centro de educación continua
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Y

MARC ANALYSIS RESEARCH CORPORATION

Curso-Seminario Intensivo

"TEMAS AVANZADOS DE ANALISIS POR ELEMENTOS FINITOS"

Marzo 22-26, 1976

Palacio de Minería
Tacuba 5, primer piso. México 1, D. F.
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P R O G R A M A

LUNES 22

8:00 - 8:45	Inscripciones	
8:45- 9:00	Apertura del curso	Octavio Rascón Chávez Timothy J. Dwyer
9:00 - 10:30	Estudio y categorización de los métodos computacionales de análisis de ingeniería.	O. C. Zienkiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Formulaciones alternativas en mecánica estructural.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Formulaciones mixtas o híbridas del método de elementos finitos.	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Métodos de incremento de tiempo	O. C. Zienkiewicz

MARTES 23

9:00 - 10:30	Flujo viscoso.	O. C. Zienckiewicz
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Problemas de ingeniería ambiental.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Ecuaciones constitutivas inelásticas	R. H. Gallagher
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Alogaritmos de análisis por el método de elementos finitos en medios inelásticos.	R. H. Gallagher

MIERCOLES 24

9:00 - 10:30	Análisis de mecánica de propagación de grietas.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis de cascarón por el método de elementos finitos.	R. H. Gallagher
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Visco-plasticidad.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Mecánica de suelos y rocas.	O. C. Zienkiewicz

JUEVES 25

9:00 - 10:30	Análisis por medio de elementos finitos en problemas de pandeo con desplazamientos grandes.	P. V. Marcal
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Análisis combinado de no-linealidad y comportamiento dinámico.	P. V. Marcal
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Revisión y crítica del programa MARC	T. J. Dwyer
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 18:00	Caso aplicación: análisis de los componentes de reactor nuclear.	P. V. Marcal

VIERNES 26

9:00 - 10:30	Ecuaciones constitutivas del concreto.	P. Ballesteros
10:30 - 11:00	Receso (café y refrescos)	
11:00 - 12:30	Aplicaciones de elementos finitos en problemas de concreto.	P. V. Marcal
12:30 - 14:30	Receso (comida por cuenta participantes)	
14:30 - 16:00	Procedimientos de solución de valores en la frontera por el método de elementos finitos.	O. C. Zienkiewicz
16:00 - 16:30	Receso (café y refrescos)	
16:30 - 17:00	El método de elementos finitos en el análisis de presas.	O. C. Zienkiewicz
17:00 - 18:00	Discusión final y preguntas.	O. C. Zienkiewicz P. V. Marcal T. J. Dwyer P. Ballesteros
18:00	C l a u s u r a	Octavio Rascón Chávez Pedro Martínez Pereda

ADVANCED TOPICS SEMINAR
MEXICO CITY
MARCH 22-26, 1976.

MONDAY, MARCH 22, 1976.

1) An Overview and Categorization of Computational Methods in Engineering Analysis	Zienkiewicz
2) Alternative Formulations is Structural Mechanics	Gallagher
3) Mixed and Hybrid F.E.M. Formulations	Gallagher
4) Time-Stopping Methods	Zienkiewicz

TUESDAY, MARCH 23, 1976.

5) Viscous Flows	Zienkiewicz
6) Environmental Problems	Gallagher or Zienkiewicz
7) Constitutive Equations for Inelasticity	Gallagher
8) F.E. Analysis Algorithms for Inelastic Analysis	Gallagher

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9) Shell Analysis by F.E.M.	Gallagher or Ballesteros
10) Fracture Mechanics Analysis	Gallagher
11) Viscoplasticity	Zienkiewicz
12) Soil and Rock Mechanics	Zienkiewicz

THURSDAY, MARCH 25, 1976.

13) F.E.M. Analysis for Buckling and Large Displacement	Marcal
14) Analysis for Combined Nonlinear and Dynamic Behavior	Marcal
15) MARC Review and Critique	Dwyer
16) Case Study: Nuclear Reactor Component Analysis	Marcal

FRIDAY, MARCH 26, 1976.

17) Constitutive equations of Concrete and Reinforced Concrete.	Ballesteros
18) Boundary Solution Procedures and the F.E.M.	Zienkiewicz
19) F.E.M. in Dam Analysis	Zienkiewicz
20) Final discussion and questions.	Zienkiewicz Gallagher Marcal Dwyer Ballesteros.

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"ADVANCED TOPICS IN FINITE ELEMENT ANALYSIS."

"TEMAS AVANZADOS DE ANALISIS POR ELEMENTOS FINITOS."

MARCH 22-26, 1976.
MARZO 22-26, 1976.

Director de la Facultad.	M. en Ing. Enrique Del Valle Calderón
Jefe de la División.	Dr. Octavio Rascón Chávez
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MONDAY, MARCH 22, 1976.

AN OVERVIEW AND CATEGORIZATION OF COMPUTATIONAL METHODS IN ENGINEERING ANALYSIS.

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O. C. Zienkiewicz

AN OVERVIEW AND CATEGORIZATION OF MODERN COMPUTATIONAL METHODS IN ENGINEERING

I. INTRODUCTION

We shall try, in this chapter, to

(a) define the finite element method in its general form, and

(b) show its different facets which to a greater or lesser extent have been used to date.

We shall see how within a broad definition, the finite difference techniques fall into a 'sub-class' of the general finite element methodology which indeed embraces many other classical approximation procedures. This generalization of the finite element concept is by no means a 'power bid' by its over-enthusiastic adherents. On the contrary, it serves, we believe, to lay a firm foundation to a wide variety of solution methods and provide expanded possibilities of application.

Before embarking on the main theme of this introduction, the reader may well ask the further question, "Does the finite element method in fact provide tools which are in all respects superior to those of the more traditional methods?" To this question the answer must be left to his intuition. However, the observation of the field of structural and solid mechanics in which finite element procedures have today 'taken over' other alternatives may provide a clue. The research developments of the last fifteen years (since the first mention of the "finite element" name was made) have become everyday practice in many stress analysis/structures situations. It is, by analogy, probable that in such fields as fluid mechanics, neutron flux analysis and many more similar revolutions are now feasible.

II. THE FINITE ELEMENT CONCEPT

The finite element method is concerned with the solution of mathematical or physical problems which are usually defined in a continuous domain either by local differential equations or by equivalent global statements. To render the problem amenable to numerical treatment, the infinite degrees of freedom of the system are discretised or replaced by a finite number of unknown parameters as indeed is the practice in other processes of approximation.

The original 'finite element' concept replaces the continuum by a number of subdomains (or elements) whose behavior is modeled adequately by a limited number of degrees of freedom and which are assembled by processes well known in the analysis of discrete systems. Often at this early stage the model of the element behavior was derived by a simple physical reasoning avoiding the mathematical statement of the problem. While one can well argue that such a approach is just or realistic as formal differential statements (which imply the possibility of an infinite subdivision of matter), we prefer to give here a more general definition embracing a wider scope.*

Thus, we define the finite element process as any approximation process in which

(a) the behavior of the whole system is approximated by a finite number, n, of parameters, a_i, i=1,....n for which

(b) the n-equations governing the behavior of the whole system, i.e.,

$$F_j(a_i) = 0 \qquad j = 1,....n \tag{1}$$

can be assembled by the simple process of addition of terms contributed from all subdomains (or elements) which divide the system into physically identifiable entities (without overlap or exclusion). Thus,

$$F_j = \sum F_j^e \tag{2}$$

where F_j^e is the element contribution to the quantity under consideration.

This broad definition allows us to include in the process both physical and mathematical approximations and if the 'elements' of the system are simple and repeatable, to derive prescriptions for calculation of their contributions to the system equations which are generally valid. Further, as

*Although in some situations, such as for instance the behavior of granular media the first approach is still one of the most promising as continuously defined constitutive relations have not yet been adequately developed.

the process is precisely analogous to that used in discrete system assembly, computer programs and experience accumulated in dealing with discrete systems can be immediately transferred.

An important practical point of the approximation has been specifically excluded here. This concerns the fact that often contributions of the elements are highly localized and only a few non-zero terms are contributed by each element. In practice this localization results in sparse and often banded equation systems reducing computer storage requirements. While most desirable in practice this feature is not essential to the definition of the finite element process.

What are then the procedures by which a finite element approximation can be made? We have already mentioned - but exclude now from further discussion here the direct physical approach and will concentrate on any problem which can be defined mathematically either by a (set) of differential equations valid in a domain Ω

$$\underset{\sim}{D}\ (\underset{\sim}{\phi}) = 0 \tag{3}$$

together with their associated boundary conditions on boundaries of the domain Γ

$$\underset{\sim}{B}\ (\underset{\sim}{\phi}) = 0 \tag{4}$$

or by a variation principle requiring stationarity (max, min or 'saddle') of some scalar functional Π

$$\Pi = \int_{\Omega} C\ (\phi)\ d\Omega + \int_{\Omega} C\ (\phi)\ d\Gamma \tag{5}$$

In both statements $\underset{\sim}{\phi}$ represents either the single unknown function or a set of unknown functions.

To clarify ideas consider a particular problem presented by seepage flow in a porous medium where ϕ is the hydraulic head (a scalar quantity).

The specific governing equation is now in a two-dimensional domain

$$D(\phi) = \frac{\partial}{\partial x}\left(K \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial \phi}{\partial y}\right) + Q = 0 \tag{6}$$

together with boundary conditions

$$B(\phi) = \phi - \bar{\phi} \qquad \text{on } \Gamma_\phi \tag{7}$$

$$B(\phi) + K\frac{\partial \phi}{\partial n} - q \qquad \text{on } \Gamma_q$$

in which K the permeability, and Q the flow input, may be functions of position (and in non-linear problems - of the gradients or values of ϕ).

An alternative formulation (for linear problems) requires stationarity (a minimum) of a functional

$$\Pi = \int_\Omega \{\frac{1}{2} K \left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2} K \left(\frac{\partial \phi}{\partial y}\right)^2 - Q \phi\} d\Omega - \int_{\Gamma_q} q\phi \, d\Gamma \tag{8}$$

for ϕ which satisfies only the first boundary condition.

In general if a functional Π exists, then an associated set of (Euler) differential equations can always be found but the reverse is not necessarily true.

To obtain a finite element approximation to the general problem defined by Equation 3 to 5 we proceed as follows:

(a) the unknown function is expanded in a finite set of assumed, known trial functions $\underset{\sim}{N}_i$ and unknown parameters $\underset{\sim}{a}_i$

i.e.

$$\hat{\underset{\sim}{\phi}} = \underset{\sim}{N}_i \underset{\sim}{a}_i = \underset{\sim}{N}\,\underset{\sim}{a} \tag{9}$$

and

(b) the approximation must be cast in a form of n equations which are defined as integrals over Ω and Γ i.e.

$$\underset{\sim}{F}_j = \int_\Omega \underset{\sim}{E}\,(\hat{\underset{\sim}{\phi}})\, d\Omega + \int_\Gamma \underset{\sim}{e}\,(\hat{\underset{\sim}{\phi}})\, d\Gamma$$

$$j = 1, \ldots . n \tag{10}$$

Immediately, we note that the basic definitions of the finite element process previously given apply as for integral functions

$$\int_{\Omega} ( ) \, d\Omega = \sum \int_{\Omega^e} ( ) \, d\Omega \tag{11}$$

and

$$\int_{\Gamma} ( ) \, d\Gamma = \sum \int_{\Gamma^e} ( ) \, d\Gamma \tag{12}$$

in which Ω^e, Γ^e represent "element" subdomains.

The problem of how the integrals of approximations are formed is thus the first, crutial, question of casting a problem in a finite element form.

III. APPROXIMATION INTEGRALS

1. Variational Principles[1,2]

If the problem is stated in terms of a stationary functional Π then the formulation is most direct. We can write the approximate form of the functional as

$$\Pi = \hat{\Pi} = \Pi \, (\hat{\underset{\sim}{\phi}}) \tag{13}$$

and for stationarity we have a set of equations

$$\underset{\sim}{F}_j = \frac{\partial \Pi}{\partial \underset{\sim}{a}_j} = 0 \tag{14}$$

which by definition of Π is already cast in an integral form. This basis of forming a finite element approximation has been and remains most popular, providing a physically meaningful variational principle exists and can be readily identified. This has led to statements of the kind that the finite element method is a 'variational process' which is, however a too limited definition, as other alternatives, often more powerful, are present. The important question of how to proceed from the differential euqation directly

in cases where a variational principle does not exist or cannot be identified remains. The answer to it lies in the reformulation by use of weighting function, or by the introduction of 'pseudo-varational' principles.

2. Weighted Integral Statements

It is obviously possible to replace the governing equations (6) or (7) by an integral statement in all respects equivalent, i.e.,

$$\int_{\Omega} \underset{\sim}{W}^{T} \underset{\sim}{D}\,(\underset{\sim}{\phi})\, d\Omega + \int_{\Gamma} \underset{\sim}{\bar{W}}^{-T} \underset{\sim}{B}\,(\underset{\sim}{\phi})\, d\Gamma \qquad (15)$$

in which $\underset{\sim}{\bar{W}}$ and $\underset{\sim}{\bar{W}}$ are completely arbitrary, 'weighting' functions. Immediately, an approximation is possible in an integral form by choosing specific functions $\underset{\sim}{W}_j$ and $\bar{\underset{\sim}{W}}_j$ and writing[3][4]

$$F_j = \int_{\Omega} \underset{\sim}{W}_j \underset{\sim}{D}\,(\underset{\sim}{\hat{\phi}})\, d\Omega + \int_{\Gamma} \bar{\underset{\sim}{W}} \underset{\sim}{B}\,(\underset{\sim}{\hat{\phi}})\, d\Gamma \qquad (16)$$

The process is known as the weighted residual method if $\underset{\sim}{D}\,(\underset{\sim}{\hat{\phi}})$ and $\underset{\sim}{B}\,(\underset{\sim}{\hat{\phi}})$ are recognized as residuals by which the approximation mises the zero value required. Classical procedures of Galerkin's method, collocation, etc. are immediately recognized.

Either form of deriving integral statements and hence the set of approximating equations can be and has been used in practice. The variational principle possesses however a unique advantage. If the function is quadratic in $\underset{\sim}{a}$ the set of approximationg equations (14) can be written as

$$\underset{\sim}{K}\,\underset{\sim}{a} + \underset{\sim}{P} = 0 \qquad (17)$$

in which $\underset{\sim}{K}$ is always a symmetric matrix ($\underset{\sim}{K}_{ij}^{T} = \underset{\sim}{K}_{ji}$). For linear differential equations the weighting processes will also result in a similar set of equations via (16) - however, these will not in general be symmetric. The user of finite difference procedures may well be acquainted with such dissimetries which present often computational difficulties. This symmetry can indeed be shown to be a precondition for the existence of a variational principle - and it will indeed be found that the Galerkin's method of weighting will yield identical equations as those derived from a variational principle whenever this exists.

Because of this (and certain other) advantages of variational formulations much work of theoretical nature has been put in to establish equivalent functionals for problems defined by differential equations or to create pseudo-variational functionals[5]-[8].

3. Psudo-Variational Principles - Constraints by Lagrange Multipliers or Penalty Functions. Adjoint Variables and Least Square Processes

Such pseudo-variational principles can be established by various means. These include constrained variational principles and the extreme cases obtained from these by the use of adjoint functions, or the application of least square parameters.

Constrained variational principles require the stationarity of some function Π subject to constraints say of the type given by some differential relations.

$$\underset{\sim}{C}\,(\underset{\sim}{\phi}) = 0 \quad \text{in} \quad \Omega \tag{18}$$

In such cases we can proceed to establish a new variational principle in either of two ways. In the first we introduce an additional set of functions $\underset{\sim}{\lambda}$ known as Lagrangian multipliers and require the stationarity of

$$\overline{\Pi} = \overline{\Pi}\left(\begin{Bmatrix}\underset{\sim}{\phi}\\ \underset{\sim}{\lambda}\end{Bmatrix}\right) = \Pi + \int_{\Omega} \underset{\sim}{\lambda}^{T}\ \underset{\sim}{C}\ d\Omega \tag{19}$$

The variation of this functional results in

$$\delta\overline{\Pi} = \delta M + \int_{\Omega} \delta\lambda^{T}\ C\ d\Omega + \int_{\Omega} \lambda^{T}\ \delta\ C\ d\Omega \tag{20}$$

which can only be true if both the stationarity of Π and the constraints (18) are satisfied.

The use of Lagrangian multipliers in practice is somewhat limited due to two drawbacks. First the additional functions λ have to be discretized thus requiring a larger number of unknowns in the final problem. Second it will always be found that, if Π is quadratic and C a linear ing to the parameters discretizing λ (this is obvious from the inspection of equation 20).

To obviate some of the difficulties associated with the use of Lagrangian multipliers it is possible to require the stationarity a modified functional based on a penalty function. For, it at the solution we require a simultaneous satisfaction of the stationarity of Π and the satisfaction of constraints we can minimize approximately

$$\bar{\Pi} = \Pi + \alpha \int_{\Omega} \mathbf{C}^T \mathbf{C} \, d\Omega \tag{21}$$

in which α is some large (positive) number 'penalizing' the error of not satisfying the constraints. As no procedure is without a drawback, we note here a purely numerical difficulty: as α becomes large the discretized equations tend to become ill-conditioned. With modern computers and high precision arithmetic penality function operations are becoming increasingly popular and their use more widespread[9], [10].

What if even a constrained variational principle does not appear to be identified? Clearly both methods given above are still applicable by putting $\Pi = 0$. Thus, we can make stationary either

$$\bar{\Pi} = \bar{\Pi}\left(\begin{matrix}\phi \\ \lambda\end{matrix}\right) = \int_{\Omega} \boldsymbol{\lambda}^T \mathbf{C} \, d\Omega \tag{22}$$

or alternatively minimize

$$\bar{\Pi} = \int_{\Omega} \mathbf{C}^T \mathbf{C} \, d\Omega \tag{23}$$

The first is equivalent to the use of adjoint functions while the latter is the straight-forward application of the least square procecures of approximation[3], [4].

The pseudo variational principle established by equation (22) in which a new, adjoint, function λ is introduced is of little practical use The resulting discretized equation systems for parameters defining ϕ and λ are entirely decoupled and indeed there is little virtue in the symmetry arising from the whole system as zero diagonal exists throughout. Nevertheless, this approach gives another interpretation of the Galerkin weighting process if similar expansions are used for ϕ and λ. The least square formulation on the other hand results in well conditioned equation systems and deserves much wider attention in the finite element literature than it has so far received[11][12].

4. Direct Integral Statements - Virtual Work

In many physical situations it is possible to formulate the problem directly in an integral form avoiding the necessity of writing down differential equations. In particular the principle of virtual work in mechanics can be so stated with greater generality than that arising from differential equations. Indeed in such cases the weighted residual form given by equations (15) arises in a form which can be obtained from this equation by the use of integration by parts. Such integration reduces the continuity requirement imposed on both functions W and N by "integrability" (to which we shall refer in next section). This due to the relaxation of requirements is known mathematically as a "weak formulation" of the problem. It is of philosophical interest to interject here a thought that perhaps such weak formulations are indeed the requirements of nature opposed to differential equations which, at certain physical discontinuities, are meaningless.

In structural mechanics virtual work principles have almost displaced the formulation based either directly on energy statements (due to the wider applicability of virtual work) or on differential equations (due to the avoidance of other complex algebraic manupulating). In the chapter of these notes dealing with "Viscous Flow" the author shows how such statements form an extremely realistic and simple approach to fluids.

In Table I we summarize the basic processes by which the integral forms of approximation can be made as a preliminary to finite element analysis.

IV. PARTIAL DISCRETIZATION

At this point it is appropriate to mention that it is often convenient to discretize the problem only partially in a manner which, say, reduces a set of differential equations in three independent variables not directly to a numerical set of equations but to a lower order differential equation, say, with only one variable. This first differential equation can then, on occasion, be solved more efficiently by exact procedures or alternative numerical solutions.

Such 'partial discretization' is particularly useful if the 'shape' of the domain in one of the independent directions is simple. This may arise if prismatic or axisymmetric shapes are considered in a three dimensional problem or if one dimension is that of time.

Considering the last case as a concrete example, the trial function expression discretizing the unknown $\underset{\sim}{\phi}$

$$\underset{\sim}{\phi} = \underset{\sim}{\phi}\,(x, y, z, t) \tag{23}$$

is made by modifying the equation (9) to

$$\underset{\sim}{\phi} = \Sigma\, \underset{\sim}{N}_i\, \underset{\sim}{a}_i = \underset{\sim}{N}\, \underset{\sim}{a} \tag{24}$$

in which

$$N_i = N_i\,(x, y, z) \tag{25}$$

i.e., is only a function of position and $\underset{\sim}{a}$ is now a set of parameters which are a function of time

$$\underset{\sim}{a} = \underset{\sim}{a}\,(t) \tag{26}$$

"Partial variations" of variation principles equation (13) or any of the weighting procedures, equation (16) in which the weighting functions do not include the independent variable t can now be used reducing the formulation to a set of ordinary differential equations.

In fluid mechanics and flow problems we shall often find such a discretization useful and the ordinary set of differential equations can often be solved efficiently by simple finite difference schemes as well as by a secondary application of the finite element methodology[13][14].

V. TRIAL FUNCTIONS

1. General Principles

So far beyond mentioning that the unknown function ϕ is expanded as in equation (9) by a set of trial functions $\tilde{N}$ no specific mention was made of the form these trial functions should take or what limitations have to be imposed on them. We shall here consider, in very general terms, some of the guidelines though by necessity the discussion cannot do justice to the problem which is crucial to the success of the finite element process. For details therefore the reader is referred to references 13 and 14 and to numerous other publications in which different trial (or shape) functions are discussed.

As the trial functions $\tilde{N}$ are constructed for practical reasons in a piecewise manner i.e., using a different definition within each "element" the question of required interelement continuity is crucial in their choice. This continuity has to be such that either the integrals of the approximation given in general by equation (10) or in particular forms by equations (14) and (16) can be evaluated directly, without any contribution arising at the element "interfaces." Alternatively, such interelement contributions must be of a kind which decreases continuously with the fineness of element subdivisions. The class of functions satisfying the first conditions shall be called conforming while the ones which satisfy only the second one are named non-conforming (admissible).

In general it is quite easy to specify the 'conformity' conditions. If the integrand contains m-th derivatives of the unknown functions ϕ then the shape functions $\tilde{N}$ have to be such that the function itself and its derivatives up to the orer m-1 are required to be continuous (C^{m-1} continuity).

In practice it is difficult to define conforming functions in a piecewise manner for any order of m greater than one and because of this many 'non-conforming' elements have originated in the past with the hope, sometimes proved a posteriori, that admissibility is achieved. The question of establishing admissibility is a difficult one and much work on this area is highly mathematical and not easy to interpret[15][16]. Some tests of admissibility have however been devised and it is essential to subject any new non-conforming element to such an examination[17][18].

In Table II we show these two main directions on which finite element approximation is based. There is however an intermediate position where interelement contribution can be evaluated without the imposition of full conformity. This arises either, if the derivatives of $\tilde{N}$ occur in a linear form in the integrals and continuity can be relaxed by one further order[18] or where in the basic formulation interface contributions are specifically inserted. The latter is the position with certain hybrid formulations[19][20] or Lagrange multiplier forms which specifically impose conformity as a constraint[21][22]. In this simple expose we shall not be further concerned with these special situations. They are taken up in more detail in two chapters by Professor Gallagher. Herein we will subsequently treat the <u>conforming</u> formulation as <u>standard</u> and the <u>non-conforming</u> one as a <u>special</u> <u>variant</u> of it.

A further condition which has to be imposed on shape function is that of "completeness," i.e., the requirement that in the limit, as the element size decreases indefinitely, the combination of trial functions should exactly reproduce the exact solution of this if available. This condition is simple to satisfy[18] if polynomial expressions are used such that the complete m-th order of polynomial is present, when m-th order derivatives exist in the integral of approximation.

To demonstrate a few simple shape functions in one and two dimensional problems we show in Figure 1 and Figure 2, some piecewise defined shapes in one and two dimensional domains in which to ensure C^0 conformity the system parameters a_i take on simply the value of the unknown function of certain points (often referred to as nodes) and which are common to more than one 'element.' With this device a simple repeatable formula can be assigned ot define N_i within any element.

It will be immediately recognized that as the parameter a_i influences the value of ϕ only in elements adjacent to a 'node' i its contribution to the integrals will be limited to elements containing that node, hence the 'banded' feature of approximating equations already referred to.

It is of interest to note that such piecewise defined functions which to many are the essence of the finite element method have been used for the first time as late as 1943 by Courant[23] despite the fact that integral approximation procedures are much older.

Today many complex forms of shape function definition exist, mostly developed in the last decade, which are capable of being piecewise defined and giving high orders of approximation. Some such functions are in fact defined not in the simple original coordinate system in which the problem is given but with a suitable transformation refer to curvilinear coordinates by means of which most complex shapes of regions can be subdivided. Figure 3 shows some such elements of an "isoparametric" kind[24] much used in practice. In all C^o continuity only is imposed and relatively simple formulation suffices.

2. Particular Example

To illustrate the process of discretization which by this time appear somewhat abstract to the reader we shall return to the specific example given by equation (6) and its associated boundary conditions (7).

Assuming that the potential ϕ (here a scalar quantity) can be approximated as

$$\hat{\phi} = \Sigma N_i a_i \tag{27}$$

in which both N_i and a_i are scalars and a_i in fact is identified with nodal values of ϕ we shall first use the variational principle of equation (8).

Substituting (27) into equation 8 and differentiating with respect to a parameter a_j gives

$$F_j = \frac{\partial \Pi}{\partial a_j} = \frac{\partial}{\partial a_j}\left[\iint_\Omega \left\{\frac{1}{2} K \left(\frac{\partial}{\partial x}(\Sigma N_i a_i)\right)^2 + \frac{1}{2} K \left(\frac{\partial}{\partial y} \Sigma N_i a_i\right)^2 - Q \Sigma N_i a_i\right\} dxdy \right.$$

$$\left. - \int_\Gamma q \, \Sigma N_i a_i \, d\Gamma\right]$$

$$= \iint_\Omega \left[K \frac{\partial N_j}{\partial x} \Sigma \frac{\partial N_i}{\partial x} a_i + K \frac{\partial N_j}{\partial y} \Sigma \frac{\partial N_i}{\partial y} a_i - Q N_j\right] dxdy$$

$$- \int_{\Gamma_q} q \, N_j \, d \qquad (28)$$

or for the whole equation system

$$\frac{\partial \Pi}{\partial a} = \underset{\sim}{K} \, \underset{\sim}{a} + \underset{\sim}{P} = 0 \qquad (29)$$

with

$$K_{ij} = \iint_\Omega K \left(\frac{\partial N_i}{\partial x} \cdot \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial N_j}{\partial y}\right) dxdy$$

$$P_j = - \iint_\Omega Q \, N_j \, dxdy - \int_{\Gamma_q} q \, N_j d \qquad (30)$$

with trial function assumed piecewise element by element it is simple to evaluate the integrals for each element obtaining their contributions K_{ij}^e and P_j^e and obtain the final equation by simple summation over all elements

$$K_{ij} = \Sigma K_{ij}^e$$

$$P_j = \Sigma P_j^e \qquad (31)$$

Alternative forms of approximation can be derived by the reader using some weighting procedures described. It can be shown that in this linear case (i.e. in which K and Q are functions of position only) identical approximation will be available by application of Galerkin weighting but that that other approximations will arise from use of alternative procedures. He will find that for instance application or least square processes (equation 22) will result in second derivatives being present and will need C^1 prime continuity trial functions with subsequent difficulties of determining such functions. He will however observe that the Galerkin process is available for non-linear problems where the simple form of the variational principle is no longer applicable.

At this stage it is of interest to insert a particular shape function and obtain in detail a typical equation for the parameter j.

Let us consider a typical internal node j = 0 on a regular mesh of triangular elements as shown in Figure 4 in which a linear interpolation is used. Assuming K and Q as constants and that the contributions of the boundary do not occur in the vicinity, the contributions of all elements are found and coefficients K_{01}^e, K_{02}^e etc. evaluated. After assembly the typical equation becomes

$$K\,(\phi_1 + \phi_3 + \phi_5 + \phi_7 - 4\,\phi_0) + Qh^2 = 0 \qquad (32)$$

The reader will recognize this as the standard finite difference equation obtained by direct, point, differencing of equation 5 and may well enquire what advantage has been here gained. Obviously, the numerical answers in this case are going to remain the same, at least if boundary of type Γ_q does not occur. Immediately, however it is important to point out that if K varied discontinuously between elements (such as may be the case at interfaces between two regions of different impermeability) the finite element method would have yielded in one operation answers which direct finite difference procedures would tackle only by the introduction of additional constraints and interface conditions. Further the variational form allows the gradient boundary condition to be incorporated directly.

Pursuing the problem further we try a rectangular element with a bilinear interpolation of ϕ as in Figure 1b. The resulting finite element equation gives

$$K\,[\phi_1 + \ldots + \phi_8] - 8\phi_0 + 6\,Qh^2 = 0 \qquad (33)$$

a form substantially different from the standard finite difference equation which although convergent to the same order of approximation reduces the truncation error. Again the same comments can be made as before regarding the advantages of the finite element approximation.

In recent years much progress has been made in the finite difference methodology, and in particular integral forms including variational principles have been used as the basis of approximation in which the differences are only applicable to the differentials occurring in the integrals[25-31]. Comparison of such processes with finite element methods have been made by Pian[32] showing some details of the problem discussed above. Such approaches eliminate some of the drawbacks of the finite differences procedures and indeed bring it close to the finite element process as will be shown in the next section. However, the difficulty of increasing the order of approximation or of using irregular meshes still preserves the advantages of finite element process.

VI. SOME ASPECTS OF NON-CONFORMITY - COMPLETING THE CIRCLE TO FINITE DIFFERENCES

As mentioned before many non-conforming elements have been implemented in practice and convergence proofs obtained. Indeed very often these non-conforming elements have proved to produce results of higher accuracy than corresponding conforming ones. What is the reason for this and is it desirable on occasion to introduce non-conformity deliberately to produce better results?

To answer this question it is of interest to consider the terms on which the performance of an element is based. It is found by mathematical reasoning that the order of convergence of a particular element is dependent only on the complete polynomial terms which occur in the expressions[18]. This indeed may be anticipated by considering the remainder terms in a local Taylor expansion of the unknown functions near the vicinity of a point of the domain.

In order to introduce conformity it is often found that either incomplete polynomial expansions are used, as for instance in the bilinear rectangle of Figure 2b in which only one quadratic term (xy) occurs in addition to a complete linear expansion or else, as in C^1 - class elements, singularities or rational fractions are introduced in addition to ordinary polynomial terms within the element. It is often the existence of such terms which causes the performance of an element to deteriorate and perhaps some means of eliminating these should be sought.

One answer to this problem has been recently supplied by Irons and Razzaque[33] who introduce the concept of substitute shape functions. The essential idea is to replace the original conforming, shape which contains superfluous high order terms of expansion or singularities by another function which is an exact polynomial and which in the least squre sense, represents the best fit to the original shape function. This if N is the original function and $\bar{N}$ its polynomial substitute of the kind

$$\bar{N} = \beta_1 + \beta_2 x + \beta_3 y \ldots \tag{34}$$

we determine the coefficients β by minimizing

$$\int_{\Omega^e} (N-\bar{N})^2 \, d\Omega \tag{35}$$

with respect to these parameters. This results in a set of equations

$$\int_{\Omega^e} (N-\bar{N}) \frac{\partial \bar{N}}{\partial \beta_i} \, d \tag{36}$$

from which β's are readily found. Clearly in general $\bar{N}$ will not be 'conforming.'

Convergence of elements derived on such substitute basis can be argued from the fact that in the limit (discounting any singularities present) the combination of either N or $\bar{N}$ is capable of representing a simple polynomial Taylor expansion in an identical manner.

The improvement of the performance of elements derived on the basis for some C^1 continuity problems in the context of plate bending has been demonstrated in reference 34 and indeed in other situations a similar improvement is expected.

An apparently alternative path to improvement of element performance has been recently demonstrated in the context of numerically integrated elements by reducing (rather than increasing) the order of numerical integration[34]. It is easy to show that one of the reasons of the success of this process is in fact its identity with the use of substitute shape functions. To show this it is of interest to record some properties of Gauss integration points used in numerical integration. Thus if n Gauss sampling points are used in an integration domain $-1 < x > 1$ then

(a) a polynomial of degrees 2n-1 is integrated exactly.

(b) The n Gauss points define uniquely a polynomial of degree n-1 which is the least square approximation of any polynomial of degree n which has the same sampling values.

Further, we can observe that if $\bar{N}$ represents a least square approximation to N then $\frac{\partial \bar{N}}{\partial x}$ is also a least square approximation to $\frac{\partial N}{\partial x}$ etc. for all derivatives.

To illustrate these properties observe that any parabolic curve N^i in x direction is represented in a least square approximation by a straight line $\bar{N}$ passing through the two Gauss points. Further, the values of $\frac{\partial N^i}{\partial x}$ and $\frac{\partial \bar{N}}{\partial x}$ obtained by one Gauss point sampling are identical, as shown in Figure 5.

In practical application one dimensional domains are of little interest, but in two or three dimensions we observe immediately that for a bilinear expansion of Figure 2b the effects of sampling N or $\frac{\partial N}{\partial x}$ at one central Gauss point is equivalent to passing a least square substitute linear expansion $\bar{N}$ as shown in Figure 6.

In second order rectangular elements used frequently in finite element analysis, Figure 7, terms such as x^2y, xy^2 arise giving first derivatives which vary parabolically in one direction. The effect of using a 2x2 Gauss sampling is to approximate to such derivatives by a bilinear expansion which eliminates the effect of these terms in a least square manner, effectively approximating to the original shape function by a complete second order expansion. The success of 2x2 Gauss integration achieved in many situations is undoubtedly due to this fact. Table III shows a typical application in which dramatic improvement of results occurs by reduction of integration.

TABLE III

Central Deflections of a Square Plate Under Lateral Uniform Load. Solution Using Four 'Parabolic' Three Dimensional Elements with Reduced Integration

(w_c - Exact Solution, ← L-Span, ← t-Thickness)

L/t =		200	10
3x3x3 Gauss point	$\frac{w}{w_c}$ =	0.60	0.85
2x2x2 Gauss point	$\frac{w}{w_c}$ =	1.00	0.98

The above remarks show that in many situations improvement of results is achieved by introduction of admissible, non-conforming, shape functions. At this stage it is of interest to examine the finite difference approximation and to how that these are in face simple applications of such non-conforming trial function assumptions.

Consider for instance a 'direct' finite difference approximation to an ordinary differential equation

$$\frac{d^2\phi}{dx^2} + Q = 0 \tag{37}$$

The standard 'local' approximation to the second derivative in the vicinity of point, i.e.

$$\frac{d^2\phi}{dx^2} = \frac{1}{h^2}(\phi_{n+1} - 2\phi_n + \phi_{n-1}) \tag{38}$$

is in fact identical to the choice of a trial function which is a parabola fitting the three consecutive values of ϕ. The governing equation well known in finite differences

$$\phi_{n+1} - 2\phi_n + \phi_{n-1} - h^2 Q = 0 \tag{39}$$

can be obtained by an integral approximation of the form given by equation 15 with a weighting function shown in Figure 8.

Clearly the shape functions chosen here are non-conforming and show discontinuities between successive elements. Convergence is dependent only on the 'interweaving' nature of those which guarantee that in the limit the discontinuity disappears.

A further much used finite difference approximation is presented in Figure 9 where, say in a variationally formulated problem, an expression for a gradient is written as

$$\frac{\partial \phi}{\partial x} = \frac{1}{2h} \left[\phi_{n+1,\, m+1} + \phi_{n+1,\, m} - \phi_{n,\, m+1} - \phi_{m,n}\right] \quad (40)$$

It is immediately evident that this is precisely the value obtained by use of the substitute shape function in Figure 6 (or simply one point integration) and indeed identical approximation will result.

Pursuing the line of thought indicates that all finite difference processes can be considered as special cases of the finite element process with non-conforming but usually admissible shape function assumptions.

The success of finite difference methods is indeed dependent on the convergence of the trial function approximation and is indeed a particular case of the finite element process which originated in a different manner. It is more than likely that the future optimal methods of numerical discretization can borrow from the successes of both procedures. The finite element methodology based on irregular subdivision of elements and often a variational formulation frees the standard finite difference analyst from his shackles of regular mesh subdivision. The finite element method may well make greater use of non-conforming assumptions for trial functions implicit in the finite difference approximation. One such interesting 'marriage' was indicated recently by Utku[36] where an interweaving mesh is used on an irregular basis in two dimensions by passing a least square quadratic surface fit of a local expansion which yields second derivatives in an 'element' association with the locality as shown in Figure 10.

Questions such as the adequacy of 'interweaving' required to obtain admissibility of such non-conforming shape functions remain yet to be answered, but, as mentioned before, pragmatic tests exist to judge whether convergence will be obtained. Problems such as instability of equation systems derived from certain function assumptions (e.g. for certain subdivisions using the one point integration rule of Figure 6 or or equivalent finite difference model of equation 40) still need to be further investigated. Nevertheless, today it can be stated that the finite element methodology represents a very considerable generalization of the finite difference ideas and hence opens the way at least to more efficient methods of solving already solvable problems - at best opens new ways to problems which previously defied analysis.

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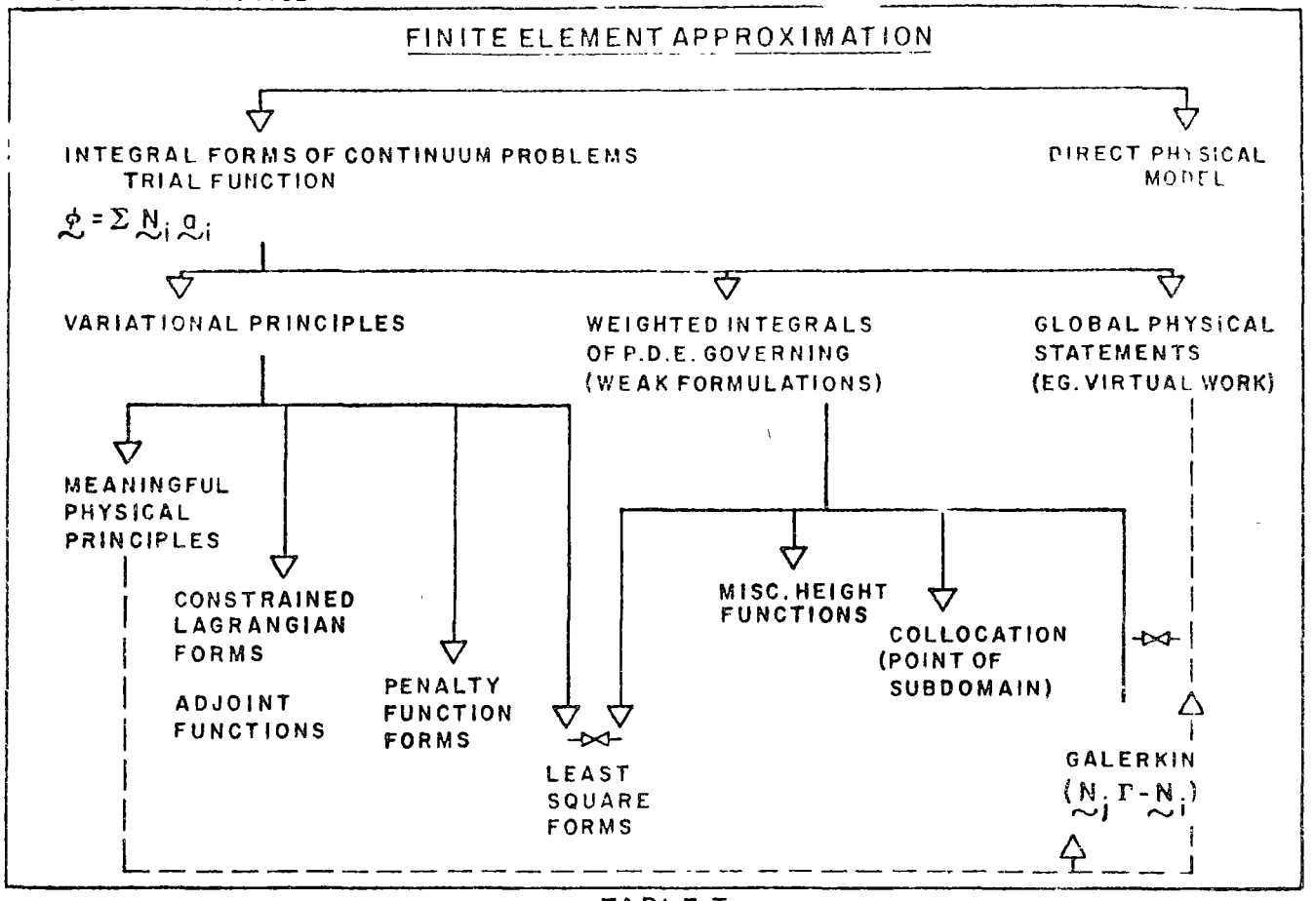


TABLE I

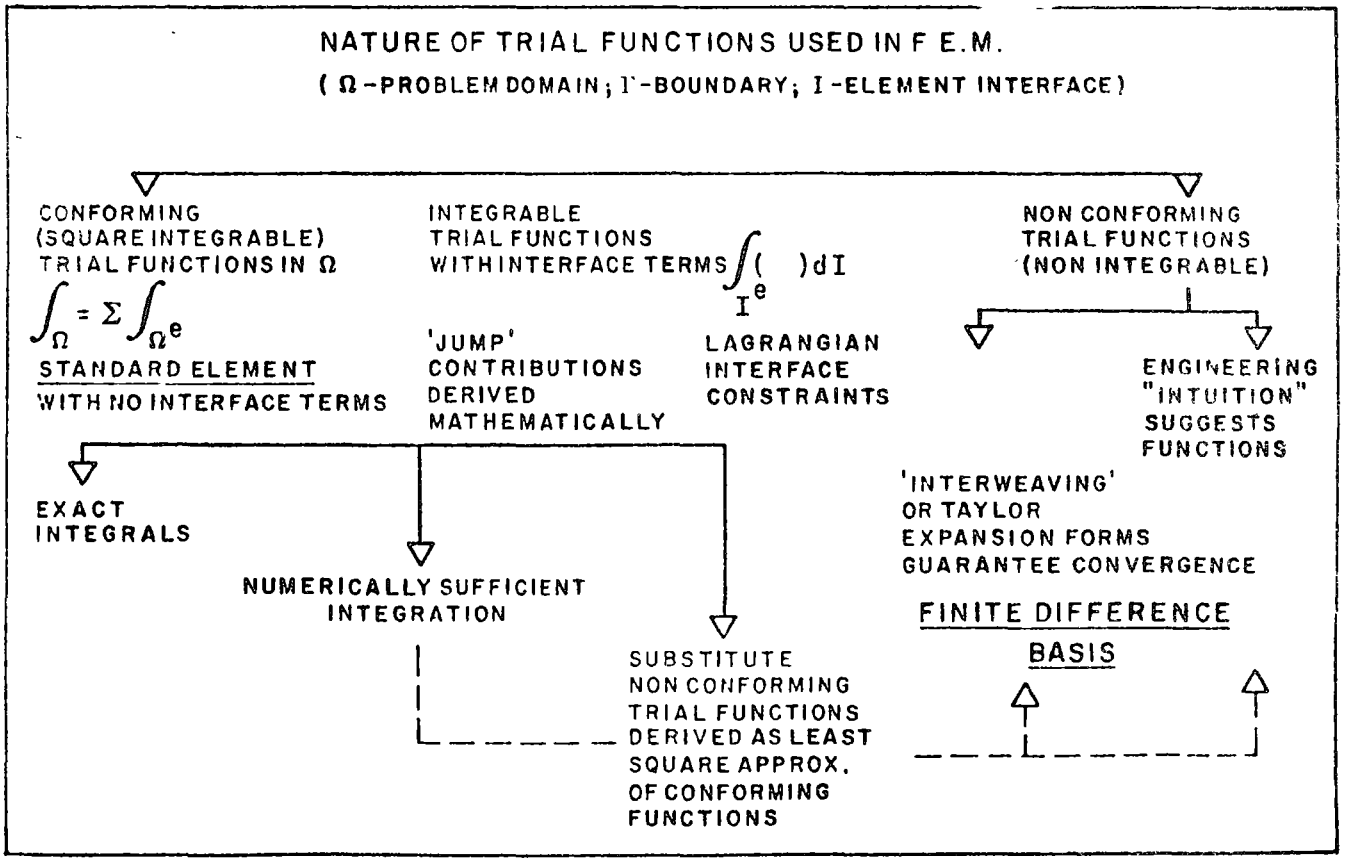


TABLE II

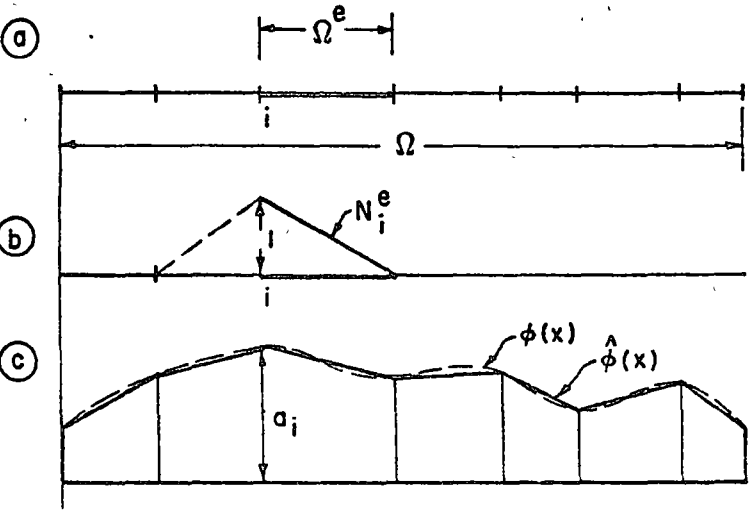


FIG. 1 ONE DIMENSIONAL, SIMPLE, LOCALIZED TRIAL FUNCTIONS OF C° CONTINUITY

(a) DOMAIN SUBDIVIDED INTO ELEMENTS

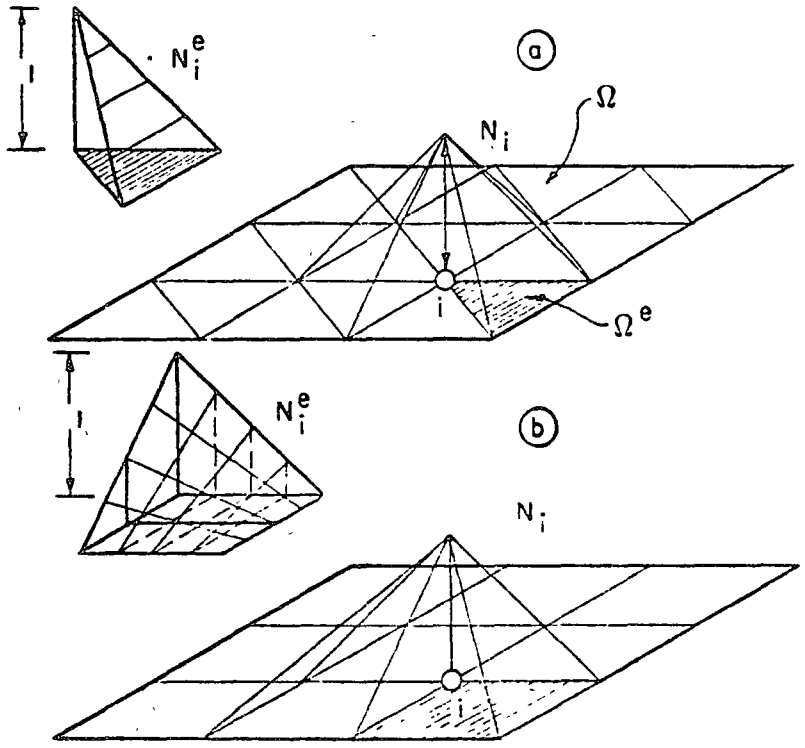
(b) LOCALIZED TRIAL FUNCTION FOR PARAMETER $a_i = \phi_i$

(c) APPROXIMATION TO AN ARBITRARY FUNCTION $\phi(x)$

FIG. 2 TWO-DIMENSIONAL LOCALIZED TRIAL FUNCTIONS OF C° CONTINUITY

(a) TRIANGULAR ELEMENTS (LINEAR EXPANSION)

(b) RECTANGULAR ELEMENTS (BI-LINEAR EXPANSION)



