerpo libre se tiene

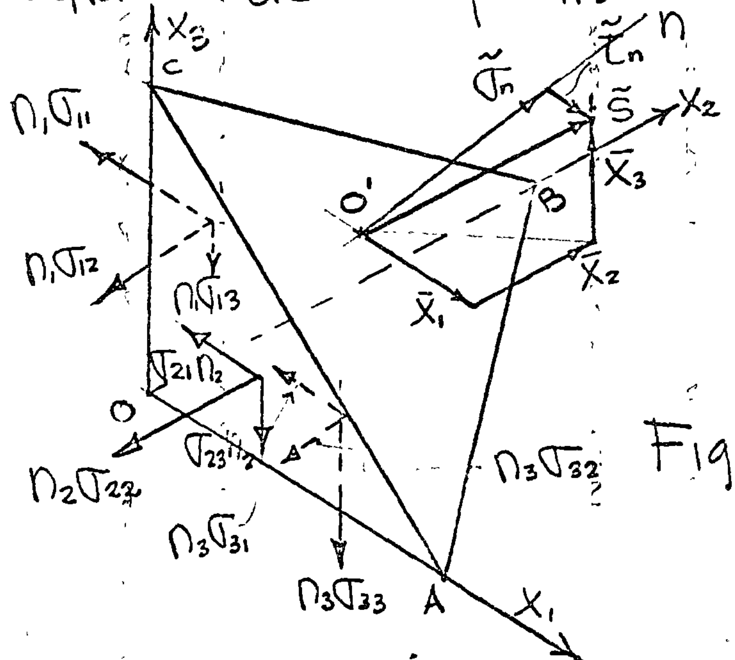


Fig. 2

En la Fig. 2 se tiene lo siguiente o'n es normal al plano ABC, formando angulos α, β y γ con respecto a los ejes coordenados $x_1, x_2,$ y x_3 respectivamente, la distancia oo' es igual a r las coordenadas de o' son x_1, x_2, x_3 por lo tanto

$$n_1 = \cos\alpha = \frac{x_1}{r}, \quad n_2 = \cos\beta = \frac{x_2}{r}, \quad n_3 = \cos\gamma = \frac{x_3}{r} \qquad (2)$$

donde $\{n_i\} = [n_1 \; n_2 \; n_3]^T$ es el vector columna de cosenos directores de la normal al plano ABC (o'n y oo'). Si el area ABC es considerada como la unidad, las proyecciones

$$\begin{aligned} n_1 &= \text{area OBC} \\ n_2 &= \text{area OAC} \\ n_3 &= \text{area OAB} \end{aligned} \qquad (2)$$

$\bar{s}$ = Esfuerzo resultante actuando sobre el plano ABC

$\{\tilde{X}_i\} = [\tilde{X}_1 \; \tilde{X}_2 \; \tilde{X}_3]^T$; proyecciones de $\bar{s}$ sobre x_i.

$\tilde{\sigma}_n$ = Proyección de $\bar{s}$ sobre la normal al plano ABC

$\tilde{\tau}_n$ = Proyección de $\bar{s}$ sobre el plano ABC.

Del equilibrio del elemento OABC se obtiene

$$\underline{X}_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$\underline{X}_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \qquad (3)$$
$$\underline{X}_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

expresando (3) matricialmente se obtiene

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (4)$$

Si no existen momentos de cuerpo, $\sigma_{ij} = \sigma_{ji}$ para $i \neq j$ y $[\sigma_{ij}] = [\sigma_{ij}]^T$ por lo que (4) puede escribirse

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (5)$$

ó

$$\{X_i\} = [\sigma_{ij}]\{n_i\} \qquad (6)$$

El esfuerzo normal al plano ABC es

$$\sigma_n = \underline{X}_1 n_1 + \underline{X}_2 n_2 + \underline{X}_3 n_3 \qquad (7)$$

ó

$$\sigma_n = \{X_i\}^T \{n_i\} \qquad (8)$$

Substituyendo (5) en (7) se obtiene

$$\sigma_n = \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 + 2(\sigma_{12} n_1 n_2 + \sigma_{23} n_2 n_3 + \sigma_{31} n_3 n_1) \qquad (9)$$

ó matricialmente de (6) y (8)

$$\sigma_n = \lfloor n_i \rfloor [\sigma_{ij}] \{n_i\} \qquad (10)$$

$$S^2 = X_1^2 + X_2^2 + X_3^2 \qquad (11)$$

$$\sigma_n^2 + \tau_n^2 = S^2 \qquad (12)$$

Esfuerzos principales. Esfuerzo principal es un valor particular del esfuerzo normal tal que $\tau_n = 0$ por lo tanto

$$\begin{aligned} X_1 &= \sigma_n n_1 \\ X_2 &= \sigma_n n_2 \\ X_3 &= \sigma_n n_3 \end{aligned} \qquad (13)$$

De (5) y (13) se obtiene

$$\begin{Bmatrix} \sigma_{n_1} n_1 \\ \sigma_{n_2} n_2 \\ \sigma_{n_3} n_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (14)$$

De donde

$$\begin{bmatrix} (\sigma_n - \sigma_{11}) & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{21} & (\sigma_n - \sigma_{22}) & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{32} & (\sigma_n - \sigma_{33}) \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = 0 \qquad (15)$$

puesto que $\{n_i\} \neq 0$, entonces el determinante

$$\begin{vmatrix} (\sigma_n - \sigma_{11}) & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{21} & (\sigma_n - \sigma_{22}) & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{32} & \sigma_n - \sigma_{33} \end{vmatrix} = 0 \qquad (16)$$

De (16) se obtiene

$$\sigma_n^3-(\sigma_{11}+\sigma_{22}+\sigma_{33})\sigma_n^2+(\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{23}^2-\sigma_{31}^2)\sigma_n$$
$$-(\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{31}^2-\sigma_{33}\sigma_{12}^2)=0 \quad (17)$$

las tres raices de la ecuación (17) nos determinan los valores de los esfuerzos principales σ_1, σ_2 y σ_3 cuyos coeficientes nos representan los invariantes de esfuerzos, dependen de σ_1, σ_2 y σ_3 independientes del sistema de ejes coordinados

$$I_1=\sigma_{11}+\sigma_{22}+\sigma_{33}\equiv\sigma_1+\sigma_2+\sigma_3$$
$$I_2=\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{23}^2-\sigma_{31}^2\equiv\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 \quad (18)$$
$$I_3=\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{13}^2-\sigma_{33}\sigma_{12}^2\equiv\sigma_1\sigma_2\sigma_3$$

donde I_1, I_2 e I_3 son los invariantes de esfuerzos, otras expresiones de invariantes pueden formarse de (18) por ejemplo (19)

$$2I_1-6I_2=(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{13}^2+\sigma_{23}^2+\sigma_{31}^2)$$

(19) se usa en la expresión de la energía de deformación, su uso se discutirá posteriormente

Despues de diagonalizar el Tensor de esfuerzo $[\sigma_{ij}]$, el elemento de la Fig. 2 se muestra en la Fig. 3, y las ecuaciones de equilibrio (5) quedan:

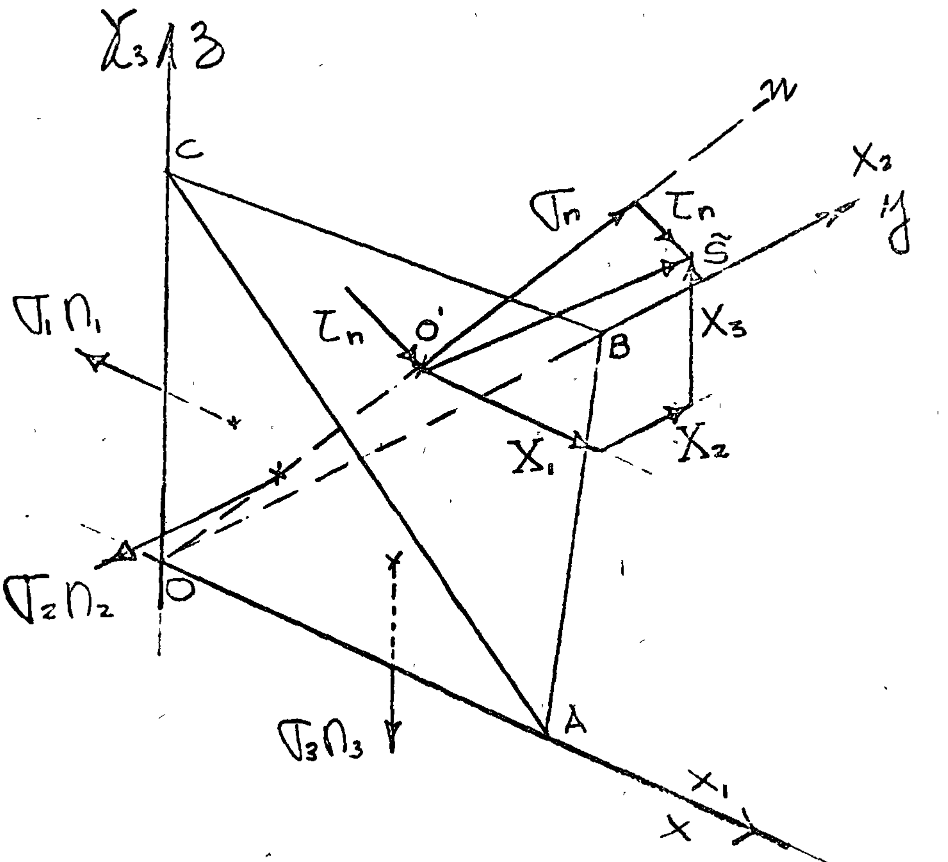


Fig. 3 Componentes del tensor de esfuerzo diagonalizado

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (20)$$

En (20) las componentes $\{X_i\}$, $\{n_i\}$, $\tilde{S}$, $\tilde{\sigma}_n$, $\bar{\tau}_n$ son diferentes a las (5) que se muestran en Fig. 2.
De geometría se conoce que

$$n_1^2 + n_2^2 + n_3^2 = 1 \qquad (21)$$

Substituyendo (20) en (21) se obtiene la ecuación

$$\frac{X_1^2}{\sigma_1^2}+\frac{X_2^2}{\sigma_2^2}+\frac{X_3^2}{\sigma_3^2}=1 \qquad (22)$$

la cual representa una superficie elipsoidal en el espacio de esfuerzos σ_i, algunos autores lo denominan elipsoide de Lamé, en la Fig. 4 se muestra su perspectiva isométrica. Para el conjunto

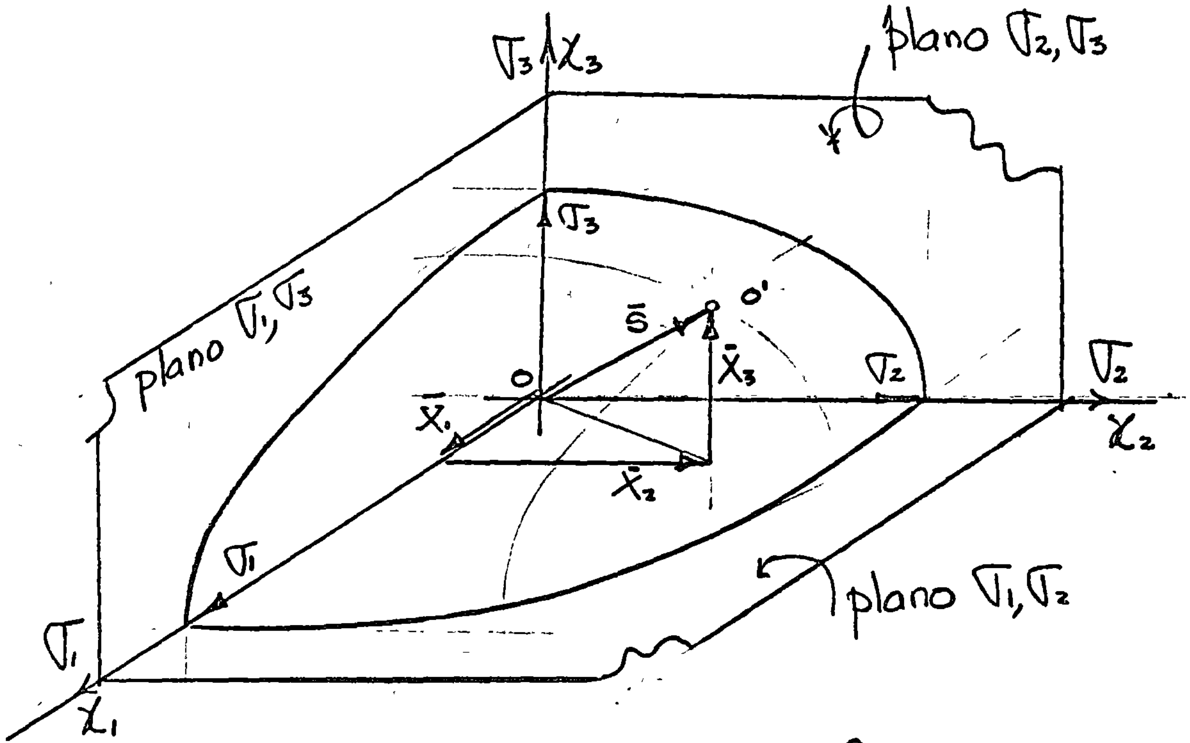


Fig. 4 Elipsoide de Lamé referido al espacio de esfuerzos σ_i, (un octagono).

de planos con cosenos directores $\{n_i\}$ a travez de o Fig. 2, le corresponde el conjunto de componentes $\{X_i\}$, los cuales junto con los esfuerzos principales σ_1, σ_2 y σ_3 forman la superficie elipsoidal de la Fig. 4.

De (20), si $\sigma_1=\sigma_2=\sigma_3=\sigma$, la superficie es esférica. si $\sigma_1\neq 0$, $\sigma_2\neq 0$ y $\sigma_3=0$ la superficie es cilindrica de sección eliptica con eje contenido en el eje σ_3. Si $\sigma_1=\sigma_2$ y $\sigma_3=0$ la superficie es cilindrica de sección circular con eje contenido en el eje σ_3, Si $\sigma_1\neq 0$ y $\sigma_2=\sigma_3=0$ la superficie son dos planos paralelos al pano σ_2,σ_3 a continuación se indican los casos particulares mencionados

$$[\sigma_{ij}]=\begin{bmatrix}\sigma&0&0\\0&\sigma&0\\0&0&\sigma\end{bmatrix} \quad (23)$$

$$X_1^2+X_2^2+X_3^2=\sigma^2 \quad (24)$$

$$\sigma\equiv S$$

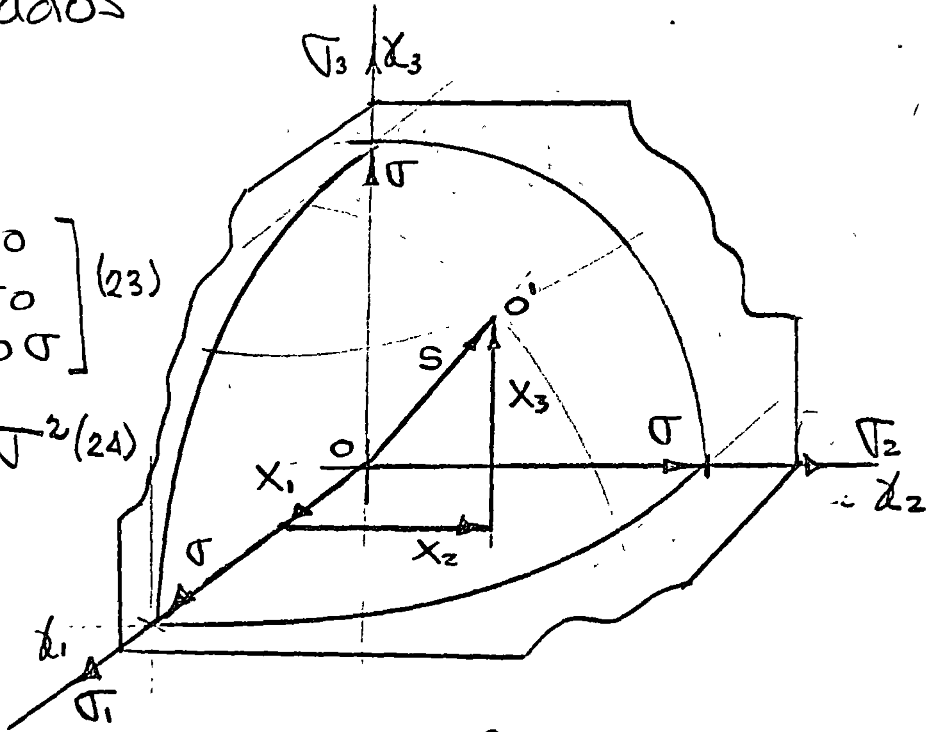


Fig. 5 Superficie esférica, equivalente a una Tensión o compresión uniforme o hidrostática

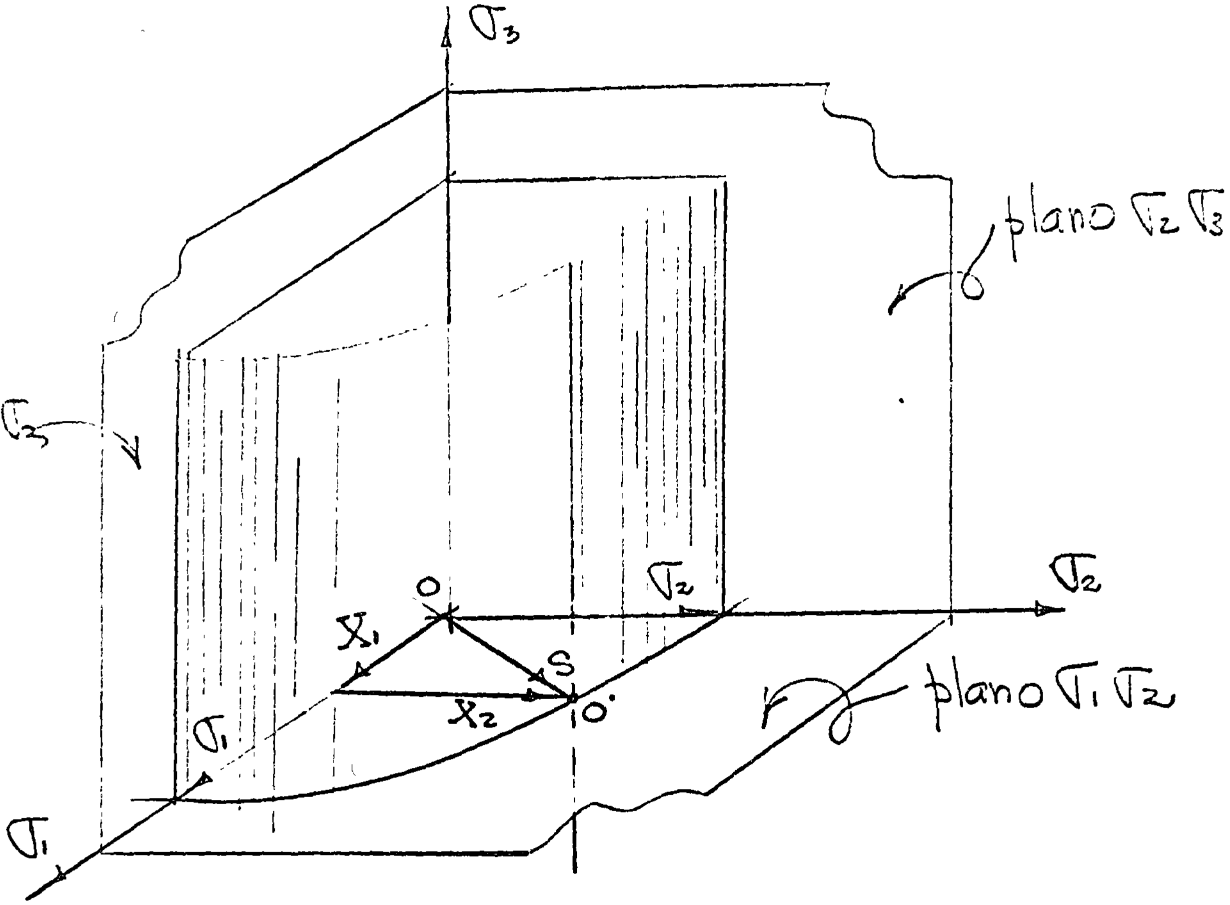


Fig. 6 Superficie cilindrica de sección eliptica directrices paralelas al eje $O\sigma_3$.

Componentes del tensor de esfuerzos: $[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (25)

Ecuación de la superficie: $\frac{X_1^2}{\sigma_1^2} + \frac{X_2^2}{\sigma_2^2} = 1$ (26)

Como caso particular de de (25) si $\sigma_1 = \sigma_2 = \sigma$ se tiene un cilindro con componentes del tensor de esfuerzos $[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (27)

y ecuación de la superficie (28)

$$X_1^2 + X_2^2 = \sigma^2$$

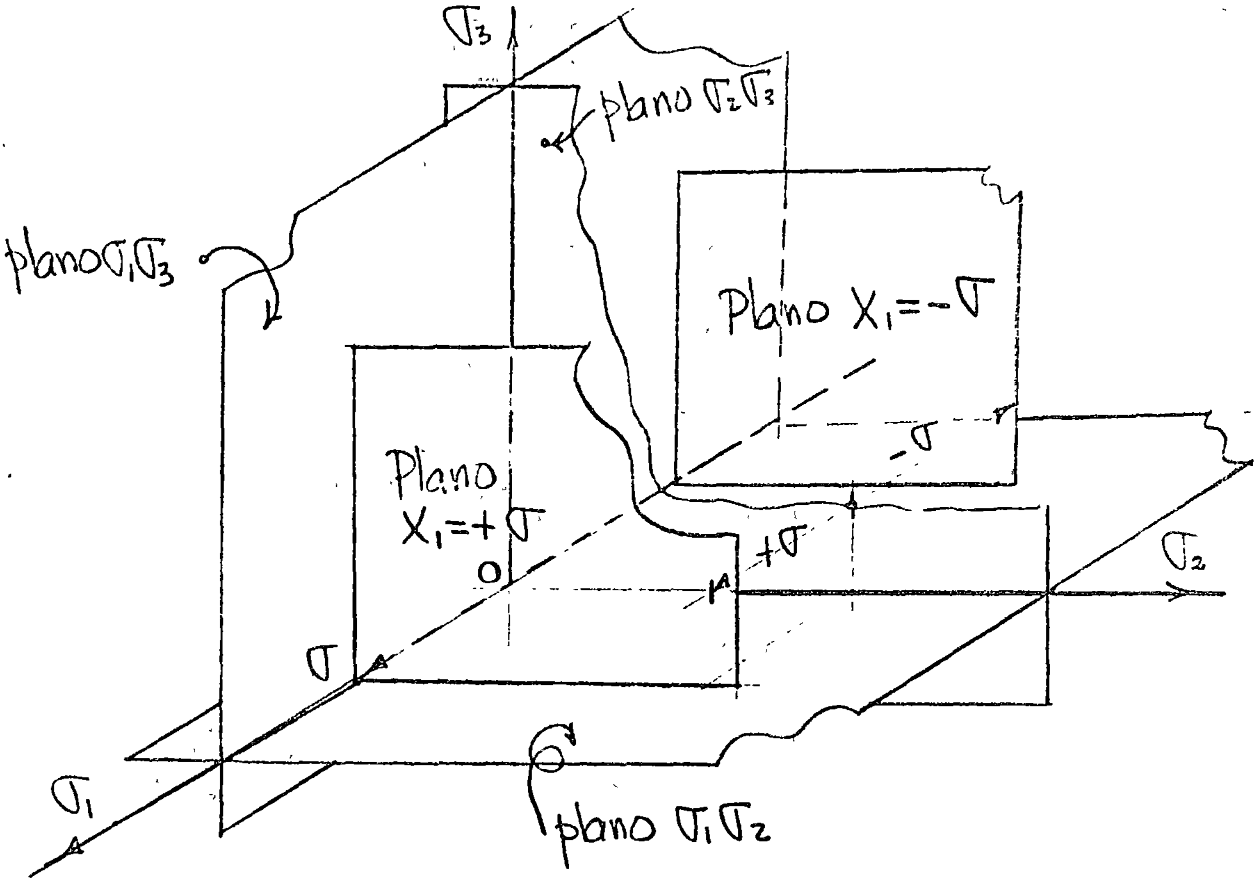


Fig. 6 Superficies planas paralelas al plano $\sigma_2\sigma_3$

Componentes del tensor de esfuerzos:

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (29)$$

Ecuación de la Superficie:

$$X_1 = \pm\sigma \qquad (30)$$

La ecuación (21) en el espacio de cosenos directores nos representa una esfera de radio unitario como se muestra en la Fig. 7

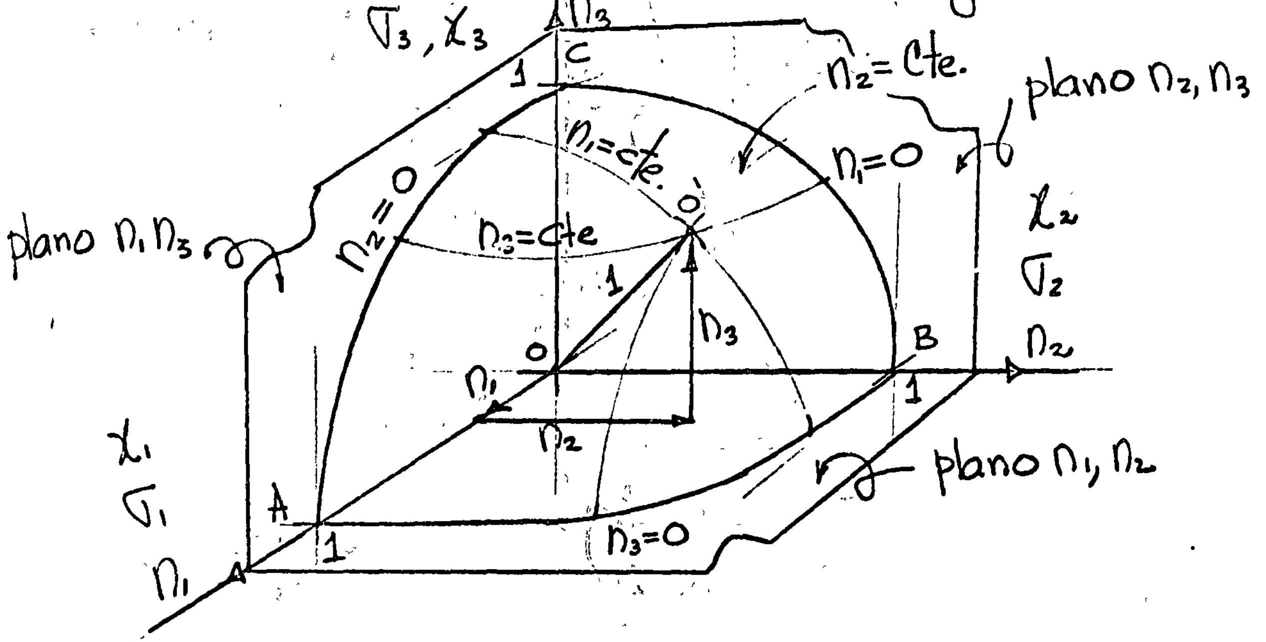


Fig. 7 Espacio de cosenos directores. un octagono de la esfera de Mohor.

$$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OO'} = 1$$

De la Fig. 3 se observa que substituyendo (20) en (7) se obtiene

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{31}$$

Substituyendo (20) y (31) en (11) y (12) se obtiene

$$\tau_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 \tag{32}$$

de las ecuaciones (31), (32) y (21) se obtiene el siguiente sistema de 3 ecuaciones con 3 incognitos no lineal en n_1, n_2 y n_3

$$\begin{bmatrix} 1 & 1 & 1 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ f(\sigma_1 n_1^2) & f(\sigma_2 n_2^2) & f(\sigma_3 n_3^2) \end{bmatrix} \begin{Bmatrix} n_1^2 \\ n_2^2 \\ n_3^2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \sigma_n \\ \tau_n^2 \end{Bmatrix} \qquad (33)$$

de (33) se obtiene

$$n_1^2 = \frac{(\sigma_2-\sigma_n)(\sigma_3-\sigma_n)+\tau_n^2}{(\sigma_2-\sigma_1)(\sigma_3-\sigma_1)} \qquad (34)$$

$$n_2^2 = \frac{(\sigma_3-\sigma_n)(\sigma_1-\sigma_n)+\tau_n^2}{(\sigma_3-\sigma_2)(\sigma_1-\sigma_2)} \qquad (35)$$

$$n_3^2 = \frac{(\sigma_1-\sigma_n)(\sigma_2-\sigma_n)+\tau_n^2}{(\sigma_1-\sigma_3)(\sigma_2-\sigma_3} \qquad (36)$$

De la Fig. 7 considerando $n_1 = \text{constante}$ de la ecuación (34) se obtiene

$$n_1^2(\sigma_2-\sigma_1)(\sigma_3-\sigma_1) = (\sigma_2-\sigma_n)(\sigma_3-\sigma_n) + \tau_n^2 \qquad (37)$$

efectuando operaciones algebraicas en (37) se obtiene

$$n_1^2(\sigma_2-\sigma_1)(\sigma_3-\sigma_1) + \left(\frac{\sigma_2-\sigma_3}{2}\right)^2 = \left[\sigma_n - \frac{\sigma_2+\sigma_3}{2}\right]^2 + \tau_n^2 = \text{Constante}$$

de donde: $r_1^2 = \left[\sigma_n - \frac{\sigma_2+\sigma_3}{2}\right]^2 + \tau_n^2 \equiv (x-a)^2 + y^2$ que es la ecuación de un circulo a una distancia $\frac{\sigma_2+\sigma_3}{2}$ del origen por lo tanto el radio r_1 que haciendo centro en $\frac{\sigma_2+\sigma_3}{2}$ localiza el punto de coordenadas $\sigma_n \tau_n$ en el diagrama de Mohor es.

$$\tau_1 = \sqrt{n_1^2(\sigma_2-\sigma_1)(\sigma_3-\sigma_1)+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2} \qquad (38)$$

Similarmente suponiendo n_2 = constante de (35) se obtiene

$$\tau_2 = \sqrt{n_2^2(\sigma_3-\sigma_2)(\sigma_1-\sigma_2)+\left(\frac{\sigma_1-\sigma_3}{2}\right)^2} \qquad (39)$$

Similarmente suponiendo n_3 = constante de (36) se obtiene

$$\tau_3 = \sqrt{n_3^2(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)+\left(\frac{\sigma_1-\sigma_2}{2}\right)^2} \qquad (40)$$

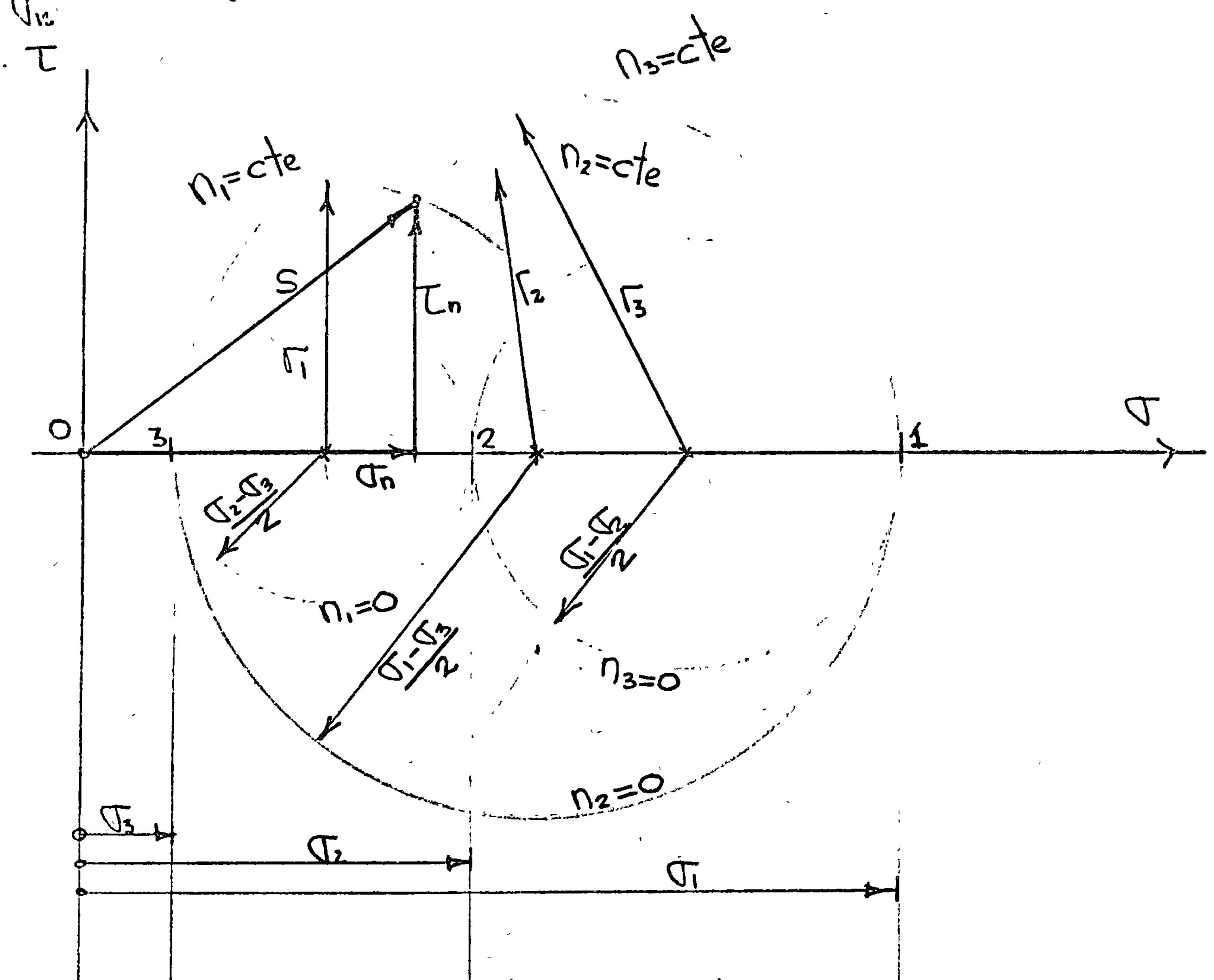


Fig. 8 Circulo de Mohor en Tres dimensiones determinar σ_n, τ_n, conociendo $\sigma_1, \sigma_2, \sigma_3$ y n_1, n_2 y n_3

2- Esfuerzos cortantes máximos, esfuerzo esférico, esfuerzo octaedral

Sean x_1, x_2, x_3 las direcciones principales (Fig. 3) y n_1, n_2, n_3 los cosenos directores de cierto plano ABC, se tiene que

$$\tau_n^2 = S^2 - \sigma_n^2 \tag{41}$$

$$S^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 \tag{42}$$

$$\sigma_n^2 = (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 \tag{43}$$

substituyendo (43) y (42) en (41) se obtiene

$$\tau_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 \tag{44}$$

Para determinar las direcciones maximas de corte de $n_3^2 = 1 - n_1^2 - n_2^2$ se elimina n_3 de (44) y se determinan

$$\frac{\partial}{\partial n_1}(\tau_n^2) = 0;\quad n_1\left[(\sigma_1-\sigma_3)n_1^2 + (\sigma_2-\sigma_3)n_2^2 - \tfrac{1}{2}(\sigma_1-\sigma_3)\right] = 0 \tag{45}$$

$$\frac{\partial}{\partial n_2}(\tau_n^2) = 0;\quad n_2\left[(\sigma_1-\sigma_3)n_1^2 + (\sigma_2-\sigma_3)n_2^2 - \tfrac{1}{2}(\sigma_2-\sigma_3)\right] = 0 \tag{46}$$

las soluciones de (45) y (46) que hacen τ_n máximo.

Si $n_2 = 0$ $\quad n_1 = \sqrt{\frac{1}{2}}$ $\quad n_3 = \sqrt{\frac{1}{2}}$

" $n_1 = 0$ $\quad n_2 = \sqrt{\frac{1}{2}}$ $\quad n_3 = \sqrt{\frac{1}{2}}$ y similarmente

" $n_3 = 0$ $\quad n_1 = \sqrt{\frac{1}{2}}$ $\quad n_2 = \sqrt{\frac{1}{2}}$

se repiten los calculos en (44) se elimina n_1 y después n_2. Conviene observar que en (45) y (46) no hay soluciones de n_1 y n_2 que sean ambos diferentes de cero, porque las expresiones dentro del parentesis no pueden anularse.

	n_1	n_2	n_3	n_1	n_2	n_3
n_1	0	0	± 1	0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$
n_2	0	± 1	0	$\pm\sqrt{\frac{1}{2}}$	0	$\pm\sqrt{\frac{1}{2}}$
n_3	± 1	0	0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$	0
	Esf. Principales $\tau_n=0$			Cortantes máximos		

Tabla 1 Cosenos directores

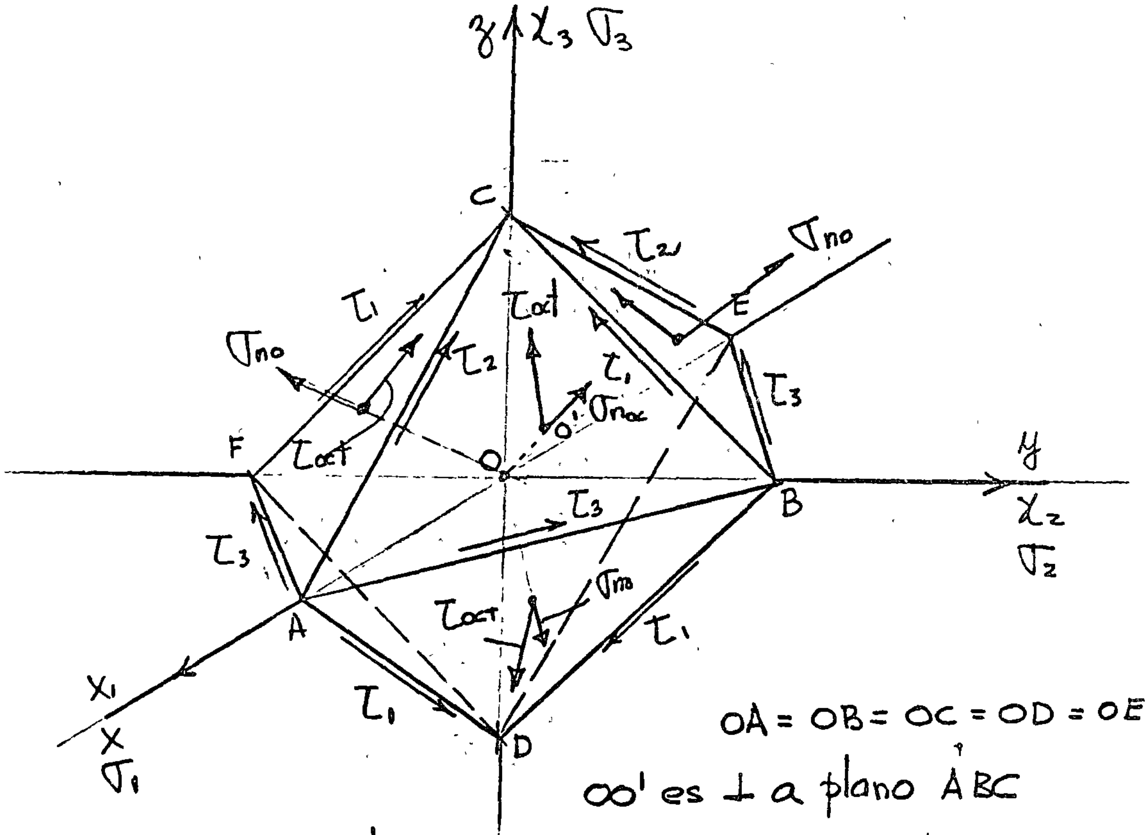
Repitiendo los calculos en (44), eliminado n_1 y determinando n_2 y n_3 tal que τ_n sea máximo y después n_2 y determinando n_1 y n_3 tal que τ_n sea máximo se obtienen los valorees

$$(\tau_{max})_1 = \tau_1 = \pm \tfrac{1}{2}(\sigma_2-\sigma_3)$$
$$(\tau_{max})_2 = \tau_2 = \pm \tfrac{1}{2}(\sigma_1-\sigma_3) \qquad (47)$$
$$(\tau_{max})_3 = \tau_3 = \pm \tfrac{1}{2}(\sigma_1-\sigma_2)$$

de (47) y (32) se puede expresar τ_n en la siguiente forma

$$\tau_n^2 = 4(n_1^2 n_2^2 \tau_3^2 + n_2^2 n_3^2 \tau_1^2 + n_1^2 n_3^2 \tau_2^2) \qquad (48)$$

Las 3 primeras columnas de la Tabla 1 dan las direcciones de los planos coordenados de las direcciones principales, para ellos $\tau_n=0$ y (32) es un minimo, las tres columnas restantes dan planos a travez de un eje principal bisectando los otros dos direcciones de esfuergos principales, substituyendo los valores de Tabla 1 en (32) se obtienen los valores de los esfuerzos cortantes maximos (47), los lados del octaedro mostrado en la Fig. 9 son las direcciones principales de cortante, y las direcciones x_1, x_2 y x_3 son la direcciones



OA = OB = OC = OD = OE = OF

OO' es ⊥ a plano ABC

Fig. 9 octaedro regular cuyos lados son las direcciones de esfuerzo cortante máximo.

principales σ_1, σ_2 y σ_3, la normal al tetaedro OABC tiene cosenos directores $n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}}$ $(\alpha=\beta=\gamma=54.76°)$ de (31) el esfuerzo normal es igual a

$$\sigma_{n_o} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{48}$$

(48) se denomina esfuerzo medio, esférico o hidrostático, el esfuerzo de corte correspondiente de (44) es

$$\tau_{oct}^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2 = \frac{1}{9}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$$

(49)

ó

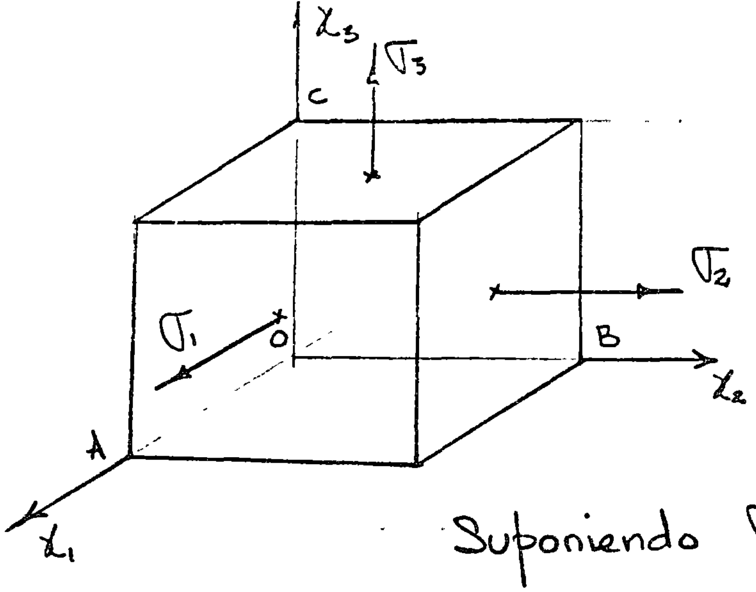
$$\tau_{OCT} = \frac{1}{3}\sqrt{\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]} \qquad (49)$$

de (48) y (49) se obtiene

$$\tau_{OCT} = \sqrt{\frac{1}{3}\left[(\sigma_1-\sigma_n)^2+(\sigma_2-\sigma_n)^2+(\sigma_3-\sigma_n)^2\right.} \qquad (50)$$

al esfuerzo de corte dado por (49) y (50) es llamado esfuerzo octaedral de corte, porque la cara donde actua es la cara ABC del octaedro regular de la Fig. 9 que tiene vertices en los ejes coordenados, se usa frecuentemente en Teoría de Plasticidad

TEORIAS DE FALLA



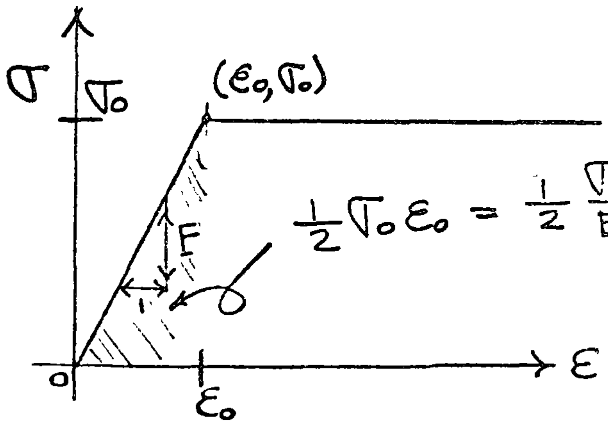
Suponiendo $\sigma_1 > \sigma_2 > \sigma_3$

Fig. 10

En la Fig. 1, después de diagonalizar las componentes del tensor de esfuerzos, se tiene

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \qquad (51)$$

se trata de obtener la superficie $f(\sigma_1 \sigma_2 \sigma_3) = 0$ en la cual el medio entra a falla plástica, a continuación se presenta el diagrama idealizado esfuerzo deformación en condiciones uniaxiales



$\frac{1}{2}\sigma_0 \varepsilon_0 = \frac{1}{2}\frac{\sigma_0^2}{E}$ = densidad energía elastica.

