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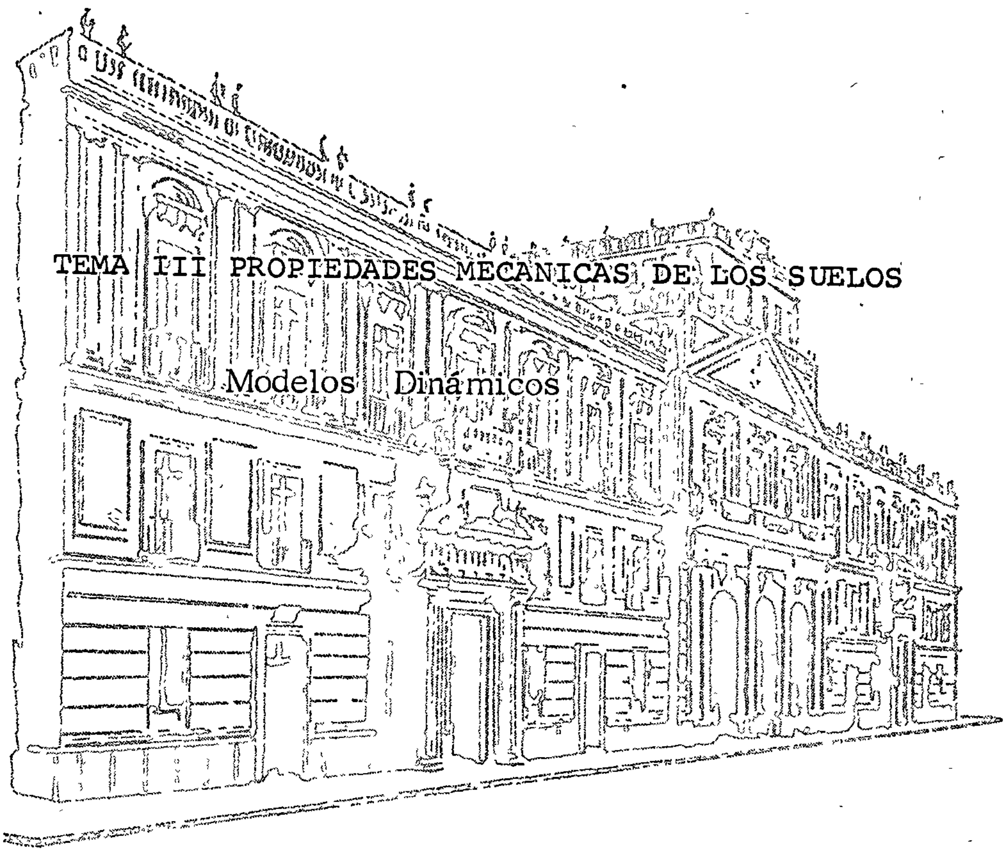
DISEÑO DE CIMENTACIONES SUJETAS A VIBRACION



TEMA III PROPIEDADES MECANICAS DE LOS SUELOS

Modelos Dinámicos

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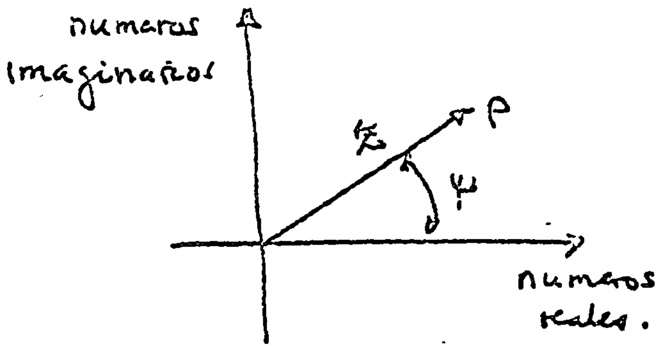
REPRESENTACION DE MOVIMIENTO ARMONICO POR EXPONENTES COMPLEJOS.

Cuando se usan exponentes complejos como $e^{i\psi}$ en lugar de funciones circulares se tienen ventajas notables en la solución de sistemas vibratorios como se podrá apreciar en los planteamientos subsecuentes

Dado z un numero complejo cuyas partes real e imaginaria. Son a y b respectivamente. En el diagrama de Argand puede ser representado por el punto P cuyo coordenadas son (a, b)



Este numero puede ser representado alternativamente como:

$z = a + ib = Z(\cos\psi + i\,\text{Sen}\,\psi) = Ze^{i\psi}$ en donde $Z = \sqrt{a^2+b^2}$

Para demostrar que $e^{i\psi} = \cos\psi + i\,\text{sen}\,\psi$. Se puede desarrollar el exponencial y luego asociar el desarrollo real e imaginario para identificar la serie del seno y coseno. Otra manera de demostrarlo es considerar $Z = 1$ esto es: $z = e^{i\psi}$.

$z = \cos\psi + i\,\text{Sen}\,\psi$.

$dz = (-\text{Sen}\,\psi + i\cos\psi)\,d\psi$.

haciendo la división $\frac{dz}{z} = \frac{(-\operatorname{Sen}\varphi + i \cos\varphi)\, d\varphi}{\cos\varphi + i \operatorname{Sen}\varphi}$.

Siguiendo las reglas de división de complejos

$$\frac{dz}{z} = \frac{(-\operatorname{Sen}\varphi + i \cos\varphi)\, d\varphi}{\cos\varphi + i \operatorname{Sen}\varphi} \times \frac{\cos\varphi - i \operatorname{Sen}\varphi}{\cos\varphi - i \operatorname{Sen}\varphi}$$

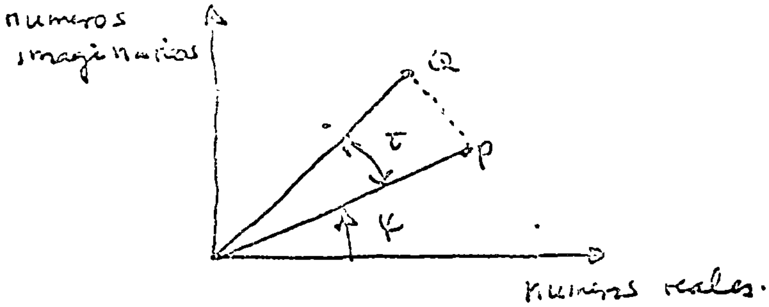
$$= i\, d\varphi.$$

integrando ambos miembros $Lz = i\varphi. \Rightarrow z = e^{i\varphi}$ o sea.

$\cos\varphi + i \operatorname{Sen}\varphi = e^{i\varphi}$. identidad de Euler

Ahora si φ se reemplaza por el argumento $\Omega t + \phi$ la representación debe ser un radio vector $\overline{OP}$ que gira a una velocidad angular Ω y su parte real e imaginaria se obtienen de las proyecciones sobre los ejes coordenados y son:
$Z \cos(\Omega t + \phi)$ y $Z \operatorname{Sen}(\Omega t + \phi)$ respectivamente.
Puede demostrarse que si una cantidad varía armonicamente como puede ser un desplazamiento su velocidad puede ponerse en esta misma forma. La identidad de Euler asegura que adición y diferenciación pueden efectuarse en los exponenciales complejos. pudiendose manejar como vectores. por ejemplo la adición de complejos se forma adicionando las partes reales e imaginarias de cada componente.
Si ahora se construye un nuevo complejo z' formado al rotar la línea que corresponde a z un angulo τ sin cambiar su longitud. se hace de la siguiente manera:

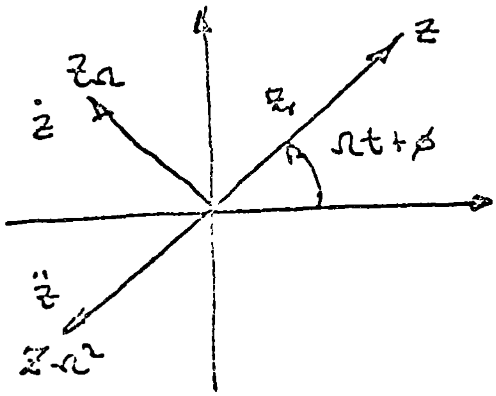
$$z' = Z e^{i(\varphi + \tau)} = z e^{i\tau}$$



la expresión anterior muestra que la generación de z' se hace multiplicando z por $e^{i\tau}$ esto llevado al complejo que depende del tiempo se tiene $Z e^{i(\Omega t+\phi)} = Z e^{i\phi} e^{i\Omega t}$ que puede ser interpretado como un vector que gira a una velocidad angular Ω y su posición inicial $Z e^{i\phi}$. Ahora diferenciando se puede obtener

$$\frac{d}{dt}\left[Z e^{i(\Omega t+\phi)}\right] = i Z\Omega e^{i(\Omega t+\phi)} = \left[Z\Omega e^{i(\phi+\pi/2)}\right] e^{i\Omega t}$$

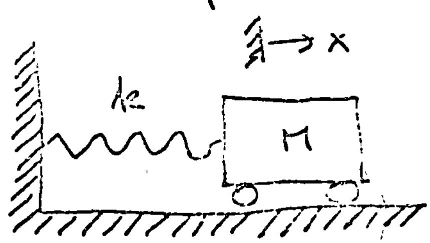
que significa que el vector velocidad tiene una longitud $Z\Omega$ pero que forma un angulo $\pi/2$ respecto al vector original de desplazamiento; una segunda diferenciación multiplicaría el modulo por Ω y formaría un angulo de $\pi/2$ con el vector velocidad.



Nota aclaratoria

$$i^2 = e^{i\pi} = -1$$

la notación compleja se aplicara ahora a la solución de un oscilador simple



cuya ecuación diferencial de movimiento es: $M\ddot{x} + Kx = 0$

o sea $\ddot{x} + p^2 x = 0$ siendo $p^2 = K/M$.

su solución es $x = A \cos pt + B \operatorname{sen} pt$

$$= R \cos (pt - \phi)$$
$$= R \operatorname{sen} (pt + \psi)$$

la forma compleja exponencial puede ponerse identificando R con $\bar{x}$ y p con Ω. Pero ni siquiera es necesario resolver la ecuación diferencial primero en terminos de funciónes trigonometricas para luego pasar a los exponenciales complejos se puede hacer directamente a partir de la ecuación caracteristica de la ecuación diferencial $\lambda^2 + p^2 = 0$ $\lambda_1 = ip$ $\lambda_2 = -ip$ por lo tanto $x = A e^{ipt} + B e^{-ipt}$

Para el caso de un oscilador mecanico en exitación senoidal o cosenoidal se resume en un solo resultado con excitación compleja exponencial.

$$M\ddot{x} + kx = F_0 e^{i\omega t}$$

Se supone una solución $x = X e^{i\omega t}$ luego $\dot{x} = i\omega X e^{i\omega t}$ y $\ddot{x} = -\omega^2 X e^{i\omega t}$ que substituida en la ecuación diferencial se obtiene:

$$-\omega^2 M X e^{i\omega t} + k X e^{i\omega t} = F_0 e^{i\omega t}.$$ como en gral.

$e^{i\omega t} \neq 0$ $$(k - \omega^2 M) X = F_0 \rightarrow X = \frac{F_0}{k - \omega^2 M}.$$

Solución $$x = \frac{F_0}{k - \omega^2 M} e^{i\omega t} = \frac{F_0/k}{1 - \frac{\omega^2}{p^2}} e^{i\omega t}.$$

Esto quiere decir que podemos separar la parte real de exitación y respuesta y la parte imaginaria de excitación y respuesta o sea si el sistema esta exitado por una forma senoidal $F_0 \operatorname{sen} \omega t$ la respuesta sera.

$X = \dfrac{Fo/k}{1-\frac{\omega^2}{p^2}}$ sen ωt y si la fuerza es cosenoidal $F_0 \cos\omega t$

la respuesta sera $x = \dfrac{Fo/k}{1-\frac{\omega^2}{p^2}} \cos\omega t$. esto es porque si

suponemos que $X = X_1 + i X_2$ luego

$$M(\ddot{X}_1 + i\ddot{X}_2) + k(X_1 + iX_2) = F_0(\cos\omega t + i \operatorname{sen}\omega t)$$

Separando estos dos sistemas se tiene:

$$M\ddot{X}_1 + kX_1 = F_0 \cos\omega t$$
$$M\ddot{X}_2 + kX_2 = F_0 \operatorname{sen}\omega t$$

X_1 corresponderá a la parte real de la solución es decir en excitación cosenoidal y X_2 a la parte imaginaria de la solución. esto indica desde luego que respuesta y excitacion estan en fase siempre y cuando $1-\frac{\omega^2}{p^2} > 0$ y con una diferencia de π si $1-\frac{\omega^2}{p^2} < 0$

EL CONCEPTO DE RECEPTANCIA MECANICA

El nombre de "receptancia" fue introducido en 1954 por los profesores W. J. Duncan y M. A. Biot para substituir la palabra de "admitancia mecanica" que se habia usado anteriormente. y diferenciar los sistemas mecanicos de los electricos.

En seguida se tratara de explicar este concepto. Si se tiene una fuerza armonica $Fe^{i\omega t}$ que actua en un punto de un sistema dinámico luego el sistema tendra un movimiento estacionario con la misma frecuencia ω y por consecuencia el punto de aplicación de la fuerza tendra un desplazamiento

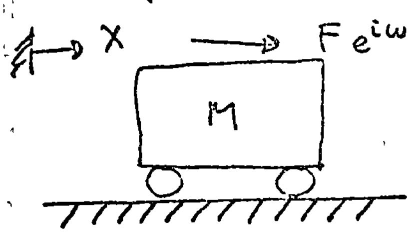
$$x = X e^{i\omega t}$$

y si la ecuación del movimiento es lineal se puede escribir

$$x = \alpha F e^{i\omega t}$$

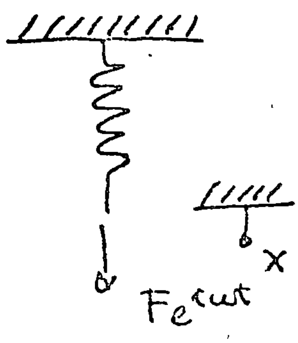
en donde α depende de la naturaleza del sistema y la frecuencia ω pero no de la amplitud F de la fuerza. el coeficiente α se denomina la receptancia directa de x. Por otro lado si x es un desplazamiento en cualquier otro punto donde no se ha aplicado la fuerza al coeficiente α se le denomina la receptancia cruzada de x.

Para ejemplificar derivemos algunas receptancias de sistemas simples. Si se tiene una masa simple que pueda moverse y se le aplica una fuerza $Fe^{i\omega t}$



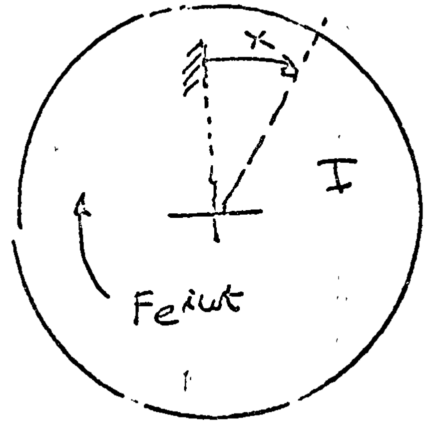
el desplazamiento se rige por la ecuación diferencial $M\ddot{x} = Fe^{i\omega t}$ ahora suponiendo la solución $x = \bar{X} e^{i\omega t}$ se tiene $-M\omega^2 \bar{X} = F$ esto es: $\alpha = -\frac{1}{M\omega^2}$

Si ahora la fuerza $Fe^{i\omega t}$ se aplica al extremo de un resorte sin masa



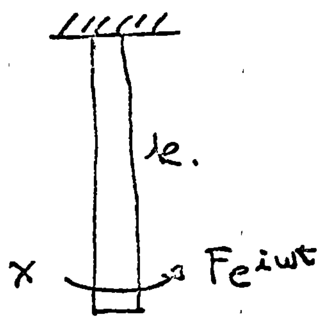
la ecuación de equilibrio será $kx = Fe^{i\omega t}$ por lo que substituyendo $x = \bar{X} e^{i\omega t}$ como solución tentativa se obtiene directamente $\alpha = \frac{1}{k}$

Desde luego el concepto de receptancia no esta restringido unicamente a desplazamientos de traslación tambien es aplicable a rotaciones es decir puede trabajarse con desplazamientos generalizados. En el caso de rotaciones desde luego la fuerza excitadora armonica $Fe^{i\omega t}$ es un par y se seguira empleando la notación de x para rotaciones. Asi podemos hablar de receptancia directa para un disco apoyado en el centro



como $\alpha = -\frac{1}{I\omega^2}$ cuando un par actua como se muestra.