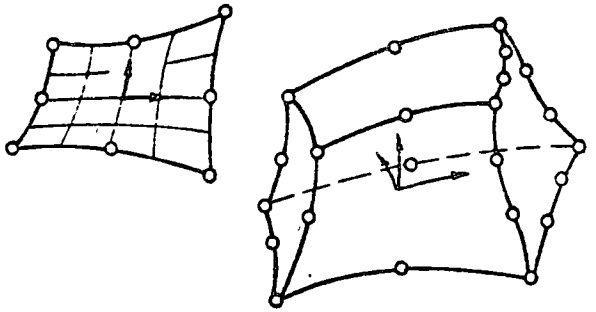


FIG. 3 SOME MORE ELABORATE ELEMENTS WITH CURVILINEAR CO-ORDINATES

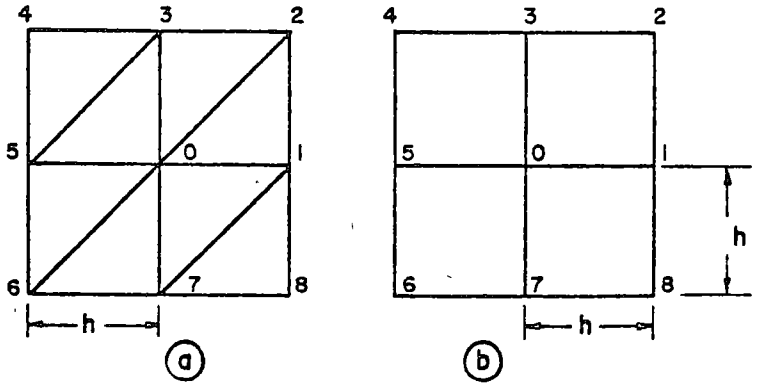


FIG. 4 REGULAR TRIANGLE AND SQUARE SUBDIVISIONS

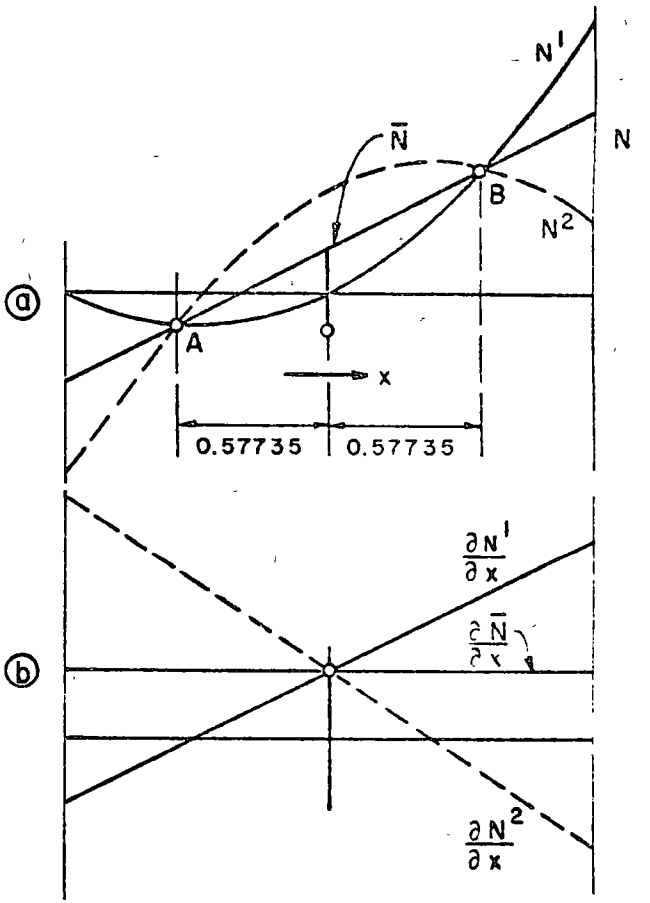


FIG. 5 A PROPERTY OF GAUSS POINTS

(a) TWO GAUSS POINTS DEFINE A STRAIGHT LINE, $\bar{N}$, WHICH IS A LEAST SQUARE APPROXIMATION OF ANY PARABOLA N^1 PASSING THROUGH THESE POINTS

(b) $\frac{\partial \bar{N}}{\partial x}$ IS A (CONSTANT) LEAST SQUARE APPROXIMATION TO $\frac{\partial N}{\partial x}$

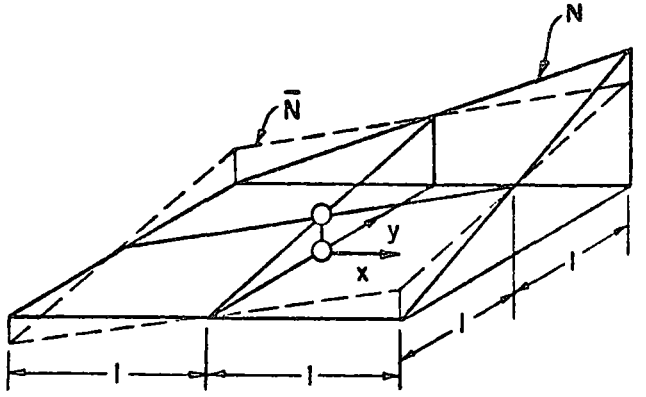


FIG. 6 ONE GAUSS POINT IN A TWO DIMENSIONAL SQUARE SAMPLES CORRECTLY $\bar{N}$ AND $\frac{\partial \bar{N}}{\partial x}$, $\frac{\partial \bar{N}}{\partial y}$ WHICH ARE LEAST SQUARE APPROXIMATIONS BY A LINEAR EXPANSION TO A BILINEAR FUNCTION N

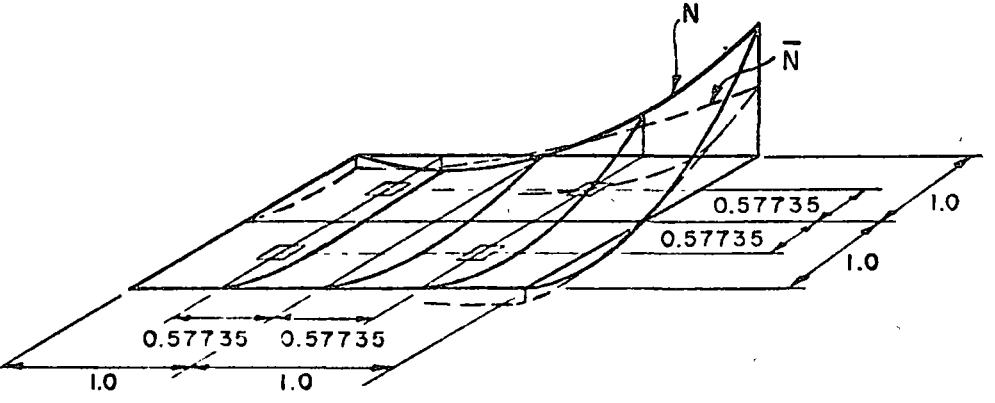


FIG. 7 FOUR GAUSS POINTS IN A TWO DIMENSIONAL SQUARE SAMPLE CORRECTLY $\bar{N}$ AND ITS DERIVITIES WHERE $\bar{N}$ IS A LEAST SQUARE APPROXIMATION BY A PARABOLIC EXPANSION TO A BI-PARABOLIC FUNCTION N

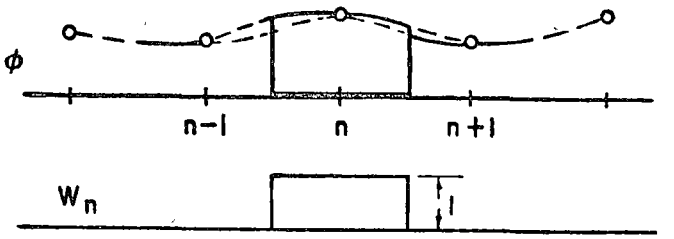


FIG. 8 EQUIVALENCE OF FINITE DIFFERENCE APPROXIMATION TO $\frac{d^2\phi}{dx^2}$ WITH A COLLATION FINITE ELEMENT APPROACH USING DISCONTINUOUS (NON-CONFORMING) TRIAL FUNCTIONS

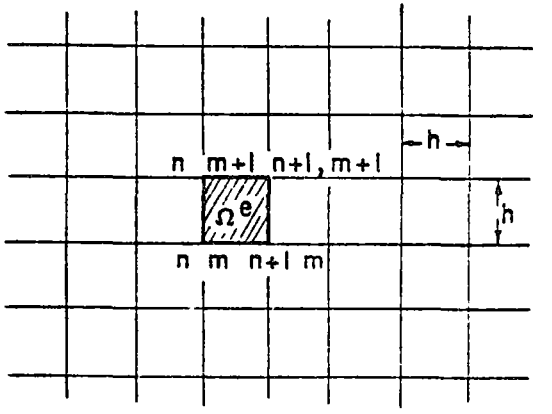


FIG. 9 AN EQUIVALENCE OF A FINITE DIFFERENCE APPROXIMATION FOR FIRST DERIVATIVE IN Ω^e TO ONE POINT NUMERICAL SAMPLING OF BILINEAR EXPANSION OF FIG. 6

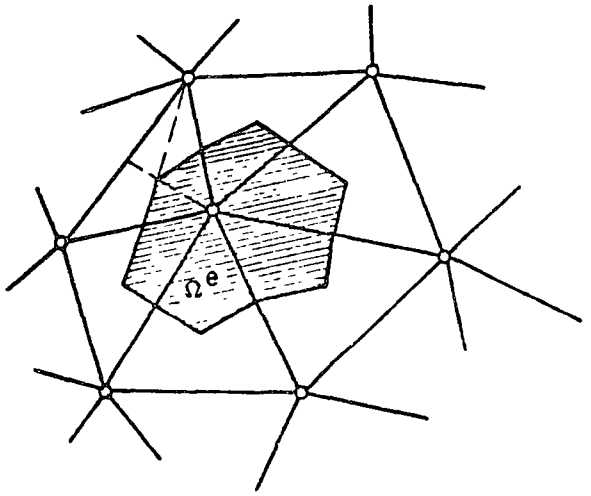


FIG. 10 NON-CONFORMING BUT INTERWEAVING AND ADMISSIBLE EXPANSION FOR ϕ IN Ω^e BY IMPOSING A LOCAL PARABOLIC EXPANSION FOR ϕ_i IN TERMS OF A LEAST SQUARE FIT SURFACE WITH NEIGHBOURING POINTS (SEE REF. 36)

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MIXED AND HYBRID F.E.M. FORMULATIONS

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MIXED VARIATIONAL PRINCIPLES AND HYBRID FORMULATIONS

I. INTRODUCTION

Alternatives to the potential or complimentary energy formulations (either in the conventional or generalized forms) are the mixed and hybrid energy formulations.

Mixed variational principles lead directly to mixed formats of the element force-displacement equations. Because the Euler equations of these functionals are the more basic equations of elasticity, with lower order derivatives, the continuity requirements on the assumed fields are of lower order than for the conventional variational principles.

Hybrid formulations involve not only the generalization of the conventional energy principles but also introduce multi-field representations of element behavior. One form of stress or displacement field is described within the element and independent stress and/or displacement fields are described on the boundary of the element. All but one of the fields is described in terms of generalized parameters; that field is given in terms of physical d.o.f. The appropriate energy expression (a generalization of either potential or complementary energy) is first formed in terms of both classes of parameters and then the stationary condition is applied to the generalized parameter set. This yields a system of equations for the generalized parameters in terms of the physical d.o.f. These equations are used to eliminate the generalized parameters from the energy expression. The resulting energy expression then contains an identifiable stiffness or flexibility matrix of conventional form.

We examine mixed formulations in some detail in the next section. Then, in subsequent sections, three alternative hybrid formulations are studied.

II. MIXED VARIATIONAL PRINCIPLES

A. Reissner Variational Principle

The simplest and most widely used mixed variational principle in finite element analysis is that which is associated with Reissner [1]. We can develop this quite readily, using the potential energy principle as a starting point.

We have

$$\pi_p = U + V \tag{1}$$

but, by definition (with U* = Complementary Energy)

$$U^* + U = \int_{Vol} \{\sigma\} \{\epsilon\} \, d(Vol) \tag{2}$$

or,

$$U = \int_{Vol} \{\sigma\} \{\epsilon\} \, d(Vol) - U^* \tag{3}$$

and, by substitution into Equation (1), we have the Reissner functional, π_R.

$$\pi_R = \int_{Vol} [\sigma] \{\epsilon\} \, d(Vol) - U^* + V \tag{4}$$

where now the functional is expressed in terms of both stresses and strains. Furthermore, we note that if independent choices are made for both stress-related and strain-related fields, then the surface integral V must account for both prescribed stresses and prescribed displacements, i.e.,

$$V = \int_{S_\sigma} \bar{q} . \Delta dS + \int_{S_\Delta} q . (\Delta - \bar{\Delta}) dS \tag{5}$$

where now S_Δ is the portion upon which the displacements $\bar{\Delta}$ are prescribed.

The Euler equations of this functional are the equilibrium and stress-displacement equations. Analysts are generally familiar with such basic equations and it is of interest to examine how one proceeds from the basic equations to the integral form. To do this, we employ the method of weighted residuals with the Galerkin criterion in the choice of weighting functions.

Let us consider the case of a uniform axial member subjected to a distributed load q. In this case the equilibrium equation is

$$\frac{d\sigma_x}{dx} + \frac{q}{A} = 0 \tag{6}$$

and, after substitution of the strain-displacement equation into the stress-strain law, we have the stress-displacement equation

$$\frac{du}{dx} - \frac{\sigma_x}{E} = 0 \tag{7}$$

In developing algebraic equations for this formulation we apply a weighting factor ψ to the equilibrium differential equation and a weighting factor ϕ to the stress-displacement differential equation

$$\int_0^L \left(\frac{d\bar{\sigma}_x}{dx} + \frac{q}{A}\right)\psi \; A dx = 0 \tag{8}$$

$$\int_0^L \left(\frac{d\bar{u}}{dx} - \frac{\bar{\sigma}_x}{E}\right)\phi \; A dx = 0 \tag{9}$$

The approximations to stress and displacement are chosen as

$$\bar{u} = \lfloor N \rfloor \{u\} \tag{10}$$

$$\bar{\sigma}_x = \lfloor \Sigma \rfloor \{\sigma\} \tag{11}$$

where the terms of [N] are the displacement shape functions and $[\Sigma]$ is employed to designate the stress shape functions. In selecting weighting factors from these terms we employ the displacement shape functions N_i for ψ and the stress shape functions Σ_i for ϕ. Consider first the weighted integral of the equilibrium differential equation. By introduction of $\psi = N_i$ we have

$$\int \left(\frac{d\bar{\sigma}_x}{dx} + \frac{q}{A}\right) N_i A dx = 0 \tag{8a}$$

and, after integration by parts of the first term, and rearrangement

$$\int_0^L \frac{dN_i}{dx} \bar{\sigma}_x \, A dx = N_i A \bar{\sigma}_x \Big|_0^L + \int_0^L q N_i dx \tag{8b}$$

We begin with potential energy (π_p) and for simplicity exclude initial strains and body forces and assume that all prescribed loads are distributed tractions $\bar{T}$. Then we have

$$\pi_p = \frac{1}{2} \int_{Vol.} \varepsilon\,[E]\,\varepsilon \; d(Vol.) - \int_{S_\sigma} \bar{T}.u \; dS \qquad (1.a)$$

where all terms are as defined previously. The volume integral indicates that the displacement fields chosen to approximate π_p will result in lack of satisfaction of the equilibrium condition $\sigma_{ij,j}=0$ and the surface integral indicates that the condition $\bar{T} = T$ will not be met along S_σ, where T denotes the edge forces given by the internal stresses σ. At the same time the functional implies that the displacement boundary condition, $u = \bar{u}$ on S_u and the strain-displacement equations $\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) = \partial\Delta$ will be satisfied throughout the volume of the structure.

Suppose we desire to work with displacement function which will not meet the displacement boundary condition and the strain-displacement equation. We can then append these conditions to the functional, using the Lagrange multiplier method. The new functional is

Introducing Eq. 11 for $\bar{\sigma}_x$ and noting that $N_i A\bar{\sigma}_x = F_i$, we have for all values of N_i

$$[\Omega_{21}]\{\sigma\} = \{F\} + \{F^d\} \tag{12}$$

where

$$[\Omega_{21}] = \left[\int_0^L \{\frac{dN_i}{dx}\}[\Sigma]\, A dx\right] \tag{12a}$$

$$\{F^d\} = \left\{\int_0^L qN_i dx\right\} \tag{12b}$$

and $\{F\}$ is the conventional listing of joint forces.

Considering next the weighted integral of the stress-displacement equation we have, upon introduction of $\phi = \Sigma_i$

$$\int_0^L (\frac{d\bar{u}}{dx} - \frac{\bar{\sigma}x}{E})\, \Sigma_i\, A dx = 0 \tag{9a}$$

and, after introduction of Eqs. (10) and (11) for $\bar{u}$ and $\bar{\sigma}_x$, we have for all Σ_i

$$[\Omega_{21}]^T\{u\} + [\Omega_{11}]\{\sigma\} = 0 \tag{9b}$$

where $[\Omega_{21}]$ is as defined by Eq. (12a) and

$$[\Omega_{11}] = -\left[\int_0^L \{\Sigma\}\frac{|\Sigma|}{E} A dx\right] \tag{12c}$$

We see that this relationship connects $\{u\}$ and $\{\sigma\}$ so it is convenient to group both Eq. (12) and Eq. (9b) into a single matrix equation

$$\left[\begin{array}{c:c} \Omega_{11} & \Omega_{12} \\ \hdashline \Omega_{21} & 0 \end{array}\right] \left\{\begin{array}{c} \sigma \\ \hdashline u \end{array}\right\} = \left\{\begin{array}{c} 0 \\ \hdashline F+F^d \end{array}\right\} \tag{13}$$

Thus, we have derived finite element equations in the mixed format.

$$\pi_{H\text{-}W} = \frac{1}{2} \int_{Vol} \underset{\sim}{\varepsilon}\,[E]\,\underset{\sim}{\varepsilon}\; d(Vol) - \int_{S_\sigma} \underset{\sim}{\bar{T}} \cdot \underset{\sim}{u}\; dS$$

$$+ \int_{S_u} (\underset{\sim}{\bar{u}} - \underset{\sim}{u})\,\underset{\sim}{\lambda}_1\; dS - \int_{Vol} (\underset{\sim}{\varepsilon} - \partial\underset{\sim}{\Delta})\;\underset{\sim}{\lambda}_2\; d(Vol) \tag{14}$$

where $\partial\underset{\sim}{\Delta}$ denotes the displacement derivatives corresponding to the strains $\underset{\sim}{\varepsilon}$. From the prior Lecture notes we can recognize that the langrange multipliers λ_1 and λ_2 must be edge tractions $\underset{\sim}{T}$ and the stresses $\underset{\sim}{\sigma}$, respectively, so we can write

$$\pi_{H\text{-}W} = \frac{1}{2} \int_{Vol} \underset{\sim}{\varepsilon}\;[E]\,\underset{\sim}{\varepsilon}\; d(Vol) - \int_{S_\sigma} \underset{\sim}{\bar{T}} \cdot \underset{\sim}{u}\; dS$$

$$+ \int_{S_u} (\underset{\sim}{\bar{u}} - \underset{\sim}{u})\;\underset{\sim}{T}\; dS - \int_{Vol} (\underset{\sim}{\varepsilon} - \partial\underset{\sim}{\Delta})\;\underset{\sim}{\sigma}\, d(Vol) \tag{14a}$$

This is the Hu-Washizu functional, containing displacements and stresses as independent variables. Note that we caused the strains to be independent of the displacements by "disconnecting" the strain-displacement equation. The subsidiary condition is that the tractions $\underset{\sim}{T}$ correspond to the stresses $\underset{\sim}{\sigma}$.

B. RELATIONSHIPS AMONG MIXED VARIATIONAL PRINCIPLES.

In the previous section we first justified the Reissner functional simply by the substitution of an identity for the strain energy into the potentional energy functional. A more elegant way to establish the Reissner functional, and all other functionals in elasticity, is to start with the potential energy functional and successively "relax" conditions on the potential energy and effect satisfaction of certain other conditons. In this way, Pian[40], and others[15, 34], have shown how it is possible to progress from the potential energy functional to the complementary energy functional with the identification of the Reissner and Hu-Washizu[15] functionals at intermediate steps.

Before detailing this process, it is useful to elaborate upon the meaning of the term "relax conditions on". One must first recognize that each integral in an energy expression can be identified (through the Euler equations of the functional) with a basic condition that is not satisfied pointwise in an approximate solution obtained with use of the functional. There are of course other conditions that exist, and it is implied that such conditions are met exactly by the functions chosen in approximation at the functional. In potential energy, for example, it is implied that the chosen displacement fields meet exactly all approoriate conditions on displacement.

Now, suppose it is expected that the satisfaction of the conditions which must be met exactly will be difficult. If it is intended that a choice be made of a function which will not exactly meet certain of these conditions, then we are "relaxing conditions on" the original functional and one must modify the original function by adding terms to it.

The Euler equations are $\bar{\underset{\sim}{T}} = \underset{\sim}{T}$ on S_σ, $\bar{\underset{\sim}{u}} = \underset{\sim}{u}$ on S_u, the equilibrium and strain-displacement equations in the interior, and the constitutive law $\underset{\sim}{\sigma} = [E]\ \underset{\sim}{\varepsilon}$ on the interior.

If we now integrate by parts the integral $\int_{Vol} \underset{\sim}{\sigma} \cdot \partial\underset{\sim}{\Delta}\ d\,(Vol)$ and substitute the result into the fourth integral on the right-side of Equation (14.a), assume that stress fields will be chosen which satisfy the equilibrium conditions and that the constitutive law is satisfied we obtain the Reissner (or Hellinger-Reissner) functional. We must eliminate the strains from $\pi_{H\text{-}W}$ by substituting $\underset{\sim}{\varepsilon} = [E]^{-1}\ \underset{\sim}{\sigma}$

$$\pi_p = -\frac{1}{2}\int_{Vol} \underset{\sim}{\sigma}\,[E]^{-1}\,\underset{\sim}{\sigma}\ d(Vol) + \int_{Vol} \underset{\sim}{\sigma} \cdot \partial\underset{\sim}{\Delta}\ d\,(Vol)$$

$$- \int_{S_\sigma} \bar{\underset{\sim}{T}} \cdot \underset{\sim}{u}\ dS - \int_{S_u} (\bar{\underset{\sim}{u}} - \underset{\sim}{u})\ \underset{\sim}{T}\ dS \qquad (15)$$

If we further assume that the strain-displacement conditions are satisfied, as well as the conditions on boundary tractions, we reach the principle of complementary energy

$$\pi_c = \frac{1}{2}\int_{Vol} \underset{\sim}{\sigma}\,[E]^{-1}\,\underset{\sim}{\sigma}\ d(Vol) - \int_{S_u} \bar{\underset{\sim}{u}} \cdot \underset{\sim}{T}\ dS \qquad (16)$$

Oden[31], in a recent paper, describes 14 different variational principles in elasticity theory. These alternatives stem from use of different variables (strains, displacements, stresses, stress functions) and conditions not explicitly treated above, such as the stress-strain law and the compatibility equations. Sandhu and Pister[18] and Kikuchi and Ando[20] have also contributed to the formulation of more general variational principles. (See also Karcher[55]).

It is important to recognize that the above integral relationships, which are the basis for finite element representations, can be developed by use of the method of Weighted Residuals with Galerkin weighting factors. This was demonstrated in the previous section, where we developed the integral of Reissner by operating on the equilibrium and stress-displacement conditions. One can consult a recent paper by Connor[28] for the development of a wide range of alternative variational principles along these lines. One advantage of this approach, which has not been exploited in practice, is that certain integrals appear which do not appear in the virtual work formulation.

C. APPLICATIONS OF THE MIXED VARIATIONAL PRINCIPLES IN FINITE ELEMENT ANALYSIS

Mixed variational principles have not gained wide popularity in finite element structural analysis. The reasons for this would appear to be the following:

(1) Unfamiliarity of practioners with related concepts.
(2) Concern that they will not fit in with established finite element programs.
(3) Lack of positive-definiteness of the algebraic equations.
(4) Larger number of solution unknowns.
(5) No clear advantage over stiffness formulation for C^0 continuity.

Still other reasons could no doubt be identified.

Mixed methods have nevertheless proved of advantage in certain problems, as outlined below. As far as is known, however, these applications have been accomplished with computer programs especially written for the purpose.

Herrmann[3,4,7,8,13] was probably the first to apply mixed variational principles to finite element analysis, although we must emphasize that Klein[29] established extensive mixed formulations of finite element analysis in a non-variational manner and there are others (e.g., all workers involved in the "transfer matrix" approach) who also dealt with mixed algebraic developments at earlier times. In one group of papers, Herrmann dealt with flat plate bending problems. The advantage of this mixed formulation is that the requirements on continuity of displacement can be reduced to C^0. This is done by modifying Reissner's principle (Equation 15) by integrating by parts the term $\int \sigma \cdot \partial\Delta \, d(\text{Vol})$. In plate bending, where the strain measure is the plate curvature (the second derivative of the displacement) the integration by parts transforms the functional into one containing only first derivatives of displacement and so only C^0 continuity is needed across element boundaries.

Figure 2 shows results obtained by Hermann[4] for the problem of a simply-supported square plate under central concentrated load. He uses a triangular element based on linear displacement and constant moment. The results are not good, and it can in fact be demonstrated[30] that identical results can be obtained with the use of a stiffness formulation that uses quadratic displacements[32]. On the other hand, Visser[5] gives results for a quadratic variation of displacement and linearly-varying bending moments and these have not been duplicated by a stiffness formulation. Other work on plates is described by Bron and Dhatt[47].

Mixed variational principles are regarded by many as especially useful for thin shell analysis, where displacements can be used to describe stretching behavior and stresses can be used to describe bending, again permitting a low order of continuity between elements. Herrmann and Campbell[7] have analyzed shells with use of flat segments and, subsequently, Herrmann and Mason formulated a curved shell element. A more sophisticated form of this has been developed by Visser[6].

Connor[10] has given a detailed account of formulations via Reissner's principle. Gould and Sen[52], Prato[11], Elias[33], and Connor and Will[12] have made effective use of mixed formulations in shell analysis.

Hermann[13] also identified the mixed variational approach as being useful for the analysis of incompressible materials. Due to a Poisson's ratio of 0.5 such materials present difficulties in potential energy (stiffness) analysis. Key[14], among others, has extended Hermann's work in this connection, and Kwang, et al[51] have applied it in soils analysis.

In concluding this section it should be noted that mixed variational formulations have been adopted quite naturally in the development of finite element fluid flow analysis. Taylor and Hood[34] and Kawahara, et al[35] each adopt the mixed approach in dealing with viscous, incompressible flow.

III. HYBRID STRESS FORMULATION

Related to the generalized variational principles are the hybrid formulations pioneered by Pian[21], which accomplish the formulation of both stiffness and flexibility matrices through specification of independent fields within and on the boundary of the element. The approach will be described for the case of the "assumed stress" hybrid method, in which a stress field is selected for the interior of the element and displacement field is independently

assigned to the element boundary. Then, the principle of minimum complementary energy is utilized to produce, directly, the element stiffness matrix.

We begin with the selection of an assumed stress field for the interior region of the element, which can be designated as

$$\{\sigma\} = [P]\ \{\beta\} \tag{20}$$

where $\{\beta\} = [\beta_1 \ldots \beta_m]$ are the undetermined parameters of the stress field and the coefficients, β_i, are functions of the spatial coordinates of the element. The surface forces $\underset{\sim}{T}$ can be expressed in terms of the parameters β by means of Equation (20), in the form

$$\underset{\sim}{T} = [R]\ \{\beta\} \tag{21}$$

The boundary displacements are now independently prescribed. The displacements along the surface are specified by a column vector $\bar{u}$ and can be related to the prescribed edge displacements, which are written in terms of the element joint displacements $\{\Delta\}$, as follows

$$\underset{\sim}{\bar{u}} = [\bar{Y}]\ \{\Delta\} \tag{22}$$

Figure 3 summarizes the above assumed fields.

We now write the complementary energy in the more appropriate form (lacking initial strain terms)

$$\pi_c = \frac{1}{2} \int_{Vol} \underset{\sim}{\sigma}\ [E]\ \underset{\sim}{\sigma}\ d(Vol) - \int_{S_u} \underset{\sim}{T}\ \underset{\sim}{\bar{u}}\ dS \tag{23}$$

where S_u, the portion of the surface on which the displacements are prescribed, is here the complete surface.

All ingredients are now available for transformation of the complementary energy functional into discrete form. We have, by substitution of (20), (21), and (22) into Equation (23)

$$\pi_c = \frac{\{\beta\}}{2}\ [H]\ \{\beta\} - \{\beta\}\ [Q]\ \{\Delta\} \tag{24}$$

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$$\pi_c = \frac{\{\beta\}}{2}\ [H]\ \{\beta\} - \{\beta\}\ [Q]\ \{\Delta\} \tag{24}$$

with

$$[H] = \left[\int_{Vol} [P]^T [E]^{-1} [P] \, d(Vol)\right] \tag{25}$$

and

$$[Q] = \left[\int_S [R]^T [\bar{T}] \, dS\right] \tag{26}$$

The stress and boundary displacement parameters have been chosen independently so that independent variation of π_c with respect to $\{\beta\}$ and $\{\Delta\}$ is called for. For $\{\beta\}$, this yields

$$[H] \{\beta\} - [Q] \{\Delta\} = 0$$

or

$$\{\beta\} = [H]^{-1} [Q] \{\Delta\} \tag{27}$$

By back-substitution into π_c (Equation 24)

$$\pi_c = -\frac{1}{2} \{\Delta\}^T [Q]^T [H]^{-1} [Q] \{\Delta\} \tag{28}$$

and it is apparent that the element stiffness matrix can be defined as

$$[k] = - [Q]^T [H]^{-1} [Q] \tag{29}$$

Due to the independence of assumptions relating to stress and displacement, any number of terms may be chosen in representation of the stress field (one generally seeks the simplest representation of edge displacement). It has been observed, however, that in order to avoid kinematic instabilities in the modeling of the complete structure the condition $m \geq n - \ell$ should be met, where m is the number of β-parameters, n is the number of displacement d.o.f., and ℓ is the number of rigid body modes of the element.

Various degrees of refinement in stress representation have been studied numerically (Ref. 21-23, 56), leading to the conclusion that no significant advantages are gained with use of more complex assumed stress fields.

IV. ASSUMED DISPLACEMENT HYBRID

We examine in this section two hybrid formulations which derive from a potential energy functional. In the first of these [24] the element interior displacement field is expressed in terms of generalized displacements and the boundary stress field is independently described in terms of joint forces. This yields an element flexibility matrix. The second hybrid formulation [25] is an extension of the above concept in that both the interior displacements and the edge (boundary) stresses are described in terms of generalized parameters, while the boundary displacements are independently described in terms of joint displacements. This leads to an element stiffness matrix.

In order to deal with independent fields a "generalization" of the potential energy principle is needed. In describing the generalization applicable to the first of the two displacement-hybrid schemes we again consider only "interior" elements, i.e., elements with no boundaries on the edges of the structure, and exclude body or initial forces. The pertinent "boundary" of the element is the complete boundary (S_n) and this is loaded by the interelement tractions $\bar{T}$. Thus, we have the modified potential energy

$$\Pi_p^m = U - \int_{S_n} \bar{T}.u \, dS \tag{30}$$

where u represents the boundary displacements consistent with the chosen interior displacement field Δ. The generalization of the conventional potential energy consists of the fact that $\bar{T}$ will be written in terms of joint force parameters so that both the displacement parameters of u (and Δ) and the joint force parameters will appear as unknowns in Π_p^m. A conventional potential energy principle involves only displacement parameters. We designate the generalized parameters of the interior displacement field as $\{a\}$ and have, for the usual polynomial representation

$$\Delta = [p] \{a\} \tag{31}$$

and, by application of the strain-displacement equations

$$\varepsilon = [C_f] \{a_f\} \tag{32}$$

where $\{a_f\}$ refers to the d.o.f. remaining after the rigid-body-motion d.o.f. $\{a_s\}$ have been eliminated through the differentiation associated with the strain-displacement equations. We also require the boundary values $\underset{\sim}{u}$ of this field, which can be obtained simply by evaluating $\underset{\sim}{\Delta}$ along the element boundaries

$$\underset{\sim}{u} = [Z]\{a\} = [Z_f \; Z_s] \begin{Bmatrix} a_f \\ a_s \end{Bmatrix} \tag{33}$$

where, for purposes of the development to follow, we have preserved the distinction between $\{a_f\}$ and $\{a_s\}$.

The final ingredient of the subject displacement-hybrid method is the description of the boundary tractions $\overline{\underset{\sim}{T}}$ in terms of joint forces $\{F_f\}$. The subscript f designates a system of joint forces exclusive of those necessary to furnish statically-determinate support of the element. This is a consequence of the requirement that the vector $\overline{\underset{\sim}{T}}$ represents a system of self-equilibrating forces for cases of zero body force. We symbolize these relationships as follows

$$\overline{\underset{\sim}{T}} = [\underset{\sim}{L}]\{F_f\} \tag{34}$$

We may now construct the discretized form of the modified potential energy. First, we observe that in constructing the work of the boundary tractions (the integral over S_n) the contribution of these self-equilibrated forces acting through rigid body displacements is zero. Since the rigid body displacements are equal to $[Z_s]\{a_s\}$ in Eq. 33, we consider only the product $[Z_f]\{a_f\}$ in treating u. With $U = \frac{1}{2}\int_{Vol} \underset{\sim}{\epsilon}[E]\,\underset{\sim}{\epsilon}\, d(Vol)$ and by substitution of Eqs. 33 and 34 into Eq. 30, we have

$$\Pi_p^m = \frac{\underline{a_f}}{2}[H]\{a_f\} - \underline{a_f}\,[J]\{F_f\} \tag{30a}$$

where

$$[H] = \int_{Vol} [C_f]^T [E] [C_f]\, d(Vol) \tag{35}$$

$$[J] = \int_{S_n} [Z_f]^T [\underset{\sim}{L}]\, dS \tag{36}$$

By variation of Eq. 30a with respect to a_f

$$[H]\{a_f\} - [J]\{F_f\} = 0$$

or

$$\{a_f\} = [H]^{-1}[J]\{F_f\}$$

By substitution into Eq. 30a.

$$\Pi_p^m = \frac{[F_f]}{2}[\delta]\{F_f\} \tag{37}$$

where the derived flexibility matrix is given by

$$[\delta] = [J]^T[H]^{-1}[J] \tag{38}$$

The second assumed displacement hybrid method[25] extends further the above concept to produce directly an element stiffness matrix. We choose here a system of interelement-compatible boundary displacements $\bar{u}$, expressed in terms of joint displacements $\{\Delta\}$. These are chosen independently of the field Δ which describes displacements within the element in terms of parameters $\{a\}$. Thus, in the general case, there is a disparity of these displacements along the element boundaries, as given by $(\bar{u}-u)$, where, as before, u are the boundary displacements consistent with Δ. This disparity can be removed by the constraint condition

$$\int_{S_n}(\bar{u} - u)dS = 0$$

Using the Lagrange multiplier concept the modified potential energy for this development becomes

$$\Pi_p^m = U - \int_{S_\sigma}\bar{T}.u\,dS - \int_{S_n}\lambda\,(\bar{u}-u)dS \tag{39}$$

where λ is the Lagrange multiplier.

The Lagrange multiplier has here the units of a load parameter and in this case it is the boundary traction associated with the gap $(\bar{u}-u)$. With this in mind we designate λ as the traction T. Also, in this development S_n is the full boundary of the structure. Since we restrict our attention to the formulation of interior elements we disregard the integral on S_σ in what follows. Thus, the modified potential energy for this development becomes

$$\pi_p^{m2} = U - \int_{S_n} T\,(\bar{u} - u)\, dS \tag{40}$$

The discretization of Eq. 40 requires field representations of ε, u, $\bar{u}$, and T. The representations of ε and u are already in hand via Eqs. 32 and 33. We must now establish appropriate representations of $\bar{u}$ and T.

It is required that $\bar{u}$ be expressed in terms of joint displacements $\{\Delta\}$, so we have (as in Equation 22)

$$\bar{u} = [\bar{Y}]\,\{\Delta\} \tag{41}$$

Also, the boundary tractions T are to be written in terms of generalized parameters $\{\beta_f\}$. We invoke the designation of parameters which exclude rigid-body terms (the subscript f) consistent with our prior discussion of the definition of boundary tractions (see Eq. 34). Hence, we denote these relationships

$$T = [Q]\,\{\beta_f\} \tag{42}$$

The above assumed fields are summarized in Figure 5.

A discretization of π_p^{m2} can now be formed by substitution of Eq. 32, the left partition of Eq. 33, Eq. 41 and 42 into Eq. 40. There results

$$\pi_p^{m2} = \frac{\lfloor a_f \rfloor}{2}\,[H]\,\{a_f\} - \underline{\beta_f}\,[X]\,\{\Delta\} + \underline{a_f}\,[\Theta]\,\{\beta_f\} \tag{43}$$

where [H] is as defined by Eq. 35 and

$$[X] = \int_{S_n} [Q]^T [\bar{Y}] dS \tag{44}$$

$$[\Theta] = \int_{S_n} [Z_f]^T [Q] dS \tag{45}$$

To develop the desired stiffness matrix we construct algebraic equations by first varying Eq. 43 with respect to $\{a_f\}$ and then with respect to $\{\beta_f\}$, yielding

$$[H] \{a_f\} + [\Theta] \{\beta_f\} = 0 \tag{46a}$$

$$- [X] \{\Lambda\} + [\Theta]^T \{a_f\} = 0 \tag{46b}$$

By solution of these for $\{a_f\}$ and $\{\beta_f\}$ in terms of $\{\Delta\}$ and by back-substitution into Eq. 43, there is obtained

$$\pi_p^{m2} = \frac{\{\Delta\}}{2} [K] \{\Delta\} \tag{47}$$

where

$$[k] = [X]^T \left[[\Theta]^T [H]^{-1} [\Theta] \right]^{-1} [X] \tag{48}$$

This is the element stiffness matrix derivable through the potential energy hybrid method.

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V. ALTERNATIVE HYBRID FORMULATIONS AND APPLICATIONS

One may perceive from the previous two sections that a very great variety of hybrid formulations is possible via the conventional and mixed variational principles. Many of these possibilities have been outlined by Pian[49], Atluri[27], Wolf[39] and others. A review of all such possibilities is beyond the scope of these notes. The writer has examined a wider range of possibilities in Ref. 48 and has illustrated their use through explicit formulation of the beam element. Extensive numerical comparisons are found in Reference 56.

Two areas in which hybrid formulations have proved effective have been in plate and shell bending analysis and in linear fracture mechanics. The bending formulations use the assumed stress hybrid approach in order to avoid the requirement of C^1 continuity displacement field. Pian summarizes work at M.I.T. in Ref. 40 while Henshall surveys work at Nottingham Univ. in Ref. 41. Other contributions have been made by Allmann[42], Chatterjee and Setlur[43], Dungar and Severn[44], Cook[23,45], Wolf[39], and Allwood and Cornes[46], Yoshida[59,60] among others. More recently, Pian, et. al.[53, 54] have given extensions of the assumed-stress hybrid concept to creep and plasticity. Other hybrid applications, to problems of linear fracture mechanics, are described in the notes devoted to that topic.

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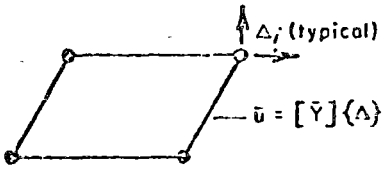
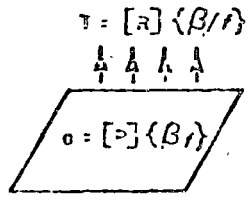


(a) Description of displacements (surface displacements expressed in terms of joint displacements $\{\Delta\}$



(b) Description of stresses (interior and surface stresses expressed in terms of generalized parameters $\{\beta_i\}$.

FIG. 1 ASSUMED STRESS HYBRID APPROACH.

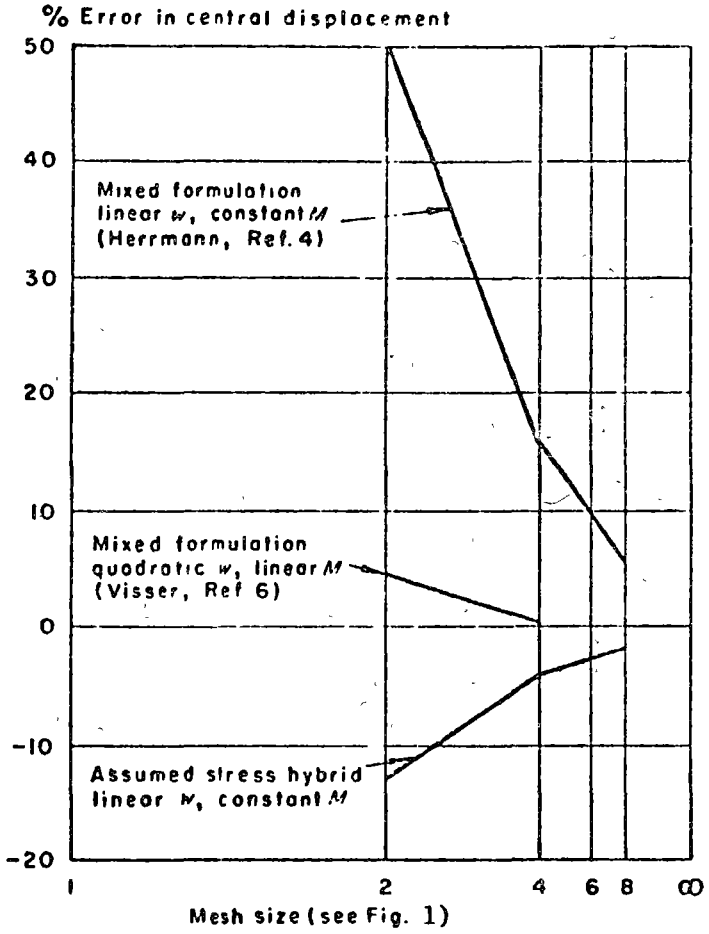
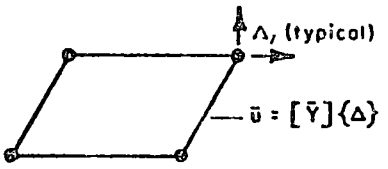
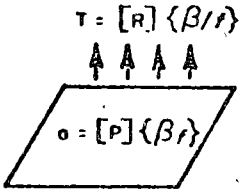

% Error in central displacement
50
40
30
20
10
0
-10
-20
Mixed formulation
linear w, constant M
(Herrmann, Ref. 4)
Mixed formulation
quadratic w, linear M
(Visser, Ref 6)
Assumed stress hybrid
linear w, constant M
1
2
4
6
8
∞
Mesh size (see Fig. 1)

FIG. 2

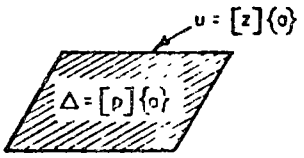


(a) Description of displacements (surface displacements expressed in terms of joint displacements $\{\Delta\}$

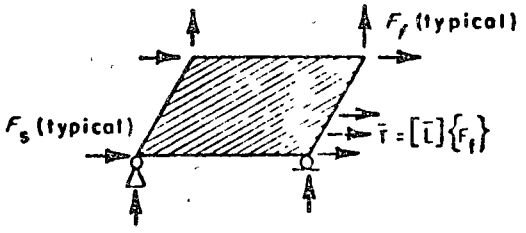


(b) Description of stresses (interior and surface stresses expressed in terms of generalized parameters $\{\beta_i\}$.

FIG. 3 ASSUMED STRESS HYBRID APPROACH.

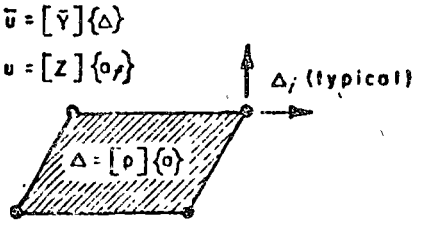


(a) Description of displacement (interior and surface displacements expressed in terms of same generalized parameters $\{\alpha\}$)

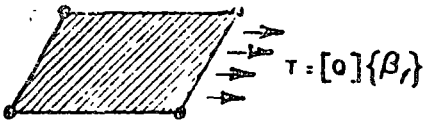


(b) Description of stress (surface forces expressed in terms of forces at joints free to displace)

FIG. 4 FIRST ASSUMED DISPLACEMENT HYBRID APPROACH.



(a) Description of displacements (interior and associated surface displacement (u) expressed in terms of generalized parameters $\{\alpha_f\}$. Prescribed surface displacements (u) expressed in terms of joint displacement $\{\Delta\}$).



(b) Description of stresses (surface forces expressed in terms of generalized parameters $\{\beta_f\}$).

FIG. 5 SECOND ASSUMED DISPLACEMENT HYBRID APPROACH.

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VISCOUS FLOWS

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VISCOUS FLOW

I. INTRODUCTION

In this chapter we are concerned with the problems of incompressible viscous flow and their discretization by the finite element process. We shall in the main concentrate our examples on creeping type problems in which dynamic terms can be neglected. The approach given here is however completely general and forms the basis of dealing with most incompressible fluid mechanics problems.

The approach to discretization taken here is via direct integral statements available from the use of virtual work principles in a manner similar to that widely practiced in solid mechanics. This has a double advantage over processes which start explicitly from the governing Navier-Stokes equations. In the first place tedious algebraic operations are avoided - in the second, a directly analogous treatment to that of solid mechanics becomes available and it is possible not only to gain a deeper insight into problems presented but also to use directly many of the available programs.

Much practical need exists for creeping flow solutions for non-Newtonian fluids in which the viscosity is strain rate dependent. All the formulations presented here are valid for such problems and indeed we shall demonstrate several applications of this, non-linear kind.

Of particular interest here are problems of metal plasticity when large deformation occurs and elastic strains are negligible. Such problems which stand on the borderline of fluid and solid mechanics occur frequently in all kinds of extrusion and forming processes and have much current interest.

II. BASIC FORMULATION FOR VISCOUS FLOW

By contrast to the solid mechanics problem where we are primarily concerned with static response and concentrate on a displacement of a material point - in fluid mechanics the main variable of interest is the velocity at a point in space. The Lagrangian description is thus predominant in solids while here we shall follow an Eulerian one. Nevertheless the similarity of general concepts is great and we shall therefore borrow heavily on the methodology used in solids (1).

Let u, v, w describe the three Cartesian components of the velocity $\underline{u}$ of a point x,y,z. Further let $\underline{b}$ denote the body forces per unit volume in part due to external causes ($\underline{b}_o$) and in part representing the dynamic acceleration effects. Thus,

$$\underline{b} = \underline{b}_o - \rho \underline{a} \qquad (1)$$

in which $\underline{a}$ stands for the acceleration of a point in the fluid and ρ is the density.

If equilibrium is considered between the internal stresses σ and body forces we have precisely the same equation set as in solid mechanics problems i.e.

$$L_1^T \sigma + b = 0 \tag{2}$$

or explicitly a set of equations

$$\frac{\partial}{\partial x}\sigma_x + \frac{\partial}{\partial y}\sigma_{xy} + \frac{\partial}{\partial z}\sigma_{xz} + b_x = 0 \tag{3}$$

etc.

with b_x being the x component of $\underline{b}$ etc.

A major difference from solid mechanics is due however to the Eulerian description of motion which even in steady state cases results in an acceleration $\underline{a}$.

Consider the acceleration component in direction x for a mass point which has a velocity $\underline{u}$ at a point of space x, y, z. Its velocity rate of change for a particle of fluid depends not only on rates of change of u with respect to time but also on the changes of position. Thus,

$$a_x = \frac{D}{Dt}u = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} \tag{4}$$

As $\frac{dx}{dt} = u$ etc.

We have a definition of the 'convective' acceleration operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \underline{u}^T\cdot\text{grad} \tag{5}$$

Here lies the major difference from solid mechanics where, as the displacements are referred to a particle and not to an element of space only the simple differentiation of displacement with respect to time suffices.

To complete the formulation of a solid mechanics problem we introduce a definition of strain in terms of displacements and a constitutive law defining a stress strain relation (1). For fluids we shall proceed similarly. First a rate of deformation, $\dot{\underline{\varepsilon}}$, is defined in terms of velocities. Thus we write

$$\dot{\underline{\varepsilon}} = \underline{L}\ \underline{u} \tag{6}$$

with $\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}$ etc. defining the operator $\underline{L}$.

If tensorial representation is preferred the equivalent definition is

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad i = 1 - 3 \tag{6a}$$

The constitutive relationship for fluids is more complex than i the solid mechanics problems, as in general stresses depend not only on rates of strain but also on the strain itself. We shall therefore restrict our attention here to incompressible flow for which the rate of volumetric straining is zero, i. e.

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with $\underline{M}^T = [1,1,1,0,0,0]$; we write

$$\dot{\varepsilon}_v = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = \dot{\varepsilon}_{ii} = \text{div}\,\underline{u} = \underline{M}^T \dot{\underline{\varepsilon}} = 0 \qquad (7)$$

for such fluids the mean stress is not defined and has to be sought from equilibrium relations. Defining the mean stress σ, or the pressure p, as

$$\sigma = -p = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \qquad (8)$$

We define the deviatoric portion of the stress, $\underline{S}$, as a function of strain rate $\dot{\underline{\varepsilon}}$ in a matrix notation as

$$\underline{S} = \underline{\sigma} - \begin{Bmatrix}1\\1\\1\\0\\0\\0\end{Bmatrix}\sigma = \underline{\sigma} + \underline{1}p = \underline{R} - \underline{f}(\dot{\underline{\varepsilon}}) \qquad (9)$$

with

$$\underline{E} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ & & \frac{2}{3} & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{bmatrix}$$

(SYM)

For a linear fluid we can write relation (9) as

$$\underline{S} = \underline{D}\dot{\underline{\varepsilon}} \qquad (11)$$

Where $\underline{D}$ is a matrix of constants. In tensorial equivalent we can rewrite above as

$$\dot{s}_{ij} = \dot{\varepsilon}_{ij} - \frac{1}{3}\dot{\varepsilon}_{ii}\delta_{ij} \qquad D_{ijkl}\dot{\varepsilon}_{kl} \qquad (10)$$

This is entirely analogous to the definition of behavior of elastic solids which are incompressible with $\underline{D}$ playing the role of the matrix of elastic constants. For isotropic linear behavior it can be readily shown that we can write

$$\underline{D} = \mu \begin{bmatrix} 2 & & & & & 0 \\ & 2 & & & & \\ & & 2 & & & \\ & & & 1 & & \\ & & & & 1 & \\ 0 & & & & & 1 \end{bmatrix} = \mu \underline{D}^0 \text{ OR } D_{ijkl} = 2\mu \qquad (11)$$

in which μ is known as viscosity and $\underline{D}^0$ is a diagonal matrix. In general μ will be a function of $\dot{\varepsilon}$ and the formulation that follows is applicable in this form to Non-Newtonian (non-linear) fluids.

The viscous flow problem is now fully defined and we can formulate its approximate solution mathematically proceeding formally by Galerkin or other weighting method. However, an examination of the equations governing the flow and of the boundary conditions, which specify either tractions or velocities at all external boundaries, permits us to adapt here the procedures used in solid mechanics. In particular the 'virtual work principle' can be applied with virtual velocities playing the part of virtual displacements used in solid mechanics.

Indeed one can conclude that the solution of a viscous flow problem is identical to the solution of an equivalent incompressible elastic problem in which displacements are replaced by velocities and the body forces described by equation 1 are inserted.

Thus, all the techniques available in the literature for the solution of one problem are available to the other -- applicability being direct if the fluid flow is sufficiently slow so that acceleration effects can be disregarded.