Fig. 11

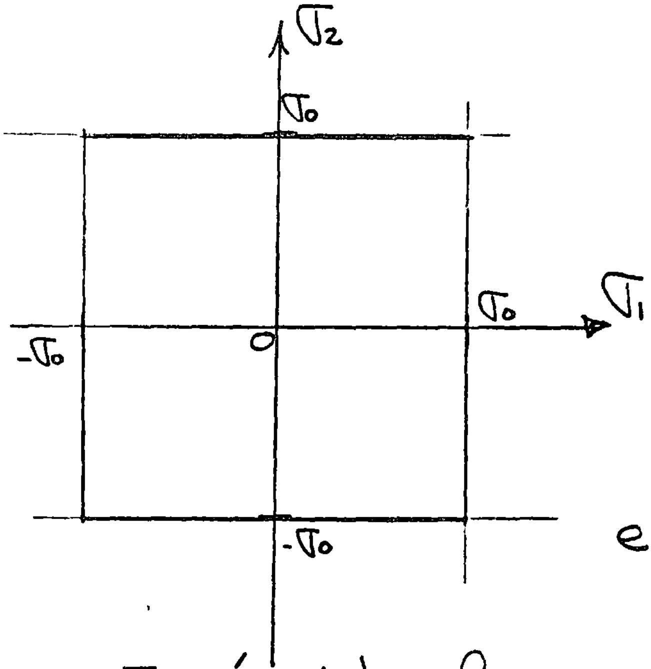
a) Teoría del Máximo esfuerzo (Rankine)

Se supone que $\sigma_1 = \sigma_0'$ o $\sigma_3 = \sigma_0'$

σ_0' esfuerzo de fluencia en tensión

σ_0'' " " " " compresión

ó σ_0' y σ_0'' pueden ser dos esfuerzos de fluencia en dos direcciones perpendiculares, suponiendo un estado plano de esfuerzos $\sigma_3 = 0$ y que $\sigma_1 = \sigma_2 = \sigma_0$ se obtiene el diagrama de esfuerzos de la Fig. 12



Planos de falla

$\sigma_1 = \pm \sigma_0$

$\sigma_2 = \pm \sigma_0$

$\sigma_3 = \pm \sigma_0$

Superficie cubica en el espacio de esfuerzos

Fig. 12 Teoría del esfuerzo máximo en esfuerzos planos

b) Teoría de la deformación maxima (Saint-Verant)

Condición triaxial de esfuerzos que alcanza la deformación de fluencia ε_0.

$$\varepsilon_0 = \frac{\sigma_0}{E} = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \qquad (52)$$

de (52) la superficie de esfuerzos referida al espacio de esfuerzos $\sigma_1, \sigma_2, \sigma_3$ es

$$f(\sigma_1 \sigma_2 \sigma_3) = (\sigma_1 - \sigma_0) - \nu(\sigma_2 + \sigma_3) = 0 \qquad (53)$$

en (53) suponiendo $\sigma_3 = 0$ y para $\sigma_1 = \sigma_2 = \sigma$ (esfuerzos planos) se obtiene para $\nu = 0.3$

$$\sigma_0 = (1-\nu)\sigma$$

$$\sigma = \frac{1}{1-\nu}\sigma_0 = \frac{1}{1-0.3}\sigma_0 = 1.43\,\sigma_0 \qquad (54)$$

Si $\sigma_1 = -\sigma_2 = \sigma_0$

$$\sigma_0 = (1+\nu)\sigma_0$$

$$\sigma = \frac{1}{1+\nu}\sigma_0 = \frac{1}{1+0.3}\sigma_0 = 0.77\,\sigma_0 \qquad (55)$$

llevando los valores (54) y (55) al plano σ_1, σ_2 del espacio de esfuerzos se obtiene las rectas de falla de la Fig. 13

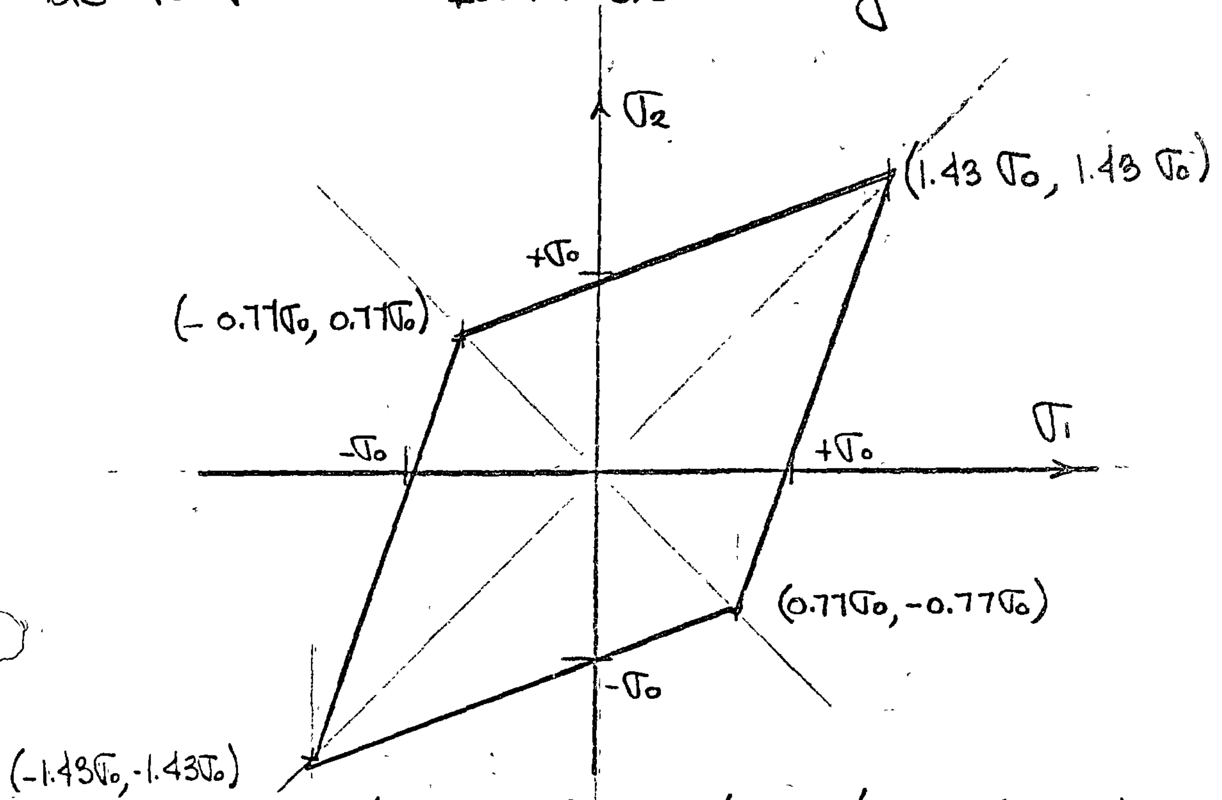


Fig.13 Teoría de deformación máxima (Saint-Venant)

c) Teoría del Esfuerzo Cortante Máximo (Coulomb)

Si $\sigma_1 > \sigma_2 > \sigma_3$ Coulomb establece que la falla se alcanza cuando

$$(\tau_2)_{max} = \frac{\sigma_1 - \sigma_3}{2} = \pm \frac{1}{2}\sigma_0 \qquad (56)$$

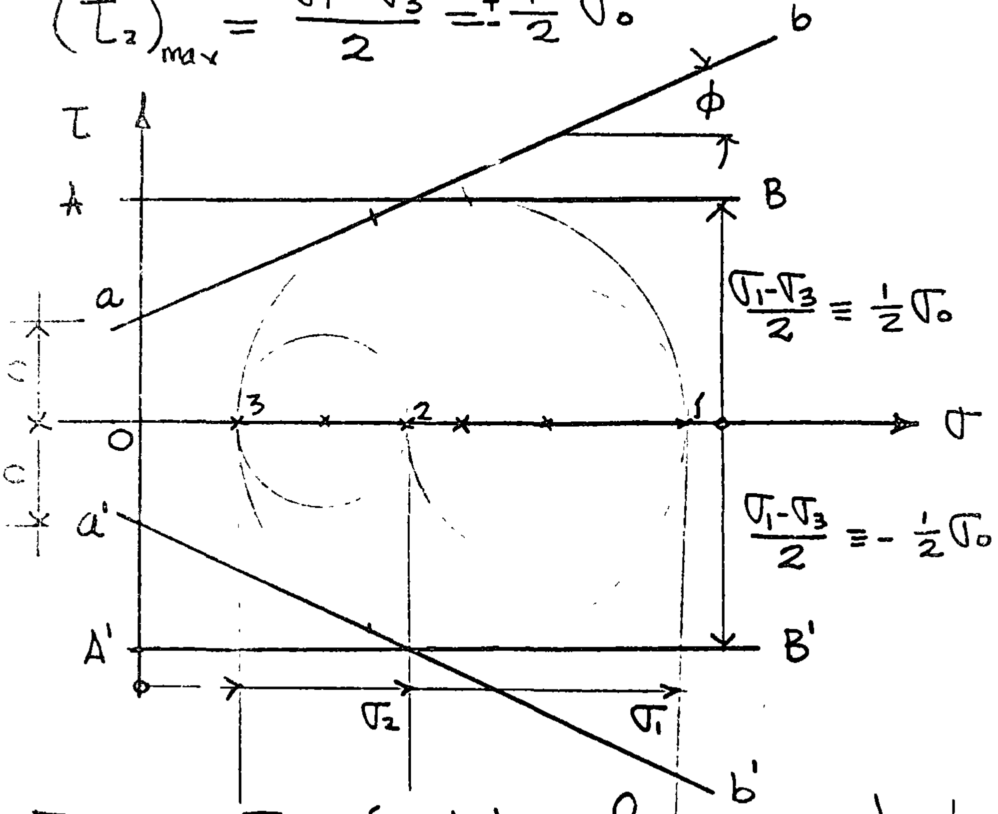


Fig. 13 Teoría del esfuerzo cortante máximo

(56) en el diagrama de Mohor establece como rectas de falla a AB y A'B' en Fig. 13 cuando el ángulo de fricción interna $\phi = 0$, y cuando $\phi > 0$ las rectas de falla son las ab y a'b' cuya ecuación es igual a

$$\tau_{max} = c + \sigma \tan\phi \qquad (57)$$

c = cohesión o resistencia al esfuerzo cortante puro
ϕ = angulo de fricción interna
σ = esfuerzo de falla.

c y ϕ son constantes constitutivas experimentales que se pueden obtener mediante una prueba triaxial de ruptura. La ecuación 56 en el plano de esfuerzos $\sigma_1 \sigma_3$ se muestra en la Fig. 14

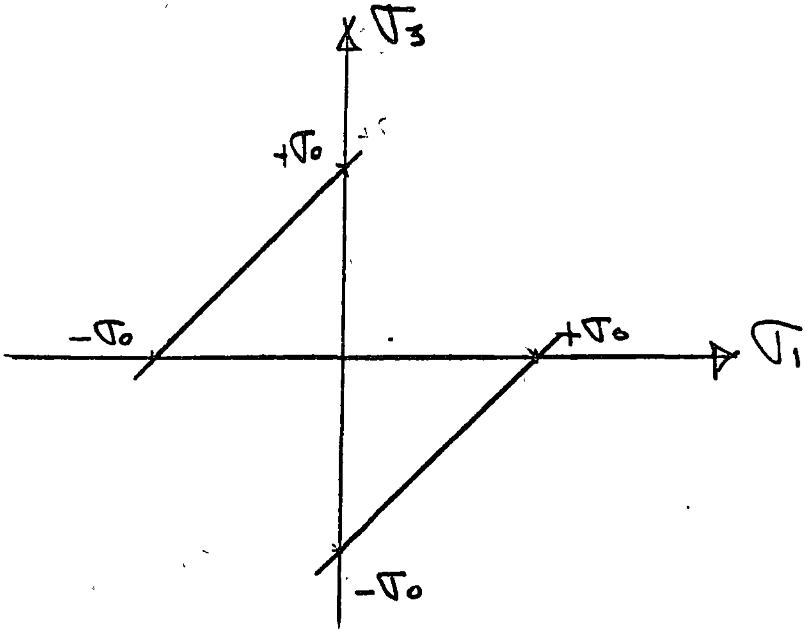


Fig. 14 Teoría del esfuerzo cortante máximo

d) Teoría de la máxima energía de deformación (Beltramí, Haig)

La densidad de energía en un medio elástico lineal viene dada por

$$U_0 = \frac{1}{2E}(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3) \quad (58)$$

de la Fig. 11 la densidad de energía hasta el límite elástico σ_0 es

$$U_0 = \frac{1}{2}\frac{\sigma_0^2}{E} \quad (59)$$

de (58) y(59) se obtiene la superficie de falla

$$f(\sigma_i) = \sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) - \sigma_0^2 = 0 \quad (60)$$

En esfuerzos planos $\sigma_3 = 0$ se obtiene

$$\frac{\sigma_1^2+\sigma_2^2}{2} - \nu\sigma_1\sigma_2 = \frac{\sigma_0^2}{2} \quad (61)$$

(61) es la ecuación de una elipse la cual en el plano de esfuerzos $\sigma_1\sigma_2$ se muestra en la Fig. 15 para el acero con $\nu = 0.3$, y las

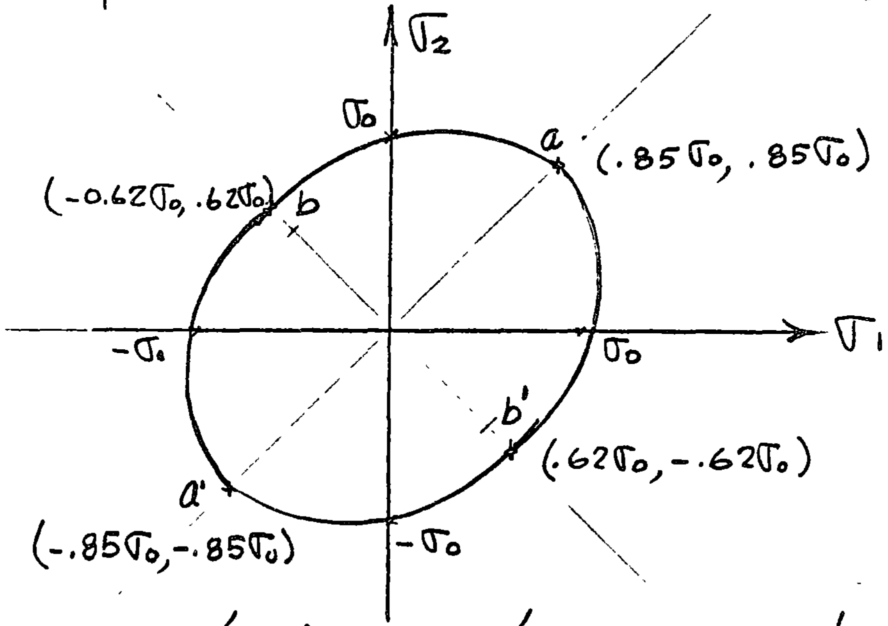


Fig. 15 Teoría de la máxima energía de deformación en el plano $\sigma_1\sigma_2$ para $\nu = 0.3$

coordenadas de los puntos a, a', b, y b'.

e) Teoría de energía máxima distorsional.
(1856, J.C. Maxwell. M.T. Huber, R.V. Mises, H. Hencky).

Los esfuerzos cortantes máximos actuan sobre el plano octaedral cuyos cosenos directores son

$\lfloor n_i \rfloor = \left[\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\right]$, y el esfuerzo normal correspondiente llamado, medio, esférico o hidrostático es

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \qquad (62)$$

la expansión volumétrica por unidad de volumen correspondiente se expresa por

$$e = e_1 + e_2 + e_3 = \frac{2(1-2\nu)}{E}\, p \qquad (63)$$

la energía por cambio unitario de volumen sea

$$U_1 = \frac{1}{2} p e \qquad (64)$$

Substituyendo (62) y (63) en (64) se obtiene

$$U_1 = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \qquad (65)$$

en un medio elástico lineal homogeneo e isotrópico la energía de deformación por unidad de volumen es

$$U_0 = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \qquad (66)$$

La densidad de energía desviatonica maxima es

$$\Delta U = U_0 - U_1 \qquad (67)$$

substituyendo (65) y (66) en (67) se obtiene

$$\Delta U = \frac{1+\nu}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right] \qquad (68)$$

el valor máximo en (68) sería sí $\sigma_2 = \sigma_3 = 0$ y (68) se transforma para $\sigma_1 = \sigma_0$ en

$$\Delta U_{max} = \frac{1+\nu}{3E}\sigma_0^2 \qquad (69)$$

por lo tanto de (68) y (69) se obtiene cuando $\Delta U = \Delta U_{max}$

$$f(\sigma_i) = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2 - 2\sigma_0^2 = 0 \qquad (70)$$

(70) es la ecuación de de un cilindro circular cuyo eje y directrices en el espacio de esfuerzo forma iguales angulos con los ejes σ_i, la intersección de (70) con el plano $\sigma_1\sigma_2$ se obtiene de (70) para $\sigma_3=0$

$$(\sigma_1-\sigma_2)^2+\sigma_1^2+\sigma_2^2-2\sigma_0 = 0 \qquad (71)$$

(71) y (61) deben ser iguales para $\nu=0.5$ material incompresible (71) representa también una elipse como en la Fig. 15 solo que las coordenadas de a, a', b y b' son para $\nu=0.3$

$a(\sigma_0, \sigma_0)$	$b(-0.577\sigma_0, 0.577\sigma_0)$
$a'(-\sigma_0, -\sigma_0)$	$b'(0.577\sigma_0, -0.577\sigma_0)$

ABCD: Teoría del esfuerzo máximo. (Rankine)

EFGH: " deformación máxima. (Saint-Venant)

——— " de máxima energía de deformación. (Beltrami)

- - - - " " " " distorcionante. (Von-Misses)

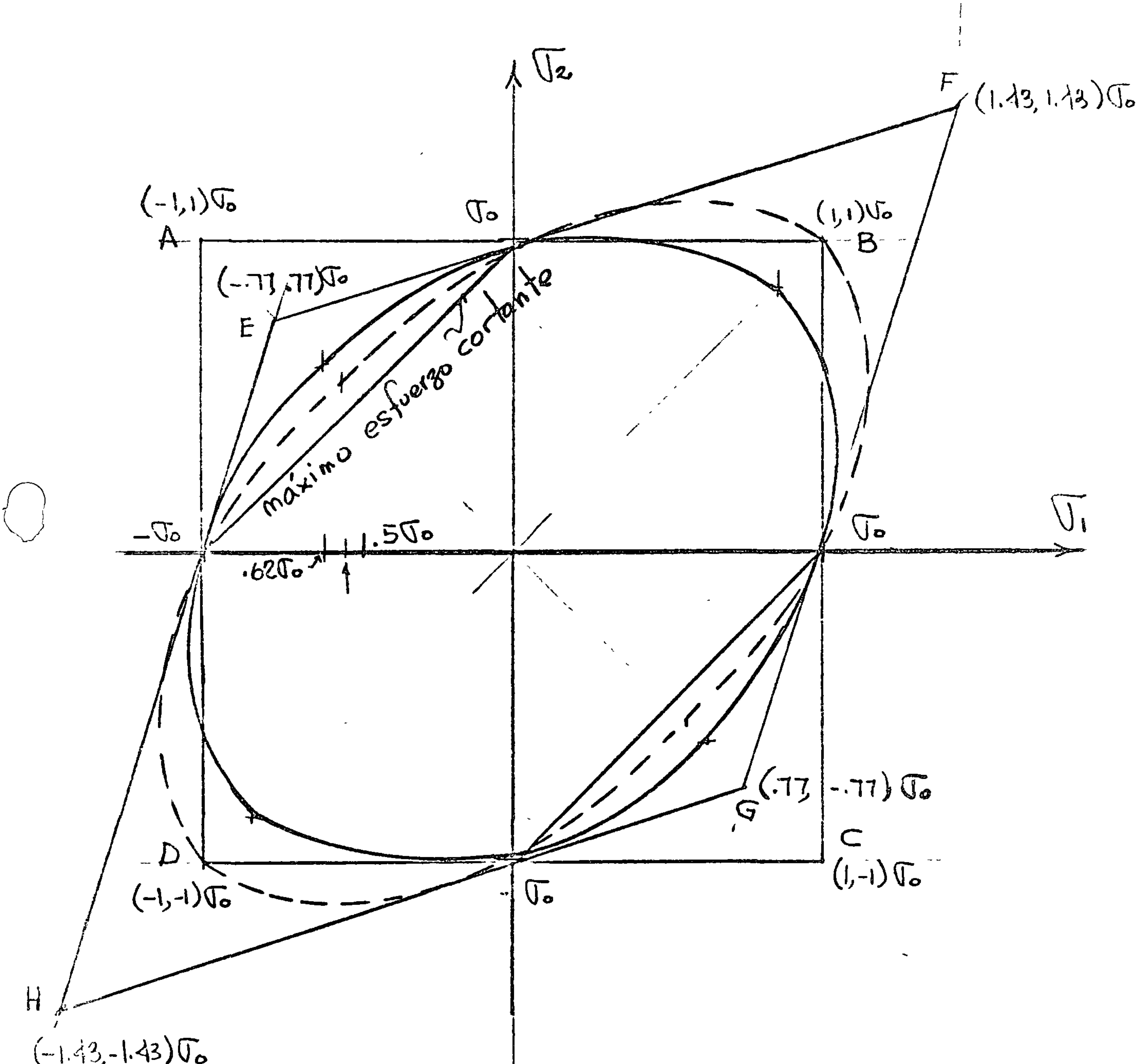
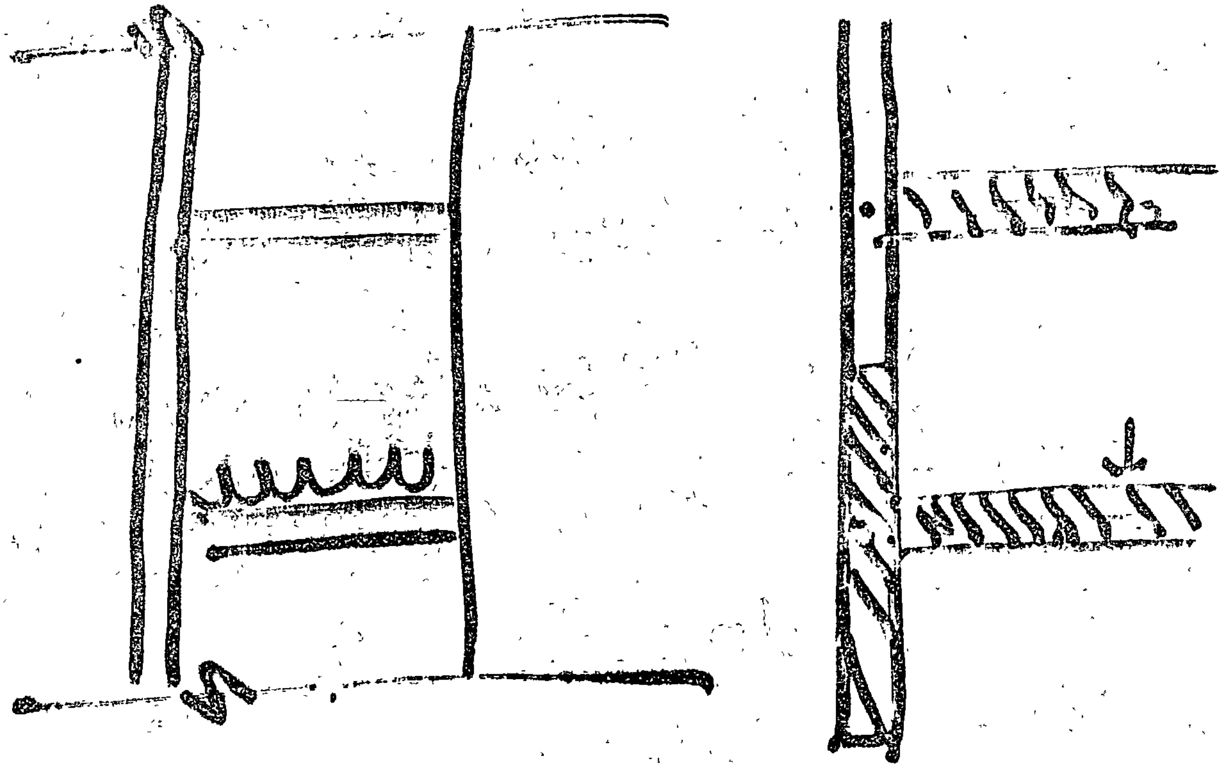
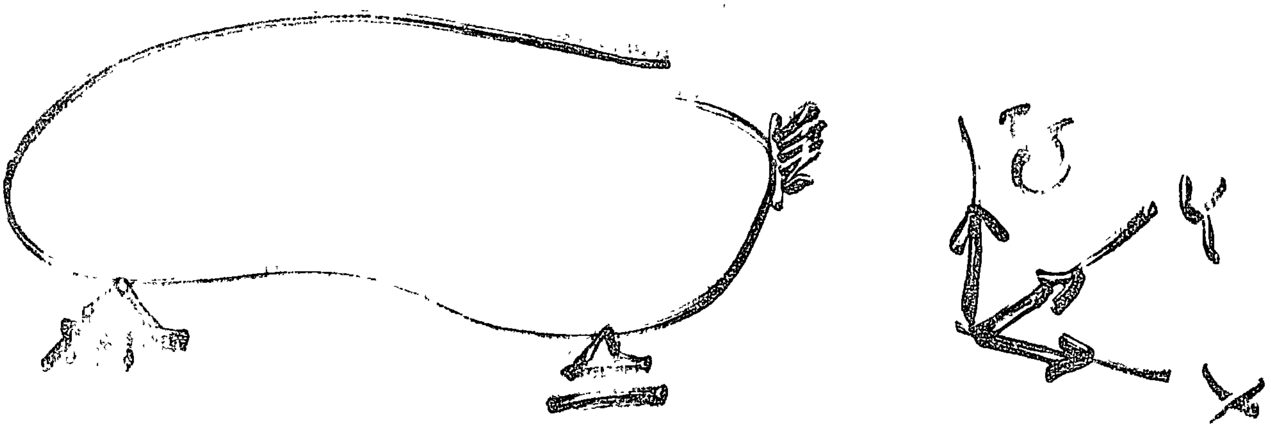


Fig. 16 Comparación entre las distintas Teorías de falla para $\nu=0.3$, $\phi=0$



El desplazamiento deberá considerarse compatible con las condiciones de apoyo



$[u, v, w]$ comp del desp en x y z

$[\delta u, \delta v, \delta w]$ " " " virtual en "

Los desp. virt. corresp. a las 6 comp. de Def. son

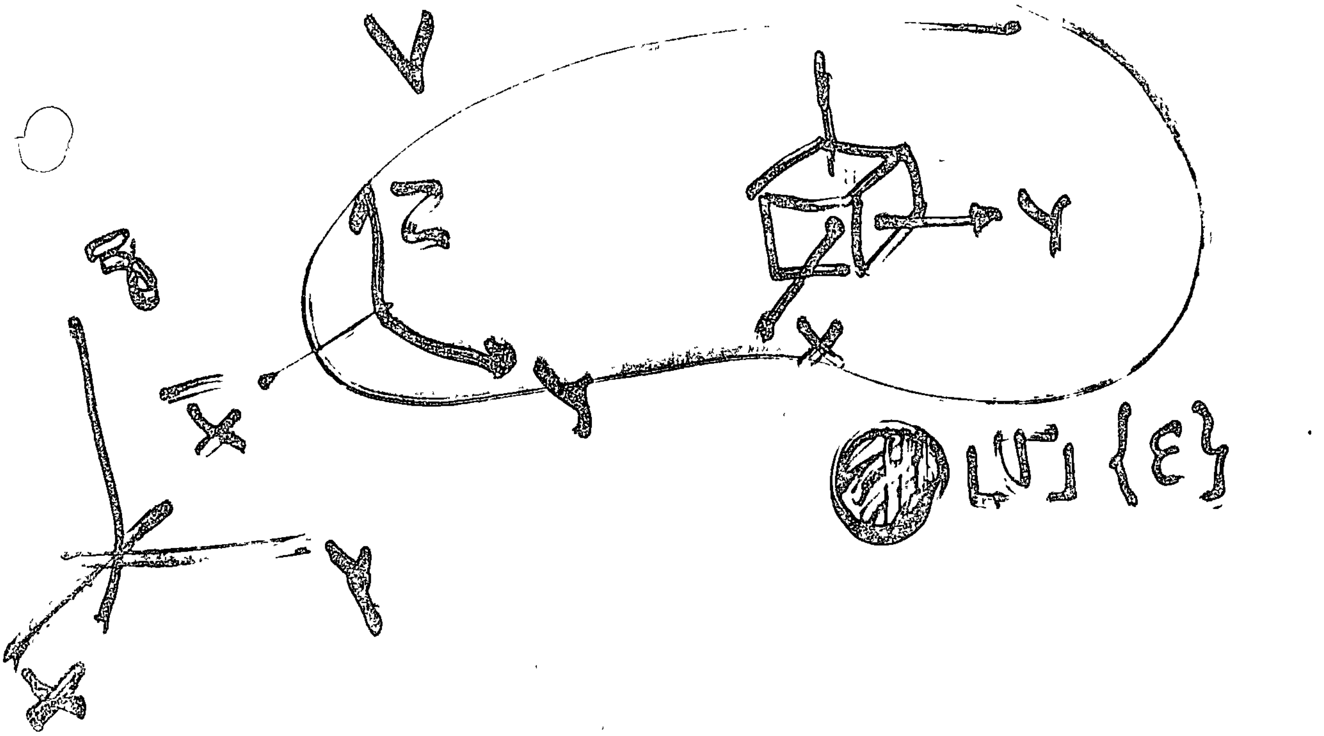
$$\delta \varepsilon_x = \frac{\partial}{\partial x}(\delta u) \qquad \delta(\gamma_{xy}) = \frac{\partial}{\partial x}(\delta v) + \frac{\partial}{\partial y}(\delta u)$$

$$\delta \varepsilon_y = \frac{\partial}{\partial y}(\delta v) \qquad \delta(\gamma_{yz}) = \frac{\partial}{\partial z}(\delta v) + \frac{\partial}{\partial y}(\delta w)$$

$$\delta \varepsilon_z = \frac{\partial}{\partial z}(\delta w) \qquad \delta(\gamma_{zx}) = \frac{\partial}{\partial x}(\delta w) + \frac{\partial}{\partial z}(\delta u)$$

①

$$\int_A (\bar{X}\,\delta u + \bar{Y}\,\delta v + \bar{Z}\,\delta w)\,dA + \int_V (X\,\delta u + Y\,\delta v + Z\,\delta w)\,dV - \int_V \delta U_0\,dV = 0$$



$[U]\{\varepsilon\}$

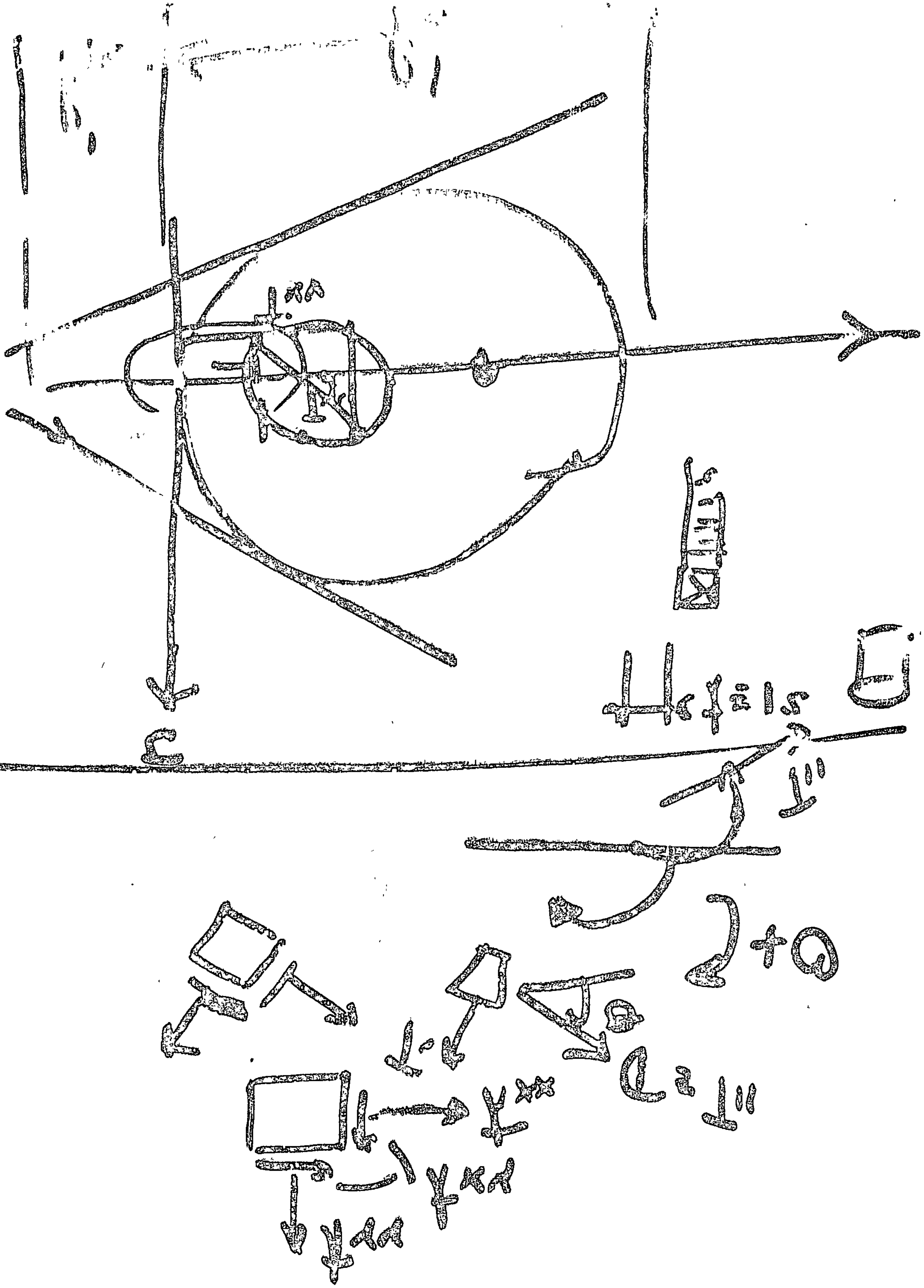
$$\boxed{\delta\underbrace{\left[\int_V \lfloor\sigma\rfloor\{\epsilon\}\,dV - \int_V \lfloor X\rfloor\{u\}\,dV - \int_A \lfloor\bar{X}\rfloor\{u\}\,dA\right]}_{\text{Energia Pot. del Sistema}} = 0}$$

$$\lfloor\sigma\rfloor = \lfloor \sigma_x\,\sigma_y \ldots \tau_{zx}\rfloor$$

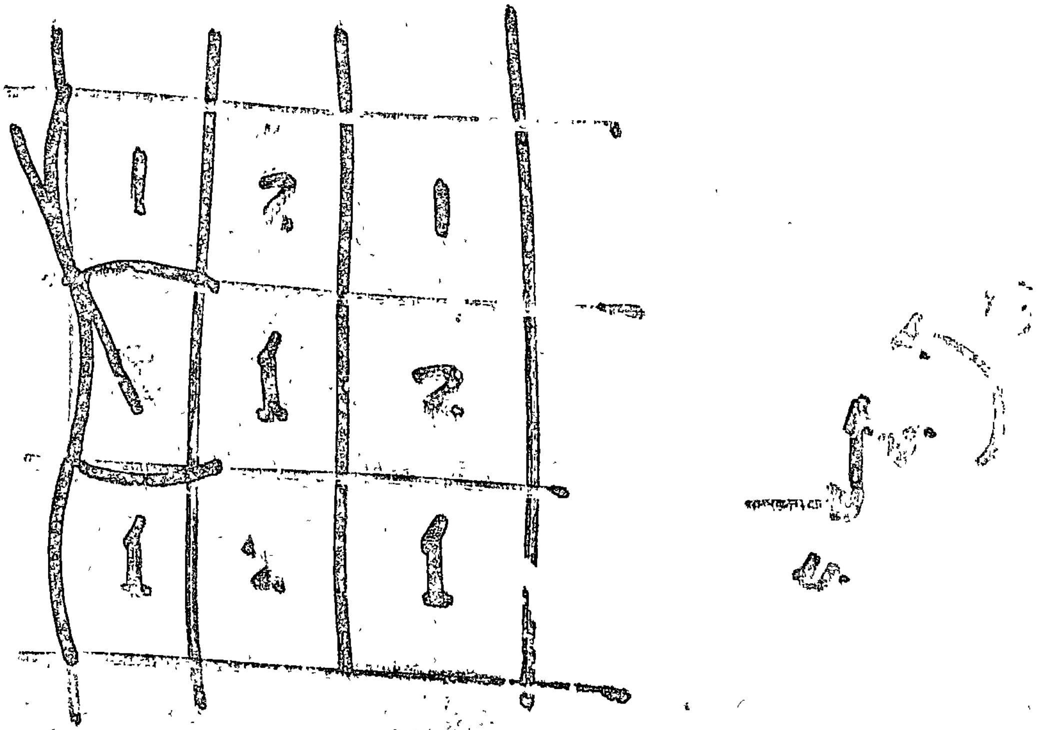
$$\{\epsilon\}^T = \lfloor \epsilon_x\,\epsilon_y \ldots \gamma_{zx}\rfloor$$

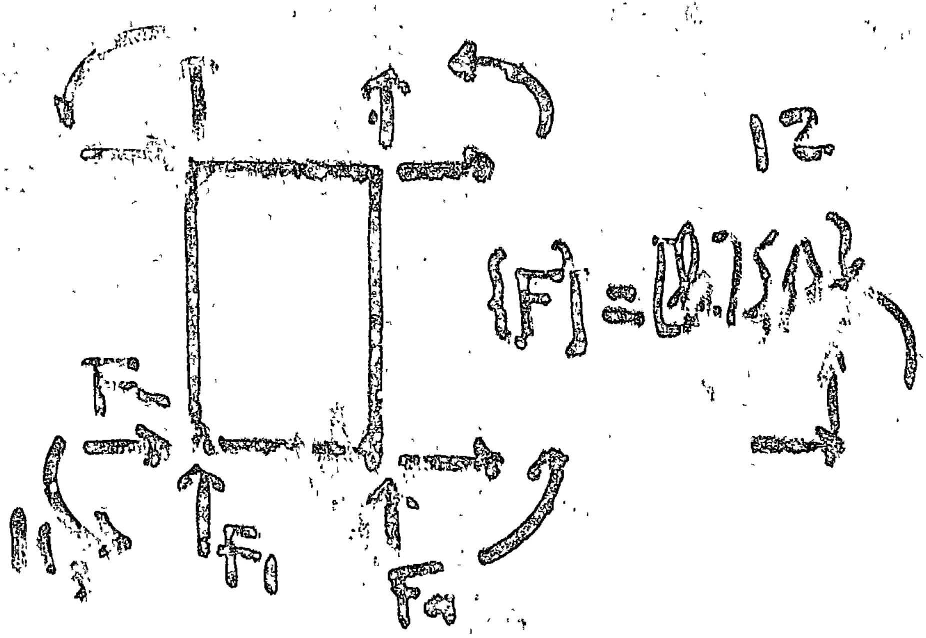
$$\lfloor X\rfloor = \lfloor X, Y, Z\rfloor$$

$$\lfloor\bar{X}\rfloor = \lfloor\bar{X}, \bar{Y}, \bar{Z}\rfloor$$



$$\left(\frac{\partial u}{\partial y}\right)_1 = \left(\frac{\partial u}{\partial y}\right)_2$$





$$\{F\} = [k]\{\Delta\}$$

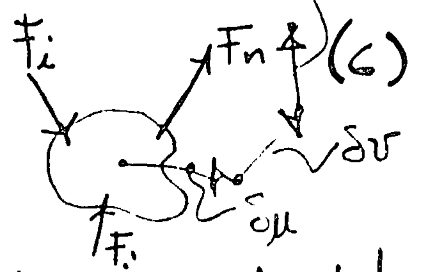
Principio del Trabajo Virtual: Si una partícula se encuentra en equilibrio, el trabajo total efectuado por todas las fuerzas actuando sobre la partícula bajo cualquier desplazamiento virtual es cero.