Similarmente podemos hablar de receptancia directa de x de una flecha sin masa rotacional de rigidez k es $\alpha = 1/k$



Considerese ahora un oscilador cuya ecuación de equilibrio sera $M\ddot{x} + kx = Fe^{i\omega t}$ en el cual la solución propuesta $x = Xe^{i\omega t}$ da como resultado $X = \frac{F}{K - M\omega^2}$ o sea que la receptancia directa sera $\frac{1}{K - M\omega^2}$

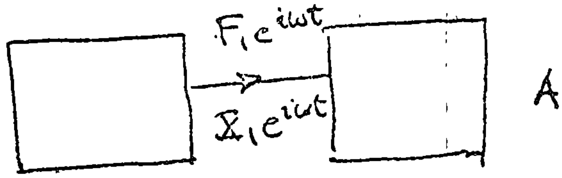
Algunas Propiedades de la receptancia

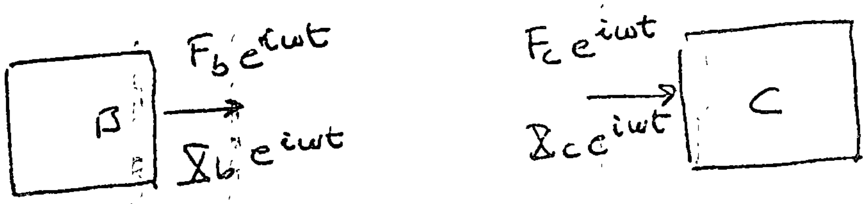
A través de los ejemplos anteriores se ha mostrado como la receptancia proporciona información de la respuesta de un sistema sujeto a cargas armonicas.

tambien se pudo ver que en la frecuencia natural del sistema la receptancia es infinita. La expresión para α fueron encontradas directamente pudiéndose prever que esta tecnica se puede extender a varios grados de libertad. Tambien se puede ver que en virtud de estas propiedades simples de la receptancia se puede analizar un sistema complejo descomponiéndolo en sus partes elementales cuya receptancia es conocida. Desde luego que para sistemas complejos la receptancia debe llevar doble índices para distinguir los directos de los cruzados. esto es α_{mn} que indica el desplazamiento en esa coordenada m y la n donde se aplica la fuerza

Un ejemplo de lo anterior es el siguiente:

Se tiene un sistema compuesto de otros dos sub-sistemas. llamados B y C





Simplificando el factor $e^{i\omega t}$ en las fuerzas y desplazamientos. se tienen las definiciones:

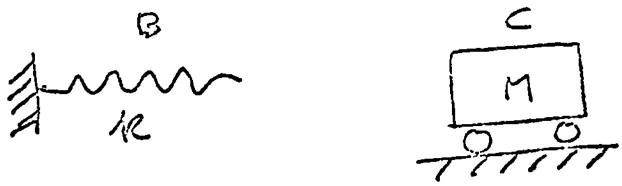
$$\beta_{11} = \frac{X_b}{F_b} \qquad \gamma_{11} = \frac{X_c}{F_c}$$

y poniendo la definición de receptancia para el sistema encadenado A. luego $\alpha_{11} = \dfrac{X_1}{F_1}$ ahora utilizando el equilibrio y la compatibilidad.

$$X_b = X_c = X_1$$
$$F_1 = F_b + F_c$$

de lo cual se sigue que $\dfrac{1}{\alpha_{11}} = \dfrac{1}{\beta_{11}} + \dfrac{1}{\gamma_{11}}$

Esto puede aplicarse al sistema del oscilador constituido por un resorte y una masa.

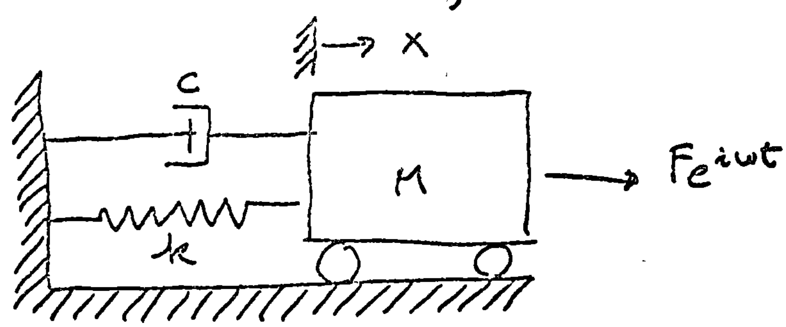


identificando términos $\beta_{11} = \frac{1}{k}$ $\quad \gamma_{11} = -\frac{1}{M\omega^2}$

esto ⇒ $\frac{1}{\alpha_{11}} = k - M\omega^2$

Sistemas con amortiguamiento viscoso.

El oscilador simple del párrafo anterior se verá ahora modificado por un amortiguador viscoso (que funciona para producir fuerza con velocidad)



La ecuación de equilibrio será:

$$M\ddot{x} + c\dot{x} + kx = Fe^{i\omega t}.$$

usando nuevamente una proposición de solución $x = X e^{i\omega t}$. Se tiene $\dot{x} = X\omega i e^{i\omega t}$; $\ddot{x} = -X\omega^2 e^{i\omega t}$.

$$(k - \omega^2 M) X + \omega i c X = F$$

$$X = \frac{F}{(k - \omega^2 M) + \omega i c}$$

luego $x = X e^{i\omega t} = \dfrac{F e^{i\omega t}}{(k - \omega^2 M) + \omega i c}$ y por lo tanto

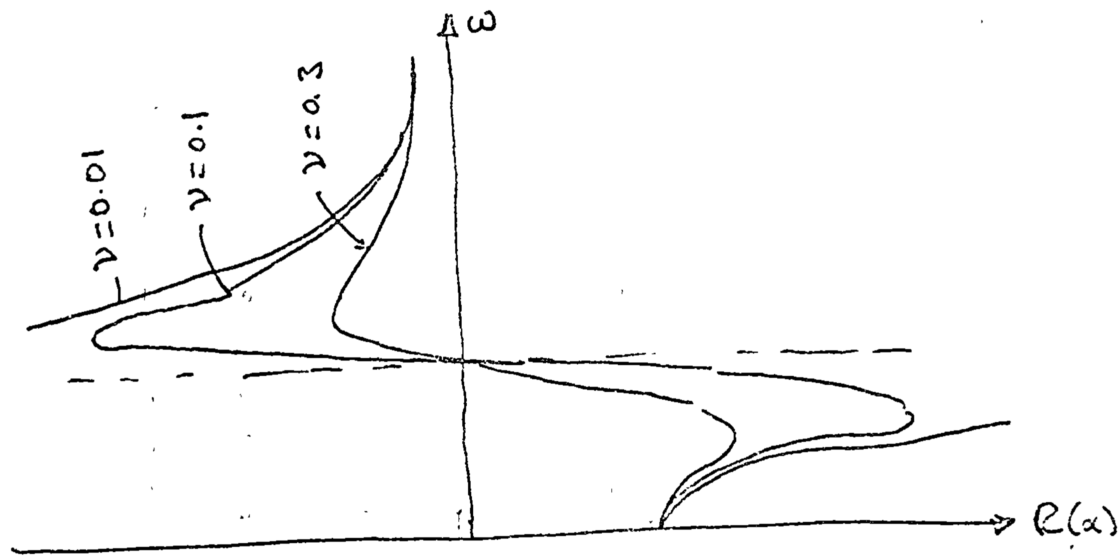
$$\alpha = \frac{1}{(k - \omega^2 M) + ic\omega}$$

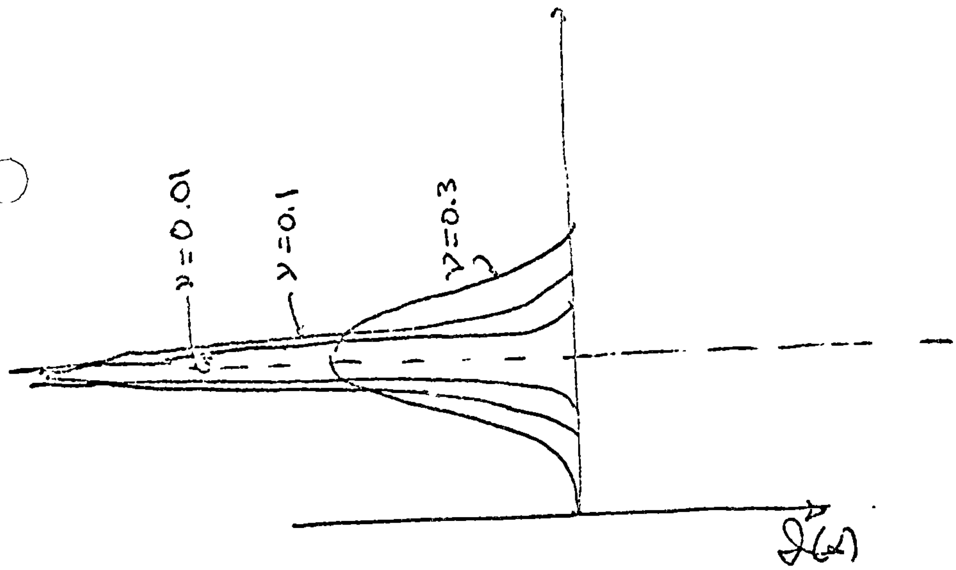
que como puede verse es una cantidad compleja.
Para examinar las implicaciones de ser compleja la receptancia se dividirá en su parte real e imaginaria.

$$\alpha = \left[\frac{k-\omega^2 M}{(k-\omega^2 M)^2 + c^2\omega^2}\right] - i\left[\frac{c\omega}{(k-\omega^2 M)^2 + c^2\omega^2}\right]$$

esto muestra que el desplazamiento x tiene una componente $\frac{(k-\omega^2 M)Fe^{i\omega t}}{(k-\omega^2 M)^2 + c^2\omega^2}$ que esta en fase con la fuerza aplicada y otra componente $\frac{c\omega Fe^{i\omega t}}{(k-\omega^2 M)^2 + c^2\omega^2}$ que esta desfasada $\pi/2$ respecto a la fuerza aplicada. esta componente se dice que esta en "cuadratura" respecto a la excitación

Las componentes en fase y en cuadratura pueden separarse y graficarse llamandose $R(\alpha)$ y $I(\alpha)$ contra la ω.





Como puede observarse en estas figuras $R(\alpha)$ posee dos lomos que por diferenciación puede demostrarse que pertenecen a los valores $\omega = p\sqrt{(1 \pm 2\nu)}$ y en donde ν es una medida adimensional del amortiguamiento dado por $\nu = \dfrac{c}{2\sqrt{kM}} = \dfrac{c}{2Mp}$ y los valores de $R(\alpha)$ en los dos maximos son:

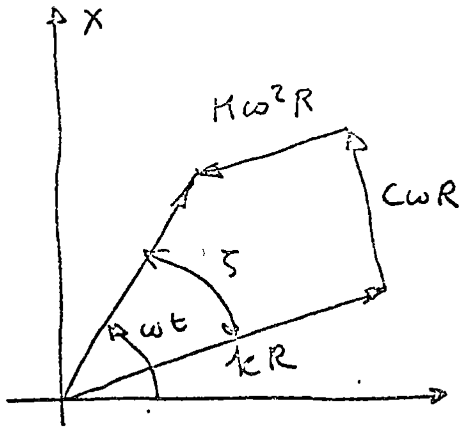
$$\frac{1}{4k\nu(1-\nu)} \quad \text{cuando} \quad \omega = p\sqrt{1-2\nu}$$

$$\frac{-1}{4k\nu(1-\nu)} \quad \text{cuando} \quad \omega = p\sqrt{1+2\nu}.$$

Algunas veces es mas conveniente expresar la receptancia en forma polar. Para esto el denominador de α puede escribirse como $\sqrt{(k-\omega^2M)^2+c^2\omega^2}\, e^{i\delta}$ donde $\delta = \tan^{-1}\left(\dfrac{c\omega}{k^2-\omega^2M}\right)$ por lo que la receptancia queda:

$$\alpha = \frac{e^{-i\delta}}{\sqrt{(k-\omega^2M)^2+c^2\omega^2}}$$

Esta cantidad puede interpretarse sabiendo que el factor $e^{-i\delta}$ significa que el vector fuerza debe rotarse contrario a las manecillas del reloj en el diagrama convencional de [illegible] un angulo ζ para dar la dirección del desplazamiento. La longitud del desplazamiento se encuentra multiplicando el vector de fuerza por el factor $\frac{1}{\sqrt{(K-\omega^2M)^2+C^2\omega^2}}$



La resultante puede tambien obtenerse con un procedimiento grafico como el mostrado en la figura. La solucion X se representa por una linea de longitud R que rota con velocidad angular ω R siendo la amplitud de X. o sea que en la notación original $R=|\bar{X}|$.

La fuerza en el resorte kX se representa mediante una linea paralela de longitud kR; la fuerza en el amortiguador $c\dot{x}$ y fuerza invertida de inercia $M\ddot{X}$ se representan por lineas de longitud $c\omega R$ y $M\omega^2R$ respectivamente. Con $\pi/2$ y π angulos sumando los tres vectores kR, $c\omega R$, $M\omega^2R$ la resultante puede obtenerse de la geometria del poligono vectorial $F^2 = [(K-\omega^2M)^2+c^2\omega^2]R^2$ o bien:

$$R = \frac{F}{\sqrt{(K-\omega^2M)^2+c^2\omega^2}}$$

como se vió en el caso de vibración forzada con amortiguamiento.

Finalmente podemos ver que si la ecuación diferencial es:

$$\ddot{x} + 2\nu p \dot{x} + p^2 x = p^2 \left(\frac{F}{k}\right) e^{i\omega t}$$

La solución estacionaria es:

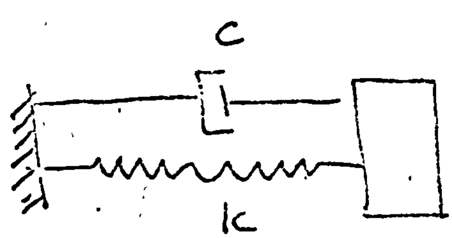
$$x = \left[\frac{1}{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^2 + 4\nu^2\frac{\omega^2}{p^2}}}\right]\frac{F}{k}\, e^{i(\omega t - \zeta)}$$

donde $\zeta = \tan^{-1}\left[\dfrac{2\nu\frac{\omega}{p}}{1-\frac{\omega^2}{p^2}}\right]$

Estos apuntes fueron sacados de The Mechanics of Vibration by R. E. D. Bishop and D. C. Jhonson Cambridge at the University Press 1960.

Como puede advertirse el método de las receptancias es el equivalente al método de flexibilidades o de los desplazamientos para problemas dinámicos y puede desde luego generalizarse a varios grados de libertad de sistemas discretos y a sistemas continuos.