III. VISCOUS FLOW - VELOCITY AS THE BASIC UNKNOWN

1. Virtual Work Statements

The equilibrium statement 3 can be replaced by an equivalent virtual work statement requiring that for any virtual velocity and strain rate changes, $\delta\underline{u}$ and $\delta\underline{\dot{\varepsilon}}$, which are compatible, external and internal rates of work are identical. Thus, we can write

$$\int_{\Omega} \delta\underline{\dot{\varepsilon}}^{T}\, \underline{\sigma}\, d\Omega - \int_{\Omega} \delta\underline{u}^{T}\, \underline{b}\, d\Omega - \int_{\Gamma_t} \delta\underline{u}^{T}\, \underline{\bar{t}}\, d\Gamma = 0 \qquad (12)$$

for any flow domain Ω in which tractions $\bar{t}$ are specified on boundary Γ_t and where $\delta\underline{u}$ is zero on boundary Γ_u where velocities are given. Further, for compatibility

$$\delta\underline{\dot{\varepsilon}} = \underline{L}\,\delta\,\underline{u} \quad \text{and} \quad \delta\underline{u} = 0 \text{ on } \Gamma_u \qquad (13)$$

As $\underline{\sigma}$ is not uniquely defined by $\underline{\dot{\varepsilon}}$ (being indeterminate due to the undefined pressure p), an additional equation needs to be written to enforce the incompressibility. As for any pressure variation δp internal work is zero due to incompressibility we can write

$$\int_{\Omega} \delta p\, \dot{\varepsilon}_v\, d\Omega = 0 \qquad (14)$$

Inserting the constitutive relation 9 to 11 we can rewrite equation 12 as

$$\int_\Omega \delta\dot{\underline{\varepsilon}}^T \underline{D}'' \dot{\underline{\varepsilon}}\, d\Omega + \int_\Omega \delta p\, \dot{\varepsilon}_v\, d\Omega - \int_\Omega \delta\underline{u}^T \underline{b}\, d\Omega \qquad (15)$$

$$- \int_{\Gamma_t} \delta\underline{u}^T \dot{\underline{t}}\, d\Gamma = 0$$

Observing that by 7

$$\dot{\varepsilon}_v = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = \underline{m}^T \dot{\underline{\varepsilon}} \qquad (15a)$$

equations 15 and 14 lead directly to a discretization. We have now several choices open. First, we can allow an unrestricted definition of the fields of $\underline{u}$ and p and proceed using both equations. Second, we can confine our attention to velocity fields which automatically satisfy incompressibility. In the latter case the equation 14 as well as the second term of equation 15 disappear and the full approximation involves only velocity parameters. We shall explore both formulations in turn.

The reader could at this stage with profit rederive the approximation equations using a straightforward Galerkin procedure, and as usual will observe the integration and interpretation difficulties present.

If p and $\underline{b}$ are for the moment treated as functions of position only it will be noted that equations 14 and 15 are in fact equivalent to a variational principle of making stationary a functional

$$\Pi = \int_\Omega \frac{1}{2} \dot{\underline{\varepsilon}}^T D\, \dot{\underline{\varepsilon}}\, d\Omega - \int_\Omega \underline{u}^T \underline{b}\, d\Omega - \int_{\Gamma_t} \underline{u}^T \underline{t}\, d\Gamma \qquad (16)$$

with respect to variations of u and subject to the constraint

$$\int_\Omega \dot{\varepsilon}_v = \int_\Omega \underline{m}^T \dot{\underline{\varepsilon}} = 0 \qquad (16a)$$

The pressure p, in equation 14 plays here the role of a Lagrangian multiplier introduced to satisfy the constraint.

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We make this remark not only because the reader will observe (in the next section that the characteristic structure of the discretized equations will contain the usual drawbacks of Langrange multiplier forms with zero diagonal terms in the matrices), but also because other way exist of enforcing the constraint 16a.

4. Discretization With Velocity and Pressure Fields

In the usual manner we describe the displacement and pressure fields by trial functions as

$$\underline{u} = \sum \underline{N}_i^u \underline{a}_i^u = \underline{N}^u \underline{a}^u$$

$$p = \sum N_i^p a_i^p = \underline{N}^p \underline{a}^p \tag{17}$$

in which $\underline{N}^u$ and $\underline{N}^p$ are appropriate shape functions, defined element by element. It will be observed from the nature of integrals involved that we require C^0 continuity for the velocity field but discontinuous functions can be used to describe the pressure field.

Writing

$$\delta\underline{\varepsilon} = \underline{L}\,\delta\underline{u} = (\underline{L}\,\underline{N}^u)\,\delta\underline{a}^u;$$

$$\delta\varepsilon_v = \underline{M}^T \delta\underline{\varepsilon} = \underline{N}^T \underline{L}\,\underline{N}^u\,\delta\underline{a}^u \tag{18}$$

$$\delta_p = \underline{N}^p\,\underline{a}^p$$

and observing that 14 and 15 are true for all variations $\delta\underline{a}$ we have from equation 15

$$\left[\int_\Omega (\underline{L}\,\underline{N}^u)^T\, \underline{D}^e\, (\underline{L}\,\underline{N}^u)\, d\Omega\right]\underline{a}^u + \left[\int_\Omega (\underline{N}^T\,\underline{L}\,\underline{N}^u)^T\,\underline{N}^p\, d\Omega\right]\underline{a}^p$$

$$- \int_\Omega \underline{N}^{u\,T}\,\underline{b}\, d\Omega - \int_{\Gamma_t} \underline{N}^{u\,T}\,\bar{\underline{t}}\, d\Gamma = 0 \tag{19}$$

and from equation 14

$$\left[\int_\Omega \underline{N}^{pT}\,\underline{M}^T\,\underline{L}\,\underline{N}^u\,d\Omega\right]\underline{a}^u = 0 \tag{20}$$

For cases of slow steady state viscous flow where $\underline{b} = \underline{b}_0$ this results in a simple symmetric set of equations which can be written as:

$$\begin{bmatrix} \underline{K}^u & \underline{K}^p \\ \underline{K}^{pT} & 0 \end{bmatrix} \begin{Bmatrix} \underline{a}^u \\ a^p \end{Bmatrix} + \begin{Bmatrix} \underline{f}^u \\ 0 \end{Bmatrix} = 0 \tag{21}$$

where

$$\underline{K}^u_{ij} = \int_\Omega (\underline{L}\,\underline{N}_i^u)^T\,\mu\,\underline{D}_0\,(\underline{L}\,\underline{N}_j^u)\,d\Omega \tag{22}$$

$$\underline{K}^p_{ij} = \int_\Omega (\underline{M}^T\,\underline{L}\,\underline{N}_i^u)^T\,\underline{N}_j^p\,d\Omega$$

$$\underline{f}_i^u = \int_\Omega \underline{N}_i^{u\,T}\,\underline{b}_0\,d\Omega - \int_{\Gamma_t} \underline{\bar{N}}_i^{u\,T}\,\underline{\bar{t}}\,d\Gamma$$

Indeed this formulation is almost identical to that used in linear, incompressible elasticity and which has been derived by imposing constraints on an energy functional (2). If μ is velocity independent 21 represents a simple linear equation system but the formulation is generally valid.

When considering unsteady state the coefficients $\underline{a}$ are time dependent.

Returning to equation 1 and the explicit expression for the acceleration of equation 5, we note that we can write

$$\underline{b} = \underline{b}_0 + \rho \frac{\partial \underline{u}}{\partial t} + \rho \underline{J}\, \underline{u} \tag{23}$$

where

$$\underline{J} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

The term $\underline{f}^u$ of equation 21 now has, in addition to the contribution given by constant body forces and boundary traction, f_0, a form

$$\underline{f}^u = \left[\int_\Omega \underline{N}^{uT} \rho\, \underline{N}^u \, d\Omega\right] \frac{d\underline{a}^u}{dt} + \left[\int_\Omega \underline{N}^{uT}\, \underline{J}\, \underline{N}^u \, d\Omega\right] \underline{a}^u \tag{24}$$

$$= \underline{M}^u \frac{d\underline{a}^u}{dt} + \underline{K}^u \underline{a}^u$$

The discretized equation 21 becomes now

$$\begin{bmatrix} (\bar{K}^{u} + \bar{K}^{u}) & K^{p} \\ K^{pT} & 0 \end{bmatrix} \begin{Bmatrix} a^{u} \\ a^{p} \end{Bmatrix} + \begin{bmatrix} M^{u} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} a^{u} \\ a^{p} \end{Bmatrix} + \begin{Bmatrix} f_{u} \\ 0 \end{Bmatrix} = 0 \qquad (25)$$

In the above the transient case can be solved by time stepping procedures but even in steady state the form matrix equations obtained by omitting the second term is non-linear and non-symmetric.

The matrix $\bar{K}^{u}$ depends on the current velocity and its form is non-symmetric. This presents difficulties in solving viscous flow problems with appreciable inertia and several alternative procedures have been used [2,3]. In some, the non-linear and non-symmetric matrix $\bar{K}^{u}$ is taken care of by repeated iteration using only the symmetric part. In other, attempts at a direct solution of the non-linear equations have been made using a non-symmetric solution scheme.

3. Discretization Using Incompressible Velocity Fields

The most usual procedure of describing incompressible velocities is by the use of stream function in two-dimensional problems or by introduction of a vector potential in three dimensions.

Thus, if we confine our attention to plane flow with u and v velocity components in x and y directions, we can write

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{OR } \underline{u} = \hat{\underline{L}}\,\psi \qquad (26)$$

It is easily verified that

$$\dot{\varepsilon}_{v} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (27)$$

In an axisymmetric flow similarly we can write for radial and axial velocity components

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial y}$$

$$v = \frac{1}{r}\frac{\partial \psi}{\partial x} \tag{28}$$

and once again incompressibility is obtained.

Finally, in three-dimensional flow we define the velocity in terms of a vector potential with three components

$$\underline{\psi}^T = [\psi_x, \psi_y, \psi_z] \tag{29}$$

as

$$\underline{u} = \text{curl } \underline{\psi} = \hat{\underline{L}}\,\underline{\psi} \tag{30}$$

where

$$\hat{\underline{L}} = \begin{bmatrix} -\frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

Again it is easily verified that incompressibility is satisfied as

$$\varepsilon_v = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{31}$$

(as $\varepsilon_v = \text{div } \underline{u}$ and div curl $\underline{\psi} = 0$).

With velocities specified on the boundaries it is possible* to determine the stream function and its normal gradient (or vector potential components) there to within an arbitrary constant. For discretization therefore we can assume an expansion for ψ

$$\underline{\psi} = \underline{N}_i \underline{a}_i = \underline{N}\,\underline{a} \tag{32}$$

and use the virtual work expression 15 with the second term dropped, (as it now is identically zero). i.e.

$$\int_\Omega \delta\underline{\varepsilon}^T \underline{D}_o \underline{\varepsilon}\, d\Omega - \int_\Omega \delta\underline{u}^T \underline{b}\, d\Omega - \int_{\Gamma_t} \delta\underline{u}^T \bar{\underline{t}}\, d\Gamma = 0 \tag{33}$$

Writing for all cases considered above

$$\underline{u} = \hat{\underline{L}}\,\underline{\psi} = \hat{\underline{L}}\,\underline{N}\,\underline{a}\;;\quad \underline{\varepsilon} = \underline{L}\,\underline{u} = \underline{L}\,\hat{\underline{L}}\,\underline{N}\,\underline{a} \tag{34}$$

with corresponding variations, we can immediately obtain the discretized form of equations for which $\underline{a}$ can be obtained as (as usual taking $\delta\underline{a}^T$ outside in 33 after substitution and equating the multiplier to zero).

$$\left[\int_\Omega (\underline{L}\,\hat{\underline{L}}\,\underline{N})^T \underline{D}_o\, (\underline{L}\,\hat{\underline{L}}\,\underline{N}) d\Omega\right] \underline{a} - \int_\Omega (\underline{L}\,\underline{N})^T \underline{b}\, d\Omega$$

$$- \int_{\Gamma_t} (\hat{\underline{L}}\,\underline{N})^T \bar{\underline{t}}\, d\Gamma = 0 \tag{35}$$

or the usual form

$$\underline{K}\,\underline{a} - \underline{f} = 0 \tag{36}$$

---

*If boundaries are simply connected

with expressions for K_{ij} and f_i apparent from 35.

Immediately an observation can be made that the shape functions $\underline{N}$ now require C^1 continuity as second derivatives operate on these in the integrals. While in two-dimensional problems the use of such functions presents little difficulty, in three-dimensions no satisfactory piecewise defined functions are available.

Confining our attention to the scalar stream function ψ defined for axisymmetric or plane problems it is readily seen that the same shape functions as used for plate bending analysis are available. It is therefore possible to use any of the numerous plate functions for solution of viscous flow.

Indeed for the linear case of slow viscous flow equations the whole formulation may be identified with plate bending equation and any standard plate bending program adapted immediately.

While in the first type of formulation (Sec. III-2) we have 'borrowed' heavily from previous methods used extensively in the solution of incompressible solids, in the second type (Sec. III-3) we have introduced standard fluid mechanics concepts for enforcing incompressibility. These appear not to have been used widely in solid mechanics and a "reverse borrowing" is obviously possible. The stream function concept can be used directly in solid mechanics and only recently has such a development been put into practice[5].

Once again inclusion of the dynamic term can be made pursuing the process of modifying the $\underline{f}$ term, as described in the previous section.

The use of stream function introduces several drawbacks into many problems. Unless the velocities are entirely prescribed on all boundaries, it is often impossible to establish, a priori, the values of stream functions on some positions of the boundary. This is particularly serious in multiple connected boundaries as are presented by flows around obstacles, etc., and to overcome the difficulties it is necessary to introduce additional constraints on the rate of boundary work. This, even in linear problems presents serious difficulties and when formulating general problems considerable thought should be given to the nature of the boundary conditions. In reference (5) and (6) we discuss this matter in detail.

IV. VISCOUS FLOW - EQUILIBRIUM AND MIXED FORMULATION

The need for enforcing incompressibility has presented some difficulties in the velocity type of formulation used in the previous section, this arising because the stresses are not completely defined by the strain rates (vide equation 9). On the other hand stresses define uniquely the strain rates and it is obviously possible to use the equivalents of equilibrium virtual work statements or of the "mixed" formulations well-known in solid mechanics with advantage. Possibilities here are enormous and have only been barely explored. We shall here restrict ourselves to a brief statement of the equilibrium formulation.

If the unknown function is the stress field $\underline{\sigma}$ which is so chosen as to satisfy exactly the equilibrium conditions then the virtual work done by 'compatible' strain rates must be zero

$$\int_{\Omega} \delta \underline{\sigma}^T \dot{\underline{\varepsilon}} \, d\Omega - \int_{\Gamma_u} \delta \underline{t}^T \bar{\underline{u}} \, d\Gamma = 0 \tag{37}$$

in which Γ_u is now the portion of the boundary on which velocities $\underline{u} = \bar{\underline{u}}$ are specified, and δt are the boundary tractions resulting from stresses $\delta\underline{\sigma}$, i.e.,

$$\delta \underline{t} = \underline{G} \, \delta \underline{\sigma} \tag{38}$$

where $\underline{G}$ is a suitable matrix of direction cosines of the normal to the surface. From equation 9-11 the strain rates are defined in terms of stress as

$$\underline{R} \, \underline{\sigma} = \mu \, \underline{D}_o \, \dot{\underline{\varepsilon}} \tag{39}$$

or

$$\dot{\underline{\varepsilon}} = \frac{1}{\mu} (\underline{D}_o^{-1} \, \underline{R}) \, \underline{\sigma} = \frac{1}{\mu} \underline{C}_o \, \underline{\sigma} \tag{40}$$

in which

$$\underline{C}_o = \begin{vmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 \\ & \frac{1}{3} & -\frac{1}{6} & 0 & 0 & 0 \\ & & \frac{1}{3} & 0 & 0 & 0 \\ \text{SYM} & & & 1 & 0 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{vmatrix}$$

To achieve an equilibrating field, stresses can be defined in terms of a stress function set ϕ as

$$\underline{\sigma} = \underline{E}\,\phi + \underline{\Omega} \tag{41}$$

Where $\underline{\Omega}$ is a particular solution which equibrates the applied body forces and $\underline{\phi}$ is so constrained as to satisfy prescribed boundary tractions.

Use of stress function in context of solid mechanics has been pioneered by Fraeijs de Veubeke and Zienkiewicz (7) with a subsequent development by Sander (8) which in a sense is an application of certain special mixed formulations. Details of the procedure are discussed elsewhere but some of the difficulties in the fluid mechanics context should now be noted.

First, if the flow is not so slow that dynamic terms can be neglected these will appear in the inhomogeneous terms defining the stress field $\underline{\Omega}$ in terms of velocities. As velocities can only be obtained by integration of the stress field similar difficulties will arise as at those encountered in dynamic of solids when equilibrating forms are used. These difficulties can be overcome as shown by Tabarrok (9) but for practical application the equilibrating formulation appears only simple in cases of slow viscous flow.

Further, in three-dimensional problems the use of the stress function necessitates again C^1 continuity to be introduced in appropriate expansions with near impossibility of achieving this in practice. Two-dimensional use of the Airy stress function is however practical and useful.

Possibilities of using a "mixed" formulation in which both stress and velocity field simultaneously appear has not yet been explored and presents a fruitful field for research.

V. SOME OTHER SOLUTION POSSIBILITIES

In preceding we have used simple virtual work or which is equivalent, Galerkin formulation. (Virtual work avoiding the difficulties of integration by parts and giving a direct physical interpretation of the various terms). Other forms of discretization are obviously possible and have been used in practice.

In some, following classical procedures of fluid dynamics, the governing equations are rewritten in terms of both stream function and vorticity. Direct approximation can then be used with Galerkin or other processes. Alternatives with the use of least square principle are possible, and can be applied directly to the equations in terms of all the variables.

A simple direct possibility can be used if we consider the analogy of fluid flow and incompressible elastic formulation. In the latter, to sidestep the difficulties involved due to incompressibility in a direct displacement formulation, the engineers have often used standard displacement programs with near incompressibility imposed. In terms of elastic constants this is equivalent to using a high Poissons ratio (say 0.49) in place of 0.5 for which a singularity arises. For simple finite element representation this usually leads to inaccurate results but with isoparametric parabolic elements good accuracy can be found with Poissons ratio as high as 0.49995 -- especially if "reduced integration" of 2x2 Gauss points is used.

This procedure is particularly simple as existing finite element programs can be used.

We show here that this formulation is equivalent to the use of the constrained variational principle of equation 16/16a and introduction of this constraint via a penalty function approach (10).

In this approach we replace the functional of equation 15 by one embodying the square of the constraint multiplied by a 'penalty' positive large number α, and seek its stationarity. Thus, we start with a functional

$$\Pi = \int_{\Omega} \frac{1}{2} \underline{\varepsilon}^T \underline{D}^0 \underline{\varepsilon} \, d\Omega - \int_{\Omega} \underline{u}^T \underline{b} \, d\Omega - \int_{\Gamma} \underline{u}^T \bar{\underline{t}} \, d\Gamma + \alpha \int_{\Omega} \dot{\varepsilon}_v^2 \, d\Omega \qquad (42)$$

Inserting expression (7) i.e.

$$\dot{\varepsilon}_v = \underline{m}^T \dot{\underline{\varepsilon}} \qquad (7)$$

we note that the new functional can be written as

$$\Pi = \int_{\Omega} \frac{1}{2} \dot{\underline{\varepsilon}}^T \bar{\underline{D}} \dot{\underline{\varepsilon}} \, d\Omega - \int_{\Omega} \underline{u}^T \underline{b} \, d\Omega - \int_{\Gamma_t} \underline{u}^T \bar{\underline{t}} \, d\Gamma \qquad (43)$$

with

$$\bar{\underline{D}} = \underline{D}^0 + 2\alpha \underline{m} \underline{m}^T \qquad (44)$$

Comparison of (42) with the solid mechanics problem immediately identifies the analogy of 2α and the bulk modulus of the solid. However, without any such physical reasoning we immediately note how a discretization can be directly achieved for the incompressible flow problem. Use of this procedure with isoparametric elements will be indicated in the next section.

VI. SOME ILLUSTRATIVE LINEAR-CREEPING FLOW APPLICATIONS

1. Entry Flow - Two and Three Dimensions

One of the first finite element solutions to such problems was achieved by Atkinson et al. with a stream function formulation[11].

In this situation the velocities have been assumed entirely specified on the external boundary and no prescribed tractions introduced. The singly connected boundary allows the stream function and its normal gradient to be determined from the velocity definitions (equations 26 and 28) and no problems arise (an arbitrary constant in the value of ψ is obtained by specifying this at some point of the boundary).

In Figure 1 we show a simple solution recently obtained by two alternative procedures but now using different boundary conditions. In the first a stream function approximation is used with a rectangular, compatible element of four degrees of freedom at each corner (introduced by Bogner et al.[12]). In the second the simple penalty function approach is used as described in the previous section and the variables are directly the velocities. Here a standard solid mechanics program (1) using parabolic isoparametric elements was employed with an effective Poisson's ratio of 0.49995 and 2x2 Gauss rule. Both element subdivisions are noted in Figure 1 and velocity distributions are virtually not distinguishable. For comparison results obtained by Lew and Fung[13] are given.

To avoid difficulties introduced at the entry section where a velocity singularity is present at (A), Figure 1, we have assumed a rapid but continuous transition of velocity near that point. Nevertheless, this near singularity introduces a considerable distrubance to the solution and quite fine subdivisions are needed for reasonable accuracy. In Figure 2 we show the effect of this subdivision on velocity development for a plane entry region. Here once again a standard displacement type program is used.

In a three-dimensional context as already indicated the use of stream function becomes impracticable. We show in the example of Figure 3, an entry velocity distribution in a rectangular conduit using again a penalty function approach, or indirectly a standard three-dimensional solid analysis program with $\nu = 0.49995$ and a 2x2x2 Gauss integration.

The reader should note again similar errors to those shown in Figure 2 when a coarse mesh is used. In Figure 4 and Figure 5 we show further properties of the velocity distribution. No alternative creeping flow solution appears to be available and for comparison results of a higher Reynolds number investigation by Carlson et al.[14,15] are shown.

2. Flow Past Asymmetrically and Non-Symmetrically Placed Obstructions in Parallel Flow

This example is illustrated in Figure 6 in which once again both approaches used in the previous example have been employed. In Figure 6ab we give the velocity (streamline pattern obtained) while in Figure 7c shear

stress and pressure gradients are compared. The asymmetric placing of the obstacle necessitates in the stream function approach a special treatment discussed in reference 6 in which two solutions for arbitrary values of stream function assumed at inner boundary are superimposed. In the u/v penalty function formulation, no such difficulty is present and direct solution is obtained from both symmetric placing of obstacle (not illustrated) and unsymmetric situations.

3. Use of Triangular Versus Rectangular Elements in Stream Function Formulation

In the previous examples we have used simple rectangular elements for the stream function formulation as with these it is simple to satisfy C^1 continuity using polynomial expressions[1][12]. The problem od devising arbitrary triangular or quadrilateral elements with such a continuity is much more complex and in the text (1) the various difficulties and their solution are discussed. One of the most satisfactory triangular elements produced is one based on the original conforming triangle of Bazeley et al.[16] using substitute smoothed (least square fit) shape function of cubic form[17][18].

To test the efficiency of such elements a simple case of Poisseuille flow between parallel plates is considered (Figure 8). On two sections AA and BB the velocity distribution is prescribed and the total pressure drop is compared with the exact solution. The rectangular elements for all subdivisions gave exact answers being totally conforming and having a complete cubic expansion available. With triangular elements an error arises and its decrease with finess of subdivision is given in Figure 8. The convergence is approximately of order h^3.

Numerous direct solutions in terms of velocity and pressures are given for creeping flow problems in reference 3.

VII. NON-NEWTONIAN FLOW

In non Newtonian fluids the viscosity μ depends in same manner as the rate of straining $\dot{\underline{\varepsilon}}$.

$$\mu = \mu(\dot{\underline{\varepsilon}}) \tag{45}$$

Here using the formulations of section II to V we have (in the absence of dynamic terms) a discretized system of equations in the form

$$\underline{K}(u)\,\underline{a} - \underline{f} = 0 \tag{46}$$

The matrix $\underline{K}$ is dependent on $\dot{\underline{\varepsilon}}$ and hence on $\underline{a}$. A simple interative procedure can now be adopted, and has been shown to converge quite rapidly even with substantially non-linear behavior.

Assuming some value of $\mu = \mu^1$ the first solution is obtained

$$\underline{a}^1 = \underline{K}_1^{-1}\,\underline{f} \tag{47}$$

and hence $\dot{\underline{\varepsilon}}$ and new value of a^1 at all points of the region is available to compute $\underline{K}_1$. The next approximation is

$$\underline{a}^2 = \underline{K}_1^{-1}\,\underline{f} \text{ etc.} \tag{48}$$

leading to a standard iterative algorithm

$$\underline{a}^n = \underline{K}_{n-1}^{-1}\,\underline{f} \tag{49}$$

It is usual to assume that viscosity is simply a function of the second strain rate invariant $\dot{\bar{\varepsilon}}$ i.e.,

$$\dot{\bar{\varepsilon}} = 2\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij} \tag{50}$$

$$= 2\dot{\varepsilon}_{xx}^2 + 2\dot{\varepsilon}_{yy}^2 + 2\dot{\varepsilon}_{zz}^2 + (2\dot{\varepsilon}_{xy})^2 + (2\dot{\varepsilon}_{yz})^2 + (2\dot{\varepsilon}_{zx})^2$$

A frequently used expression is of a form

$$\mu = \mu_0\,(\dot{\bar{\varepsilon}})^{n-1} \tag{51}$$

with index $n < 1$.

Some solutions for a fluid of this type are obtained by Palit and Fenner[19] using stream function formulation and 'incompatible' triangular elements.

In Figure 9 solutions for two-dimensional flow around a cylinder with n=1 and n=0.5 are compared for an assumed uniform entry flow velocity distribution at AA. The study illustrates well the local effects of non-linearity on velocity\(stream lines) changes.

In this problem once again a stream function formulation is employed, and triangular elements with nine degrees of freedom are used to deal with the general boundary shape.

VIII. PLASTIC OR VISCOPLASTIC BEHAVIOR OF EXTRUDED METALS - A CASE OF NON-NEWTONIAN FLOW

A particularly interesting case of non-Newtonian creeping flow is that of a Bingham fluid or its generalization, the visco-plastic material.

Such materials behave as solids exhibiting a zero rate of straining for stresses which, are in some measure, below a threshhold or yield value. When this yield is exceeded flow begins at a rate which is a function of the excess stress.

Let $F(\underline{\sigma}) = 0$ represent this yield condition and we shall assume therefore this if $F(\underline{\sigma}) < 0$ no flow occurs. Assuming further that the various components of strain rate are proportional to the gradients of F with respect to these (i.e. the so-called associated plasticity condition) we can describe with some generality the behavior of the material by writing

$$\dot{\underline{\varepsilon}} = \frac{1}{\underline{\mu}} \langle F^n \rangle \frac{\partial F}{\partial \underline{\sigma}} \tag{52}$$

or in tensorial form

$$\dot{\varepsilon}_{ij} = \frac{1}{\underline{\mu}} \langle F^n \rangle \frac{\partial F}{\partial \underline{\sigma}_{ij}} \tag{52}$$

where $\langle \cdot \rangle$ means that

$$\begin{aligned} \langle F \rangle &= 0 \quad \text{if} \quad F < 0 \\ \langle F \rangle &= F \quad \text{if} \quad F > 0 \end{aligned} \tag{53}$$

and μ is some "viscosity" parameter.

It appears that we have here once again a case of viscous flow with a variable viscosity dependent now on the current stresses. It is however possible to reduce the problem to that discussed in the previous section where viscosity is a function of the strain rate (as in equation 44).

To do this we shall find it convenient to use a tensorial notation and use the concept of deviatoric stress. Thus, equation 10 defining viscosity

$$\underline{S} = \mu \underline{D} \varepsilon \tag{54a}$$

becomes in tensor notation

$$S_{ij} = \sigma_{ij} - \delta_{ij} \sigma_{ij} = 2\mu\varepsilon_{ij} \tag{54b}$$

Taking the yield conditions defined simply by the second invariant of S_{ij} - i.e. the so-called von Mises yield criterion

$$F = \sqrt{\frac{1}{2} S_{ij} S_{ij}} - \frac{1}{\sqrt{3}} Y \tag{55}$$

in which Y is the yield stress in simple tension, we find that

$$\frac{\partial F}{\partial \sigma_{ij}} = \frac{\partial F}{\partial S_{ij}} = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{2} S_{ij} S_{ij}}} S_{ij} \tag{56}$$

and that we can write equation 52b as

$$\dot{\varepsilon}_{ij} = \frac{1}{2\mu} \left\langle \sqrt{\frac{1}{2} S_{ij} S_{ij}} - \frac{1}{\sqrt{3}} Y \right\rangle^n \frac{1}{\sqrt{\frac{1}{2} S_{ij} S_{ij}}} S_{ij} \tag{57}$$

Comparison with the definition of viscosity in equation 54b gives

$$\frac{1}{\mu} = \frac{1}{\bar{\mu}} \frac{\langle \sqrt{\frac{1}{2} S_{ij} S_{ij}} - Y \rangle^n}{\sqrt{\frac{1}{2} S_{ij} S_{ij}}} \tag{58}$$

but we note that by 54b

$$\sqrt{\frac{1}{2} S_{ij} S_{ij}} = \mu \sqrt{2 \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}} = \mu \dot{\bar{\varepsilon}} \tag{59}$$

Inserting this in equation 58 we see that μ can be obtained in terms of $\dot{\bar{\varepsilon}}$. For n=1 it can easily be verified that

$$\mu = \frac{1}{\sqrt{3}} \frac{Y + \bar{\mu} \dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}} \tag{60}$$

and the general form identical with that discussed in previous section is obtained (equation 45).

Computationally the expression 60 appears to present a difficulty as with $\mu \to \infty$ $\dot{\bar{\varepsilon}} \to 0$. However, this problem is readily overcome by limiting the upper value of μ to some large number.

The viscoplastic model used is one of a class suggested by Perzyna (20) and is of quite wide applicability (21). It is interesting to note that as the coefficient $\bar{\mu} \to 0$ the viscoplastic and plasticity formulation become identical. The solution procedure is therefore applicable to problems of both plastic and viscoplastic kind.

The solution of such plastic and visco-plastic problems is of great importance in the working (forming and extrusion) processes of metals and plastics and treatment proposed here is capable of providing simply such solutions. In another paper (reference 6) the authors outline some applications

and illustrations of this process. Figures 10 and 12 show a solution to a simple plastic extrusion process through a die with 50% reduction. With $\bar{\mu} = 0$ assumed the stress distribution and hence the forces required to sustain extrusion are independent of velocity of the piston U, and good comparison is obtained with results available for this case from simple slip line solutions. For different values of $\bar{\mu}$ both the piston force and the velocity pattern depend strongly on the value of U. This dependence is shown in Figure 1.

IX. QUASI-STATIC TRANSIENT SOLUTIONS FREE SURFACE PROBLEMS

If the process of viscous flow is sufficiently slow for dynamic effects to be ignored we may still be faced with a transient (time variable) problem. If at a particular configuration with a boundary Γ_t on which tractions are prescribed resulting velocities are not found to be entirely parallel to the surface, then clearly the configuration of the problem will change with time. A free surface is a particular example of such a problem which frequently arises in slow viscous processes such as indentation etc.

The treatment of such problems is relatively simple. Once the velocities of Γ_t are found in the manner previously described, by a time stepping extrapolation, the free surface change can be predicted. Using a straightforward Eulerian prediction the change of position of a boundary in a direction $\underline{S}$ in a time increment can be predicted as

$$\Delta S = V_s \Delta t$$

where V_s represents the corresponding velocity component.

Once the new position of the boundary is known a new solution can be readily obtained and the process continued.

In Figure 14 we show successive stages of an indentation process (Figure 13) in an ideally plastic material solved using triangular elements and a stream function formulation. In Figure 15 the variations of indenting forces with depth of penetration is indicated. Again here the solution is independent of velocity of indentation as $\bar{\mu} = o$ was assumed. For true viscoplastic materials the solution could be obtained equally easily and would show (as in the previous example) a velocity dependence.

An iteration of the type

$$\underline{a}^n = \underline{K}_o^{-1} \underline{f}(\underline{a}^{n-1}) \tag{63}$$

is effective however only at very low velocities and at higher ones does not converge. Alternatively we use the non-linear form

$$\underline{K}\,\underline{a} + \underline{f}_o = 0 \tag{64}$$

in which

$$\underline{K} = \underline{K}(\underline{a}) \tag{65}$$

and iterate as

$$\underline{a}^n = \underline{K}(\underline{a}^{n-1})\ \underline{f}_o \tag{66}$$

This process has proved effective for quite high velocities (Reynolds) numbers) but is obviously more costly requiring repeated inversion of a non-symmetric matrix which has to be recalculated at each stage.

Formulations used so far in this problem include both the approaches outlined in previous sections, i.e. the use of velocities and pressures as unknown or the use of stream function (in two-dimensional situations).

Reference 3 shows some problems solved using the first type of formulation and an interpolation of C_o continuity using isoparametric, parabolic elements.

In Non-Newtonian situations it is likely that relatively small effects of inertia will only be present. For such situations a combination of interation of the type given by equations 61-63 combined with incorporation of changes in $\underline{K}_o$, as in equation 47-49 proves effective and indeed can be accomplished at little more cost than either non-linearity treated independently. Examples of such application will be published elsewhere.

The process of adjusting the free surface profile in successive time steps resembles that used previously for transient seepage solutions. .22), (23). Recently improvements in that context have been made by the introduction of a predictor-corrector process in free surface extrapolation by Sandhu and permit large time steps to be taken without instability. Using such a process in the quasi-static viscous problem may well improve its efficiency.

A note should be made here of the necessity of periodic mesh shape adjustments as the process continues. These are necessitated by drastic surface configuration changes which are clearly indicated on Figure 14.

X INCLUSION OF DYNAMIC EFFECTS - NAVIER STOKES PROBLEM

It was shown in section III-2 how the inclusion of the dynamic term in the virtual work discretization gave rise to a non-linearity and to non-symmetric matrices. The lack of symmetry is due to the absence of a variational principle for full viscous flow problems.

The problem presented is known to have unique solution at low Reynolds numbers but for large Reynolds number steady state solution apparently do not exist and the flow becomes turbulent, a fact found experimentally. Solution to the steady state problem will thus be only sought for fairly low velocity flows. With the transient $\frac{\partial u}{\partial t}$ (vide equation 5) included, in principle a solution could be obtained for any velocity but due to the nature of turbulence and its rapid velocity changes numerical errors could well be expected unless both time and space subdivisions are very fine.

In the low velocity steady state problem we can proceed numerically in two ways: either by isolating the non-linearity in the "force" terms and writing equations corresponding to 24 as

$$\underline{\bar{K}}_o \ \underline{a} + \underline{\bar{f}} = 0 \qquad (61)$$

in which $\underline{\bar{K}}_0$ is the symmetric, constant, matrix and $\underline{\bar{f}}$ is dependent on velocity and hence

$$\underline{\bar{f}} = \underline{\bar{f}}(a) \qquad (62)$$

XI. CONCLUDING REMARKS

In this chapter we have indicated very many alternatives possible for the solution of viscous flow problems with Newtonian or non-Newtonian viscosity. Some of the possibilities have not yet been explored, and many others have not been mentioned here. We hope however that sufficient indication of the possible applications and relative merits of various approaches has been given.

From the physical viewpoint we have limited the discussion to several relatively simple situations.

Possibilities and indeed the need for extending the solution technique to new situations are being explored. Here we mention a few:

(a) Coupled thermal/flow problems where viscosity is temperature dependent.

(b) Problems where surface tension effects are present.

(c) Problems in which straining history causes anisotropy of flow and changes in properties.

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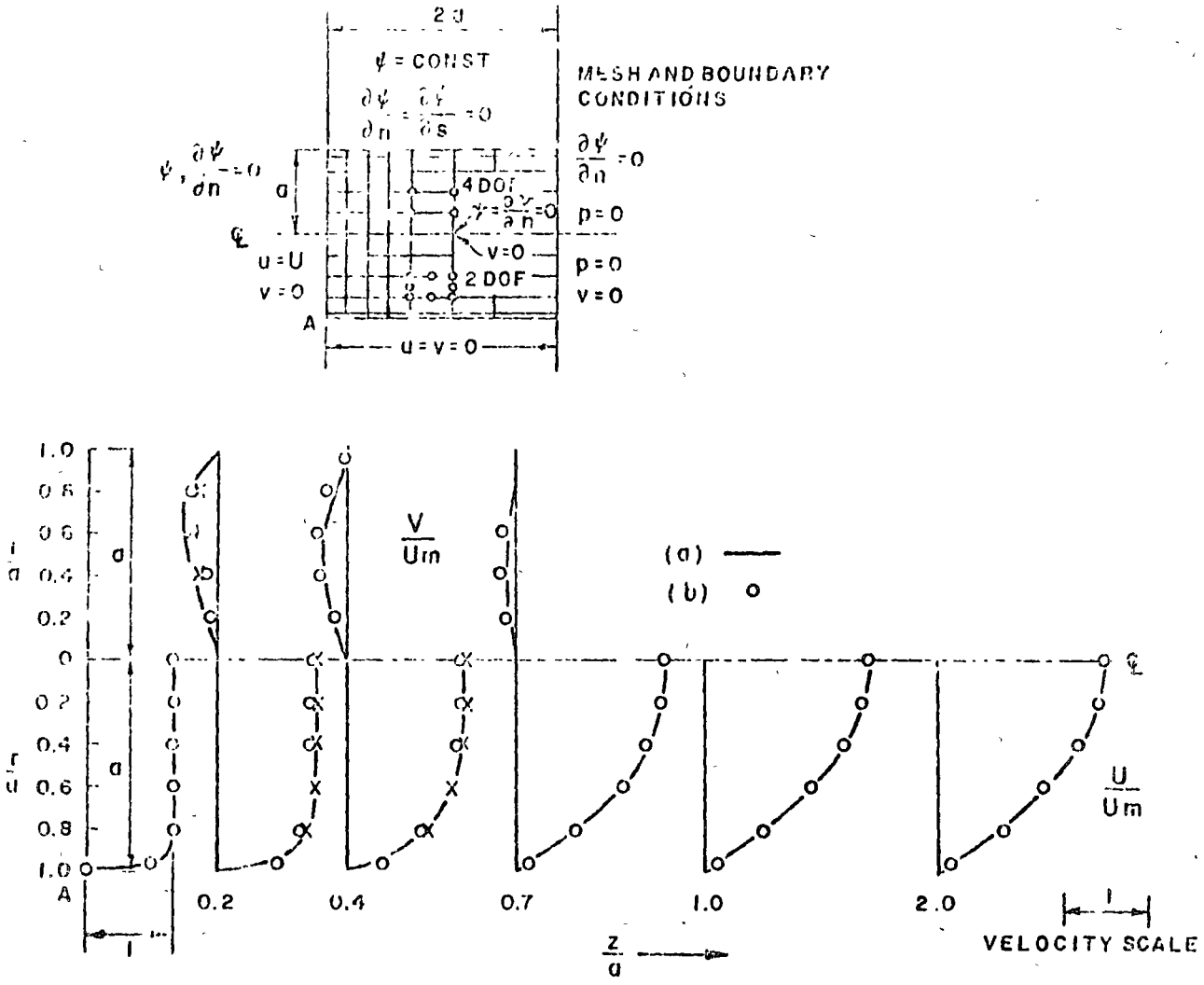

FIG. 1 VELOCITY PROFILE DEVELOPMENT IN AN AXIALLY SYMMETRIC FLOW (ENTRY FLOW)

(a) BY STREAM FUNCTION APPROACH

(b) BY U-V- PENALTY FUNCTION APPROACH (ν=0.49995)

(c) FINITE DIFFERENCE SOLUTION (LEW AND FUNG 1970) X

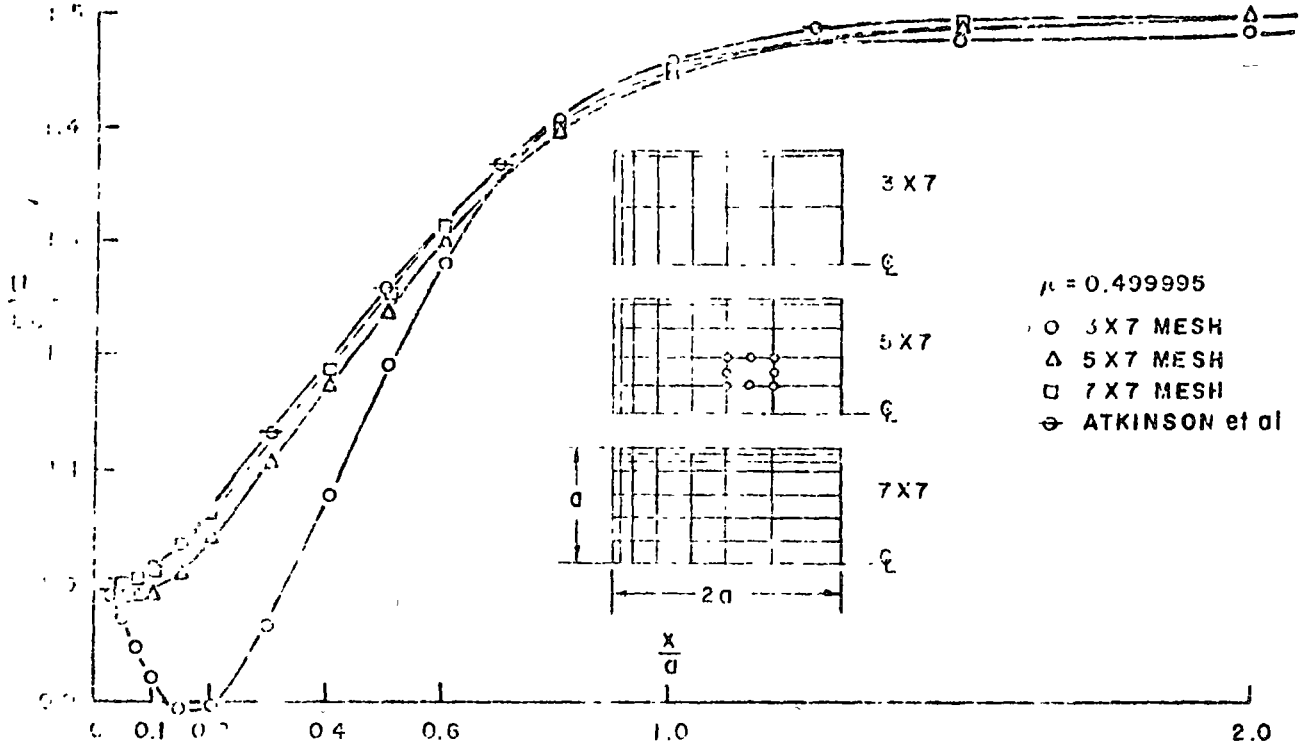

FIG 2 EFFECT OF MESH SUBDIVISION ON AXIAL VELOCITY DEVELOPEMENT ON THE PLANE ENTRY FLOW (ON ℄). PARABOLIC ELEMENTS. PENALTY FUNCTION APPROACH.

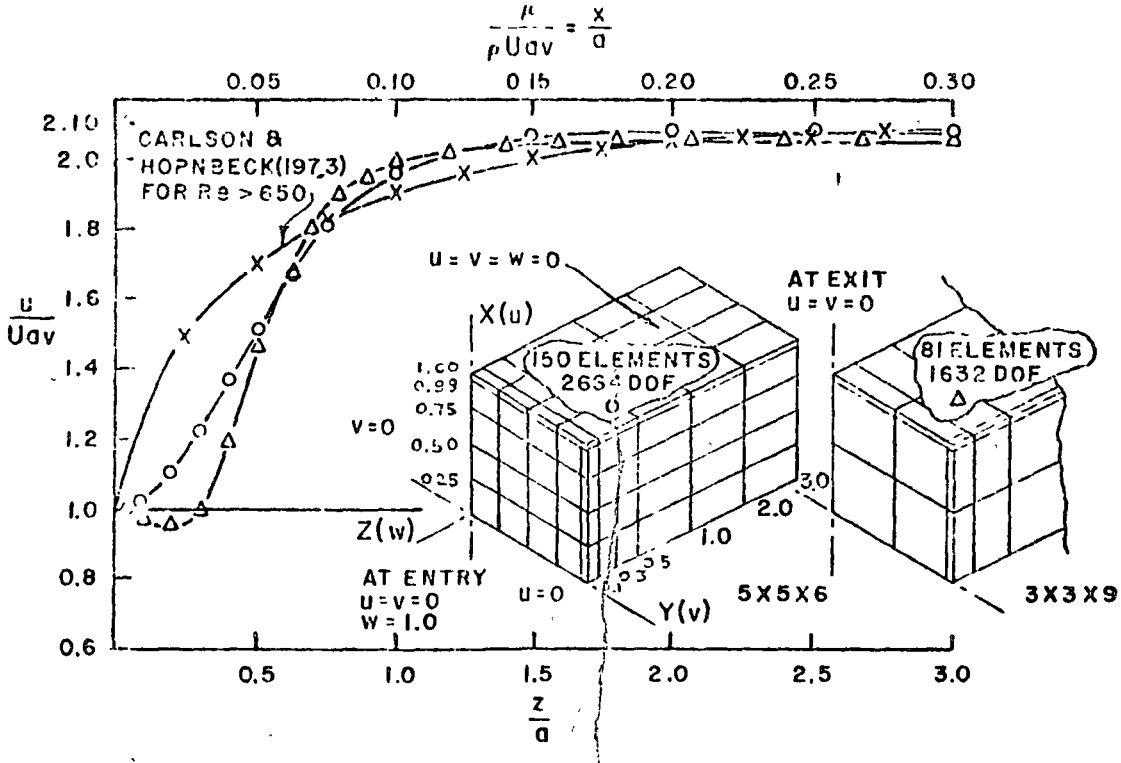

FIG 3 AXIAL VELOCITY DEVELOPMENT AT DUCT CENTRE LINE (CREEPING FLOW)

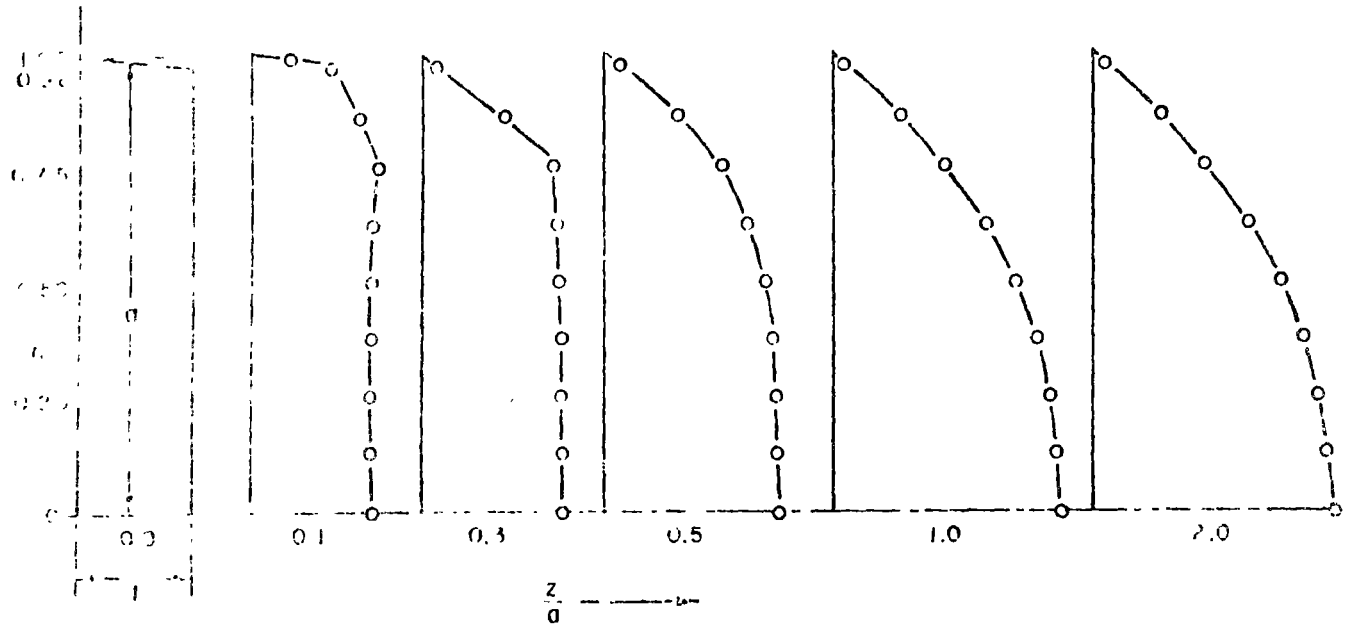

FIG. 4 AXIAL VELOCITY PROFILES ON CENTRE LINE SECTION IN FIG. 3

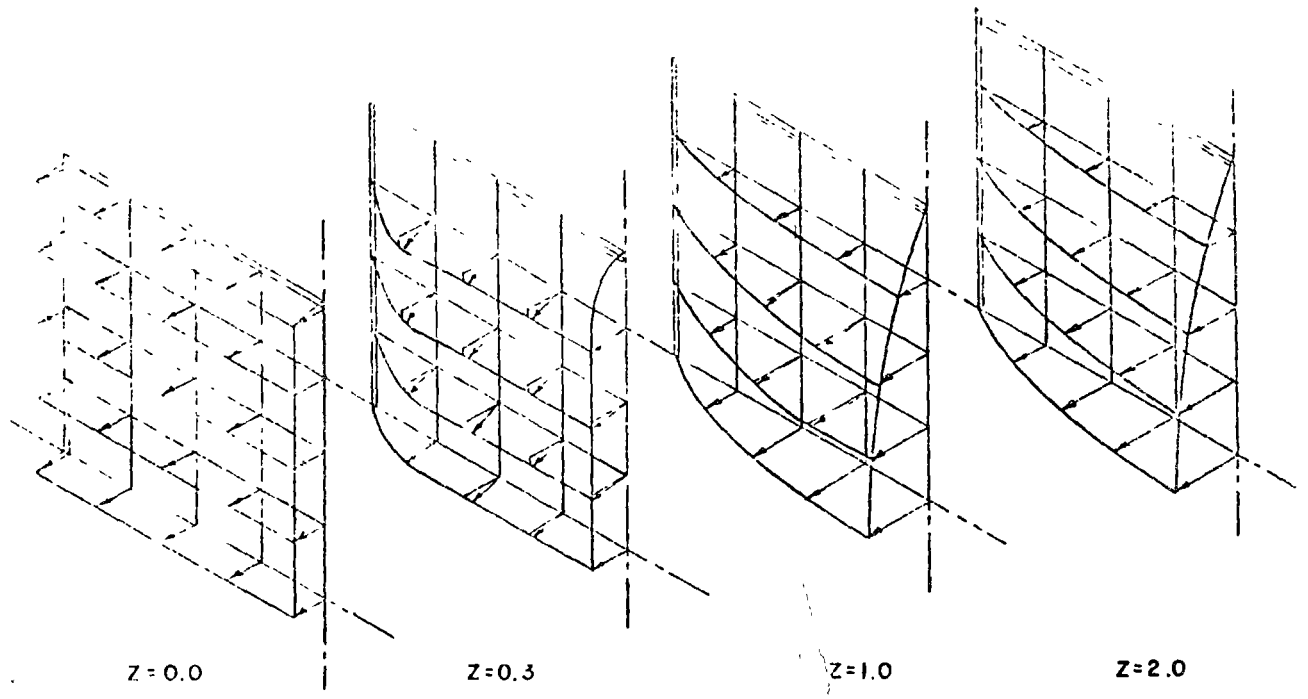

FIG. 5 THREE DIMENSIONAL ENTRY FLOW PROBLEM (SQUARE DUCT)
ISOMETRIC VIEW OF VELOCITY DISTRIBUTION

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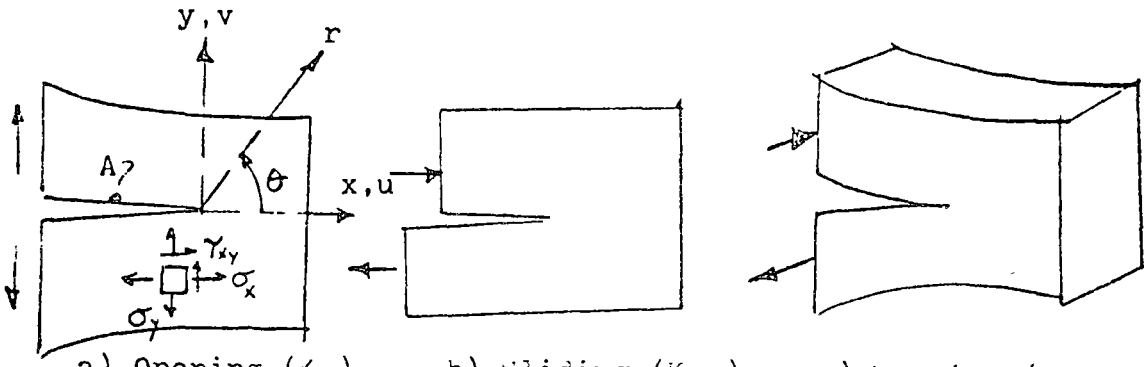

a).Opening (K_I) b).Sliding (K_{II}) c) Tearing (K_{III})

FIGURE 1. BASIC MODES OI CRACK EXTENSION

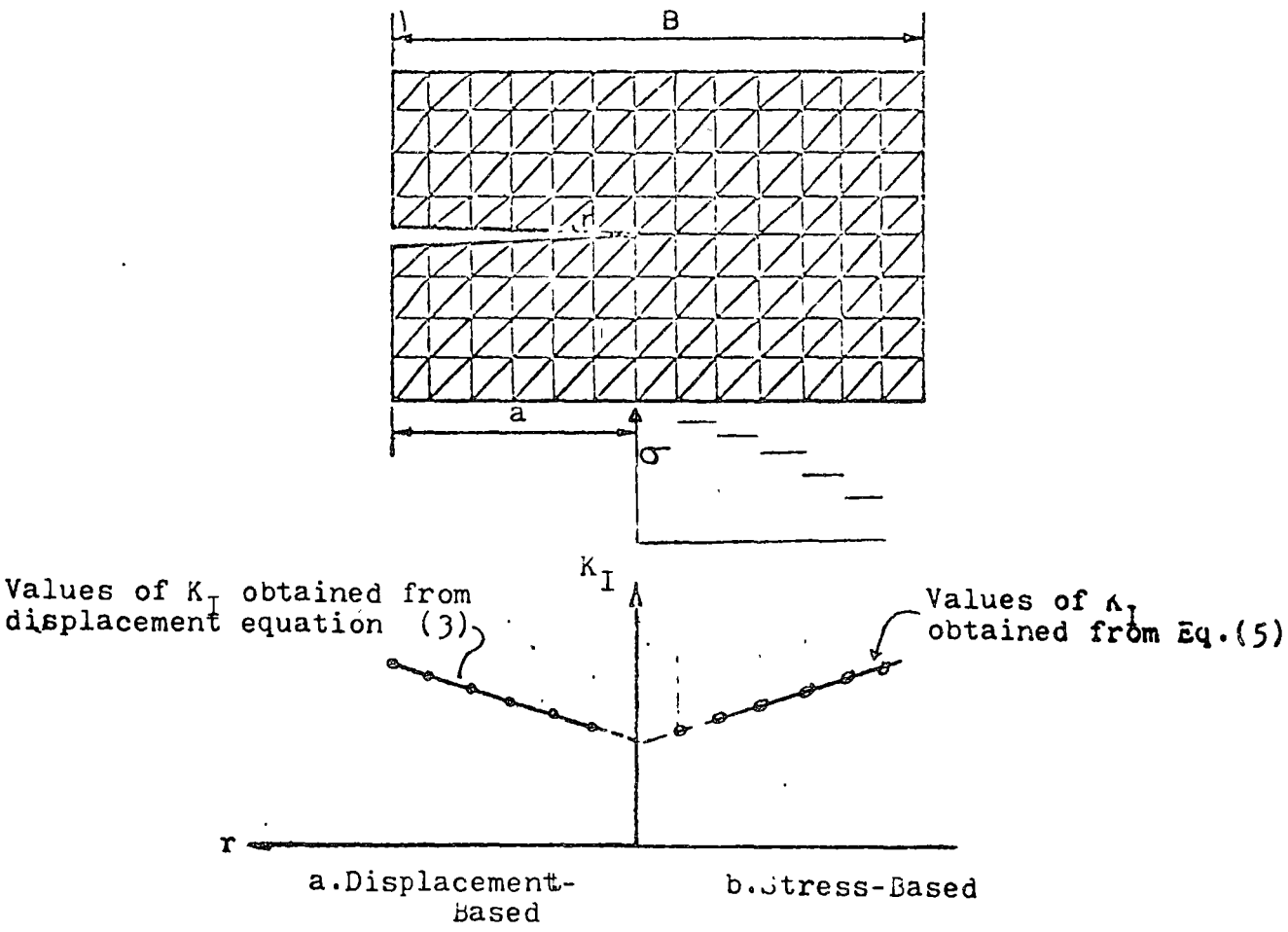

FIGURE 2. EXTRAPOLATION METHODS.