$\delta u, \delta v, \delta w$; componentes del desplazamiento virtual en las direcciones x, y y z

$\Sigma F_x, \Sigma F_y \ \Sigma F_z$; sumas de fuerzas en x, y y z que actuan sobre la part. o cuerpo rigido

El principio del desplazamiento virtual establece que

$$\delta u \sum_1^n F_{xi} = 0$$
$$\delta v \sum_1^n F_{yi} = 0$$
$$\delta w \sum_1 F_{zi} = 0$$



(6) se satisfacen para cualquier desplazamiento virtual si el sistema está en equilibrio

En un cuerpo elástico en reposo constituye consistuye un sistema de partículas sobre cada una actua un conjunto de fuerzas en equilibrio. En cualquier desp. virtual el trabajo virtual sobre cada partícula es cero, y por lo tanto el trabajo total virtual de todas las fuerzas del sistema es cero. El cual deberá tomarse compatible con las condiciones de continuidad y apoyo puesto que las fuerzas de superficie y los esfuerzos no cambian durante el desp. virtual pequeño, el símbolo variacional δ se puede sacar fuera del signo integral

$$⑨\quad \delta\Big[\underbrace{\int_V U_0\,dV}_{\substack{\text{energía de Def.}\\ \text{''}\\ \text{energía Pot. de Def.}}} - \underbrace{\int_V (Xu+Yv+Zw)\,dV}_{\substack{\text{Energía Pot. de}\\ \text{las fuerzas de cuerpo}}} - \underbrace{\int_A (\bar{X}u+\bar{Y}v+\bar{Z}w)\,dA}_{\substack{\text{Energía pot.}\\ \text{de las fuerzas}\\ \text{de superficie}}}\Big]=0$$

$$\boxed{\delta\Big[\underbrace{\#\int_V \lfloor\sigma\rfloor\{\varepsilon\}\,dV - \int_V \lfloor X\rfloor\{u\}\,dV - \int_A \lfloor\bar{X}\rfloor\{u\}\,dA}_{\text{Energía potencial total del Sistema}}\Big]=0}\quad ⑩$$

$\lfloor\sigma\rfloor = [\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}]$

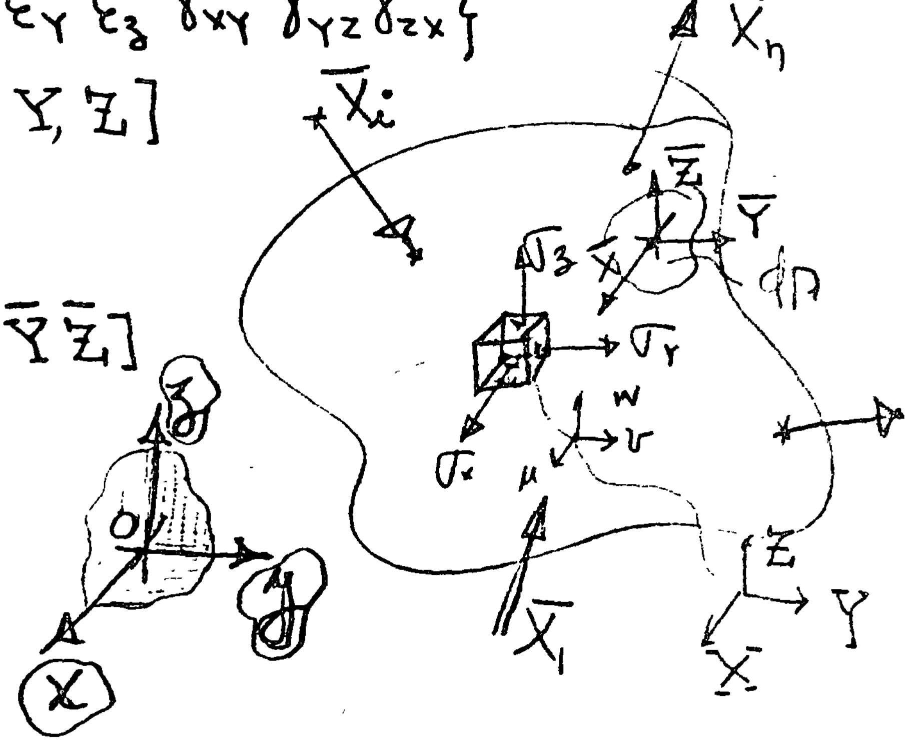
$\{\varepsilon\}^T = \{\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \gamma_{xy}\ \gamma_{yz}\ \gamma_{zx}\}^T$

$\lfloor X\rfloor = [X, Y, Z]$

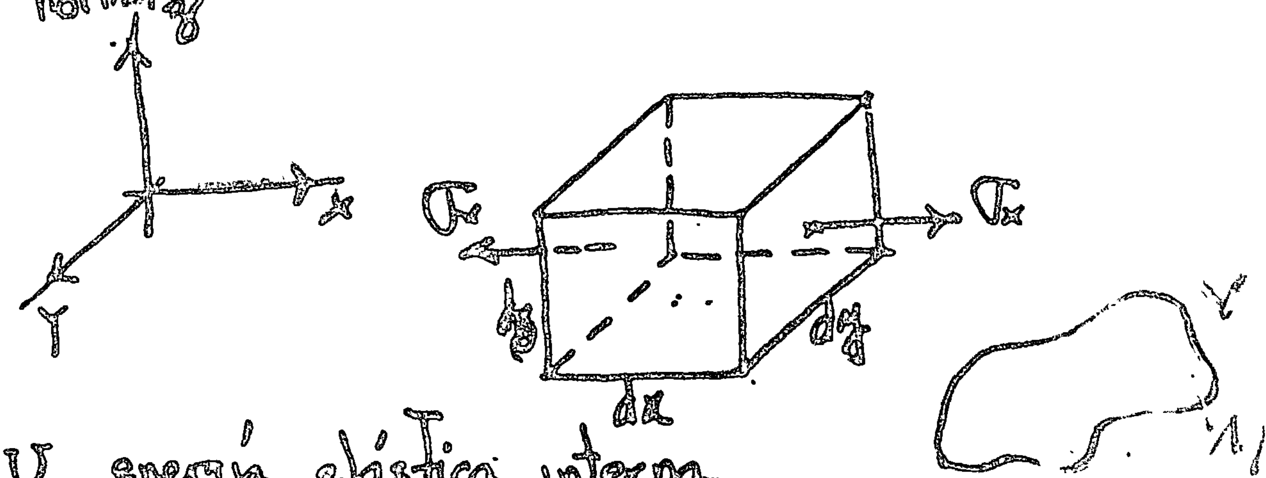
$\{u\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$

$\lfloor\bar{X}\rfloor = [\bar{X}\ \bar{Y}\ \bar{Z}]$

u

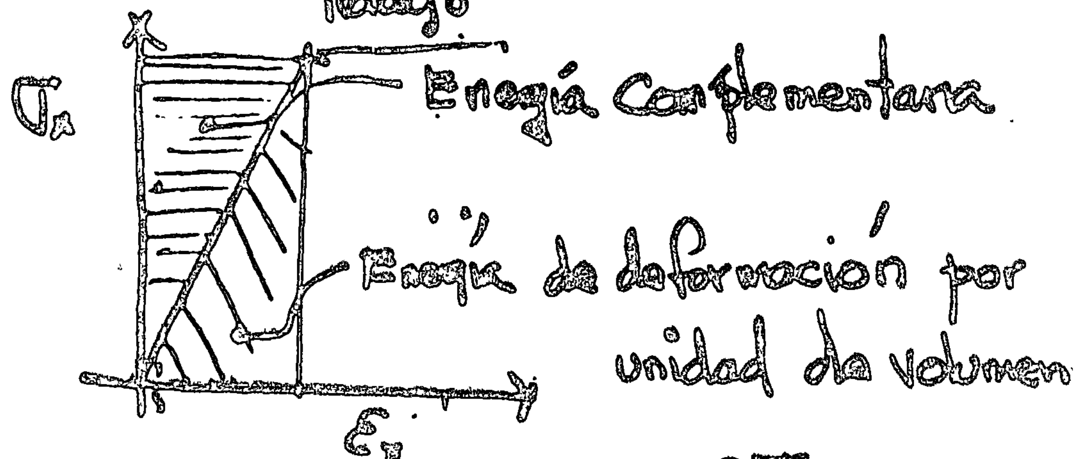


Energía Elástica de Deformación por esf. normal



U energía elástica interna

$$dU = \underbrace{\underbrace{\tfrac{1}{2}\sigma_x\, dy\, dz}_{\text{Fuerza promedio}} \times \underbrace{e_x\, dx}_{\text{distancia}}}_{\text{Trabajo}} = \tfrac{1}{2}\sigma_x e_x \underbrace{dx\,dy\,dz}_{V} \quad (1)$$



Para un cuerpo elástico perfecto no hay disipación de energía, y el trabajo hecho por un elemento es almacenado como energía de deformación interna recuperable

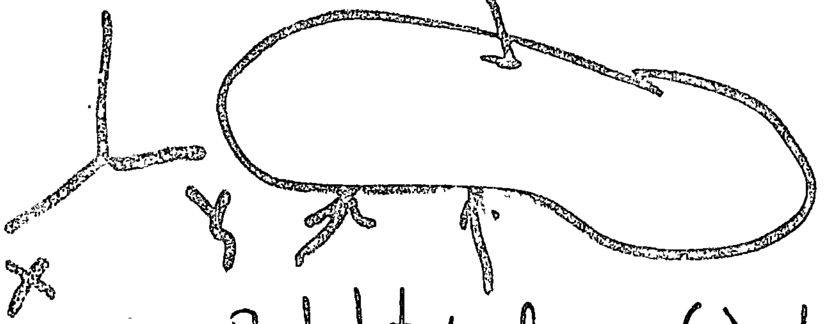
De (1) la densidad de energía

$$\frac{dU}{dV} = U_0 = \frac{\sigma_x \epsilon_x}{2} \quad (2)$$

$$\frac{dU}{dV} = \underline{U_0} = \frac{1}{2}\sigma_x \epsilon_x + \frac{1}{2}\sigma_y \epsilon_y + \frac{1}{2}\sigma_z \epsilon_z + \frac{1}{2}\underline{\tau_{xy}\gamma_{xy}} + \frac{1}{2}\underline{\tau_{yz}\gamma_{yz}} + \frac{1}{2}\underline{\tau_{zx}\gamma_{zx}} \qquad (5)$$

Expresando (5) matricialmente se obtiene

$$U_0 = \frac{1}{2}\left[\sigma_x\ \sigma_y\ \sigma_z\ \tau_{xy}\ \tau_{yz}\ \tau_{zx}\right]\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix} = \frac{1}{2}\lfloor\sigma\rfloor\{\epsilon\} \qquad (6)$$



Substituyendo en (5) la ley generalizada de Hooke (7)

$\{\epsilon\} = [C]\{\sigma\}$

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} & \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} & \gamma_{yz} &= \frac{\tau_{yz}}{G} \qquad (7) \\ \epsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} & \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned}$$

se obtiene

$$U_0 = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) \qquad (8)$$

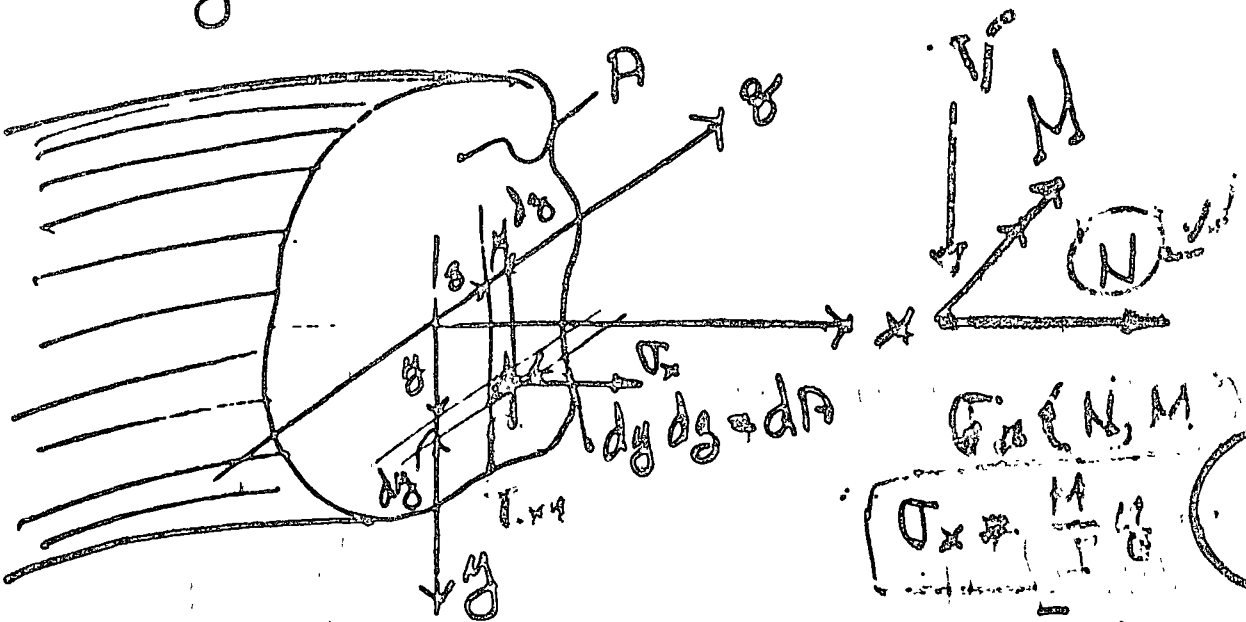
Para materiales elásticos lineales homogeneos e isotropicos se puede obtener una expresión similar a 8) en términos de las deformaciones en lugar de los esfuerzos, la energía total se obtiene de

$$U = \iiint U_0\, dx\, dy\, dz \qquad (9)$$

Energía de deformación para barras cargadas axialmente

$\sigma_x = \frac{N}{A} = \frac{\text{carga axial}}{\text{sección transversal}}$, $A = \iint_A dy\,dz$ 14

N y A son funciones de x solamente



$\sigma_x(N, M)$

$\boxed{\sigma_x = \frac{M}{I}y}$ $\frac{N}{A}$

Por lo tanto (13) se reduce a [de (14) y (13)]

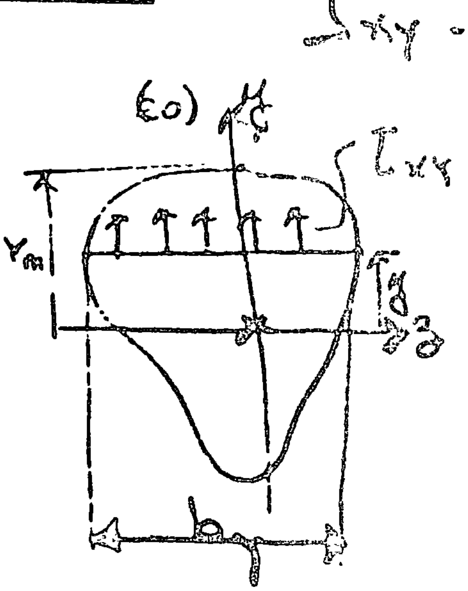
$$U_N = \iiint_V \frac{\sigma_x^2}{2E}\,dV = \iiint \frac{N^2}{2A^2E}\,dx\,dy\,dz$$

$$= \int_L \frac{N^2}{2A^2E}\underbrace{\left[\iint_A dy\,dz\right]}_{A} dx = \int_L \frac{N^2}{2EA}\,dx$$

$$\boxed{U_N = \int_L \frac{N^2}{2EA}\,dx} \longrightarrow \qquad (15)$$

Energía de Deformación por Cortante

En este caso $\tau_{xy} = \dfrac{VQ_Y^{Ym}}{b_Y I_z}$ (20)



V = Cortante en la sección

$Q_Y^{Ym} = \int_y^{Ym} y\,dA$ = momento estático de y a y_m

b = ancho a la altura y de los ejes centroidales

I = Momento de Inercia de la sección

$Q(y)$
$b(y)$

Subst (20) en (13)

$$U_V = \iiint \frac{1}{2G}\left(\frac{VQ_Y^{Ym}}{bI}\right)^2 dx\,dy\,dz = \int \frac{V^2}{2GI^2}\underbrace{\left[\iint\left(\frac{Q_Y^{Ym}}{b}\right)^2 dy\,dz\right]}_{f} dx$$

$$\boxed{U_V = \int_L \frac{V^2}{2GI^2}\left[\iint\left(\frac{Q_Y^{Ym}}{b}\right)^2 dy\,dz\right] dx} \quad (21)$$

$\dfrac{A}{A_s}$

$\dfrac{\partial U}{\partial M} = 0$

La expresión total de la energía de deformación será: $U = U_N + U_M + U_T + U_V$ o sea

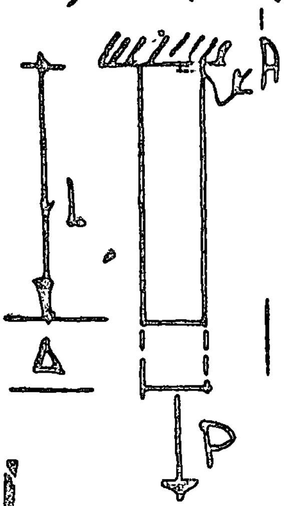
$$\boxed{U = \int_L \left\{\frac{N^2}{2EA} + \frac{M^2}{2EI} + \frac{M_T^2}{2GJ} + \frac{V^2}{2GI^2}\left[\iint\left(\frac{Q_Y^{Ym}}{b}\right)^2 dy\,dz\right]\right\} dx} \quad (22)$$

$$W_e + W_i = 0 \quad (24)$$

$U = - W_i$ las deformaciones siempre se oponen a las fuerzas internas. Es importante considerar la aplicación gradual de las cargas de cero a su valor total por lo tanto We será 1/2 Fuerza total por el desplazamiento

Ejemplos

a) Determine la deflexión de la viga mostrada



$W_e = \frac{1}{2} P\Delta$ y de (22)

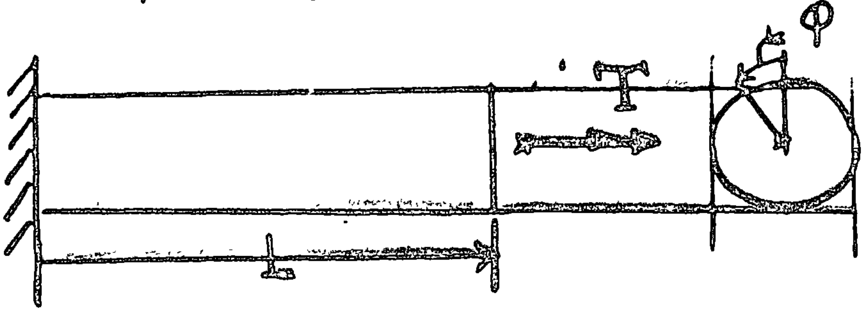
$$U = \frac{1}{2EA}\int_0^L N^2 dx$$

$$U = \frac{P^2}{2EA}\int_0^L dx = \frac{P^2L}{2EA}$$

De (23) $\frac{1}{2}\cancel{P}\Delta = \frac{P^{\cancel{2}}L}{2EA}$

$$\boxed{\Delta = \frac{PL}{AE}}$$ Ley de Hooke

b) Determine la rotación en el extremo de una flecha de sección circular



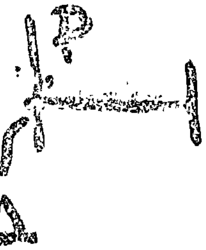
obtiene

$$U_{corte} = \iiint_V \frac{\tau^2}{2G}\,dx\,dy\,dz = \frac{1}{2G}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left\{\frac{P}{2I}\left[\left(\frac{h}{2}\right)^2 - y^2\right]\right\}^2 L\,b\,dy$$

$$= \frac{P^2Lb}{8GI^2}\times\frac{h^5}{30} = \frac{P^2Lbh^5}{240G}\left(\frac{12}{bh^3}\right)^2 = \frac{3P^2L}{5AG}$$

donde $A = bh$ sección Transversal. Entonces

$$W_e = U = U_{FLEXION} + U_{CORTE}$$



$$\frac{P\Delta}{2} = \frac{P^2L^3}{6EI} + \frac{3P^2L}{5AG}$$ de donde

$$\Delta = \underbrace{\frac{PL^3}{3EI}}_{\text{Flexión}} + \underbrace{\frac{6PL}{5AG}}_{\text{Corte}} \qquad (24)$$

El término debido al cortante se puede interpretar

$$\tau_{av} = \frac{P}{A} = \frac{V}{A}$$ corte promedio

puesto que τ varía parabólicamente $\frac{6}{5}$ representa un factor de corrección numérico por lo tanto

$$\Delta_{corte} = \gamma_s L = \alpha\frac{\tau_{av}}{G}L = \alpha\frac{VL}{AG} = \frac{6}{5}\frac{PL}{AG}$$

el valor α depende de la forma de la sección, en general V puede variar con x. De (24)

$$\Delta = \frac{PL^3}{3EI}\left(1 + \frac{3E}{10G}\frac{h^2}{L^2}\right) \qquad (25)$$

suponiendo acero estructural

$$\frac{E}{G} = 2\underset{1.2}{(1+\nu)} \doteq 2.5$$ y (25) queda

$$\gamma_{\theta+\frac{\pi}{2}}=\frac{\partial v}{\partial x}\text{sen}^2\theta-\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\text{sen}\,\theta\cos\theta-\frac{\partial u}{\partial y}\cos^2\theta \quad (15.3)$$

La deformación unitaria de corte en las direcciones PQ y PN es:

$$\gamma_\theta=\gamma_\theta-\gamma_{\theta+\frac{\pi}{2}} \quad (15.4)$$

Substituyendo (15.2) y (15.3) en (15.4) se obtiene

$$\gamma_\theta=\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)(\cos^2\theta-\text{sen}^2\theta)+\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)2\,\text{sen}\,\theta\cos\theta$$

ó

$$\boxed{\frac{1}{2}\gamma_\theta=\frac{1}{2}\gamma_{xy}(\cos^2\theta-\text{sen}^2\theta)+(\varepsilon_y-\varepsilon_x)\,\text{sen}\,\theta\cos\theta} \quad (15.5)$$

Comparando (15.1) y (15.5) con (14.1) se observa que son ecuaciones similares si se reemplaza:

σ por ε_θ
τ por $\frac{\gamma_\theta}{2}$
σ_x por ε_x
σ_y por ε_y
τ_{xy} por $\frac{\gamma_{xy}}{2}$
α por θ

$$\sigma=\sigma_x\cos^2\alpha+\sigma_y\,\text{sen}^2\alpha+2\tau_{xy}\,\text{sen}\,\alpha\cos\alpha$$
$$\tau=\tau_{xy}(\cos^2\alpha-\text{sen}^2\alpha)+(\sigma_y-\sigma_x)\,\text{sen}\,\alpha\cos\alpha$$

$$\varepsilon_\theta=\varepsilon_x\cos^2\theta+\varepsilon_y\,\text{sen}^2\theta+\gamma_{xy}\,\text{sen}\,\theta\cos\theta$$
$$\frac{\gamma_\theta}{2}=\frac{\gamma_{xy}}{2}(\cos^2\theta-\text{sen}^2\theta)+(\varepsilon_y-\varepsilon_x)\,\text{sen}\,\theta\cos\theta$$

16. Medición de deformaciones. Las deformaciones unitarias en superficies son medidas por medio de resistencias électricas (Medidores de deformación) pegadas a la superficie, existen diversas formas de resistencias electricas, cuando la deformación ocurre la resistencia electrica varía, por lo que conociendo esta ley de variación, la deformación puede ser medida electricamente.

El uso de los medidores de deformación es simple cuando las direcciones principales son conocidas; se colocan medidores, uno en cada dirección y medidas directas de e_1 y e_2 son hechos. Los esfuerzos principales σ_1, σ_2 se calculan de la Ley de Hooke y la relación de Poisson con $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, $\sigma_3 = 0$

$$\left.\begin{aligned} \sigma_1 &= \frac{E}{1-\nu^2}(\varepsilon_1 + \nu\varepsilon_2) \\ \sigma_2 &= \frac{E}{1-\nu^2}(\varepsilon_2 + \nu\varepsilon_1) \end{aligned}\right\} \text{ — } (16.1)$$

Si consideramos las direcciones principales las x, y, las equaciones (15.1) y (15.5) se transforman a:

$$\left.\begin{aligned}\varepsilon_\theta &= \varepsilon_1 \cos^2\theta + \varepsilon_2 \operatorname{sen}^2\theta \\ \tfrac{1}{2}\gamma_\theta &= -(\varepsilon_1 - \varepsilon_2)\operatorname{sen}\theta\cos\theta\end{aligned}\right\} \quad (16.2)$$

ó

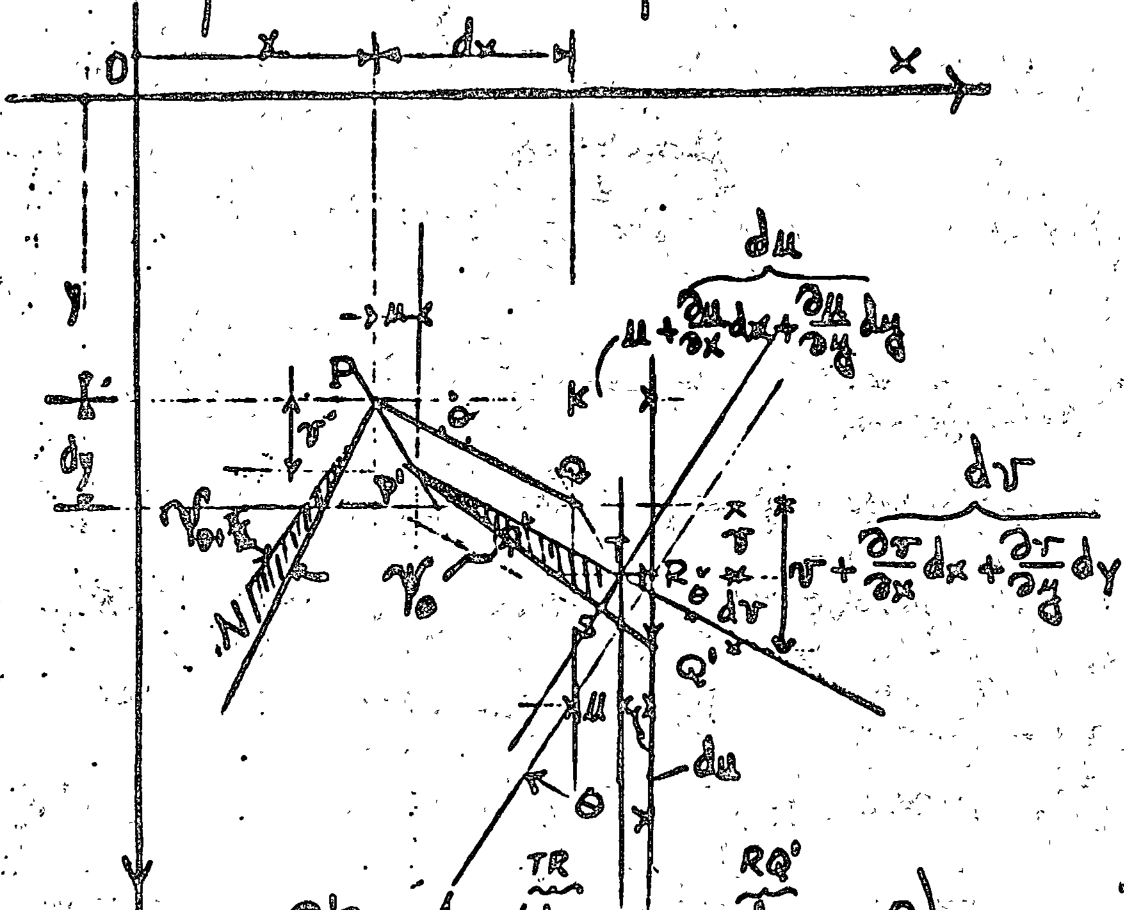
$$\left.\begin{aligned}\varepsilon_\theta &= \tfrac{1}{2}(\varepsilon_1+\varepsilon_2) + \tfrac{1}{2}(\varepsilon_1-\varepsilon_2)\cos 2\theta \\ \tfrac{1}{2}\gamma_\theta &= -\tfrac{1}{2}(\varepsilon_1-\varepsilon_2)\operatorname{sen} 2\theta\end{aligned}\right\} \quad (16.3)$$

(16.3) son representadas por el punto P de la figura (16.1). Si θ toma el valor de ϕ P corresponde al punto A de la Figura (16.1)

EL PROBLEMA ES TRAZAR EL CIRCULO DE MOHR CUANDO LAS TRES ABCISAS ε_ϕ, $\varepsilon_{\phi+\alpha}$, $\varepsilon_{\phi+\alpha+\beta}$ y los angulos α, y β son conocidos.

$O'\varepsilon_\phi$ eje auxiliar sobre el que se trazan ε_ϕ $\varepsilon_{\phi+\alpha}$, $\varepsilon_{\phi+\alpha+\beta}$.
D punto seleccionado al azar soble la vertical $\varepsilon_{\phi+\alpha}$.
De alli se trazan rectas con los angulos α y β con la vertical hasta intersectar las verticales ε_ϕ y $\varepsilon_{\phi+\alpha+\beta}$ el circulo a traves de A D y C es el circulo requerido

15. Deformaciones en el punto. Cuando las componentes de deformacion e_x, ε_y, γ_{xy} en un punto (x.y) son conocidas, la elongacion unitaria en cualquier dirección, la disminución del angulo recto o la deformación unitaria por corte, en cualquier dirección pueden encontarse.



$$e_\theta = \frac{Q'S}{PQ} = \frac{1}{ds}\left(\overset{TR}{du\cos\theta} + \overset{RQ'}{dv\,\mathrm{sen}\,\theta}\right)$$

$$e_\theta = \cos\theta\,\frac{du}{ds} + \mathrm{sen}\,\theta\,\frac{dv}{ds}$$

$$= \cos\theta\left(\frac{\partial u}{\partial x}\times\frac{dx}{ds} + \frac{\partial u}{\partial y}\times\frac{dy}{ds}\right) + \mathrm{sen}\,\theta\left(\frac{\partial v}{\partial x}\,\frac{dx}{ds} + \frac{\partial v}{\partial y}\,\frac{dy}{ds}\right)$$

1 Cuadrica de Esfuerzos de Cauchy, superficies de esfuerzos, Esfuerzos principales, Invariantes

Las componentes del tensor de esfuerzo en notación índice e Ingeniería son

$$[\sigma_{ij}] = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix} = \begin{bmatrix}\sigma_{x} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_{y} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_{z}\end{bmatrix} \qquad (1)$$

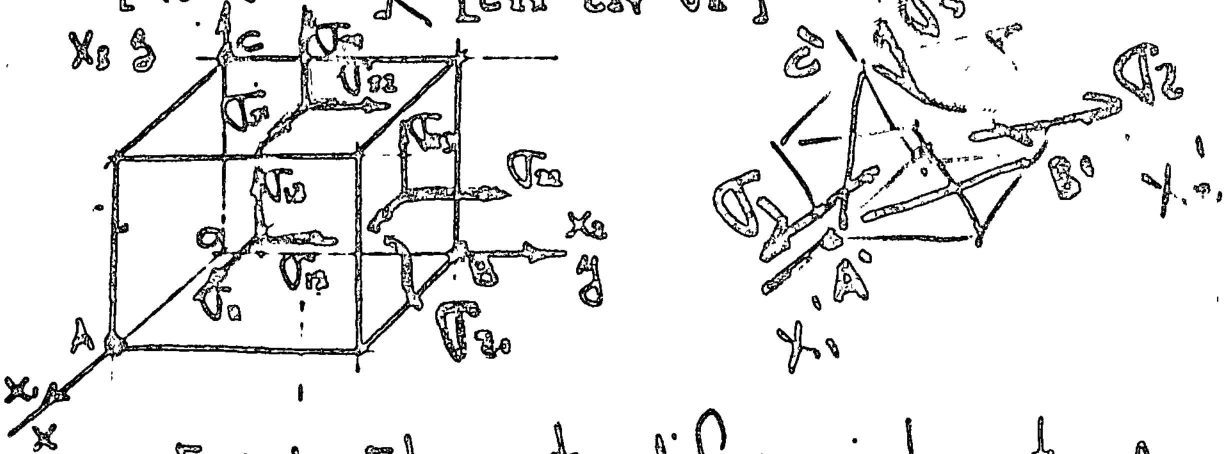


Fig. 1 Elemento diferencial, actuando los esfuerzos $[\sigma_{ij}]$.

Llevando un plano a travez de ABC y considerando su diagrama de cuerpo libre se tiene

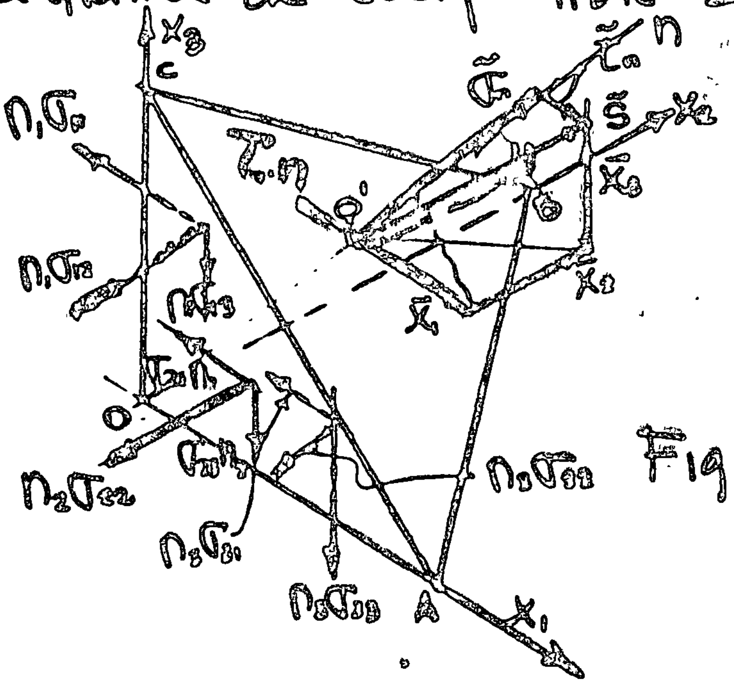


Fig. 2

ABC = 1

$$n_1 = \frac{OBC}{ABC} = \cos \alpha$$

$$\sigma_{ij} = \sigma_{ji}$$

$$X_1 = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$X_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \qquad (3)$$
$$X_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$$

expresando (3) matricialmente se obtiene

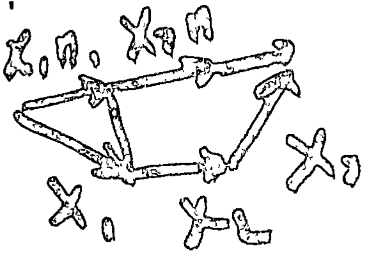
$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (4)$$

no polar

~~Si no existen momentos de cuerpo~~, $\sigma_{ij} = \sigma_{ji}$ ~~para~~ $i \neq j$

y $[\sigma_{ij}] = [\sigma_{ij}]^T$ por lo que (4) puede escribirse

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \qquad (5)$$



ó $$\{X_i\} = [\sigma_{ij}]\{n_i\} \qquad (6)$$

El esfuerzo normal al plano ABC es

$$\tilde{\sigma}_n = \tilde{X}_1 n_1 + \tilde{X}_2 n_2 + \tilde{X}_3 n_3 \qquad (7)$$

ó $$\sigma_n = \{X_i\}^T \{n_i\} \qquad (8)$$

Substituyendo (5) en (7) se obtiene

$$\sigma_n = \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 + 2(\sigma_{12} n_1 n_2 + \sigma_{23} n_2 n_3 + \sigma_{31} n_3 n_1) \qquad (9)$$

ó matricialmente de (6) y (8)

$$\sigma_n = \cancel{[\sigma_{ij}]^T \{n_i\}^T} \{n_i\} = \{n_i\}^T [\sigma_{ij}]^T \{n_i\} \qquad (10)$$

De (6) se obtiene

$$\sigma_n^3 - \underbrace{(\sigma_{11}+\sigma_{22}+\sigma_{33})}_{I_1}\sigma_n^2 + (\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{23}^2-\sigma_{31}^2)\sigma_n$$
$$-(\sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{31}^2-\sigma_{33}\sigma_{12}^2) = 0 \quad (17)$$

las tres raices de la ecuación (7) nos determinan los valores de los esfuerzos principales σ_1, σ_2 y σ_3 cuyos coeficientes nos representan los invariantes de esfuerzos, dependen de σ_1, σ_2 y σ_3 independientes del sistema de ejes coordinados

$$I_1 = \sigma_{11}+\sigma_{22}+\sigma_{33} \equiv \sigma_1+\sigma_2+\sigma_3$$
$$I_2 = \sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}-\sigma_{12}^2-\sigma_{23}^2-\sigma_{31}^2 \equiv \sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1 \quad (18)$$
$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33}+2\sigma_{12}\sigma_{23}\sigma_{31}-\sigma_{11}\sigma_{23}^2-\sigma_{22}\sigma_{13}^2-\sigma_{33}\sigma_{12}^2 \equiv \sigma_1\sigma_2\sigma_3$$

donde I_1, I_2 e I_3 son los invariantes de esfuerzos, otras expresiones de invariantes pueden formarse de (18) por ejemplo (19)

$$2I_1-6I_2 = (\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2+6(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2)$$

(19) se usa en la expresión de la energía de deformación, su uso se discutirá posteriormente

Substituyendo (20) en (21) se obtiene la ecuación

$$\frac{X_1^2}{\sigma_1^2}+\frac{X_2^2}{\sigma_2^2}+\frac{X_3^2}{\sigma_3^2}=1 \qquad (22)$$

la cual representa una superficie elipsoidal en el espacio de esfuerzos σ_i, algunos autores lo denominan elipsoide de Lamé, en la Fig. 4 se muestra su perspectiva isométrica. Para el conjunto

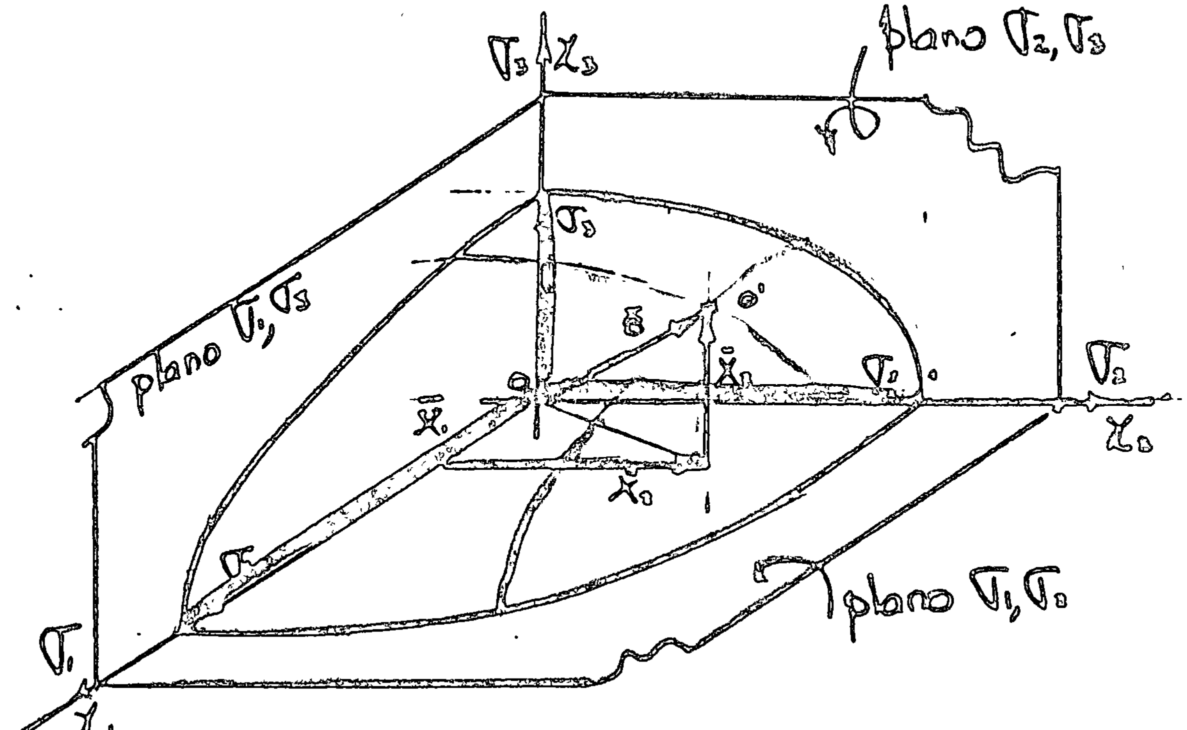


Fig. 4 Elipsoide de Lamé referido al espacio de esfuerzos σ_i, (un octagono).

de planos con cosenos directores $\{n_i\}$ a travez de o Fig. 2, le corresponde al conjunto de componentes $\{X_i\}$, los cuales junto con los esfuerzos principales σ_1, σ_2 y σ_3 forman la superficie elipsoidal de la Fig. 4.