Vamos en seguida a ver una analogía de un grado de libertad aplicando el método de receptancias con semi espacio que se verá posteriormente. Se trata de identificar el resorte y el amortiguador del sistema siguiente:



en el que se supone que el cuerpo rígido que ...

amortiguadores y al resorte carece de masa.

La ecuación de equilibrio será $kx + c\dot{x} = 0$ y si tiene excitación $Fe^{i\omega t}$ será $kx + c\dot{x} = Fe^{i\omega t}$. Suponiendo como siempre $x = X e^{i\omega t}$ se tiene $\dot{x} = i\omega c X e^{i\omega t}$ que sustituidos en la ecuación con excitación se tiene:

$$kXe^{i\omega t} + i\omega c X e^{i\omega t} = Fe^{i\omega t} \quad \text{o sea.}$$

$$(k + i\omega c)X = F \rightarrow X = \frac{F}{k + i\omega c}$$

$$; \ x = \frac{F}{k + i\omega c} e^{i\omega t}$$

Para separar la parte real de la imaginaria se multiplica el numerador y el denominador por el conjugado del denominador. esto es:

$$X = \frac{F}{k + i\omega c} \frac{k - i\omega c}{k - i\omega c} = \frac{F(k - i\omega c)}{k^2 + \omega^2 c^2}$$

$$= \frac{Fk}{k^2 + \omega^2 c^2} - i\frac{F\omega c}{k^2 + \omega^2 c^2} = (f_1 + i f_2)F$$

Si suponemos que lo que se conoce son las funciones f_1 y f_2 y se quiere identificar a k y c luego

$$\frac{k}{k^2 + \omega^2 c^2} = f_1 \quad y \quad -\frac{\omega c}{k^2 + \omega^2 c^2} = f_2$$

elevando al cuadrado cada expresión y sumandolas:

$$\frac{k^2}{(k^2 + \omega^2 c^2)^2} = f_1^2 \qquad \frac{\omega^2 c^2}{(k^2 + \omega^2 c^2)^2} = f_2^2$$

$$f_1^2 + f_2^2 = \frac{k^2 + \omega^2 c^2}{(k^2 + \omega^2 c^2)^2} = \frac{1}{k^2 + \omega^2 c^2}$$

o sea $\quad k(f_1^2 + f_2^2) = f_1$

$$k = \frac{f_1}{f_1^2 + f_2^2}$$

$$c = -\frac{f_2}{f_1^2 + f_2^2}\,\frac{1}{\omega}$$

Puedemos advertir varias propiedades

1).- f_1 y f_2 dependen de la frecuencia de excitación

2).- f_2 es negativa.

3).- La c al deducirse depende del recíproco de la frecuencia y la k no depende directamente de ω.

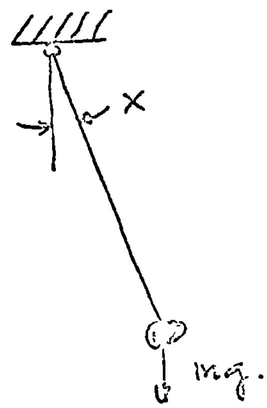
Ejercicios

1.- Un punto se mueve en el diagrama de Argand tal que su posición en el tiempo t está dado por.

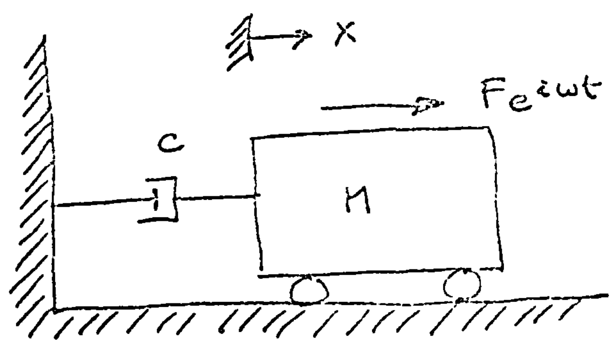
$$z = r_1 e^{i\omega t} + r_2 e^{-i\omega t}$$

demuestre que la trayectoria es una elipse con semi ejes mayor $\frac{r_1+r_2}{2}$ y menor $\frac{r_1-r_2}{2}$

2.- Encuentre la receptancia directa X del péndulo simple

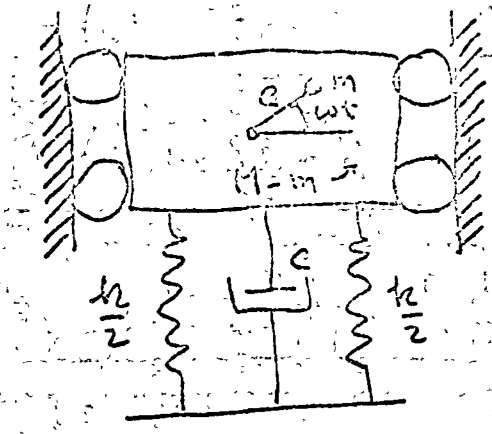


3.- Calcule la receptancia del siguiente sistema:



4.- La figura que se presenta a continuación es un modelo de una máquina de masa total M que vibra debido a un brazo de longitud e que sujeta una masa m. y que rota con una velocidad angular constante ω. Encuentre el desplazamiento estacionario de la forma

$$X = R e^{i(\omega t - \delta)}$$



e
m
ωt
M-m
c
k/2
k/2

and will be called the *damped natural circular frequency*. For systems with less than 40 per cent critical damping, the reduction in natural frequency is less than 10 per cent. For greater values of damping, the reduction in natural frequency is more pronounced.

Referring to Fig. 2-10, the amplitudes of two successive peaks of oscillation are indicated by z_1 and z_2. These will occur at times t_1 and t_2, respectively. Evaluating Eq. (2-34) at t_1 and t_2 we get

$$z_1 = \exp(-\omega_n D t_1)(C_3 \sin \omega_d t_1 + C_4 \cos \omega_d t_1) \tag{2-36a}$$

$$z_2 = \exp(-\omega_n D t_2)(C_3 \sin \omega_d t_2 + C_4 \cos \omega_d t_2) \tag{2-36b}$$

However, $t_2 = t_1 + 2\pi/\omega_d$. Thus, $\omega_d t_2 = \omega_d t_1 + 2\pi$ and hence

$$\sin \omega_d t_2 = \sin (\omega_d t_1 + 2\pi) = \sin \omega_d t_1$$

Thus, the ratio of peak amplitudes is given by

$$\frac{z_1}{z_2} = \exp[-\omega_n D(t_1 - t_2)] = \exp\left(\omega_n D \frac{2\pi}{\omega_d}\right) \tag{2-37}$$

Substitution of Eq. (2-35) gives

$$\frac{z_1}{z_2} = \exp\left(\frac{2\pi D}{\sqrt{1 - D^2}}\right) \tag{2-38}$$

The *logarithmic decrement* is defined as the natural logarithm of two successive amplitudes of motion, or

$$\delta = \ln \frac{z_1}{z_2} = \frac{2\pi D}{\sqrt{1 - D^2}} \tag{2-39}$$

It can be seen that one of the properties of viscous damping is that the decay of vibrations is such that the amplitude of any two successive peaks is a constant ratio. Thus the logarithmic decrement can be obtained from any two peak amplitudes z_1 and z_{1+n} from the relationship

$$\delta = \frac{1}{n} \ln \frac{z_1}{z_{1+n}} \tag{2-40}$$

It is also important to note that if the peak amplitude of vibration is plotted on a log scale against the cycle number on an arithmetic scale, the points will [illegible] straight line if the damping [illegible] of the viscous type as [illegible] in Eq. (2-26)

Forced Vibrations—Undamped

We shall next consider the response of the spring-mass system to the application of a harmonic force Q of amplitude Q_o, as shown in Fig. 2-11a. Using Newton's second law, we find the differential equation of motion to be

$$m\ddot{z} + kz = Q_o \sin \omega t \tag{2-41}$$

The solution to this equation includes the solution for free vibrations, Eq. (2-16), along with the solution which satisfies the right-hand side of Eq. (2-41). In order to obtain a physical feeling for the problem, the particular

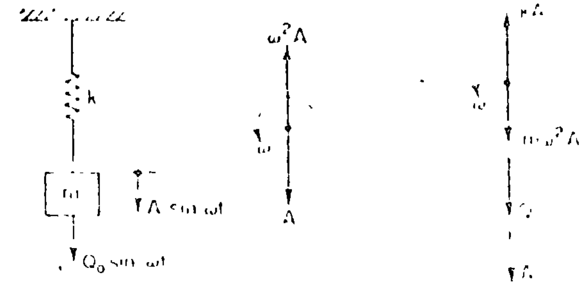


Figure 2-11. Forced vibrations of a single-degree-of-freedom system without damping.

(a) System (b) Motion Vectors (c) Force Vectors

solution will be obtained using the concept of rotating vectors. Since the applied force is harmonic, it is reasonable to assume that the internal spring force and inertia force will also be harmonic. Thus, the motion of the system will be of the form

$$z = A \sin \omega t \tag{2-42}$$

which is represented graphically in Fig. 2-11b. The forces acting on the mass are shown in Fig. 2-11c. The spring force acts opposite to the displacement and the inertia force acts opposite to the direction of acceleration. The exciting-force vector of amplitude Q_o is shown acting in phase with the displacement vector. Thus, from equilibrium requirements,

$$Q_o + m\omega^2 A - kA = 0 \tag{2-43}$$

giving

$$A = \frac{Q_0}{k - m\omega^2} = \frac{\frac{Q_0}{k}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{2-44}$$

The complete solution obtained from Eqs. (2-44) and (2-16) is

$$z = \frac{\dfrac{Q_0}{k}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \sin \omega t + C_1 \sin \omega_n t + C_2 \cos \omega_n t \tag{2-45}$$

For a real system, the vibrations associated with the last two terms of Eq. (2-45) will eventually vanish because of damping, leaving the so-called *steady-state solution:*

$$z = \frac{\dfrac{Q_0}{k}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \sin \omega t \tag{2-46}$$

Investigation of Eq. (2-44) shows that for $\omega < \omega_n$, A is positive, and that for $\omega > \omega_n$, A is negative. However, by noting that $-A \sin \omega t = A \sin (\omega t - \pi)$, the amplitude of motion can always be taken as positive by introducing a phase angle between force and displacement equal to π for $\omega > \omega_n$. If the amplitude A is divided by the static displacement produced on the system by a force of amplitude Q_0, the *dynamic magnification factor M* is obtained:

$$M = \frac{A}{\dfrac{Q_0}{k}} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \tag{2-47}$$

This is plotted in Fig. 2-12 along with the relationship for the phase angle between force and displacement. The magnification factor becomes infinite when $\omega = \omega_n$, because no damping is included in the model. An important feature to point out in the solution is that for $\omega < \omega_n$ the exciting force is in phase with the displacement and opposes the spring force. For $\omega > \omega_n$ the exciting force is 180° out of phase with the displacement and opposes the inertia force. At $\omega = \omega_n$ the inertia force and spring force balance, and the exciting force increases the amplitude of motion without bound.

Forced Vibrations—Damped

The introduction of viscous damping into the single-degree-of-freedom model provides a system which closely approximates the properties of many

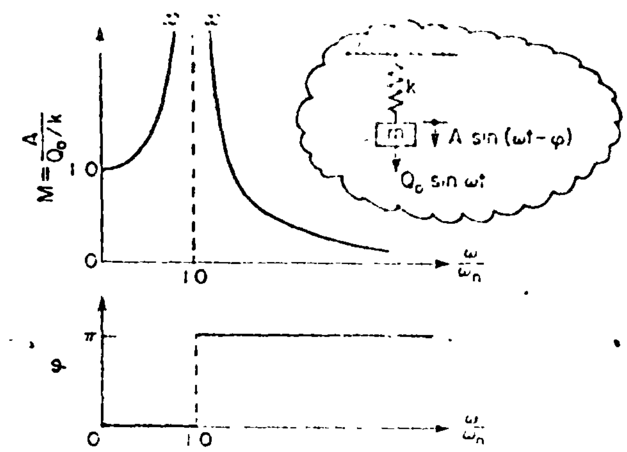
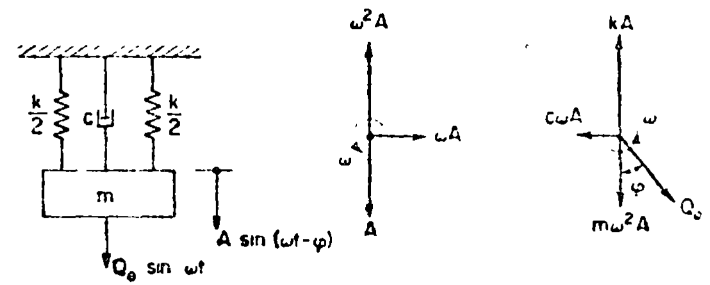


Figure 2-12. Dynamic magnification factor and phase angle between force and displacement of an undamped single-degree-of-freedom system

real systems, since damping is always present in one form or another. Although the use of a viscous-type damping is for mathematical convenience, there are surprisingly few instances where it does not provide a satisfactory model. Figure 2-13a shows the system to be analyzed. Again, using the reasoning described for the undamped case, the particular solution to the differential equation

$$m\ddot{z} + c\dot{z} + kz = Q_o \sin \omega t \tag{2-48}$$

may be obtained using the concept of rotating vectors. The displacement, velocity, and acceleration vectors are shown in Fig. 2-13b. In this problem



(a) System. (b) Motion Vectors (c) Force Vectors

Figure 2-13. Forced vibrations of a single-degree-of-freedom system with viscous damping the displacement is assumed as

$$z = A\sin(\omega t - \varphi) \qquad (2\text{-}49)$$

Hence, when the force vectors are drawn as in Fig. 2-13c, the exciting force will be φ degrees ahead of the displacement vector. In this case the existence of the phase angle is apparent since the damping force $c\omega A$ is 90° out of phase with the spring and inertia forces. Summation of the vectors in the horizontal and vertical directions provides two equations with A and φ as unknowns:

$$kA - m\omega^2 A - Q_o\cos\varphi = 0 \qquad (2\text{-}50a)$$

$$c\omega A - Q_o\sin\varphi = 0 \qquad (2\text{-}50b)$$

Solving for A and φ gives

$$A = \frac{Q_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \qquad (2\text{-}51)$$

$$\tan\varphi = \frac{c\omega}{k - m\omega^2} \qquad (2\text{-}52)$$

Substitution of the expressions for D and ω_n and rearrangement gives

$$M = \frac{A}{\frac{Q_0}{k}} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2D\frac{\omega}{\omega_n}\right]^2}} \qquad (2\text{-}53)$$

$$\tan\varphi = \frac{2D\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad (2\text{-}54)$$

which are the dynamic magnification factor and phase angle between force and displacement for steady-state vibration. These equations are plotted in Fig. 2-14 for various values of D and will be referred to as the response curves for *constant-force–amplitude* excitation. Constant-force amplitude implies that Q_o is independent of ω. It is noticed that the frequency at which the maximum amplitude occurs is not the undamped natural circular frequency ω_n, but a frequency slightly less than ω_n. The frequency at maximum amplitude, f_m, will be referred to as the *resonant frequency* for constant-force amplitude and is given by the expression

$$f_m = f_n\sqrt{1 - 2D^2} \qquad (2\text{-}55)$$

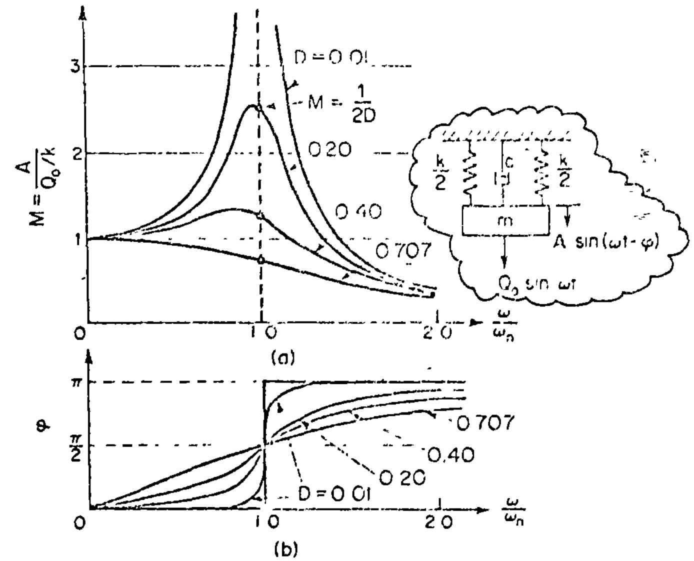


Figure 2-14 Response curves for a viscously damped single-degree-of-freedom system

The value of M at this frequency is given by

$$M_{max} = \frac{1}{2D\sqrt{1 - D^2}} \qquad (2\text{-}56)$$

Inspection of the above equations reveals that for $D = 1/\sqrt{2}$, $f_m = 0$ and the maximum response is the static response. The curves showing the variation of φ with ω/ω_n have the properties that the point of maximum slope occurs at the resonant frequency and all curves have a value of $\pi/2$ at $\omega = \omega_n$. Figure 2-14a is given in more detail in Appendix Fig. A-1.