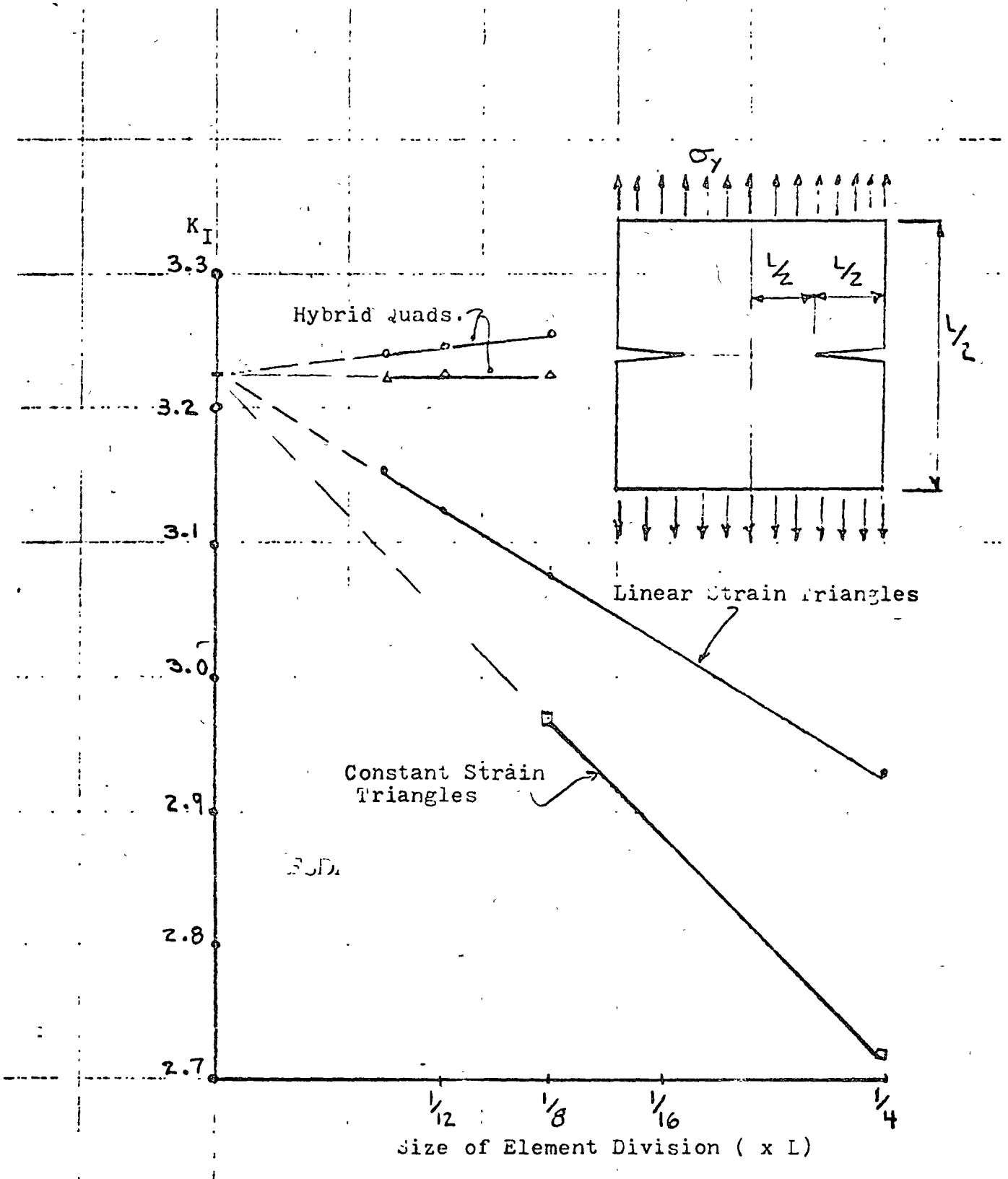

FIGURE 3. Comparison of Convergence Rates for Various Types of Elements in Calculation Stress Intensity Factor (Ref. 64).

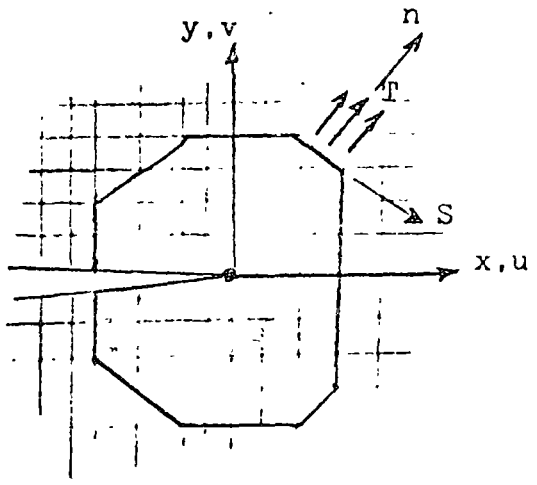

FIGURE 4. ISOLATED REGION FOR J-INTEGRAL CALCULATION (SHADED)

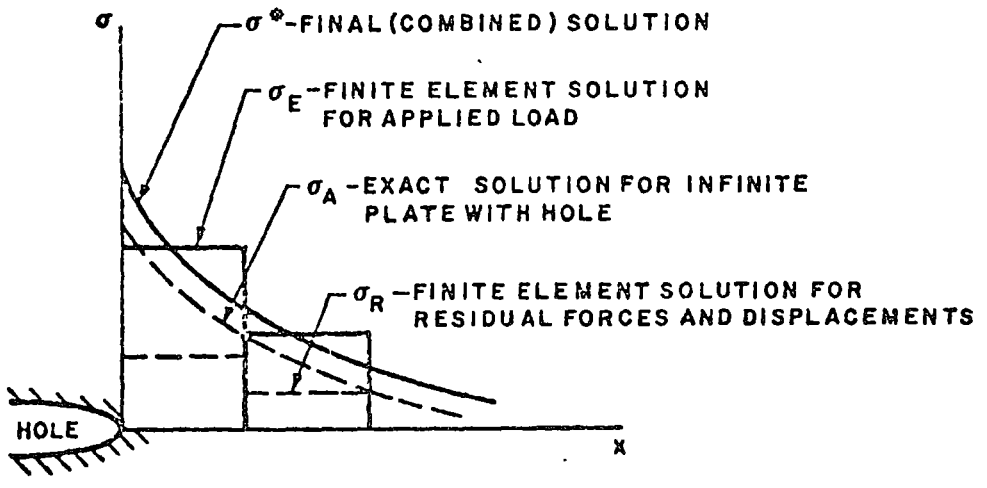

FIG. 5 SUPERPOSITION METHOD (REFERENCE 19)

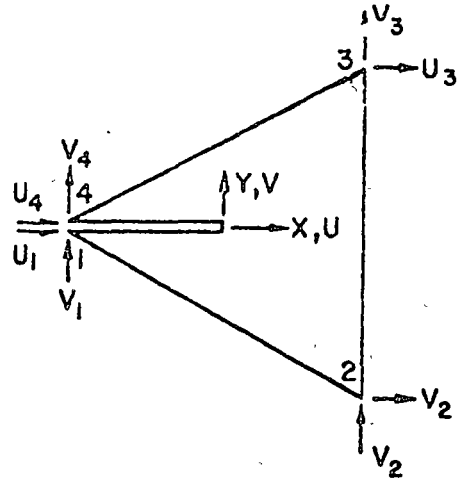

FIG. 6 TRIANGULAR SINGULARITY ELEMENT (REFERENCE 21)

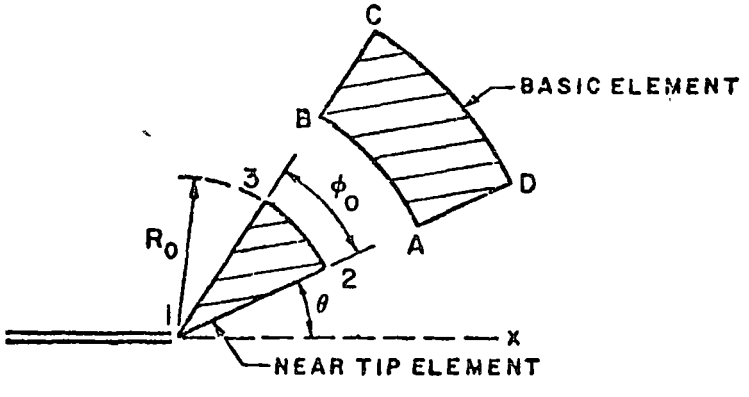

FIG. 7 SINGULARITY ELEMENTS IN POLAR COORDINATES

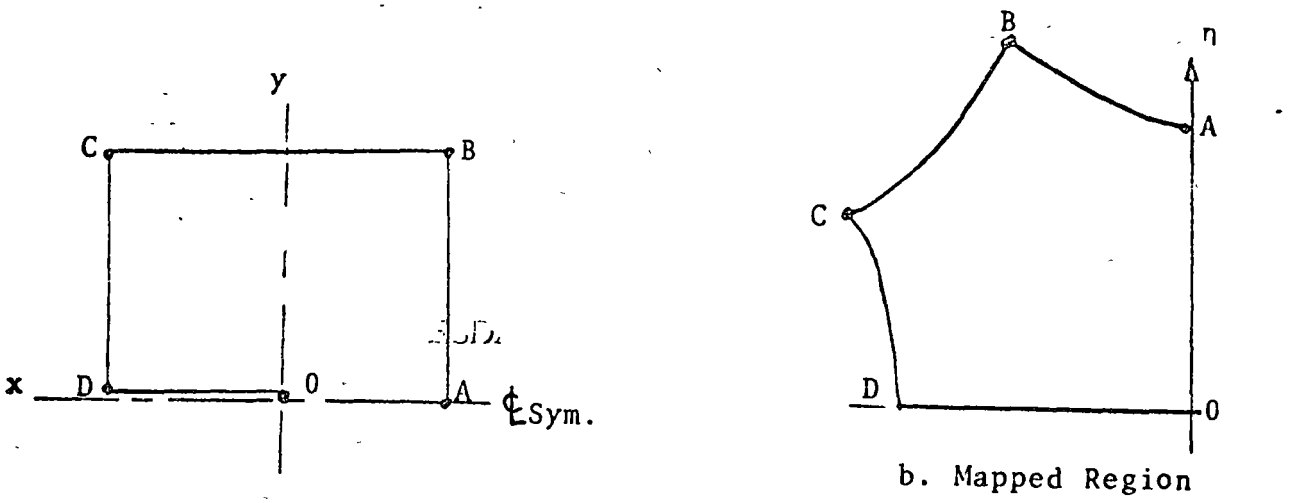

FIGURE 8. CONFORMAL MAPPING OF A SINGULARITY
(REF. 51)

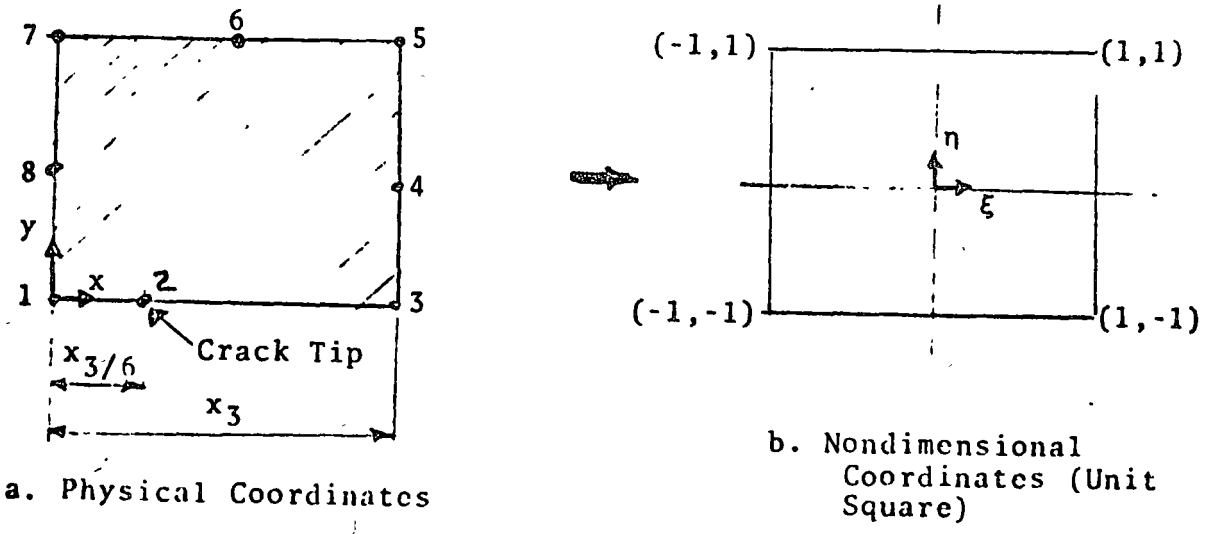

FIGURE 9. SINGULARITY CONDITIONS ON QUADRATIC ISOPARAMETRIC ELEMENT
(REFS. 49 AND 50)

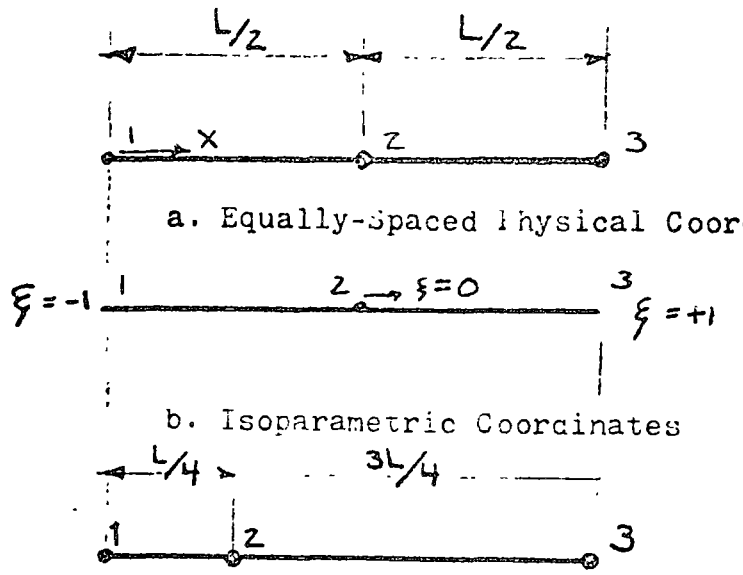

FIGURE 10. UNIAXIAL ISOPARAMETRIC COORDINATES FOR REPRESENTATION OF SINGULARITIES.

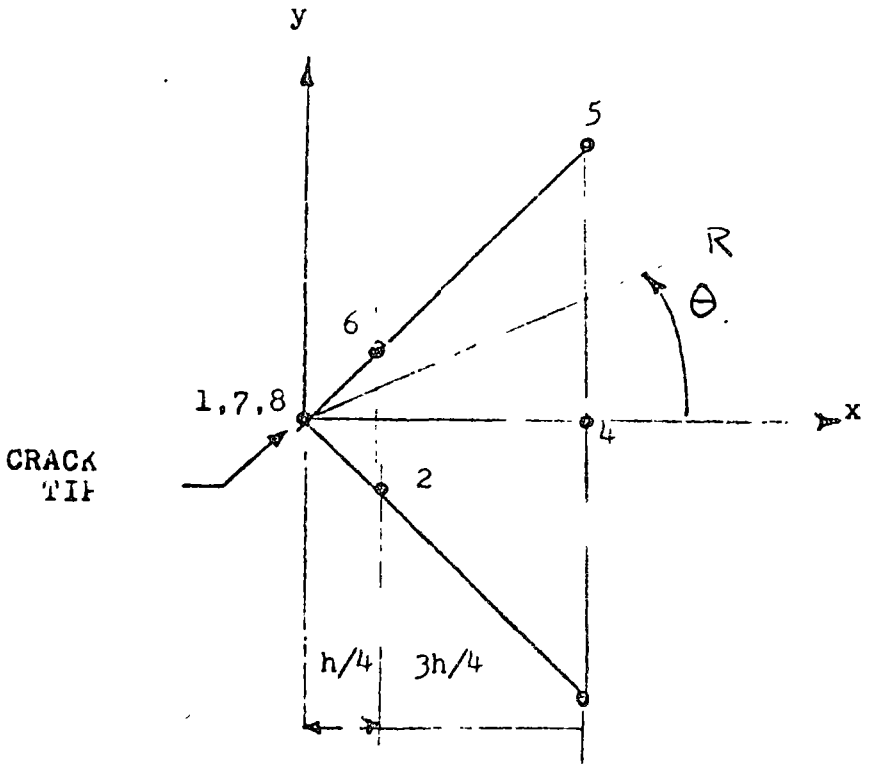

FIGURE 11. TRIANGULAR ISOPARAMETRIC ELEMENT (Formed by Coalescing Points 1,7,8).

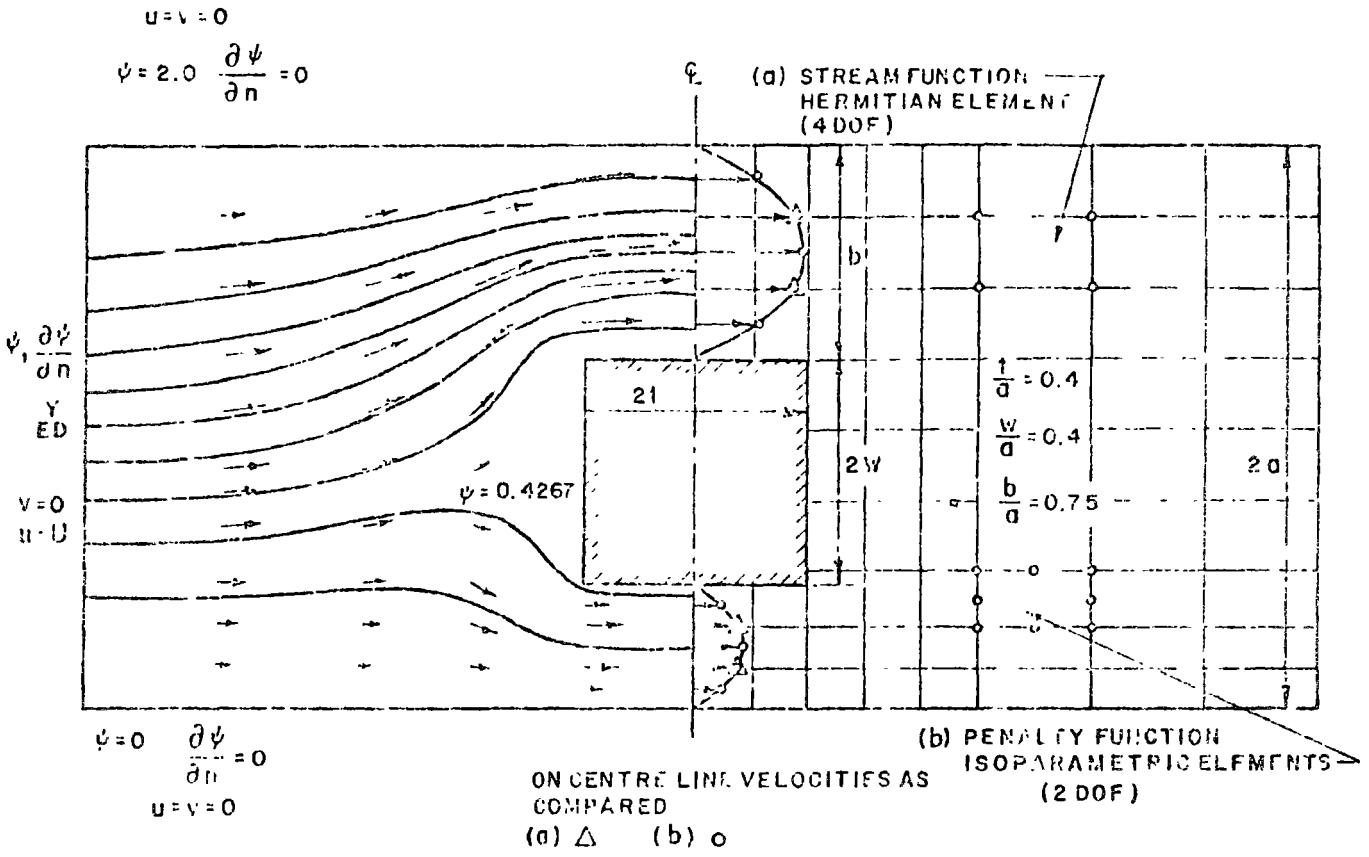

FIG. 6 TWO DIMENSIONAL FLOW WITH A SQUARE OBSTRUCTION PLACED ASSYMMETRICALLY
(a) STREAM FUNCTION APPROACH–STREAM LINES SHOWN
(b) PENALTY FUNCTION APPROACH–VELOCITY VECTORS SHOWN

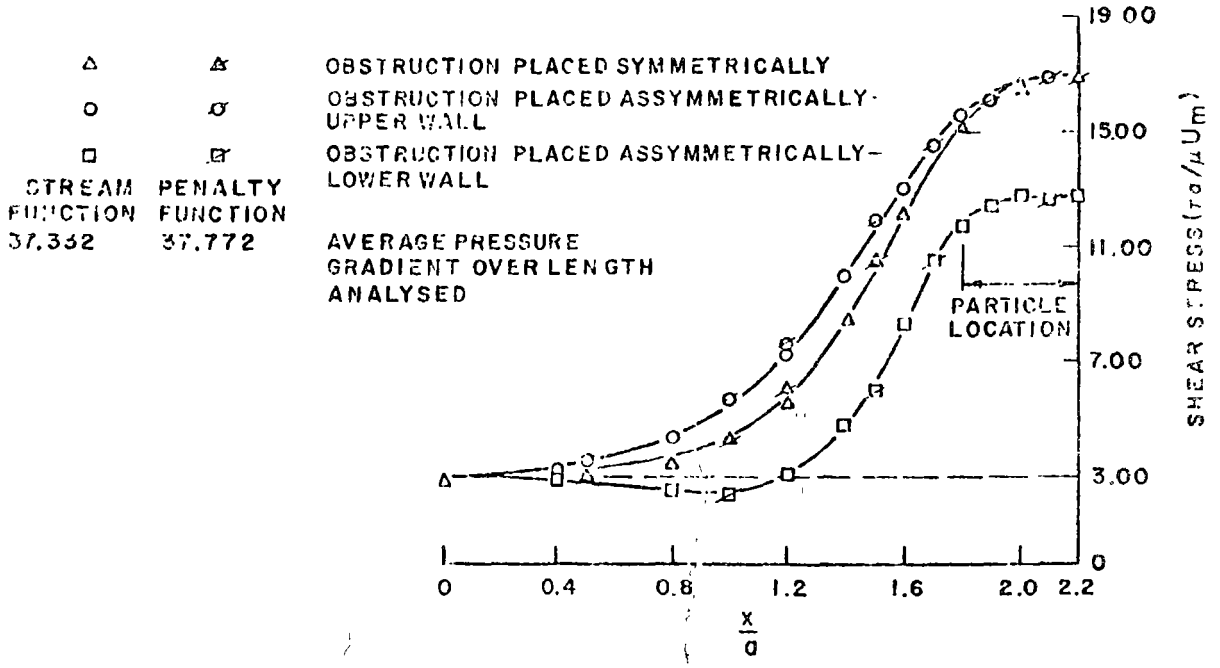

FIG. 7 SHEAR STRESS DISTRIBUTION OF THE WALL

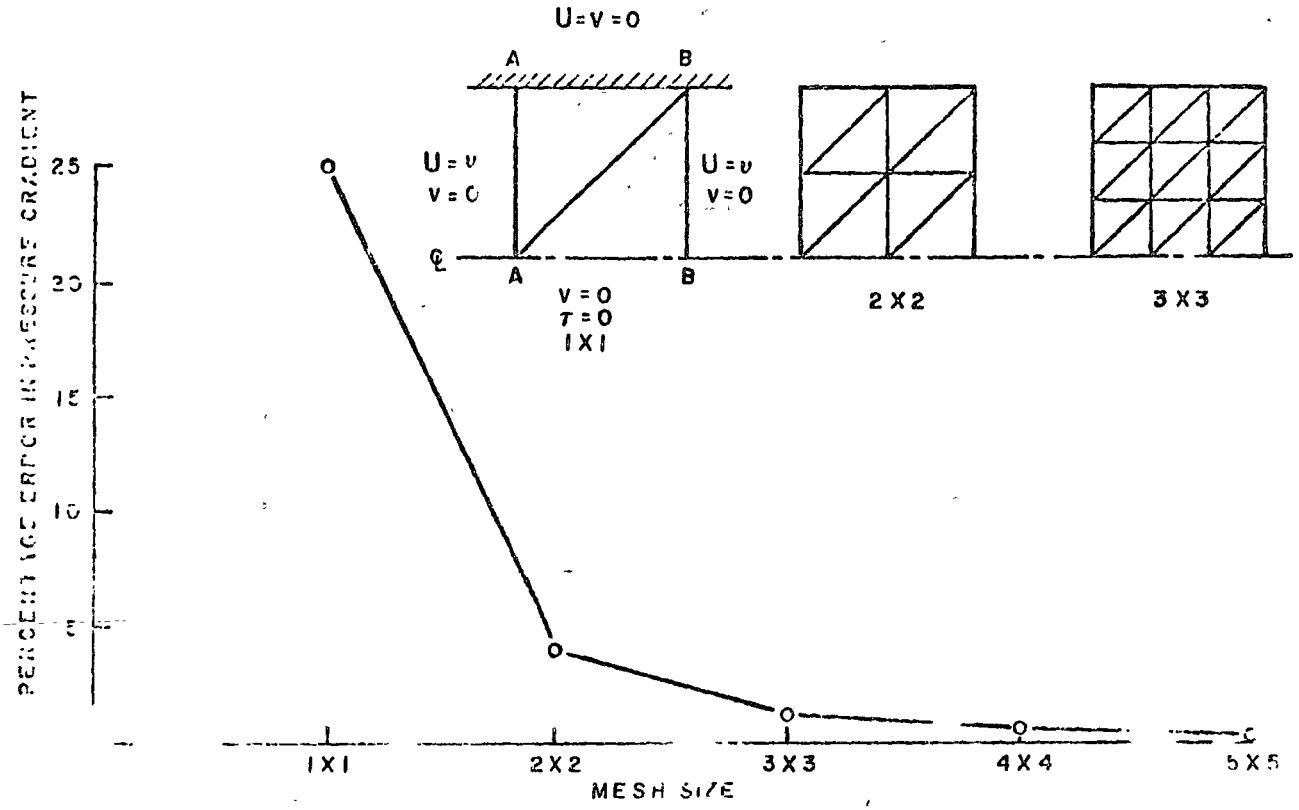

FIG. 8. CONVERGENCE OF TRIANGULAR ELEMENTS – PLANE POISSEUILLE FLOW. PARABOLIC VELOCITY PRESCRIBED ON AA AND BB

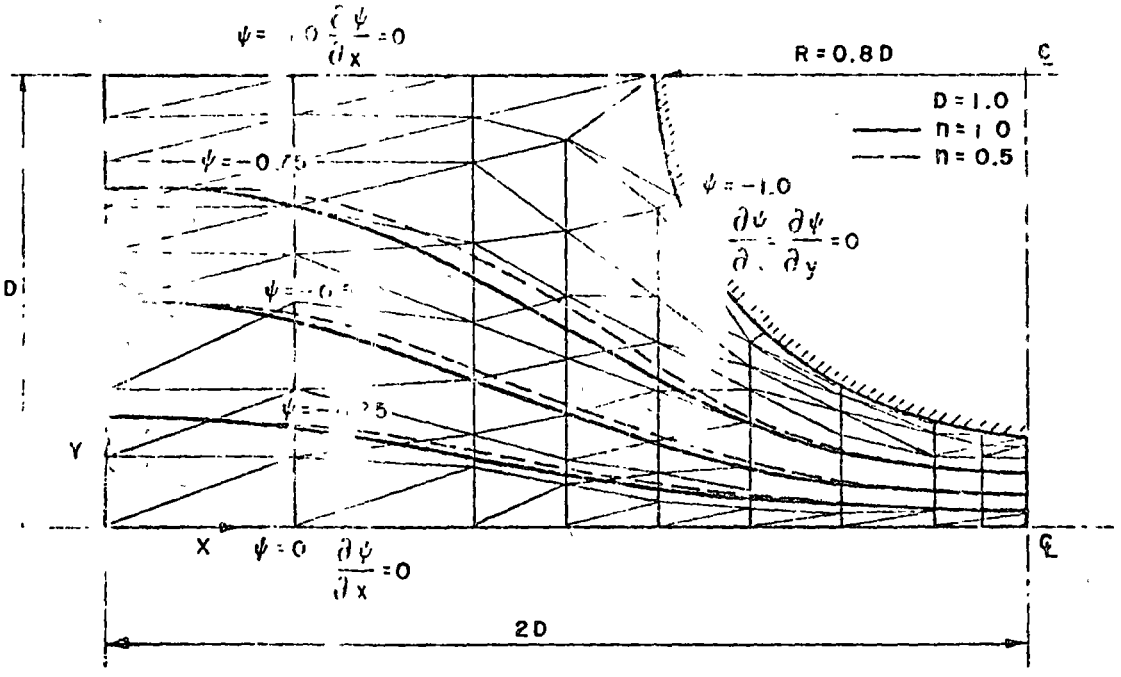

FIG. 9 STREAM LINES FOR FLOW AROUND A SERIES OF CYLINDERS IN A PLANE PARALLEL FLOW – NON-NEWTONIAN FLUID – STREAM LINES

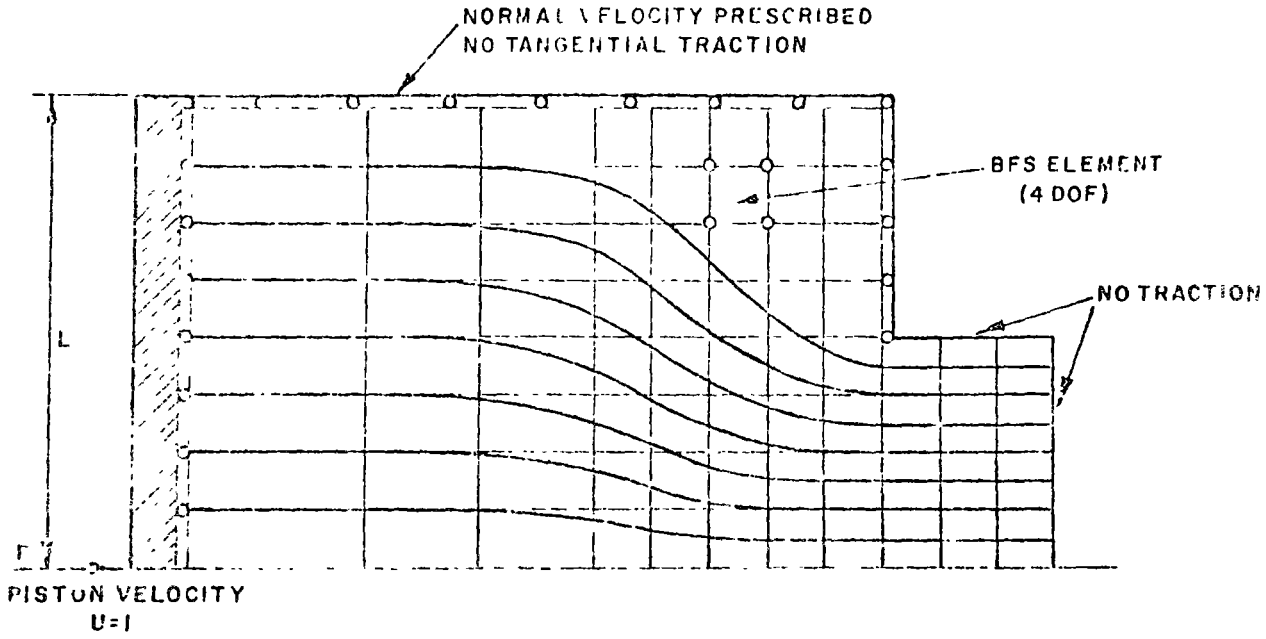

FIG. 10 ELEMENT SUBDIVISION AND RESULTING STREAM LINE PATTERN FOR THE EXTRUSION PROBLEM (PLANE STRAIN) STREAM FUNCTION FORMULATION

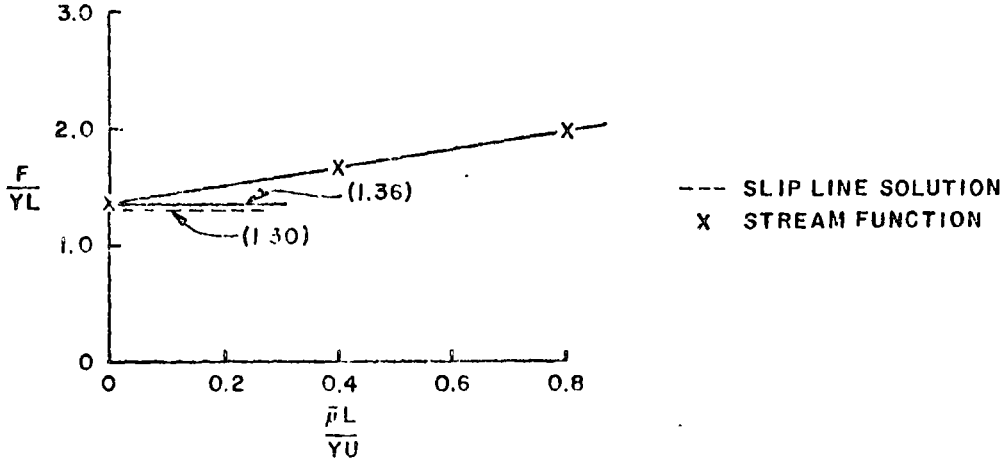

FIG. 11 VARIATION OF EXTRUSION FORCE IN EXAMPLE OF FIG. 10 WITH $\bar{\mu}$ FOR A VISCO-PLASTIC MATERIAL (U=1)

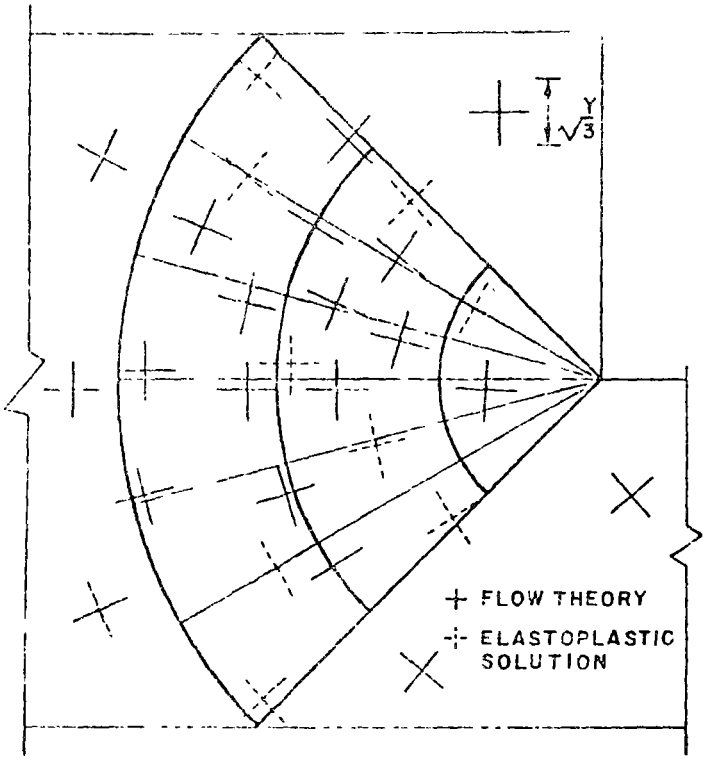

FIG.12 DIRECTIONS, MAGNITUDES OF MAXIMUM SHEAR STRESSES FROM FLOW AND ELASTO-PLASTIC SOLUTIONS SUPERIMPOSED ON THE SLIP-LINE FIELD

Y=UNIAXIAL YIELD STRESS

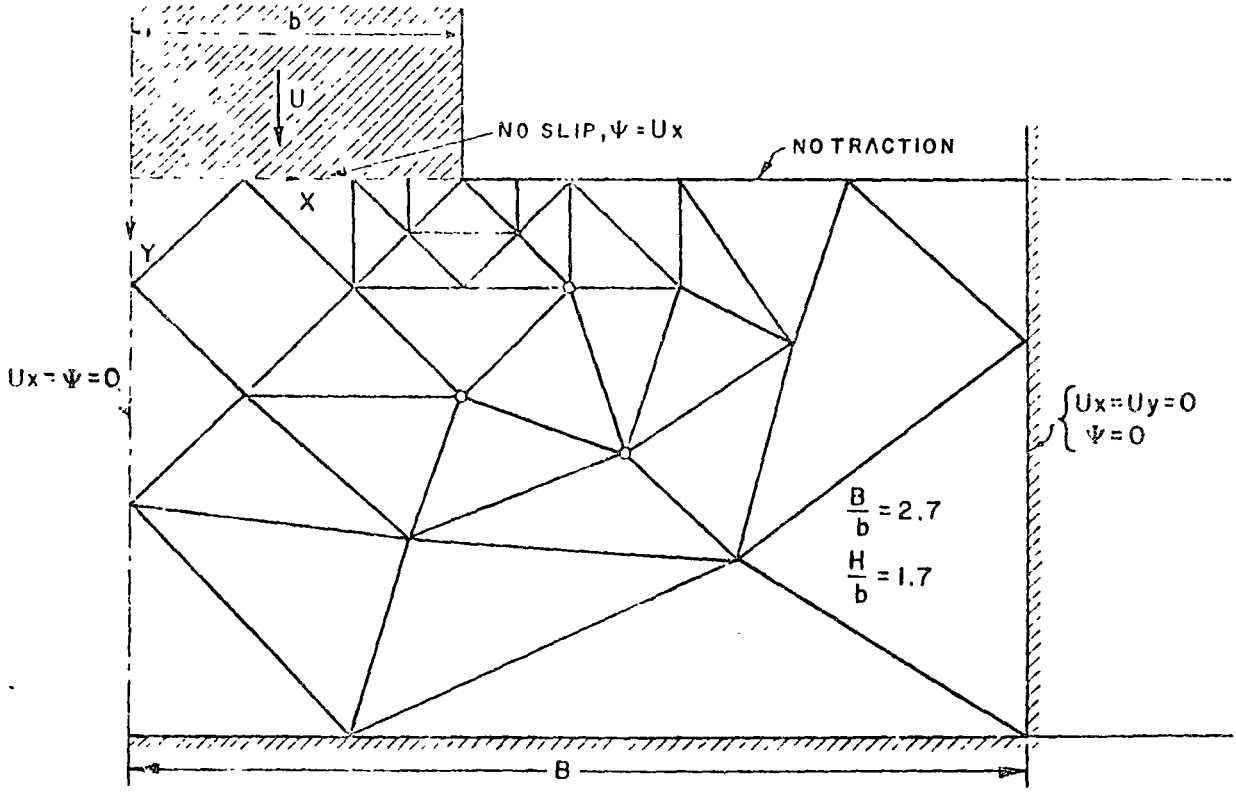

FIG.13 NON STEADY STATE PROBLEM OF PUNCH INDENTATION (40 ELEMENTS, 29 NODES) IDEALLY PLASTIC MATERIAL $\bar{\mu}=0$

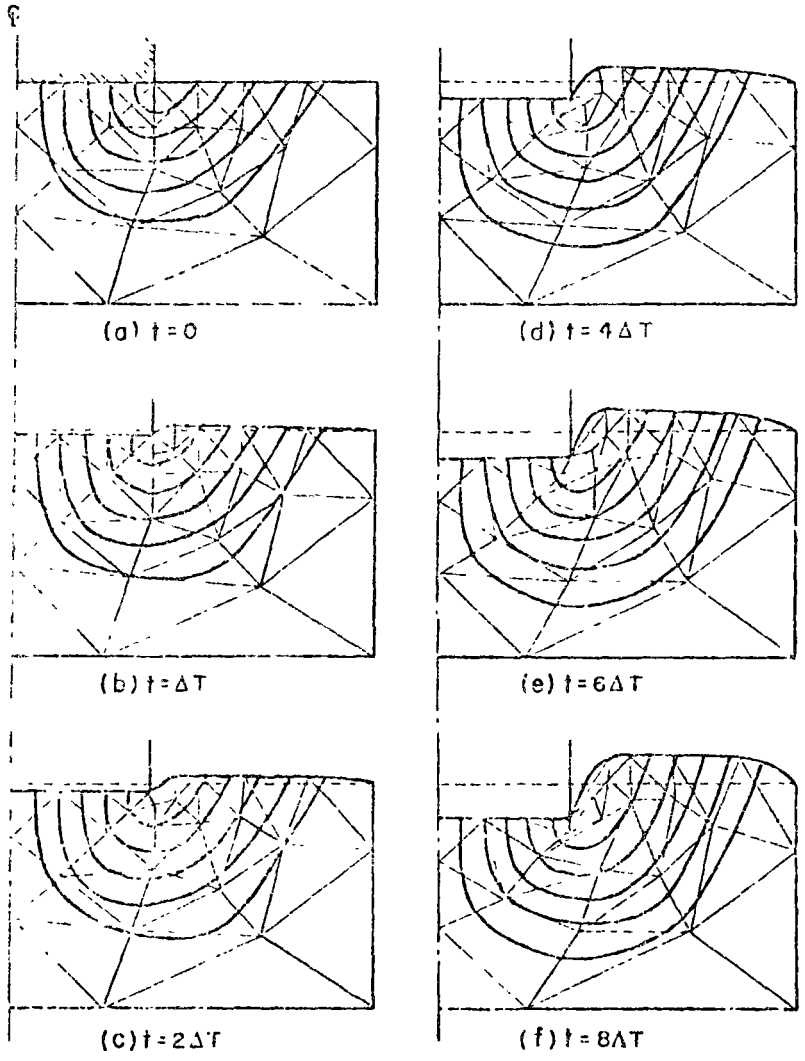

FIG. 14 STREAM LINES FOR THE PUNCH INDENTATION PROBLEM

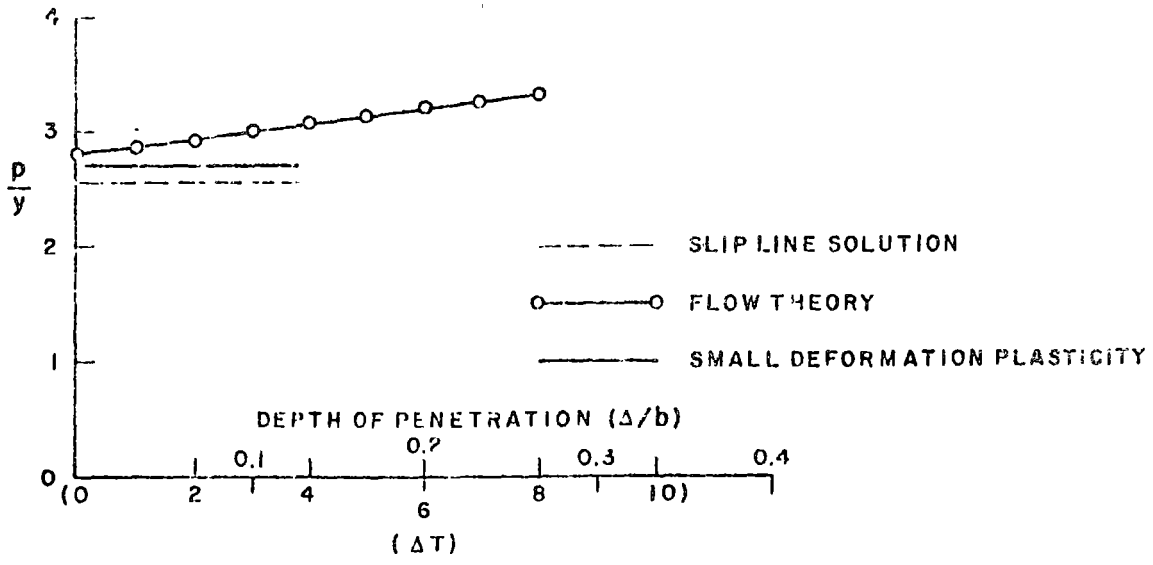

FIG. 15 INDENTING PRESSURE VERSUS PENETRATION DEPTH Δ

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ENVIRONMENTAL PROBLEMS

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ENVIRONMENTAL PROBLEMS ASSOCIATED WITH FLUID FLOW

R. H. Gallagher and D. L. Young

I. INTRODUCTION

The topic of computational fluid mechanics does not have a long history. By the 1930's there were only a limited number of papers on this topic, a condition that was due not only to the absence of high speed computational facilities but also to difficulties in dealing with the inherent nonlinearity of most problems of interest. Thus, computational fluid mechanics has developed in tandem with the finite element method. Very many applications of the latter to flow problems have been recorded, as described in References 1-4. As these references demonstrate, a detailed review of the complete field would require a complete text. We therefore limit our attention, in these lecture notes, to the application of the finite element method to environmental problems associated with fluid flow.