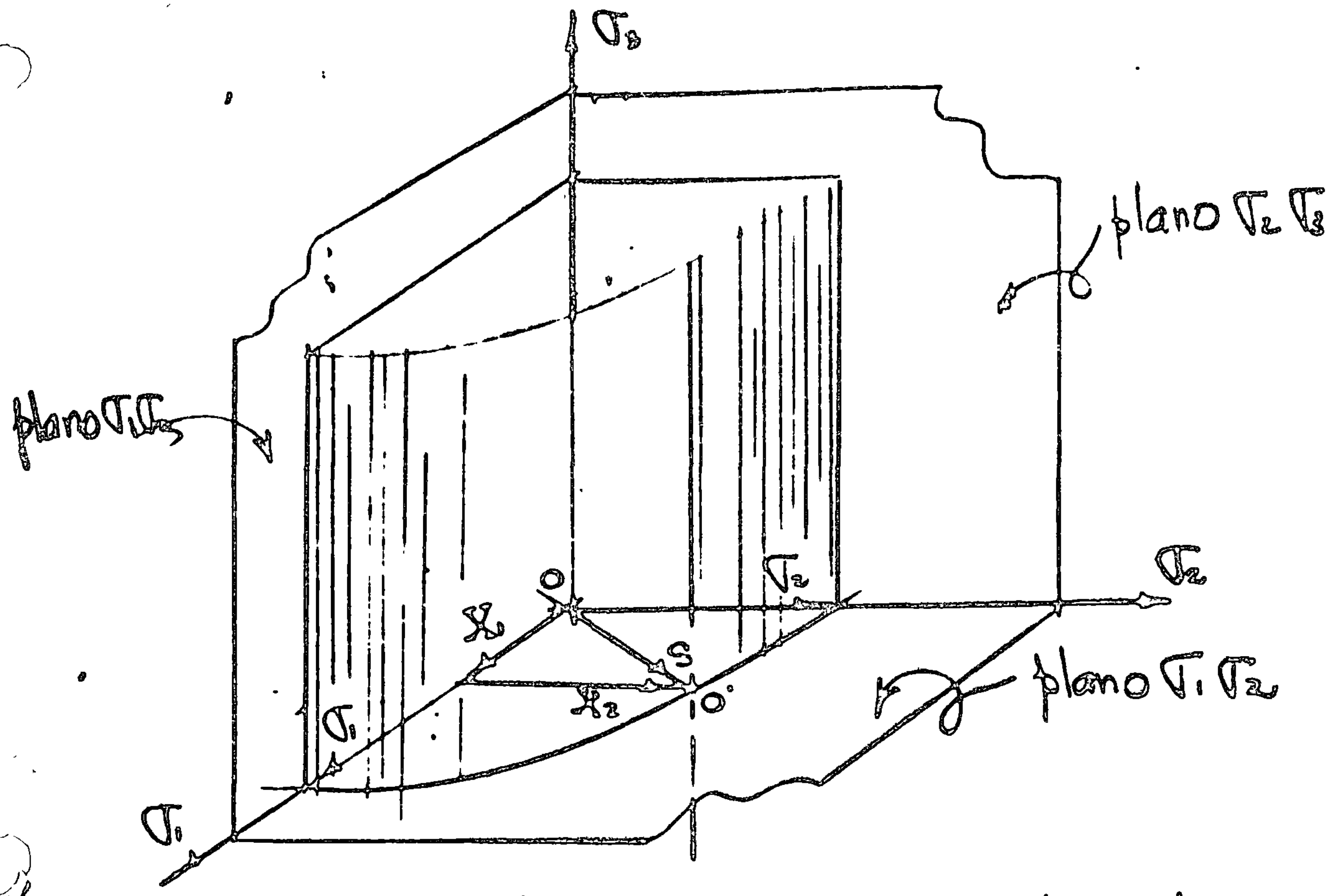


Fig. 6 Superficie cilíndrica de sección elíptica directrices paralelas al eje $O\sigma_3$.

Componentes del tensor de esfuerzos: $[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (25)

Ecuación de la superficie: $\frac{X_1^2}{\sigma_1^2} + \frac{X_2^2}{\sigma_2^2} = 1$ (26)

Como caso particular de de (25) si $\sigma_1 = \sigma_2 = \sigma$ se tiene un cilindro con componentes del tensor de esfuerzos $[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (27)

y ecuación de la superficie (28)

$$X_1^2 + X_2^2 = \sigma^2$$

La ecuación (21) en el espacio de cosenos directores nos representa una esfera de radio unitario como se muestra en la Fig. 7

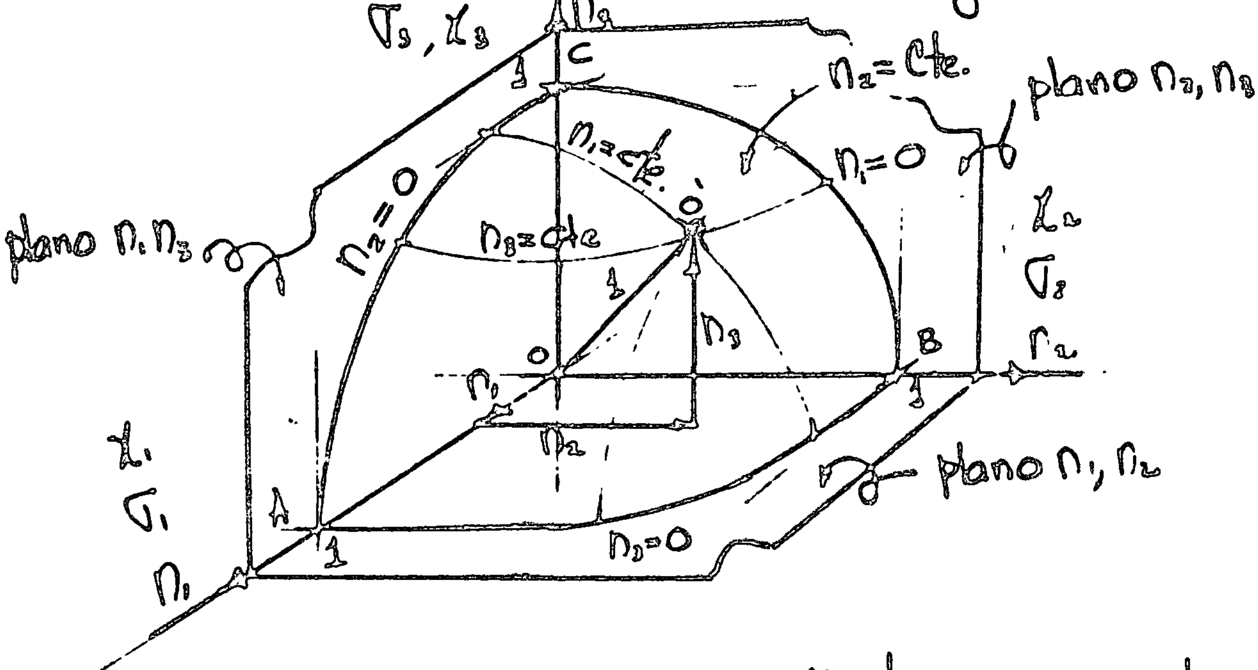


Fig. 7 Espacio de cosenos directores. un octagono de la esfera de Mohor.

$\overline{OA} = \overline{OB} = \overline{OC} = \overline{OO'} = 1$

De la Fig. 3 se observa que substituyendo (20) en (7) se obtiene

$$\sigma_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \qquad (31)$$

Substituyendo (20) y (31) en (11) y (12) se obtiene

$$\tau_n^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2 - (\sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2)^2 \qquad (32)$$

de las ecuaciones (31), (32) y (21) se obtiene el siguiente sistema de 3 ecuaciones con 3 incognitas no lineal en $\underline{n_1}$, $\underline{n_2}$ y $\underline{n_3}$

$$\tau_1 = \sqrt{n_1^2(\sigma_2-\sigma_1)(\sigma_3-\sigma_1)+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2} \qquad (38)$$

Similarmente suponiendo n_2 = constante de (35) se obtiene

$$\tau_2 = \sqrt{n_2^2(\sigma_3-\sigma_2)(\sigma_1-\sigma_2)+\left(\frac{\sigma_1-\sigma_3}{2}\right)^2} \qquad (39)$$

Similarmente suponiendo n_3= constante de (36) se obtiene

$$\tau_3 = \sqrt{n_3^2(\sigma_1-\sigma_3)(\sigma_2-\sigma_3)+\left(\frac{\sigma_1-\sigma_2}{2}\right)^2} \qquad (40)$$

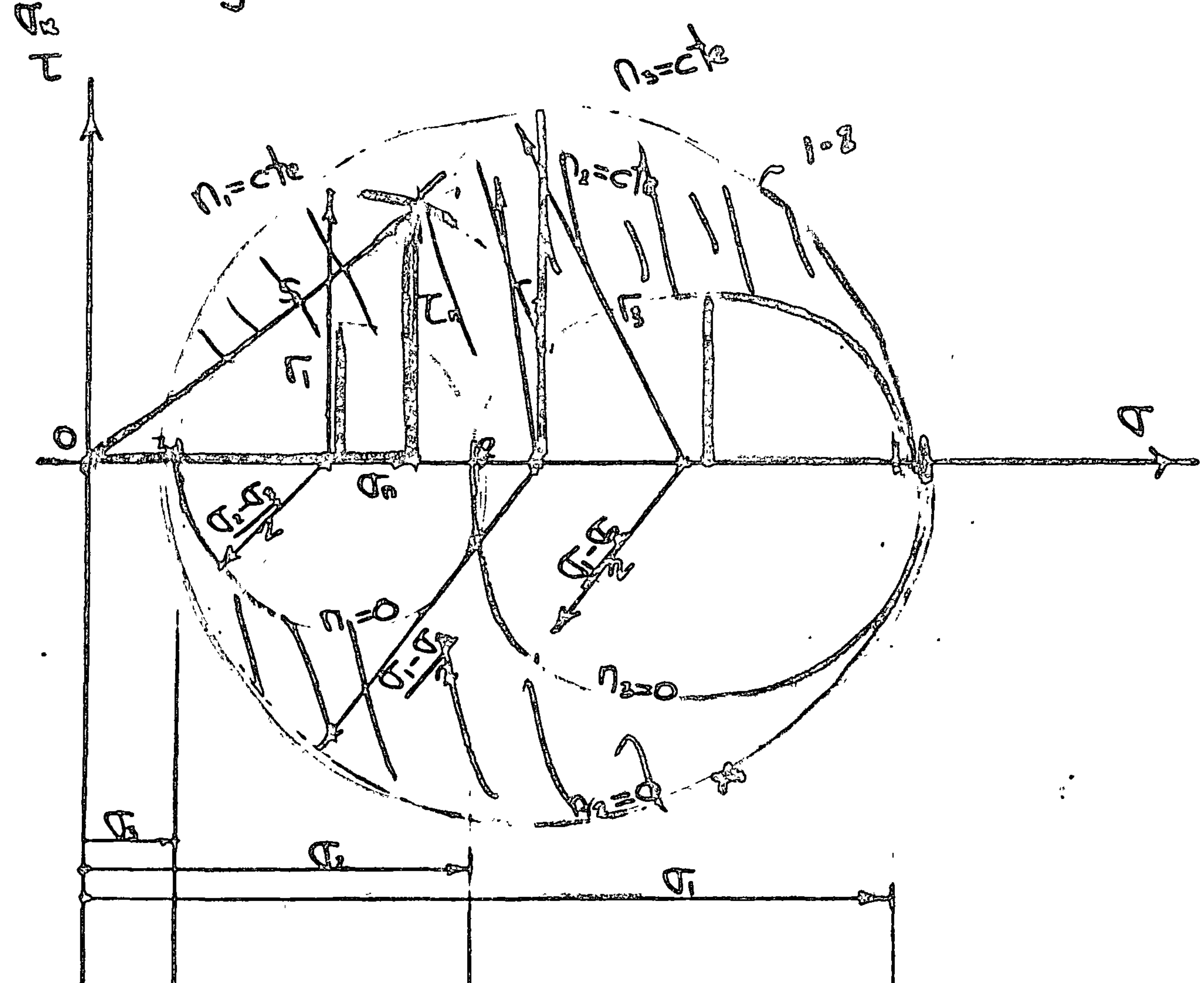


Fig. 8 Circulo de Mohor en Tres dimensiones determinar σ_n, τ_n, conociendo $\sigma_1, \sigma_2, \sigma_3$ y n_1, n_2 y n_3

no hay soluciones de n_1 y n_2 que sean ambos diferentes de cero, porque las expresiones dentro del parentesis no pueden anularse.

	n_1	n_2	n_3	n_1	n_2	n_3
n_1	0	0	±1	0	$\pm\sqrt{\frac{1}{2}}$	$\pm\sqrt{\frac{1}{2}}$
n_2	0	±1	0	$\pm\sqrt{\frac{1}{2}}$	0	$\pm\sqrt{\frac{1}{2}}$
n_3	±1	0	0	$\pm\sqrt{\frac{1}{2}}$	±0	$\pm\sqrt{\frac{1}{2}}$
	Esf. Principales $\tau_n=0$			Cortantes máximos		

Tabla 1 Cosenos directores

Repitiendo los calculos en (44), eliminado n_1 y determinando n_2 y n_3 tal que τ_n sea máximo y después n_2 y determinando n_1 y n_3 tal que τ_n sea máximo se obtienen los valorees

$$(\tau_{max})_1 = \tau_1 = \pm\frac{1}{2}(\sigma_2-\sigma_3) \tag{47}$$
$$(\tau_{max})_2 = \tau_2 = \pm\frac{1}{2}(\sigma_1-\sigma_3)$$
$$(\tau_{max})_3 = \tau_3 = \pm\frac{1}{2}(\sigma_1-\sigma_2)$$

de (47) y (32) se puede expresar τ_n en la siguiente forma

$$\tau_n^2 = 4(n_1^2 n_2^2 \tau_3^2 + n_2^2 n_3^2 \tau_1^2 + n_1^2 n_3^2 \tau_2^2) \tag{48}$$

Las 3 primeras columnas de la Tabla 1 dan las direcciones de los planos coordenados de las direcciones principales, para ellos $\tau_n=0$ y (32) es un minimo, las tres columnas restantes dan planos a travez de un eje principal bisectando los otros dos direcciones de esfuergos principales, substituyendo los valores de Tabla 1 en (32)

ó
$$\tau_{oct} = \frac{1}{3}\sqrt{[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]} \qquad (49)$$

de (48) y (49) se obtiene

$$\tau_{oct} = \sqrt{\frac{1}{3}[(\sigma_1-\sigma_n)^2+(\sigma_2-\sigma_n)^2+(\sigma_3-\sigma_n)^2} \qquad (50)$$

al esfuerzo de corte dado por (49) y (50) es llamado esfuerzo octaedral de corte, porque la cara donde actua es la cara ABC del octaedro regular de la Fig. 9 que tiene vertices en los ejes coordenados, se usa frecuentemente en Teoría de Plasticidad

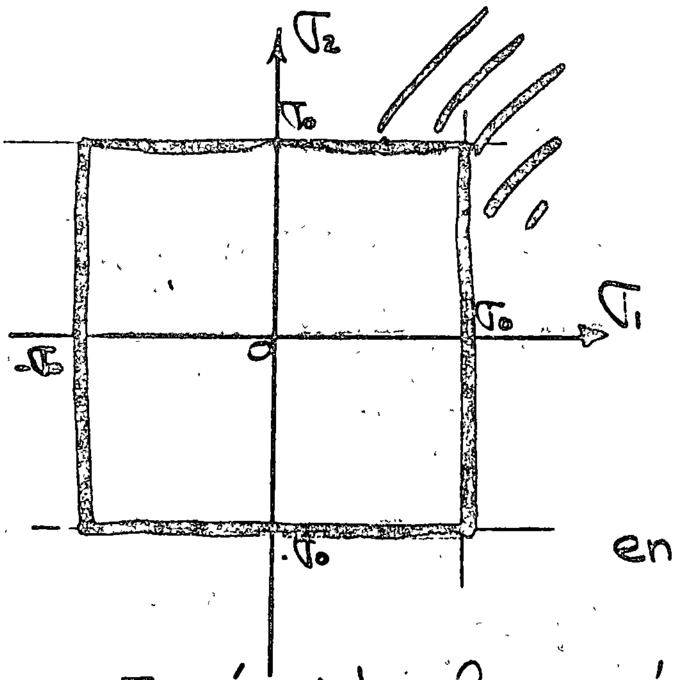
a) Teoría del Máximo esfuerzo (Rankine).

Se supone que $\sigma_1 = \sigma_0'$ o $\sigma_3 = \sigma_0'$

σ_0' esfuerzo de fluencia en tensión

σ_0'' " " " " compresión

ó σ_0' y σ_0'' pueden ser dos esfuerzos de fluencia en dos direcciones perpendiculares, suponiendo un estado plano de esfuerzos $\sigma_3 = 0$ y que $\sigma_1 = \sigma_2 = \sigma_0$ se obtiene el diagrama de esfuerzos de la Fig. 12



Planos de falla

$\sigma_1 = \pm \sigma_0$

$\sigma_2 = \pm \sigma_0$

$\sigma_3 = \pm \sigma_0$

Superficie cubica en el espacio de esfuerzos

Fig. 12 Teoría del esfuerzo máximo en esfuerzos planos

b) Teoría de la deformación maxima (Saint-Venant)

Condición triaxial de esfuerzos que alcanza la deformación de fluencia ε_0.

$$\varepsilon_0 = \frac{\sigma_0}{E} = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \qquad (52)$$

de (52) la superficie de esfuerzos referida al

c) Teoría del Esfuerzo Cortante Máximo (Coulomb)

Si $\sigma_1 > \sigma_2 > \sigma_3$, Coulomb establece que la falla se alcanza cuando

$\tau = c + \sigma\,\mathrm{Tan}\,\phi$

$$(\tau_2)_{max} = \frac{\sigma_1 - \sigma_3}{2} = \pm\frac{1}{2}\sigma_0 \qquad (56)$$

$\phi = 0$

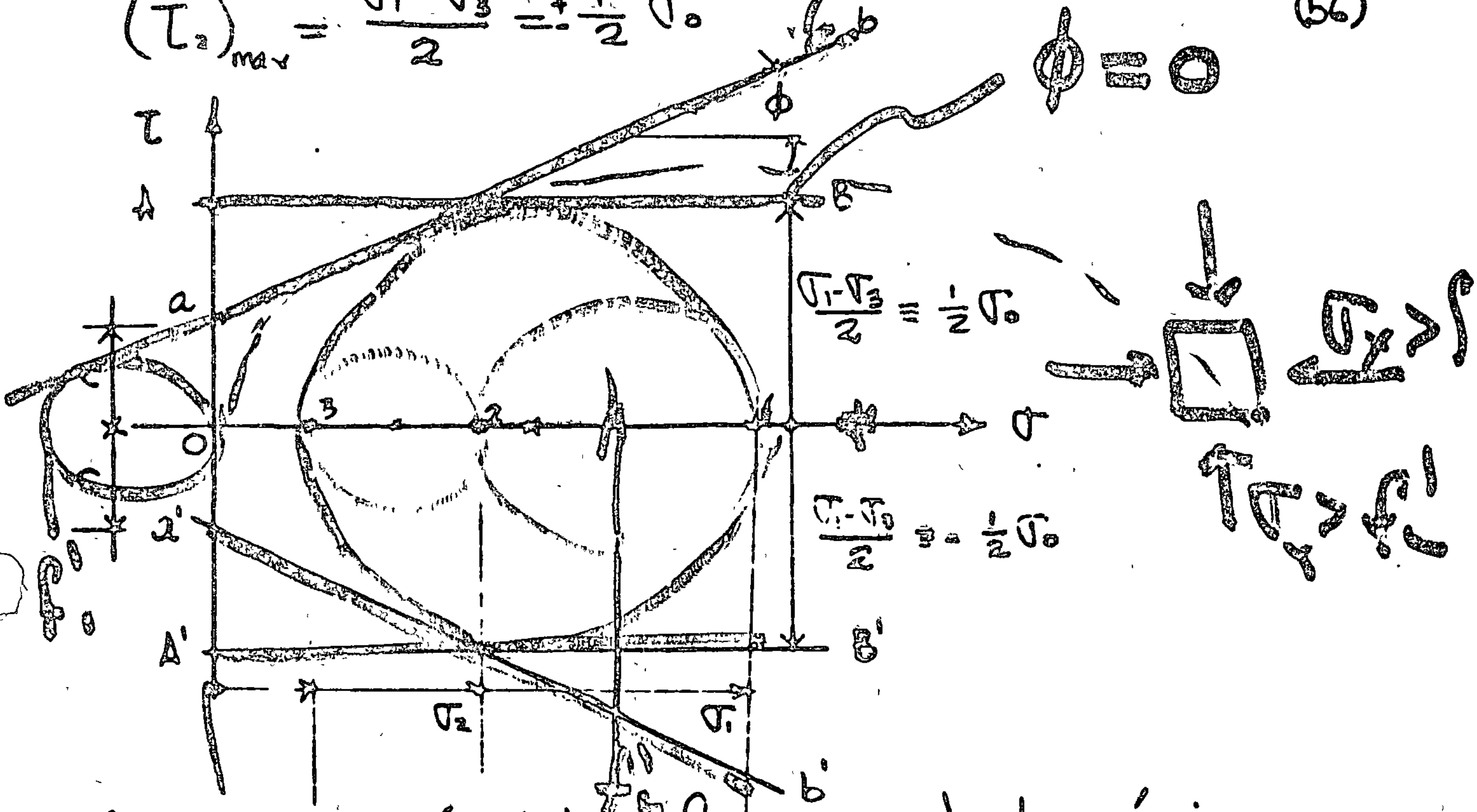


Fig. 13 Teoría del esfuerzo cortante máximo

(56) en el diagrama de Mohor establece como rectas de falla a AB y A'B' en Fig. 13 cuando el ángulo de fricción interna $\phi = 0$, y cuando $\phi > 0$ las rectas de falla son las ab y a'b' cuya ecuación es igual a

$$\underline{\tau_{max} = c + \sigma\,\mathrm{Ton}\,\phi} \qquad (57)$$

c = cohesión o resistencia al esfuerzo cortante puro
ϕ = angulo de fricción interna
σ = esfuerzo de falla.

$$\underset{\sigma_{ij}}{f(\sigma_i)} = \sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) - \sigma_0^2 = 0 \quad (60)$$

En esfuerzos planos $\underline{\sigma_3} = 0$ se obtiene

$$\frac{\sigma_1^2+\sigma_2^2}{2} - \nu\sigma_1\sigma_2 = \frac{\sigma_0^2}{2} \quad (61)$$

(61) es la ecuación de una elipse la cual en el plano de esfuerzos $\sigma_1\sigma_2$ se muestra en la Fig. 15 para el acero con $\underline{\nu = 0.3}$, y los coordenadas de los puntos $a, a', b,$ y b'.

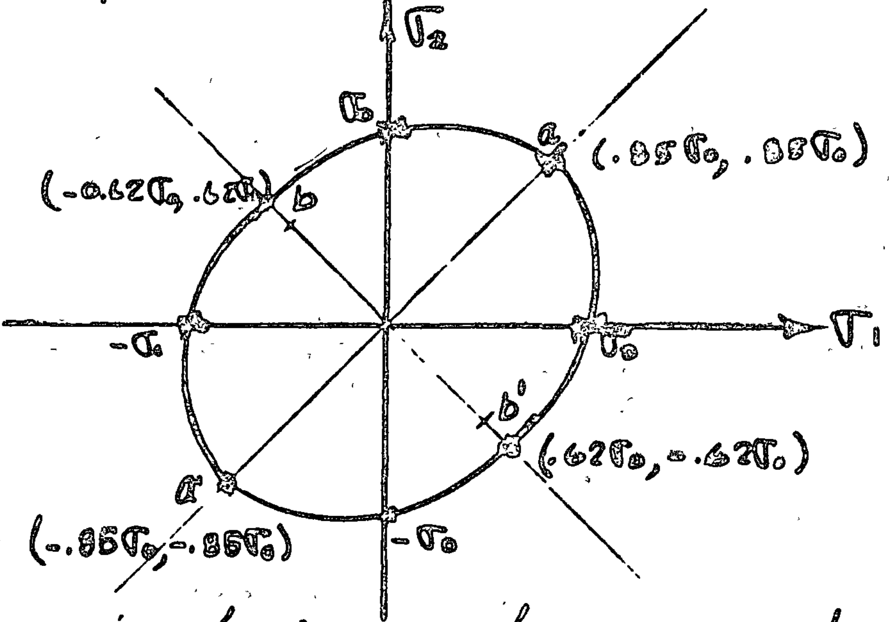


Fig. 15 Teoría de la máxima energía de deformación en el plano $\sigma_1\sigma_2$ para $\nu = 0.3$

e) Teoría de energía máxima distorsional

(1856, J.C. Maxwell, M.T. Huber, R.V. Mises, H. Hencky).

Los esfuerzos cortantes máximos actuan sobre el plano octaedral cuyos cosenos directores son

$$\Delta U = \frac{1+\nu}{6E}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right] \qquad (68)$$

el valor máximo en (68) sería si $\sigma_2=\sigma_3=0$ y (68) se transforma para $\sigma_1=\sigma_0$ en

$$\Delta U_{max} = \frac{1+\nu}{3E}\sigma_0^2 \qquad (69)$$

por lo tanto de (68) y (69) se obtiene cuando $\Delta U = \Delta U_{max}$

$$f(\sigma_i) = (\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2 - 2\sigma_0^2 = 0 \qquad (70)$$

(70) es la ecuación de de un cilindro circular cuyo eje y directrices en el espacio de esfuerzos forma iguales angulos con los ejes σ_i, la intersección de (70) con el plano $\sigma_1\sigma_2$ se obtiene de (70) para $\sigma_3=0$

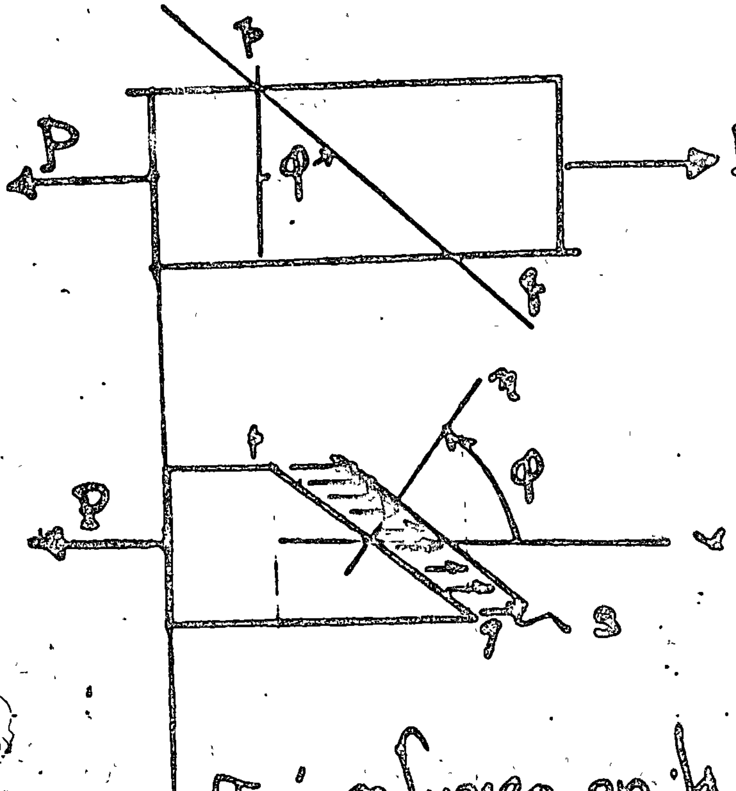
$$(\sigma_1-\sigma_2)^2+\sigma_1^2+\sigma_2^2-2\sigma_0 = 0 \qquad (71)$$

(71) y (61) deben ser iguales para $\nu=\underline{0.5}$ material incompresible (71) representa también una elipse como en la Fig. 15 solo que las coordenadas de a, a', b y b' son para $\nu=0.3$.

$a(\sigma_0, \sigma_0)$ $b(-0.577\sigma_0, 0.577\sigma_0)$

$a'(-\sigma_0, -\sigma_0)$ $b'(0.577\sigma_0, -0.577\sigma_0)$

CIRCULO DE MOHR



Area fg = $\frac{A}{\cos\varphi}$

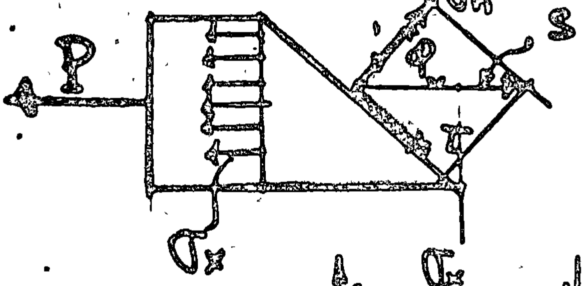
A sección transversal

$$\left.\begin{array}{l} s = \frac{P}{A/\cos\varphi} = \frac{P\cos\varphi}{A} \\ s = \sigma_x . \cos\varphi \end{array}\right\} (1)$$

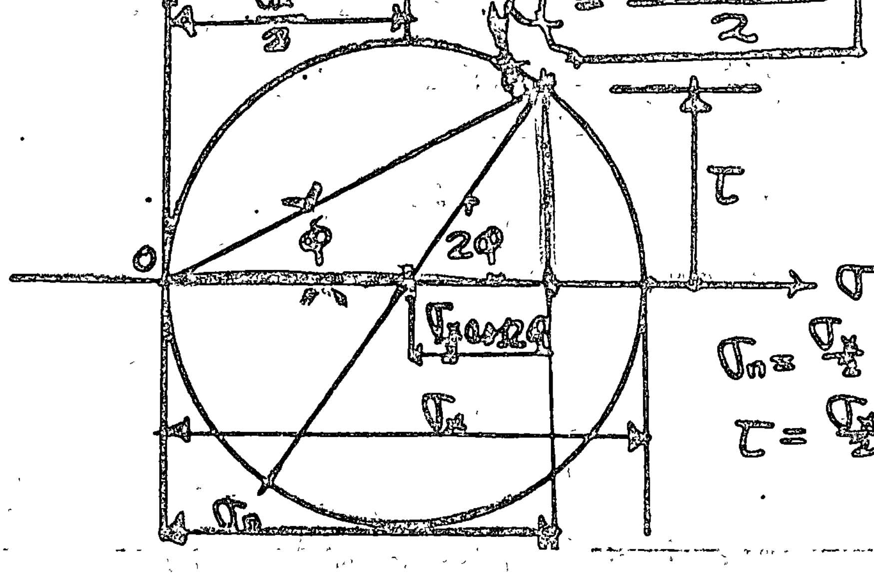
σ_x' esfuerzo en la sección recta

$\sigma_n = s \cos\varphi = \sigma_x \cos^2\varphi$ (2)

$\tau = s \operatorname{sen}\varphi = \sigma_x \operatorname{sen}\varphi \cos\varphi = \frac{\sigma_x \operatorname{sen} 2\varphi}{2}$



$$\boxed{\begin{array}{l}\sigma_n = \sigma_x \cos^2\varphi \\ \tau = \frac{\sigma_x \operatorname{sen} 2\varphi}{2}\end{array}}$$ (2) (2)

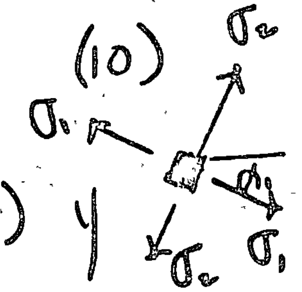


$\sigma_n = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 2\varphi = \sigma_x \cos^2\varphi$

$\tau = \frac{\sigma_x}{2} \operatorname{sen} 2\varphi$

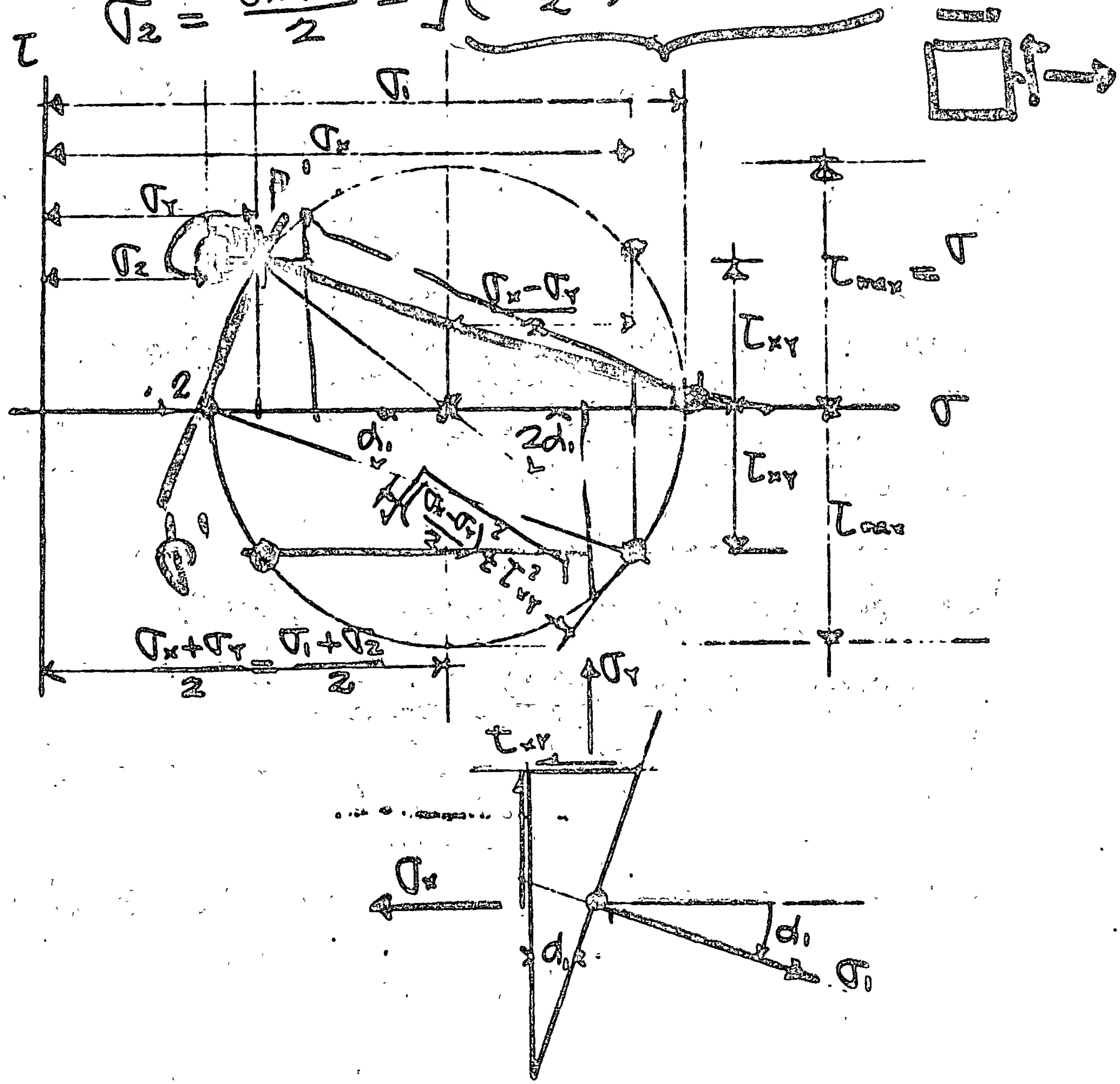
Subst: (9) en (6) ;

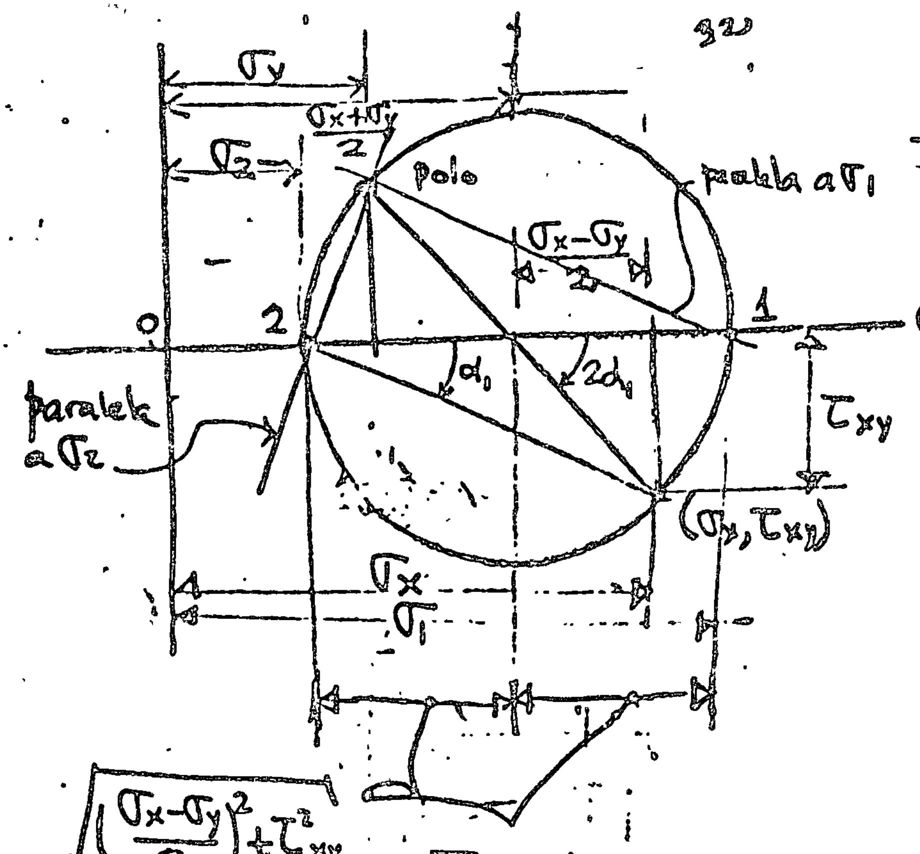
$$\sigma_1 = \frac{\sigma_x+\sigma_Y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_Y}{2}\right)^2 + \tau_{xY}^2} \qquad (10)$$



reemplazando α_1 por $\alpha_1 + \frac{\pi}{2}$ en (9) y subst. en (6) se obtiene

$$\sigma_2 = \frac{\sigma_x+\sigma_Y}{2} - \underbrace{\sqrt{\left(\frac{\sigma_x-\sigma_Y}{2}\right)^2 + \tau_{xY}^2}} \qquad (11)$$





$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$ y actua a 45° con respecto a σ_1

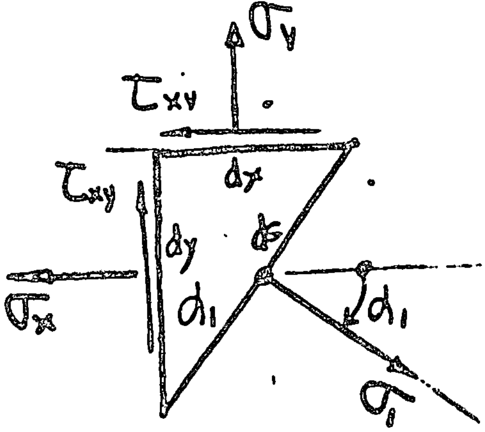
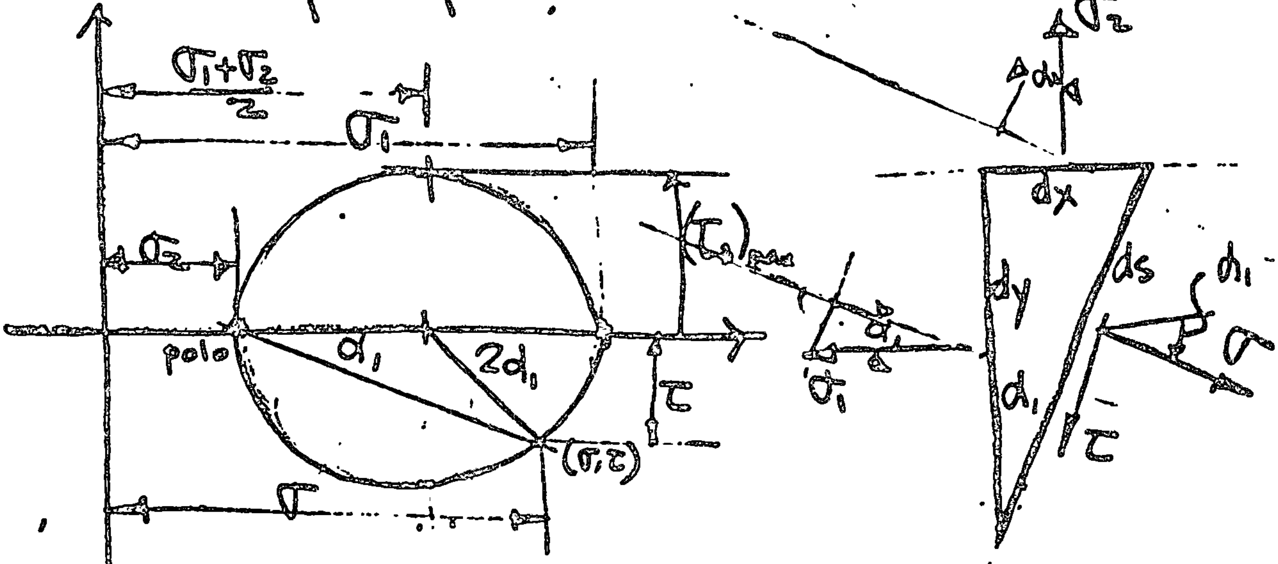


Fig. 14.2

Determinación del circulo de Mohr partiendo de las direcciones principales:



$$\sigma = \sigma_1 \cos^2\alpha_1 + \sigma_2 \operatorname{sen}^2\alpha_1 = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\alpha_1 \quad (14.5)$$

$$\tau = \frac{1}{2}(\sigma_2 - \sigma_1)\operatorname{sen} 2\alpha_1$$

$$(\sigma_x + \sigma_y) = \sigma_1 + \sigma_2 \quad (14.6)$$

APENDICE A

INSTRUCTIVO DEL PROGRAMA PARA EL ANALISIS DE ESTRUCTURAS MURO - MARCO

Se describe cómo se preparan los datos que servirán para el análisis de estructuras muro-marco mediante el programa de computadora descrito del cap 4. Los datos para el programa se proporcionan mediante tarjetas perforadas.

A1. Datos de entrada

A1.1. Tarjeta título (13A6). De la columna 1 a la 78 se puede perforar cualquier información alfanumérica con objeto de identificar los problemas que se van a procesar en una corrida.

A1.2 Tarjeta de archivos (4I5).

Columnas	
1 - 5	No. del disco de barras
6 - 10	No. del disco de gráficas
11 - 15	No. del disco de fuerzas internas
16 - 20	No. del disco de cuadrados

Según el listado del programa (Apéndice B), los números de los discos son 10, 15, 20 y 25 respectivamente.

A1.3 Tarjeta de problemas (I5). Se especifica el número de estructuras que se desee analizar en una corrida del programa.

A1.4 Paquete de tarjetas para cada problema. Las instrucciones A1.4.1 a A1.4.20 serán suficientes para definir un problema.

Se repetirán tantas veces según se especifique en el inciso A1.3

A1.4.1 Tarjeta título (13A6). De la columna 1 a la 78 se perfora cualquier información alfanumérica que permite identificar el problema en particular que se está analizando.