Rotating-Mass–Type Excitation

For many systems the vibrations are produced by forces from unbalanced rotating masses. A common type vibration generator, shown in Fig.

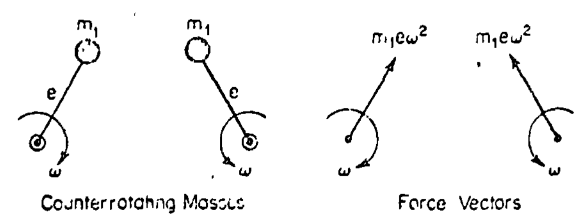


Figure 2-15 Forces produced by two counterrotating masses

2-15, consists of two counterrotating eccentric masses m_1 at an eccentricity e. The phase relationship between the masses is such that they both reach their top position simultaneously. Each mass produces a rotating-force vector equal to $m_1 e\omega^2$. Addition of these two vectors results in the cancellation of the horizontal components and the addition of the vertical components. The vibratory force is thus

$$Q = m_e e\omega^2 \sin \omega t \tag{2-57}$$

where $m_e = 2m_1$ = total eccentric mass. In contrast to the constant-force-amplitude case discussed previously, the rotating-mass-type force has an amplitude proportional to the square of the frequency of oscillation. The solution to the damped single-degree-of-freedom system acted upon by the force defined by Eq. (2-57) can be obtained by a substitution of $m_e e\omega^2$ for Q_o in Eq. (2-51). Note that

$$\frac{m_e e\omega^2}{k} = \frac{m_e}{m}\frac{m}{k} e\omega^2 = \frac{m_e e}{m}\left(\frac{\omega}{\omega_n}\right)^2 \tag{2-58}$$

Thus, the quantity

$$\frac{A}{\dfrac{m_e e}{m}} = \left(\frac{\omega}{\omega_n}\right)^2 M \tag{2-59}$$

where M is the dynamic magnification factor for the constant-force–amplitude case. The expression for φ obviously remains the same. Figure 2-16 is a

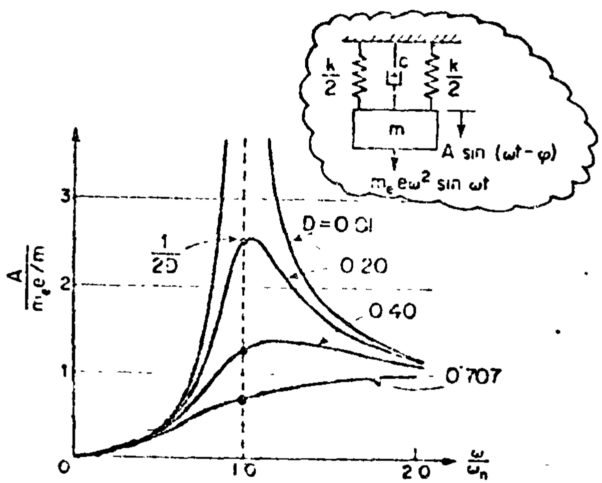


Figure 2-16 Response curves for rotating mass type excitation of a viscously damped single degree-of-freedom system.

plot of Eq. (2-59) for various values of D. The curves are similar in appearance to those obtained for the constant-force-amplitude case. An important difference is that the resonant frequency occurs above the undamped natural frequency and is given by

$$f_{mr} = f_n \frac{1}{\sqrt{1 - 2D^2}} \tag{2-60}$$

The ordinate at f_{mr} is given by

$$\left[\frac{A}{\dfrac{m_e e}{m}}\right]_{\max} = \frac{1}{2D\sqrt{1 - D^2}} \tag{2-61}$$

It should be pointed out that m is the *total* vibrating mass and *includes* the mass m_e. The physical significance of the quantity $m_e e/m$ can be interpreted in two ways. When the eccentric mass is rotating at frequency ω_n, it is producing a force having an amplitude of $m_e e\omega_n^2$. If this force amplitude is divided by the spring constant of the system, the quantity $m_e e/m$ is obtained. From another, more practical viewpoint, it can be observed that in Fig. 2-16 the amplitude approaches the value $m_e e/m$ as the frequency increases beyond the resonant condition. This is related to the physical phenomenon that a rotating mass, if unrestrained, will tend to rotate about its center of gravity. For this case the vibration amplitude is e, since $m_e = m$. However, for most systems m_e represents only part of the total mass resulting in a limiting vibration amplitude of (m_e/m)e. This phenomenon is the basis for adding more mass to a system vibrating above its resonant frequency in order to reduce its vibration amplitude.

Up to this point there have been two natural frequencies (undamped and damped) and two resonant frequencies (constant-force–amplitude and rotating-mass) associated with a single-degree-of-freedom system. As a comparison, all four of these have been plotted in terms of D in Fig. 2-17. For values of D less than 0.2, all frequencies are within 5 per cent of the undamped natural frequency. For higher values the differences between the frequencies become large. For $D \geq 0.707$, no peak exists in the curves for forced vibration; for $D \geq 1.0$, oscillatory motion does not exist for damped free vibrations.

Geometrical Shape of Resonance Curves

If an experiment is run to determine the reponse curve of a single-degree-of-freedom system, it is possible to deduce properties of the system from the shape of the curve. Some of the significant points have already been noted in the previous discussion, but the ones given here are also useful. It is

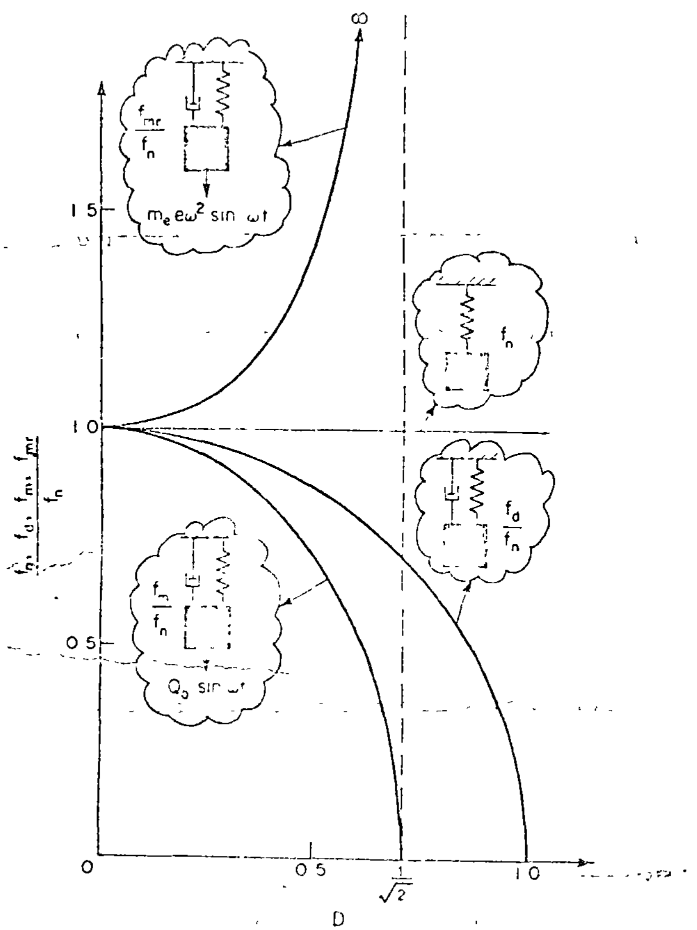


Figure 2-17 Frequencies of a single-degree-of-freedom system

apparent from the fact that the curve starts at zero amplitude that Fig. 2-18a is the response curve for a rotating-mass-type excitation. If a line is drawn from the origin tangent to the response curve, its point of tangency coincides with the undamped natural frequency of the system. In addition, for any line drawn from the origin intersecting the curve at two points the following relationship exists:

$$f_1 \cdot f_2 = f_n^2 \tag{2-62}$$

where f_1 and f_2 are the frequencies at the points where the line intersects the curve. Thus, from a single experimental curve several independent calculations can be made and an average used to obtain the undamped natural frequency of the system. From this it is possible to calculate other properties such as m_e [illegible]

Figure 2-18b is a response curve for a single-degree-of-freedom system acted upon by a constant-force amplitude, as indicated from the finite value of A at $f = 0$. A classical method of measuring damping makes use of the relative width of the curve. Using the quantities indicated on the curve, the logarithmic decrement can be calculated from

$$\delta = \frac{\pi}{2} \frac{f_2^2 - f_1^2}{f_m^2} \sqrt{\frac{A^2}{A_{max}^2 - A^2}} \frac{\sqrt{1 - 2D^2}}{1 - D^2} \tag{2-63}$$

The equation must obviously be evaluated by trial and error, since the expression involves D on the right-hand side. When D is small, the last term

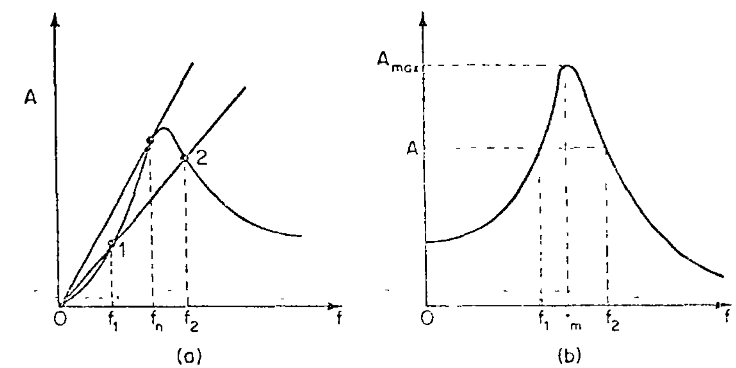


Figure 2-18 Geometric shapes of response curves for the determination of damping

can be taken as equal to 1.0. An extremely simplified expression is obtained if D is small and A is chosen equal to $0.707A_{max}$. Then

$$\delta = \pi \frac{\Delta f}{f_m} \tag{2-64}$$

where $\Delta f = f_2 - f_1$.

Response Due to Motion of the Support

In many cases the vibration of a system is not due to forces acting directly on the mass but from motion of the base. The solution to this problem will also be obtained making use of vector representation of motion and forces. Figure 2-19a shows the problem to be analyzed. The motion of the base is taken as $A_1 \sin \omega t$ and the response of the mass m is assumed to be $A \sin (\omega t - \varphi_1)$. The motion vectors along with the associated force vectors are shown in Fig. 2-19b and c respectively. It is noted [illegible]

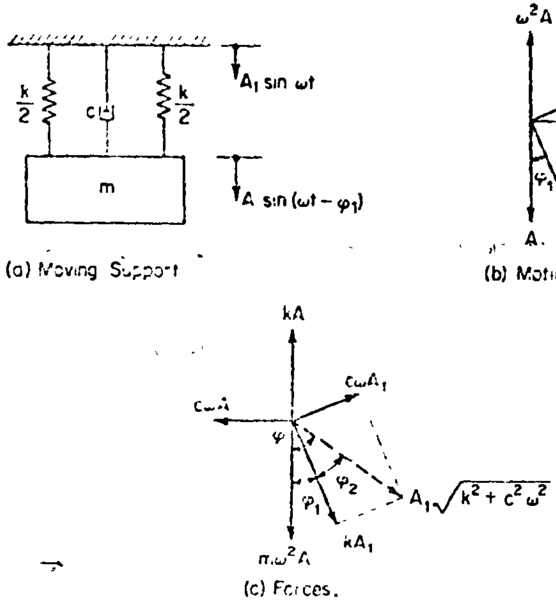


Figure 2-19 Motion and force vectors for the case of a moving support.

support causes a force to be applied to the mass in the same direction as the displacement, whereas the opposite is true for a displacement of the mass. The force vectors are obtained from the motion vectors, as shown in Fig. 2-19c. If the force components from the motion of the support are resolved into a resultant, the same vector diagram as in Fig. 2-13c is obtained. Hence, the solution for A may be obtained by substituting $A_1\sqrt{k^2 + c^2\omega^2}$ for Q_o in Eq. (2-51), giving

$$A = \frac{A_1\sqrt{k^2 + c^2\omega^2}}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \tag{2-65}$$

which reduces to

$$\frac{A}{A_1} = \frac{\sqrt{1 + \left(2D\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2D\frac{\omega}{\omega_n}\right]^2}} \tag{2-66}$$

The expression for φ_1 is found from the relationships shown in Fig. 2-19c. The angle φ is given by Eq. (2-52) and from the vector diagram

$$\varphi_2 = \tan^{-1}\frac{c\omega A_1}{kA_1} = \tan^{-1}\frac{c\omega}{k} \tag{2-67}$$

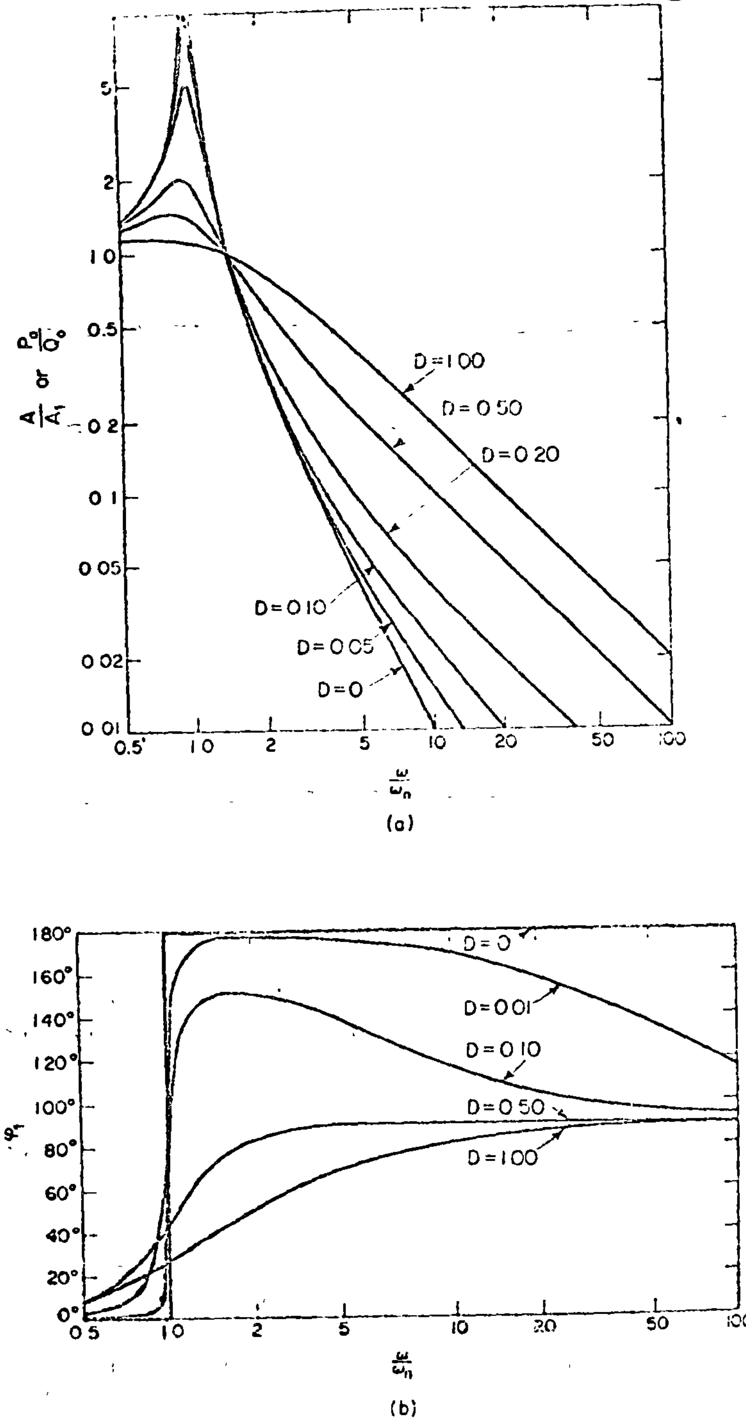


Figure 2-20 Solutions for motion of the support or force transmission

Also $\varphi_1 = \varphi - \varphi_2$ and, from the trigonometric expression for the tangent of the difference of two angles,

$$\tan \varphi_1 = \frac{2D\left(\frac{\omega}{\omega_n}\right)^3}{1 - \left(\frac{\omega}{\omega_n}\right)^2(1 - 4D^2)} \tag{2-68}$$

Equations (2-66) and (2-68) have been plotted in Fig. 2-20 for various values of D. In problems of vibration isolation of sensitive equipment, the above solution affords a guide to the solution of designing the supporting system if the input from the base is known.