A large number of conditions can be grouped under the heading of "environmental problems". In the present case we refer principally to the transport of heat or the concentration of a substance through a body of water. The processes of convection and diffusion participate in the transport process. Velocities appear in the convective terms and, although the most vigorous treatment of the problem will involve coupling of velocity and temperature (or concentration) equations, practical considerations may require independent solutions for the two fields. We therefore include in our review a discussion of solutions for flow velocities alone for lake and stream situations.

A study of the literature of topic under review discloses that although the problems are basically three-dimensional, no numerical solutions of this scale have yet been attempted. Simplifying assumptions are customarily made about one of the dimensions and the problem is reduced to one of analysis in the plane. Indeed, some investigations make assumptions regarding two of the dimensions and study the velocity in one dimension together with the temperature.

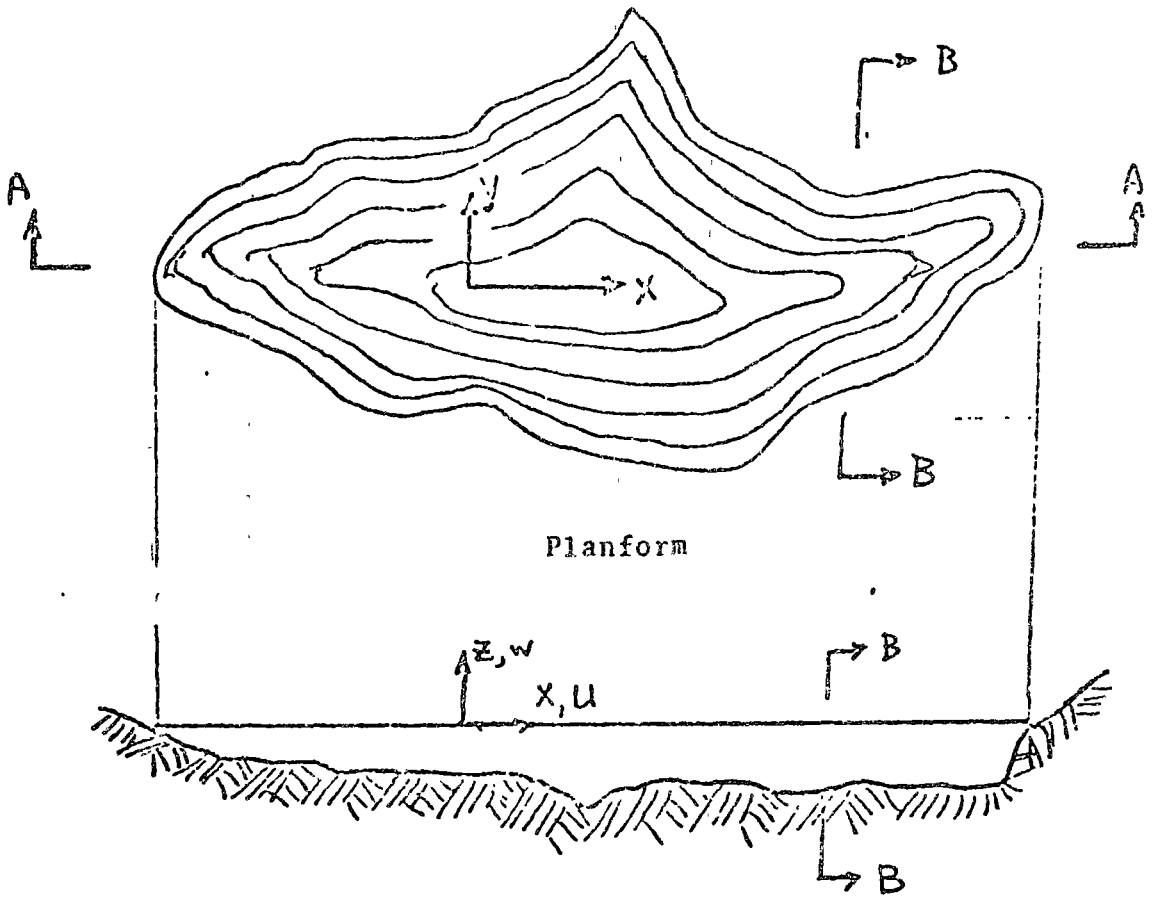
In view of the above circumstance, these notes are categorized with respect to the different types of two-dimensional situations. Only the cases of flow in planform and that of flow on the narrow cross-section of a lake or similar body of water are treated in this review.

First we define the coordinate systems associated with the respective types of problems. Then, separate sections are devoted to each type.

II. COORDINATE SYSTEMS AND GOVERNING EQUATIONS

Figure 1 illustrates the body of water and the associated coordinate systems. The body of water we have in mind is a lake, although cases will be treated which refer to streams and estuaries. In the latter circumstances the flow is predominantly in the y-direction.

The planform (x-y plane) is the basis for analysis of wind-driven circulation and of flow through basins and estuaries. The assumptions that are invoked are discussed in some detail later, but, for the present we simply note that they are directed to elimination of the z-coordinate from the problem. The maximum



Section A-A

Figure 1 Lake-Planform and Side Views

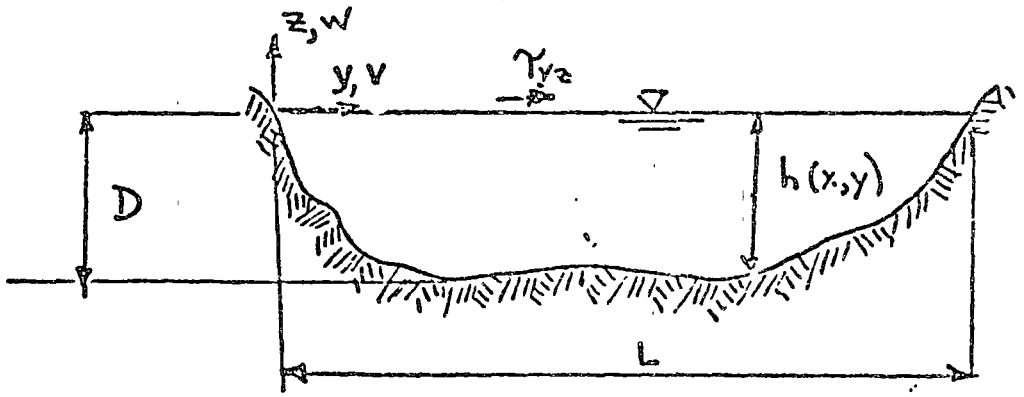


Figure 2 Narrow-Cross-Section (Section B-B)
(Vertical and Horizontal Dimensions have been Exaggerated in Comparison with Fig. 1)

z-dimension is very much smaller than the y- and x-dimensions. The latter may be of approximately the same magnitude.

The narrow cross-section (x-z) is intended to represent the section of a lake or similar body of water. Here, the x- and z-dimensions are of similar magnitude and the y-direction is very large. Finally, we have the side view (y-z), which refers principally to flow in streams and estuaries and which is often reduced to just the y-direction.

Physically, the velocity and temperature fields are determined through the conservation of mass, momentum and energy. The governing equations are

Continuity (Conservation of mass)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

x-Momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fu = \frac{1}{\rho_o}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(K^M_{xx}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K^M_{xy}\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K^M_{xz}\frac{\partial u}{\partial z}\right) \qquad (2)$$

and correspondingly for y and z. (In the z-direction there is no Coriolis force (- fu) but a buoyancy term $(\frac{\rho}{\rho_o} g_x)$ must be added to the left side.)

Temperature (Conservation of Energy)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial}{\partial x}\left(K^H_x\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K^H_y\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K^H_z\frac{\partial T}{\partial z}\right) \qquad (3)$$

In these equations u, v, and w are the x, y, and z-direction velocities, g is the gravitational acceleration, p is pressure, ρ is density, ρ_o is reference density, T is temperature. f is the Coriolis parameter and K^M and K^H (with appropriate directional

subscripts) are the viscosity and diffusivity. The above equations are supplemented by equations of state, such as density as a function of temperature, pressure and concentration of a substance, and the viscosities and diffusivities as a function of "stability parameters", e.g., Richardson Number, Prandtl Number, Monin-Obukhov Length, etc.

These are very general equations. Consequently, in fact, the following approximations may be introduced, depending on the analyst's interests and goal.

1) The Boussinesq approximation: The variation of density is small, so that the fluid can be treated as an incompressible fluid. The variation of density is only considered in the buoyancy term $\frac{\rho}{\rho_o} g$ in the z-momentum equation.

2) The shallow water approximation: The inertia forces are negligible compared to the other forces. Also, the w-component is much smaller than the horizontal components, so that the pressure is hydrostatic ($g = -\frac{1}{\rho}\frac{\partial p}{\partial z}$, where g is the acceleration due to gravity).

3) The eddy viscosity and diffusivity approximations: Since the stratification is almost perpendicular to the gravitational force, it is customary to assume that the horizontal eddy viscosity (K^M_{xx}, etc.) and diffusivity (K^H_x) are approximated by constants, while the vertical ones are functions of the gradients of density and velocity. The exact relationships are still hot debate. In practical analysis, the determination must come from semi-empirical stratified turbulent theory. (Monin-Yaglom, Ref. 5)

III. PLANFORM ANALYSIS. WIND-DRIVEN CIRCULATION AND FLOW THROUGH BASINS

The cross-section shown in Figure 2 defines the basic geometric parameters of this development, which is due to Liggett and Hadjitheodourou[6] in its fundamental theoretical form. The origin of coordinates is fixed at the surface of the lake, with velocity $w = 0$ at $z = o$. (The 'rigid lid' assumption). The physical properties of the lake, including the eddy viscosity and the mass density per unit volume, are assumed to be constant and the Coriolis parameter is also assumed constant. The pressure is taken to vary hydrostatically. The surface wind stresses τ_{xz} and τ_{yz} are prescribed. Under these assumptions the momentum equations take the form

$$-fv = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2} \cdot K_o^M \tag{4}$$

$$fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} \cdot K_o^M \tag{5}$$

$$g = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} \tag{6}$$

The continuity equation is unmodified.

A stream function ψ is defined as follows

$$\frac{\partial \psi}{\partial y} = \bar{u}h \tag{7}$$

$$-\frac{\partial \psi}{\partial x} = \bar{v}h \tag{8}$$

in which $\bar{u}$ and $\bar{v}$ are depthwise averages of the component velocities. After combination of the above equations, with consideration of the boundary conditions (zero velocity on all solid surfaces and $\tau_{xz} = K_o^M \frac{\partial u}{\partial x}$, $\tau_{yz} = K_o^M \frac{\partial v}{\partial y}$) one obtains

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + A(x,y)\,\frac{\partial\psi}{\partial x} + B(x,y)\,\frac{\partial\psi}{\partial y} + C(x,y) = 0 \qquad (9)$$

The terms A(x,y), B(x,y) and C(x,y) are those which result from consideration of the varying depth and, as indicated, are functions of the planform coordinates x and y. Thus, they account for the varying depth.

Transformation of Eq. (9) into the finite element form, given in detail in Ref. 7, is accomplished by means of the Galerkin method. The approximation of ψ is by means of the trial function $\bar{\psi}$, which has the form

$$\bar{\psi} = N_i\psi_i = \lfloor N \rfloor \{\psi\} \qquad (10)$$

wherein the N_i are the shape functions and $\{\psi\}$ are the nodal values of the stream function. Applying now the weighted residual concept

$$\int_A \{N\}\left(\frac{\partial^2 \lfloor N \rfloor}{\partial x^2} + \frac{\partial^2 \lfloor N \rfloor}{\partial y^2} + \frac{A\partial \lfloor N \rfloor}{\partial x} + \frac{B\partial \lfloor N \rfloor}{\partial y} + C\right) dA\ \{\psi\} = 0 \qquad (11)$$

Next, integration by parts in the plane is applied to reduce the order of the derivatives appearing in this integral and to produce boundary terms. One obtains

$$\int_A \left[-\frac{\partial}{\partial x}\{N\}\,\frac{\partial \lfloor N \rfloor}{\partial x} - \frac{\partial}{\partial y}\{N\}\,\frac{\partial \lfloor N \rfloor}{\partial y} + A\{N\}\,\frac{\partial \lfloor N \rfloor}{\partial x} + B\{N\}\,\frac{\partial \lfloor N \rfloor}{\partial y}\{\psi\} + \{N\}C\right] dA + \oint \{N\}\,\frac{\partial \lfloor N \rfloor}{\partial n}\{\psi\}\, dS = 0 \qquad (12)$$

The values of $\{\psi\}$ are zero on the entire exterior boundary and the closure integrals along interelement boundaries vanish if continuity is preserved across these boundaries. Thus, the contour integral term is excluded from subsequent consideration.

Evaluation of the remaining integrals for all i then yields the following system of element equations

$$[k^e]\{\psi\} = \{r^c\} \tag{13}$$

in which

$$[k^e] = \int_A \left(- \frac{\partial\{N\}}{\partial x} \frac{\partial [N]}{\partial x} - \frac{\partial\{N\}}{\partial y} \frac{\partial [N]}{\partial y} + A\{N\} \frac{\partial [N]}{\partial x} + B\{N\} \frac{\partial [N]}{\partial y}\right) dA$$

$$\{r^e\} = - \int \{N\} C \, dA \tag{14}$$

It should be noted that due to the terms $A\{N\} \frac{\partial [N]}{\partial x}$ and $B\{N\} \frac{\partial [N]}{\partial y}$ the resulting algebraic equations will be nonsymmetric.

The equations for the complete lake are constructed from the equations of the elements by imposing the condition of stream function continuity at each element joint. Thus, the global equations are, by simple addition of all coefficients with like subscripts

$$[K]\{\psi\} = \{R\} \tag{15}$$

After solution for $\{\psi\}$ the other variables, such as averaged velocities and pressure gradients, can subsequently be evaluated by back substitution.

Numerical solutions to Eq. (15) have been obtained for both simple test problems and for Lake Ontario[7]. Since field data is not available for Lake Ontario the convergence of the solution has been studied with use of higher-order elements[8]. Cheng[9] has analyzed Lake Erie, using a formulation which excludes consideration of variable depth. Tong[10] includes this factor in a finite element formulation based on Welander's theory,[11] which does not differ significantly from the theory cited above.

If a stream function is adopted as the dependent variable, as is done in the formulations discussed previously, the presence of islands raises a basic complication in the definition of the boundary conditions at the node points of the island shore line. The stream function is zero at points on the shore of the lake but takes on a constant, undefined value on each of the islands. Thus, as Tong[10] proposes, the values of the stream function on a given island are set equal to a single value that is determined in the solution process. This substantially contracts the number of unknowns in the equations to be solved.

Cheng[9] adopts a different approach to the treatment of islands. The system of global equations is first assembled without consideration of the islands and their boundary conditions. We denote this solution as $\{\Psi_o\}$. Then, in succession, 'unit' solutions $\{\Psi_i\}$ (i=1, .. M, where M is the number of islands) are obtained for $\Psi_j = 1$ for node points on the respective islands. Finally, an M x M system of equations must be solved to give the amplitudes G_i (i=1..M) which apply to the unit solutions. The complete solution is then represented by

$$\{\Psi\} = \{\Psi_o\} + \sum_{i=1}^{M} G_i\{\Psi_i\} \qquad (16)$$

The determination of the planform distribution by transport of temperature in a lake or basin with known flow is also a problem of major practical importance, especially for cooling ponds and similar basins. Temperature distributions have been determined for such conditions by Loziuk, Anderson, and Belytschko[12,13]. Tong[10] presented a more general development

along these lines which permits the finite element calculation of any concentration of substance in a lake. We outline the latter in this section.

If we define ϕ as the average concentration across the depth (h) of the substance under study, the governing differential equation can be written as

$$\rho\left(\frac{\partial(h\phi)}{\partial t} + \frac{\bar{u}}{h}\frac{\partial(h\phi)}{\partial x} + \frac{\bar{v}}{h}\frac{\partial(h\phi)}{\partial y}\right) = \frac{\partial}{\partial x} K_x^H \frac{\partial(h\phi)}{\partial x} + \frac{\partial}{\partial y} K_y^H \frac{\partial(h\phi)}{\partial y} + Q \tag{17}$$

where K_x^H and K_y^H are the eddy diffusivity coefficients and Q is a source or sink term. Now, the approximation of ϕ can be written in the form of the trial function

$$\bar{\phi} = N_i \phi_i = {}_{\lfloor}N_{\rfloor} \{\phi\} \tag{18}$$

where $\{\phi\}$ represents nodal values of $h\phi$ and ${}_{\lfloor}N_{\rfloor}$ is the relevant set of shape functions. When the analysis is performed for temperature, with a single temperature across the depth of the lake, $T = h\phi$.

Application of the Galerkin approach can again be made to construct element equations. Using Eqs. (17) and (18), one obtains

$$[h]\{\dot{\phi}\} + [s]\{\phi\} = \{Q\} \tag{19}$$

where

$$[h] = \left[\int_A \{N\}\, {}_{\lfloor}N_{\rfloor}\, dA\right] \tag{20}$$

$$[s] = \left[\int_A \left[\{N\}\left(\frac{\bar{u}}{h}\frac{\partial\, {}_{\lfloor}N_{\rfloor}}{\partial x} + \frac{\bar{v}}{h}\frac{\partial\, {}_{\lfloor}N_{\rfloor}}{\partial y}\right) + K_x^H \frac{\partial\{N\}}{\partial x}\frac{\partial\, {}_{\lfloor}N_{\rfloor}}{\partial x} + K_y^H \frac{\partial\{N\}}{\partial y}\frac{\partial\, {}_{\lfloor}N_{\rfloor}}{\partial y}\right] dA\right] \tag{21}$$

The vector {Q} accounts for the source or sink terms and any prescribed boundary conditions. Finally, by assembly of the global equations from the element equations

$$[H]\{\dot{\phi}\} + [S]\{\phi\} = \{\hat{Q}\} \qquad (22)$$

where [H], [S] and $\{\hat{Q}\}$ correspond to [h], [s] and {Q}.

The idealization for transport analysis is done in the same way as for flow analysis. After calculation of the velocities in the flow analysis the values obtained are used in the formation of the matrix [s].

Loziuk, et al[12,13] apply the above approach to various practical problems, including an actual lake with irregular boundary. Available field data indicate a reasonable level of agreement with the analysis results. Tong[10] calculates the diffusion of a substance in a rectangular basin containing a circular island.

Solutions for transient flow governed by the shallow water equations have been given by Connor and Wang[24]. By integrating across the depth and assuming uniform velocity and hydrostatic pressure over the depth they establish equations in terms of nodal values of flux and elevation. Solutions are given for harmonic forcing of a rectangular basin and for tidal circulation in Massachusetts Bay.

Taylor and Davis[26] have developed finite element representations of tidal propagation in estuaries. The unknowns in these equations are the node point velocities and elevations. Surface runoff, described by means of the shallow-water equations, has been studied by Al-Mashidani and Taylor (Ref. 30).

They treat a one-dimensional case, with velocity and surface elevation as problem unknowns.

Taylor and Davis (26) and Adey and Brebbia (27) have studied dispersion in estuaries. Ref. 27 uses known values of velocity and solves for the concentration. Taylor and Davis, on the other hand, solve for concentration, velocity, and surface elevation.

Planform (x-y) finite element analysis of a rather different environmental problem has been persented by Mercer and Pinder (29). They examine heat transport in the liquid and solid phases in a ground-water flow system. The finite element equations to be solved consist of two sets, one being a flow equation in terms of pressure and the second being a temperature equation. The solutions are marched in time.

IV. CROSS-SECTION ANALYSIS

The motivation for cross-section analysis (x-z) has principally been the prediction of thermal stratification, although attempts have also been made to deal with more basic phenomena in viscous flow.

Thermal stratification is widely believed to exert an important influence on lake flow phenomena through its effects on density variations and other physical factors. In many lakes uniform temperature conditions are realized in winter and, as summer atmospheric conditions approach, a rise in temperature occurs in the upper regions of the lake. The peak is reached in these regions towards the end of summer. Since the rise in

temperature penetrates to only a limited depth (say 20 to 40 feet) the lower portions of the lake are not affected, and a somewhat 'stratified' temperature profile prevails. The heated upper region is known as the epilimnion while the unheated lower region is termed the hypolimnion.

The problem to be solved is the vertical temperature profile. There is an influence, however, of the action of the wind and this produces a two-dimensional problem.

Liggett and Bedford (Ref. 14) and Bedford (Ref. 15) have dealt with the steady-state problem of a two-dimensional cavity containing a nonhomogeneous fluid subjected to surface shear. No consideration was given to eddy viscosity and diffusivity variations. The latter was accounted for by Young, Liggett, and Gallagher (Ref. 16) and the results demonstrate that stratification, as well as circulation patterns, can be predicted with the proper empirical definition of these variations. Skiba, Unny and Weaver (Ref. 17), Debongnie (Ref. 18), and Kawahara, et al (Ref. 19) have studied cavity flow without the consideration of temperature. Coupled velocity-temperature solutions are also described by Zienkiewicz, Gallagher, and Hood (Ref. 20). In the following we describe the development of Young, Liggett and Gallagher.[16]

The physical properties which enter into the differential equations of the problem are the eddy viscosity and the eddy diffusivity. The eddy viscosity and diffusivity in the horizontal direction (K^M_{xx} etc. and K^H_x) can realistically be taken as constant. Values of these coefficients are customarily taken as the same

magnitude as those which are measured under neutral stratification. The vertical eddy viscosity and diffusivity (K^M_{xz} etc. and K^H_z) vary highly within the whole basin, however, and are dependent on such factors as the turbulence level in the surface layer, the depth, the local density gradient and the overall motion with respect to the specified geometry.

No satisfactory theory for the prediction of these variations from the more basic environmental and physical parameters is presently available and dependence must be placed on empirical relationships. In this work the relationships employed are extended forms of those proposed by Sundaram and Rehm[21], as follows

$$K^M_{xz} = K^M_{zz} = K^M_o (1 - \sigma_m Ri) \tag{23}$$

$$K^H_z = K^H_o (1 - \sigma_h Ri) \tag{24}$$

where Ri, the Richardson number, is

$$Ri = - \frac{g z^2 \frac{\partial \rho}{\partial z}}{\rho_o U^2} \tag{25}$$

in which U is a characteristic velocity, σ_m and σ_h are empirical constants, and K^M_o and K^H_o are the vertical eddy viscosity and diffusivity under neutral stratification. The continuity equation, with the assumption of incompressibility, simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{26}$$

In defining the relevant forms of the momentum equations we assume that Boussinesq's approximation applies (ρ is taken as constant except when multiplied by g, i.e., in buoyancy terms).

Thus,

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho_o} \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left(K_o^M \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z} \left(K_z^M \frac{\partial u}{\partial z}\right) \quad (27)$$

$$u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho_o} \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left(K_o^M \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z} \left(K_z^M \frac{\partial w}{\partial z}\right) - \frac{\rho}{\rho_o} g \quad (28)$$

where p is the local pressure.

The diffusion-advection of temperature is given by

$$u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \frac{\partial}{\partial x} \left(K_o^H \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial z} \left(K_z^H \frac{\partial T}{\partial z}\right) \quad (29)$$

Finally, the equation of state can be written more explicitly as

$$\rho = \rho_0 [1 - \beta(T - T_o)] \quad (30)$$

in which β is the coefficient of volumetric expansion (assumed constant) and T_o is the point about which the true relationship is linearized.

We introduce the stream function ψ in place of u and v, such that

$$u = \frac{\partial \psi}{\partial z} \quad , \quad w = - \frac{\partial \psi}{\partial x} \quad (31)$$

The resulting two differential equations, which replace Eqs. (27-30), can then be written in terms of nondimensional variables as follows

$$D_1(\psi,\rho) = - \nabla^4 \psi + Re \frac{\partial(\nabla^2 \psi, \psi)}{\partial(x,z)} - Re\ Ri_o \frac{\partial \rho}{\partial x}$$
$$+ \sigma_m Ri_o \left\{ \frac{\partial^2}{\partial z^2} \left(z^2 \frac{\partial \rho}{\partial z} \frac{\partial^2 \psi}{\partial z^2}\right) \right.$$
$$\left. + \frac{\partial^2}{\partial x \partial z} \left(z^2 \frac{\partial \rho}{\partial z} \frac{\partial^2 \psi}{\partial x \partial z}\right) \right\} = 0 \quad (32)$$

and,

$$D_2(\psi,\rho) = - \nabla^2 \rho + Re\ Pr \frac{\partial(\rho,\psi)}{\partial(x,z)}$$
$$+ \sigma_h Ri_o \frac{\partial}{\partial z} \left\{ \left(z \frac{\partial \rho}{\partial z}\right)^2 \right\} = 0 \quad (33)$$

where $Re = UH/K_o^M$ is the Reynolds number, $Pr = K_o^M/K_o^H$ is the turbulent Prandtl number under neutral stratification, and $Ri_o = -\Delta\rho gH/\rho_o U^2$ is the overall Richardson number. H is the depth of the cavity. All parameters and variables have been nondimensionalized, e.g., x and z have been divided by H.

To transform the above into a finite element representation we adopt shape function approximations for ψ and ρ and use the Galerkin method. Thus, with $\psi = N_i\psi_i$ and $\rho = Q_i\rho_i$ we have the following weighted integrals

$$\int_A N_i[D_1(\overline{\psi},\overline{\rho})]dA = 0 \tag{34}$$

$$\int_A Q_i[D_2(\overline{\psi},\overline{\rho})]dA = 0 \tag{35}$$

This leads to the following set of nonlinear algebraic equations

$$S^1_{ij}\psi_j + Re\ S^2_{ijk}\psi_j\psi_k + Re\ Ri_o\ S^3_{ij}\rho_j - \sigma_m\ Ri_o\ S^4_{ijk}\psi_j\rho_k + P_i = 0 \tag{36}$$

$$S^5_{ij}\rho_j + Re\ Pr\ S^6_{ijk}\psi_j\rho_k - \sigma_h\ Ri_o\ S^7_{ijk}\rho_j\rho_k = 0 \tag{37}$$

The multipliers Re, Ri_o, Pr, σ_m and σ_h have been preserved in these representations to enable identification of the source of each term.

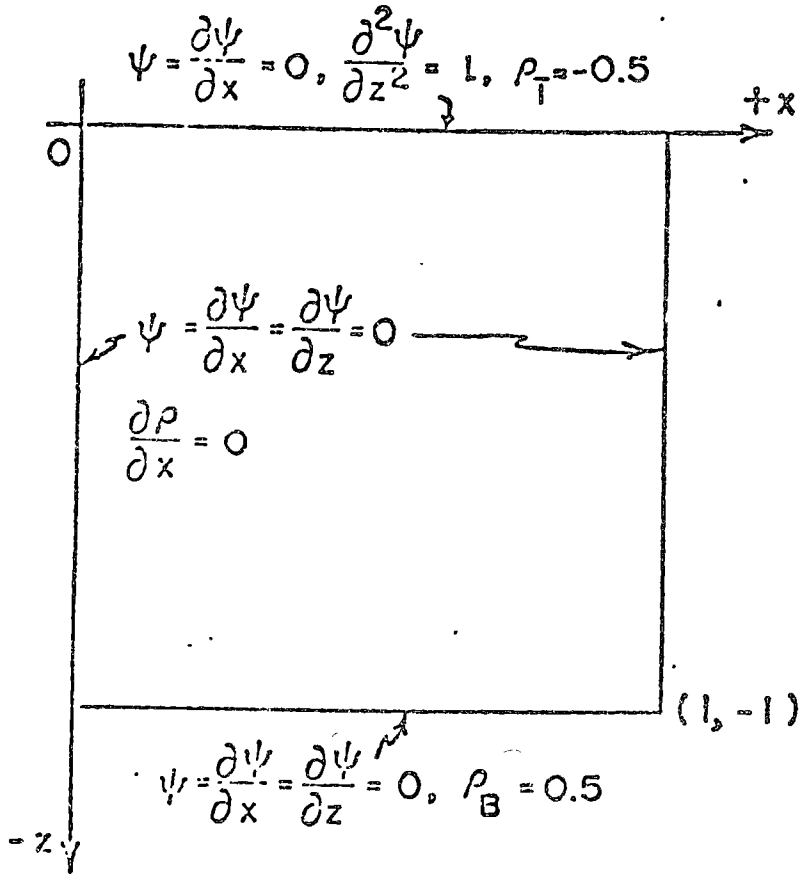
The specific algebraic form of the coefficients $S^1_{ij},\ldots,S^7_{ijk}$ is obtained after performance of the integration indicated in Eqs. (34) and (35).

The global representation is obtained by summation, from the coefficients of the above element equations, of all coefficients with like subscripts. The resulting equations are of

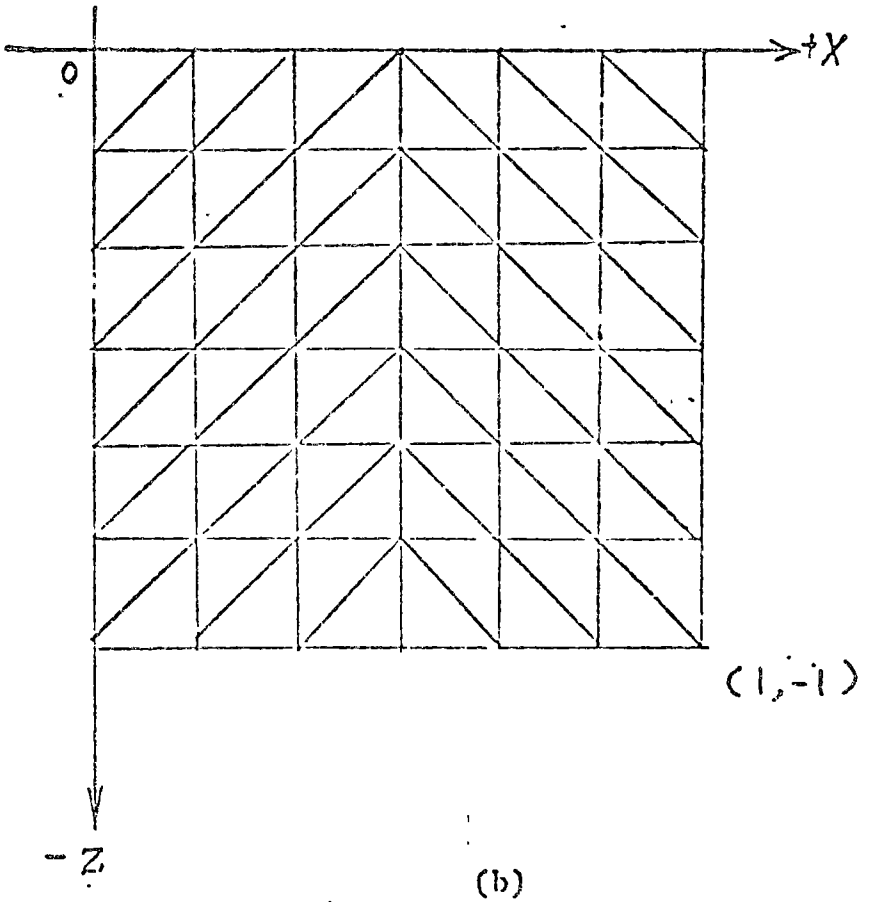
a form identical to that of Eqs. (36) and (37). The Newton-Raphson approach is adopted as the method of solution of these coupled nonlinear equations.

Numerical calculations were performed for the square cavity of Fig. 3a for the boundary conditions shown and for various assumed vertical formulations of the eddy viscosity and diffusivity. The finite element representation consisted of 72 elements arrayed in the 6x6 gridwork. (Fig. 3b).

Steady state calculations have been performed for Re = 1 to Re = 1000, Gr = 0 to Gr = 10000, and Pr = 1 to Pr = 10 where Gr is the Grashof number (The Grashof number is $Gr = Re^2 Ri_o$.). Additional numerical experiments were performed to test the sensitivity of the solution on the assumed behavior of the eddy viscosity and eddy diffusivity. Ten such computations were performed, all using Re = 100, Pr = 1, $Ri_o = 1$ but different choices of σ_h and σ_m and also different assumptions as to the form of the depthwise variation of K_{xz}^M and K_z^H as summarized in Table 1. The values of the characteristic numbers represent, of course, an infinite variety of physical data, but the following are typical: $\rho_B = 1.0$ gr/cm^3, $\rho_T = 0.9999$ gr/cm^3 ($T_T - T_B \simeq 4°C$), $\tau_x = 1.0$ dyne/cm^2, $K_o^M = K_o^H = 100$ cm^2/sec, H = 10m, and g = 980 cm/sec^2. These are approximately equivalent to the experimental data of Sundaram et al[22]. However, in the present case the boundary conditions have been chosen so that a steady-state solution exists, a condition relaxed in some subsequent computations.


$\psi = \frac{\partial \psi}{\partial x} = 0, \frac{\partial^2 \psi}{\partial z^2} = 1, \rho_T = -0.5$
+x
0
$\psi = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = 0$
$\frac{\partial \rho}{\partial x} = 0$
(1, -1)
$\psi = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = 0, \rho_B = 0.5$
-z

(a)


+X
0
(1, -1)
-Z

(b)

Figure 3

TABLE 1

Summary of Computations
for Re = 100, Pr = 1, Ri_o = 1

RUN NO.	1	2	3	4	5
	σ_h	σ_m	Depth-Dependent	Cut-off Below the First Cut-off Depth	Plot Symbols
1	0	0	---	---	○
2	0.1	0.035	No	---	△
3	0.2	0.07	Yes	---	
4	0.3	0.1	Yes	---	
5	0.5	0.15	Yes	---	⊡
6	0.8	0.25	Yes	---	
7	1.0	0.35	No	No	⧋
8	1.0	0.35	Yes	No	□
9	1.0	0.35	Yes	Yes	◇
10	2.0	0.70	Yes	No	⊙

Conclusions resulting from the first set of steady-state runs are shown in Table 2 (Ref.15). In these runs eddy viscosity and eddy diffusivity were held constant. A typical picture of streamlines and isopycnals is shown in Figure 4. The influence of the stratification on the circulation is obvious. Additional runs could probably have elicited a specific relationship between the formation of multiple, closed circulation cells and the three parameters, Re, Pr, and Gr. However, such a relationship was not pursued since it would undoubtedly be altered with different geometries and since the eddy viscosity and eddy diffusivity relationships probably have a large effect.

That effect has been tested in ten subsequent runs which are summarized in Figures 5 and 6. In these cases the same sort of cell structure formed as shown in Figure 4, but with considerable variation in the details of the velocity, shape and size of the cells, and the density distribution.

The latter computations show that the density structure continues to have a large effect on the velocity structure and also the velocity structure greatly alters the density distribution. With the eddy viscosity and eddy diffusivity formulation that Sundaram and Rehm[21] found necessary in their one-dimensional analysis, the surface shear alone is sufficient to form a thermocline type of structure. This result is quite different, but does not conflict with, those of previous investigators who have used a one-dimensional analysis. In those previous investigations the thermocline structure formed over a period of time (several weeks) while unsteady heat inputs were applied. We have shown, however, that given an initial inhomogeniety in density, a wind shear is quite sufficient

Table 2. Summary of Results

	Streamfunction Results	Density Field Results
Re	An increasing Re reflects an increase in wind shear, the primary source of kinetic energy. Momentum transport is increasingly by convection, which discourages the formation of cells. The primary vortex center moves downstream and toward the surface with decreasing momentum diffusion.	Density transport is increasingly by advection. High Re encourages the accumulation of a thickening region of light homogeneous fluid at the top of the downwind portion of the cavity. Vertical and horizontal density gradients rise with the loss of diffusion.
Gr	At all but the lowest levels of kinetic energy (Re = 1.0) an increasing Gr encourages the growth of cells, by increasing the strength of the applied vertical density gradients. The angle of tilt between the cells diminishes with increasing Gr.	An increasing applied density difference or buoyancy strength seeks to create a linearly varying vertical density profile. Additional flow field energy is required for the readjustment as reflected in the increasing Gr.
Pr	By severely increasing the density gradients, an increased Pr encourages the formation of cells. The angle of tilt between cells increases with decreasing density diffusion.	By further diminishing density diffusion with increasing Pr an additional accumulation of lighter homogeneous fluid occurs. Also an additional increase in the energy absorbing vertical and horizontal density gradients is noted.
Low Aspect Ratio (shallow cavity)	The unimportance of horizontal diffusion results in a more exaggerated density and flow field response to any change in the Re, Pr, and Gr. Therefore cells form at lower Gr, or Pr. Vortex centers are convected farther apart horizontally with increasing transport by inertia. For a given wind and applied density difference, the vertical density gradient is more severe and thereby requires more kinetic energy to overcome.	
High Aspect Ratio (deep cavity)	Homogeneous circulation cells appear when momentum diffusion can't penetrate the wall shear resistance. Each cell rotates in the opposite direction to and is O(10) less intense than the cell above.	All test cases were homogeneous.

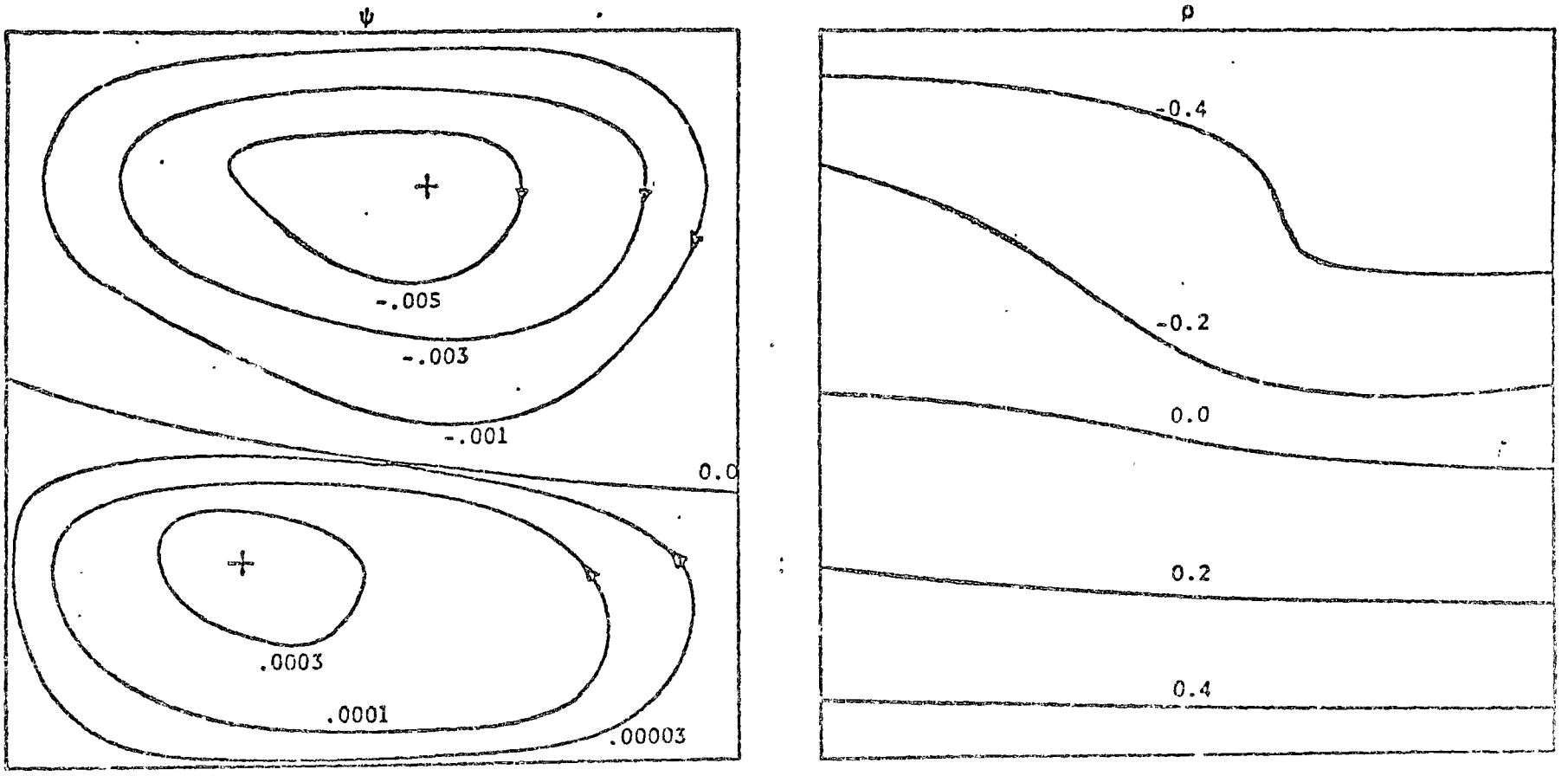


Figure 4. Streamline and Isopycnic Contours; Shear Driven, O(1) Cavity, Re = 100, Gr = 1000, Pr = 5.0

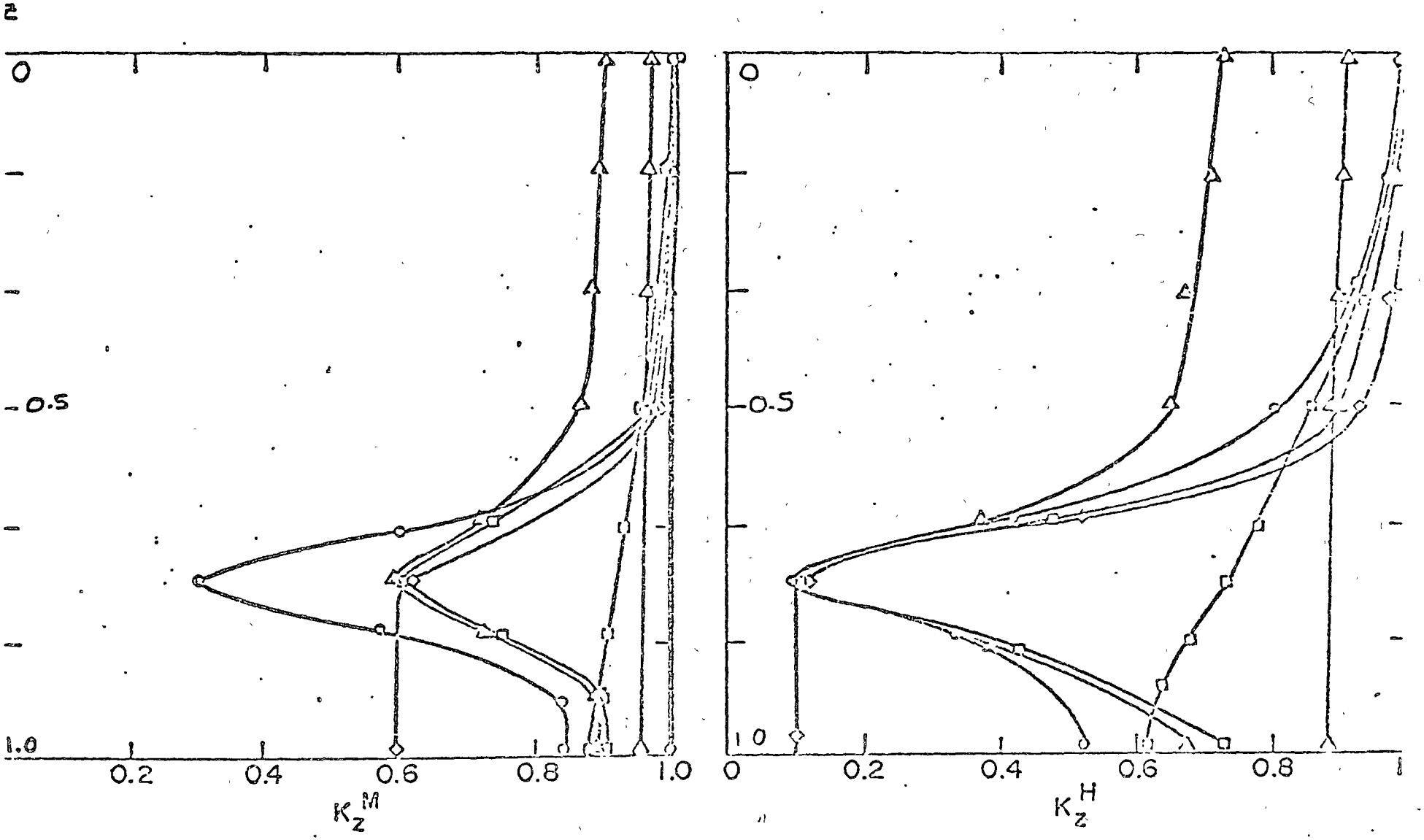
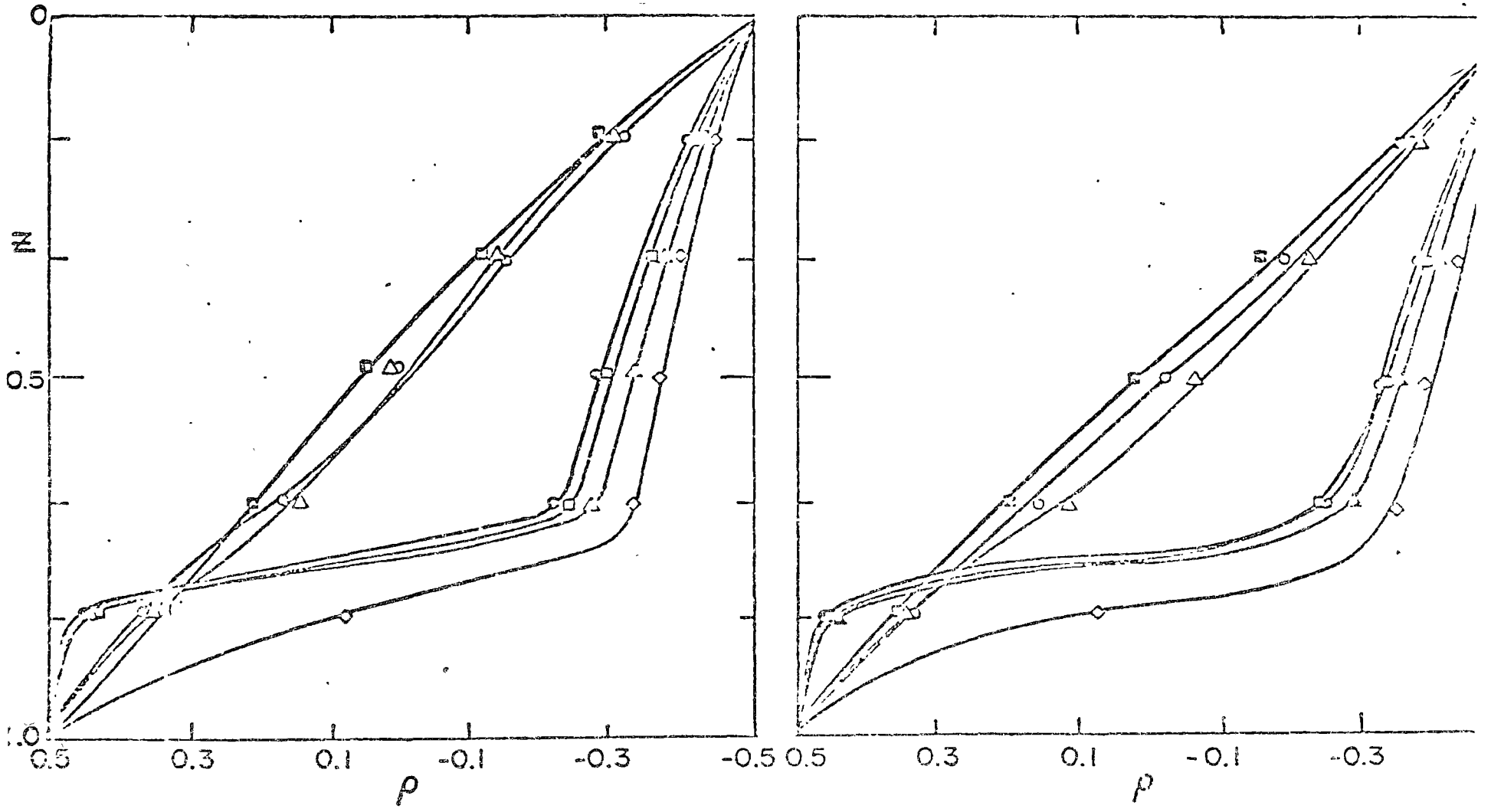


Figure 5 Vertical Distribution of Eddy Viscosity (K_z^M) and Eddy Diffusivity (K_z^H) at x = 0.8333.

(See Table 1 for definition of symbols)



a. x = 0.1667

b. x = 0.8333

Figure 6 Density (ρ) versus Depth (z) at x = 0.1667

(See Table 1 for definition of symbols)

to form the thermocline. Unsteady computations, shown below, indicate the time scale involved in such a formation and also the extent of the feedback which influences the current structure.

Fewer transient computations of cavity flow were made due to the computer costs. A total of five runs with Reynolds numbers of 100 and 1000 and Richardson numbers of 1 and 10 were made. Two values of σ_h were used. Figure 7 indicates the results of one of these calculations. In all cases in which the motion begins from rest, the entire cavity begins to circulate as a whole; that is, the cavity forms a single circulation cell. As time progresses the flow may break up into two or more cells, as is indicated in Figure 7. At the same time the density distribution is altered to show the typical thermocline shape.