Columnas

1 - 5	identificador del material
6 - 15	módulo de Young (ton/m^2)
16 - 25	coeficiente de Poisson
26 - 35	peso volumétrico (ton/m^3)

A1.4.4 Tarjetas de secciones (2I5,4F10.0)

Las barras de la estructura pueden tener distintas secciones, por ejemplo, circular, rectangular u otras. Para identificar la sección utilizada en cada barra se asigna a ésta un número entero empezando por uno, el que se denomina identificador de la sección y habrá tantas tarjetas como tipos de sección se especifiquen en la instrucción A1.4.2

En cada tarjeta se perforará la siguiente información

Columnas

1 - 5	identificador de la sección
6 - 10	indicador del tipo de sección. Existe internamente un catálogo de secciones transversales numeradas: 0 especial 1 T 2 rectangular 3 circular

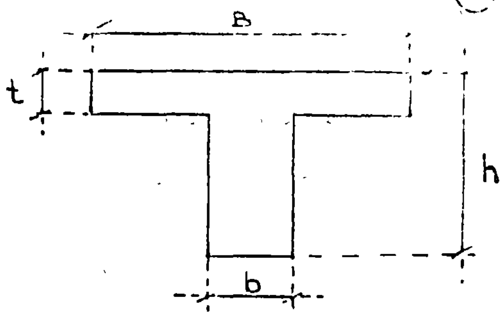
El indicador tomará cualquiera de esos valores según el tipo de sección.

Dependiendo del número asignado al indicador que define el tipo de sección transversal de la barra, la información que determina tales secciones se perforará en el resto de la tarjeta de la forma siguiente.

a) Sección T. Indicador del tipo de sección = 1

columnas

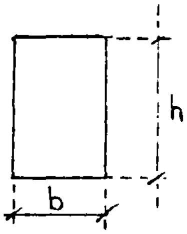
11 - 20	B(cm)



21 - 30	b(cm)
31 - 40	t(cm)
41 - 50	h(cm)

b) Sección rectangular. Indicador del tipo de sección = 2

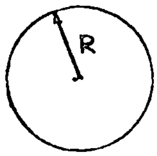
Columnas



11 - 20	b(cm)
21 - 30	h(cm)

c) Sección circular. Indicador del tipo de sección = 3

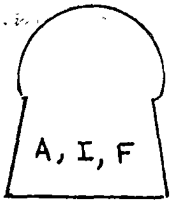
Columnas



11 - 20	R(cm)

d) Sección especial. Indicador del tipo de sección = 0

Columnas



11 - 20	A área transversal (cm^2)
21 - 30	I momento de inercia (cm^4)
31 - 40	F factor de forma

Ejemplo A.1

Tiene la finalidad de ilustrar las instrucciones A1.1 a A1.4.4 de la estructura mostrada en la fig A.1.1.

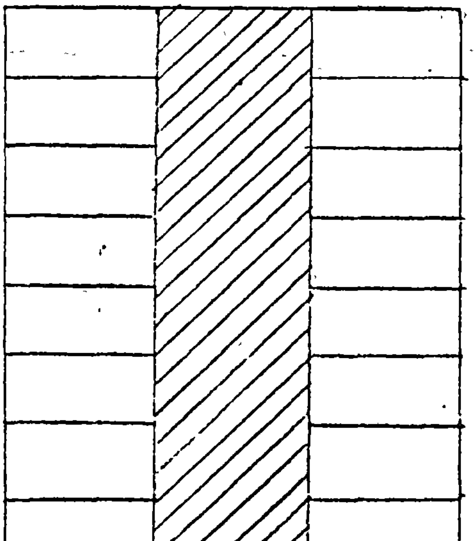


Fig. A.1.1 MURO-MARCO A

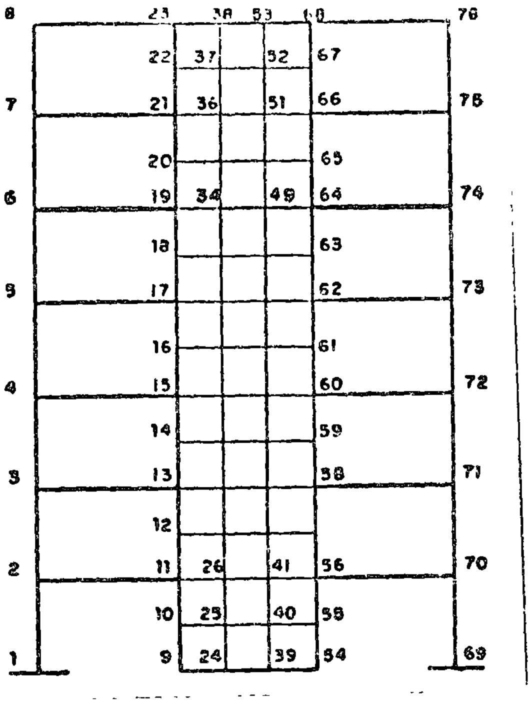


Fig. A.1.3 Alternativa 2
Se numeran en la dirección vertical los nudos, siendo la diferencia máxima 16

De las dos alternativas presentadas, la 1, por ser la de diferencia menor es la mejor opción. Respecto a la numeración de las barras o de los cuadrados no importa su orden; sin embargo, a fin de utilizar las opciones para generación de datos es necesario numerar, en orden secuencial, los elementos que tengan propiedades comunes como se ilustra en el ejemplo A.3.

Suponiendo que en la estructura solo participa un material y tres tipos de secciones para la estructura dada, la codificación de la instrucción será: (Fig A.1.2)

28	42	1	76	3	
5	10	15	20	25	80

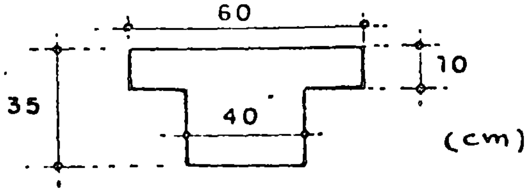
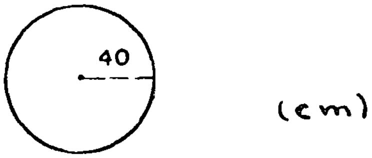
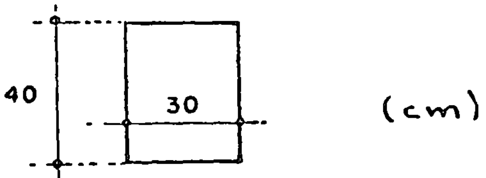
En el caso de tratarse de una estructura formada por dos materiales, la instrucción se codificará como:

28	42	2	76	3	
5	10	15	20	25	80

vii) Instrucción A1.4.3

Si la estructura de la fig A.1.2 está formada por un solo material, por ejemplo, concreto con las propiedades siguientes, resulta:

Tabla A.1.1

Nº	SECCION
1	
2	
3	

Una vez ejemplificadas las instrucciones A1.1 a A1.4.4 se continúa con el instructivo.

A1.4.5 Tarjetas de coordenadas para los puntos nodales (I5, 2F10.0, 2I5)

Contienen las coordenadas de cada punto nodal referidas a un sistema cartesiano global. Las unidades son metros, y en general se requiere una tarjeta para cada punto nodal. El orden debe ser secuencial

Columnas

1 - 5	No. del punto nodal
6 - 15	abscisa (m)
16 - 25	ordenada (m)

Se ha incluido la alternativa de poder generar ciertas coordenadas a partir de los datos del primero y último punto de un grupo que cumpla con las condiciones siguientes:

i) Los puntos de este grupo son equidistantes y están sobre una recta.

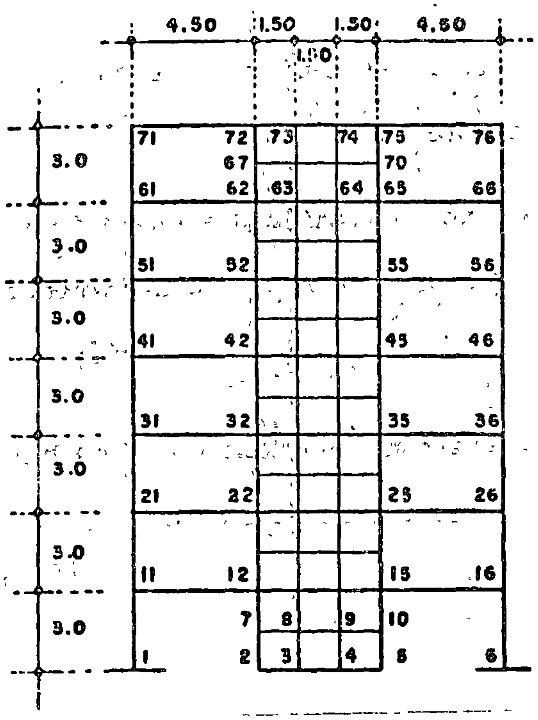


Fig A.2.1

MURO - MARCO A

Se observa que los grupos de los puntos nodales 1 al 71, 2 a 72, 3 a 73, 4 a 74, 5 a 75 y 6 a 76 cumplen con las condiciones i y ii, o sea que son equidistantes y están sobre una misma recta, y la diferencia entre puntos sucesivos es constante. Por tanto, se puede utilizar la opción de generación con los datos del primero y último puntos de los grupos mencionados, cuyos datos resultan:

1	0.0	0.0		10
71	0.0	21.0	10	
2	4.5	0.0		5
72	4.5	21.0	5	
3	6.0	0.0		5
73	6.0	21.0	5	
4	7.5	0.0		5
74	7.5	21.0	5	
5	9.0	0.0		5
75	9.0	21.0	5	
6	13.5	0.0		10
76	13.5	21.0	10	
5	15	25	30	35

26 - 30 indicador del tipo de apoyo en el nudo I

31 - 35 indicador del tipo de apoyo en el nudo J

Los indicadores anteriores tomarán los valores asociados a la condición de apoyo de la barra:

0 el apoyo es continuo

1 el apoyo es articulado

36 - 40 Indice de generación. Se emplea cuando se requiere utilizar la opción de generación de datos, toma los siguientes valores:

0 indica no generación.

1 indica generación. Con este valor se establece que el número de barras comprendido entre la tarjeta anterior y ésta, poseen las características siguientes:

i) La numeración de los nudos debe seguir la siguiente regla:

$$I_n = I_{n-1} + IGC$$

$$J_n = J_{n-1} + IGC$$

donde n es el número de la barra; I, J los nudos de la barra y IGC es una constante (ejemplo A.3).

ii) El grupo de las barras debe ser numerado en forma secuencial; ser construídas con el mismo material y poseer la misma longitud, sección transversal y tipos de apoyos (ejemplo A.3).

Ejemplo A.3

La instrucción A1.4.7 se ejemplificará usando los datos de la fig A.1.2 con la información adicional para las barras indicadas en la tabla A.2.1

1	11	12	1	
2	21	22	1	
3	31	32	1	
4	41	42	1	
5	51	52	1	
6	61	62	1	
7	71	72	1	
8	15	16	1	
9	25	26	1	
10	35	36	1	
11	45	46	1	
12	55	56	1	
13	65	66	1	
14	75	76	1	
15	1	11	2	
16	11	21	2	
17	21	31	2	
18	31	41	2	
19	41	51	2	
20	51	61	2	
21	61	71	2	
22	6	16	3	
23	16	26	3	
24	26	36	3	
25	36	46	3	
26	46	56	3	
27	56	66	3	
28	66	76	3	

5 10 15 20 25 35 40 50 60 70

Para establecer la opción de generación conviene hacer el análisis siguiente:

En la fig A.1.2 se observa que los grupos de las barras 1 a 7, 8 a 14, 15 a 21 y 22 a 28 cumplen con las condiciones i y ii de la instrucción A1.4.7 para generación de datos en barras, es decir, que cada grupo de barras está numerado en forma secuencial y sus nudos observan la regla

11 - 20	espesor dominante de los cuadrados que forman el muro (m).

A1.4.10 Tarjetas de tipos de cuadrados (80I1). Contiene los índices de los tipos de elementos (Tipo 1 y Tipo 2; fig 2.3.3).

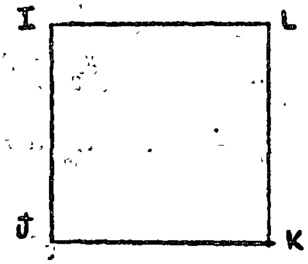
Estos índices toman los siguientes valores: (ejemplo A.4).

0	Elemento tipo 1
1	Elemento tipo 2

A1.4.11 Tarjetas de elementos cuadrados (6I5, F10.0, 2I5). Se requiere una tarjeta por cada cuadrado; contiene la información relativa a la geometría y material. Además, se tiene implementada la opción para generar los datos de un grupo de elementos, que tengan características idénticas, con los datos del primero y último elemento de este grupo, mediante un indicador que se explica enseguida.

Columnas

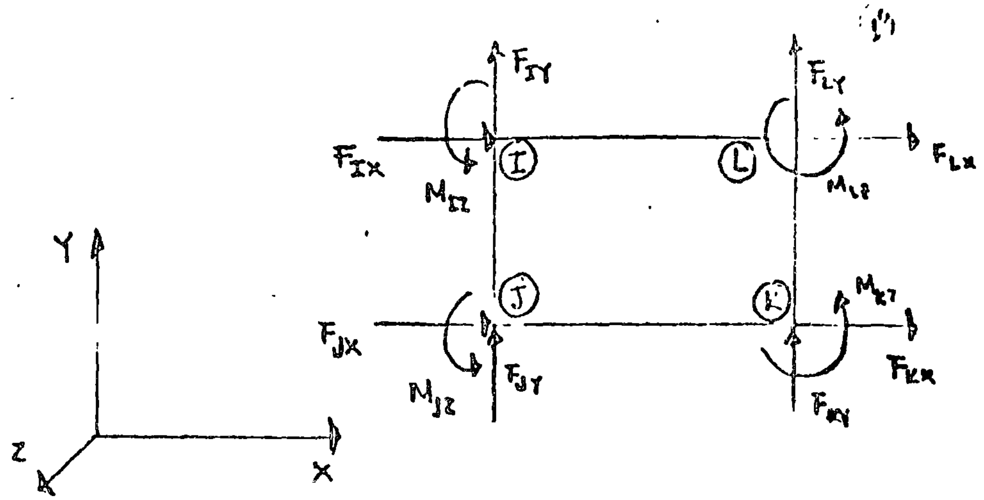
1 - 5	No. del elemento
6 - 10	Punto nodal I
11 - 15	Punto nodal J
16 - 20	Punto nodal K
21 - 25	Punto nodal L



La numeración I, J, K, L asignada a los nudos del elemento cuadrado se debe proporcionar en esta instrucción en sentido contrario a las manecillas de un reloj, para un sistema derecho empezando siempre por I.

Columnas

26 - 30	identificador del material. Se puede omitir en el caso de tratarse de un solo material.
31 - 40	espesor del elemento (m) Este valor se puede omitir cuando el elemento tenga el espesor dominante y el programa internamente le asigna



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mediante los valores siguientes:

0 no se requiere calcular los valores de las fuerzas actuantes en los puntos nodales del elemento.

1 Si se requiere calcular tales valores.

Ejemplo A.4.

Utilizando el muro-marco A y la numeración de la fig A.1.2, ejemplificar el empleo de las instrucciones A1.4.9 a la A1.4.11.

i) Instrucción A1.4.9

La fuerza de gravedad forma un ángulo con un eje x global (fig A.1.2) de 270. grados y suponiendo un espesor de 0.15 m para todo el muro se tiene

270.	0.15	
10	20	80

ii) Instrucción A1.4.10

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1													
				5					10					15					20					25					30	35	40	80

iii) Instrucción A1.4.11

Suponiendo un solo tipo de material y que no se quiere calcular las fuerzas equilibrantes, utilizando la opción de generación, los datos codificados que

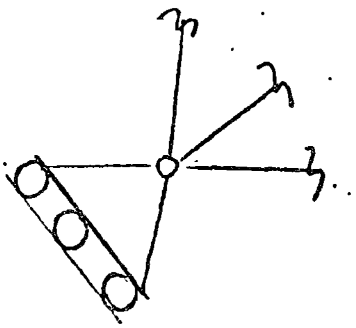
1.4.13 Tarjetas de nudos frontera restringidos (10(I5,3I1))

En una tarjeta se perforan hasta diez grupos de valores que definen el tipo de restricción para un nudo. El primer valor de este grupo corresponde al número del nudo restringido, y los siguientes tres son los valores indicadores del tipo de restricción correspondientes a los componentes de desplazamiento u, v y θ_x respectivamente y tomarán los siguientes valores

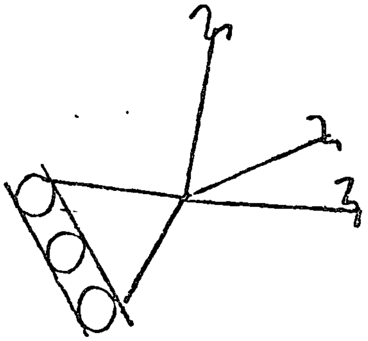
1 componente de desplazamiento restringido

0 componente de desplazamiento libre

Es frecuente encontrar nudos en los que el desplazamiento no se restringe en dirección horizontal o vertical sino en direcciones inclinadas (figs a y b)



a) Extremos de barras articulados



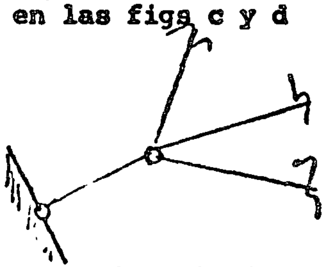
b) Extremos de barras continuos

Es costumbre reemplazar estos casos por una barra orientada en la dirección en que se restringe el movimiento, de longitud usual y una área de la sección transversal muy grande (A → ∞). Las condiciones de de frontera y el momento de inercia de la sección transversal se especifican en las figs c y d

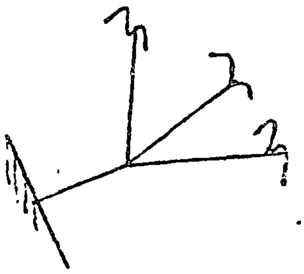
A → ∞

I Arbitrario

L usual



c) apoyo idealizado para la fig a



d) apoyo idealizado para la fig b

A → ∞

I → 0

L usual

indicador especificado en la instrucción A1.4.12. Si no se necesita el cálculo de rigideces, es decir, cuando el indicador vale 1 se continúa con la instrucción A1.4.15.

Si solamente se requiere el cálculo de rigideces, es decir, cuando el indicador vale -1 se continúa con la instrucción A1.4.20.

A1.4.14.1 Tarjeta de control (2I5)

Contiene:

Columnas

1 - 5	No. de niveles de la estructura MURO-MARCO
6 - 10	No. máximo de puntos nodales por nivel

A1.4.14.2 Tarjeta título (13A6)

De las columnas 1 a 78 se escribe un encabezado alfanumérico para indicar que se va a calcular las rigideces de entrepiso.

A1.4.14.3 Tarjeta de puntos nodales por nivel (16I5)

Se perfora el arreglo que contiene el número de puntos nodales por nivel, empezando por el primero (ejemplo A.6).

A1.4.14.4 Tarjeta con numeración de los nudos por nivel (16I5)

Un grupo de tarjetas por cada nivel contiene información respecto a la numeración de los puntos nodales en ese nivel. Habrá tantos grupos de tarjetas como número de niveles especificado en la instrucción A1.4.14.1; a la primera tarjeta corresponde el primer nivel.

A1.4.14.5 Tarjeta de alturas de entrepiso (8F10.0)

En cada tarjeta se perforan hasta ocho valores con las alturas de los entrepisos, en metros, empezando por la del primer entrepiso.

A1.4.14.6 Tarjeta de pesos por nivel (8F10.0)

En cada tarjeta máximo se perforan ocho valores de la carga que actúa en cada nivel a la estructura estudiada. La carga se especifica en to

i) Instrucción A1.4.14

Debido a que en este caso se requiere el cálculo de rigideces de entrepiso, se codificarán las instrucciones A1.4.14.1 a A1.4.14.7 con lo cual es posible definir los datos de rigideces de entrepiso.

ii) Instrucción A1.4.14.1

5	10	80
7	6	

iii) Instrucción A1.4.14.2

	78	80
DATOS PARA EL CALCULO DE RIGIDECES DE ENTREPISO		

iv) Instrucción A1.4.14.3

5	10	15	20	25	30	35	80
6	6	6	6	6	6	6	

v) Instrucción A1.4.14.4

5	10	15	20	25	30	80
11	12	13	14	15	16	
21	22	23	24	25	26	
31	32	33	34	35	36	
41	42	43	44	45	46	
51	52	53	54	55	56	
61	62	63	64	65	66	
71	72	73	74	75	76	

vi) Instrucción A1.4.14.5

10	20	30	40	50	60	70	80
4.0	3.5	3.5	3.0	3.0	3.0	3.0	

vii) Instrucción A1.4.14.6

10	20	30	40	50	60	70	80
685.	685.	685.	676.	676.	676.	522.	

A1.4.17.2 Paquete de tarjetas que definen las cargas en las barras.

Las instrucciones A1.4.17.2.1 a A1.4.17.2.2 serán suficientes para proporcionar los datos de las cargas en las barras. Las instrucciones se repiten para cada barra y en el orden dado en la A1.4.17.1.

Es frecuente encontrar estructuras donde las cargas son iguales para un grupo de barras de igual longitud y condiciones de apoyo. En tal caso, existe la posibilidad de aprovechar los datos de cargas y los cálculos realizados para una barra y asignárselos a las barras restantes. La forma de utilizar la opción se explica en las instrucciones siguientes.

A1.4.17.2.1 Tarjeta de control (2I5)

Contiene la siguiente información.

Columnas	
1 - 5	No. de cargas intermedias
6 - 10	Indicador de generación de cargas. Toma el valor cero cuando no hay posibilidad de utilizar la opción de generación, es decir, cuando no hay un grupo con cargas iguales. En el caso de haberlo, el indicador tomará el valor del número de barras restantes con cargas iguales a la barra en cuestión. Una limitación adicional consistirá en que la numeración del grupo de barras deberá ser secuencial

A1.4.17.2.2 Tarjeta de tipos de carga (I5, 3F10.2)

Existe un catálogo interno de tipos de cargas, por lo tanto, para cada carga se identifican el tipo de carga y los datos que la definen en una tarjeta. Las cargas están referidas a los ejes locales de cada barra y son positivas en las direcciones positivas de los ejes locales de referencia. A continuación se indic la forma de especificar cada tipo

A1.4.18 Paquete de tarjetas de nudos cargados

Se define mediante la instrucción A1.4.18.1 según el número de nudos cargados especificado en la instrucción A1.4.16. En caso de ser cero, se continúa con la A1.4.19

A1.4.18.1 Tarjetas de cargas en los nudos (I5, 3F10.0, 2I5). El número de tarjetas que integran el grupo es igual al número de nudos cargados dado en la instrucción A1.4.16, es decir, una tarjeta para cada nudo cargado; además contiene la siguiente información:

Columnas	
1 - 5	No. del nudo cargado
6 - 15	fuerza paralela al eje x global, en ton
16 - 25	fuerza paralela al eje y global, en ton
26 - 35	par concentrado respecto al eje z global, en ton-m

La convención de signos para fuerzas es positivo (+) cuando el sentido de éstas es el mismo que el sentido positivo de los ejes coordenados; para momentos es positivo (+) si giran en sentido contrario a las manecillas de un reloj.

En aquellos casos en que las cargas en nudos sean idénticas para un grupo de nudos, se puede utilizar la opción de generación de cargas en nudos si se cumplen las siguientes condiciones:

i) Las cargas son las mismas para un grupo de nudos

ii) La numeración de los nudos deberá ser secuencial, o bien el número de cada nudo igual al anterior más una constante, que se denota mediante el indicador IC.

En caso de que la numeración sea secuencial, se puede omitir dicho indicador.

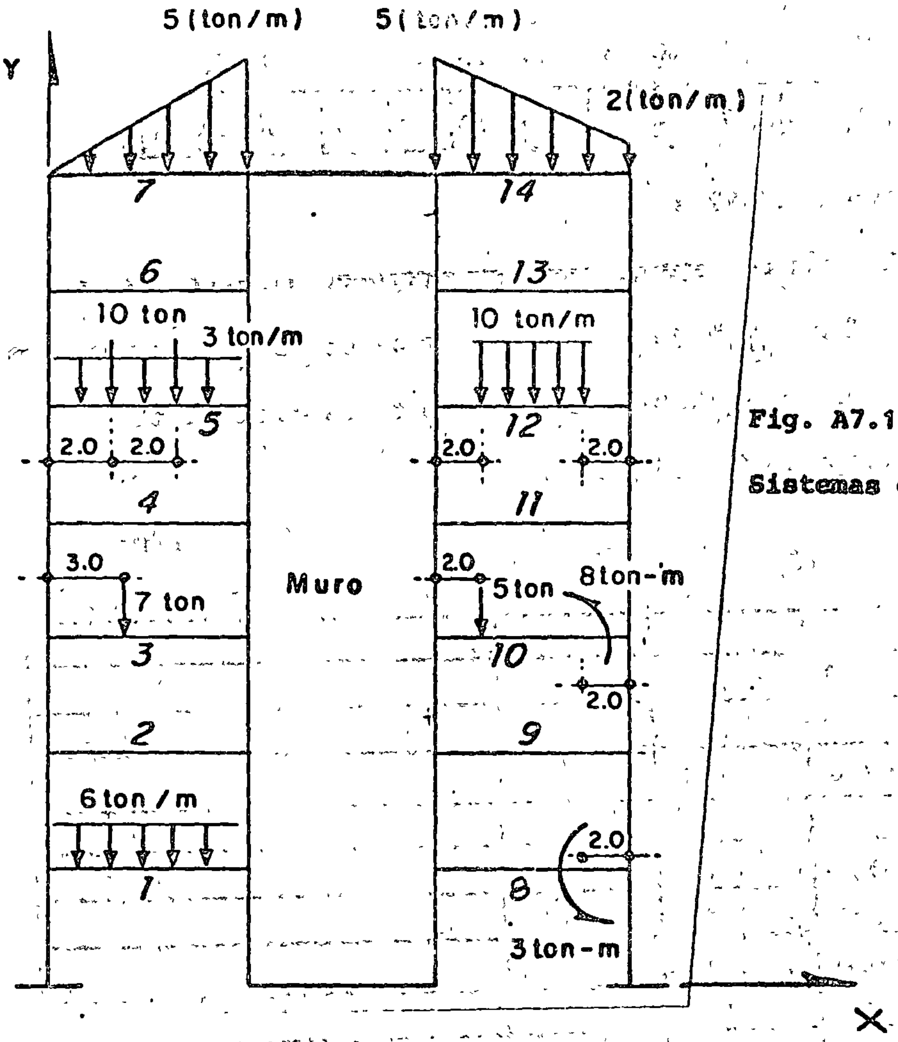


Fig. A7.1 Muro-marco A

Sistemas de cargas I.

i) Instrucción A1.4.15

80

ANALISIS DEL MURO-MARCO A ANTE EL SISTEMA DE CARGAS

ii) Instrucción A1.4.16

5 10 80

8

iii) Instrucción A1.4.17

Como el número de barras cargadas es distinto de cero, se sigue con las instrucciones A1.4.17.1 a A1.4.17.2

iv) Instrucción A1.4.17.1

Supóngase que se requieren las gráficas de elementos mecánicos de todas las barras cargadas

b) Ilustración con el marco de la fig A7.2.

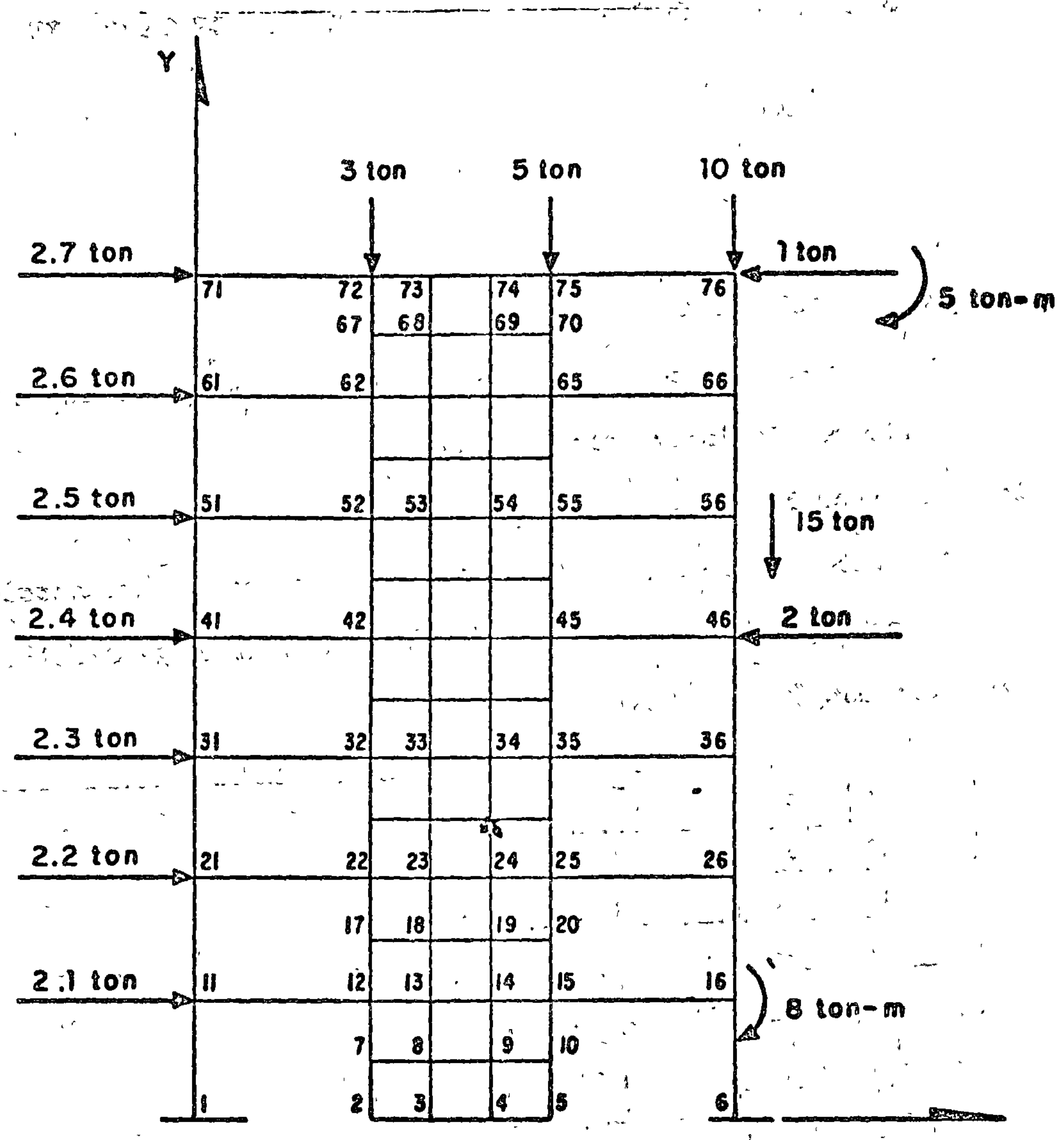


Fig A7.2 Muro-marco A

Sistemas de cargas II

c) Ilustración con el muro-marco de la fig A6.3

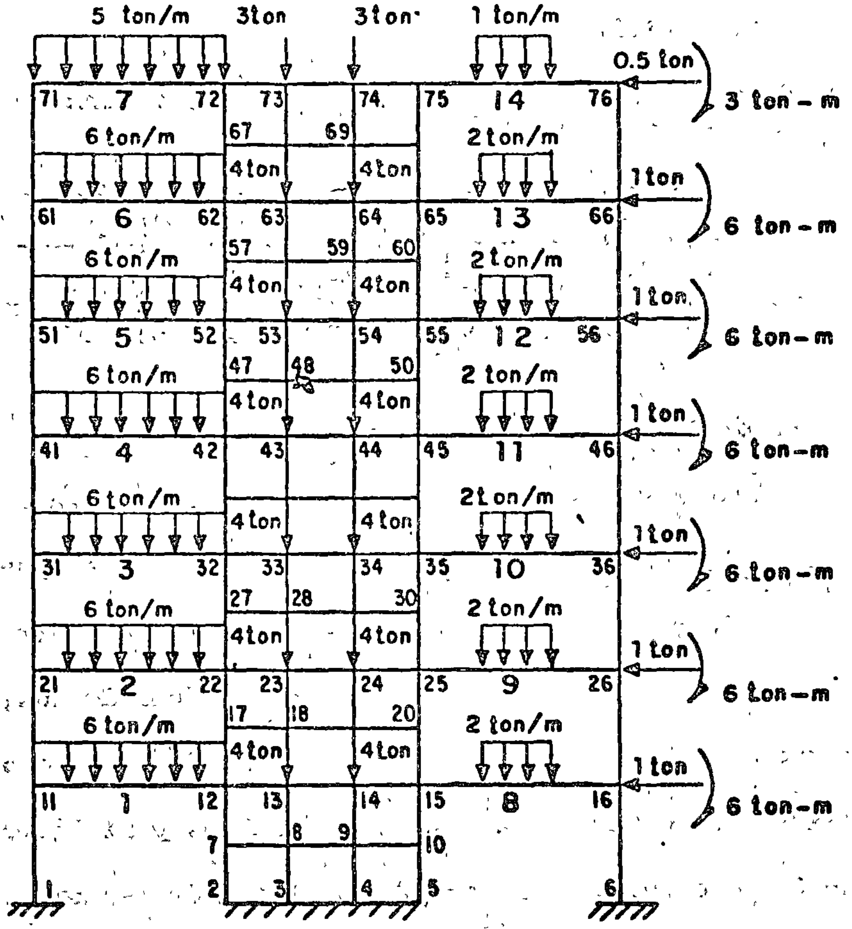


Fig A6.3 Muro-marco A

Sistema de cargas III

i) Instrucción A1.4.15

ANALISIS DEL MURO-MARCO A ANTE EL SISTEMA DE CARGAS III

80

ii) Instrucción A1.4.16

5	10	80
14	21	

vii) Instrucción A1.4.18

Como el número de nudos cargados es distinto de cero, se definen las cargas de estos mediante la instrucción A1.4.18.1.

En este caso se puede utilizar la opción de generación, ya que los grupos de nudos 16, 26, 36, 46, 56 y 66; 13, 23, 33, 43, 53 y 63; 14, 24, 34, 44, 54 y 64, tienen la misma condición de carga y su numeración difiere en una constante que es 10. Por tanto, basta definir la carga en los nudos 16, 13 y 14 y mediante el indicador IFUER que vale 5, señalar que los cinco nudos siguientes tienen la misma carga. La codificación queda como sigue:

5	15	25	35	40	45	80
13		-4.0		5	10	
14		-4.0		5	10	
16	1.0		6.0	5	10	
76	0.5		3.0			
73		-3.0		1		

Finite-Element Method for Water-Distribution Networks

Anthony G. Collins and Robert L. Johnson

A contribution submitted to the JOURNAL on Oct 13, 1973, and revised Sep 24, 1974, by Anthony G Collins, pollution cont engr, ACI Environics, Melbourne, Australia, and Robert L Johnson (Active Member, AWWA), assoc prof Dept of Civ Engrg, Lehigh Univ., Bethlehem, Pa

Over the past two decades, the finite-element method has been increasingly used in a variety of engineering fields including structural analysis, solid mechanics, and soil mechanics. The method uses the relationship between the basic properties of each discrete element to define the behavior of that element. A solution for the response of the overall system, subject to a set of boundary conditions, is provided by solving a set of compatible simultaneous equations by matrix solution techniques.

The equivalence of structural systems and pipe networks has long been recognized, and there are many examples of concurrent application of solution techniques or the exchange of solution methods. The system of equations for structural problems is normally linear and hence suitable to matrix solution.

Although a specific pipe-network problem can be defined using a finite-element approach,[1] the actual solution of the network problem becomes very difficult because of the nonlinear constitutive equations relating the flow and head loss in each pipe or element. Indeed, matrix solution of the pipe-network problem[2] has been achieved using extensive numerical analysis and graph theory but without recognition of the advantages of the finite-element method

The successful application of the finite-element method to pipe-network problems shows that the method is not only superior to conventional Hardy Cross solution techniques but that the further advantages of complete network representation, simplified input data, and unlimited network size can be obtained. Although not specific to a finite-element solution, the program developed allows for solution by either the Hazen-Williams or the Darcy-Weisbach flow-head-loss relationships The program also can consider the effect of temperature variations on head loss throughout the network

The computational algorithm used to arrive at the unique solution for an easily solved linear system equivalent to the true nonlinear system for the pipe networks was maintained in an extremely simple form in this article so that the advantages of the finite-element method could be readily observed Undoubtedly, further application of numerical-analysis techniques would improve the efficiency of the method

There are two specific reasons for the development of this method. First, a computer program, PAWDS,[3,4] which uses the Hardy Cross solution[5] method of balancing flow for pipe-network problems, is used in undergraduate courses at Lehigh Univ in Bethlehem, Pa This Hardy Cross method or various refinements of the loop method (balancing heads in loops) were, in 1973, still used extensively in undergraduate education, engineering practice, and research.[6 7]

The Hardy Cross approach to pipe-network analysis uses as a boundary condition either the fact that the algebraic sum of flows at any joint is zero (balancing flows) or that the algebraic sum of the head loss around any loop is zero (balancing heads). These are simply adaptations of the classical conservation of mass and conservation of energy, respectively Depending upon the criteria used, a correction is applied to the assumed pipe flows or assumed piezometric heads until convergence to a solution is obtained. This classical iteration procedure is quite satisfactory for most well-conditioned pipe systems. However, it has been pointed out that convergence to a solution is not necessarily guaranteed.

> There appears to be nothing inherent in either the electric analyzer with ordinary resistors or the Hardy Cross method which will consistently produce convergence of the errors toward zero with subsequent adjustments [8]

The PAWDS program used at Lehigh Univ. was plagued by convergence problems typical of the Hardy Cross method. Dillingham and Cleasby[9] point out that when using the balancing-heads method, a pipe or pipes with high resistance to flow compared with others in the network can result in calculated flow corrections larger and in the opposite direction to the currently assumed flow This will often cause a divergence in the computations, and no solution can be obtained. When the method of balancing flows is used. Dillingham[9] points out that if a large pipe of short length and relatively low flow exists, many iterations are necessary before an appreciable change in piezometric head is obtained if the value of the assumed piezometric head is incorrect. These situations are very practical in their nature. Typical pipe-distribution networks[10] have these exact conditions

An extensive discussion of the convergence problems of the Hardy Cross method and the PAWDS program in particular is not intended in this article, but the existence of these problems should be noted.

The second reason for developing the solution technique was because of the existence of a very efficient finite-element program, GENFEM, developed by Desai [11] The advantage of this program is its completely general nature and hence easy adaptation for the pipe-network problem The mathematical basis and the method of application of the finite-element method is described in detail to allow easy adaption of other existing finite-element programs for use in solving water-distribution-network problems

Application of the Finite-Element Method

Mathematical basis. When the finite-element method is applied to a structural problem, the structure is subdivided into discrete elements. Each of these elements must satisfy three conditions

1. Equilibrium of forces must be maintained.
2 Compatibility must be maintained.
3. The force-displacement relationship specified by the geometric and elastic properties of the discrete element must be satisfied

The force F in the member or element is related to the displacement u and the element properties or stiffness K by Eq (1).

$$F = K\,u \qquad (1)$$

The sum of the forces in the members at each node of the structure is zero except where an external force is applied By combining Eq (1) for all the elements in the structure into an equation of identical form to Eq (1) and solving for displacements, the equilibrium of the system is satisfied.

An equivalent set of conditions for a pipe network exists; hence, the ability to draw the analogy:

1. The algebraic sum of the flows at any joint or node must be zero.
2 The value of the piezometric head at a joint or node is the same for all pipes connected to that joint
3. The flow-head-loss relationship (such as Darcy-Weisbach or Hazen-Williams) must be satisified for each element or pipe

The conditions for a pipe network deal with scalar quantities, whereas the structural conditions deal with vector quantities. The analogy is drawn between the magnitudes of the equivalent quantities as the vector aspects of the flow have no meaning for the network problem.

For a direct application of the finite-element method involving a matrix solution,

a linear relationship is required to define the element or pipe. Hence at this point, a relationship of the form of Eq (2) between flow q, head loss h and the hydraulic properties of the pipe c will be assumed

$$q = ch \quad (2$$

The method of solution to make Eq (2 equivalent to established nonlinear flow-head-loss relationships will be described subsequently.

The head loss h in Eq (2) is the difference between the piezometric head H of the nodes or joints at each end of the element or pipe contributing to the sum of the flows Q at that joint. The pipe-system matrix is assembled by writing the equations for the sum of the flows Q at each joint since this value is known to be either zero or equal to the imposed external flow or demand. Alternatively, if the piezometric head is specified at a joint, the sum of the pipe flows is implicitly defined. The resulting set of simultaneous equations can be combined into matrix form defining the entire pipe system in terms of the sum of flows Q at a joint and of the piezometric heads H at the joints. This matrix has the form

$$Q = CH \quad (3)$$

When the matrix is solved, the piezometric heads at all joints are obtained. The difference in piezometric heads between two joints, which is the head loss, can be substituted into Eq (2) to calculate the flow in the pipe between those two joints. The direction of flow is automatically preserved by taking the sign of the difference of the piezometric heads into account.

The finite element representing the pipe is of the simplest form possible. Each element is one dimensional and has one degree of freedom at each node or joint. To apply the summation of the flows at a joint successfully, a convention must be adopted. Flow into a joint is taken as positive, and flow out of a joint is negative.