Force Transmission

When a force is applied to a mass, it is sometimes necessary to consider the force transmitted to the support. This essentially involves the computation of the resultant of the spring force and the damping force caused by the relative motion between the mass and its support. From Fig. 2-21 the transmitted force P_o is equal to

$$P_o = \sqrt{(kA)^2 + (c\omega A)^2} = A\sqrt{k^2 + c^2\omega^2} \tag{2-69}$$

Substitution of Eq. (2-51) into the above gives, upon simplification,

$$\frac{P_o}{Q_o} = \frac{\sqrt{1 + \left(2D\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2D\frac{\omega}{\omega_n}\right]^2}} \tag{2-70}$$

which is exactly the same as the relationship for A/A_1 obtained for the case involving motion of the support.

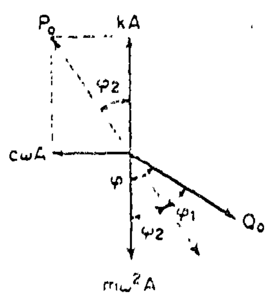


Figure 2-21. Phase relationship between applied force and transmitted force

The phase relationship between Q_o and the force transmitted to the support may be derived from Fig. 2-21, if we note that P_o is opposite in direction to the force applied to the mass. It can be seen that

$$\tan \varphi_2 = \frac{c\omega A}{kA} = \frac{c\omega}{k} \tag{2-71}$$

and $\varphi_1 = \varphi - \varphi_2$, giving exactly the same expression as Eq. (2-68).

In the presentation of the solutions to different cases involving the single-

degree-of-freedom system, it is conventional to express the relationships in terms of the dimensionless parameters of $c/(2\sqrt{km})$ and $\omega\sqrt{m/k}$ which separate the effects of damping and frequency. However, in design or analysis problems the damping factor is probably the least used. In most cases it is the mass or spring constant of

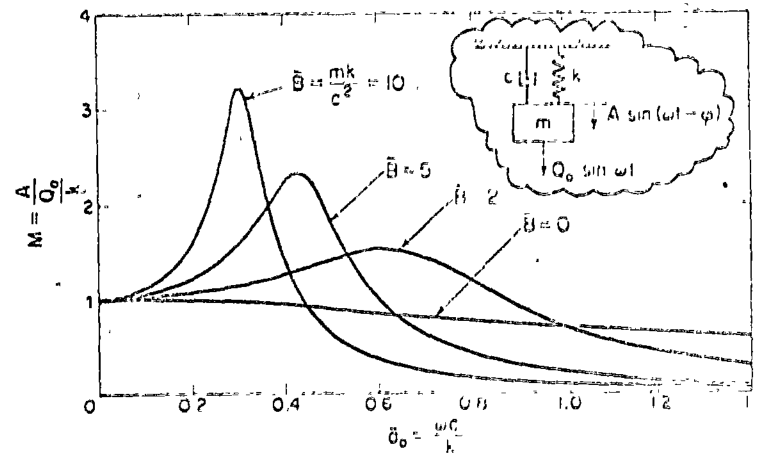


Figure 2-22. Response curves for a single-degree-of-freedom system with the effect of mass and frequency separated (after Lysmer, 1965)

the system which can most easily be changed or adjusted. Using curves such as those in Fig. 2-14, it is a very cumbersome task to determine the exact effect of changes in m and k, since they are interrelated between the abscissa and each of the family of curves. Lysmer (1965) separated the effects of mass and frequency and derived the set of relations given below (and shown in Fig. 2-22) in terms of a *dimensionless-frequency factor*

$$\bar{a}_o = \frac{\omega c}{k} \tag{2-72}$$

and *mass factor*

$$B = \frac{mk}{c^2} \tag{2-73}$$

The dynamic magnification factor, phase angle, and other quantities for the *constant-force-amplitude case* are as follows:

$$M = \frac{1}{\sqrt{(1 - B\bar{a}_0^2)^2 + \bar{a}_0^2}} \tag{2-74}$$

$$\tan \varphi = \frac{\bar{a}_o}{1 - B\bar{a}_0^2} \tag{2-75}$$

$$M_{max} = \frac{B}{\sqrt{B - \frac{1}{4}}} \tag{2-76}$$

$$\bar{a}_{o,m} = \frac{\sqrt{B - \frac{1}{2}}}{B} \tag{2-77}$$

2.4 Phase-Plane Analysis of Single-Degree-of-Freedom Systems

The previous section, covering the solutions to problems involving the single-degree-of-freedom system, made use of the method of rotating vectors in the formulation of the solutions. This concept is useful for visualization of the graphical procedures which will be described next. When the response of a vibrating system is plotted graphically in terms of z and $\dot{z}/\omega_n$, we obtain a curve referred to as the phase-plane trajectory. This curve is very useful for problems involving transient motion, since it allows the engineer to "see" how the properties of the system affect its response to impact or transient loads. The phase-plane method can also be applied to systems with nonlinear properties such as friction damping, nonlinear spring forces, and many others (see Jacobsen and Ayre, 1958).

Free Vibrations of Spring-Mass Systems

From Eq. (2-16) we have the solution to free vibrations of a spring-mass system. By combining terms in the equation we can write the solution as

$$z = \sqrt{C_1^2 + C_2^2}\cos(\omega_n t - \varphi) \tag{2-78}$$

Differentiation with respect to time and division by ω_n gives

$$\frac{\dot{z}}{\omega_n} = -\sqrt{C_1^2 + C_2^2}\sin(\omega_n t - \varphi) \tag{2-79}$$

Squaring Eqs. (2-78) and (2-79) and adding gives

$$z^2 + \left(\frac{\dot{z}}{\omega_n}\right)^2 = C_1^2 + C_2^2 \tag{2-80}$$

which is the equation of a circle with its center at the origin and having a radius of $\sqrt{C_1^2 + C_2^2}$. The constants C_1 and C_2 have been expressed in terms of initial conditions by Eqs. (2-19). Plotting Eqs. (2-78) and (2-79) on coordinates of z and $\dot{z}/\omega_n$, as shown in Fig. 2-23, gives a point starting at z_0 and $\dot{z}_0/\omega_n$ traveling clockwise on the circular arc, described by Eq. (2-80), and moving with an angular velocity of ω_n. Thus, at any time t, the angular distance traveled around the circle is $\omega_n t$. The quantities z or $\dot{z}/\omega_n$ can be obtained as a function of time t and plotted by extending lines from the phase-plane, as shown in Fig. 2-23. The [illegible] free

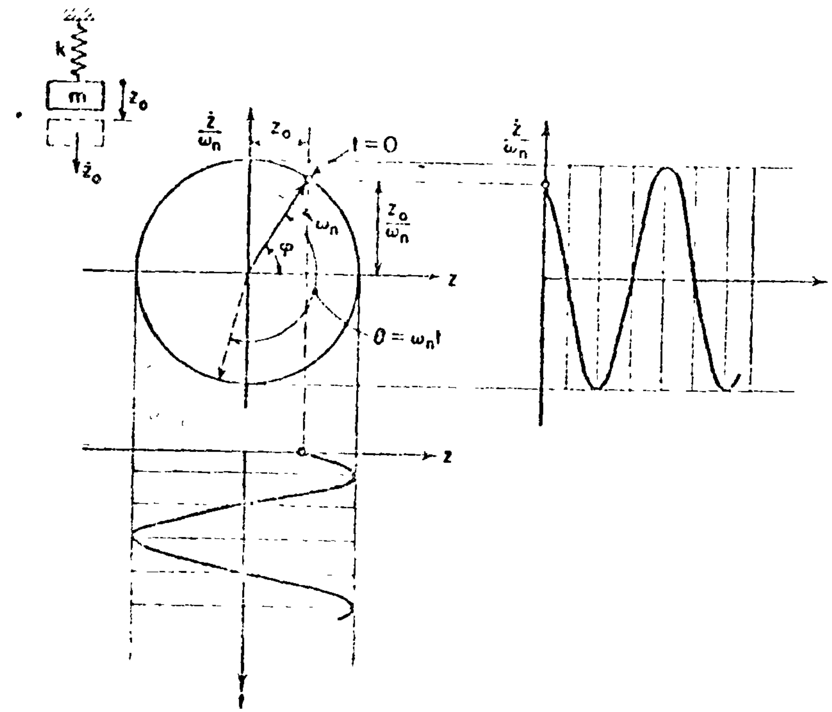


Fig. 2-23. Phase-plane solution for free-vibrations of an undamped single-degree-of-freedom system.

vibrations of a spring-mass system can be seen immediately from Fig. 2-23. For example, the maximum displacement and velocity in terms of the initial displacement and velocity are related to the radius of the circle. That is,

$$z_{max} = \left(\frac{\dot{z}}{\omega_n}\right)_{max} = \sqrt{z_0^2 + \left(\frac{\dot{z}_0}{\omega_n}\right)^2} \tag{2-81}$$

Vibrations from a Step Function

If a constant force Q is instantaneously applied to a simple spring-mass system, the equation governing the motion is

$$[illegible]$$

the soil mass with the amount of compaction decreasing with radial distance from the line of penetration. The pattern of compacted columns to be developed by this method depends upon the required relative density, the characteristics of the original deposit, and the equipment available. D'Appolonia (1953) has given some criteria which are useful for estimating the required spacings for Vibroflot penetrations.

In these discussions of compaction of cohesionless soils, it was emphasized that a particular value of relative density is desired. For normal machinery vibrations, it is usually satisfactory if the supporting soil is compacted to 70–75 per cent relative density (see D'Appolonia, 1953). However, for foundations subjected to intensive vibrations or earthquake loadings, higher values of D_r may be required. The final choice of required D_r at any particular site must be established by the designer.

7 THEORIES FOR VIBRATIONS OF FOUNDATIONS ON ELASTIC MEDIA

7.1 Introduction

In this chapter various solutions for the dynamic behavior of foundations supported by an elastic medium are presented and discussed. The principal elastic medium considered is the homogeneous, isotropic, elastic semi-infinite body which is often called simply the "elastic half-space" in following sections. After accepting the assumptions involved in considering a footing resting on the elastic half-space, it is possible to develop mathematical solutions for the dynamic response of footings thus supported. Several solutions which demonstrate the importance of the geometrical variables and types of exciting forces are presented to form a basis for the design procedures which will be treated in Chap. 10.

In Chap. 6 it was demonstrated that soils may be considered to behave approximately as elastic materials for small amplitudes of strain. Furthermore, fairly routine methods for evaluating the "elastic" constants of soils both in the laboratory and in the field have been developed. The elastic soil constants obtained at a given site can be introduced into the appropriate theory to provide an estimate of the dynamic response of a particular foundation. Consequently, theories based on the concepts of elastic media engineering value which increases as our knowledge and confidence in estimating the effective values of the elastic soil properties increases.

Following the evaluations of footings resting on the elastic half-space, the change in the dynamic response is considered when the elastic half-space is replaced by an elastic layer resting on a rigid sub-base. Because the theories for this situation are not as well developed, the results are useful primarily to indicate general trends for the dynamic response. Finally, a short section on the influence of piles on the dynamic response of foundations is included to indicate a method of approach which may be useful as a rough guide to design for this condition.

There are also several theories and design procedures which treat the behavior of foundations resting on nonlinear media (Lorenz, 1950, 1953; Novak, 1960; Weissmann, 1966, and Funston and Hall, 1967, for example). These methods will not be considered in this chapter, but they may be considered as supplemental reading. Several of these procedures will become more useful in the future as methods for identifying the nonlinear "elastic" behavior of soils under higher strain amplitudes becomes available.

7.2 *Lamb (1904) and the Dynamic Boussinesq Problem*

The paper by Lamb (1904) has been mentioned previously in Chap. 3 in connection with the theoretical development of ground motions associated with the Rayleigh wave. The Lamb paper is also the cornerstone of theoretical solutions developed from the assumption of an oscillator resting on the surface of a homogeneous, isotropic, elastic, semi-infinite body. In this paper Lamb first studied the response of the elastic half-space as it was excited by oscillating vertical forces acting along a line. Thus, he established the solution for two-dimensional wave propagation. He extended this study to include the condition of oscillating forces acting in a horizontal direction on a line on the surface and for either the vertical or horizontal line source acting at an interior point within the body. The locations of these oscillating line loads are described in Fig. 7-1. He also showed how a series of vertically oscillating forces acting at different frequencies could be combined to produce a single pulse acting on a line on the surface. This pulse was then applied to the surface to produce the surface displacements associated with the compression, shear, and Rayleigh waves.

Lamb followed through the same line of reasoning for the three-dimensional case in which a single oscillating force acted at a point on the surface and within the half-space, and again he developed the solution for both the steady-state oscillation and the transient-pulse loading. It is the oscillating vertical force at the surface, which has often been termed the "dynamic

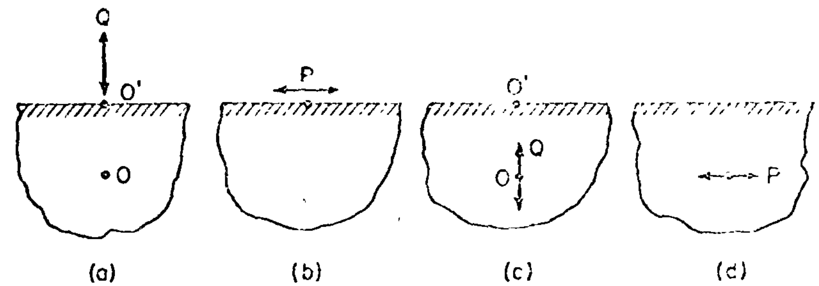


Figure 7-1. Lamb's problems for steady-state oscillating force or pulse loading acting at a point (three-dimensional) or along a line (two-dimensional). (a) For vertical loading at the surface. (b) For horizontal loading at the surface (c) For vertical loading within the body (d) For horizontal loading within the body

Boussinesq loading," which forms the basis for the study of oscillations of footings resting on the surface of the half-space. By integration of the solution for the oscillating vertical force over a finite area of the surface, the contact pressure produced on the half-space by the oscillating footing can be described, and the dynamic response of the footing on the half-space can be evaluated.