The flow does not change from a state of rest to the final cell formation montonically. Instead the velocities increase rapidly to a value not far from the steady state value and then oscillate about this value. The frequency of oscillation is near the Brunt-Vaisala frequency. Other characteristics of the flow, the density gradient, the cell location, and the streamline positions, show similar damped oscillations.

The number of cells can be calculated, using certain gross approximations, from the theory of Turner[32] as expanded for this problem by Young[31]. This theory has been compared with the transient and steady-state computations with rough agreement. The difficulty in the application of such theories to real lakes (or even cavities) is that all the factors, the most important being the density distribution, and the interaction of those factors cannot be considered adequately. Results indicate that multiple cells are

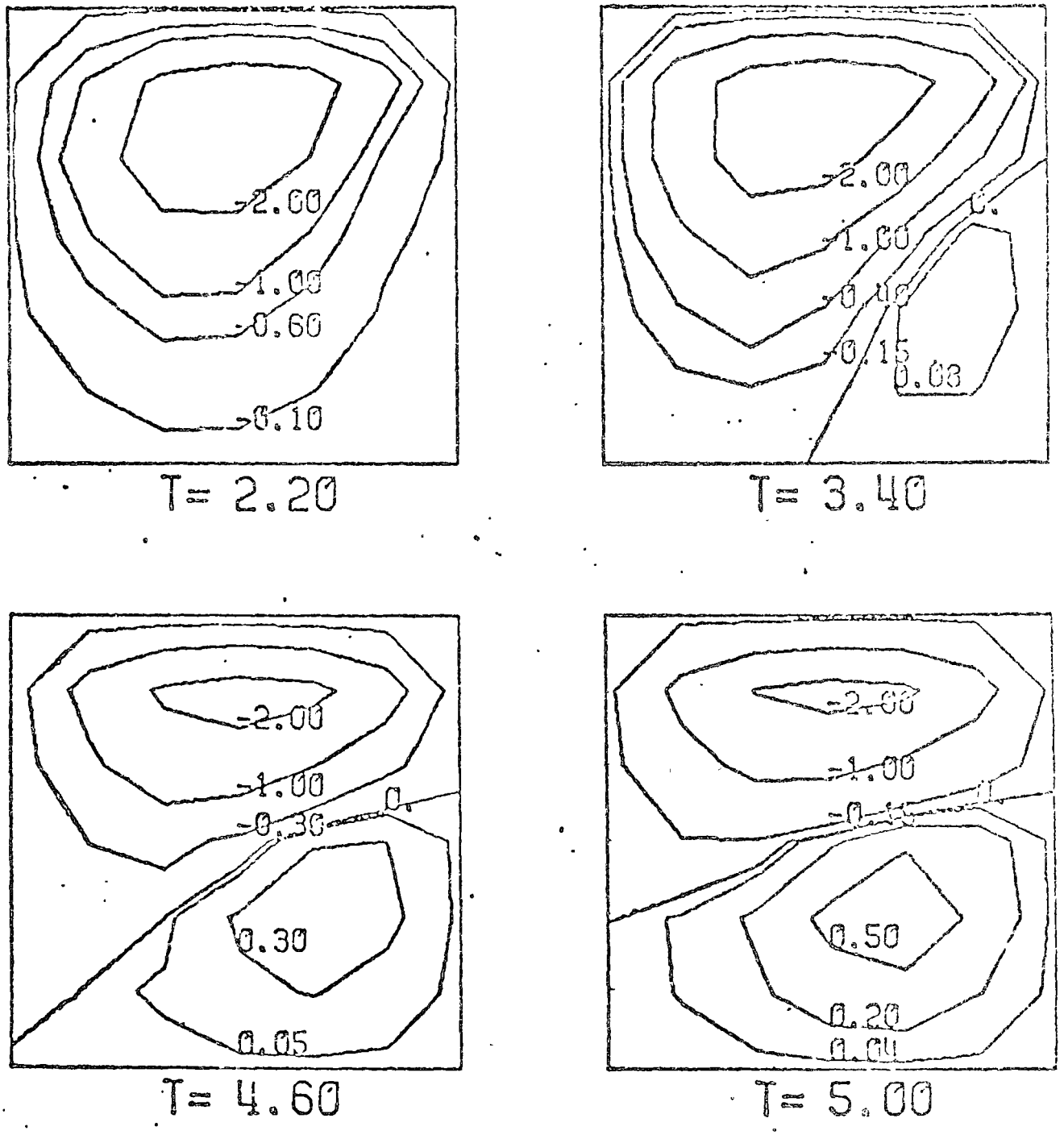
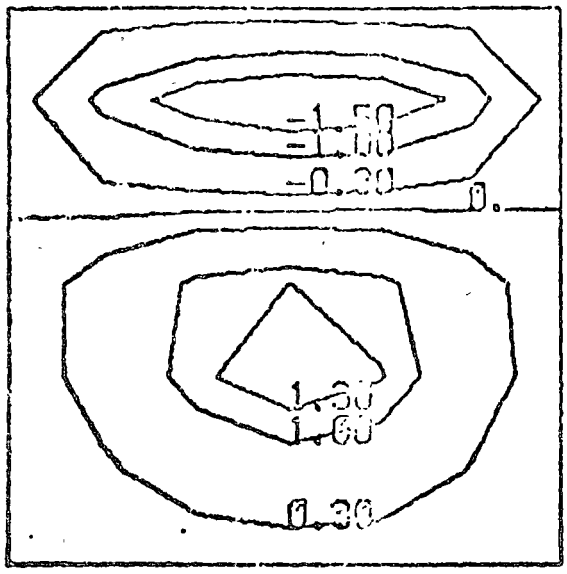
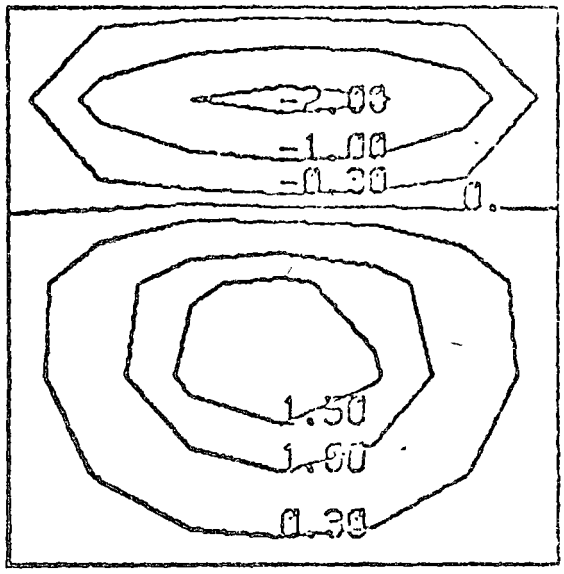


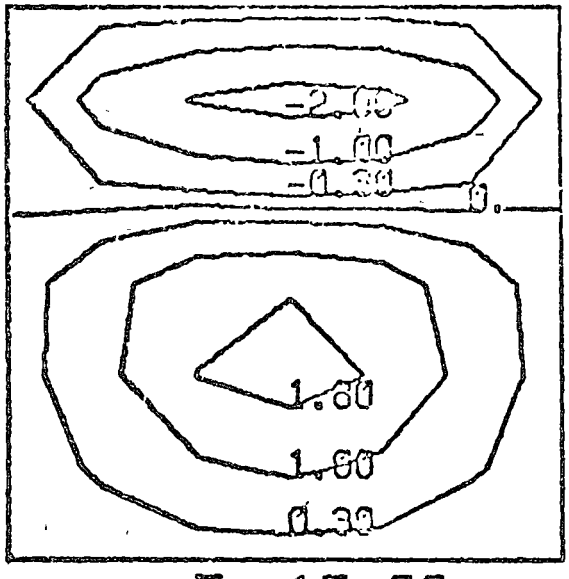
Figure 7 Streamline Pattern for Stratified Flow at Various (Dimensionless) Times



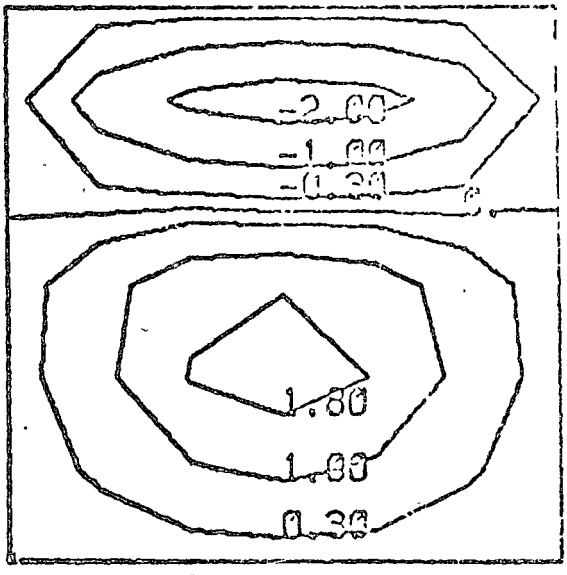
T= 7.00



T= 10.00



T= 15.00



T= 25.00

Figure 7 (continued)

likely to form in the case of a diffuse metalimmion whereas a sharp thermocline promotes two cell circulation. However, the feedback of the currents into the density structure has not been considered, and this feedback may alter the density distribution, thus changing the results of the theory.

Variable viscosity, especially a formulation which is strongly dependent on the density gradient, has a great effect on the ability of the current structure to alter the density distribution. A reduction of viscosity in zones of steep density gradient promotes the oscillations observed previously and increases their amplitude (but does not affect the frequency). The reduction in viscosity also increases the time to equilibrium significantly in those cases in which a steady state exists.

A particularly striking result of the transient calculations is the time scale involved in thermocline formation. A shear applied at the surface may alter the density distribution and create a thermocline-like structure in a few hours versus the weeks involved in the one-dimensional computations. Thus the entire process of the development, maintenance, and erosion of the thermocline is a complex process strongly influenced by the current structure. The "physical constants" (i.e., eddy diffusivity) derived for the one-dimensional analysis have, in reality, little physical meaning when the current structures is neglected.

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CONSTITUTIVE EQUATIONS FOR INELASTICITY

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5. CONSTITUTIVE EQUATIONS FOR PLASTICITY AND CREEP

R. H. Gallagher

I. INTRODUCTION

Analyses for material nonlinearities have been of increasing importance in recent years in conjunction with nuclear powerplant design. A fairly complete finite element approach to the analysis of such problems has developed and at the present time these capabilities are routinely operational in many large design offices. It is the purpose of these notes to present the relevant basic relationships in finite element inelastic analysis, to describe the established computational algorithms, and to identify some significant new developments and directions of future work. As in any nonlinear aspect of analysis, where the costs of current procedures are substantially greater than for elastic analysis, new developments are pursued vigorously for the sake of improvements in computational efficiency.

It is customary to divide the problems of inelastic analysis into the categories of time-independent and time-dependent behavior. Time-independent inelastic behavior occurs when a structure is loaded in such a way that strains in excess of the elastic limit are sustained. In time-dependent behavior permanent deformation accumulates with the passage of time even under stresses which are well within the elastic limit. This division into two categories is realistic for many design situations. Noting that there are design situations where this division is unrealistic and that some finite element analysis work has been addressed to more general material behavior, we nevertheless follow these divisions in this and the next lecture.

A subsequent lecture, dealing with "viscoplasticity", erases these distinctions.

In extending the finite element method from elastic to inelastic analysis two considerations stand foremost: (1) the establishment of inelastic stress-strain (constitutive) relationships, and (2) the formulation of reliable and efficient algorithms for the solution of the associated large-order systems of nonlinear algebraic equations. The first of these topics is examined in the present section.

II. TIME-INDEPENDENT PLASTICITY

1. UNIAXIAL STRESS-STRAIN BEHAVIOR

This section is devoted to an outline of those considerations which are relevant to our formulation of finite element time-independent inelastic analysis. Detailed developments of plasticity theory can be found in Refs. 1-5. In constructing the relationship between stress and strain in the inelastic range of multiaxial states of stress one must define (a) the condition for yield, (b) the general form of the desired stress-strain law, and (c) a criterion for work hardening. Preliminary to these operations, however, an understanding must be gained of uniaxial stress-strain behavior into the inelastic range because the theory of multiaxial plasticity is largely concerned with the analytical transformation of such behavior into two- and three-dimensions.

The "basic" uniaxial stress-strain relationship, obtained in a standard tension test, is given by the solid curve in Fig. 1. Engineering strains (the specimen elongation divided by the nominal length of the specimen), rather than true strains, are represented.

Test specimens, at a certain strain, are reduced in area in a small region ("necking") and it is this area which is used in the calculation of true stresses. The nominal stress, however, is usually represented in figures such as Fig. 1. The point A is the elastic limit, which is herein assumed to be synonymous with the yield point.

Mild steel, of the type generally used in civil engineering structures, displays a somewhat different stress-strain diagram (see Fig. 2) in that it has a flat yield region. This phenomenon is attributed to slip along slip planes of the crystals.

Materials which demonstrate very little plastic deformation before breaking (such as cast iron, beryllium, rock) are called brittle materials. This behavior is shown in Fig. 3. It has been demonstrated experimentally that rocks tend to become brittle when subjected to large hydrostatic pressure.

Consider the (Fig. 1) application of load past the yield point to point B. The load is then removed and the strains decrease linearly until at zero load (point C) the residual total strain is $\overline{OC}$. If the load is then reapplied, the straight line $\overline{CB}$ is retraced. For stresses higher than σ_B, the original stress-strain curved (solid line) is followed.

If compressive loading were applied beyond point C, it would be expected that strains would continue to develop linearly up to a point. The point at which the stress-strain relationship again becomes non-linear is often assumed to be equally distant from C as the length BC. Thereafter, for continued compressive loading, Figure 1 shows the curve BCB' being followed, where the curve CBD

is identical in shape to CB'D'. The analytical representation of this behavior is termed an isotropic hardening representation.

Often, when a real metal is deformed in tension and the load is then reduced and progressed to compression yield, it will be found that the yield stress in compression is considerably less than that of tension. This is termed the Bauschinger effect and if it is taken into account the length $\overline{CB'}$, in Figure 1, would be considerably less than $\overline{CB}$. Theoretical plasticity has attempted to account for effect through what is termed a kinematic hardening representation.

Plastic deformation is physically anisotropic. The process of slip on a crystal plane is clearly directional. As a consequence, any initial isotropy which may have been present is usually destroyed by plastic deformation. From the point of view of the dislocation theory, slip is an irreversible process; every slip produced a new material. These changes are revealed in the Bauschinger effect and in the anisotropy of materials after plastic deformation.

2. YIELD CRITERION-MULTIAXIAL STRESS

In order to be able to solve multiaxial stress problems, one must be able to relate increments in stress $\{d\sigma\}$ with increments in the corresponding strains $\{d\varepsilon\}$ by extrapolating from a simple uniaxial tension test into the multidimensional situation.

The definition of a yield criterion, the combination of stresses that brings about yield, is an initial step in the formulation of the incremental elastoplastic stress-strain law. Experiments have shown that the shear stress is the major cause of yielding. Moreover, hydrostatic pressures of the order of the yield stress have little

influence on yielding of metals. The chosen form of the yield criterion must reflect these considerations. Although alternative forms are available and have been used in finite element analysis the Von Mises criterion has overwhelmingly been the choice. This is due to the continuous nature of the function defining this criterion and because it holds reasonably well for the commonly-encountered ductile materials. It states that yield occurs when the effective stress ($\bar{\sigma}$) equals the yield stress ($\bar{\sigma}_y$) measured in a uniaxial test. For an isotropic material

$$\bar{\sigma} = \frac{1}{\sqrt{2}} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{\frac{1}{2}} \qquad (1)$$

where σ_1, σ_2, and σ_3 are the principal stresses. A Poisson's ratio of $\frac{1}{2}$ has been introduced, consistent with incompressible material behavior, since yielding is assumed to be an incompressible phenomenon.

$\bar{\sigma}$ can also be expressed in terms of non-principal stresses,

$$\bar{\sigma} = \frac{1}{\sqrt{2}} [(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6 (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{\frac{1}{2}} \qquad (2)$$

This can also be written in terms of the second invariant (J_2) of the stress deviation $\bar{\sigma} = \sqrt{J_2}$ (3)

The corresponding expression for effective plastic strain, $\bar{\varepsilon}^p$, is given in differential form, as a function of increments in the principal plastic strains. Thus

$$d\bar{\varepsilon}^p = \frac{\sqrt{2}}{3} [(d\varepsilon_1^p - d\varepsilon_2^p)^2 + (d\varepsilon_2^p - d\varepsilon_3^p)^2 + (d\varepsilon_3^p - d\varepsilon_1^p)^2]^{\frac{1}{2}} \qquad (4)$$

The plastic strains here have the usual meaning, that is they represent the differences between the total strains and the elastic strains as obtained by Hooke's law. Expressed in terms of increments of non-principal strains,

$$d\bar{\varepsilon}^p = \frac{\sqrt{2}}{3}\left[(d\varepsilon_x^p - d\varepsilon_y^p)^2 + (d\varepsilon_y^p - d\varepsilon_z^p)^2 + (d\varepsilon_z^p - d\varepsilon_x^p)^2 + \frac{3}{2}\left(d\gamma_{xy}^{p\,2} + d\gamma_{xz}^{p\,2} + d\gamma_{zx}^{p\,2}\right)\right]^{\frac{1}{2}} \qquad (5)$$

Note that according to these definitions neither $\bar{\sigma}$ nor $d\bar{\varepsilon}^p$ can assume negative values.

With these expressions in mind, it is possible to conduct a multiaxial test in which the stresses are statically known at all times, and to measure the corresponding incremental plastic strains. From them the effective stresses and effective plastic strains may be calculated and plotted as in Figure 4. The effective plastic strain $\bar{\varepsilon}^p$ is the integral of Eq. 4 (or Eq. 5) taken along the loading path so that all of the increments of plastic strain are included. As a special case, for uniaxial stress where the only stress component different from zero is σ_x, $\bar{\sigma} = \sigma_x$ and $\bar{\varepsilon}^p = \varepsilon_x^p$. Thus Figure 4 may be obtained directly from Fig. 1 by first subtracting out the elastic strains σ/E from the total strains ε. (Note that unloading along the line BC corresponds to zero change in effective strain). According to the theory, a variety of loading combinations could be investigated experimentally, and for a given material, all should give the same $\bar{\sigma}$ vs $\bar{\varepsilon}^p$ curve.

3. FLOW RULE

Two concepts are central to the establishment of the elasto-plastic material stiffness matrix $[E^{ep}]$: the slope (H') of the

tangent to the effective stress-effective plastic strain ($\bar{\sigma}$-$\bar{\epsilon}^p$) diagram, otherwise known as the hardening coefficient, and the "flow rule", or description of the differential changes in the plastic strain components $\{d\epsilon^p\}$ as a function of the rates of change of stress.

With respect to H', it is apparent from Figure 4 that

$$H' d\bar{\epsilon}^p = d\bar{\sigma} \tag{6}$$

Furthermore, the total differential $d\bar{\sigma}$ can be written as

$$d\bar{\sigma} = \frac{\partial\bar{\sigma}}{\partial\sigma_1} d\sigma_1 + \frac{\partial\bar{\sigma}}{\partial\sigma_2} d\sigma_2 + \frac{\partial\bar{\sigma}}{\partial\sigma_3} d\sigma_3 = \left[\frac{\partial\bar{\sigma}}{\partial\sigma}\right] \{d\sigma\} \tag{7}$$

so that, from Eq. 6

$$H' d\bar{\epsilon}^p = \left[\frac{\partial\bar{\sigma}}{\partial\sigma}\right] \{d\sigma\}$$

For the flow rule we choose the commonly employed Prandtl-Reuss representation, with isotropic hardening

$$\{d\epsilon^p\} = d\bar{\epsilon}^p \left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\} \tag{9}$$

The significance of this representation is described in Figure 5, illustrating the "stress-space" for the two-dimensional case. The solid curve gives the yield surface (locus of all points (stress states) causing yield) as defined by the Von Mises criterion, Equation 1.

Equation 9 expresses the condition that the direction of inelastic straining be normal to the yield surface and is therefore alternatively termed the normality condition.

To elaborate upon Eq. 9 algebraically, we consider the typical term $d\epsilon_i^p$ in the vector $\{d\epsilon^p\}$. $\bar{\sigma}$ is given by Eq. 1 and performing on it the required operation $\frac{\partial\bar{\sigma}}{\partial\sigma_1}$, we obtain

$$d\epsilon_1^p = \frac{d\bar{\epsilon}^p}{\bar{\sigma}} \left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\right] \tag{10}$$

Consider now the differential form of the familiar stress-strain law with the plastic strains interpreted as initial strains

$$\{d\sigma\} = [E]\{d\epsilon\} - [E]\{d\epsilon^p\} \tag{11}$$

which becomes, after substitution of Equation 9

$$\{d\sigma\} = [E]\{d\epsilon\} - [E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\} d\bar{\epsilon}^p \tag{12}$$

Multiplying through by $\left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor$

$$\left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor\{d\sigma\} = \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\{d\epsilon\} - \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\} d\bar{\epsilon}^p \tag{13}$$

and, with use of Equation 8, in place of the left-hand side

$$H' d\bar{\epsilon}^p = \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\{d\epsilon\} - \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\} d\bar{\epsilon}^p \tag{14}$$

By rearrangement

$$d\bar{\epsilon}^p = \frac{\left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\{d\epsilon\}}{H' + \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}} \tag{15}$$

Finally, by substitution of this expression into Equation 12, we obtain

$$\{d\sigma\} = [E^{ep}]\{d\epsilon\} \tag{16}$$

where

$$[E^{ep}] = [E] - [E^p] \tag{17}$$

with [E] the conventional elastic material stiffness matrix and

$$[E^p] = \frac{[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}\left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]}{\left(H' + \left\lfloor\frac{\partial\bar{\sigma}}{\partial\sigma}\right\rfloor[E]\left\{\frac{\partial\bar{\sigma}}{\partial\sigma}\right\}\right)} \tag{18}$$

It is of interest to observe that $[E^{cp}]$ (and $[E^{p}]$) are symmetric and that the case of perfect plasticity, where $H' = 0$, causes no difficulty.

In isotropic hardening the yield surface expands uniformly about the origin in stress space. In kinematic hardening the yield surface translates as a rigid body in stress space, maintaining the shape and orientation of the yield surface. The yield surface in kinematic hardening is given by

$$\hat{\sigma} = \frac{1}{2} (s_{ij}-\alpha_{ij}) (s_{ij}-\alpha_{ij}) - k^2 \qquad (19)$$

where $s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$ (δ_{ij} is the Kronecker delta) (s_{ij} is a deviatoric stress component), α_{ij} is the translation of the yield locus, and $\hat{\sigma}$ is a constant. By supplanting $\bar{\sigma}$ by $\hat{\sigma}$ in the previous development we can establish the incremental stress-strain relationships for this condition.

Mroz [6] has produced a workhardening representation that is similar to kinematic hardening. Hunsaker, et al [7] have recently conducted an evaluation of various workhardening rules, including isotropic, kinematic, and Mroz's, with a mind towards finite element applications.

III. LINEAR VISCOELASTICITY

Linear viscoelasticity attempts to deal with time-dependent material behavior by establishing mathematical forms of the constitutive relationships involving time (t) and derivatives with respect to time that are linear in the stresses and strains. Many thorough accounts of this topic, including both the representation

of the constitutive relationships and methods of structural analysis based on these relationships, are available (see Ref. 8-12). The following is a brief outline of considerations in representation of viscoelastic constitutive equations, which are important to the methods of viscoelastic finite element analysis to be described subsequently.

Before discussing specific forms of viscoelastic constitutive relationships, it is essential to distinguish between such expressions when written for stress in terms of strain, and vice versa. When stress is written in terms of strain, strain rates, and time, the variation in stress with respect to time for constant strain can be established. This variation, portrayed schematically in Fig. 6a, is termed relaxation, so that the terms relating stress to strain are collectively designated as the relaxation modulus. Conversely, when strain is expressed in terms of stress, derivatives with respect to time, and time, the case of constant applied stress (Fig. 6b.) produced creep behavior and the constitutive relationships define the creep compliance.

Measured (experimental) data are usually obtained in the creep compliance format. The principle of minimum potential energy, the almost universally employed approach to finite element analysis, involves the relaxation modulus format for the constitutive relationships. This presents no practical difficulty for the usual approach to finite element viscoelastic analysis (the time increment-initial strain method) since the coefficients of the constitutive relationships are defined "instantaneously" as numerical values determined separately by reference to the functional (differential

or integral) form of these relationships.

Viscoelastic constitutive relationships may be assumed to be represented by mechanical models consisting of springs and dashpots. This leads directly to the differential form of the constitutive relationship. The simplest representations are those due to Maxwell and Kelvin respectively (see Fig. 7).

The Maxwell model consists of a spring and dashpot in series, representing the following analytical expression for strain versus stress and time (Fig. 7a).

$$\varepsilon = \frac{\sigma}{E} + \frac{\sigma}{\eta} t \tag{20}$$

Deficiencies of this representation include the linearity of the strain versus time variation and a failure to represent any "recovery" of viscoelastic strain upon removal of load. Recovery is an experimentally observed phenomenon. By combining a spring and dashpot in parallel (Fig. 7b), the behavior is represented as

$$\sigma = E\varepsilon + \eta \frac{d\varepsilon}{dt} \tag{21}$$

This representation is also deficient because it does not account for the initial elastic strain, as was done in the Maxwell model. Thus, to obtain features of both models it is feasible to tie together four elements (Fig. 7c), to yield

$$\sigma + p_1 \frac{d\sigma}{dt} + p_2 \frac{d^2\sigma}{dt^2} = q_1 \frac{d\varepsilon}{dt} + q_2 \frac{d^2\varepsilon}{dt^2} \tag{22}$$

where p_1, ... q_2 are material constants.

Successively more sophisticated and realistic viscoelastic constitutive relationships can be formed by combining still more springs and dashpots. The most general viscoelastic relationship is then of the form:

$$\sum_{o}^{m} p_k \frac{d^k\sigma}{dt^k} = \sum_{o}^{n} q_k \frac{d^k\varepsilon}{dt^k} \tag{23}$$

where p_o, ... p_k, ... p_m, q_o, ... q_k, ... q_n are material constants.

For the purposes of analysis one seeks a direct viscoelastic relationship in either the creep compliance [$\varepsilon(t) = J(t)\sigma_o$] or relaxation modulus [$\sigma(t) = Y(t)\varepsilon_o$] format, each referring to an initially applied stress (σ_o) or strain which is held constant; rather than the differential form of Eq. 23. This is conveniently accomplished by use of Laplace transform techniques. The analysis problem requires consideration of a time-history of stress and strain intensities, however, and for these cases the concept of the hereditary integral is introduced. As in the case of constant stress or strain, either creep or relaxation formats of this integral may be written. For creep we have

$$\varepsilon\ (t) = \sigma(t)J(o) + \int_{o}^{t} \sigma(t') \frac{dJ(t-t')}{d(t-t')}\, dt' \tag{24}$$

where t' is the time parameter to measure the stress variation and t measures time from the start of viscoelastic deformation.

Any attempt to introduce computational economies by direct use of the functional form of the constitutive relationships in the stiffness equations and integration thereof in time confronts formidable difficulties. Inversion of the creep compliance to define the

relaxation modulus can be costly. Procedures for this are given in Refs. 15-16 and an illustration of the related computational expense is presented by White.[17]

To avoid the difficulties of the formalized representation of viscoelastic constitutive equations and yet retain the hereditary nature of the phenomenon, while dealing with a form appropriate to finite element analysis, the scheme employed in Ref. 18 can be considered. With reference to the Kelvin model (Eq. 21) for a single element (element i) we have

$$\frac{d\varepsilon^c}{dt} = \frac{\sigma}{\eta_i} - \frac{E_i}{\eta_i}\varepsilon^c \tag{21a}$$

Here, since the elastic strain is not represented, we have set $\varepsilon = \varepsilon^c$. For a series of Kelvin models (1, ... i, ... r) and a finite increment of time (Δt), the increment of creep strain is

$$\Delta\varepsilon^c = [\sum_{i=i}^{r}(\frac{\sigma}{\eta_i} - \frac{E_i}{\eta_i}\varepsilon_i^c)]\Delta t \tag{21b}$$

and the values of σ and ε_i^c for this interval are taken as those existing at the start of the interval.

The generalization of the above to multiaxial states of stress is accomplished straightforwardly, adopting the assumption that linear creep occurs only for the deviatoric components of stress, so that Poisson's ratio equals one-half.

IV. CREEP

Although many efforts have been launched by material scientists in recent years to gain a more complete understanding of creep behavior in metals,[31] many questions remain unanswered and reliable theoretical procedures are not yet available for the

calculation of creep representations from more fundamental physical properties. Thus, dependence is placed upon mathematical representations drawn from test data.

Comprehensive studies of both the creep response of materials and of procedures for structural analysis in the presence of this response are given in Refs. 19-27. In the latter context these references deal almost exclusively with classical analysis procedures whose results are of extremely limited applicability. Nevertheless, common considerations underlie both classical and numerical methods and dependence must be placed by the reader on these references for detailed development of that which will be sketched in the following.

Three questions require study of the definition of appropriate constitutive relationships for creep analysis: (1) the form of uniaxial creep data and its dependence upon such factors as time and temperature, (2) the generalization of uniaxial creep data to multiaxial states of stress, and (3) the manner in which creep strains are accumulated under varying stress and temperature histories.

In discussing (1), uniaxial creep data, it is useful to refer to the original representation of E. N. Costa de Andrade[28] shown in Fig. 8. Here, the creep strain is plotted as a function of time for a given stress level and temperature. This behavior is approximated by an expression of the form

$$\varepsilon^c = \beta t^m + \gamma t \qquad (25)$$

where β, γ, and m are material constants. By differentiation with respect to time

$$\frac{d\varepsilon^c}{dt} = \dot{\varepsilon}^c = m\beta t^{m-1} + \gamma = \dot{\varepsilon}_1^c + \dot{\varepsilon}_2^c \qquad (26)$$

The exponent m is less than one so that for short times the first expression predominates while for long time the second term, γ, is of greatest value. The early portion, governed by βt^m, is termed the primary phase of creep. The portion governed by γt is characterized as secondary creep. The third, or tertiary phase, leading to creep rupture generally represents a highly nonlinear form of behavior where the usual engineering definition of strain does not suffice for valid characterization. Due to the high cost of creep analysis it is usually not feasible to conduct analyses into the tertiary phase with representation of phenomena encountered in this phase.

For analysis purposes it is necessary to express the material constants m, β, and γ as functions of stress and, if possible, temperature. This is desirable even for finite element analysis, which does not require such functional representation, because of the extensive data tabulations needed to account for all ranges of behavior. One of the most popular basic forms of the steady-state creep rate ε_1^c, is that due to Norton[29]

$$\varepsilon_2^c = B\sigma^n \qquad (27)$$

where B and n are material parameters (If this equation is employed as a representation of primary creep then B is a function of time).

Bailey generalized Eq. 27 to account for temperature dependence by defining B as follows (see Refs. 23-25 for discussion of this work)

$$B = De^{bt}(\varepsilon^c)^{-p} \qquad (28)$$

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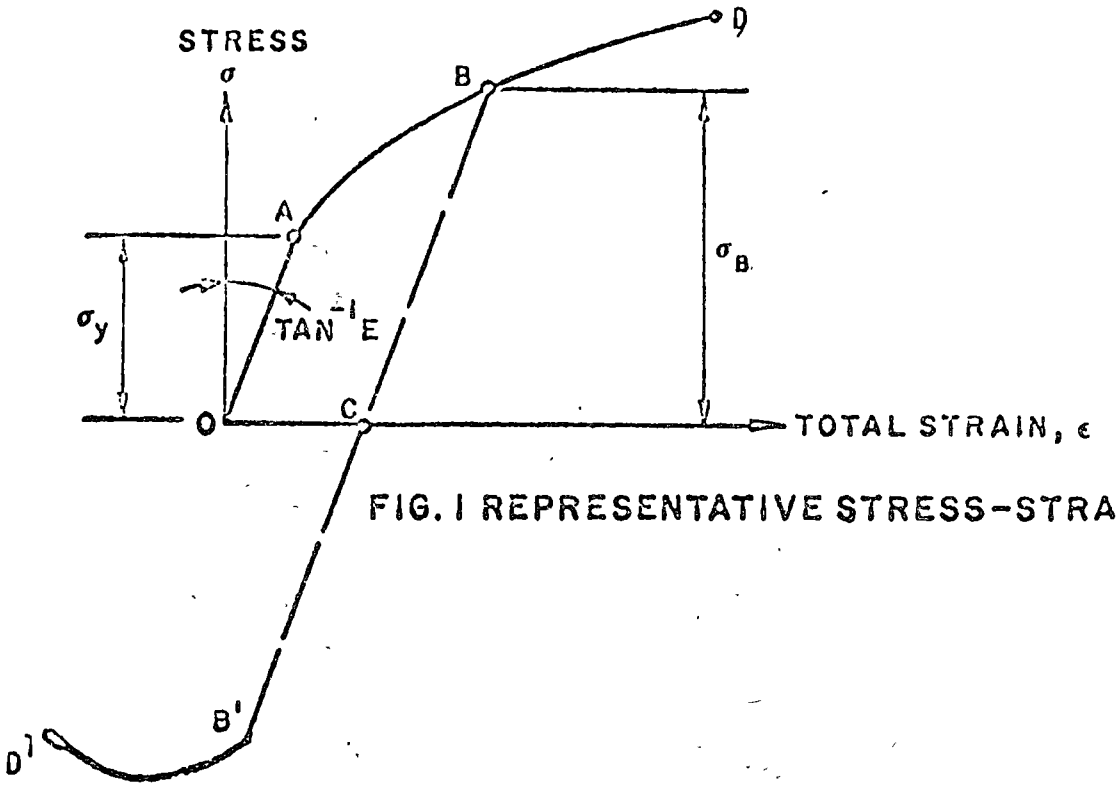


FIG. I REPRESENTATIVE STRESS-STRAIN CURVE

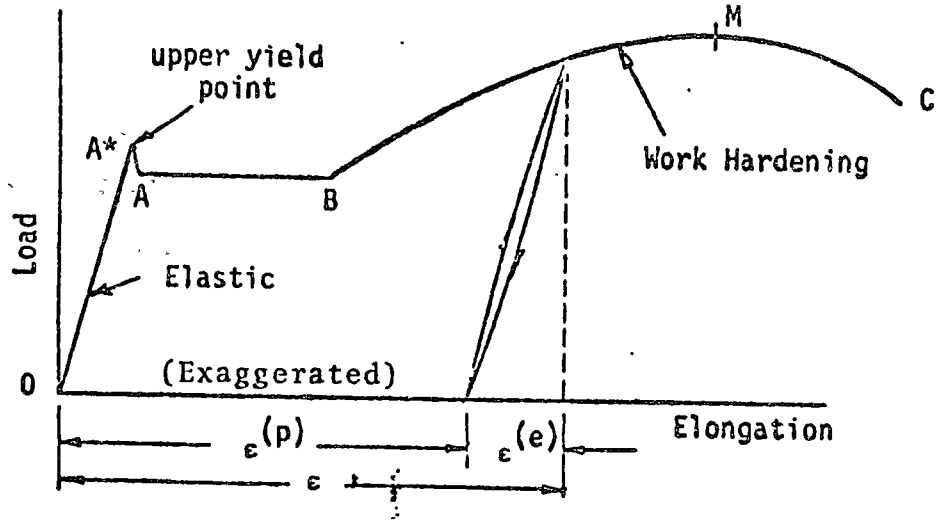


FIG. 2 STRESS-STRAIN CURVE FOR MILD STEEL

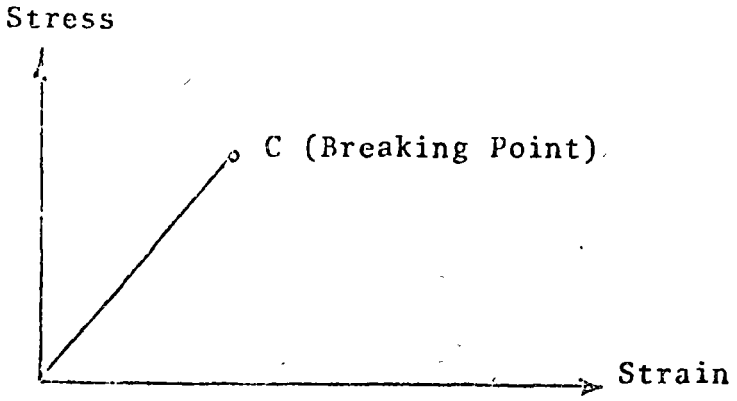


FIG. 3 STRESS-STRAIN RELATIONSHIP FOR A BRITTLE MATERIAL

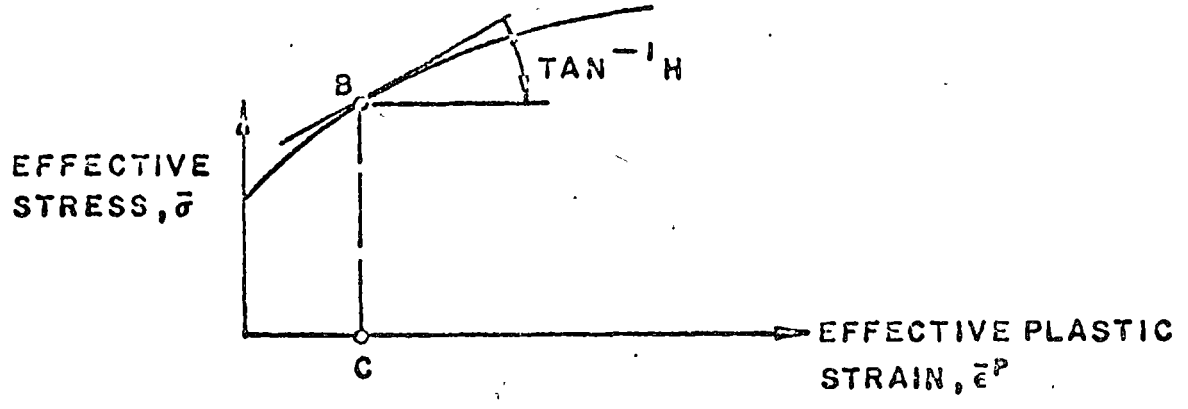


FIG. 4 EFFECTIVE STRESS–EFFECTIVE PLASTIC STRAIN CURVE

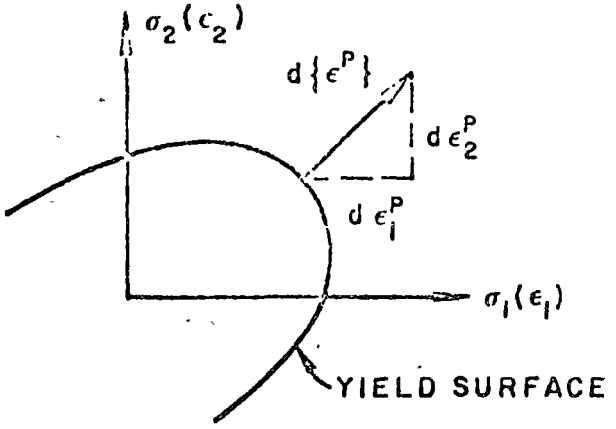
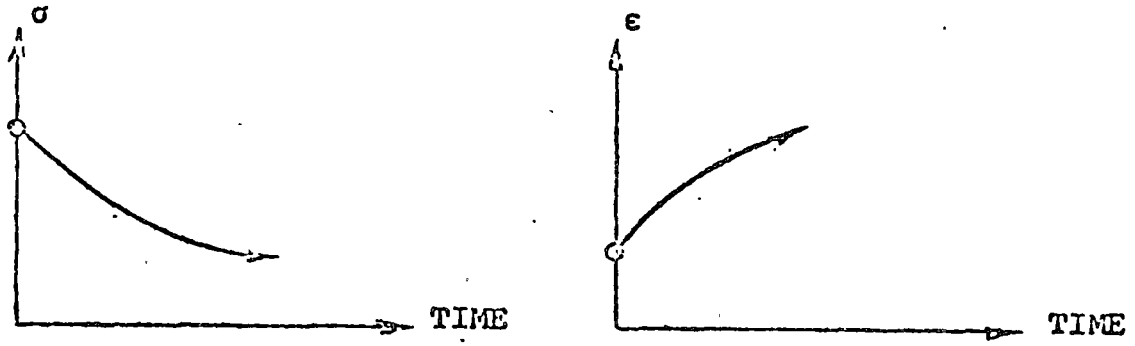


FIG 5. YIELD SURFACE AND NORMALITY CRITERION
2-D STRESS SPACE



(a) RELAXATION (b) CREEP

Fig. 6. Schematic Form of Basic Representation of Time-Dependent Material Behavior

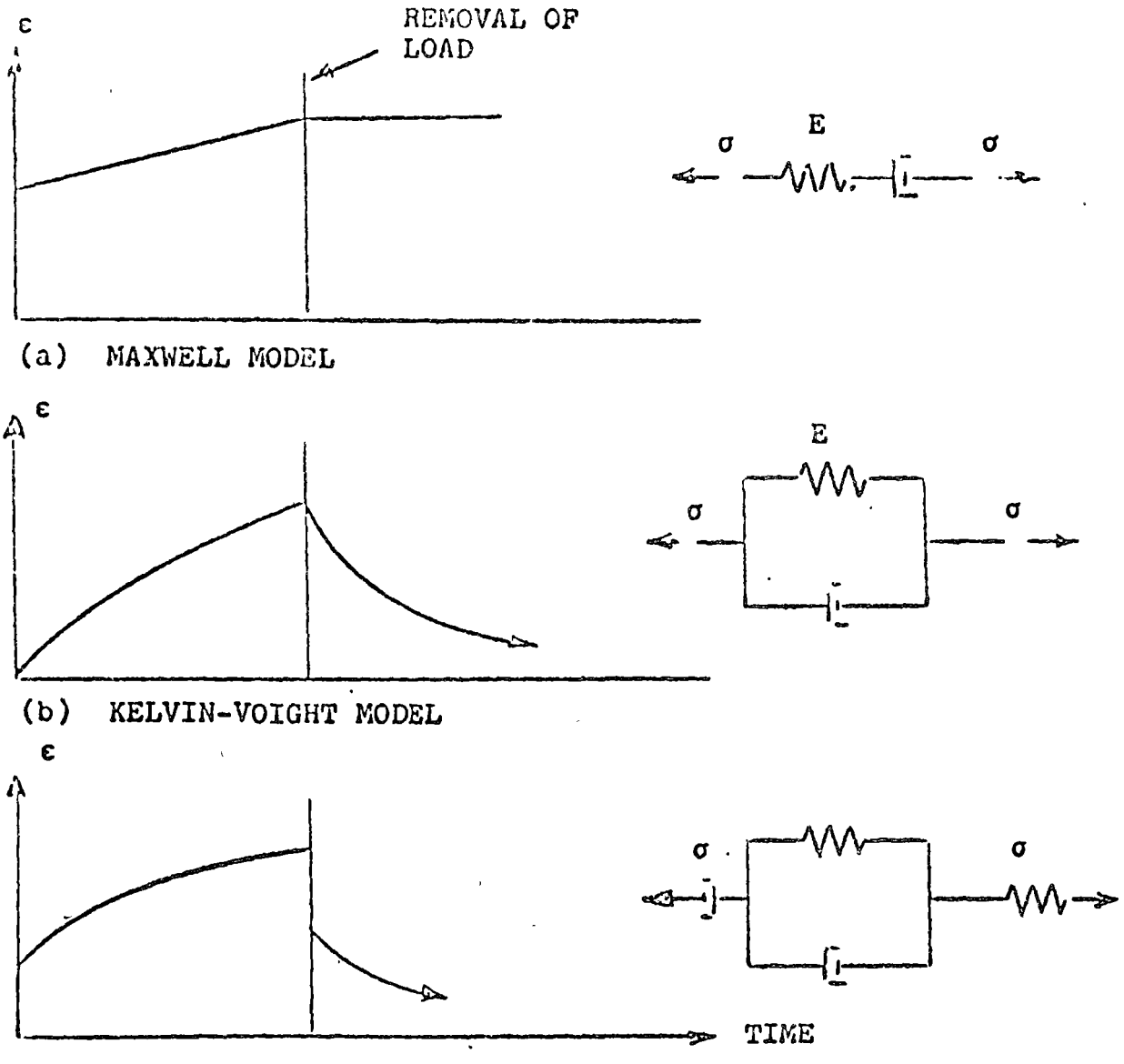


Fig. 7. Spring-Dashpot Models of Viscoelastic Behavior.

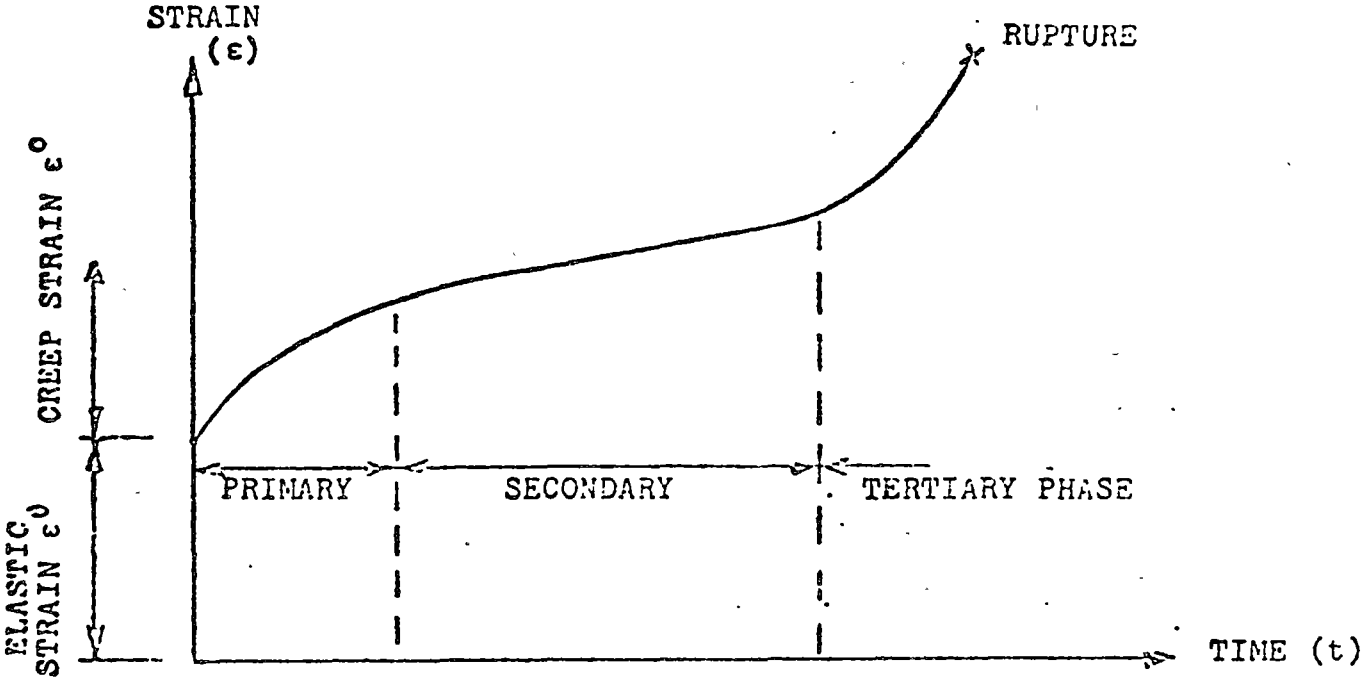
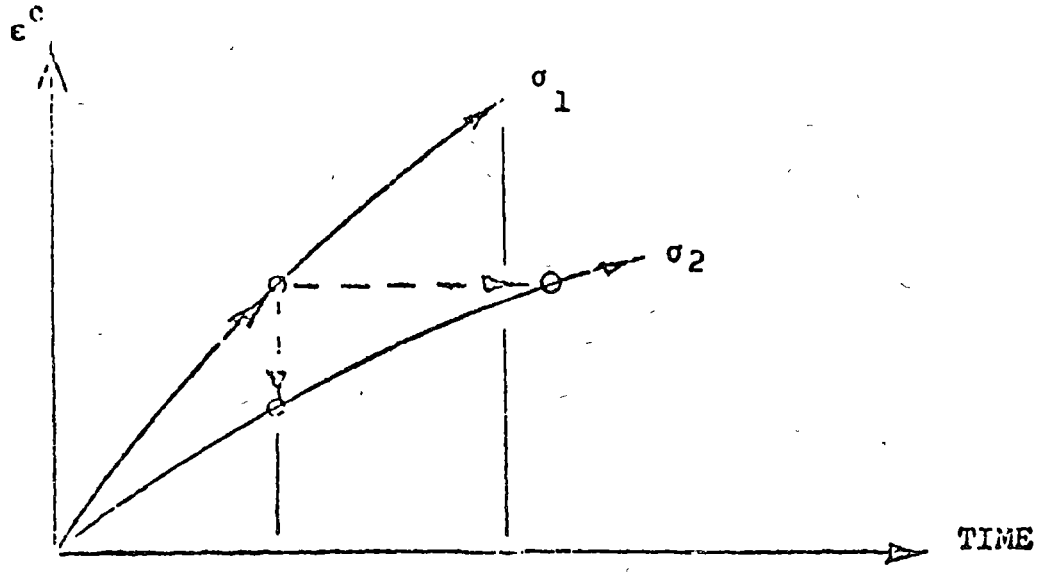
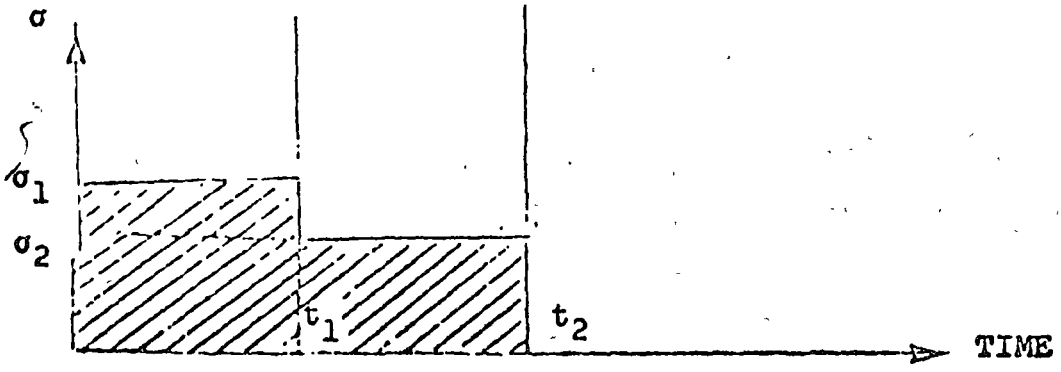


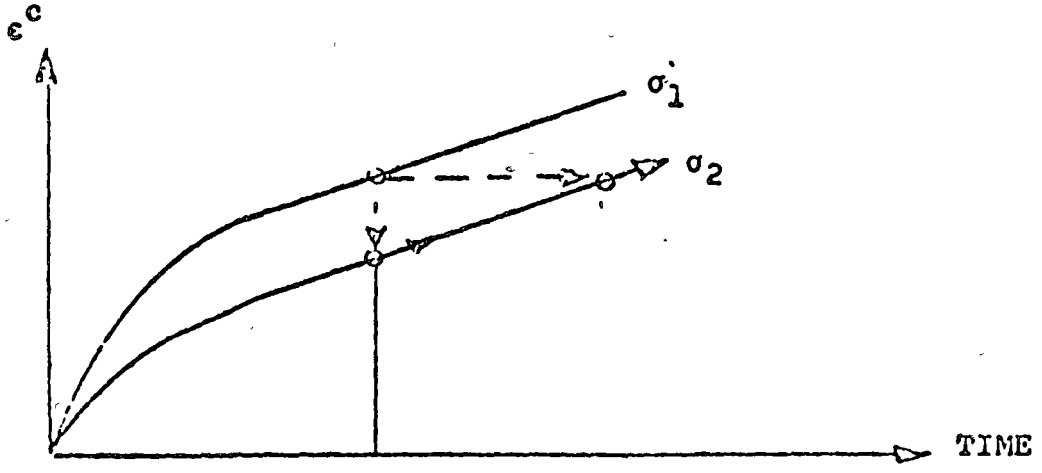
Fig. 8. Andrade Representation of Constant-Stress Creep Behavior



(a) ACCUMULATION OF CREEP STRAIN IN PRIMARY CREEP



(b) TIME-VARYING STRESS



(c) ACCUMULATION OF CREEP STRAIN IN SECONDARY CREEP

Fig. 9. Comparison of Rules for Accumulation of Creep Strain

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F.E. ANALYSIS ALGORITHMS FOR INELASTIC ANALYSIS.