The analysis of a simple pipe network, Fig. 1, is used to show the application of the finite-element method. Using the condition that the sum of the pipe flows (q_a, q_b, ...) in or out of a joint must satisfy the equilibrium flow criteria (Q_1, Q_2, . .) (i.e., the boundary conditions) at that joint, one can write the following equations:

$$Q_1 = q_a + q_d \quad (4)$$
$$Q_2 = q_a + q_b \quad (5)$$
$$Q_3 = q_b + q_c + q_f \quad (6)$$
$$Q_4 = q_c + q_d + q_e \quad (7)$$
$$Q_5 = q_e + q_f \quad (8)$$

The individual pipe flows can be expressed by Eq (2) noting that the head loss h is equal to the difference in the piezometric heads of the joints at each end of the particular pipe.

$$q_a = \pm C_a(H_1 - H_2) \quad (9)$$
$$q_b = \pm C_b(H_2 - H_3) \quad (10)$$
$$q_c = \pm C_c(H_3 - H_4) \quad (11)$$
$$q_d = \pm C_d(H_1 - H_4) \quad (12)$$
$$q_e = \pm C_e(H_4 - H_5) \quad (13)$$
$$q_f = \pm C_f(H_3 - H_5) \quad (14)$$

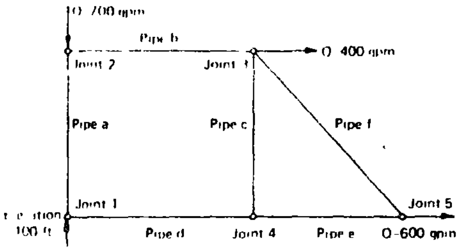


Fig. 1. Example Problem — Analysis of a Simple Pipe Network

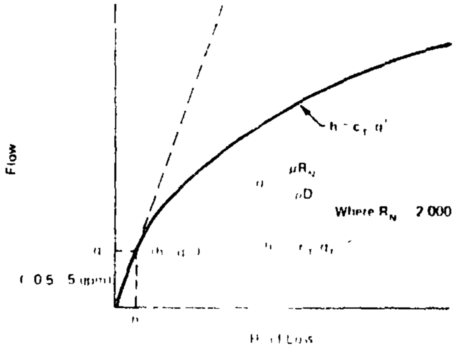


Fig. 2. Typical Flow-Head Loss Relationship

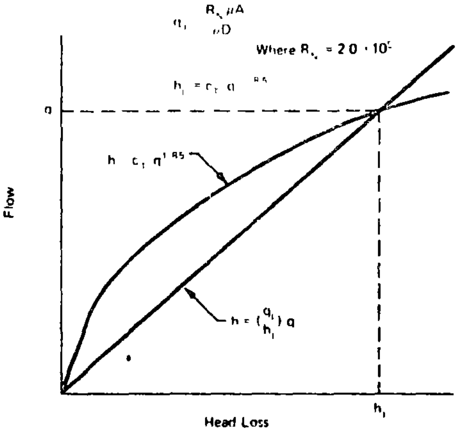


Fig. 3. Initial Value of Pipe Coefficient c

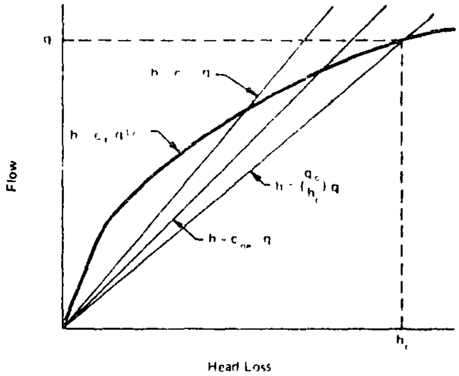


Fig. 4. Correction of Pipe Coefficient c

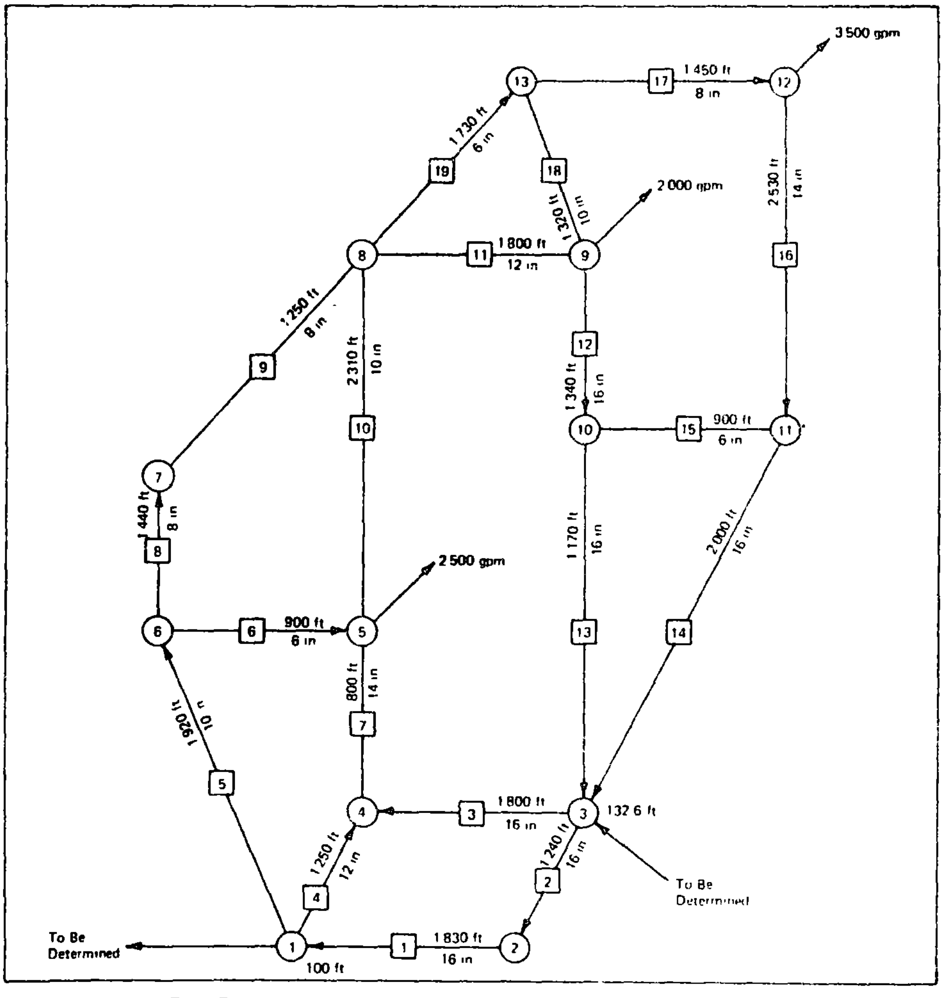


Fig. 5. Comparative Example Using PAWDS and GENFEM

○ —joint numbers; ☐ —pipe numbers; length in feet, diameter in inches

Equations (4–8) can now be written in terms of the pipe coefficients (C_a,C_b, . .) and the piezometric heads (H_1,H_2, . .) Consistency of flow directions is taken into account by assuming the flow is away from the joint being considered, that is, the piezometric head at the other joints is subtracted from the piezometric head at the joint being considered

$$Q_1 = C_a(H_1 - H_2) + C_d(H_1 - H_4) \qquad (15)$$
$$Q_2 = C_a(H_2 - H_1) + C_b(H_2 - H_3) \qquad (16)$$
$$Q_3 = C_b(H_3 - H_2) + C_c(H_3 - H_4) + C_f(H_3 - H_5) \qquad (17)$$
$$Q_4 = C_c(H_4 - H_3) + C_d(H_4 - H_1) + C_e(H_4 - H_5) \qquad (18)$$
$$Q_5 = C_e(H_5 - H_4) + C_f(H_5 - H_3) \qquad (19)$$

Equations (15–19) can be combined into the matrix form of Eq (3) to yield Eq (20). (See below.)

For this particular example, the following boundary conditions are given.

$H_1 = 100$ ft
$Q_2 = 700$ gpm
$Q_3 = 400$ gpm
$Q_4 = 0$ gpm
$Q_5 = 600$ gpm

Substituting these values into Eq (20) gives Eq (21) as the final form for solution (See below)

The values of the coefficients (C_a,C_b, . . .) for each pipe are determined by the procedure to be outlined. The unknowns, H_2, H_3, H_4, H_5 and Q_1, can be obtained by solving the matrix Eq (21). The flows in the individual pipes can be found, as previously stated, by substituting into the defining equations (Eq. [9-14] for this example) after the piezometric heads have been found for each joint

Method of application. For the successful application of the finite-element method, the constitutive equations used to relate flow and head loss must be linear or the matrix solution cannot be applied In reality, the relationship is nonlinear and varies with the equation chosen. The application of the finite element method is accomplished by using a linear equation (Eq [2]) as the defining flow-head loss relationship and the successive correction of the pipe coefficient c until a unique solution is found satisfying both the equivalent linear relationship and a real nonlinear relationship such as the Hazen–Williams equation or the Darcy-Weisbach equation. When this unique solution has been found for all pipes the network distribution has been solved

The program, GENFEM, allows a choice of the Darcy-Weisbach equation or the Hazen–Williams equation The Hazen–Williams equation will be used for purposes of explanation, although the method is identical for both equations. The relationship used to define flow versus head loss is shown in Fig 2. The transition from laminar to turbulent conditions for pipe flow occurs at a Reynolds Number R_N of approximately 2 000. R_N is defined by the pipe diameter D, and the dynamic viscosity μ, the density ρ and the flow velocity V of the fluid flowing.

$$R_N = \frac{\rho V D}{\mu} \qquad (22)$$

The flow q_T at which transition occurs, corresponding to a R_N of 2 000, is given by

$$q_T = VA = \frac{2\,000\,\mu A}{\rho D} \qquad (23)$$

For flows less than q_T, the flow vs head-loss relationship is linear. To avoid a discontinuity in the defining relationship because of the transition region between laminar and turbulent flow, the linear relationship is obtained by simply joining the origin to the point (h_T,q_T) with a straight line. The coordinate h_T is found from a substitution of the flow q_T into the turbulent flow equation. The linear portion of the graph, the laminar region, does not enter into the calculations of a practical problem The value of q_T ranges from 0.5 to 5 gpm for 6–16-in diameter pipes whereas typical flows range from 200 to 5000 gpm for these size pipes, well into the turbulent range.

The Hazen–Williams equation relates the head loss h to the pipe diameter D, the pipe length L, the Hazen-Williams coefficient C_{HW}, the flow q and a coefficient c' for unit conversion.

$$h = c' \frac{L}{D^{4.87}} \left(\frac{q}{C_{HW}}\right)^{1.85} \qquad (24)$$

This equation can be rewritten for a particular pipe by grouping terms into one constant c_T.

$$h = c_T q^{1.85} \qquad (25)$$

The solution technique can be divided into three steps. The first step is to select an initial value of the pipe coefficient c for each pipe and combine these to yield the matrix pipe coefficients C The system matrix is then solved for the value of the piezometric head at each joint Secondly, the individual pipe flows q are calculated by use of Eq (2) using the differences between the determined piezometric heads. These flows are then substituted into Eq (25) and since (c_T) for each pipe is known, the pipe head losses are calculated. If the pipe head losses obtained from Eq (25) correspond to those obtained from the matrix solution, then the unique solution satisfying both the Hazen–Williams equation and the linear Eq (2) has been found The third and final step required is to change the value of c to converge the problem to a solution if there is a difference between the head losses calculated by the two methods

A more detailed explanation of each of these steps follows. The initial value of the pipe coefficient c_1 is chosen to correspond to R_N of 200 000 in each pipe, a typical value for a practical problem. The flow (q_1) is then calculated from the Reynolds Number relationship, Eq (26):

$$q_1 = VA \frac{200\,000\,\mu A}{\rho D} \qquad (26)$$

The value of the head loss h_1 corresponding to this flow q_1 is calculated from Eq (25):

$$h_1 = c_T q_1^{1.85} \qquad (27)$$

The pipe coefficient is then found from Eq (2) as shown in Fig. 3.

$$c_1 = \frac{q_1}{h_1} \qquad (28)$$

This initial value of the pipe coefficient c_1 for each pipe is then combined, according to the geometry of the network into the pipe coefficients C_1 used in the matrix description of the network system The matrix is then solved to yield the first estimate of the piezometric heads at each joint

The allowable deviation between a pipe head loss determined from the matrix solution of the joint piezometric heads and the corresponding value from the Hazen-Williams equation is a variable and can be specified for a particular case taking into account the type of problem and the degree of precision desired for the solution.

The third step, adjusting the value of c, was developed with two criteria in mind The solution should converge reasonably rapidly, yet the technique should remain simple During the checking procedure, the flow q_c for each pipe calculated via Eq (2), and the matrix solution is used to determine the head loss h_c from the Hazen–Williams equation The first procedure used in the development of the program was to obtain the correction of the c value for each pipe by assuming that the point h_c, q_c was the unique solution and thus the correct linear relationship was defined by a straight line joining this

---

$$\begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_f & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \qquad (20)$$

$$\begin{vmatrix} Q_1 \\ 700 \\ 400 \\ 0 \\ 600 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_c & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} 100 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \qquad (21)$$

point to the origin and defined by Eq (29)

$$h = (\frac{q_c}{h_c}) \quad q \qquad (29)$$

The new value of c was then set equal to $\frac{q_c}{h_c}$. When all the pipe coefficients were corrected in a similar way, the flow distribution obviously was altered, and this method proved to be an overcorrection when the matrix was resolved To dampen this overcorrection effect, an averaging technique was introduced The corrected value of c is taken to be the mean of the c value defined by Eq (29) and the value of c used to obtain the matrix solution This method of correcting c is shown in Fig 4. The averaging method reduced the number of cycles required for convergence by approximately one third

Example Problems

It is not the intent of this article to present extensive comparison of different network problems since any comparison of computer programs must take into account ease of input data and flexibility of use as well as efficiency of computer time Two example problems are discussed to point out some apparent potential advantages of the finite element approach

An example problem[3] shown in Fig. 5, consisting of nineteen pipes and thirteen joints, was solved using the PAWDS program and the GENFEM program. The PAWDS program solved the system in eighteen iteration cycles and 1 07 s. The GENFEM program solved the network in fifteen iteration cycles and 4.73 s Obviously, this example does not indicate a preferential method, but is included so that it can be considered with the next example to show the effect of increased system size.

The second example problem with 75 pipes and 57 joints was also solved with both programs. The particular problem had been submitted by an undergraduate student and would not converge in the allowed time using the PAWDS program Both the time limit and the iteration cycle limit were increased, and the problem eventually converged by the use of the PAWDS program after 16 048 iteration cycles and 768 s The same problem was solved with the GENFEM program after twenty iteration cycles and 22.2 s This problem highlights the apparent lack of convergence problems for the finite-element method and also shows, when compared with the first example problem, that for the finite-element method, the number of iteration cycles to convergence appears virtually independent of the number of pipes and joints.

Discussion

The finite-element method is not restricted to a pipe as the only element. Any type of hydraulic element can be included that can be defined by a flow-head loss relationship When the pipe network is relatively small, such as in an industrial plant piping system, the fittings may become major head loss contributors. The head loss h across any of these elements is usually considered to be directly proportional to the velocity head by a coefficient k corresponding to the type of element.

$$h = k \quad \frac{V^2}{2g} \qquad (30)$$

This can be easily converted to the required form, that is, in terms of flow q knowing the area A of the element.

$$h = \frac{k}{2g A^2} \quad q^2 \qquad (31)$$

A pump can be included in the system since a pump merely provides a "head gain" or negative head loss The use of a pump element requires a flow-head loss relationship (the head-capacity curve for the pump). If the information is not available as an equation relating discharge and head, the pump information could be provided in tabular form. In this form the program would use linear interpolation between any two data points.

The basic finite-element method is not restricted to pipes flowing full In practice, water systems often contain open channels in the headwater sections. These open channels or even pipes flowing partially full can be included for analysis The only requirement for an element is that the flow can be related to the head loss.

The range of hydraulic elements that can be included is limitless, provided a flow-head loss relationship for each element is known. An exact system representation can be obtained by introducing a combination of all the hydraulic elements causing head loss or gain in the system. While discussing this point one should remember that methods such as the Hardy Cross method of balancing heads technique[5,7] require that all pipes are part of a loop. Typically, reservoirs or elevated tanks have one pipe connecting to the distribution system and an artificial pipe must be introduced to make a loop so that the Hardy Cross balancing heads solution method can be applied.

The use of high-resistance artificial pipes in the network has, in fact, often contributed to convergence difficulties. To simulate the actual conditions, the artificial pipes introduced to form loops are often of small diameter and high resistance so that they carry an insignificant flow and hence can be neglected. As was pointed out earlier, this is the exact condition that creates convergence problems. The finite-element method does not require the use of artificial pipes to complete a loop since the connectivity of the elements is defined explicitly by the system matrix Consequently, tree-type systems are readily solved with this procedure. In fact, the GENFEM program was used very successfully during the 1974 spring semester at Lehigh Univ to analyze a transmission system problem that was almost entirely tree type with approximately ten loops included

The input data required for the program is equivalent to other solution techniques with the major exception that loop data does not need to be included. The distribution network is defined by input of the number of the pipe and the joints to which it is connected Another advantage over some solution methods is that any number of points of known pressure can be preselected

With loop-solution methods, all pipe and joint information must be available to the program at the same time This puts a definite limit on the size of the system that can be solved. The finite-element program, GENFEM, however, can operate on blocks of data Thus, there is virtually no limit to the size of the network that can be solved. The element and nodal information can be stored on magnetic tapes or other devices and then read from the storage device in blocks, operated upon, and returned to storage on the tapes This feature must gain greater significance as water-distribution networks become larger and more interdependent

As stated previously, although not being specific to a finite-element program, the program developed has two additional features worth noting. First, the program GENFEM provides a choice of two established flow-head loss relationships. The Hazen-Williams equation has already been stated as Eq (24). The Darcy-Weisbach equation relates the same variables and includes the friction factor f.

$$h = (f \frac{L}{D\, 2g A^2}) \quad q^2 \qquad (32)$$

An explicit expression[12] for the friction factor f is used rather than the classical implicit Colebrook and White equation[12] that requires an iterative solution The friction factor f is expressed in terms of the Reynolds Number R_N and the relative roughness κ where κ is the ratio of the absolute roughness e to the pipe diameter D.

$$f = 0\ 094\kappa^{0\ 255} + 0\ 53\kappa + 88\kappa^{0\ 44}\ R_N^{-1\ 62\kappa^{0\ 134}} \qquad (33)$$

Second, allowance for changes in temperature has been included since temperature appreciably affects the viscosity and to a negligible degree, the density of water The viscosity of water over a temperature range of 5C-30C varies from 0.0152 poise to 0.8004 centipoise. An algorithm[13] in terms of temperature T in degrees Celsius is used to define the viscosity μ in poise.

$$\frac{1}{\mu} = 2\ 1482\ ([T - 8\ 435] + \sqrt{8078\ 4 + [T - 8\ 435^2]} - 120) \qquad (34)$$

The program is written so that the temperature can be specified for each pipe. Any set of temperature conditions can be investigated for a particular circumstance.

Practical adaptation of the finite-element method should require a minimum of computer programming since most engineering firms and universities have finite-element programs readily available. The ease of modifying these programs depends upon the generality of their nature.

Summary

The many advantages of the finite-element method have been documented. Most of these advantages hold true in a comparison of any loop method to the finite-element method The major advantage is the speed of convergence and the apparent lack of convergence problems of the proposed method over the Hardy Cross balancing flows method Other important advantages are the ability to include in the analysis all types of hydraulic elements, the choice of flow-head loss relationships, the lack of artificial loops, the ease of adaption of existing finite-element programs, the unlimited network size, and finally, the ability to account for temperature effects

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Finite-Element Method for Water-Distribution Networks

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A contribution submitted to the JOURNAL on Oct 13, 1973, and revised Sep 24, 1974, by Anthony G. Collins, pollution cont engr, ACI Environics, Melbourne, Australia, and Robert L Johnson (Active Member, AWWA), assoc prof Dept of Civ Engrg, Lehigh Univ, Bethlehem, Pa

Over the past two decades, the finite-element method has been increasingly used in a variety of engineering fields including structural analysis, solid mechanics, and soil mechanics. The method uses the relationship between the basic properties of each discrete element to define the behavior of that element. A solution for the response of the overall system, subject to a set of boundary conditions, is provided by solving a set of compatible simultaneous equations by matrix solution techniques.

The equivalence of structural systems and pipe networks has long been recognized, and there are many examples of concurrent application of solution techniques or the exchange of solution methods The system of equations for structural problems is normally linear and hence suitable to matrix solution.

Although a specific pipe-network problem can be defined using a finite-element approach,[1] the actual solution of the network problem becomes very difficult because of the nonlinear constitutive equations relating the flow and head loss in each pipe or element. Indeed, matrix solution of the pipe-network problem[2] has been achieved using extensive numerical analysis and graph theory but without recognition of the advantages of the finite-element method

The successful application of the finite-element method to pipe-network problems shows that the method is not only superior to conventional Hardy Cross solution techniques but that the further advantages of complete network representation, simplified input data, and unlimited network size can be obtained. Although not specific to a finite-element solution, the program developed allows for solution by either the Hazen-Williams or the Darcy-Weisbach flow-head-loss relationships The program also can consider the effect of temperature variations on head loss throughout the network

The computational algorithm used to arrive at the unique solution for an easily solved linear system equivalent to the true nonlinear system for the pipe networks was maintained in an extremely simple form in this article so that the advantages of the finite-element method could be readily observed Undoubtedly, further application of numerical-analysis techniques would improve the efficiency of the method

There are two specific reasons for the development of this method First, a computer program, PAWDS,[3,4] which uses the Hardy Cross solution[5] method of balancing flow for pipe-network problems, is used in undergraduate courses at Lehigh Univ in Bethlehem, Pa. This Hardy Cross method or various refinements of the loop method (balancing heads in loops) were, in 1973, still used extensively in undergraduate education, engineering practice, and research [6,7]

The Hardy Cross approach to pipe-network analysis uses as a boundary condition either the fact that the algebraic sum of flows at any joint is zero (balancing flows) or that the algebraic sum of the head loss around any loop is zero (balancing heads) These are simply adaptations of the classical conservation of mass and conservation of energy, respectively. Depending upon the criteria used, a correction is applied to the assumed pipe flows or assumed piezometric heads until convergence to a solution is obtained. This classical iteration procedure is quite satisfactory for most well-conditioned pipe systems. However, it has been pointed out that convergence to a solution is not necessarily guaranteed.

> There appears to be nothing inherent in either the electric analyzer with ordinary resistors or the Hardy Cross method which will consistently produce convergence of the errors toward zero with subsequent adjustments [8]

The PAWDS program used at Lehigh Univ. was plagued by convergence problems typical of the Hardy Cross method. Dillingham and Cleasby[9] point out that when using the balancing-heads method, a pipe or pipes with high resistance to flow compared with others in the network can result in calculated flow corrections larger and in the opposite direction to the currently assumed flow. This will often cause a divergence in the computations, and no solution can be obtained. When the method of balancing flows is used. Dillingham[9] points out that if a large pipe of short length and relatively low flow exists, many iterations are necessary before an appreciable change in piezometric head is obtained if the value of the assumed piezometric head is incorrect These situations are very practical in their nature. Typical pipe-distribution networks[10] have these exact conditions

An extensive discussion of the convergence problems of the Hardy Cross method and the PAWDS program in particular is not intended in this article, but the existence of these problems should be noted.

The second reason for developing the solution technique was because of the existence of a very efficient finite-element program, GENFEM, developed by Desai.[11] The advantage of this program is its completely general nature and hence easy adaptation for the pipe-network problem The mathematical basis and the method of application of the finite-element method is described in detail to allow easy adaption of other existing finite-element programs for use in solving water-distribution-network problems.

Application of the Finite-Element Method

Mathematical basis. When the finite-element method is applied to a structural problem, the structure is subdivided into discrete elements. Each of these elements must satisfy three conditions.

1. Equilibrium of forces must be maintained.
2 Compatibility must be maintained
3. The force-displacement relationship specified by the geometric and elastic properties of the discrete element must be satisfied.

The force F in the member or element is related to the displacement u and the element properties or stiffness K by Eq (1).

$$F = K u \qquad (1)$$

The sum of the forces in the members at each node of the structure is zero except where an external force is applied. By combining Eq (1) for all the elements in the structure into an equation of identical form to Eq (1) and solving for displacements, the equilibrium of the system is satisfied

An equivalent set of conditions for a pipe network exists, hence, the ability to draw the analogy:

1 The algebraic sum of the flows at any joint or node must be zero.
2 The value of the piezometric head at a joint or node is the same for all pipes connected to that joint.
3. The flow-head-loss relationship (such as Darcy-Weisbach or Hazen-Williams) must be satisfied for each element or pipe

The conditions for a pipe network deal with scalar quantities, whereas the structural conditions deal with vector quantities. The analogy is drawn between the magnitudes of the equivalent quantities as the vector aspects of the flow have no meaning for the network problem.

For a direct application of the finite-element method involving a matrix solution,

a linear relationship is required to define the element or pipe Hence at this point, a relationship of the form of Eq (2) between flow q, head loss h and the hydraulic properties of the pipe c will be assumed.

$$q = ch \qquad (2$$

The method of solution to make Eq (2 equivalent to established nonlinear flow-head-loss relationships will be described subsequently.

The head loss h in Eq (2) is the difference between the piezometric head H of the nodes or joints at each end of the element or pipe contributing to the sum of the flows Q at that joint The pipe-system matrix is assembled by writing the equations for the sum of the flows Q at each joint since this value is known to be either zero or equal to the imposed external flow or demand. Alternatively, if the piezometric head is specified at a joint, the sum of the pipe flows is implicitly defined. The resulting set of simultaneous equations can be combined into matrix form defining the entire pipe system in terms of the sum of flows Q at a joint and of the piezometric heads H at the joints. This matrix has the form

$$Q = CH \qquad (3)$$

When the matrix is solved, the piezometric heads at all joints are obtained. The difference in piezometric heads between two joints, which is the head loss, can be substituted into Eq (2) to calculate the flow in the pipe between those two joints. The direction of flow is automatically preserved by taking the sign of the difference of the piezometric heads into account.

The finite element representing the pipe is of the simplest form possible. Each element is one dimensional and has one degree of freedom at each node or joint. To apply the summation of the flows at a joint successfully, a convention must be adopted. Flow into a joint is taken as positive, and flow out of a joint is negative.

The analysis of a simple pipe network, Fig. 1, is used to show the application of the finite-element method Using the condition that the sum of the pipe flows ($q_a, q_b, \ldots$) in or out of a joint must satisfy the equilibrium flow criteria ($Q_1, Q_2, \ldots$) (i.e., the boundary conditions) at that joint, one can write the following equations:

$$Q_1 = q_a + q_d \qquad (4)$$
$$Q_2 = q_a + q_b \qquad (5)$$
$$Q_3 = q_b + q_c + q_f \qquad (6)$$
$$Q_4 = q_c + q_d + q_e \qquad (7)$$
$$Q_5 = q_e + q_f \qquad (8)$$

The individual pipe flows can be expressed by Eq (2) noting that the head loss h is equal to the difference in the piezometric heads of the joints at each end of the particular pipe.

$$q_a = \pm C_a(H_1 - H_2) \qquad (9)$$
$$q_b = \pm C_b(H_2 - H_3) \qquad (10)$$
$$q_c = \pm C_c(H_3 - H_4) \qquad (11)$$
$$q_d = \pm C_d(H_1 - H_4) \qquad (12)$$
$$q_e = \pm C_e(H_4 - H_5) \qquad (13)$$
$$q_f = \pm C_f(H_3 - H_5) \qquad (14)$$

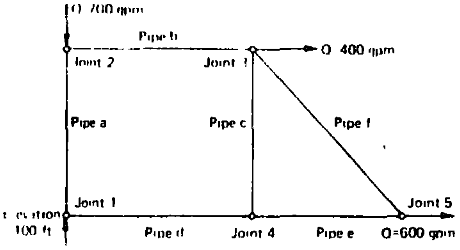

Fig. 1. Example Problem — Analysis of a Simple Pipe Network

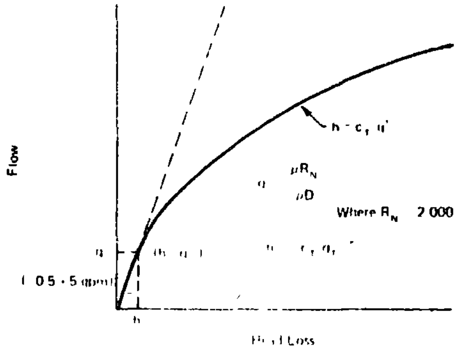

Fig. 2. Typical Flow-Head Loss Relationship

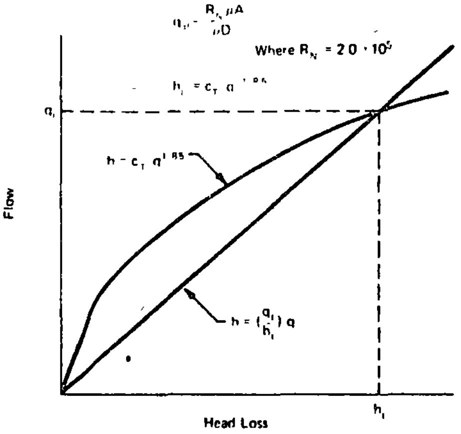

Fig. 3: Initial Value of Pipe Coefficient c

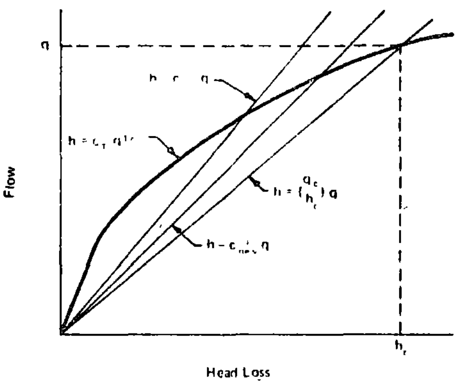

Fig. 4. Correction of Pipe Coefficient c

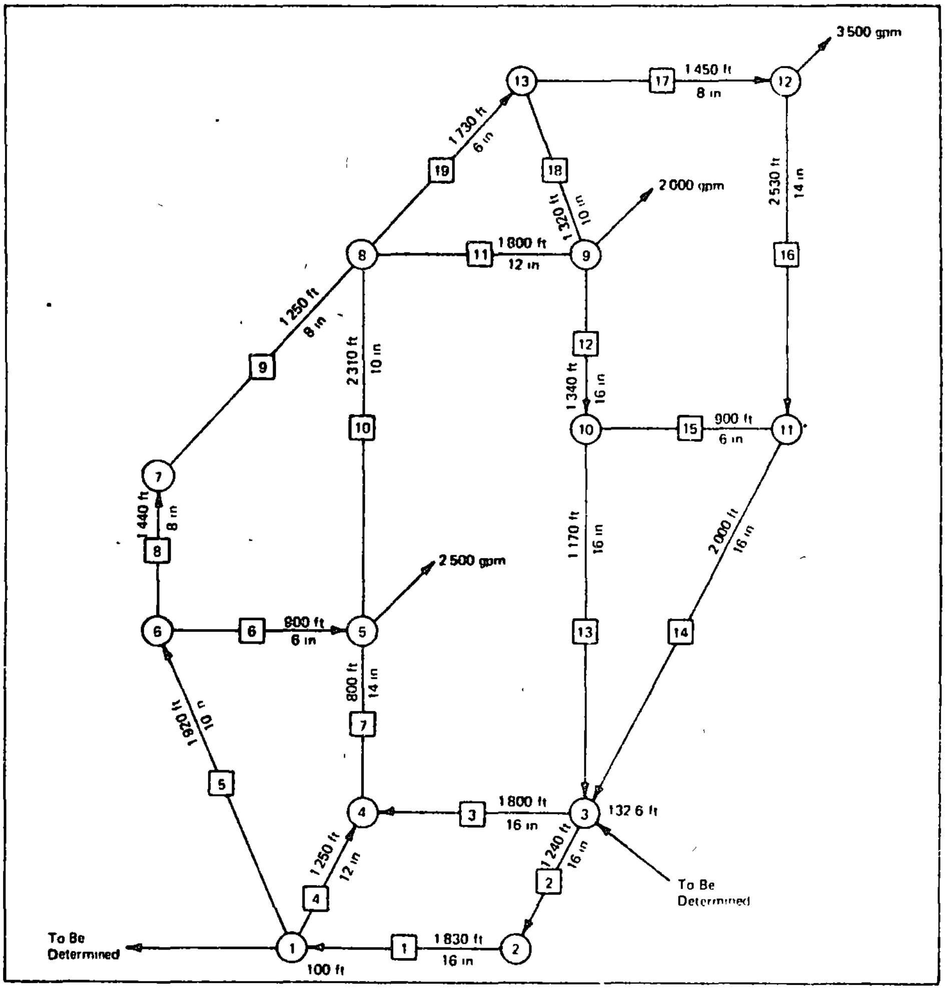

Fig. 5. Comparative Example Using PAWDS and GENFEM

○ *—joint numbers;* □ *—pipe numbers; length in feet, diameter in inches*

Equations (4-8) can now be written in terms of the pipe coefficients (C_a, C_b, ..) and the piezometric heads (H_1, H_2,) Consistency of flow directions is taken into account by assuming the flow is away from the joint being considered, that is, the piezometric head at the other joints is subtracted from the piezometric head at the joint being considered.

$$Q_1 = C_a(H_1 - H_2) + C_d(H_1 - H_4) \quad (15)$$
$$Q_2 = C_a(H_2 - H_1) + C_b(H_2 - H_3) \quad (16)$$
$$Q_3 = C_b(H_3 - H_2) + C_c(H_3 - H_4) + C_f(H_3 - H_5) \quad (17)$$
$$Q_4 = C_c(H_4 - H_3) + C_d(H_4 - H_1) + C_e(H_4 - H_5) \quad (18)$$
$$Q_5 = C_c(H_5 - H_4) + C_f(H_5 - H_3) \quad (19)$$

Equations (15-19) can be combined into the matrix form of Eq (3) to yield Eq (20). (See below.)

For this particular example, the following boundary conditions are given.

$H_1 = 100$ ft
$Q_2 = 700$ gpm
$Q_3 = 400$ gpm
$Q_4 = 0$ gpm
$Q_5 = 600$ gpm

Substituting these values into Eq (20) gives Eq (21) as the final form for solution (See below)

The values of the coefficients ($C_a, C_b, \ldots$) for each pipe are determined by the procedure to be outlined. The unknowns, H_2, H_3, H_4, H_5 and Q_1, can be obtained by solving the matrix Eq (21). The flows in the individual pipes can be found, as previously stated, by substituting into the defining equations (Eq. [9-14] for this example) after the piezometric heads have been found for each joint.

Method of application. For the successful application of the finite-element method, the constitutive equations used to relate flow and head loss must be linear or the matrix solution cannot be applied. In reality, the relationship is nonlinear and varies with the equation chosen The application of the finite element method is accomplished by using a linear equation (Eq [2]) as the defining flow-head loss relationship and the successive correction of the pipe coefficient c until a unique solution is found satisfying both the equivalent linear relationship and a real nonlinear relationship such as the Hazen-Williams equation or the Darcy-Weisbach equation. When this unique solution has been found for all pipes the network distribution has been solved.

The program, GENFEM, allows a choice of the Darcy-Weisbach equation or the Hazen-Williams equation The Hazen-Williams equation will be used for purposes of explanation, although the method is identical for both equations. The relationship used to define flow versus head loss is shown in Fig 2 The transition from laminar to turbulent conditions for pipe flow occurs at a Reynolds Number R_N of approximately 2 000. R_N is defined by the pipe diameter D, and the dynamic viscosity μ, the density ρ and the flow velocity V of the fluid flowing.

$$R_N = \frac{\rho V D}{\mu} \quad (22)$$

The flow q_T at which transition occurs, corresponding to a R_N of 2 000, is given by

$$q_T = VA = \frac{2\,000\,\mu A}{\rho D} \quad (23)$$

For flows less than q_T, the flow vs head-loss relationship is linear. To avoid a discontinuity in the defining relationship because of the transition region between laminar and turbulent flow, the linear relationship is obtained by simply joining the origin to the point (h_T, q_T) with a straight line. The coordinate h_T is found from a substitution of the flow q_T into the turbulent flow equation. The linear portion of the graph, the laminar region, does not enter into the calculations of a practical problem. The value of q_T ranges from 0.5 to 5 gpm for 6-16-in. diameter pipes whereas typical flows range from 200 to 5000 gpm for these size pipes, well into the turbulent range

The Hazen-Williams equation relates the head loss h to the pipe diameter D, the pipe length L, the Hazen-Williams coefficient C_{HW}, the flow q and a coefficient c' for unit conversion.

$$h = c' \frac{L}{D^{4\,87}} \left(\frac{q}{C_{HW}}\right)^{1\,85} \quad (24)$$

This equation can be rewritten for a particular pipe by grouping terms into one constant c_T.

$$h = c_T q^{1\,85} \quad (25)$$

The solution technique can be divided into three steps. The first step is to select an initial value of the pipe coefficient c for each pipe and combine these to yield the matrix pipe coefficients C The system matrix is then solved for the value of the piezometric head at each joint. Secondly, the individual pipe flows q are calculated by use of Eq (2) using the differences between the determined piezometric heads These flows are then substituted into Eq (25) and since (c_T) for each pipe is known, the pipe head losses are calculated. If the pipe head losses obtained from Eq (25) correspond to those obtained from the matrix solution, then the unique solution satisfying both the Hazen-Williams equation and the linear Eq (2) has been found The third and final step required is to change the value of c to converge the problem to a solution if there is a difference between the head losses calculated by the two methods

A more detailed explanation of each of these steps follows. The initial value of the pipe coefficient c_1 is chosen to correspond to R_N of 200 000 in each pipe, a typical value for a practical problem The flow (q_1) is then calculated from the Reynolds Number relationship, Eq (26).

$$q_1 = VA \frac{200\,000\,\mu A}{\rho D} \quad (26)$$

The value of the head loss h_1 corresponding to this flow q_1 is calculated from Eq (25).

$$h_1 = c_T q_1^{1\,85} \quad (27)$$

The pipe coefficient is then found from Eq (2) as shown in Fig 3

$$c_1 = \frac{q_1}{h_1} \quad (28)$$

This initial value of the pipe coefficient c_1 for each pipe is then combined, according to the geometry of the network into the pipe coefficients C_1 used in the matrix description of the network system The matrix is then solved to yield the first estimate of the piezometric heads at each joint

The allowable deviation between a pipe head loss determined from the matrix solution of the joint piezometric heads and the corresponding value from the Hazen-Williams equation is a variable and can be specified for a particular case taking into account the type of problem and the degree of precision desired for the solution.

The third step, adjusting the value of c, was developed with two criteria in mind. The solution should converge reasonably rapidly, yet the technique should remain simple During the checking procedure, the flow q_c for each pipe calculated via Eq (2), and the matrix solution is used to determine the head loss h_c from the Hazen-Williams equation The first procedure used in the development of the program was to obtain the correction of the c value for each pipe by assuming that the point h_c, q_c was the unique solution and thus the correct linear relationship was defined by a straight line joining this

---

$$\begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_f & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \quad (20)$$

$$\begin{vmatrix} Q_1 \\ 700 \\ 400 \\ 0 \\ 600 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_c & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} 100 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \quad (21)$$

point to the origin and defined by Eq (29)

$$h = (\frac{q_c}{h_c}) \quad q \tag{29}$$

The new value of c was then set equal to $\frac{q_c}{h_c}$. When all the pipe coefficients were corrected in a similar way, the flow distribution obviously was altered, and this method proved to be an overcorrection when the matrix was resolved To dampen this overcorrection effect, an averaging technique was introduced. The corrected value of c is taken to be the mean of the c value defined by Eq (29) and the value of c used to obtain the matrix solution This method of correcting c is shown in Fig 4. The averaging method reduced the number of cycles required for convergence by approximately one third

Example Problems

It is not the intent of this article to present extensive comparison of different network problems since any comparison of computer programs must take into account ease of input data and flexibility of use as well as efficiency of computer time Two example problems are discussed to point out some apparent potential advantages of the finite element approach

An example problem[3] shown in Fig. 5, consisting of nineteen pipes and thirteen joints, was solved using the PAWDS program and the GENFEM program. The PAWDS program solved the system in eighteen iteration cycles and 1.07 s. The GENFEM program solved the network in fifteen iteration cycles and 4.73 s. Obviously, this example does not indicate a preferential method, but is included so that it can be considered with the next example to show the effect of increased system size.

The second example problem with 75 pipes and 57 joints was also solved with both programs. The particular problem had been submitted by an undergraduate student and would not converge in the allowed time using the PAWDS program. Both the time limit and the iteration cycle limit were increased, and the problem eventually converged by the use of the PAWDS program after 16 048 iteration cycles and 768 s The same problem was solved with the GENFEM program after twenty iteration cycles and 22.2 s. This problem highlights the apparent lack of convergence problems for the finite-element method and also shows, when compared with the first example problem, that for the finite-element method, the number of iteration cycles to convergence appears virtually independent of the number of pipes and joints.

Discussion

The finite-element method is not restricted to a pipe as the only element. Any type of hydraulic element can be included that can be defined by a flow-head loss relationship When the pipe network is relatively small, such as in an industrial plant piping system, the fittings may become major head loss contributors The head loss h across any of these elements is usually considered to be directly proportional to the velocity head by a coefficient k corresponding to the type of element.

$$h = k \quad \frac{V^2}{2g} \tag{30}$$

This can be easily converted to the required form. that is, in terms of flow q knowing the area A of the element

$$h = \frac{k}{2gA^2} \quad q^2 \tag{31}$$

A pump can be included in the system since a pump merely provides a "head gain" or negative head loss The use of a pump element requires a flow-head loss relationship (the head-capacity curve for the pump) If the information is not available as an equation relating discharge and head, the pump information could be provided in tabular form In this form the program would use linear interpolation between any two data points

The basic finite-element method is not restricted to pipes flowing full. In practice, water systems often contain open channels in the headwater sections. These open channels or even pipes flowing partially full can be included for analysis The only requirement for an element is that the flow can be related to the head loss.

The range of hydraulic elements that can be included is limitless, provided a flow-head loss relationship for each element is known. An exact system representation can be obtained by introducing a combination of all the hydraulic elements causing head loss or gain in the system While discussing this point one should remember that methods such as the Hardy Cross method of balancing heads technique[5,7] require that all pipes are part of a loop. Typically, reservoirs or elevated tanks have one pipe connecting to the distribution system and an artificial pipe must be introduced to make a loop so that the Hardy Cross balancing heads solution method can be applied.