Lamb also noted the condition of *dynamic reciprocity*, which is an extension of Maxwell's law of reciprocal deflections to dynamic conditions. Maxwell's law for the usual static case may be stated as follows: The deflection at point 1 in an elastic body due to a unit value of load at point 2 in that body is equal precisely to the deflection at point 2 due to a unit value of load applied at point 1. In using this relation it is necessary to consider the component of each deflection which is in the direction of the applied force at the point under consideration, such that the product of the load and deflection gives work. In the dynamic case, Lamb demonstrated that the horizontal displacement produced at a point on the surface of a semi-infinite elastic body by an oscillating unit vertical force at that point has the same value as the vertical displacement at the same point caused by an oscillating horizontal unit force acting at the point. He noted further that this dynamic reciprocity could be used to evaluate the dynamic motion within an elastic body caused by a point load on the surface by considering the displacements at the surface developed by an oscillating point force acting within the body. This concept is illustrated in Figs. 7-1a and 7-1c, where the vertical displacement at point O caused by the vertical load Q acting at O', shown in Fig. 7-1a, is equal to the vertical displacement at point O' caused by the vertical load Q acting at O as shown in Fig. 7-1c

7.3 Vertical Oscillation of Footings Resting on the Surface of the Elastic Half-Space

Reissner (1936)

During the early 1930s the Deutschen Forschungsgesellschaft fur Bodenmechanik (DEGEBO) investigated the use of mechanical oscillators to evaluate soil properties in the field (see, for example, Hertwig, Früh, and Lorenz, 1933). Because of this activity, E. Reissner attempted to provide a theory for evaluating the dynamic response of a vibrating footing as it was influenced by properties of the soil. He chose the semi-infinite homogeneous, isotropic, elastic body (elastic half-space) to represent the soil mass. The parameters needed to describe the properties of this elastic body were the shear modulus G, the Poisson's ratio ν, and the mass density ρ $(=\gamma/g)$. The vibrating footing was represented by an oscillating mass which produced a periodic vertical pressure distributed uniformly over a *circular area of radius* r_o on the surface of the half-space.

With the elastic half-space as the mathematical model, Reissner developed an analytical solution for the periodic vertical displacement z_o at the center of the circular loaded area of the surface. He obtained this solution by integration of Lamb's 1904 solution over a circular area. The mathematical treatment will not be repeated here, but it may be found in the original paper or in the papers by Quinlan (1953) or Sung (1953). The vertical displacement is expressed by

$$z_o = \frac{P_o \exp(i\omega t)}{G r_o}(f_1 + i f_2) \tag{7-1}$$

in which

P_o = amplitude of the total force applied to the circular contact area,
ω = circular frequency of force application
G = shear modulus of the half-space,
r_o = radius of the circular contact area,
f_1, f_2 = Reissner's "displacement functions."

In Eq. (7-1) both the displacement and the force are positive in the downward direction. The expressions for f_1 and f_2 are complicated functions of Poisson's ratio and a dimensionless frequency term a_o, described by

$$a_o = \omega r_o \sqrt{\frac{\rho}{G}} = \frac{\omega r_o}{v_S} \tag{7-2}$$

In Eq. (7-2) v_S is the velocity of propagation of the shear wave in the elastic body.

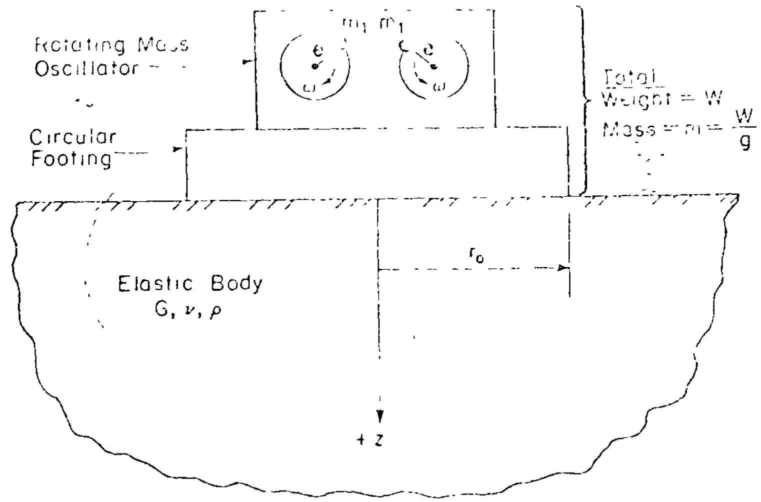


Figure 7-2 Rotating mass oscillator with circular footing resting on semi-infinite elastic body

Reissner also established a second dimensionless term designated as the "mass ratio" b, which is described by

$$\mathbf{b} = \frac{m}{\rho r_o^3} \tag{7-3}$$

in which m is the total mass of the vibrating footing and exciting mechanism which rests on the surface of the elastic half-space (see Fig. 7-2). Equation (7-3) essentially describes a relation between the mass of the rigid body which undergoes vertical motion and a particular mass of the elastic body.

Reissner established expressions for the amplitude of oscillator motion,

$$A_z = \frac{Q_o}{G r_o}\sqrt{\frac{f_1^2 + f_2^2}{(1 - \mathbf{b} a_o^2 f_1)^2 + (\mathbf{b} a_o^2 f_2)^2}} \tag{7-4}$$

the phase angle φ between the external force, $Q = Q_o \exp(i\omega t)$, and the displacement z_o was expressed as

$$\tan \varphi = \frac{f_2}{-f_1 + \mathbf{b} a_o^2 (f_1^2 + f_2^2)} \tag{7-5}$$

and the input power required was expressed as

$$PR = \frac{Q_o^2}{\sqrt{\rho}\, G r_o} \cdot \frac{a_o f_1}{(1 - \mathbf{b} a_o^2 f_1)^2 + (\mathbf{b} a_o^2 f_2)^2}$$

The external oscillating force amplitude Q_o may either be a constant (i.e., independent of the frequency ω) or it may be a function of the frequency of excitation. For the rotating-mass–type exciter with a total mass of m_e acting at a radius designated as the eccentricity e, this force is

$$Q_o = m_e e \omega^2 \tag{7-7}$$

For the two-mass oscillator shown in Fig. 7-2, the total eccentric mass m_e is equal to $2m_1$.

Reissner's theory formed the basis for nearly all further analytical studies of oscillators resting on the half-space, although his theory did not receive immediate adoption by engineers working in the field of soil dynamics because his theoretical results did not completely agree with the results of field tests. There are several reasons for this, including (1) permanent settlements developed during many tests, thereby violating the conditions assumed for an elastic medium; (b) the amplitudes of motion sustained by the model field vibrators were so large (as required for the insensitive recording instruments then available) that the accelerations were often on the order of $2g$ to $3g$, which allowed the vibrator to jump clear of the ground and to act as a hammer; (3) the assumption of a uniformly distributed pressure at the oscillator–soil contact zone was not realistic, and (4) there was an error in the calculation of f_2 which influenced the numerical value of the results. Nevertheless, the study by Reissner is the classic paper in this field.

Quinlan (1953) and Sung (1953)

Two papers which appeared at the same time extended Reissner's solution to consider the effects of changes in pressure distribution over the circular area of contact on the surface of the half-space. Quinlan established the equations for oscillating contact pressures which vary across a diameter of the contact area with a parabolic distribution, with a uniform distribution, and with the distribution corresponding to a rigid base. He developed solutions only for the rigid-base case. Sung also established the basic equations for the three pressure distributions and presented solutions for each case. The pressure distributions are identified as

(a) Rigid Base (approximation)

$$\sigma_z = \frac{P_o \exp(i\omega t)}{2\pi r_o \sqrt{r_o^2 - r^2}} \quad \text{for } r \le r_o \qquad \sigma_z = 0 \quad \text{for } r > r_o \tag{7-8}$$

(b) Uniform

$$\sigma_z = \frac{P_o \exp(i\omega t)}{\pi r_o^2} \quad \text{for } r \le r_o \qquad \sigma_z = 0 \quad \text{for } r > r_o \tag{7-9}$$

(c) Parabolic

$$\sigma_z = \frac{2P_o(r_o^2 - r^2)\exp(i\omega t)}{\pi r_o^4} \quad \text{for } r \le r_o \qquad \sigma_z = 0 \quad \text{for } r > r_o \tag{7-10}$$

Sung's solutions described the displacement at the center of the circular area loaded by the three pressure distributions. For the parabolic and uniform-pressure distributions the loaded surface developed larger displacements at the center than at the edges—a displacement pattern which can be developed only by a flexible footing. The rigid-base pressure distribution produced uniform displacement of the loaded surface under static conditions. Thus, the three pressure distributions developed three different shapes of surface displacements.

After determining these center displacements, Sung established the dynamic response of a mass supported on the half-space for each type of contact-pressure distribution by considering that the center of gravity of the mass moved the same distance as the center of the loaded area. This assumption produced exaggerated response curves for the parabolic and uniform-pressure distributions because the center point has greater displacement than the average. However, these response curves are instructive from a qualitative standpoint for visualizing the influence of contact-pressure distribution on the vibration response of the system.

Figure 7-3a shows the amplitude-frequency response curves corresponding to the three pressure distributions in an oscillator-soil system for which $b = 5$, $\nu = \frac{1}{4}$, and the exciting force is caused by a rotating-mass exciter (see Eq. 7-7). From Fig. 7-3a it is evident that as the load is progressively concentrated nearer the center of the loaded area, the peak amplitude of motion increases and the dimensionless frequency a_o at which this peak amplitude occurs is lowered. An improvement on the presentation of the response of footings which develop these pressure distributions was presented by Housner and Castellani (1969). They determined response curves based on a weighted average displacement which was based on the work done by the total dynamic force. The peaks of the corresponding response curves for the weighted average displacements are shown in Fig. 7-3a as the solid circles designated as $\bar{P}$, $\bar{U}$, and $\bar{R}$. As might be anticipated, the response curves for the weighted average displacements for the parabolic and uniform displacements are closer to the curve for the rigid-base condition [illegible]

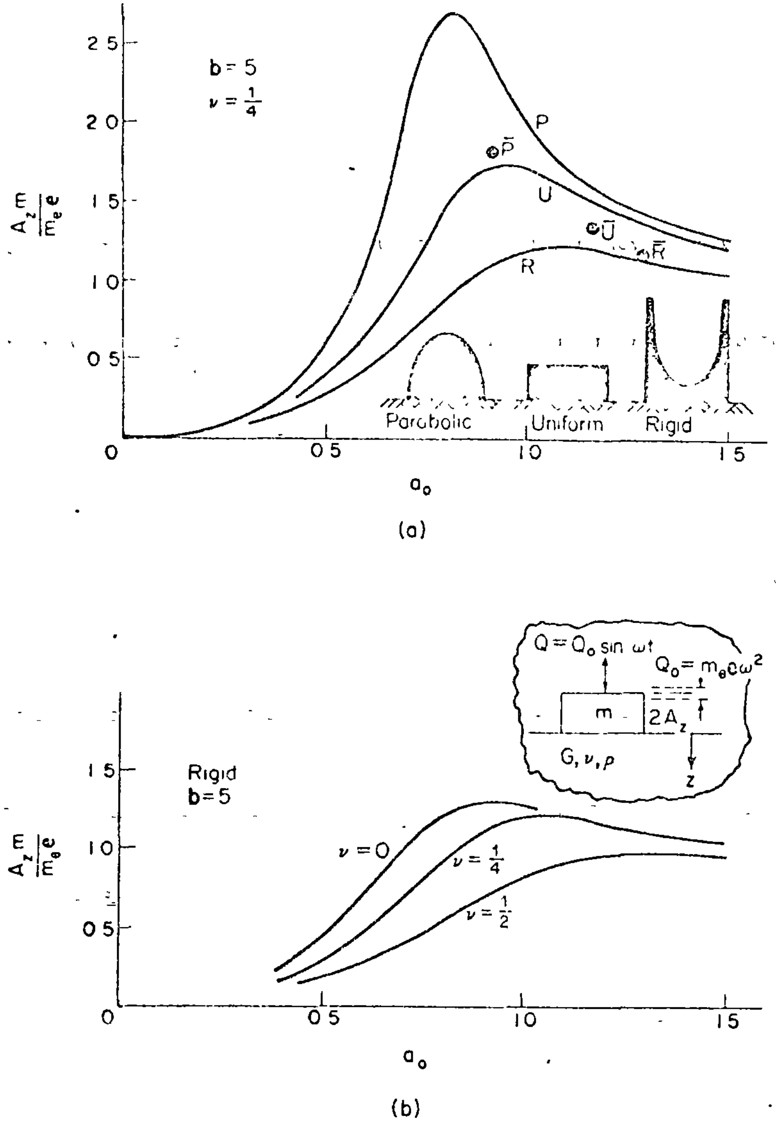


Figure 7-3 Effect of pressure distribution and Poisson's ratio on theoretical response curves for vertical footing motion (*after Richart and Whitman, 1967*)

Sung's curves; however, the trend is still the same. This diagram should indicate to a designer that he may influence the dynamic response of a foundation by his control of the flexibility of the foundation pad. A practical application of this type of control has been described by F[illegible]dis (1957), who

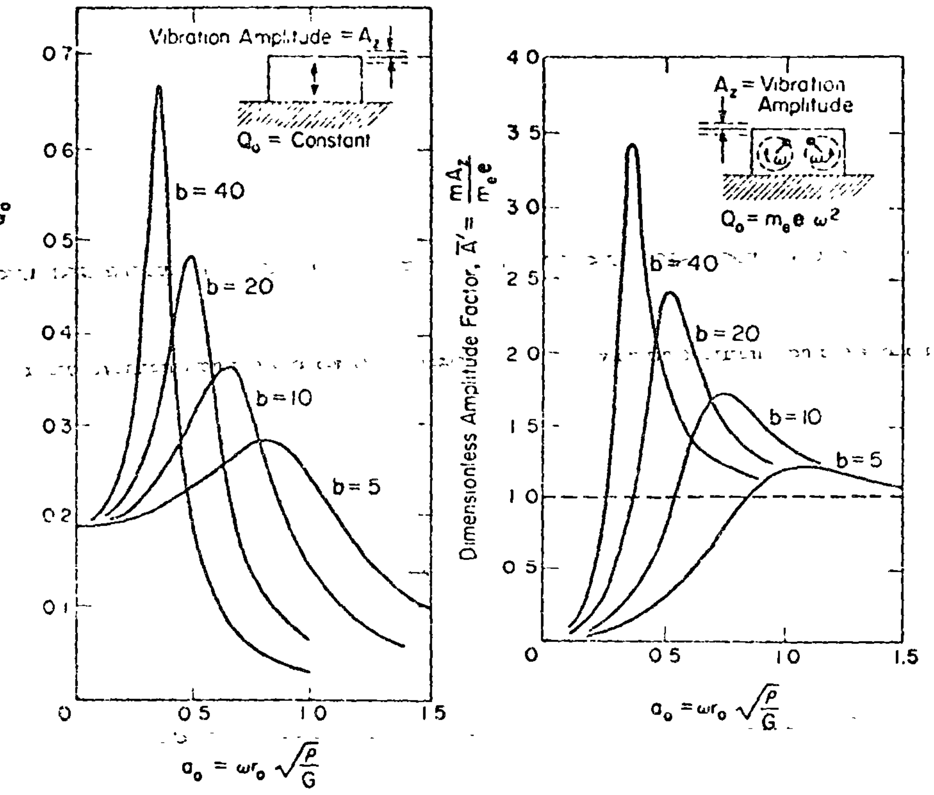
indicated that adjustment of the resonant frequency of an engine or compressor foundation could be accomplished by post-tensioning a prestressed concrete base pad. By changing the camber of the base pad, the major part of the soil-contact pressure could be moved toward or away from the edges.