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6. ELEMENT AND GLOBAL FORMULATIONS FOR INELASTIC ANALYSIS: SOLUTION ALGORITHMS

R. H. Gallagher

I. INTRODUCTION

The purpose of these notes is to describe algorithms for inelastic finite element analysis which are regarded by many as "standard". These algorithms are found in widely-distributed computer programs and have been tested over the past ten years in extensive practical applications.

The topic of inelastic finite element analysis has been the subject of a number of review papers in recent years, including Refs. 1-7. Applications in practice are dealt with these references and, except for a relatively few citations, are excluded from the present coverage.

A convenient division of time-independent inelastic analysis is into the tangent stiffness and initial stress algorithms. These are treated separately in the next two sections. Then, two sections are devoted to the newer developments in the use of mathematical programming and complementary energy concepts. Finally, algorithms for viscoelastic and creep analysis are described.

II. TANGENT STIFFNESS ALGORITHM

The tangent stiffness algorithm represents direct utilization of incremental plasticity concepts. It is an approach which involves revision of the elastic element stiffness matrix to form an elastoplastic element stiffness matrix $[k_{ep}]$ that accounts for the inelastic material properties. To accomplish this transformation one merely supplants the elastic material stiffness matrix by the elastoplastic material stiffness matrix $[E_{ep}]$ in the familiar element

stiffness formula

$$[k_{ep}] = \int_{Vol} [D]^T [E_{ep}] [D] \, d\,(Vol) \qquad (1)$$

The basic scheme in the tangent stiffness algorithm is as follows:

1) An elastic analysis is performed for an arbitrary load intensity and this intensity is scaled to the level at which the yield criterion is satisfied.
2) An increment of loading is selected for the first passage into the inelastic range.
3) An estimate is made of the stress and strain increments caused by the load increment of (2), plastified elements are identified, and estimated elastoplastic (tangent) stiffnesses $[k_{ep}]$ are calculated for such elements and incorporated in the system stiffness matrix. (The computational procedure for this step is amplified below.)
4) The load increment of step (2) is applied to the revised system stiffness matrix of step (3) and stress and strain increments ($\{d\sigma\}$ and $\{d\varepsilon\}$) are calculated.
5) Stresses and accumulated plastic strains are updated consistent with the results of prior step.
6) Select another increment of load and repeat steps 2-5. Continue process of load incrementation until the maximum value of load is reached or until limited by collapse or similar phenomena.

The determination of a collapse situation has been interpreted, by some authors[10,11], to occur when the change in effective plastic

strain is negative.

In step (3), the estimated (or calculated) stresses are employed in the computation of a new effective stress, $\bar{\sigma}$. Using it, a new $\bar{\varepsilon}^p$ is read from the effective stress-effective plastic strain curve, as obtained from a simple tension test. The quantity $d\bar{\varepsilon}^p$ is the increase in $\bar{\varepsilon}^p$. (If the new $\bar{\sigma}$ happens to be smaller than the old, corresponding to unloading, $d\bar{\varepsilon}^p$ will of course be zero). The values $\bar{\sigma}$ and $d\bar{\varepsilon}^p$ are substituted into Eq. 19, Chapter 5, together with the stress values σ_1, σ_2, σ_3, and the material tangential stiffness matrix is formed and employed in construction of $[k_{ep}]$ via Eq. 1.

In the case of the general nonlinear hardening situation it is in theory necessary to iterate within a given load increment (steps 3 and 4) to establish a consistency of the tangential stiffness of yielded elements. It is normally sufficient, however, to work with an estimated tangent stiffness for the interval. This bears a relationship to the chosen size of load increment, about which comments are given below.

A major aspect in the definition of load increments is the manner in which new plastified elements are introduced into the stiffness matrix. Some authors [12] present a careful procedure in which the load increment is adjusted to bring in plastified elements one at a time. Others, however, apply relatively large increments and delineate an approximate way of accounting for the "transitional" elements, those which enter the plastic range during a load increment. In one proposed approach,[13] if m is the proportion of strain increment to cause yield during the increment,

a "weighted" stress-strain relation for the transition region can be taken as

$$[E^{ep}] = \left[[I] - \frac{(1-m)[E] \left\langle \frac{\partial \bar{\sigma}}{\partial \sigma} \right\rangle \left\lfloor \frac{\partial \bar{\sigma}}{\partial \sigma} \right\rfloor}{H' + \left\lfloor \frac{\partial \bar{\sigma}}{\partial \sigma} \right\rfloor [E] \left\langle \frac{\partial \bar{\sigma}}{\partial \sigma} \right\rangle} \right] [E] \qquad (2)$$

III. INITIAL STRAIN ALGORITHMS

The basic concept of "initial strain" procedures is to define a "reference" elastic material stiffness, with associated "reference" elastic strains, and to treat the departures from linearity as initial strains. Many forms of this approach have been proposed, encompassing various degrees of approximation.[10, 11, 14-16] The approach described here[10] includes representation of incremental plasticity theory.

To establish the form of element equations for initial strain analysis, reference is made to Eq. 17, Chapter 5, from which $[E^{ep}] = [E] - [E^p]$. Substituting this expression into the familiar expression for an element stiffness matrix, we have*

$$[k_{ep}] = \int_{Vol} [B]^T [E] [B] \, d(Vol) - \int_{Vol} [B]^T [E^p] [B] \, d(Vol) \qquad (3)$$

The first term on the right-hand-side is the linear elastic stiffness matrix [k]. The second term on the right-hand-side gives what may be termed the element plastic stiffness matrix

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*[B] is the strain-displacement transformation matrix, i.e. $\{\varepsilon\} = [B] \{\Delta\}$.

$$[k_p] = - \int_{Vol} [B]^T [E^p] [B] \, d(Vol) \qquad (4)$$

and the product of $[k_p]$ by the element displacements gives a vector of forces

$$\{F_p\} = [k_p] \{\Delta\} \qquad (5)$$

In the initial strain approach an estimated value of $\{\Delta\}$ (or, synonymously, of strains $\{\varepsilon\}$ since $\{\varepsilon\} = [B]\{\Delta\}$ is employed to construct the vector $\{F_p\}$) and this vector is treated as if it were the set of initial forces arising from "initial" strains $\{\varepsilon_p\}$.

In describing the algorithm for the inelastic analysis of the complete structure, we first assume that the load level at which inelastic deformation is initiated has been identified and that an increment of load has been selected for the first excursion into the plastic range. The computation then progresses as follows (see Figure 1):

1) Apply the first load increment and determine $\{\Delta\sigma'\}_1$ and $\{\Delta\varepsilon'\}_1$ elastically (primes denote elastic computations.
2) Add $\{\Delta\sigma'\}_1$ to the stresses existing at the start of interval ($\{\sigma_o\}$) to form $\{\sigma'\}_1$.
3) Calculate a first estimate of the stress change due to elastoplastic behavior within the interval from $\{\Delta\sigma\}_1 = [E^{ep}]\{\Delta\varepsilon'\}_1$.
4) The discrepancy ($\{\Delta\sigma''\}_1$) between the elastic stress and the stress estimate of step (3), $\{\Delta\sigma''\}_1 = \{\Delta\sigma'\}_1 - \{\Delta\sigma\}_1$, can be regarded as being supported by "body forces". Compute the element initial forces $\{F^i\}_1$ due to this stress supported by body forces,

$$\{P^i\}_1 = \int_{Vol} [B]^T \{\Delta\sigma''\}_1 \, d(Vol) \qquad (6)$$

Form a global vector of such forces, $\{P^i\}_1$. Also, define the current stress and strain

$$\{\sigma\}_1 = \{\sigma'\}_1 - \{\Delta\sigma''\}_1 = \{\sigma\}_o + \{\Delta\sigma\}_1$$

$$\{\varepsilon\}_1 = \{\varepsilon\}_o + \{\Delta\varepsilon'\}_1$$

5) Calculate the changes in displacement, stress and strain due to $\{P^i\}_1$ as $\{\Delta u\}_1 = [K]^{-1} \{P^i\}_1$. One then determines the stress and strain increments

$$\{\Delta\sigma'\}_2 = [S] \{\Delta u\}_1 - \{\Delta\sigma''\}_1$$

$$\{\Delta\varepsilon'\}_2 = [B'] \{\Delta u\}_1$$

One repeats steps (2)-(5) until the stress change computed in step 5 is acceptably small. The load is then incremented again.

Particular note should be taken of the relationship between Eqs. (3) and (6). In accordance with a constitutive law, $\{\sigma\} = [E] \{\varepsilon\}$ and by the relationship between strain and displacement ($\{\varepsilon\} = [B] \{\Delta\}$) we have $\{\sigma\} = [E] [B] \{\Delta\}$. Thus, a stiffness equation may alternatively be written

$$\{F\} = [k] \{\Delta\} = [\int_{Vol} [B]^T [E] [B] \, d(Vol)] \{\Delta\}$$

$$= [\int_{Vol} [B]^T \{\sigma\} \, d(Vol)] \{\Delta\}$$

Thus, in Eq. (6), the stresses are applied directly to define joint forces and for this reason Zienkiewicz, et al [16] have termed this the "initial stress" approach.

Nayak and Zienkiewicz[17] have refined this approach considerably and have interpreted it as a modified Newton-Raphson method of nonlinear analysis. Comparisons with alternative schemes are presented in [18]. Yamamoto[19] has presented studies of the rate of convergence of this method.

IV. MATHEMATICAL PROGRAMMING

Inelastic analysis has, for some time, been recognized as a mathematical programming problem, i.e., the problem of minimizing a function (the objective function) subject to constrain conditions which are expressed as inequalities. Livesley [20] describes the background of this approach as it applies to the plastic design of beams and frames. Recently, these ideas have drawn the attention of those interested in the application of inelastic analysis of continua via the finite element method.[21, 22]

The minimum principles of plasticity, which are generalizations of corresponding principles in elasticity, deal with stress and strain-rates and define the quantities to be minimized. For perfectly plastic materials inequality constraints derive from the condition $\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \leq 0$, for $f =$ a constant, where f is the yield criterion and $\dot{\sigma}_{ij}$ is the stress-rate. Equality constraints may be present in the usual manner of constraints in elastic analysis. The total problem is therefore one of minimizing a quadratic function of a system of unknowns, subject to linear constraint conditions. This is termed a quadratic programming problem.

D. Donato and Maier [23] have exploited the quadratic programming approach in finite element plastic analysis. Sayegh and Rubenstein [24] also presented a development in the context of the

finite element. One advantage of this approach would appear to be the applicability of general algorithms for quadratic programming to the inelastic problem.

V. COMPLEMENTARY PROCEDURES

Complementary procedures in finite element analysis, i.e., procedures that are founded in assumed stress fields which satisfy the conditions of equilibrium have not made significant inroads into the practice of elastic finite element analysis. Nevertheless, these procedures hold certain promise for inelastic finite element analysis and for this reason they have recently drawn considerable interest.

A complementary energy formulation in which stress functions are chosen as primary (joint) unknowns has special advantage because of the correspondence of the assumed functions and resulting equation coefficients with particular aspects of displacement-based formulations. Rybicki and Schmit [25] are apparently the first to have applied these ideas to elastoplastic analysis. They deal with orthotropic plane stress and employ the Prandtl-Reuss incremental stress-strain relations in an "initial strain" format.

The element represented in Rybicki and Schmit's development is a rectangle with a 36 degrees-of-freedom representation of the Airy stress function (fifth order Hermitian polynomial interpolation). Clearly, the strain field in any such element can vary from elastic to plastic in a complex manner and must be treated via numerical integration of the element initial forces. The use of an element with this numer of degees-of-freedom enables direct treatment of all boundary conditions but is likely to be more sophisticated than is required by overall structural idealization requirements.

A simpler representation, defined also for plane stress and in terms of the Airy stress function, has been presented by the writer and Dhalla [26]. This formulation of the rectangular element, with 16 degrees-of-freedom, requires careful attention in the treatment of stress (force) boundary conditions.

An alternative complementary approach, based upon direct representation of stress parameters, has been explored by Belytschko et al. [27-29] in a series of papers pertaining to both plane stress and flexure. The advantage of element matrices that are identical to those in conventional stiffness analysis are lost in this scheme, but the matrices required are nevertheless simple in form.

VI. VISCOELASTIC ANALYSIS

The linear form of time-dependent material behavior, which is conventionally termed viscoelasticity, has been studied analytically for more than a century and the development of related analysis tools has progressed continuously to the present high level of capability. Nonlinear time-dependent material behavior, or nonlinear creep, has on the other hand been identified in analytical form only since 1910 and progress towards general analysis capabilities is measured from the late 1950's.

The time-dependent behavior of metals is characterized by nonlinear creep. This does not entirely discount an interest in finite element-based solutions of viscoelastic deformation, since procedures formulated for viscoelasticity form a basis for creep analysis procedures. Certain other viscoelastic analysis procedures may prove useful for future creep analysis developments.

Published finite element viscoelastic analysis procedures are

in each case cast in the form of the stiffness approach but a variety of distinct treatments are represented. The treatments discussed here are (1) the time increment-initial strain method, (2) the correspondence principle, and (3) the integral approach.

The time increment-initial strain method is applicable to all classes of finite element analyses for time-dependent material response and is also the basis for most of the practical nonlinear creep analysis. Thus, it is given close attention in the following. This procedure was first introduced for the nonlinear creep problem [15] and was subsequently employed by Zienkiewicz et al.[30] for viscoelasticity.

In this procedure the time-history of loading (and temperature, if this also varies with time) is represented by a series of constant load intervals, as portrayed in Fig. 2. The viscoelastic deformation accumulated at the close of a given interval is treated as an initial strain in a determination of the stress prevailing in the subsequent interval.

The complete algorithm, from time zero (t_o), is of the form:

(a) Calculate the elastic stress distribution $\{\sigma\}_o$ at t_o, based on $\{P\}_o = [K]\{\Delta\}$, and any initial strain due to temperature, if present.

(b) Refer to the constitutive relationships and calculate the change in time-dependent strain $\{\Delta\varepsilon^c\}_1$ in the first interval Δt_1 assuming $\{\sigma\}_o$ to be constant within this interval.

(c) Solve the elastic problem at the close of the first interval, at $t_1 = t_o + \Delta t_1$, using $\{\Delta\varepsilon^c\}_1$ to form an initial force vector $\{F^i\}$ (and for temperature T_1, if present)

$$\{P_1\}_1 = [K]\{\Delta\}_1 - \{P^i\}_0 \qquad (7)$$

Calculate the elastic stresses $\{\sigma\}_1$ from this solution.

(d) Calculate the increment in time-dependent strain $\{\Delta\varepsilon^c\}_2$ for the second time interval Δt_2 by reference to the constitutive relationships and add to the already-sustained time-dependent strain.

(e) Employ step (c) for the close of the second interval, using the total time-dependent strain to calculate $\{P^i\}_2$.

(f) Repeat steps (d) and (e) for succeeding intervals.

It is important to note the significance of the selection of the time interval with respect to solution accuracy, even for a viscoelastic analysis. The above procedure implies that stress does not change during the selected interval. One may approximate the change in stress, however, and form an average stress for the interval in order to minimize the error. Questions of interval selection will again be taken up in the next Section.

The viscoelastic constitutive relationship represented by Eq. of Chapter 5 is the basis for steps (b) and (d) in Ref. 30. The problem of a prism and a reinforced concrete cylinder, for which exact solutions are available, and the more practical cases of a solid propellant rocket engine and a tunnel lining, for which no comparison solutions are available, are solved in Ref. 30.

The correspondence principle of viscoelasticity has been employed by Webber[33] and by Booker and Small[39] in finite element analysis. In accordance with this principle, a Laplace transform is taken of the elastic solution and the elastic constants; these

combinations are dependent upon the chosen form of the viscoelastic constitutive law. The Laplace inverse of the resulting relationships gives the viscoelastic solution.

"Integral procedures" in finite element viscoelastic analysis operate upon the integral form of the constitutive relationships. The latter are in the class of Eq. (24) of the notes on Constitutive Relations for Plasticity and Creep. Rashid[31], Taylor, Pister and Goodreau[32], Henrichson[40], and Ueda, et al[42], among others, have taken this approach. The various developments differ principally in the manner of approximation of the hereditary integrals.

VII. CREEP ANALYSIS

Finite element creep analysis has, to date, depended principally upon use of the incremental-initial strain method. Exceptions have included the works of Cyr, Teeter and Stocks[44] and Sharifi and Yates[8] who cast the creep laws into such a form as to be useful for tangent stiffness analysis.

The work in incremental-initial strain creep analysis is summarized in Table 1 and is discussed in the following. Its earliest application in finite element creep analysis appears to have been that of the writer, Padlog, and Bijlaard[15]. Chronologically, the next significant contribution to finite element creep analysis was made by Lansing, Jensen, and Falby[45]. This work, based upon the matrix force method and the representation of a stiffened sheet in plane stress subjected to time-varying loads and temperatures by means of axial force members and shear panels, is noteworthy for its contribution of the most significant test data yet reported. (An extensive development of complex-structure creep test data has

recently been completed and is reported in Ref. 38.

The extension of the subject procedures to axisymmetric solids, together with useful information regarding convergence of the solution process in time, is given by Greenbaum and Rubenstein[46] and by Sutherland[47]. Both papers employ the simplest form of axisymmetric solid element, the triangle with linear displacement fields (joints only at the vertices).

References 46 and 47 identify two limitations on the time-step length in finite element creep analysis and establish tolerances for each of these within the confines of the problem they study. The first limitation pertains to change of stress within the time interval. First-order creep analysis features the assumption of constant stress within the time interval. To keep the error due to this source within acceptable bounds, it would appear appropriate to limit the change in stress to 5%. Sutherland[47] employs both 5% and 10% as the upper limits in the creep analysis of a flow duct using the creep law $\dot{\varepsilon}_c = B\sigma^{6.35}$. The differences in the results did not prove significant.

The second limitation applies to the change in creep strain in a given increment in time. It has been found[45] that if the change in creep strain equals the elastic strain, the solutions for succeeding intervals oscillate and diverge. Thus the change in creep strain is limited to some fraction of the elastic strain[46, 47]. Alternatively, upon attainment of constancy of the stress field with time, extrapolation may be attempted to estimate the displacements for a subsequent time.

A key aspect of any creep analysis program is the automatic selection of time increments, subject to the above criteria. The MARC program, for example, asks for an arbitrary initial estimate of the time interval. This is doubled in the next increment if the above criteria are met or halved (and the increment repeated) if they are not. Cormeau[49] has recently published a more rigorous development of this topic.

VIII. CONCLUDING REMARKS

Procedures that have been discussed in these notes have been incorporated in many widely-distributed finite element analysis programs. Nearly all of these programs employ the more straightforward of the foregoing methods, e.g., tangent stiffness in time-independent plasticity analysis, the incremental-initial strain procedure in creep analysis. Since these programs are used throughout international industry there have been literally hundreds of practical applications and very many of these have been described in the literature. It would be beyond the scope of these notes to attempt a review of these papers.

It is possible to obtain a clear picture of the inelastic procedures employed in the widely-distributed programs by reference to either the program user's manuals or the open literature. The procedures of the MARC and ANSYS programs are to be found in the relevant user's manuals. ASKA capabilities in this respect are detailed by Balmer, et al in Ref. 50. Other programs are discussed in Refs. 51-53.

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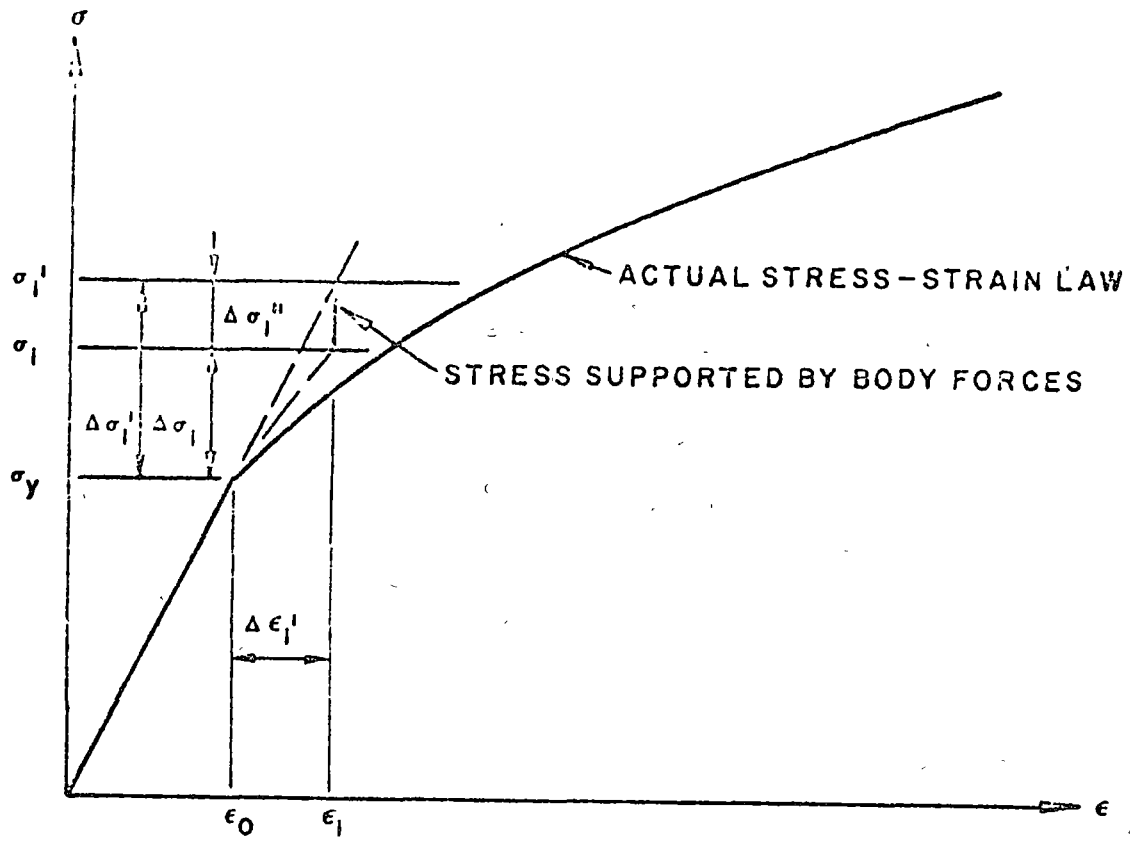

FIG. 1 INITIAL STRESS ALGORITHM

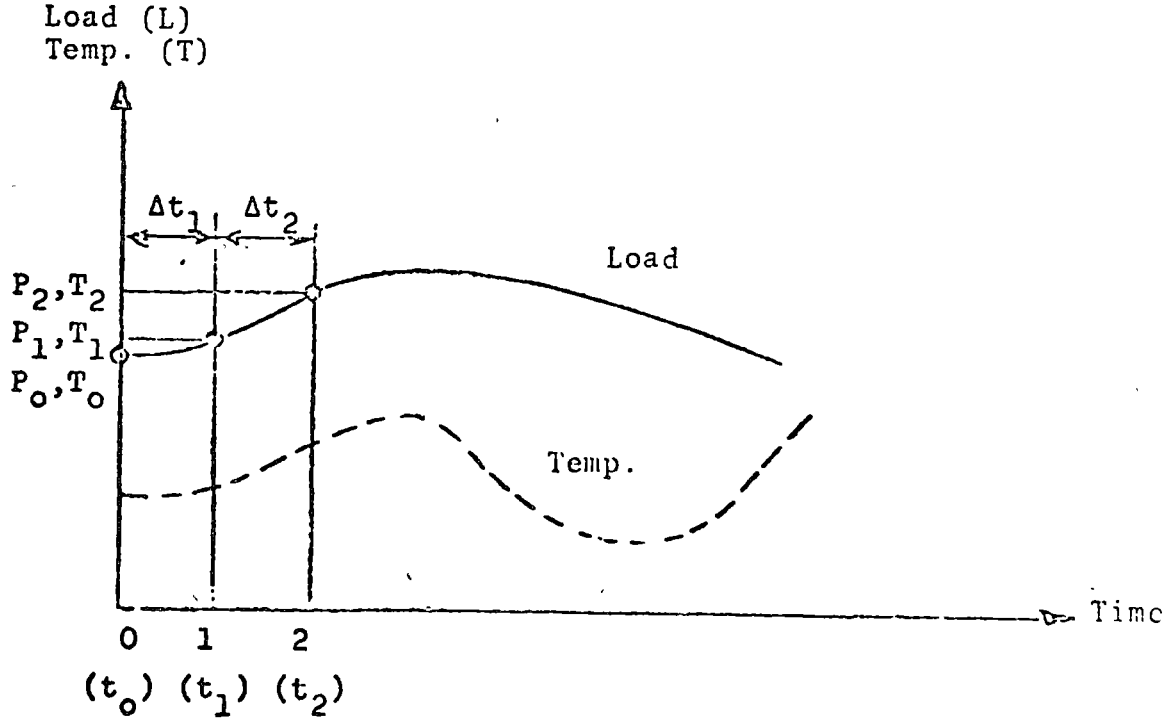

Fig. 12. Representation of Time-Varying Loads and Temperatures.

Table 1. Applications of Time Incremental - Initial Strain Method of Finite-Element Creep Analysis

Reference	Authors	Creep Law	Types of Element	Problems Solved
15	Gallagher, Bijlaard, and Padlog	$\dot{\varepsilon}^c = B\sigma^n$ Strain hardening	Triangle and rectangle in plane stress	Finite-width plate with central hole
45	Lansing, Jensen and Falby	$\varepsilon^c = at^\gamma (e^{B\sigma} - 1)$ Strain hardening	Axial member and shear panel	Stiffened rectangular plate
46	Greenbaum and Rubenstein	$\varepsilon^c = B\sigma^{3.61}$ Strain hardening and time hardening	Triangle: axisymmetric	Thick-walled cylinder
47	Sutherland	$\dot{\varepsilon} = B\sigma^{6.35}$ Strain hardening	Triangle: axisymmetric, plane strain, plane stress	Axisymmetric cylinder Flow duct beam
48	Swanson and Patterson	$\dot{\varepsilon}^c = C_1 e^{C_2/T} \times (\sinh \frac{C_3\sigma - C_4}{T})$ C_1 = material constants		LEMBR fuel duct

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"ADVANCED TOPICS IN FINITE ELEMENT ANALYSIS."

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SHELL ANALYSIS BY F.E.M.

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SHELL ELEMENT

R. H. Gallagher

1. INTRODUCTION

Shell structures have come to fruition in the twentieth century. The thin-shell roof emerged as a practical means for spanning large distances in the 1920's. This emergence was due to a multiplicity of factors - the ability to form and reinforce shallow depths of concrete, architectural imagination, and the development of analytical tools to insure the structural integrity of the completed design. Availability of high-strength sheet metal, at about the same time, had similar impact on structural form in aerospace and mechanical engineering design.

Analytical tools for thin shell structures were first developed much earlier, over a century ago. Lamé and Clapeyron (Ref. 1) established the fundamental theory for shell membrane action in 1826. H. Aron (Ref. 2) considered bending behavior in 1874, but the first general theory was not advanced until 1888, by A.E.H. Love (Ref. 3). Subsequent theoretical efforts have been directed towards improvements of Love's formulation and the solution of the associated differential equations. Such solutions were not easily obtained in the early era of shell analysis. Indeed, one of the earliest solution procedures of practical applicability, developed by Carl Ziess in Jena, Germany in 1924, was patented. This was an important factor in the aforementioned emergence at that time of thin-shell concrete roof structures. Analytical formulations have followed rapidly in the interim, and papers dealing with shell analysis number in the thousands. Coordinated treatments of progress accomplished in thin shell theory can be found in the texts by Flügge (Ref. 4) and Kraus (Ref. 5), among others.

Available analytical solutions to thin-shell structural problems are nevertheless limited in scope and in general do not apply to arbitrary shapes, load conditions, irregular stiffening and support conditions, cutouts, and many other aspects of practical design. The finite element method has consequently come to the fore as an approach to thin shell

structural analysis because of its facility to deal with these complications. Like shell structures themselves, the finite element method has come to fruition in the twentieth century and it promises to be the most powerful and widely used approach to the design analysis of such structures.

One of the first surveys of finite element representations for thin-shell analysis was published in 1969 (Ref. 6). A significant number of other surveys, including Refs. 7-9, have since appeared. These demonstrate that a large number of formulations have preferred to follow the well-established path of minimum potential energy (displacement-based relationships) and that extreme difficulty is encountered in attempting to satisfy the desired conditions, e.g., interelement displacement continuity, zero strain under rigid body motion. In consequence, the problem of finite element thin-shell analysis has been a motivating factor in the development of alternative formulative procedures, such as mixed and hybrid methods.

There are three distinct approaches to the finite element representation of thin shell structures: (a) in "faceted" form, with flat elements, (b) via elements formulated on the basis of curved shell theory, and (c) by means of three-dimensional (solid) elements. In the following two sections we deal with the pros and cons of flat-plate representations and solid elements. Developments in the literature are largely concerned with elements formulated on the basis of curved-shell theory. In later sections, therefore, we examine some of the underlying factors of curved shell elements. We then amplify these factors for displacement-based formulations, generalized potential energy, and alternative (mixed and hybrid) procedures.

2. FLAT PLATE ELEMENTS

It is first necessary to clarify the type of representation found in this class. See Fig. 1c, which portrays a

triangular element. This element is formed of the superposition of stretching behavior (Fig. 1a) and bending behavior (Fig. 1b). Any discussion of the suitability of flat plate elements in thin-shell analysis must first consider the matter of available element formulations in the component membrane and bending behaviors, Figs. 1a and 1b, respectively.

It suffices to say that acceptable triangular membrane (plane stress) element formulations are available for a various degrees of higher-order elements, with the use of node point degrees-of-freedom in terms of either the displacement function itself (C^0 representations) or in terms of the function and various orders of its derivatives (C^1 representations). For bending, however, simple formulations of acceptable accuracy are not possible for the triangle. One may employ an elegant, highly accurate, but computationally expensive formulation such as that which is based on a complete quintic polynomial (21-term polynomial)[10], or a simpler formulation based on a complete cubic polynomial with constraints imposed to enforce interelement continuity[11]. Alternatively, one may attempt the use of special variational principles which lead to mixed and hybrid element formulations[12,13]. These and other ideas in flat plate bending formulations are developed in some detail in Ref. 14.

Given acceptable element force-displacement relationships, a number of difficulties and shortcomings are present in the application of the elements in analysis of the complete shell. These include: (a) the exclusion of the coupling of stretching and bending within the elements, (b) the difficulty of treating junctions where all elements are co-planar, and (c) the presence of "discontinuity" bending moments, which do not appear in the continuously-curved actual structure, at the element juncture lines.

Straight finite elements represent the behavior of curved structures in the limit and errors due to the exclusion of bending-stretching coupling in the elements can be made small by use of a refined finite element network.

When elements are co-planar a null stiffness corresponding to rotation about the axis normal to the plane will indeed be present. One may define coordinate axes in the plane (which is generally at an angle with the global axes) and eliminate this rotational degree-of-freedom. This may be awkward in practical application since many different such planes may appear in the structure. Conversely, for small angles between the elements, dependence may be placed on this angle to maintain solution stability. The stiffness equations approach singularity as the elements approach co-planarity, but numerical evidence discloses that this angle can be quite small, given the number of significant figures carried by modern digital computers.

In order to illustrate the presence along element juncture lines of calculated bending moments which do not appear in continuously-curved actual structures one can consider the special case of flat elements, that of the axisymmetric truncated cone element, as was done in Ref. 15. Consider such elements employed in the idealization of a pressurized hemisphere. There are no bending moments in the actual structure, only membrane stresses $\sigma_\phi = \sigma_\theta = pr/2t$, where p = pressure, r = radius, t = thickness. When the stiffness equations are used to solve for the joint forces and moments, however, it is found that meridional bending moments are determined.

The source of this inconsistency can be visualized by consideration of the finite element solution for a structure that actually consists of truncated cones. In this case the finite element geometric model is exact. Also, meridional bending moments are indeed present at the juncture lines. In other words, the truncated cone idealization is quite appropriate for the geometric form it represents exactly. If the same idealization is applied in approximation of a meridionally-curved shell the juncture line moments constitute the error due to this approximation. To remove the error one can employ curved, rather than flat elements.

The above difficulty can be circumvented if the solved-for displacements are substituted into stress-displacement equations written for the curved shells. Furthermore, this difficulty is associated with regions of the structure where membrane stress predominates. The question discussed here might not be important where bending is the predominant mode of actual behavior.

It can be seen, therefore, that although there are pitfalls in the application of flat plate elements in the analysis of curved shell structures the difficulties can be surmounted through various artifices and additional computational expense. The problem that remains is to determine if the solution accuracy is properly balanced by the computational cost, given that alternative approaches (as discussed in subsequent sections) are available.

3. SOLID ELEMENTS

Figure 2 shows a solid (three-dimensional) element based on quadratic displacement fields. Because of the isoparametric mode of geometric description, also with quadratic functions, this element can be used in description of thin shell structures. This idea has strong appeal since it seemingly permits one to dispense with the assumptions of special theories, such as shell theory, and the controversies which surround them. Certain assumptions must be invoked, however, if curved thin-shell solutions are to be achieved by this mode of representation (Ref. 46).

The basic assumptions to be made when the 20-node brick element in Fig. 2 is employed in curved thin-shell analysis are that no strain occurs across the element thickness and that the direct (ξ- and η- direction) strains vary linearly in that direction. This enables elimination of the node points in the middle surface and results in the equality of normal displacements (w) of corresponding points on the upper and lower surfaces of the element. Also, the ξ- and η-direction displacements, u and v, of nodal points on the top surface can be expressed in terms of the u and v displacements

of the corresponding node on the bottom surface and of the rotations θ_ξ and θ_η of the line between these two nodes. Thus, the element node point displacements are u, v, w, θ_ξ and θ_η at each of eight bottom-surface locations (four vertices, four midside locations). Note that transverse shear deformation is permitted, because the angular displacements are not tied to the slope of the middle surface.

Numerical evidence has shown that the above assumptions are not always sufficient to recapture the behavior of curved thin shell structures. This inadequacy can be explained by the case of a planar rectangular element intended to represent beam flexure (Fig. 3a). The element is formulated on the basis of linear displacements which give the displacement pattern of Fig. 3b for the pure bending state. The displacements for pure bending, however, are as shown in Fig. 3c where the upper and lower surfaces are curved, not straight. This discrepancy can also be explained as resulting from an overemphasis of the shear strain energy.

Various schemes have been proposed to account for the above difficulty. Ref. 16, using the three-dimensional generalization of the element shown in Fig. 3a (i.e., a "trilinear", eight-noded brick element with straight sides), adds displacement modes ("bubble modes") to describe the curvatures referred to in Fig. 3c. Difficulties have been encountered with non-rectangular forms of this element, however, in consequence of interelement displacement discontinuities that are introduced.

The most popular and most effective scheme (Ref. 17) involves the use of lower-order numerical integration of the strain energy of the element of Fig. 2. The complications of geometric representation and of the quadratic displacement fields requires the evaluation of the strain energy, for the purpose of constructing the element stiffness coefficients, through use of numerical integration. Accurate evaluation of the strain energy would suggest a 3x3x3 system of integration points. Choice of a 2x2x2 system, however, more properly represents the shear strain energy.

The above formulation has proved highly effective in practical applications. The 20-node brick element can be found in most of the widely-distributed finite element programs. If one is able to specify the constraints cited previously and a reduced order of numerical integration of the stiffness coefficients, then a curved-thin-shell analysis is at hand without the complications of classical shell theory.

The element of Fig. 2 possesses a disadvantage in common with flat plate and curved thin shell elements, that of node points at the vertices. It may be difficult to attach such elements to their neighbors at lines of abrupt curvature change, e.g., at the juncture of a sphere and cylinder. A development which is intended to surmount this problem has been presented by Irons (Ref. 18) who excludes node point angular displacement continuity at the vertices, but rather enforces it at points along the sides. He terms the latter "Loof" nodes.

4. CURVED THIN SHELL ELEMENTS - SHELL THEORY

In this and the following two sections we turn to comments that are intended to place in perspective the developments which appear in later sections dealing with specific curved element formulations. These comments have been categorized under the headings of "shell theory", "geometric representation" and "displacement fields".

A substantive review of shell theory with particular reference to finite element formulations has been given by Morris[19]. Other contributions have been made by Cantin (Ref. 20) and Dawe (Ref. 21). In discussing shell theories it is useful to distinguish between "shallow" and "nonshallow" formulations. We comment first on nonshallow shell theories since they are more generally applicable.

The usual approach in nonshallow shell theory is to describe behavior with reference to curvilinear coordinates (α_1, α_2) in the middle surface of the shell (Fig. 4). One may then describe the in-surface stress and deformational behavior in

terms of "membrane" stress resultants (N_{11}, N_{22}, N_{12}), and direct strains (ε_{11}, ε_{22}, ε_{12}). Moment parameters (M_{11}, M_{22}, M_{12}) and curvatures (κ_{11}, κ_{22}, κ_{12}) are defined with respect to the radial behavior in curvilinear coordinates.

The basic complexity of the shell analysis problem led early theorists to establish strain-displacement relationships with different types of approximations, and to date no one of these formulations has received exclusive acceptance. The significant differences between the respective shell theories, for linear conditions, arise in conjunction with the strain-displacement expression for the twisting curvature κ_{12}. A thorough examination by Koiter[22] of available theories discloses that certain formulations do not properly account for the condition of zero strain under rigid-body motion in the representation of this term.

It is essential that the strain-displacement equations for finite element shell analysis meet all conditions related to rigid-body motion. This is a requirement apart from the requirement associated with rigid-body motion existing in the choice of displacement fields, and errors in the satisfaction of one will reinforce errors in the other. Hence, theories that satisfy these requirements should be chosen. Popular choices for shells of general curvature have been those due to Koiter[22] or Budiansky and Sanders[23]. The designation "consistent" shell theory has been applied to them because they are consistent with respect to the basic Love-Kirchoff hypotheses (i.e., normals remain normal, neglect of transverse shear deformation, etc.). Koiter shows, however, that the "inconsistency" of certain other formulations is manifest in terms of order t/R (t = shell thickness, R = radius of curvature) which have no significance on numerical results. A lucid development and summary of these and other aspects of thin-shell theory is found in the text by Kraus (Ref. 5).

Another caution regarding deep-shell finite element representations concerns the definition of angular displacements. For curved shells the angular displacement is the first

derivative of the radial displacement plus a term equal to a displacement component divided by the appropriate radius of curvature of the shell. Thus, when adjacent elements with different radii of curvature are joined, one must be careful to evaluate properly the angular displacements. Serious errors may be introduced in such cases when the angular displacement is evaluated simply as the first derivative of radial displacement and continuity of angular displacement is imposed on this measure.

The situation in shallow-shell analysis parallels that of deep-shell theory. Alternative formulations of the strain-displacement equations have appeared and these differ in the expression for twisting curvature. Again, certain theories are not free of strain under rigid-body motion, but this does not appear to be of as much significance as in deep-shell theory if the element is indeed shallow.

Although it is appealing to expect that shallow-shell theory is entirely satisfactory because the individual finite element of a deep shell is also shallow, it is possible that the global solution will be that of a shallow shell. This circumstance depends upon the particular form of shallow shell theory employed. Lindberg, et al. (Refs. 72 and 41) discuss this point in some detail.

5. CURVED THIN SHELL ELEMENTS - GEOMETRIC REPRESENTATION

The problem of geometric representation in finite element shell analysis is one of its most important aspects and yet has drawn very little attention to date. Many shell element formulations pertain to specific geometric configurations (e.g., cylindrical shell elements) where no considerations arise with respect to definition of geometric parameters or the transformations needed to connect elements with differing curvatures. In the general case, however, the problem arises of defining not only the x, y and z coordinates at the element nodes, but also the physical slopes and curvature parameters.

A mathematical basis for the representation of curved surfaces has been realized in the work of Coons (Ref. 25), Ferguson (Ref. 26), Bezier (Ref. 27), and others. These techniques are variants on the method wherein a surface in three dimensions is cross-hatched with spline curves and the regions between the curves are generated by "surface patches" which have continuity properties in two dimensions analogous to those exhibited by splines with respect to one dimension. The principal advantage of this spline-like approach is that one can achieve a specified degree of continuity in the representation.

The surface patch technique, which was developed initially with computer graphics in mind, has been applied to finite element thin-shell analysis in Ref. 28 and elsewhere. Others (e.g., Refs. 29 and 30) have used the isoparametric approach to curved elements in describing thin-shell geometry.

Problems related to geometric representation are reduced when the element formulation is based on shallow-shell theory. This eliminates difficulty in the establishment of curvilinear coordinate transformations.

It should be kept in mind that the approximation of geometry may produce solution errors which are far more serious than the errors due to the choice of displacement fields. Note has also been taken of this source of error in our discussion of the use of flat plate elements. A subtle consequence of this approximation arises in the elastic instability analysis of thin shells since the critical loads to cause instability are, under certain conditions, sensitive to imperfections in structural geometry.

6. CURVED SHELL ELEMENTS - DISPLACEMENT FIELDS

In treating the topic of displacement fields for element formulations we perhaps imply that the minimum potential energy-assumed displacement approach is the only viable means for the establishment of element relationships. This, of course, is not the case and approaches based on assumed stress

parameters and on a mixture of stress and displacement fields have already shown promise and will surely play an increasing role in the future. We will discuss such alternatives in Sections 8 and 9. Nevertheless, current theory and practice is mainly tied to assumed displacement formulations and most developments in this paper take that approach. We therefore give some introductory views on the choice of curved shell element displacement fields.

Three problems which may prove to be of concern in the definition of displacement functions for curved thin-shell elements are: (a) the retention of the required degree of interelement displacement continuity, (b) the representation of constant states of strain, and (c) the assurance of the presence of zero-strain-energy modes of rigid-body displacement. Each of these is meaningful for displacement fields expressed in terms of curvilinear coordinates. When local rectangular coordinate axes are employed only items (a) and (b) exist as problems.

We can illustrate, in a simple way, the source of the above difficulties by referring to the circular arc element of Fig. 5. Displacements in the curvilinear coordinate directions are designated by u' (circumferential) and w' (radial), respectively. The rectangular axes are designated as x (horizontal) and z (vertical), with corresponding displacements u and v. When a z-direction rigid-body-motion is imposed on the element the displacement at point 1 is fully described by u'_1. At point 2, however, the displacement is described by a combination of u'_2 and w'_2. This combination involves trigonometric functions of the arc angle of the element (θ). Displacement functions defined in this manner for the more difficult case of shells, especially doubly-curved shells, are rather more complex. Consequently, there is a preference among many individuals for conventional polynomial fields which merely approximate the "rigid-body-motion condition". It can be shown[31] that satisfaction of the condition is approached in the limit by higher-order polynomials[32].

An interesting approach to the satisfaction of the condition of zero strain under rigid-body motion has been developed by Cantin (Ref. 33) and employed by Fonder and Clough (Ref. 34). This approach recognizes that it may be most expedient to define, initially, displacement fields which do not meet the rigid-body-motion condition. Then, by use of transformation relationships, the chosen displacement fields are modified so that they satisfy this condition. Care must be exercised so that singular or ill-conditioned transformations are recognized.

The inclusion of all constant states of strain is also a requirement whose satisfaction is related to the presence of trigonometric functions in the displacement fields, where now their presence may have adverse effect. We can see this by recollecting that certain components of the strains derive from differentiation of displacement. When the latter contain trigonometric functions the differentiation produces other trigonometric functions, rather than constants. Undifferentiated displacements in the strain-displacement equations give rise to the same condition. Ashwell and Sabir, in Refs. 35-37. show how this problem can be surmounted by integrating the strain-displacement equations to produce a field which satisfies the rigid-body-motion condition. Satisfaction of the rigid-body-motion condition might not be achieved in some of the more difficult situations, however.

The condition of interelement displacement continuity is not easily satisfied. The problems which are present in the representation of flat plate flexure, i.e., the provision of C^1 continuity when the conventional Kirchoff theory is employed, are even more serious in shell formulations. This is due to the additional terms in the strain-displacement equations.

One ongoing debate in curved thin-shell element formulations concerns the relative order of polynomials used to describe the in-surface and radial displacements, respectively. Some authors (e.g., Cowper, Lindberg, and Olson, Ref. 41) argue that

one should adhere to the notion that each field should be of one order less than the highest derivative in which it appears in the strain energy expression, this being the "classical" condition on piecewise approximating functions in the finite element method. It then transpires that in-surface displacements are to be of lower order than radial displacements in the subject development.

Dawe[38], through study of arch elements, has concluded that expansions of the same order are desirable and that this order should be no less than quintic. This reasoning is based mainly on substantially improved stress predictions for the quintic versus cubic polynomials. The predictions were not improved when cubic polynomials were used for the in-surface representation and a quintic polynomial for the radial component. Dawe also notes the usefulness of the higher-order terms in the in-surface components in approximating the rigid-body-motion condition.

Others, including Thomas and the writer (Ref. 39), choose the same order of polynomial for all components, but use only cubic expansions. The motivation for this is detailed in Ref. 39, but we can say in brief that it stems principally from a desire to use the "extra" terms of the in-surface expansions to approximate more closely the constant strain condition. Numerical evidence of the suitability of this approach is also found in Ref. 39.

To be sure, a number of elegant developments of thin shell finite elements have managed to surmount the difficulties cited above through exploitation of advantageous coordinate systems, special construction of the strain-displacement equations, and the adoption of appropriate displacement expansions. Two noteworthy developments along these lines are due to Argyris and Scharpf, in the SHEBA element (Ref. 40) and the formulation by Dupuis (Ref. 30).

Nevertheless, a very great number of theorists and practitioners have sought and continue to seek formulations in terms of simple expansions and familiar coordinate systems.

Approximations, the more important of which are described above, are wittingly adopted with the idea that the loss in accuracy is more than compensated for by reduced computational cost. Numerical experimentation plays an important role in determining if such approximations are admissible and, if they are, in the assessment of relative merit. A paper by Morley (Ref. 51) addresses itself to the first part of this question.

7. CURVED SHELL ELEMENTS - DISPLACEMENT BASED

This section is devoted to an examination of the more sophisticated and accurate displacement-based curved shell elements now employed in practice. These represent a synthesis and progression from the very many developments reviewed in Refs. 6-9. Moreover, we limit our attention to triangular elements in view of their generality in description of geometric form.

The first of the curved thin-shell finite elements discussed in this section is due to Cowper, Lindberg, and Olson (Refs. 24, 41). Originally formulated in terms of shallow shell theory (Ref. 24), this element was extended (Ref. 41) to cover nonshallow applications. A restricted quintic polynomial (containing a complete quartic) was chosen for the normal displacement field, w, and a complete cubic field for each of the in-plane displacements, u and v. By imposing cubic normal rotations along each edge, the derivation satisfies interelement continuity. The centroidal values of u and v can be condensed from the stiffness matrix. The resulting stiffness matrix then has 36 degrees of freedom, 12 at each of the corner nodes, consisting of u, u_x, u_y, v, v_x, v_y, w, w_x, w_y, w_{xx}, w_{xy} and w_{yy}.

The choice of displacement fields was motivated by the view that the consistency of the order of the error term in the strain energy expression for a general displacement configuration is more important than achieving a zero error for one particular displacement mode. Because the second

differential of w and only the first differential of u and v are present in the strain energy expression, the complete quartic expansion for w is chosen to be one order higher than that for u and v. No explicit consideration has been made in the derivation for rigid-body modes of displacement.

A more recent development by Dawe[38] is very similar to the foregoing except that quintic polynomials are used for all three displacement components. The complete 21-term quintic expansion is reduced to 18-terms by imposition of constraints that remove the degrees-of-freedom on the sides (between the vertices). The motivation for Dawe's choice of displacement components was discussed in the previous section. Numerical comparisons given in Ref. 38 disclose the superiority of this choice of displacement components for deep, thin shells.