The use of high-resistance artificial pipes in the network has, in fact, often contributed to convergence difficulties. To simulate the actual conditions, the artificial pipes introduced to form loops are often of small diameter and high resistance so that they carry an insignificant flow and hence can be neglected. As was pointed out earlier, this is the exact condition that creates convergence problems. The finite-element method does not require the use of artificial pipes to complete a loop since the connectivity of the elements is defined explicitly by the system matrix. Consequently, tree-type systems are readily solved with this procedure. In fact, the GENFEM program was used very successfully during the 1974 spring semester at Lehigh Univ to analyze a transmission system problem that was almost entirely tree type with approximately ten loops included

The input data required for the program is equivalent to other solution techniques with the major exception that loop data does not need to be included. The distribution network is defined by input of the number of the pipe and the joints to which it is connected Another advantage over some solution methods is that any number of points of known pressure can be preselected

With loop-solution methods, all pipe and joint information must be available to the program at the same time This puts a definite limit on the size of the system that can be solved. The finite-element program, GENFEM, however, can operate on blocks of data. Thus, there is virtually no limit to the size of the network that can be solved. The element and nodal information can be stored on magnetic tapes or other devices and then read from the storage device in blocks, operated upon, and returned to storage on the tapes This feature must gain greater significance as water-distribution networks become larger and more interdependent

As stated previously, although not being specific to a finite-element program, the program developed has two additional features worth noting. First, the program GENFEM provides a choice of two established flow-head loss relationships. The Hazen–Williams equation has already been stated as Eq (24). The Darcy–Weisbach equation relates the same variables and includes the friction factor f.

$$h = (f\frac{L}{D\,2g\,A^2}) \quad q^2 \tag{32}$$

An explicit expression[12] for the friction factor f is used rather than the classical implicit Colebrook and White equation[12] that requires an iterative solution. The friction factor f is expressed in terms of the Reynolds Number R_N and the relative roughness κ where κ is the ratio of the absolute roughness e to the pipe diameter D.

$$f = 0\,094\kappa^{0\,255} + 0\,53\kappa + 88\kappa^{0\,44}\,R_N{}^{-1\,62\kappa^{0\,134}} \tag{33}$$

Second, allowance for changes in temperature has been included since temperature appreciably affects the viscosity and to a negligible degree, the density of water The viscosity of water over a temperature range of 5C–30C varies from 0 0152 poise to 0.8004 centipoise An algorithm[13] in terms of temperature T in degrees Celsius is used to define the viscosity μ in poise.

$$\frac{1}{\mu} = 2\,1482\,([T - 8\,435] + \sqrt{8078\,4 + [T - 8\,435^2]} - 120) \tag{34}$$

The program is written so that the temperature can be specified for each pipe. Any set of temperature conditions can be investigated for a particular circumstance.

Practical adaptation of the finite-element method should require a minimum of computer programming since most engineering firms and universities have finite-element programs readily available. The ease of modifying these programs depends upon the generality of their nature

Summary

The many advantages of the finite-element method have been documented. Most of these advantages hold true in a comparison of any loop method to the finite-element method The major advantage is the speed of convergence and the apparent lack of convergence problems of the proposed method over the Hardy Cross balancing flows method Other important advantages are the ability to include in the analysis all types of hydraulic elements, the choice of flow-head loss relationships, the lack of artificial loops, the ease of adaption of existing finite-element programs, the unlimited network size, and finally, the ability to account for temperature effects

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Finite-Element Method for Water-Distribution Networks

Anthony G. Collins and Robert L. Johnson

A contribution submitted to the JOURNAL on Oct 13, 1973, and revised Sep 24, 1974, by Anthony G Collins, pollution cont engr, ACI Environics, Melbourne, Australia, and Robert L. Johnson (Active Member, AWWA), assoc prof Dept of Civ Engrg, Lehigh Univ, Bethlehem, Pa.

Over the past two decades, the finite-element method has been increasingly used in a variety of engineering fields including structural analysis, solid mechanics, and soil mechanics. The method uses the relationship between the basic properties of each discrete element to define the behavior of that element. A solution for the response of the overall system, subject to a set of boundary conditions, is provided by solving a set of compatible simultaneous equations by matrix solution techniques.

The equivalence of structural systems and pipe networks has long been recognized, and there are many examples of concurrent application of solution techniques or the exchange of solution methods The system of equations for structural problems is normally linear and hence suitable to matrix solution.

Although a specific pipe-network problem can be defined using a finite-element approach,[1] the actual solution of the network problem becomes very difficult because of the nonlinear constitutive equations relating the flow and head loss in each pipe or element. Indeed, matrix solution of the pipe-network problem[2] has been achieved using extensive numerical analysis and graph theory but without recognition of the advantages of the finite-element method

The successful application of the finite-element method to pipe-network problems shows that the method is not only superior to conventional Hardy Cross solution techniques but that the further advantages of complete network representation, simplified input data, and unlimited network size can be obtained. Although not specific to a finite-element solution, the program developed allows for solution by either the Hazen-Williams or the Darcy-Weisbach flow-head-loss relationships The program also can consider the effect of temperature variations on head loss throughout the network

The computational algorithm used to arrive at the unique solution for an easily solved linear system equivalent to the true nonlinear system for the pipe networks was maintained in an extremely simple form in this article so that the advantages of the finite-element method could be readily observed Undoubtedly, further application of numerical-analysis techniques would improve the efficiency of the method

There are two specific reasons for the development of this method First, a computer program, PAWDS,[3,4] which uses the Hardy Cross solution[5] method of balancing flow for pipe-network problems, is used in undergraduate courses at Lehigh Univ in Bethlehem, Pa This Hardy Cross method or various refinements of the loop method (balancing heads in loops) were, in 1973, still used extensively in undergraduate education, engineering practice, and research.[6 7]

The Hardy Cross approach to pipe-network analysis uses as a boundary condition either the fact that the algebraic sum of flows at any joint is zero (balancing flows) or that the algebraic sum of the head loss around any loop is zero (balancing heads) These are simply adaptations of the classical conservation of mass and conservation of energy, respectively Depending upon the criteria used, a correction is applied to the assumed pipe flows or assumed piezometric heads until convergence to a solution is obtained. This classical iteration procedure is quite satisfactory for most well-conditioned pipe systems. However, it has been pointed out that convergence to a solution is not necessarily guaranteed

> There appears to be nothing inherent in either the electric analyzer with ordinary resistors or the Hardy Cross method which will consistently produce convergence of the errors toward zero with subsequent adjustments [8]

The PAWDS program used at Lehigh Univ. was plagued by convergence problems typical of the Hardy Cross method. Dillingham and Cleasby[9] point out that when using the balancing-heads method, a pipe or pipes with high resistance to flow compared with others in the network can result in calculated flow corrections larger and in the opposite direction to the currently assumed flow This will often cause a divergence in the computations, and no solution can be obtained. When the method of balancing flows is used Dillingham[9] points out that if a large pipe of short length and relatively low flow exists, many iterations are necessary before an appreciable change in piezometric head is obtained if the value of the assumed piezometric head is incorrect. These situations are very practical in their nature. Typical pipe-distribution networks[10] have these exact conditions

An extensive discussion of the convergence problems of the Hardy Cross method and the PAWDS program in particular is not intended in this article, but the existence of these problems should be noted

The second reason for developing the solution technique was because of the existence of a very efficient finite-element program, GENFEM, developed by Desai.[11] The advantage of this program is its completely general nature and hence easy adaptation for the pipe-network problem The mathematical basis and the method of application of the finite-element method is described in detail to allow easy adaption of other existing finite-element programs for use in solving water-distribution-network problems

Application of the Finite-Element Method

Mathematical basis. When the finite-element method is applied to a structural problem, the structure is subdivided into discrete elements. Each of these elements must satisfy three conditions.

1. Equilibrium of forces must be maintained
2. Compatibility must be maintained.
3. The force-displacement relationship specified by the geometric and elastic properties of the discrete element must be satisfied.

The force F in the member or element is related to the displacement u and the element properties or stiffness K by Eq (1).

$$F = K\ u \tag{1}$$

The sum of the forces in the members at each node of the structure is zero except where an external force is applied By combining Eq (1) for all the elements in the structure into an equation of identical form to Eq (1) and solving for displacements, the equilibrium of the system is satisfied

An equivalent set of conditions for a pipe network exists, hence, the ability to draw the analogy:

1. The algebraic sum of the flows at any joint or node must be zero.
2. The value of the piezometric head at a joint or node is the same for all pipes connected to that joint.
3. The flow-head-loss relationship (such as Darcy-Weisbach or Hazen-Williams) must be satisified for each element or pipe.

The conditions for a pipe network deal with scalar quantities, whereas the structural conditions deal with vector quantities. The analogy is drawn between the magnitudes of the equivalent quantities as the vector aspects of the flow have no meaning for the network problem.

For a direct application of the finite-element method involving a matrix solution,

linear relationship is required to define the element or pipe Hence at this point, a relationship of the form of Eq (2) between flow q, head loss h and the hydraulic properties of the pipe c will be assumed.

$$q = ch \qquad (2$$

The method of solution to make Eq (2 equivalent to established nonlinear flow-head-loss relationships will be described subsequently.

The head loss h in Eq (2) is the difference between the piezometric head H of the nodes or joints at each end of the element or pipe contributing to the sum of the flows Q at that joint The pipe-system matrix is assembled by writing the equations for the sum of the flows Q at each joint since this value is known to be either zero or equal to the imposed external flow or demand. Alternatively, if the piezometric head is specified at a joint, the sum of the pipe flows is implicitly defined The resulting set of simultaneous equations can be combined into matrix form defining the entire pipe system in terms of the sum of flows Q at a joint and of the piezometric heads H at the joints This matrix has the form

$$Q = CH \qquad (3)$$

When the matrix is solved, the piezometric heads at all joints are obtained. The difference in piezometric heads between two joints, which is the head loss, can be substituted into Eq (2) to calculate the flow in the pipe between those two joints. The direction of flow is automatically preserved by taking the sign of the difference of the piezometric heads into account

The finite element representing the pipe is of the simplest form possible Each element is one dimensional and has one degree of freedom at each node or joint. To apply the summation of the flows at a joint successfully, a convention must be adopted. Flow into a joint is taken as positive, and flow out of a joint is negative

The analysis of a simple pipe network, Fig. 1, is used to show the application of the finite-element method. Using the condition that the sum of the pipe flows $(q_a, q_b, \ldots.)$ in or out of a joint must satisfy the equilibrium flow criteria $(Q_1, Q_2, \ldots)$ (i.e., the boundary conditions) at that joint, one can write the following equations:

$$Q_1 = q_a + q_d \qquad (4)$$
$$Q_2 = q_a + q_b \qquad (5)$$
$$Q_3 = q_b + q_c + q_f \qquad (6)$$
$$Q_4 = q_c + q_d + q_e \qquad (7)$$
$$Q_5 = q_e + q_f \qquad (8)$$

The individual pipe flows can be expressed by Eq (2) noting that the head loss h is equal to the difference in the piezometric heads of the joints at each end of the particular pipe.

$$q_a = \pm C_a(H_1 - H_2) \qquad (9)$$
$$q_b = \pm C_b(H_2 - H_3) \qquad (10)$$
$$q_c = \pm C_c(H_3 - H_4) \qquad (11)$$
$$q_d = \pm C_d(H_1 - H_4) \qquad (12)$$
$$q_e = \pm C_e(H_4 - H_5) \qquad (13)$$
$$q_f = \pm C_f(H_3 - H_5) \qquad (14)$$

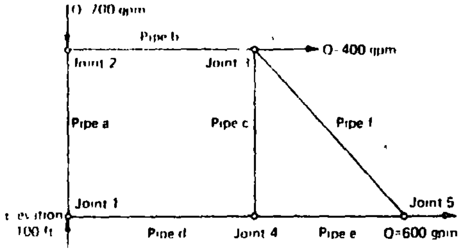

Fig. 1. Example Problem — Analysis of a Simple Pipe Network

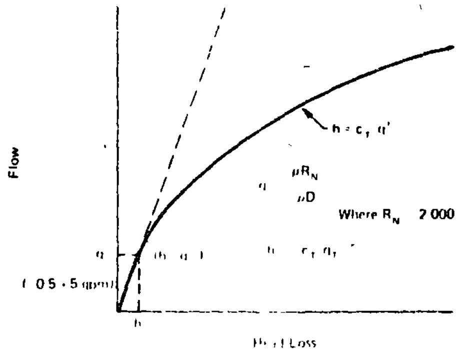

Fig. 2. Typical Flow-Head Loss Relationship

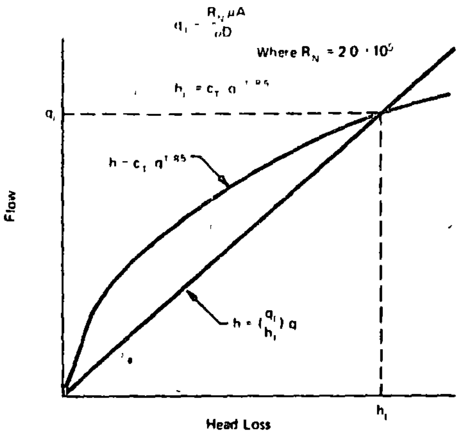

Fig. 3: Initial Value of Pipe Coefficient c

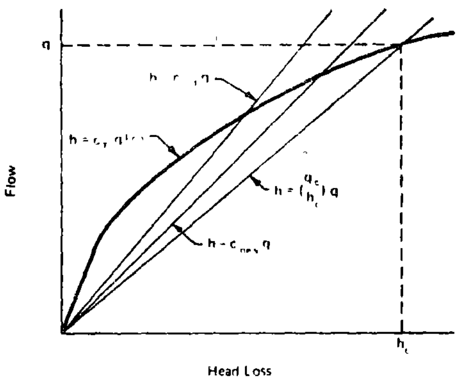

Fig. 4. Correction of Pipe Coefficient c

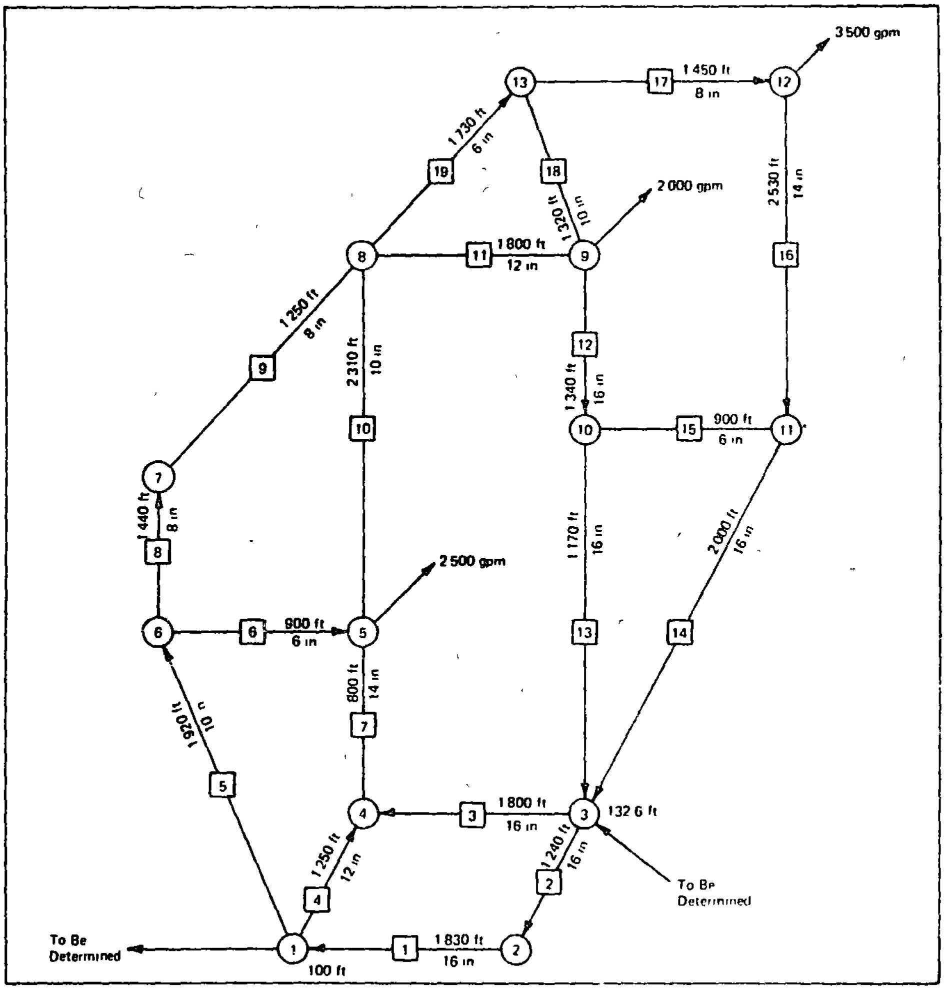

Fig. 5. Comparative Example Using PAWDS and GENFEM

○ —*joint numbers;* □ —*pipe numbers; length in feet, diameter in inches* tions (4-8) can now be written in terms of the pipe coefficients ($C_a, C_b, \ldots$) and the piezometric heads ($H_1, H_2, \ldots$). Consistency of flow directions is taken into account by assuming the flow is away from the joint being considered, that is, the piezometric head at the other joints is subtracted from the piezometric head at the joint being considered

$$Q_1 = C_a(H_1 - H_2) + C_d(H_1 - H_4) \quad (15)$$

$$Q_2 = C_a(H_2 - H_1) + C_b(H_2 - H_3) \quad (16)$$

$$Q_3 = C_b(H_3 - H_2) + C_c(H_3 - H_4) + C_f(H_3 - H_5) \quad (17)$$

$$Q_4 = C_c(H_4 - H_3) + C_d(H_4 - H_1) + C_e(H_4 - H_5) \quad (18)$$

$$Q_5 = C_e(H_5 - H_4) + C_f(H_5 - H_3) \quad (19)$$

Equations (15-19) can be combined into the matrix form of Eq (3) to yield Eq (20). (See below)

For this particular example, the following boundary conditions are given.

$H_1 = 100$ ft
$Q_2 = 700$ gpm
$Q_3 = 400$ gpm
$Q_4 = 0$ gpm
$Q_5 = 600$ gpm

Substituting these values into Eq (20) gives Eq (21) as the final form for solution (See below)

The values of the coefficients ($C_a, C_b, \ldots$) for each pipe are determined by the procedure to be outlined. The unknowns, H_2, H_3, H_4, H_5 and Q_1, can be obtained by solving the matrix Eq (21). The flows in the individual pipes can be found, as previously stated, by substituting into the defining equations (Eq. [9-14] for this example) after the piezometric heads have been found for each joint

Method of application. For the successful application of the finite-element method, the constitutive equation used to relate flow and head loss must be linear or the matrix solution cannot be applied. In reality, the relationship is nonlinear and varies with the equation chosen. The application of the finite element method is accomplished by using a linear equation (Eq [2]) as the defining flow-head loss relationship and the successive correction of the pipe coefficient c until a unique solution is found satisfying both the equivalent linear relationship and a real nonlinear relationship such as the Hazen-Williams equation or the Darcy-Weisbach equation. When this unique solution has been found for all pipes the network distribution has been solved

The program, GENFEM, allows a choice of the Darcy-Weisbach equation or the Hazen-Williams equation The Hazen-Williams equation will be used for purposes of explanation, although the method is identical for both equations. The relationship used to define flow versus head loss is shown in Fig 2 The transition from laminar to turbulent conditions for pipe flow occurs at a Reynolds Number R_N of approximately 2 000 R_N is defined by the pipe diameter D, and the dynamic viscosity μ, the density ρ and the flow velocity V of the fluid flowing.

$$R_N = \frac{\rho V D}{\mu} \quad (22)$$

The flow q_T at which transition occurs, corresponding to a R_N of 2 000, is given by

$$q_T = VA = \frac{2\,000\,\mu A}{\rho D} \quad (23)$$

For flows less than q_T, the flow vs head-loss relationship is linear To avoid a discontinuity in the defining relationship because of the transition region between laminar and turbulent flow, the linear relationship is obtained by simply joining the origin to the point (h_T, q_T) with a straight line. The coordinate h_T is found from a substitution of the flow q_T into the turbulent flow equation. The linear portion of the graph, the laminar region, does not enter into the calculations of a practical problem The value of q_T ranges from 0.5 to 5 gpm for 6-16-in. diameter pipes whereas typical flows range from 200 to 5000 gpm for these size pipes, well into the turbulent range.

The Hazen-Williams equation relates the head loss h to the pipe diameter D, the pipe length L, the Hazen-Williams coefficient C_{HW}, the flow q and a coefficient c' for unit conversion.

$$h = c' \frac{L}{D^{4.87}} \left(\frac{q}{C_{HW}}\right)^{1.85} \quad (24)$$

This equation can be rewritten for a particular pipe by grouping terms into one constant c_T.

$$h = c_T q^{1.85} \quad (25)$$

The solution technique can be divided into three steps. The first step is to select an initial value of the pipe coefficient c for each pipe and combine these to yield the matrix pipe coefficients C The system matrix is then solved for the value of the piezometric head at each joint Secondly, the individual pipe flows q are calculated by use of Eq (2) using the differences between the determined piezometric heads These flows are then substituted into Eq (25) and since (c_T) for each pipe is known, the pipe head losses are calculated If the pipe head losses obtained from Eq (25) correspond to those obtained from the matrix solution, then the unique solution satisfying both the Hazen-Williams equation and the linear Eq (2) has been found The third and final step required is to change the value of c to converge the problem to a solution if there is a difference between the head losses calculated by the two methods

A more detailed explanation of each of these steps follows. The initial value of the pipe coefficient c_1 is chosen to correspond to R_N of 200 000 in each pipe, a typical value for a practical problem The flow (q_1) is then calculated from the Reynolds Number relationship, Eq (26).

$$q_1 = VA = \frac{200\,000\,\mu A}{\rho D} \quad (26)$$

The value of the head loss h_1 corresponding to this flow q_1 is calculated from Eq (25):

$$h_1 = c_T q_1^{1.85} \quad (27)$$

The pipe coefficient is then found from Eq (2) as shown in Fig 3

$$c_1 = \frac{q_1}{h_1} \quad (28)$$

This initial value of the pipe coefficient c_1 for each pipe is then combined, according to the geometry of the network into the pipe coefficients C_1 used in the matrix description of the network system The matrix is then solved to yield the first estimate of the piezometric heads at each joint

The allowable deviation between a pipe head loss determined from the matrix solution of the joint piezometric heads and the corresponding value from the Hazen-Williams equation is a variable and can be specified for a particular case taking into account the type of problem and the degree of precision desired for the solution

The third step, adjusting the value of c, was developed with two criteria in mind The solution should converge reasonably rapidly, yet the technique should remain simple During the checking procedure, the flow q_c for each pipe calculated via Eq (2), and the matrix solution is used to determine the head loss h_c from the Hazen-Williams equation The first procedure used in the development of the program was to obtain the correction of the c value for each pipe by assuming that the point h_c, q_c was the unique solution and thus the correct linear relationship was defined by a straight line joining this

$$\begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_f & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \quad (20)$$

$$\begin{vmatrix} Q_1 \\ 700 \\ 400 \\ 0 \\ 600 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_c & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} 100 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \quad (21)$$

..it to the origin and defined by Eq (29)

$$h = (\frac{q_c}{h_c}) \quad q \tag{29}$$

The new value of c was then set equal to $\frac{q_c}{h_c}$. When all the pipe coefficients were corrected in a similar way, the flow distribution obviously was altered, and this method proved to be an overcorrection when the matrix was resolved To dampen this overcorrection effect, an averaging technique was introduced The corrected value of c is taken to be the mean of the c value defined by Eq (29) and the value of c used to obtain the matrix solution This method of correcting c is shown in Fig 4. The averaging method reduced the number of cycles required for convergence by approximately one third.

Example Problems

It is not the intent of this article to present extensive comparison of different network problems since any comparison of computer programs must take into account ease of input data and flexibility of use as well as efficiency of computer time Two example problems are discussed to point out some apparent potential advantages of the finite element approach

An example problem[3] shown in Fig 5, consisting of nineteen pipes and thirteen joints, was solved using the PAWDS program and the GENFEM program. The PAWDS program solved the system in eighteen iteration cycles and 1.07 s. The GENFEM program solved the network in fifteen iteration cycles and 4.73 s. Obviously, this example does not indicate a preferential method, but is included so that it can be considered with the next example to show the effect of increased system size

The second example problem with 75 pipes and 57 joints was also solved with both programs. The particular problem had been submitted by an undergraduate student and would not converge in the allowed time using the PAWDS program. Both the time limit and the iteration cycle limit were increased, and the problem eventually converged by the use of the PAWDS program after 16 048 iteration cycles and 768 s. The same problem was solved with the GENFEM program after twenty iteration cycles and 22.2 s This problem highlights the apparent lack of convergence problems for the finite-element method and also shows, when compared with the first example problem, that for the finite-element method, the number of iteration cycles to convergence appears virtually independent of the number of pipes and joints.

Discussion

The finite-element method is not restricted to a pipe as the only element. Any type of hydraulic element can be included that can be defined by a flow-head loss relationship When the pipe network is relatively small, such as in an industrial plant piping system, the fittings may become major head loss contributors The head loss h across any of these elements is usually considered to be directly proportional to the velocity head by a coefficient k corresponding to the type of element.

$$h = k \quad \frac{V^2}{2g} \tag{30}$$

This can be easily converted to the required form, that is, in terms of flow q knowing the area A of the element.

$$h = \frac{k}{2gA^2} \quad q^2 \tag{31}$$

A pump can be included in the system since a pump merely provides a "head gain" or negative head loss. The use of a pump element requires a flow-head loss relationship (the head-capacity curve for the pump). If the information is not available as an equation relating discharge and head, the pump information could be provided in tabular form. In this form the program would use linear interpolation between any two data points.

The basic finite-element method is not restricted to pipes flowing full In practice, water systems often contain open channels in the headwater sections. These open channels or even pipes flowing partially full can be included for analysis. The only requirement for an element is that the flow can be related to the head loss.

The range of hydraulic elements that can be included is limitless, provided a flow-head loss relationship for each element is known. An exact system representation can be obtained by introducing a combination of all the hydraulic elements causing head loss or gain in the system While discussing this point one should remember that methods such as the Hardy Cross method of balancing heads technique[5,7] require that all pipes are part of a loop. Typically, reservoirs or elevated tanks have one pipe connecting to the distribution system and an artificial pipe must be introduced to make a loop so that the Hardy Cross balancing heads solution method can be applied.

The use of high-resistance artificial pipes in the network has, in fact, often contributed to convergence difficulties. To simulate the actual conditions, the artificial pipes introduced to form loops are often of small diameter and high resistance so that they carry an insignificant flow and hence can be neglected As was pointed out earlier, this is the exact condition that creates convergence problems. The finite-element method does not require the use of artificial pipes to complete a loop since the connectivity of the elements is defined explicitly by the system matrix. Consequently, tree-type systems are readily solved with this procedure. In fact, the GENFEM program was used very successfully during the 1974 spring semester at Lehigh Univ to analyze a transmission system problem that was almost entirely tree type with approximately ten loops included

The input data required for the program is equivalent to other solution techniques with the major exception that loop data does not need to be included. The distribution network is defined by input of the number of the pipe and the joints to which it is connected Another advantage over some solution methods is that any number of points of known pressure can be preselected

With loop-solution methods, all pipe and joint information must be available to the program at the same time This puts a definite limit on the size of the system that can be solved The finite-element program, GENFEM, however, can operate on blocks of data. Thus, there is virtually no limit to the size of the network that can be solved. The element and nodal information can be stored on magnetic tapes or other devices and then read from the storage device in blocks, operated upon, and returned to storage on the tapes This feature must gain greater significance as water-distribution networks become larger and more interdependent

As stated previously, although not being specific to a finite-element program, the program developed has two additional features worth noting. First, the program GENFEM provides a choice of two established flow-head loss relationships. The Hazen–Williams equation has already been stated as Eq (24). The Darcy–Weisbach equation relates the same variables and includes the friction factor f.

$$h = (f \frac{L}{D\,2gA^2}) \quad q^2 \tag{32}$$

An explicit expression[12] for the friction factor f is used rather than the classical implicit Colebrook and White equation[12] that requires an iterative solution The friction factor f is expressed in terms of the Reynolds Number R_N and the relative roughness κ where κ is the ratio of the absolute roughness e to the pipe diameter D.

$$f = 0\;094\kappa^{0\;255} + 0\;53\kappa + 88\kappa^{0\;44}\;R_N^{-1\;62\kappa^{0\;134}} \tag{33}$$

Second, allowance for changes in temperature has been included since temperature appreciably affects the viscosity and to a negligible degree, the density of water. The viscosity of water over a temperature range of 5C–30C varies from 0.0152 poise to 0.8004 centipoise An algorithm[13] in terms of temperature T in degrees Celsius is used to define the viscosity μ in poise.

$$\frac{1}{\mu} = 2\;1482\,([T - 8\;435] + \sqrt{8078\;4 + [T - 8.435^2]} - 120) \tag{34}$$

The program is written so that the temperature can be specified for each pipe Any set of temperature conditions can be investigated for a particular circumstance.

Practical adaptation of the finite-element method should require a minimum of computer programming since most engineering firms and universities have finite-element programs readily available The ease of modifying these programs depends upon the generality of their nature

Summary

The many advantages of the finite-element method have been documented. Most of these advantages hold true in a comparison of any loop method to the finite-element method The major advantage is the speed of convergence and the apparent lack of convergence problems of the proposed method over the Hardy Cross balancing flows method Other important advantages are the ability to include in the analysis all types of hydraulic elements, the choice of flow-head loss relationships, the lack of artificial loops, the ease of adaption of existing finite-element programs, the unlimited network size, and finally, the ability to account for temperature effects.

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Finite-Element Method for Water-Distribution Networks

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A contribution submitted to the JOURNAL on Oct 13, 1973, and revised Sep 24, 1974, by Anthony G Collins, pollution cont engr, ACI Environics, Melbourne, Australia, and Robert L Johnson (Active Member, AWWA), assoc prof Dept of Civ Engrg, Lehigh Univ, Bethlehem, Pa

Over the past two decades, the finite-element method has been increasingly used in a variety of engineering fields including structural analysis, solid mechanics, and soil mechanics. The method uses the relationship between the basic properties of each discrete element to define the behavior of that element. A solution for the response of the overall system, subject to a set of boundary conditions, is provided by solving a set of compatible simultaneous equations by matrix solution techniques.

The equivalence of structural systems and pipe networks has long been recognized, and there are many examples of concurrent application of solution techniques or the exchange of solution methods The system of equations for structural problems is normally linear and hence suitable to matrix solution

Although a specific pipe-network problem can be defined using a finite-element approach,[1] the actual solution of the network problem becomes very difficult because of the nonlinear constitutive equations relating the flow and head loss in each pipe or element. Indeed, matrix solution of the pipe-network problem[2] has been achieved using extensive numerical analysis and graph theory but without recognition of the advantages of the finite-element method.

The successful application of the finite-element method to pipe-network problems shows that the method is not only superior to conventional Hardy Cross solution techniques but that the further advantages of complete network representation, simplified input data, and unlimited network size can be obtained. Although not specific to a finite-element solution, the program developed allows for solution by either the Hazen-Williams or the Darcy-Weisbach flow-head-loss relationships The program also can consider the effect of temperature variations on head loss throughout the network

The computational algorithm used to arrive at the unique solution for an easily solved linear system equivalent to the true nonlinear system for the pipe networks was maintained in an extremely simple form in this article so that the advantages of the finite-element method could be readily observed Undoubtedly, further application of numerical-analysis techniques would improve the efficiency of the method

There are two specific reasons for the development of this method First, a computer program, PAWDS,[3,4] which uses the Hardy Cross solution[5] method of balancing flow for pipe-network problems, is used in undergraduate courses at Lehigh Univ in Bethlehem, Pa This Hardy Cross method or various refinements of the loop method (balancing heads in loops) were, in 1973, still used extensively in undergraduate education, engineering practice, and research.[6,7]

The Hardy Cross approach to pipe-network analysis uses as a boundary condition either the fact that the algebraic sum of flows at any joint is zero (balancing flows) or that the algebraic sum of the head loss around any loop is zero (balancing heads) These are simply adaptations of the classical conservation of mass and conservation of energy, respectively. Depending upon the criteria used, a correction is applied to the assumed pipe flows or assumed piezometric heads until convergence to a solution is obtained. This classical iteration procedure is quite satisfactory for most well-conditioned pipe systems. However, it has been pointed out that convergence to a solution is not necessarily guaranteed.

> There appears to be nothing inherent in either the electric analyzer with ordinary resistors or the Hardy Cross method which will consistently produce convergence of the errors toward zero with subsequent adjustments [8]

The PAWDS program used at Lehigh Univ. was plagued by convergence problems typical of the Hardy Cross method. Dillingham and Cleasby[9] point out that when using the balancing-heads method, a pipe or pipes with high resistance to flow compared with others in the network can result in calculated flow corrections larger and in the opposite direction to the currently assumed flow This will often cause a divergence in the computations, and no solution can be obtained. When the method of balancing flows is used Dillingham[9] points out that if a large pipe of short length and relatively low flow exists, many iterations are necessary before an appreciable change in piezometric head is obtained if the value of the assumed piezometric head is incorrect. These situations are very practical in their nature. Typical pipe-distribution networks[10] have these exact conditions

An extensive discussion of the convergence problems of the Hardy Cross method and the PAWDS program in particular is not intended in this article, but the existence of these problems should be noted

The second reason for developing the solution technique was because of the existence of a very efficient finite-element program, GENFEM, developed by Desai.[11] The advantage of this program is its completely general nature and hence easy adaptation for the pipe-network problem The mathematical basis and the method of application of the finite-element method is described in detail to allow easy adaption of other existing finite-element programs for use in solving water-distribution-network problems.

Application of the Finite-Element Method

Mathematical basis. When the finite-element method is applied to a structural problem, the structure is subdivided into discrete elements. Each of these elements must satisfy three conditions·

1. Equilibrium of forces must be maintained.
2 Compatibility must be maintained.
3 The force-displacement relationship specified by the geometric and elastic properties of the discrete element must be satisfied.

The force F in the member or element is related to the displacement u and the element properties or stiffness K by Eq (1).

$$F = K u \tag{1}$$

The sum of the forces in the members at each node of the structure is zero except where an external force is applied By combining Eq (1) for all the elements in the structure into an equation of identical form to Eq (1) and solving for displacements, the equilibrium of the system is satisfied

An equivalent set of conditions for a pipe network exists, hence, the ability to draw the analogy·

1. The algebraic sum of the flows at any joint or node must be zero.
2. The value of the piezometric head at a joint or node is the same for all pipes connected to that joint
3. The flow-head-loss relationship (such as Darcy-Weisbach or Hazen-Williams) must be satisified for each element or pipe

The conditions for a pipe network deal with scalar quantities, whereas the structural conditions deal with vector quantities. The analogy is drawn between the magnitudes of the equivalent quantities as the vector aspects of the flow have no meaning for the network problem.

For a direct application of the finite-element method involving a matrix solution,

a linear relationship is required to define the element or pipe Hence at this point, a relationship of the form of Eq (2) between flow q, head loss h and the hydraulic properties of the pipe c will be assumed

$$q = ch \qquad (2)$$

The method of solution to make Eq (2) equivalent to established nonlinear flow-head-loss relationships will be described subsequently

The head loss h in Eq (2) is the difference between the piezometric head H of the nodes or joints at each end of the element or pipe contributing to the sum of the flows Q at that joint The pipe-system matrix is assembled by writing the equations for the sum of the flows Q at each joint since this value is known to be either zero or equal to the imposed external flow or demand Alternatively, if the piezometric head is specified at a joint, the sum of the pipe flows is implicitly defined. The resulting set of simultaneous equations can be combined into matrix form defining the entire pipe system in terms of the sum of flows Q at a joint and of the piezometric heads H at the joints This matrix has the form

$$\mathbf{Q} = \mathbf{C}\mathbf{H} \qquad (3)$$

When the matrix is solved, the piezometric heads at all joints are obtained. The difference in piezometric heads between two joints, which is the head loss, can be substituted into Eq (2) to calculate the flow in the pipe between those two joints The direction of flow is automatically preserved by taking the sign of the difference of the piezometric heads into account

The finite element representing the pipe is of the simplest form possible Each element is one dimensional and has one degree of freedom at each node or joint To apply the summation of the flows at a joint successfully, a convention must be adopted. Flow into a joint is taken as positive, and flow out of a joint is negative

The analysis of a simple pipe network, Fig. 1, is used to show the application of the finite-element method Using the condition that the sum of the pipe flows $(q_a, q_b, \ldots)$ in or out of a joint must satisfy the equilibrium flow criteria $(Q_1, Q_2, \ldots)$ (i.e., the boundary conditions) at that joint, one can write the following equations:

$$Q_1 = q_a + q_d \qquad (4)$$
$$Q_2 = q_a + q_b \qquad (5)$$
$$Q_3 = q_b + q_c + q_f \qquad (6)$$
$$Q_4 = q_c + q_d + q_e \qquad (7)$$
$$Q_5 = q_e + q_f \qquad (8)$$

The individual pipe flows can be expressed by Eq (2) noting that the head loss h is equal to the difference in the piezometric heads of the joints at each end of the particular pipe.

$$q_a = \pm\, C_a(H_1 - H_2) \qquad (9)$$
$$q_b = \pm\, C_b(H_2 - H_3) \qquad (10)$$
$$q_c = \pm\, C_c(H_3 - H_4) \qquad (11)$$
$$q_d = \pm\, C_d(H_1 - H_4) \qquad (12)$$
$$q_e = \pm\, C_e(H_4 - H_5) \qquad (13)$$
$$q_f = \pm\, C_f(H_3 - H_5) \qquad (14)$$

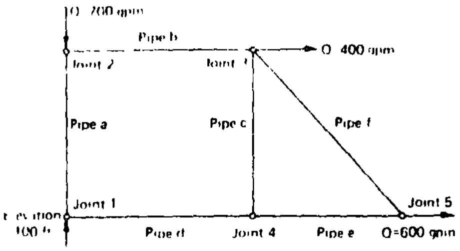

Fig. 1. Example Problem — Analysis of a Simple Pipe Network

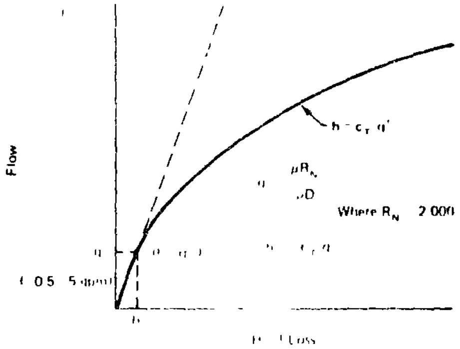

Fig. 2. Typical Flow-Head Loss Relationship

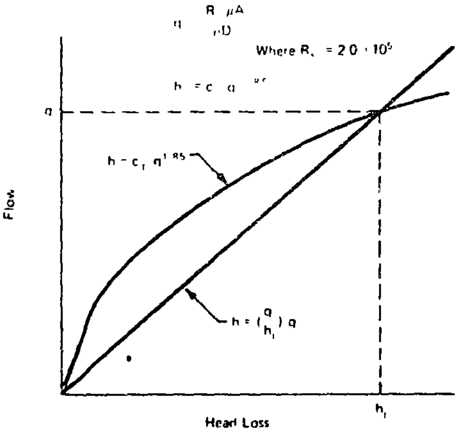

Fig. 3. Initial Value of Pipe Coefficient c

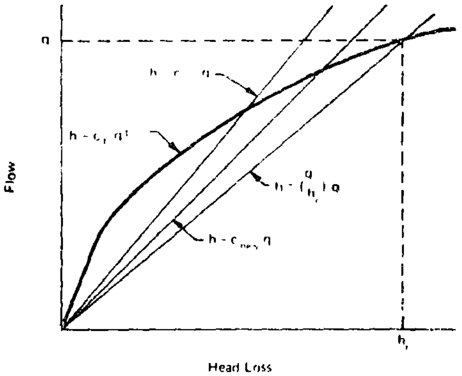

Fig. 4. Correction of Pipe Coefficient c

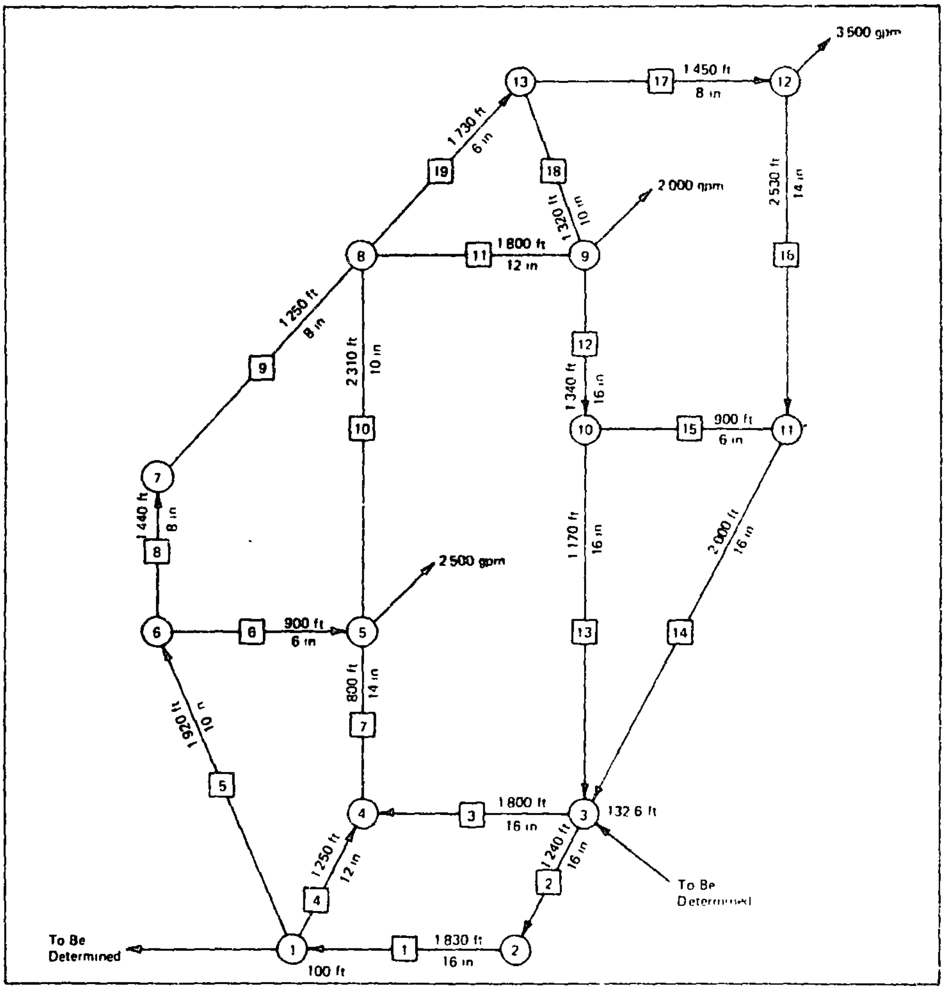

Fig. 5. Comparative Example Using PAWDS and GENFEM

○ —joint numbers; □ —pipe numbers; length in feet, diameter in inches

Equations (4-8) can now be written in terms of the pipe coefficients ($C_a, C_b, \ldots$) and the piezometric heads ($H_1, H_2, \ldots$) Consistency of flow directions is taken into account by assuming the flow is away from the joint being considered, that is, the piezometric head at the other joints is subtracted from the piezometric head at the joint being considered

$$Q_1 = C_a(H_1 - H_2) + C_d(H_1 - H_4) \qquad (15)$$

$$Q_2 = C_a(H_2 - H_1) + C_b(H_2 - H_3) \qquad (16)$$

$$Q_3 = C_b(H_3 - H_2) + C_c(H_3 - H_4) + C_f(H_3 - H_5) \qquad (17)$$

$$Q_4 = C_c(H_4 - H_3) + C_d(H_4 - H_1) + C_e(H_4 - H_5) \qquad (18)$$

$$Q_5 = C_e(H_5 - H_4) + C_f(H_5 - H_3) \qquad (19)$$

Equations (15-19) can be combined into the matrix form of Eq (3) to yield Eq (20). (See below.)