Figure 7-3b illustrates the influence of a change in Poisson's ratio ν of the elastic body on the steady-state response for the rigid-base condition and $b = 5$. This diagram shows that the amplitude of motion is greater and the frequency at maximum amplitude is lower when $\nu = 0$. Because some solutions for other modes of oscillation are available *only* for the case of $\nu = 0$, it is useful to have this guide to indicate how the solution for $\nu = 0$ might vary from a more realistic condition of $\nu = 0.33$ or 0.40. Generally, the solution for $\nu = 0$ would represent the "worst case" of a greater motion at a lower frequency.

Sung established values for the displacement functions f_1 and f_2 (used in Eq. 7-1) for values of ν of 0, $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$ for each of the three base-pressure distributions over the range of a_0 from 0 to 1.5. These displacement functions were then introduced into Eq. (7-4) to evaluate the amplitude-frequency (A_z vs. a_0) response curves for different values of the mass ratio b. Figure 7-7 illustrates the influence of the mass ratio b on the shape of the amplitude-frequency response curves for the case of the *rigid-base* pressure distribution and Poisson's ratio of $\frac{1}{4}$. By taking the values of the dimensionless amplitudes $\bar{A}$ or $\bar{A}'$ and the frequency factor a_0 at the peak of each response curve, a series of curves can be established to relate $\bar{A}_m$ (or $\bar{A}'_m$), a_{om}, and b, which are useful for establishing the maximum amplitude of motion and the frequency at which it occurs. These curves have been prepared by Richart (1962) for use in analysis or design. The same information is presented in Fig. 7-11 in a more convenient form.

By comparing Fig. 7-4a with Fig. 2-14a, and Fig. 7-4b with Fig. 2-16, we note that the shapes of these response curves are quite similar. The curves for the lower b-values correspond to the curves with large damping ratios D. This is a graphical illustration that vertical oscillation of a rigid footing on an elastic semi-infinite body includes a significant loss of energy by radiation of elastic waves from the footing throughout the half-space. This loss of energy through propagation of elastic waves was defined in Sec. 3.3 as *geometrical damping*. Because most footing systems which undergo vertical vibrations lead to b-values of less than 10, *vertical oscillations are usually highly damped and extreme amplitudes of motion do not occur.*

In Sung's study he assumed that the pressure distribution remained constant throughout the range of frequencies considered. Actually, the rigid-base pressure distribution which correctly predicts a uniform displacement of the loaded surface under static conditions does not produce uniform displacement under dynamic conditions. Bycroft (1956) evaluated the weighted average of the displacements beneath the footing and established



(a) For Constant Amplitude of Exciting Force

(b) For Exciting Force Amplitude Dependent on Exciting Frequency.

Figure 7-4. Amplitude vs. frequency relations for vertical oscillation of a rigid circular footing on an elastic half-space ($\nu = \frac{1}{4}$). (*After Richart, 1962.*)

better values for the displacement functions f_1 and f_2. His values of the displacement functions for the rigid base and for $\nu = 0$, $\frac{1}{4}$, and $\frac{1}{2}$ are given in Fig. 7-5.

Several important points may be observed in Fig. 7-5. At $a_o = 0$ (static case), $f_2 = 0$ and the value of f_1 must produce the correct static displacement when introduced into Eq. (7-1). For the rigid circular footing, the static displacement is

$$z_s = \frac{P_o(1 - \nu)}{4Gr_o} \tag{7-11}$$

It should be noted that the f_1 and f_2 terms are evaluated only over the range

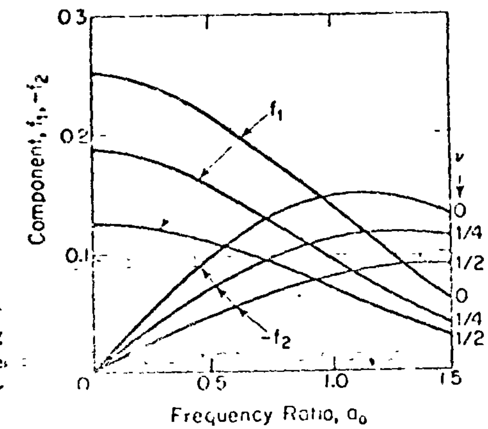


Figure 7-5. Displacement functions for rigid circular footing vibrating vertically on the surface of an elastic half-space (*after Bycroft, 1956*).

of ($0 < a_o < 1.5$). This is the practical range over which the resonance peaks may occur in the response curves and is satisfactory for most steady-state resonance studies. Finally, the displacement function f_2 essentially describes *damping* in the system (see Eq. 7-16).

Hsieh's Equations

By a reorganization of Reissner's basic equations, Hsieh (1962) was able to improve the presentation of the expression for "geometrical damping" which developed from the elastic theory. He considered first a weightless, rigid, circular disk of radius r_o resting on the surface of the elastic half-space (Fig. 7-6a). The disk was subjected to a vertical periodic loading of

$$P = P_o \exp(i\omega t) \tag{7-12}$$

From Eq. (7-1) it was shown that the vertical displacement is

$$z = \frac{P_o \exp(i\omega t)}{Gr_o}(f_1 + if_2) \tag{7-1}$$

By differentiating with respect to time, he obtained

$$\frac{dz}{dt} = \frac{P_o\omega \exp(i\omega t)}{Gr_o}(if_1 - f_2) \tag{7-13}$$

Thus,

$$f_1\omega z - f_2\frac{dz}{dt} = \frac{P_o\omega}{Gr_o}(f_1^2 + f_2^2)\exp(i\omega t) = \frac{P\omega}{Gr_o}(f_1^2 + f_2^2)$$

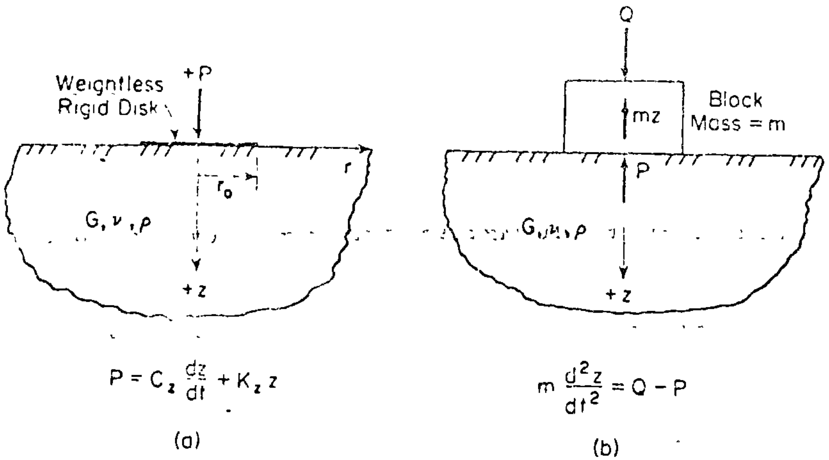


Figure 7-6. Notation for Hsieh's equations.

or

$$P = -\frac{Gr_o}{\omega}\frac{f_2}{(f_1^2+f_2^2)}\frac{dz}{dt} + Gr_o\frac{f_1}{f_1^2+f_2^2}z \tag{7-14}$$

Equation (7-14) can be simplified to

$$P = C_z\frac{dz}{dt} + K_z z \tag{7-15}$$

after substituting

$$C_z = \frac{r_o^2}{a_o}\sqrt{G\rho}\left(\frac{-f_2}{f_1^2+f_2^2}\right) \tag{7-16}$$

and

$$K_z = Gr_o\frac{f_1}{f_1^2+f_2^2} \tag{7-17}$$

Note that because both C_z and K_z include f_1 and f_2, they also depend on a_o and ν.

Now, if a rigid cylindrical footing of base radius r_0 and total weight W is placed on the half-space and set into vertical oscillation by an external periodic force Q, we may write an expression for its dynamical equilibrium as

$$\frac{W}{g}\frac{d^2z}{dt^2} = Q - P \tag{7-18}$$

After substituting Eq. (7-15) and $m = W/g$, this reduces to

$$m\frac{d^2z}{dt^2} + C_z\frac{dz}{dt} + K_z z = Q = Q_o \exp(i\omega t) \tag{7-19}$$

Equation (7-19) has the same general form as the equilibrium equation for the damped-single-degree-of-freedom system (see Eq. 2-48). The major difference is that the *damping term* C_z and the *spring-reaction term* K_z are both functions of the frequency of vibration. However, Eq. (7-16) shows clearly that the geometrical damping in the elastic system is governed by the displacement function f_2. Equation (7-17) demonstrates that the static displacement and elastic-spring response of the system is governed by the term f_1.

Hsieh included in this study a description of the frequency-dependent damping and spring functions for horizontal oscillation, rocking, and torsional oscillations and demonstrated their use in establishing equations for coupled oscillations. This will be considered further in Sec. 7.8.

Lysmer's Analog

To approximate the dynamic response of a rigid circular footing to vertical motion, Lysmer (1965) considered a footing made up of a series of concentric rings. By applying uniform pressures of different magnitudes on each ring, he was able to develop a constant deflection under the footing and to evaluate the dynamic response of the footing to a periodic exciting force. In the process of developing his solution, Lysmer found several notations to be convenient for simplification of the presentation. The displacement function

$$f = f_1 + if_2 \tag{7-20}$$

includes Poisson's ratio, but if it is multiplied by a factor $4/(1-\nu)$, a new displacement function

$$F = \frac{4}{1-\nu}f = F_1 + iF_2 \tag{7-21}$$

is obtained which is essentially independent of ν. Figure 7-7 illustrates the way in which Bycroft's displacement functions collapse onto a nearly common curve when modified by Eq. (7-21). Using this notation, Lysmer calculated values for F_1 and F_2 over the range of frequency ratio ($0 < a_o < 8.0$) and, with an approximation, extended this to $a_o \to \infty$. The curves in Fig. 7-8 show the F_1 and F_2 curves over the range of ($0 < a_o < 8$). It is useful to note that previous analytical solutions only considered the displacement functions up to a_o of 1.5 and did not clearly identify the peak of the F_2 curve.

With the displacement function F as noted in Eq. (7-21), and the positive directions of the force P, and the displacement z designated as downward,

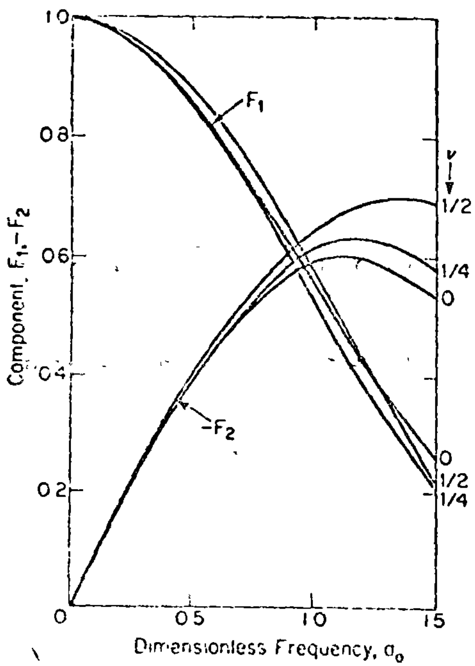


Figure 7-7. Variation of components of F with Poisson's ratio (*after Lysmer and Richart, 1966*).

Eq. (7-1) becomes

$$z = \frac{P}{k_z} F \tag{7-22}$$

wherein the *directions of the applied force P and the resulting displacement are the same for static loading*. The spring factor k_z can be developed from Eq. (7-17) after substitution of the terms from Eq. (7-21).

Lysmer further noted that by introducing a *modified dimensionless mass ratio*

$$\mathrm{B}_z = \frac{1-\nu}{4}\,\mathrm{b} = \frac{1-\nu}{4}\,\frac{m}{\rho r_o^3} \tag{7-23}$$

for the vertical vibration of the rigid circular footing, the influence of Poisson's ratio was essentially eliminated. Then he developed response curves by introducing his modified expressions F (Eq. 7-21) and B_z (Eq. 7-23) into Eq. (7-4). After these substitutions, Eq. (7-4) can be expressed as

$$A_z = \frac{(1-\nu)Q_o}{4Gr_o} M \tag{7-24}$$

in which M is the magnification factor by which the equivalent static displacement produced by Q_o is multiplied to give the displacement amplitude

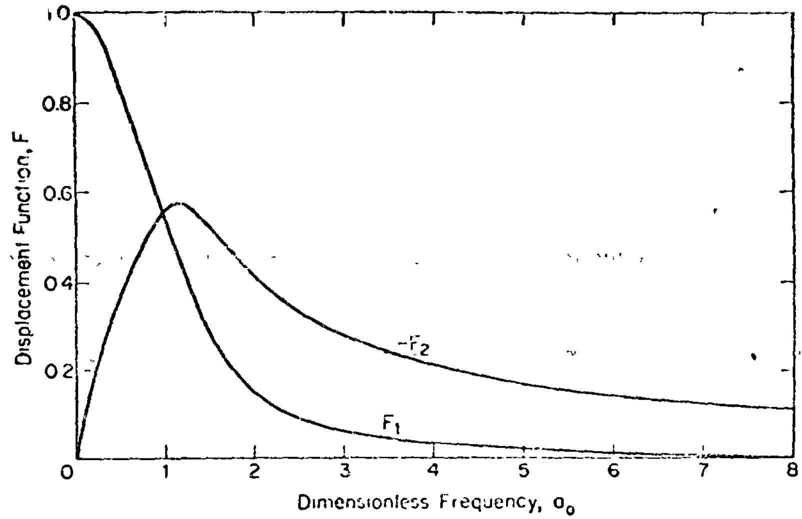


Figure 7-8 Displacement function F for vertical vibration of a weightless rigid circular disk ($m = 0$) (*after Lysmer and Richart, 1966*)

A_z. The response curves corresponding to a *constant-force* excitation Q_o are shown in Fig. 7-9 for several values of B_z as solid curves. Figure 7-10 includes response curves which are produced by the *rotating-mass excitation* ($Q_o = m_e e\omega^2$, Eq. 7-7). After substituting Eq. (7-7) into Eq. (7-24) and simplifying,

$$A_z = \frac{m_e e}{m} M a_o^2 \mathrm{B}_z = \frac{m_e e}{m} M_r \tag{7-25}$$

in which M_r is the magnification factor by which the quantity $m_e e/m$ is multiplied to give the displacement amplitude A_z. From Figs. 7-9 and 7-10, the values of M and M_r at the peak of each response curve and the value of a_o at the peak can be established. These values may then be plotted as B_z vs. a_{om} (Fig 7-11a) or B_z vs. M_m or M_{rm} (Fig. 7-11b). Figures 7-11a and 7-11b provide a simple means for evaluating the maximum amplitude of vertical motion of a rigid circular footing and the frequency at which this occurs for both the constant force and rotating-mass excitation.