The SHEBA element, formulated by Argyris and Scharpf (Ref. 40) bears a relationship to the work described above. Complete quintic polynomials (21 terms each) are utilized for all three displacement components. Interelement continuity is assured by the use of midside nodes having all three normal derivatives as degrees of freedom. The corner nodes have 18 degrees of freedom each: 3 translational displacement, all six first derivatives and all 9 second derivatives. In contrast to Dawe's formulation, the condition of zero strain under rigid-body motion is satisfied exactly. This is done by using the same interpolating procedure for the surface location as for the displacements. The natural strain concept plays a central role in the shell theory employed. Published results (Ref. 42) demonstrate that 'SHEBA' produces extremely accurate solutions.

Another widely-used triangular thin shell finite element to be commented upon here has been published by Dupuis and Goel (Ref. 43). This formulation satisfies all necessary requirements for convergence of the potential energy. Rational functions are used for the displacement functions as well as to define the position of the shell above a datum plane. The use of rational functions other than simple polynomials is

described in the text by Zienkiewicz (Ref. 44) where they are called singular functions. In this element the rigid-body-motion condition is satisfied in the same way as for the SHEBA element above, i.e., through choice of coordinate. Two levels of sophistication of the element are available, the first having the required continuity of first derivatives, whereas the second has continuity of second derivatives, which are of course excessive for the imposition of the variational principle. Dupuis and Goel conclude that the second level is the most efficient.

In a later paper Dupuis (Ref. 30) has used the same approach to develop an almost identical element. The latter derivation utilizes cubic rational functions which are somewhat simpler than the previous ones. The element has nine degrees of freedom at each of the corner nodes. The element is incorporated in the MARC program (Ref. 45) and has therefore enjoyed intensive and widespread evaluation and application in practice.

Each of the above elements is complex from the standpoint of formulative effort. Also, the bandwidth of the algebraic equations of the problem, when expressed as a percentage of the total number of structural freedoms, is very high for an idealization using such elements, in comparison to simpler elements, and the effort required to solve a system of equations is proportional to the square of the bandwidth. One must also take into account the effort required to construct the individual element stiffness coefficients. Nevertheless, accuracy studies and the desire for solution reliability indicate the necessity of developments of this type when the analyst seeks to employ a conventional potential energy approach. In the next two sections we discuss the establishment of simpler formulations by use of modified potential energy approaches and alternative variational principles.

8. GENERALIZED POTENTIAL ENERGY

The idea of generalized potential energy is to employ simple, nonconforming displacement fields in the element stiffness formulation and "restore" continuity by imposition of that condition on the interelement boundary. Many different forms of this approach are possible but in the discussion which follows we restrict our attention to a scheme given by Harvey and Kelsey (Ref. 11). The scheme has been used in the triangular thin shell formulation by Thomas and the writer (Ref. 39).

The latter element is shown in Fig. 6. The shell middle surface corresponds to the orthogonal curvilinear system α - β. All three displacement components are described by complete cubic polynomials in triangular coordinates

$$u = \lfloor N \rfloor \{u\}, \quad v = \lfloor N \rfloor \{v\}, \quad w = \lfloor N \rfloor \{w\} \tag{1}$$

where

$$\{u\}^T = \lfloor u_1 \ (\frac{\partial u}{\partial \alpha})_1 \ (\frac{\partial u}{\partial \beta})_1 \ \cdots \ (\frac{\partial u}{\partial \beta})_3 \ u_4 \rfloor \tag{2}$$

and similarly for $\{v\}$ and $\{w\}$. $\lfloor N \rfloor$ contains the standard shape functions of a complete cubic corresponding to these degrees-of-freedom.

Eq. (1) does not permit continuity of the angular displacement across element boundaries and leads to very poor results even for flat-plate flexure. To resolve this, Harvey and Kelsey (Ref. 11) introduced the notion of a constraint condition to "restore" continuity. That is, if A and B are neighboring elements and the subscript n denotes the normal direction, the relative angular displacement (θ) of adjacent edges at their midpoint can be set equal to zero

$$\theta_n^{A-B} = \theta_n^A - \theta_n^B = 0 \tag{3}$$

Differentiation of the displacement field (Eq. (1)) gives θ^A and θ^B in terms of the joint displacements $\{\Delta\} = \lfloor u \ v \ w \rfloor^T$. Applying this to each boundary in each displacement component gives the set of algebraic constraint equations

$$[C]\{\Delta\} = 0 \tag{4}$$

To deal with this one may employ the Lagrange multiplier technique, as follows. Each row of Eq. (4) is multiplied by a parameter λ_i, where the subscript i identifies the row. For all rows of Eq. (4) we then have $\lfloor\lambda\rfloor[C]\{\Delta\}$, where the row matrix $\lfloor\lambda\rfloor$ lists the parameters λ_i. This is added to the potential energy, resulting in the augmented value $\bar{\Pi}$,

$$\bar{\Pi} = \frac{1}{2}\lfloor\Delta\rfloor[K]\{\Delta\} - \lfloor\Delta\rfloor\{P\} + \lfloor\lambda\rfloor[C]\{\Delta\} \qquad (5)$$

where [K] and {P} are the stiffness matrix and applied load vector, respectively. After variation of $\bar{\Pi}$ with respect to both {Δ} and {λ} one obtains

$$\begin{bmatrix} K & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} \Delta \\ \lambda \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \qquad (6)$$

Solution of this equation gives the displacements {Δ} and the values {λ}. The latter are the force quantities corresponding to the displacement components at the interelement displacement discontinuities.

It is noteworthy that the constraint condition can be directly incorporated in the variational principle and represented in the discretized system as a corrective element boundary stiffness matrix. Kikuchi and Ando (Ref. 47) refer to this as the "simplified hybrid displacement method" and have recently (Ref. 48) applied it to thin shell problems.

The numerical comparisons found in Refs. 39 and 47-50 demonstrate the accuracy in displacement computations of the above approaches in a very wide range of problems. Accuracy in stress calculations is largely untested. We should note in this connection, that "extra" stress information is obtained from the solution since, as noted above, the Lagrange multipliers represent the internal forces that are conjugate to the displacements whose continuity is restored by the constraint conditions.

9. ALTERNATIVE APPROACHES - MIXED AND HYBRID

The difficulties attendant upon the formulation of displacement-based shell elements have given rise to many studies

of alternative variational principles. These include the hybrid and mixed functionals, the "discrete-Kirchoff" approach, and generalized variational principles.

We have discussed generalized variational principles in thin-shell analysis in the previous section. These are, in fact, forms of mixed variational principles. As noted in reference to the work of Kikuchi and Ando (Refs. 47 and 48), when the constraints on a conventional variational principle are appended to the basic functional by use of the Lagrange multiplier concept, the Lagrange multipliers can be identified as the variables conjugate to the variables of the basic functional. Thus, in the case of potential energy, where the variables of the basic functional are strains, the Lagrange multipliers of the appended terms are stresses (or edge tractions). The total, augmented functional then contains both strains (in the form of displacement derivatives) and boundary tractions.

The most widely-used mixed functional in thin-shell finite element analysis, derived directly as such, is the Reissner-Hellinger integral. This can be interpreted as the complementary energy functional supplemented by constraints which enforce boundary stress continuity. It is employed because the functional can be written in such a form that bending is characterized by C^0 continuity requirements. Thus, in early developments of this functional in thin-shell finite element analysis, by Prato (Ref. 52), Connor and Will (Ref. 53) and Herrmann and Mason (Ref. 54), among others, the radial (bending) behavior is dealt with in this way while a conventional potential energy (assumed displacement) approach is taken for in-surface behavior. More recently, Tahiani and Lachance (Ref. 55) used the Reissner-Hellinger approach in the full development. In all of these developments rather simple assumed stress and displacement fields, generally of linear or quadratic form, are employed.

Various theorists have established a connection between mixed and hybrid formulations. In the view of the writer,

however, there are important distinctions between them under the following conditions. Mixed variational principles are written directly in terms of stresses and displacements and both of these are approximated by physical, node point parameters. Both parameters appear as solution parameters. Hybrid formulations, which have been pioneered by Pian and are discussed by him elsewhere in these proceedings, may be based on modified forms of the conventional variational principles. One field, say the stresses, are approximated by expressions in terms of generalized parameters while the other field is written in terms of physical node point parameters. The generalized parameters are eliminated from the element formulation, using the stationary condition of the functional, resulting in element relationships which have the same algegraic form as element stiffness matrices. Many alternative hybrid formulations are possible.

The advantage of the hybrid approach in thin-shell formulations is that representations can be developed for the condition of only C^o continuity. The most extensive work in this area seems to have been done by the group at Nottingham University, England. A review of these efforts and work done elsewhere has been given by Edwards and Webster (Ref. 56).

Finally, we take note of the discrete-Kirchoff concept, which is also intended to eliminate the C^1 continuity requirement in the selection of assumed displacement fields. If the Kirchoff assumption ("normals remain normal") is not invoked one can write the flexural energy expression in terms of first derivatives of angular displacements, θ. One then assumes independent, C^o-continuous expansions for the angular (θ) and radial (w) displacements. The problem is singular, however, in the absence of transverse shear deformation, which is the usual circumstance in thin-shell analysis. To render the problem non-singular one may invoke the Kirchoff condition (e.g., $\theta_x = \frac{\partial w}{\partial x}$ for plates) at discrete points. As many such conditions must be introduced as there are rank deficiencies in the basic formulation.

Wempner, et al. (Ref. 57) appear to have introduced the above ideas and applied them to thin-shell formulation. Subsequently, the same concepts were adapted to this purpose by Key and Beisinger (Ref. 29), Dhatt (Ref. 58) and Bonnes, et al. (Ref. 59).

10. AXISYMMETRIC THIN SHELL FORMULATIONS

A review of the linear finite element analysis of curved thin shells must also consider the special case of shells of revolution. Many structures take this form and if geometric symmetry is taken into account, it is possible to realize very significant computational economies. Early attempts to formulate satisfactory axisymmetric thin shell elements (Refs. 60-62), which were among the earliest of finite element thin shell studies of any kind, encountered problems which were met subsequently in amplified form with general shell elements. An account of these developments has been given by Gould and Sen (Ref. 63).

Consequences of the failure to deal properly with rigid body motion in the chosen displacement fields was studied by Mebane and Stricklin (Ref. 31) and Murray (Ref. 32), using the axisymmetric thin shell element. This type of element was also used in basic studies of the significance of additional terms in the displacement expansions.

The trend in recent years has been towards formulations with isoparametric representations of geometry and higher-order displacement fields, particularly in the in-surface components. Either or both of these characteristics is reflected in the work of Zudans (Ref. 64), Chan and Firman (Ref. 65), Gould and Sen (Ref. 63), Webster (Ref. 66) and Adelman, et al. (Ref. 67). Ahmad et al. (Ref. 68) and Cole, Abel and Billington (Ref. 69) have demonstrated the effectiveness of the axisymmetric element with a cross-section in the form of an isoparametric planar domain. In this formulation, reduced integration may be exploited to represent properly the transverse shear strain energy.

11. CONCLUDING REMARKS

We have attempted, in this paper, to review some of the key features of finite element thin-shell analysis. The component aspects, including choices of variational principle, geometric representation, and displacement (or stress) fields, could each be the subjects of extensive reviews. Developments based on assumed displacement fields no longer appear with the frequency they once did, but much work will surely appear in the future with respect to geometric representation and alternative variational principles.

What do the developments to date mean to the practitioner? This depends most strongly on the finite element computer programs he employs. If he works with marketed programs such as MARC or ASKA, the curved thin shell elements such as the Dupuis formulation[30] and SHEBA[40] are at his disposal and represent a high degree of sophistication, accuracy, and reliability. Other widely-distributed programs which do not have curved shell elements may have isoparametric solid elements with the reduced integration option. Such elements are also effective in thin-shell analysis. As more programs become capable of accommodating alternative approaches we can expect to see the emergence of mixed, hybrid, generalized variational, discrete Kirchoff, and perhaps other less orthodox formulations.

The different element formulations in the references cited are verified mainly through comparisons with classical analytical solutions. There is a growing need for experimental data for comparison studies of more complex shell situations, for which no alternative solutions are available. The growing field of study of the mathematical convergence characteristics of the finite element method also deserves application to the shell analysis problem, and a start has been made in this direction in Refs. 70 and 71.

The comparisons of the different element formulations do not often account for the total analysis costs. Tradeoffs between element formulative efforts and the size and

configuration of the equations to be solved are in need of study. Comparisons of the alternatives, to be valid, must include not only the operational costs to reach a desired level of solution accuracy, but should also reflect an amortization of costs to develop the associated software.

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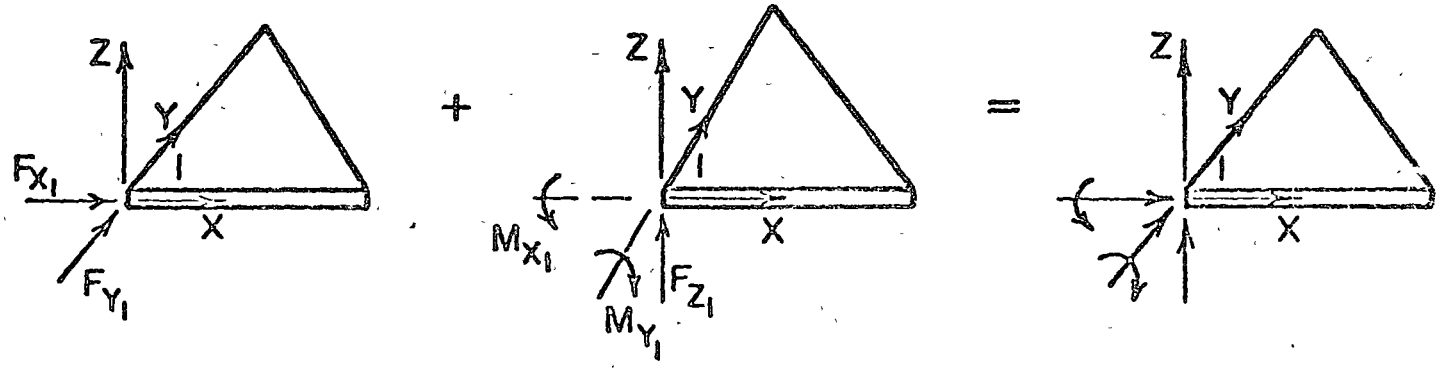


a. Membrane b. Flexure c. Combined for Shell Analysis

FIGURE I. FLAT PLATE ELEMENT IN SHELL ANALYSIS

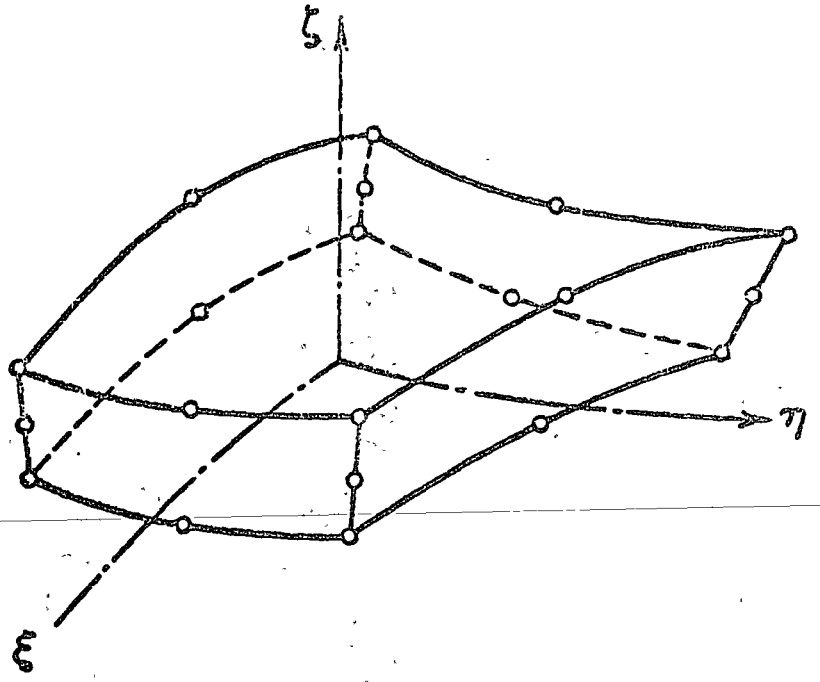
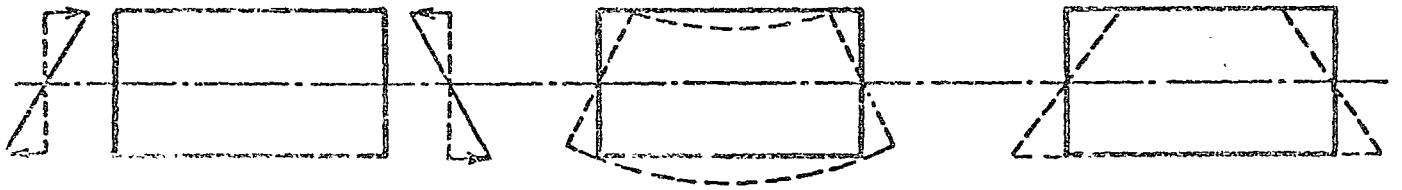


FIGURE 2. ISOPARAMETRIC SOLID ELEMENT FOR SHELL ANALYSIS. (20 NODE POINTS)



(a) Pure Bending (b) Exact Displacements (c) Linear Displacement Element

FIGURE 3. REPRESENTATION OF PURE BENDING BY PLANE STRESS ELEMENT.

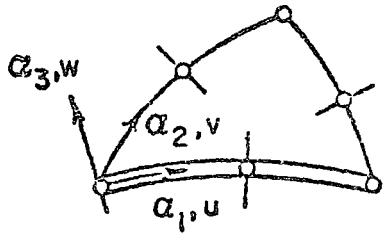


FIGURE 4. CURVILINEAR COORDINATES.

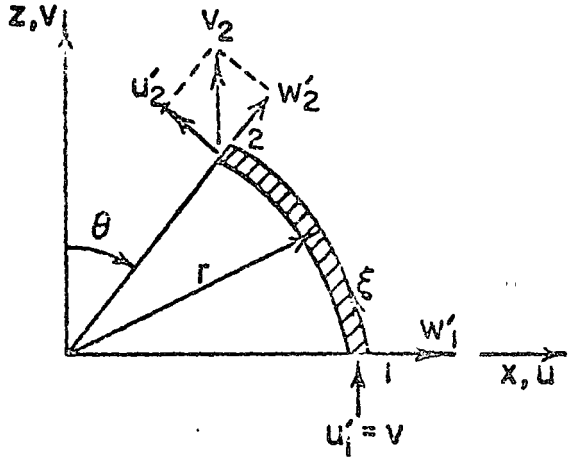
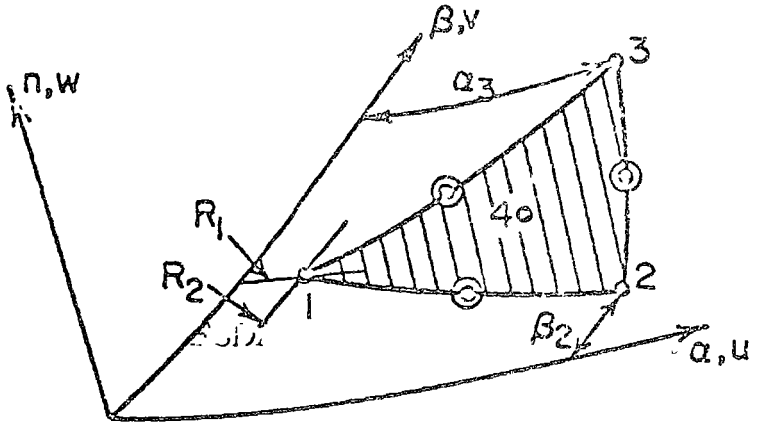


FIGURE 5. RIGID BODY MOTION CONSIDERATIONS.



◎ Points of applications of constraint conditions

FIGURE 6. TRIANGULAR SHELL ELEMENT FOR GENERALIZED POTENTIAL ENERGY FORMULATION.

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FRACTURE MECHANICS ANALYSIS

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THE FINITE ELEMENT METHOD IN FRACTURE MECHANICS ANALYSIS

R. H. GALLAGHER

10. THE FINITE ELEMENT METHOD IN FRACTURE MECHANICS ANALYSIS

R. H. Gallagher

I. INTRODUCTION

Despite the relative novelty of finite element analysis as a tool for fracture mechanics studies--the earliest papers appeared as recently as 1969--a wide range of alternative formulations have been published. The objective of these notes, therefore, is a classification and review of the various published methods of finite element fracture mechanics analysis and, to a limited extent, an assessment of their advantages and limitations. These notes update a state-of-the-art review of the topic by the writer (Ref. 1) in 1971 and incorporate information found in reviews by Rice and Tracey (Ref. 2), Pian (Ref. 64), and Apostal, et al (Ref. 65). A symposium on computational fracture mechanics (Ref. 66) is also represented. The present work concentrates its attention upon papers published in the open literature, to which most readers will have access. It should be observed that as in any topic which continues under development there is also a substantial literature in the form of company and institute reports.

The present topic is conveniently divided into "linear" (or elastic) and "inelastic" fracture mechanics. Linear fracture mechanics is a well-established design analysis tool, so it is natural that it has been given the most attention in the literature. We therefore devote most of our effort to this side of the subject. Brief comments are given with reference to inelastic finite element fracture-mechanics analysis.

Because of the scope and diversity of contributions to the area of finite element linear fracture mechanics analysis, it i necessary that certain classifications be made of the area. The author prefers the following classifications: (1) direct methods, (2) energy methods, (3) the superposition approach and (4) singularity function formulations. Further subdivisions will be identified within the respective sections dealing with these classifications. These classifications are examined after a brief review of the basic notions of linear fracture mechanics.

II. LINEAR FRACTURE MECHANICS

Linear fracture mechanics is concerned with the determination of the length at which a crack will propagate rapidly, under circumstances of a brittle nature, due to specified load and geometric conditions. The concepts stem from work by Griffith (Ref. 71) on purely brittle materials, and were generalized by Irwin (Ref. 3) to cases of ductile materials where the conditions at the crack are such that the inelastic deformation characteristic of ductile behavior is confined to a relatively small zone.

The elastic stress field at the tip of the crack can be characterized by a parameter K_j (j = I, II, or III), known as the stress intensity factor. The magnitude of K_j depends on the distribution and intensity of the applied loads and the geometry of the structure. Any combination of these conditions which will give a stress intensity factor equal to or greater than the experimentally determined critical value K_{j_c} will cause failure. The various modes are shown in Fig. 1.

In the mode I case (K_I) the stress is applied in the direction normal to the crack and tends to open the crack. Mode II is a sliding mode, due to applied shear, and Mode III is a bending, or tearing mode.

The solution for the stresses and displacements in the vicinity of a sharp crack in the Mode I and II conditions for plane strain or plane stress conditions in isotropic materials can be established by conformal mapping techniques (Ref. 70). The series expansion of the solution for stress contains the terms $r^{-1/2}$, $r^{1/2}$, $r^{3/2}$, etc., where r is the radius coordinate in a polar coordinate system originating at the crack tip (see Fig. 1a). The first term, $r^{-1/2}$, predominates at the tip of the crack so only this term is retained in the customary expressions for the stresses and displacements in the vicinity of the crack tip. These are, for the Mode I condition:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{1a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{1b}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right) \tag{1c}$$

$$u = \frac{K_I}{4G}\left(\sqrt{\frac{r}{2\pi}}\right)\left[(2\kappa-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] \tag{1d}$$

$$v = \frac{K_I}{4G}\left(\sqrt{\frac{r}{2\pi}}\right)\left[(2\kappa+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right] \tag{1e}$$

where G is the shear modulus and μ is Poisson's ratio, and $\kappa = (3 - 4\mu)$ for plane strain and $(3 - \mu)/(1 + \mu)$ for plane stress. Formulations can be constructed in the same general form for other modes of fracture.

The basic consideration in design, from the standpoint of linear fracture mechanics, is the relationship between stress intensity factor and the strain energy release rate, G, which is defined as the change in strain energy in the structure per unit length of crack extension. This relationship has been established by Irwin (Ref. 3) for plane strain and plane stress conditions, as

$$G = \frac{(\kappa+1)}{8G} (K_I^2 + K_{II}^2) \tag{2}$$

There are other parameters which relate strain energy to stress intensity factors, such as the J-integral (Ref. 18). These will be taken up in the section dealing with energy methods.

Extensive catalogs of solutions for the stress intensity factors for various simple load and geometric conditions are available (Refs. 5, 72). These are not adequate, however, for many of the complicated conditions found in practice, where the load and geometry may be highly irregular and the material can in general be anisotropic. The finite element method is a logical approach to the solution of such problems.

III. DIRECT METHODS

We define direct methods as those which involve the performance of a conventional finite element analysis (one which is based on simple elements and simple polynomial functions) with a high degree of refinement in the region of the crack tip. They determine the stress intensity factor by interpretation of the calculated stresses or displacements. Interpretation of some form is needed since, due to its "finite" nature, the conventional finite element representation is incapable of producing a direct evaluation of the crack tip stress intensity factor.

We consider first the direct methods which operate on the calculated node point displacements. One approach, the crack-opening-displacement method (Kobayashi, et al, Ref. 7) deals with the displacement of a point close to the crack tip, such as Point A in Figure 1a. From Equation (1e) we can write the expression for this displacement in the form

$$v_A = \frac{K_I}{G}\sqrt{\frac{r}{2\pi}}\, f_1(\theta,\mu) \tag{3a}$$

Solving for K_I we have

$$K_I = \frac{Gv_A\sqrt{2\pi}}{\sqrt{r}\cdot f_1(\theta,\mu)} \tag{3b}$$

Miyamoto (Ref. 8) employs the same approach in three-dimensional linear fracture mechanics analysis. He uses tetrahedronal elements.

In an attempt to reduce the amount of grid idealization needed to obtain an acceptable answer, Chan, et al [9] apply an extrapolation scheme. One form of this scheme is shown in Fig. 2. A radial line at a fixed value of θ is first established and by finite element analysis the displacements at the node points that

lie on this line are calculated. Then, using Equation 2 a stress intensity factor (K_I) is calculated at each of the node points along the line. By extrapolation of a line through the so-determined points on a K_I vs. radial distance plot, using points away from the crack tip, the value of K_I at zero radius is established. It should be noted that the use of this method is not restricted to cases where conventional elements are employed in the analysis. Subsequently, in our discussion of the use of isoparametric elements with singularities, this method will again be employed.

Alternative to the interpretation of displacement values to obtain K_I is the interpretation of stresses for this purpose. The interpretation of stress is effected in the manner of the above-described procedures. The radial stress field in the vicinity of the crack tip is, by appropriate combination of Eq. (1a)-(1c)

$$\sigma_r = \frac{K_I f_2(\theta)}{\sqrt{2\pi r}} \tag{4}$$

and solving for K_I:

$$K_I = \frac{\sigma_r \sqrt{2\pi r}}{f_2(\theta)} \tag{5}$$

This expression is evaluated at various locations along a given radius and the results extrapolated in order to define K_I as r approaches zero. Fig. 2 shows how the stresses might be interpreted. This procedure is due to Chan, et al [9] and has also been applied by Watwood [11]. The commonly-employed constant-stress triangular elements give an erratic basis for interpretation. If linearly varying stress elements were used the problem

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of interpretation would be reduced somewhat but the number of degrees of freedom needed for an adequate solution would still be quite large.

An important operational improvement to the above schemes is the "zooming" technique used by Miyamoto (8) and Becker, et al (41). The analysis is first performed for a coarse mesh and the stresses and displacements on the boundary of a local surrounding the crack are determined. The local region is then analyzed with a more refined grid. The localization may then be applied to still smaller regions. This approach gives a very significant improvement to computational efficiency of the extrapolation schemes. Indeed, this technique is used to considerable advantage when coupled to other general approaches to be discussed subsequently.

Another approach which one is tempted to take in the interests of improving the efficiency of the direct methods is to use the simplest type of elements in the regions away from the crack tip and to use higher-order elements in the vicinity of the crack tip. Tong and Pian (Ref. 68) and Fried and Yang (40) have shown that increasing the order of the interpolation functions inside the element will not result, with a given mesh, in an indefinite increase in the rate of convergence. They demonstrate, however, that the rate of convergence of a given order of interpolation polynomial can be recaptured by a proper, nonuniform, spacing of the mesh near the singularity.

Fig. 3, taken from Reference 68, illustrates the above point. The problem shown is a square panel with side cracks under uniform tension σ_y. Each analysis point represents the solution for the

strain energy for a uniform spacing of an element of a given type. The types of elements employed are constant and linear strain triangles and two types of hybrid-stress rectangular elements. The convergence rate in all cases is a linear function of the element size. In contrast, for problems without stress singularities, the convergence rate is a function of the element size, being of higher order as the order of the element approximating polynomial increases.

IV. ENERGY METHODS

A number of distinct approaches can be found under this heading, including the total energy method and the line integral method.

The total energy method involves a relationship between the stress intensity factor and the rate of change of strain energy with respect to crack length ($\frac{dU}{da}$). As was already noted, according to fracture theory the latter is equal to the strain energy release rate G_j, i.e.

$$G = \frac{dU}{da} \tag{6}$$

For isotropic plane stress Eq. (2) becomes

$$G = \frac{1}{E}\left(K_I^2 + K_{II}^2 + K_{III}^2\right) \tag{7}$$

Then, for example, for mode I plane stress conditions we have

$$\frac{dU}{da} = \frac{K_I^2}{E} \tag{8}$$

Expressing $\frac{dU}{da}$ in terms of finite differentials $\frac{\delta U}{\delta a}$ and solving for K_I, we have

$$K_I = \sqrt{\frac{E(\delta U)}{\delta a}} \tag{9}$$

Hence, in this procedure (see Figure 4) one performs an analysis for a stipulated crack length (a) and computes the strain energy U. The strain energy is obtained from the solution simply as $\frac{1}{2}\lfloor\Delta\rfloor\{P\}$, where $\lfloor\Delta\rfloor$ and $\{P\}$ are the node point displacement and load vectors. The crack is then opened a distance δa and a new strain energy $U + \delta U$ is calculated.

Finite element fracture mechanics analyses based on this idea have been studied and described by Anderson, et al (Ref. 10),

Watwood (Ref. 11), Dixon, et al (Refs. 12, 13), Mowbray (Ref. 14), Blackburn (Ref. 15), Hellan (Ref. 16), and Fuhring (Ref. 17).

As noted above, two solutions are needed which differ only in the increment of crack size. The resulting difference in strain energy is not as sensitive to grid refinement in the vicinity of the crack as is the determination of stress intensity factors by the "direct" methods. Thus, a relatively coarse mesh can be used in each analysis.

Computational procedures which improve the efficiency of the total energy method by requiring a full solution for only one gridwork are those which are based on an approximation to the solutions for the gridwork that has been changed on account of the crack extension. Designating the change in the global stiffness matrix ([K]) due to the opening of the crack by a distance δa as $[\delta K]$, the stiffness equations of the altered gridwork are

$$[[K] + [\delta K]]\{\{\Delta\} + \{\delta\Delta\}\} = \{P\}$$

or

$$[K]\{\Delta\} + [\delta K]\{\Delta\} + [K]\{\delta\Delta\} + [\delta K]\{\delta\Delta\} = \{P\}$$

where $\{\delta\Delta\}$ is the change in the solution. Now, since $[K]\{\Delta\} = \{P\}$ and with discard of the product $[\delta K]\{\delta\Delta\}$, one obtains

$$\{\delta\Delta\} = - [K]^{-1}[\delta K]\{\Delta\} \qquad (10)$$

Only a small portion of $[\delta K]$ is populated since the crack extension affects only a few elements in the global representation. Both Parks (Ref. 44) and Hellan (Refs. 16, 73) have exploited this idea. Hellan (Ref. 44) embeds this scheme in the algorithm for solution of [K], resulting in further efficiencies.

Other approaches to more efficient analysis by the total energy method have been devised. An important aspect of most

current applications of the work by Aamodt, et al (43) is the use of a "multilevel condensation" technique in reducing the analysis degrees-of-freedom to a relatively small number. In this approach "superelements" are formed of a large number of the basic elements. The interior degrees-of-freedom are eliminated so that the superelement is represented only in terms of the boundary points of the superelement. The superelements are combined to give new, higher-level, superelements and the process of condensation is again applied. There is a similarity to the work of Miyamoto (Ref. 8) and Becker, et al (41) in that the final analysis is performed for a few number of elements and the efficiency of analysis is enhanced considerably.

Various authors (Anderson, et al (1), Watwood (11), Fuhring (17), and Aamodt, Bergan, and Klem (43)) among others have conducted numerical studies which compared the total energy and direct methods, resulting in the conclusion that the energy approach is more efficient.

The line integral, or path-independent integral method was introduced into linear fracture mechanics by Rice (18). Similar to the local energy method, but with no restriction on the shape of the region studied, a region containing the crack tip is delineated as in Figure 4 . Rice (18) shows that the following integral, taken over the boundary (Γ) of the region is equal to a constant J:

$$J = \int_{\Gamma} (U dy - T \cdot \frac{\partial u}{\partial x} ds) \tag{12}$$

where U is the strain energy density, T is the Traction vector defined according to the outward normal along Γ, and u is the displacement vector. ds is the element of arc along Γ. The stress intensity factor for plane strain is related to J by the expression

$$K_I = \left[\frac{JE}{(1-\mu^2)}\right]^{1/2} \tag{13}$$

To use the finite element method one must delineate an isolated region around the crack and determine the above line integral. A finite element application of this concept is described by Chan, et al (9). It has also been applied in the evaluative study of Anderson, et al (10), by Leverentz (60), Becker, et al (41), and by Kobayashi, et al (62).

V. SUPERPOSITION METHODS

The superposition approach seeks the computation of the stress intensity factor through a linear combination of classical singularity solutions and a coarse finite element grid. The approach has been used successfully in fracture mechanics calculations by Yamamoto, et al (Ref. 19). Similar ideas have been discussed by Morley in Ref. 20 and Ando in Ref. 69.

The concept is to define the classical solution for a problem as close to the problem of interest as possible. For example, in the present case, the solution for a crack in an

infinite plate subjected to biaxial stress is first obtained. This solution will not satisfy all conditions for the actual problem and disparities can be interpreted as body forces and edge loadings. The body forces and edge loadings are applied to the finite element analysis in reversed direction and the results from the finite element and classical solutions are superimposed.

The general scheme is sketched in Figure 5. The classical solution is denoted by σ_a, while finite element solutions are given by σ_r and σ_e. The appropriate superposition of solutions yields the "exact" answer σ^*.

VI. SINGULARITY FUNCTION FORMULATIONS

a. Outline of Alternative Approaches

One of the most appealing approaches, from both a theoretical and computational view, is to formulate an element containing a singularity to be employed in the region of the crack tip. Regions beyond the crack tip are represented by conventional elements. Many attempts to exploit this concept have appeared.

Byskov (Ref. 2) was perhaps the first to publish a development along these lines, using singularity formulations due to Muskhelishvili (Ref. 22). His element is shown in Figure 6. Many alternative schemes have been proposed by Rao, et al (Ref. 23), Blackburn (Ref. 15), Tracey (Ref. 24), Benzley (Ref. 25), Robinson (Ref. 26), Walsh (Refs. 27, 28), Pian (Ref. 29), Apostal (Ref. 30), Aberson and Anderson (59), among others. We examine here four different categories of this general approach.

b. Embodiment of a Singularity Element in a Conventional Grid

It is of interest to examine in some detail Walsh's approach to incorporating a singularity element in a mesh that contains conventional finite elements. The element itself is conventional in that he first chooses displacement fields (but which include a singularity) and the basic element stiffness matrix

is then constructed from the usual stiffness formulas. Special relationships are added to these, however, to minimize the incompatibility between the crack tip element and the conventional elements outside of it. Blackburn (15) has formulated a singularity element and combined it with the total energy method.

The structure is divided into two regions: an "outer" region consisting of conventional finite elements, and an "inner" region consisting of a single element encompassing the crack. The degrees-of-freedom of points in the outer region, excluding the points on the interface of the two regions, are designated as Δ_o. The interface d.o.f. are Δ_i. The expressions for displacement of a medium continuing a crack will be characterized by parameters which are the stress intensity factor and rigid body motion terms (see Eq. 1c, 1d). We designate these as $\{\Delta_s\}$, noting that they contain the stress intensity factors K_I and/or K_{II}.

In view of the above, the stiffness equations for the outer region can be written as

$$\begin{Bmatrix} P_o \\ P_i \end{Bmatrix} = \left[\begin{array}{c|c} K_{oo} & K_{oi} \\ \hline K_{io} & K_{ii} \end{array}\right] \begin{Bmatrix} \Delta_o \\ \Delta_i \end{Bmatrix} \tag{14}$$

Also, by evaluation of Equation (1c) at the juncture points we can write

$$\{\Delta_i\} = [\Gamma]\{\Delta_s\} \tag{15}$$

Then using this in a transformation of the degrees-of-freedom in Equation (14) we have

$$\begin{Bmatrix} P_o \\ F_s^o \end{Bmatrix} = \left[\begin{array}{c|c} K_{oo} & K_{oi}\Gamma \\ \hline \Gamma^T K_{oi} & \Gamma^T K_{ii}\Gamma \end{array}\right] \begin{Bmatrix} \Delta_o \\ \Delta_s \end{Bmatrix} \tag{16}$$

where F_s^o represents the generalized forces of the outer region corresponding to the degrees-of-freedom Δ_i. Counterpart stiffness relationships between the generalized forces F_s^s of the

inner region (singularity element) and these d.o.f. can also be constructed, of the form

$$\{F_s^s\} = [K_{ss}]\{\Delta_s\} \tag{17}$$

Finally, then, we obtain the global stiffness using the condition $\{P_s\} = \{F_s^o\} + \{F_s^s\}$.

$$\begin{Bmatrix} P_o \\ P_s \end{Bmatrix} = \left[\begin{array}{c|c} K_{oo} & K_{o1}\Gamma \\ \hline \Gamma^T K_{io} & \Gamma^T K_{11}\Gamma + K_{ss} \end{array}\right] \begin{Bmatrix} \Delta_o \\ \Delta_s \end{Bmatrix} \tag{18}$$

Solution of these equations gives the stress intensity factors directly, as they are included in Δ_s.

We should note that in the above approach there is a significant degree of interelement displacement nonconformity. Nevertheless, results of reasonable accuracy are reported in Refs. 27 and 28.

The elements of Tracey (24) and Benzley (25) can be regarded as falling in this category. Each, however, constructs the singular element displacement field in such a way that displacements on boundaries that connect to conventional elements will conform to the displacement fields of the latter.

c. Conformal Mapping

Conformal mapping, which has been employed effectively in classical solutions to stress intensity factor problems, can also be useful in finite element solutions to such problems. The related ideas and their application to problems have been given by Cheng (51) and Armen, et al (52). To describe these ideas we show the rectangular region of Fig. 8a with a slit extending from point O to point D. In the case where a solution is sought to the equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

where ϕ is the independent variable in the region, a mapping

from the x-y to the ξ-η (complex) plane by use of the transformation

$$\zeta = z^{\frac{1}{2}}$$

where $\zeta = \xi + i\eta$, $z = x + iy$. The mapping is as shown in Figure 8b and the equation to be solved is

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0$$

Appropriate boundary conditions must of course be defined and these must also be transformed.

It is apparent that the transformed problem would be difficult to solve by means of classical methods because of the irregular boundaries and associated boundary conditions of the transformed region. It is not difficult, however, to solve the transformed problem by the finite element method, using isoparametric elements. This is the approach taken in Refs. 51 and 52. Ref. 52, in particular, gives extensive numerical results for fracture mechanics problems.

d. Isoparametric Element

It has long be recognized that the transformation from physical to isoparametric coordinates can be singular if node points along the edges of members are positioned in a certain way. For example, in Fig. 9a, when the side node of a rectangular element with bi-quadratic fields is located one-quarter of the edge length from the corner (as for point 2), then the transformation will be singular at the corner. Henshall (Ref. 50), by studying this more closely, and also independently by Barsoum (Ref. 49) have shown that the singularity is that of $1/\sqrt{r}$, which is identical to the form of singularity in fracture mechanics. Thus, by locating the node point 1 at the crack tip, as in Figure 9a, it is possible to solve for the stress intensity factor without development of a special element. Neale (Ref. 61) reports the combination of this approach with the J-integral method.

To examine this more closely we follow the development by Barsoum (49) and consider the case of a side of total length L with points 1, 2, 3 (Fig. 10a) and point 2 midway between the ends. Introducing the nondimensional coordinate $\xi = -1 + \frac{2x}{L}$, which has values of -1, 0, and 1 at points 1, 2, and 3, respectively, the coordinate x can be written in shape function form as

$$x = -\frac{1}{2}\xi(1-\xi)x_1 + (1-\xi^2)x_2 + \frac{1}{2}\xi(1+\xi)x_3$$

Now, the same expression will hold for other locations of x_2 if the definition of ξ is properly revised. To do this we need only express x_2 in terms of L and substitute into the above equation and solve for ξ. Thus, for $x_2 = L/4$ (and $x_1 = 0$, $x_3 = L$)

$$x = (1-\xi^2)\frac{L}{4} + \frac{1}{2}\xi(1+\xi)L$$

and, solving for ξ

$$\xi = \left(-1 + 2\sqrt{\frac{x}{L}}\right)$$

or $$x = \frac{L}{4}(\xi+1)^2$$

In an isoparametric formulation the transformation from physical to isoparametric coordinates is based upon the inverse of the matrix of derivatives of the former with respect to the latter.

For the present example the only derivative present is $\frac{dx}{d\xi}$, which is, from the above, $\frac{L}{2}(1+\xi)$. This gives $\frac{dx}{d\xi} = 0$ at $\xi = -1$, so that the transformation is singular at point 1. To investigate the consequences of this singularity we can write the displacement field (u) in terms of the shape functions given above, then express the shape functions in terms of x, and differentiate to give strain. This will disclose that the strain singularity at point 1 is of the $1/\sqrt{r}$ type.

In more recent work, Hibbitt (Ref. 77) and Barsoum (Ref. 5) have cast doubts upon the adequacy of the quadrilateral isoparametric element for the above application. Hibbitt shows that the stress singularity is defined only along the boundary and that the strain energy (and, therefore, stiffness) is singular at the crack tip. Barsoum shows that by coalescing points (Fig. 11) to form a triangle the stress singularity is defined on all radial lines through the crack tip and that the strain energy is non-singular at the crack tip. Thus, triangular isoparametric elements are to be preferred when this approach is taken.

It should be noted that in the implementation of the above approach one can calculate the displacements (or stresses) which are then interpreted in the manner of the "direct" methods discussed earlier.

e. Assumed Stress Hybrid Method

Pian, et al (Ref. 24) use the hybrid stress approach in their work. In applying the hybrid stress approach to the formulation of a singularity element we expand the characterization of the internal stress field to account for the singular stress field.

$$\{\sigma\} = [P_o]\{\beta_o\} + [P_s]\{\beta_s\}$$

where $[P_o]$ represents the coefficients of a simple (usually polynomial) expansion in the element, $[P_s]$ contains the coefficients from the singularity stress field, and $\{\beta_s\}$ contains the stress intensity factors. For example, if modes I and II are both included in the analysis

$$\{\beta_s\} = \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix}$$

It follows that the surface tractions ($\underset{\sim}{T}$) are similarly described

$$\underset{\sim}{T} = [R_o]\{\beta_o\} + [R_s]\{\beta_s\}$$

As before, the edge displacements are simply

$$\underset{\sim}{\bar{u}} = [\bar{Y}]\{\Delta\}$$

Introducing these expressions into Equation (23) of the lecture notes on "Mixed Variational Principles and Hybrid Formulations" we obtain

$$\Pi_c = \frac{1}{2} [\beta_o][H_o]\{\beta_o\} + [\beta_o][H_{os}]\{\beta_s\}$$

$$+ \frac{1}{2} [\beta_s][H_s]\{\beta_s\} - [\beta_o][Q_o]\{\Delta\} - [\beta_s][Q_s]\{\Delta\}$$

where $[H_o] = [\int [P_o]^T [E]^{-1} [P_o] d(Vol)]$

$[H_{os}] = [\int [P_o]^T [E]^{-1} [P_s] d(Vol)]$

$[H_s] = [\int [P_s]^T [E]^{-1} [P_s] d(Vol)]$

$[Q_o] = [\oint [R_o]^T [\bar{Y}] dS]$

$[Q_s] = [\oint [R_s]^T [\bar{Y}] dS]$

The stationary condition on Π_c with respect to $\{\beta_o\}$ is imposed and the result is used to eliminate $\{\beta_s\}$ from Π_c. The stationary condition on Π_c with respect to $\{\beta_s\}$ and $\{\Delta\}$ then gives equations of the form

$$\begin{bmatrix} K & M^T \\ M & N \end{bmatrix} \begin{Bmatrix} \Delta \\ \beta_s \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution of these equations gives the stress intensity factors directly.

A simpler and seemingly more effective hybrid approach has been presented by Tong, Pian and Lasry (Ref. 47). Recognizing that the singularity displacement fields they will choose will represent satisfaction of both continuity and equilibrium conditions within the element, they construct a functional (Π_m) which represents the approximation of these conditions on the element boundary.

$$\Pi_m = \int T \cdot \bar{u} dS - \frac{1}{2} \int T \cdot u dS$$

where T represents the element edge tractions due to the assumed stress field, u are the corresponding edge displacements, and

$\bar{u}$ are boundary displacements that are independently chosen in such a way that interelement compatibility is satisfied. These fields can be discretized as follows

$$T = [R]\{\beta\}$$

$$u = [Z]\{\beta\}$$

where $\{\beta\}$ contains the relevant stress intensity factors as degrees-of-freedom and

$$\bar{u} = [\bar{Y}]\{\Delta\}$$

one then obtains, after substitution of these in Π_m and imposition of the stationary conditions, the following stiffness matrix

$$[K] = [Q]^T[H]^{-1}[Q]$$

where $[Q]$ is as previously defined and now

$$[H] = \frac{1}{2}\left[\int ([T]^T[Z] + [Z]^T[R])d(Vol)\right]$$

Other investigators (Apostal (1) and Atluri, et al (48)) have applied the "assumed displacement" hybrid method, which depends on assumed displacement fields, rather than stress fields, within the element.

VII. INELASTIC ANALYSIS

Finite element analysis possesses special advantages in elastoplastic analysis where opportunities for classical solutions are very limited and some form of numerical analysis is required for virtually all circumstances. A description of this aspect of finite element analysis was given in prior lecture notes. The significance of elastoplastic studies in the prediction of fracture is reviewed by Rice and Tracey (Ref. 2). Our

purpose here is merely to cite efforts towards the analysis of inelastic behavior in the region of the crack tip.

An exposition of the direct method in inelastic fracture mechanics has been given by Aamodt and Bergan (53, 54).

The representation described by Levy, et al (31), formulated for elasto-plastic fracture mechanics analysis, represents an alternative singularity formulation. The "basic" element, shown in Figure 7, consists of a transformation edge-displacement field (see Raju and Rao (32) for stiffness coefficient details). The singularity is achieved by allowing the joints a and b to coalesce into joint 1 as the element is used as a "near tip element" with joints 1-2-3. This scheme yields a $\frac{1}{r}$ singularity of the type associated with inelastic crack tip phenomena, in contrast to the $1/\sqrt{r}$ singularity of linear fracture mechanics. The inelastic analysis procedure used was the tangent stiffness method.

Armen, et al (52) have used the conformal mapping procedure in inelastic fraction mechanics analysis.

Another attempt at the representation of the crack tip singularity is due to Hilton and Hutchinson (33), who employ an analytical model of the singularity in a circular region surrounding the crack tip and represent the structure outside this region with conventional finite elements. Swedlow (34) and Tuba (35) perform finite element elastoplastic analysis for cracked and notched plates, basing their representations entirely upon conventional elements. Walton, Woodman, and Ellison (Ref. 36) use simple triangular elements and the "initial strain" method of inelastic analysis in studies of fatigue-crack growth. Miyamoto (Refs. 6 and 8) and Yokobori and Kamei (Ref. 37), on the other hand, combine conventional elements and the tangent stiffness approach.

VIII. CONCLUDING REMARKS

The following comments, which apply only to the linear fracture mechanics methods are intended as a summary of the more apparent advantages and limitations of the respective methods,

must principally pertain to computational efficiency, rather than accuracy, since improved accuracy is always possible at increased cost. Furthermore, account must be taken of the possibility of required additional research or major modifications of existing computer programs to establish working capabilities of practical analysis.

The direct methods are appealing in that any available computer program can be employed "as is." Available evidence reflects unfavorably on the economics of such methods, however, if special steps are not taken to deal with the isolated region of the crack. The use of 200 to 1000 elements has been found necessary if an accuracy within 5% is to be realized in a plane stress problem. This degree of refinement is economically unacceptable for a simple geometry and load condition and is unquestionably so for more complex geometries and three-dimensional stress situations. This disadvantage is counterbalanced if "zooming" or "substructuring" capabilities are available in the program being used.

Among the direct methods, the extrapolation procedure on displacements is clearly preferable when the element formulations are based upon assumed displacement fields. Considerable reliability is introduced through the device of extrapolation; many more elements must be employed in the crack-opening-displacement method to obtain as accurate a result. Extrapolation upon stresses may prove more profitable if the finite element model is an "equilibrium field" model (see first set of lecture notes).

Among the energy methods, the total energy scheme is the simplest to apply. Various studies (Refs. 9, 10, 13) have effected numerical comparisons of procedures in the direct and energy-based categories. Many of these studies rely upon the simplest type of element (triangle with linear edge displacements). It is of interest to note that Anderson, et al (10) found that the order of merit of methods in the two categories could be interchanged by appropriate definition of the finite element grid (node spacing).

For users of widely-available computer programs, the total energy method is immediately applicable. Computational efficiency can be greatly enhanced through efficient computational algorithms which can effect a change of crack length without reanalysis (Parks (44), Hellen (16)). Superposition methods are highly efficient from the view of computer costs.

Singularity function elements represent the most promising of all approaches. In general, these require that the program into which they are to be incorporated be able to accommodate procedures for numerical integration of the individual element stiffness coefficients. Ref. (46) gives a limited comparison of energy-based and singularity function methods. It is also possible to combine the energy-based and singularity-function methods as was done by Blackburn (15) and Neale (61).

It is clear that relative merit of the many proposed procedures depends on the computer programs available for their application. A review of such programs has been presented by Benzley and Parks (45) and by Apostal, et al (Ref. 65).

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