For this particular example, the following boundary conditions are given.

$H_1 = 100$ ft
$Q_2 = 700$ gpm
$Q_3 = 400$ gpm
$Q_4 = 0$ gpm
$Q_5 = 600$ gpm

Substituting these values into Eq (20) gives Eq (21) as the final form for solution (See below)

The values of the coefficients ($C_a, C_b, \ldots$) for each pipe are determined by the procedure to be outlined. The unknowns, H_2, H_3, H_4, H_5 and Q_1, can be obtained by solving the matrix Eq (21). The flows in the individual pipes can be found, as previously stated, by substituting into the defining equations (Eq [9-14] for this example) after the piezometric heads have been found for each joint

Method of application. For the successful application of the finite-element method, the constitutive equations used to relate flow and head loss must be linear or the matrix solution cannot be applied. In reality, the relationship is nonlinear and varies with the equation chosen. The application of the finite element method is accomplished by using a linear equation (Eq [2]) as the defining flow-head loss relationship and the successive correction of the pipe coefficient c until a unique solution is found satisfying both the equivalent linear relationship and a real nonlinear relationship such as the Hazen-Williams equation or the Darcy-Weisbach equation. When this unique solution has been found for all pipes the network distribution has been solved

The program, GENFEM, allows a choice of the Darcy-Weisbach equation or the Hazen-Williams equation The Hazen-Williams equation will be used for purposes of explanation, although the method is identical for both equations The relationship used to define flow versus head loss is shown in Fig 2. The transition from laminar to turbulent conditions for pipe flow occurs at a Reynolds Number R_N of approximately 2 000 R_N is defined by the pipe diameter D, and the dynamic viscosity μ, the density ρ and the flow velocity V of the fluid flowing.

$$R_N = \frac{\rho V D}{\mu} \qquad (22)$$

The flow q_T at which transition occurs, corresponding to a R_N of 2 000, is given by

$$q_T = VA = \frac{2\,000\,\mu A}{\rho D} \qquad (23)$$

For flows less than q_T, the flow vs head-loss relationship is linear. To avoid a discontinuity in the defining relationship because of the transition region between laminar and turbulent flow, the linear relationship is obtained by simply joining the origin to the point (h_T, q_T) with a straight line. The coordinate h_T is found from a substitution of the flow q_T into the turbulent flow equation The linear portion of the graph, the laminar region, does not enter into the calculations of a practical problem The value of q_T ranges from 0.5 to 5 gpm for 6-16-in. diameter pipes whereas typical flows range from 200 to 5000 gpm for these size pipes, well into the turbulent range

The Hazen-Williams equation relates the head loss h to the pipe diameter D, the pipe length L, the Hazen-Williams coefficient C_{HW}, the flow q and a coefficient c' for unit conversion.

$$h = c' \frac{L}{D^{4.87}} \left(\frac{q}{C_{HW}}\right)^{1.85} \qquad (24)$$

This equation can be rewritten for a particular pipe by grouping terms into one constant c_T.

$$h = c_T q^{1.85} \qquad (25)$$

The solution technique can be divided into three steps. The first step is to select an initial value of the pipe coefficient c for each pipe and combine these to yield the matrix pipe coefficients C The system matrix is then solved for the value of the piezometric head at each joint Secondly, the individual pipe flows q are calculated by use of Eq (2) using the differences between the determined piezometric heads. These flows are then substituted into Eq (25) and since (c_T) for each pipe is known, the pipe head losses are calculated If the pipe head losses obtained from Eq (25) correspond to those obtained from the matrix solution, then the unique solution satisfying both the Hazen-Williams equation and the linear Eq (2) has been found The third and final step required is to change the value of c to converge the problem to a solution if there is a difference between the head losses calculated by the two methods

A more detailed explanation of each of these steps follows. The initial value of the pipe coefficient c_1 is chosen to correspond to R_N of 200 000 in each pipe, a typical value for a practical problem. The flow (q_1) is then calculated from the Reynolds Number relationship, Eq (26)·

$$q_1 = VA = \frac{200\,000\,\mu A}{\rho D} \qquad (26)$$

The value of the head loss h_1 corresponding to this flow q_1 is calculated from Eq (25)·

$$h_1 = c_T q_1^{1.85} \qquad (27)$$

The pipe coefficient is then found from Eq (2) as shown in Fig 3

$$c_1 = \frac{q_1}{h_1} \qquad (28)$$

This initial value of the pipe coefficient c_1 for each pipe is then combined, according to the geometry of the network into the pipe coefficients C_1 used in the matrix description of the network system The matrix is then solved to yield the first estimate of the piezometric heads at each joint

The allowable deviation between a pipe head loss determined from the matrix solution of the joint piezometric heads and the corresponding value from the Hazen-Williams equation is a variable and can be specified for a particular case taking into account the type of problem and the degree of precision desired for the solution

The third step, adjusting the value of c, was developed with two criteria in mind The solution should converge reasonably rapidly, yet the technique should remain simple During the checking procedure, the flow q_c for each pipe calculated via Eq (2), and the matrix solution is used to determine the head loss h_c from the Hazen-Williams equation The first procedure used in the development of the program was to obtain the correction of the c value for each pipe by assuming that the point h_c, q_c was the unique solution and thus the correct linear relationship was defined by a straight line joining this

$$\begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_f & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \qquad (20)$$

$$\begin{vmatrix} Q_1 \\ 700 \\ 400 \\ 0 \\ 600 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_c & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} 100 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \qquad (21)$$

point to the origin and defined by Eq (29)

$$h = (\frac{q_c}{h_c}) \quad q \qquad (29)$$

The new value of c was then set equal to $\frac{q_c}{h_c}$ When all the pipe coefficients were corrected in a similar way, the flow distribution obviously was altered, and this method proved to be an overcorrection when the matrix was resolved. To dampen this overcorrection effect, an averaging technique was introduced. The corrected value of c is taken to be the mean of the c value defined by Eq (29) and the value of c used to obtain the matrix solution This method of correcting c is shown in Fig 4. The averaging method reduced the number of cycles required for convergence by approximately one third

Example Problems

It is not the intent of this article to present extensive comparison of different network problems since any comparison of computer programs must take into account ease of input data and flexibility of use as well as efficiency of computer time. Two example problems are discussed to point out some apparent potential advantages of the finite element approach.

An example problem[3] shown in Fig 5, consisting of nineteen pipes and thirteen joints, was solved using the PAWDS program and the GENFEM program. The PAWDS program solved the system in eighteen iteration cycles and 1.07 s. The GENFEM program solved the network in fifteen iteration cycles and 4.73 s. Obviously, this example does not indicate a preferential method, but is included so that it can be considered with the next example to show the effect of increased system size.

The second example problem with 75 pipes and 57 joints was also solved with both programs. The particular problem had been submitted by an undergraduate student and would not converge in the allowed time using the PAWDS program. Both the time limit and the iteration cycle limit were increased, and the problem eventually converged by the use of the PAWDS program after 16 048 iteration cycles and 768 s. The same problem was solved with the GENFEM program after twenty iteration cycles and 22.2 s This problem highlights the apparent lack of convergence problems for the finite-element method and also shows, when compared with the first example problem, that for the finite-element method, the number of iteration cycles to convergence appears virtually independent of the number of pipes and joints.

Discussion

The finite-element method is not restricted to a pipe as the only element. Any type of hydraulic element can be included that can be defined by a flow-head loss relationship When the pipe network is relatively small, such as in an industrial plant piping system, the fittings may become major head loss contributors The head loss h across any of these elements is usually considered to be directly proportional to the velocity head by a coefficient k corresponding to the type of element

$$h = k \quad \frac{V^2}{2g} \qquad (30)$$

This can be easily converted to the required form, that is, in terms of flow q knowing the area A of the element.

$$h = \frac{k}{2gA^2} \quad q^2 \qquad (31)$$

A pump can be included in the system since a pump merely provides a "head gain" or negative head loss. The use of a pump element requires a flow-head loss relationship (the head-capacity curve for the pump) If the information is not available as an equation relating discharge and head, the pump information could be provided in tabular form In this form the program would use linear interpolation between any two data points

The basic finite-element method is not restricted to pipes flowing full. In practice, water systems often contain open channels in the headwater sections These open channels or even pipes flowing partially full can be included for analysis The only requirement for an element is that the flow can be related to the head loss.

The range of hydraulic elements that can be included is limitless, provided a flow-head loss relationship for each element is known. An exact system representation can be obtained by introducing a combination of all the hydraulic elements causing head loss or gain in the system While discussing this point one should remember that methods such as the Hardy Cross method of balancing heads technique[5,7] require that all pipes are part of a loop. Typically, reservoirs or elevated tanks have one pipe connecting to the distribution system and an artificial pipe must be introduced to make a loop so that the Hardy Cross balancing heads solution method can be applied.

The use of high-resistance artificial pipes in the network has, in fact, often contributed to convergence difficulties. To simulate the actual conditions, the artificial pipes introduced to form loops are often of small diameter and high resistance so that they carry an insignificant flow and hence can be neglected As was pointed out earlier, this is the exact condition that creates convergence problems. The finite-element method does not require the use of artificial pipes to complete a loop since the connectivity of the elements is defined explicitly by the system matrix. Consequently, tree-type systems are readily solved with this procedure. In fact, the GENFEM program was used very successfully during the 1974 spring semester at Lehigh Univ to analyze a transmission system problem that was almost entirely tree type with approximately ten loops included

The input data required for the program is equivalent to other solution techniques with the major exception that loop data does not need to be included. The distribution network is defined by input of the number of the pipe and the joints to which it is connected Another advantage over some solution methods is that any number of points of known pressure can be preselected

With loop-solution methods, all pipe and joint information must be available to the program at the same time This puts a definite limit on the size of the system that can be solved The finite-element program, GENFEM, however, can operate on blocks of data Thus, there is virtually no limit to the size of the network that can be solved The element and nodal information can be stored on magnetic tapes or other devices and then read from the storage device in blocks, operated upon, and returned to storage on the tapes This feature must gain greater significance as water-distribution networks become larger and more interdependent

As stated previously, although not being specific to a finite-element program, the program developed has two additional features worth noting. First, the program GENFEM provides a choice of two established flow-head loss relationships. The Hazen-Williams equation has already been stated as Eq (24). The Darcy-Weisbach equation relates the same variables and includes the friction factor f.

$$h = (f \frac{L}{D\,2g\,A^2}) \quad q^2 \qquad (32)$$

An explicit expression[12] for the friction factor f is used rather than the classical implicit Colebrook and White equation[12] that requires an iterative solution The friction factor f is expressed in terms of the Reynolds Number R_N and the relative roughness κ where κ is the ratio of the absolute roughness e to the pipe diameter D.

$$f = 0.094\kappa^{0.255} + 0.53\kappa + 88\kappa^{0.44} R_N^{-1.62\kappa^{0.134}} \qquad (33)$$

Second, allowance for changes in temperature has been included since temperature appreciably affects the viscosity and to a negligible degree, the density of water. The viscosity of water over a temperature range of 5C–30C varies from 0.0152 poise to 0.8004 centipoise. An algorithm[13] in terms of temperature T in degrees Celsius is used to define the viscosity μ in poise.

$$\frac{1}{\mu} = 2.1482\,([T - 8.435] + \sqrt{8078.4 + [T - 8.435^2]} - 120) \qquad (34)$$

The program is written so that the temperature can be specified for each pipe. Any set of temperature conditions can be investigated for a particular circumstance.

Practical adaptation of the finite-element method should require a minimum of computer programming since most engineering firms and universities have finite-element programs readily available. The ease of modifying these programs depends upon the generality of their nature.

Summary

The many advantages of the finite-element method have been documented. Most of these advantages hold true in a comparison of any loop method to the finite-element method The major advantage is the speed of convergence and the apparent lack of convergence problems of the proposed method over the Hardy Cross balancing flows method Other important advantages are the ability to include in the analysis all types of hydraulic elements, the choice of flow-head loss relationships, the lack of artificial loops, the ease of adaption of existing finite-element programs, the unlimited network size, and finally, the ability to account for temperature effects.

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Finite-Element Method for Water-Distribution Networks

Anthony G. Collins and Robert L. Johnson

A contribution submitted to the JOURNAL on Oct 13, 1973, and revised Sep 24, 1974, by Anthony G Collins, pollution cont engr, ACI Environics, Melbourne, Australia, and Robert L Johnson (Active Member, AWWA), assoc prof Dept of Civ Engrg, Lehigh Univ, Bethlehem, Pa.

Over the past two decades, the finite-element method has been increasingly used in a variety of engineering fields including structural analysis, solid mechanics, and soil mechanics. The method uses the relationship between the basic properties of each discrete element to define the behavior of that element. A solution for the response of the overall system, subject to a set of boundary conditions, is provided by solving a set of compatible simultaneous equations by matrix solution techniques.

The equivalence of structural systems and pipe networks has long been recognized, and there are many examples of concurrent application of solution techniques or the exchange of solution methods The system of equations for structural problems is normally linear and hence suitable to matrix solution

Although a specific pipe-network problem can be defined using a finite-element approach,[1] the actual solution of the network problem becomes very difficult because of the nonlinear constitutive equations relating the flow and head loss in each pipe or element Indeed, matrix solution of the pipe-network problem[2] has been achieved using extensive numerical analysis and graph theory but without recognition of the advantages of the finite-element method

The successful application of the finite-element method to pipe-network problems shows that the method is not only superior to conventional Hardy Cross solution techniques but that the further advantages of complete network representation, simplified input data, and unlimited network size can be obtained. Although not specific to a finite-element solution, the program developed allows for solution by either the Hazen-Williams or the Darcy-Weisbach flow-head-loss relationships The program also can consider the effect of temperature variations on head loss throughout the network

The computational algorithm used to arrive at the unique solution for an easily solved linear system equivalent to the true nonlinear system for the pipe networks was maintained in an extremely simple form in this article so that the advantages of the finite-element method could be readily observed Undoubtedly, further application of numerical-analysis techniques would improve the efficiency of the method.

There are two specific reasons for the development of this method First, a computer program, PAWDS,[3,4] which uses the Hardy Cross solution[5] method of balancing flow for pipe-network problems, is used in undergraduate courses at Lehigh Univ in Bethlehem, Pa This Hardy Cross method or various refinements of the loop method (balancing heads in loops) were, in 1973, still used extensively in undergraduate education, engineering practice, and research.[6,7]

The Hardy Cross approach to pipe-network analysis uses as a boundary condition either the fact that the algebraic sum of flows at any joint is zero (balancing flows) or that the algebraic sum of the head loss around any loop is zero (balancing heads) These are simply adaptations of the classical conservation of mass and conservation of energy, respectively Depending upon the criteria used, a correction is applied to the assumed pipe flows or assumed piezometric heads until convergence to a solution is obtained This classical iteration procedure is quite satisfactory for most well-conditioned pipe systems. However, it has been pointed out that convergence to a solution is not necessarily guaranteed.

> There appears to be nothing inherent in either the electric analyzer with ordinary resistors or the Hardy Cross method which will consistently produce convergence of the errors toward zero with subsequent adjustments [8]

The PAWDS program used at Lehigh Univ was plagued by convergence problems typical of the Hardy Cross method. Dillingham and Cleasby[9] point out that when using the balancing-heads method, a pipe or pipes with high resistance to flow compared with others in the network can result in calculated flow corrections larger and in the opposite direction to the currently assumed flow. This will often cause a divergence in the computations, and no solution can be obtained. When the method of balancing flows is used. Dillingham[9] points out that if a large pipe of short length and relatively low flow exists, many iterations are necessary before an appreciable change in piezometric head is obtained if the value of the assumed piezometric head is incorrect. These situations are very practical in their nature. Typical pipe-distribution networks[10] have these exact conditions

An extensive discussion of the convergence problems of the Hardy Cross method and the PAWDS program in particular is not intended in this article, but the existence of these problems should be noted.

The second reason for developing the solution technique was because of the existence of a very efficient finite-element program, GENFEM, developed by Desai[11] The advantage of this program is its completely general nature and hence easy adaptation for the pipe-network problem The mathematical basis and the method of application of the finite-element method is described in detail to allow easy adaption of other existing finite-element programs for use in solving water-distribution-network problems

Application of the Finite-Element Method

Mathematical basis. When the finite-element method is applied to a structural problem, the structure is subdivided into discrete elements Each of these elements must satisfy three conditions:

1. Equilibrium of forces must be maintained.
2. Compatibility must be maintained
3. The force-displacement relationship specified by the geometric and elastic properties of the discrete element must be satisfied.

The force F in the member or element is related to the displacement u and the element properties or stiffness K by Eq (1).

$$F = K u \qquad (1)$$

The sum of the forces in the members at each node of the structure is zero except where an external force is applied. By combining Eq (1) for all the elements in the structure into an equation of identical form to Eq (1) and solving for displacements, the equilibrium of the system is satisfied

An equivalent set of conditions for a pipe network exists, hence, the ability to draw the analogy:

1. The algebraic sum of the flows at any joint or node must be zero.
2. The value of the piezometric head at a joint or node is the same for all pipes connected to that joint.
3. The flow-head-loss relationship (such as Darcy-Weisbach or Hazen-Williams) must be satisfied for each element or pipe

The conditions for a pipe network deal with scalar quantities, whereas the structural conditions deal with vector quantities. The analogy is drawn between the magnitudes of the equivalent quantities as the vector aspects of the flow have no meaning for the network problem.

For a direct application of the finite-element method involving a matrix solution,

a linear relationship is required to define the element or pipe Hence at this point, a relationship of the form of Eq (2) between flow q, head loss h and the hydraulic properties of the pipe c will be assumed.

$$q = ch \quad (2$$

The method of solution to make Eq (2 equivalent to established nonlinear flow-head-loss relationships will be described subsequently.

The head loss h in Eq (2) is the difference between the piezometric head H of the nodes or joints at each end of the element or pipe contributing to the sum of the flows Q at that joint The pipe-system matrix is assembled by writing the equations for the sum of the flows Q at each joint since this value is known to be either zero or equal to the imposed external flow or demand. Alternatively, if the piezometric head is specified at a joint, the sum of the pipe flows is implicitly defined. The resulting set of simultaneous equations can be combined into matrix form defining the entire pipe system in terms of the sum of flows Q at a joint and of the piezometric heads H at the joints. This matrix has the form

$$Q = CH \quad (3)$$

When the matrix is solved, the piezometric heads at all joints are obtained. The difference in piezometric heads between two joints, which is the head loss, can be substituted into Eq (2) to calculate the flow in the pipe between those two joints. The direction of flow is automatically preserved by taking the sign of the difference of the piezometric heads into account.

The finite element representing the pipe is of the simplest form possible. Each element is one dimensional and has one degree of freedom at each node or joint. To apply the summation of the flows at a joint successfully, a convention must be adopted. Flow into a joint is taken as positive, and flow out of a joint is negative

The analysis of a simple pipe network, Fig. 1, is used to show the application of the finite-element method Using the condition that the sum of the pipe flows $(q_a, q_b, \ldots)$ in or out of a joint must satisfy the equilibrium flow criteria $(Q_1, Q_2, \ldots)$ (i.e., the boundary conditions) at that joint, one can write the following equations:

$$Q_1 = q_a + q_d \quad (4)$$
$$Q_2 = q_a + q_b \quad (5)$$
$$Q_3 = q_b + q_c + q_f \quad (6)$$
$$Q_4 = q_c + q_d + q_e \quad (7)$$
$$Q_5 = q_e + q_f \quad (8)$$

The individual pipe flows can be expressed by Eq (2) noting that the head loss h is equal to the difference in the piezometric heads of the joints at each end of the particular pipe.

$$q_a = \pm C_a(H_1 - H_2) \quad (9)$$
$$q_b = \pm C_b(H_2 - H_3) \quad (10)$$
$$q_c = \pm C_c(H_3 - H_4) \quad (11)$$
$$q_d = \pm C_d(H_1 - H_4) \quad (12)$$
$$q_e = \pm C_e(H_4 - H_5) \quad (13)$$
$$q_f = \pm C_f(H_3 - H_5) \quad (14)$$

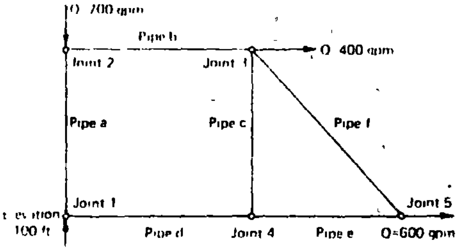

Fig. 1. Example Problem — Analysis of a Simple Pipe Network

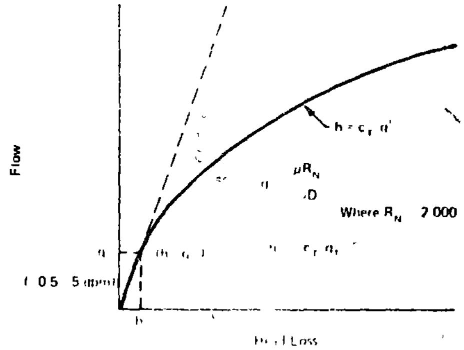

Fig. 2. Typical Flow-Head Loss Relationship

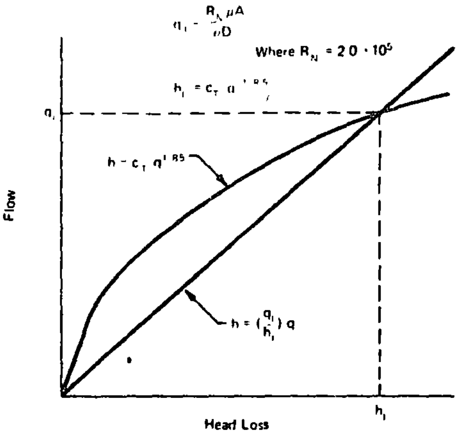

Fig. 3: Initial Value of Pipe Coefficient c

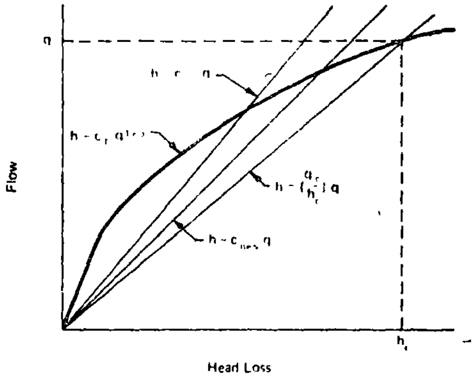

Fig. 4. Correction of Pipe Coefficient c

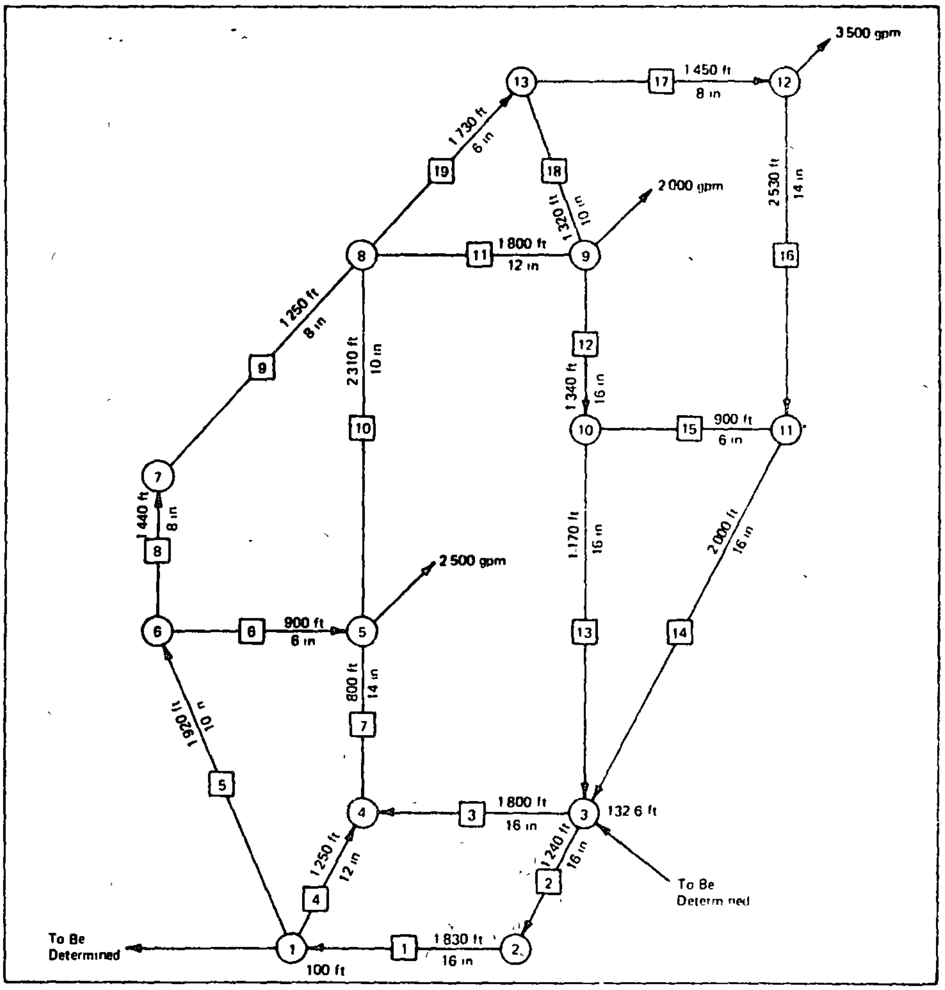

Fig. 5. Comparative Example Using PAWDS and GENFEM

○ —joint numbers; □ —pipe numbers; length in feet, diameter in inches

Equations (4-8) can now be written in terms of the pipe coefficients ($C_a, C_b, \ldots$) and the piezometric heads ($H_1, H_2, \ldots$) Consistency of flow directions is taken into account by assuming the flow is away from the joint being considered, that is, the piezometric head at the other joints is subtracted from the piezometric head at the joint being considered

$$Q_1 = C_a(H_1 - H_2) + C_d(H_1 - H_4) \tag{15}$$

$$Q_2 = C_a(H_2 - H_1) + C_b(H_2 - H_3) \tag{16}$$

$$Q_3 = C_b(H_3 - H_2) + C_c(H_3 - H_4) + C_f(H_3 - H_5) \tag{17}$$

$$Q_4 = C_c(H_4 - H_3) + C_d(H_4 - H_1) + C_e(H_4 - H_5) \tag{18}$$

$$Q_5 = C_c(H_5 - H_4) + C_f(H_5 - H_3) \tag{19}$$

Equations (15-19) can be combined into the matrix form of Eq (3) to yield Eq (20). (See below)

For this particular example, the following boundary conditions are given.

$H_1 = 100$ ft
$Q_2 = 700$ gpm
$Q_3 = 400$ gpm
$Q_4 = 0$ gpm
$Q_5 = 600$ gpm

Substituting these values into Eq (20) gives Eq (21) as the final form for solution (See below)

The values of the coefficients ($C_a, C_b, \ldots$) for each pipe are determined by the procedure to be outlined. The unknowns, H_2, H_3, H_4, H_5 and Q_1, can be obtained by solving the matrix Eq (21). The flows in the individual pipes can be found, as previously stated, by substituting into the defining equations (Eq [9-14] for this example) after the piezometric heads have been found for each joint.

Method of application. For the successful application of the finite-element method, the constitutive equation used to relate flow and head loss must be linear or the matrix solution cannot be applied. In reality, the relationship is nonlinear and varies with the equation chosen. The application of the finite element method is accomplished by using a linear equation (Eq [2]) as the defining flow-head loss relationship and the successive correction of the pipe coefficient c until a unique solution is found satisfying both the equivalent linear relationship and a real nonlinear relationship such as the Hazen-Williams equation or the Darcy-Weisbach equation. When this unique solution has been found for all pipes the network distribution has been solved.

The program GENFEM, allows a choice of the Darcy-Weisbach equation or the Hazen-Williams equation The Hazen-Williams equation will be used for purposes of explanation, although the method is identical for both equations. The relationship used to define flow versus head loss is shown in Fig. 2 The transition from laminar to turbulent conditions for pipe flow occurs at a Reynolds Number R_N of approximately 2 000. R_N is defined by the pipe diameter D, and the dynamic viscosity μ, the density ρ and the flow velocity V of the fluid flowing.

$$R_N = \frac{\rho V D}{\mu} \tag{22}$$

The flow q_T at which transition occurs, corresponding to a R_N of 2 000, is given by

$$q_T = VA = \frac{2\,000\,\mu A}{\rho D} \tag{23}$$

For flows less than q_T, the flow vs head-loss relationship is linear. To avoid a discontinuity in the defining relationship because of the transition region between laminar and turbulent flow, the linear relationship is obtained by simply joining the origin to the point (h_T, q_T) with a straight line The coordinate h_T is found from a substitution of the flow q_T into the turbulent flow equation The linear portion of the graph, the laminar region, does not enter into the calculations of a practical problem The value of q_T ranges from 0.5 to 5 gpm for 6-16-in. diameter pipes whereas typical flows range from 200 to 5000 gpm for these size pipes, well into the turbulent range.

The Hazen-Williams equation relates the head loss h to the pipe diameter D, the pipe length L, the Hazen-Williams coefficient C_{HW}, the flow q and a coefficient c' for unit conversion.

$$h = c' \frac{L}{D^{4.87}} \left(\frac{q}{C_{HW}}\right)^{1.85} \tag{24}$$

This equation can be rewritten for a particular pipe by grouping terms into one constant c_T.

$$h = c_T q^{1.85} \tag{25}$$

The solution technique can be divided into three steps. The first step is to select an initial value of the pipe coefficient c for each pipe and combine these to yield the matrix pipe coefficients C The system matrix is then solved for the value of the piezometric head at each joint. Secondly, the individual pipe flows q are calculated by use of Eq (2) using the differences between the determined piezometric heads. These flows are then substituted into Eq (25) and since (c_T) for each pipe is known, the pipe head losses are calculated If the pipe head losses obtained from Eq (25) correspond to those obtained from the matrix solution, then the unique solution satisfying both the Hazen-Williams equation and the linear Eq (2) has been found The third and final step required is to change the value of c to converge the problem to a solution if there is a difference between the head losses calculated by the two methods

A more detailed explanation of each of these steps follows The initial value of the pipe coefficient c_1 is chosen to correspond to R_N of 200 000 in each pipe, a typical value for a practical problem. The flow (q_1) is then calculated from the Reynolds Number relationship, Eq (26)

$$q_1 = VA \; \frac{200\,000\,\mu A}{\rho D} \tag{26}$$

The value of the head loss h_1 corresponding to this flow q_1 is calculated from Eq (25):

$$h_1 = c_T\, q_1{}^{1.85} \tag{27}$$

The pipe coefficient is then found from Eq (2) as shown in Fig 3

$$c_1 = \frac{q_1}{h_1} \tag{28}$$

This initial value of the pipe coefficient c_1 for each pipe is then combined, according to the geometry of the network into the pipe coefficients C_1 used in the matrix description of the network system The matrix is then solved to yield the first estimate of the piezometric heads at each joint

The allowable deviation between a pipe head loss determined from the matrix solution of the joint piezometric heads and the corresponding value from the Hazen-Williams equation is a variable and can be specified for a particular case taking into account the type of problem and the degree of precision desired for the solution.

The third step, adjusting the value of c, was developed with two criteria in mind The solution should converge reasonably rapidly, yet the technique should remain simple During the checking procedure, the flow q_c for each pipe calculated via Eq (2), and the matrix solution is used to determine the head loss h_c from the Hazen-Williams equation. The first procedure used in the development of the program was to obtain the correction of the c value for each pipe by assuming that the point h_c, q_c was the unique solution and thus the correct linear relationship was defined by a straight line joining this

$$\begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_f & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \tag{20}$$

$$\begin{vmatrix} Q_1 \\ 700 \\ 400 \\ 0 \\ 600 \end{vmatrix} = \begin{bmatrix} C_a + C_d & -C_a & 0 & -C_d & 0 \\ -C_a & C_a + C_b & -C_b & 0 & 0 \\ 0 & -C_b & C_b + C_c + C_f & -C_c & -C_f \\ -C_d & 0 & -C_c & C_c + C_d + C_e & -C_e \\ 0 & 0 & -C_c & -C_e & C_e + C_f \end{bmatrix} \begin{vmatrix} 100 \\ H_2 \\ H_3 \\ H_4 \\ H_5 \end{vmatrix} \tag{21}$$

point to the origin and defined by Eq (29)

$$h = (\frac{q_c}{h_c}) \quad q \qquad (29)$$

The new value of c was then set equal to $\frac{q_c}{h_c}$. When all the pipe coefficients were corrected in a similar way, the flow distribution obviously was altered, and this method proved to be an overcorrection when the matrix was resolved To dampen this overcorrection effect, an averaging technique was introduced. The corrected value of c is taken to be the mean of the c value defined by Eq (29) and the value of c used to obtain the matrix solution This method of correcting c is shown in Fig. 4. The averaging method reduced the number of cycles required for convergence by approximately one third

Example Problems

It is not the intent of this article to present extensive comparison of different network problems since any comparison of computer programs must take into account ease of input data and flexibility of use as well as efficiency of computer time Two example problems are discussed to point out some apparent potential advantages of the finite element approach

An example problem[3] shown in Fig. 5, consisting of nineteen pipes and thirteen joints, was solved using the PAWDS program and the GENFEM program. The PAWDS program solved the system in eighteen iteration cycles and 1.07 s. The GENFEM program solved the network in fifteen iteration cycles and 4 73 s. Obviously, this example does not indicate a preferential method, but is included so that it can be considered with the next example to show the effect of increased system size.

The second example problem with 75 pipes and 57 joints was also solved with both programs The particular problem had been submitted by an undergraduate student and would not converge in the allowed time using the PAWDS program. Both the time limit and the iteration cycle limit were increased, and the problem eventually converged by the use of the PAWDS program after 16 048 iteration cycles and 768 s The same problem was solved with the GENFEM program after twenty iteration cycles and 22.2 s This problem highlights the apparent lack of convergence problems for the finite-element method and also shows, when compared with the first example problem, that for the finite-element method, the number of iteration cycles to convergence appears virtually independent of the number of pipes and joints.

Discussion

The finite-element method is not restricted to a pipe as the only element. Any type of hydraulic element can be included that can be defined by a flow-head loss relationship When the pipe network is relatively small, such as in an industrial plant piping system, the fittings may become major head loss contributors. The head loss h across any of these elements is usually considered to be directly proportional to the velocity head by a coefficient k corresponding to the type of element.

$$h = k \quad \frac{V^2}{2g} \qquad (30)$$

This can be easily converted to the required form, that is, in terms of flow q knowing the area A of the element

$$h = \frac{k}{2g A^2} \quad q^2 \qquad (31)$$

A pump can be included in the system since a pump merely provides a "head gain" or negative head loss. The use of a pump element requires a flow-head loss relationship (the head-capacity curve for the pump). If the information is not available as an equation relating discharge and head, the pump information could be provided in tabular form In this form the program would use linear interpolation between any two data points.

The basic finite-element method is not restricted to pipes flowing full. In practice, water systems often contain open channels in the headwater sections. These open channels or even pipes flowing partially full can be included for analysis The only requirement for an element is that the flow can be related to the head loss.

The range of hydraulic elements that can be included is limitless, provided a flow-head loss relationship for each element is known. An exact system representation can be obtained by introducing a combination of all the hydraulic elements causing head loss or gain in the system. While discussing this point one should remember that methods such as the Hardy Cross method of balancing heads technique[5,7] require that all pipes are part of a loop. Typically, reservoirs or elevated tanks have one pipe connecting to the distribution system and an artificial pipe must be introduced to make a loop so that the Hardy Cross balancing heads solution method can be applied.

The use of high-resistance artificial pipes in the network has, in fact, often contributed to convergence difficulties. To simulate the actual conditions, the artificial pipes introduced to form loops are often of small diameter and high resistance so that they carry an insignificant flow and hence can be neglected. As was pointed out earlier, this is the exact condition that creates convergence problems. The finite-element method does not require the use of artificial pipes to complete a loop since the connectivity of the elements is defined explicitly by the system matrix Consequently, tree-type systems are readily solved with this procedure. In fact, the GENFEM program was used very successfully during the 1974 spring semester at Lehigh Univ to analyze a transmission system problem that was almost entirely tree type with approximately ten loops included

The input data required for the program is equivalent to other solution techniques with the major exception that loop data does not need to be included. The distribution network is defined by input of the number of the pipe and the joints to which it is connected Another advantage over some solution methods is that any number of points of known pressure can be preselected

With loop-solution methods, all pipe and joint information must be available to the program at the same time This puts a definite limit on the size of the system that can be solved The finite-element program, GENFEM, however, can operate on blocks of data. Thus, there is virtually no limit to the size of the network that can be solved The element and nodal information can be stored on magnetic tapes or other devices and then read from the storage device in blocks, operated upon, and returned to storage on the tapes This feature must gain greater significance as water-distribution networks become larger and more interdependent

As stated previously, although not being specific to a finite-element program, the program developed has two additional features worth noting. First, the program GENFEM provides a choice of two established flow-head loss relationships. The Hazen-Williams equation has already been stated as Eq (24) The Darcy-Weisbach equation relates the same variables and includes the friction factor f.

$$h = (f \frac{L}{D\,2g\,A^2}) \quad q^2 \qquad (32)$$

An explicit expression[12] for the friction factor f is used rather than the classical implicit Colebrook and White equation[12] that requires an iterative solution. The friction factor f is expressed in terms of the Reynolds Number R_N and the relative roughness κ where κ is the ratio of the absolute roughness e to the pipe diameter D.

$$f = 0\,094\kappa^{0\,255} + 0\,53\kappa + 88\kappa^{0\,44}\, R_N{}^{-1\,62\kappa^{0\,134}} \qquad (33)$$

Second, allowance for changes in temperature has been included since temperature appreciably affects the viscosity and to a negligible degree, the density of water. The viscosity of water over a temperature range of 5C-30C varies from 0.0152 poise to 0 8004 centipoise. An algorithm[13] in terms of temperature T in degrees Celsius is used to define the viscosity μ in poise.

$$\frac{1}{\mu} = 2\,1482\,([T - 8\,435] + \sqrt{8078\,4 + [T - 8\,435^2]} - 120) \qquad (34)$$

The program is written so that the temperature can be specified for each pipe. Any set of temperature conditions can be investigated for a particular circumstance.

Practical adaptation of the finite-element method should require a minimum of computer programming since most engineering firms and universities have finite-element programs readily available. The ease of modifying these programs depends upon the generality of their nature.

Summary

The many advantages of the finite-element method have been documented. Most of these advantages hold true in a comparison of any loop method to the finite-element method The major advantage is the speed of convergence and the apparent lack of convergence problems of the proposed method over the Hardy Cross balancing flows method. Other important advantages are the ability to include in the analysis all types of hydraulic elements, the choice of flow-head loss relationships, the lack of artificial loops, the ease of adaption of existing finite-element programs, the unlimited network size, and finally, the ability to account for temperature effects.

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Marc Analysis Research Corporation

MARC APPLICATION SUMMARY

THERMAL AND ELASTIC ANALYSIS OF A PISTON

A piston was analyzed by MARC Analysis Research Corporation under combined thermal and pressure loading that simulated normal operating conditions. The idealized piston mesh is shown in a perspective plot in Figure 1.

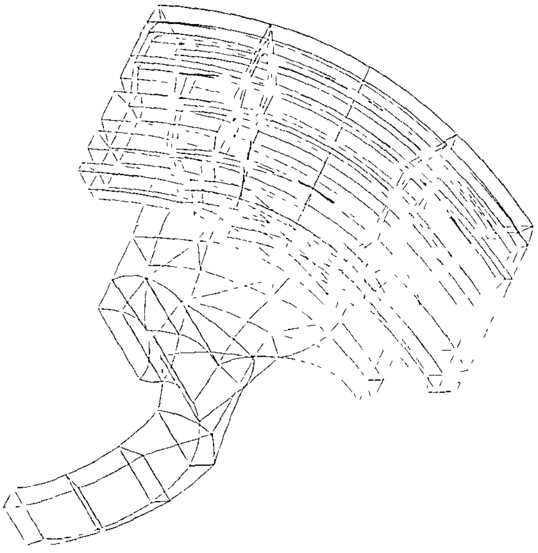

Figure 1

A linear elastic analysis indicated that the most highly stressed areas were at the wrist pin-pin bore interface and at the oil cooling channel surface, just inside the ring land area at the top of the piston.

The MARC system was used to generate the model mesh, the thermal data and the stress analysis results. One hundred and twenty-eight isoparametric twenty node brick elements were used to model the piston and the piston pin. Special modeling considerations included use of an elastic foundation stiffness in place of the crank rod and tying constraints for the interaction of the pin and the piston. The final model resulted in 1002 node points with a total of 2673 reduced degrees of freedom. The maximum nodal half-bandwidth of the optimized mesh was 175. Figure 2 is an isotherm plot of the upper piston surface.

The thermal data for this analysis was generated using the MARC system transient heat transfer capability. Figure 3, a plot of the Mises equivalent stress in the piston top, demonstrates the MARC graphic capabilities to distill and present results in the most straight-forward manner.

MARC ANALYSIS RESEARCH CORPORATION

MARC Analysis Research Corporation has offices in Providence, Rhode Island, and in Palo Alto, California. Dr. Pedro V. Marcal is President, and he is located in the Palo Alto office. The company is oriented toward providing problem-solving services to the engineering community through lease or through the data-center offering of the MARC Program, as well as through complete problem solution via our consulting groups in Palo Alto and Providence and through the MARC-sponsored finite-element-technology and MARC-usage courses. The staff is equally divided between the Palo Alto and Providence offices, and hence will give short turn-around on problems that may arise. In addition, Mr. Patrick Stuart, manager of MARC European Operations, is in Stuttgart, West Germany (address on back side) in order to better serve our European customers. A brochure describing the MARC Analysis Research Corporation is available on request.