After studying the variations of the effective damping and spring factors with frequency (a_o) as obtained from the elastic-half-space theory, Lysmer discovered that constant values of these quantities (i.e., independent of a_o) could be used. He chose the spring constant equal to the static value

$$k_z = \frac{4Gr_o}{1-\nu} \tag{7-26}$$

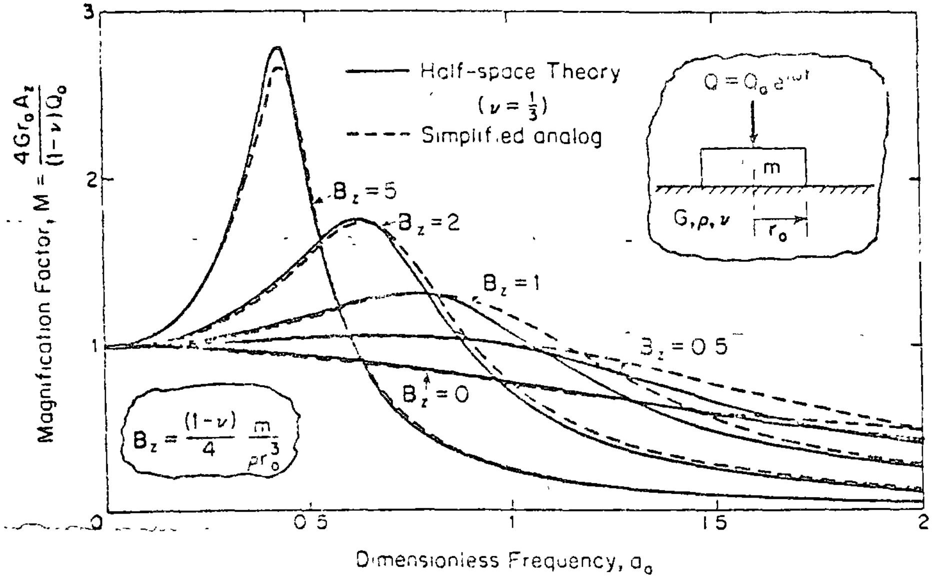


Figure 7-9. Response of rigid circular footing to vertical force developed by constant force excitation (*from Lysmer and Richart, 1966*).

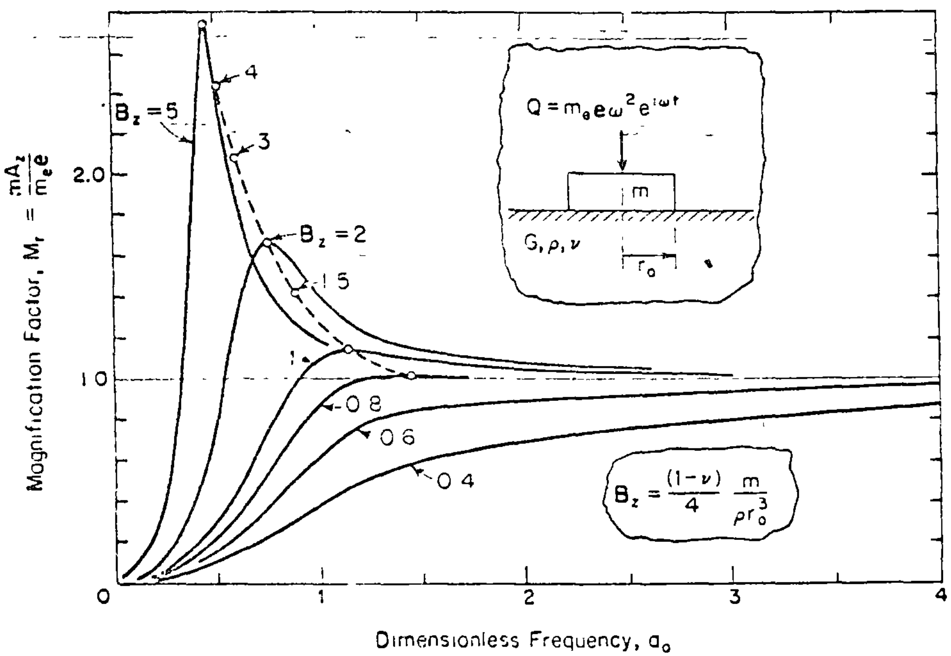
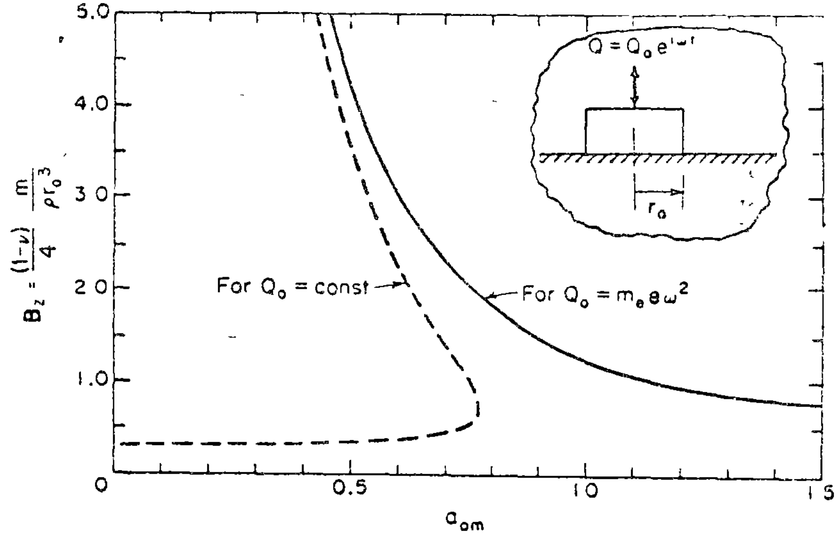
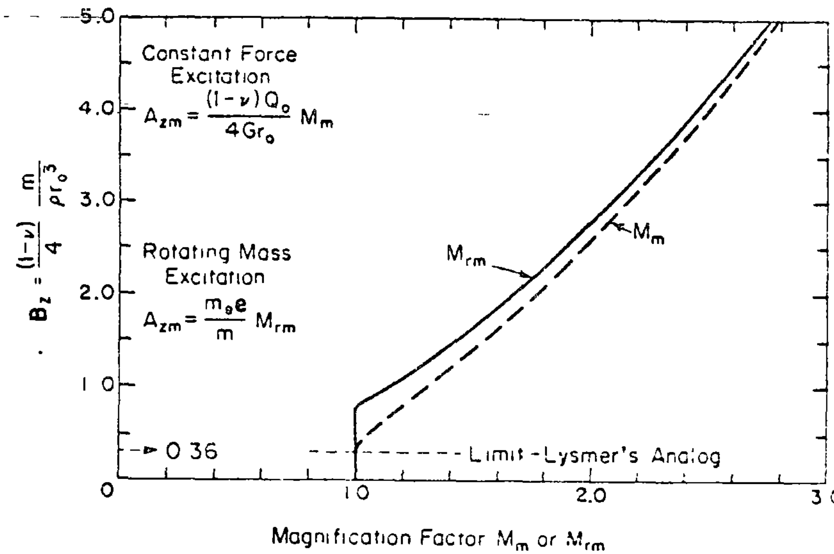


Figure 7-10. Response of rigid circular footing to vertical force developed by rotating mass excitation ($Q_0 = m_e e \omega^2$)



(a)



(b)

Figure 7-11. Vertical oscillation of rigid circular footing on elastic half-space (a) Mass ratio vs dimensionless frequency at resonance. (b) Mass ratio vs magnification factor at resonance.

and found the best fit for the damping term in the range ($0 < a_o < 1.0$) to be

$$c_z = \frac{3.4 r_o^2}{(1-\nu)} \sqrt{\rho G} \tag{7-27}$$

When these values of spring and damping constants were introduced into Eqs. (2-72), (2-73), and (2-74), the steady-state response curves shown as the dashed curves in Fig. 7-9 were obtained. Because the agreement is so remarkable, it is sufficient, for all practical purposes, to use the approximate expressions given in Eqs. (7-26) and (7-27) in the equation of equilibrium (Eq. 7-19) for vertical oscillation of the rigid circular footing on the elastic half-space. The equation of motion for Lysmer's analog is

$$m\ddot{z} + \frac{3.4 r_o^2}{(1-\nu)} \sqrt{\rho G}\, \dot{z} + \frac{4 G r_o}{(1-\nu)} z = Q \tag{7-28}$$

Using the spring constant k_z (Eq. 7-26) and damping constant c_z (Eq. 7-27), the functions corresponding to the single-degree-of-freedom system can be established from the procedures described in Chap. 2. The expressions for resonant frequencies depend on the damping ratio D, which is obtained by dividing the damping constant c_z (Eq. 7-27) by the critical damping c_c. For the vertical oscillation of the rigid circular footing, the critical damping is obtained by substituting Eq. (7-26) into Eq. (2-31):

$$c_c = 2\sqrt{k_z m} = 2\sqrt{\frac{4 G r_o m}{(1-\nu)}} \tag{7-29}$$

then

$$D = \frac{c_z}{c_c} = \frac{0.425}{\sqrt{\mathbf{B}_z}} \tag{7-30}$$

after substituting $\mathbf{B}_z$ for the expressions noted in Eq. (7-23). For excitation by a force of *constant amplitude* Q_o, the resonant frequency is

$$f_m = \frac{1}{2\pi}\sqrt{\frac{k_z}{m}}\sqrt{1-2D^2} = \frac{1}{2\pi}\frac{v_S}{r_o}\frac{\sqrt{\mathbf{B}_z - 0.36}}{\mathbf{B}_z} \tag{7-31}$$

When the *exciting force is a function of the frequency* (Eq. 7-7), Lysmer's expression for the resonant frequency is

$$f_{mr} = \frac{v_S}{2\pi r_o}\sqrt{\frac{0.9}{\mathbf{B}_z - 0.45}} \tag{7-32}$$

Note that these approximations give good answers only for $\mathbf{B}_z \geq 1$.

By substituting the value of D from Eq. 7-30 into Eqs. [illegible] we can establish expressions for the maximum amplitude of oscillation as

$$A_{zm} = \frac{Q_o(1-\nu)}{4 G r_o} \frac{\mathbf{B}_z}{0.85\sqrt{\mathbf{B}_z - 0.18}} \tag{7-33a}$$

for constant-force excitation and

$$A_{zm} = \frac{m_e e}{m} \frac{\mathbf{B}_z}{0.85\sqrt{\mathbf{B}_z - 0.18}} \tag{7-33b}$$

for the rotating-mass excitation. The phase angle φ is determined from

$$\tan \varphi = \frac{0.85 a_o}{\mathbf{B}_z a_o^2 - 1} \tag{7-34}$$

The most important result of Lysmer's study was establishing the bridge between the elastic-half-space theory and the mass-spring-dashpot system and providing values for the damping and spring constants. Now that the results for the vertically loaded rigid footing can be expressed by Eq. (7-28) we can evaluate the response of this system to either periodic or transient excitation. Further discussions of the development of this study and the use of Eq. (7-28) for conditions of transient loading were given by Lysmer (1965) and by Lysmer and Richart (1966). The response of rigid footings to transient vertical loadings will be discussed in Chap. 10, where Eq. (7-28) will be included in the graphical phase-plane method.

Example of Footing Subjected to Steady-State Vertical Oscillation

The report by Fry (1963) contains data from field vibration tests of circular concrete footings which varied from about 5 to 16 ft in diameter. For one series of tests, these model footings rested on the surface of a [illegible] bed of loess, classified as a silty clay (CL). For this soil the [illegible] was 117 lb/ft³ and the properties needed for dynamic analyses [illegible] through seismic and steady-state-vibration tests as described in [illegible] These were found to be v_S = 460 ft/sec, G = 5340 lb/in² [illegible]

The footings were excited into vertical vibration by [illegible] with eccentric weights arranged as shown in Fig. 10-5 [illegible] weight was 339 lb, giving a total unbalanced weight of [illegible] dead weight of the oscillator was [illegible] The total [illegible] included the dead weight of the oscillator, the weight of the concrete footing, and the weight of the lead ballast which was rigidly attached to the footing.

To illustrate the method for calculation of the amplitude and frequency of resonant vibration, consider a footing with a 62-in. diameter ($r_o = 31$ in. $=$ 2.583 ft) which has a total weight W of 30,970 lb. Then, the mass ratio for this test condition is

$$\mathbf{B}_z = \frac{1-\nu}{4}\frac{W}{\gamma r_o^3} = \frac{1-0.355}{4}\frac{30{,}970}{117(2.583)^3} = 2.48$$

From Fig. 7-11a the dimensionless frequency at maximum amplitude a_{om} is 0.67. Then the frequency at maximum amplitude may be obtained from Eq. (7-2) after rearranging and substituting quantities:

$$f_{mr} = \frac{\omega}{2\pi} = \frac{a_{om}v_S}{2\pi r_o} = \frac{0.67 \times 460}{2\pi(2.583)} = 19.0 \quad \text{cycles/sec}$$

The approximate value of resonant frequency obtained from Eq. (7-32) is

$$f_{mr} = \frac{v_S}{2\pi r_o}\sqrt{\frac{0.90}{\mathbf{B}_z - 0.45}} = 18.9 \quad \text{cycles/sec}$$

The amplitude of vertical oscillation depends on $\mathbf{B}_z$ and the magnitude of the exciting force. From Fig. 7-11b the magnification factor M_{rm} is found to be 1.86 for $\mathbf{B}_z = 2.48$. Then for the test in which the radius of eccentricity of the unbalanced weights was 0.105 in., the amplitude of motion is calculated as

$$A_{zm} = \frac{m_e e}{m} M_{rm} = \frac{W_e e}{W} M_{rm} = \frac{1356 \times 0.105}{30{,}970} 1.86 = 0.0086 \quad \text{in.}$$

The approximate solution obtained from Eq. (7-33b) is

$$A_{zm} = \frac{m_e e \mathbf{B}_z}{m(0.85)\sqrt{\mathbf{B}_z - 0.18}} = 0.0088 \quad \text{in.}$$

Vertical Oscillation of Rigid Rectangular Footing

Analytical solutions for vertical oscillating loads on a rectangular zone of the surface of the elastic half-space have also been developed by integrating

Lamb's solution. Sung (1953a) developed the mathematical expressions for a uniformly distributed oscillating load acting on a rectangular area but did not obtain numerical values. Kobori (1962) and Thomson and Kobori (1963) followed the same procedure and obtained the displacement functions f_1 and f_2 for the case of a uniformly distributed load over the rectangular surface area. They evaluated these functions only in terms of the displacement at the *center* of the loaded zone, which produced results indicating negative damping at some values of the frequency ratio. (Negative damping cannot occur for this vibrating system.) A recent paper by Elorduy, Nieto, and Szekely (1967) presented solutions for the vertical oscillation of a *rigid rectangular* base with a length $2c$ and a width $2d$ on the surface of the elastic half-space. By superposing the effects of uniform loading on square elements of the surface, they were able to produce uniform displacement of the loaded area. They found, as did Lysmer, that the pressure distribution required to maintain this uniform displacement varied with the frequency of oscillation. They evaluated several of these distributions of pressure and also computed the displacement functions f_1 and f_2 for a square ($c/d = 1$) and rectangular ($c/d = 2$) loading area on an elastic half-space for which the Poisson's ratio was [illegible]. These functions are shown as the solid line in Fig. 7-12. Also shown in Fig. 7-12 are the corresponding curves from Sung (1953) and Bycroft (1956) after the radius had been adjusted to give a circular area equal to that for the square or rectangle. Because these curves are approximately the same, for all practical purposes it is satisfactory to use the solution for a circular rigid base of the same area to represent the case of vertical oscillation of a rigid rectangular base (for c/d up to 2.0).

The limiting condition of a rectangular foundation occurs when the oscillating body is treated as an infinitely long rigid strip. Quinlan (1953) has given a solution for the resonant frequency of such a strip footing of width $2d$ oscillating vertically in response to excitation of the rotating-mass-type excitation (Eq. 7-7). For this two-dimensional case the mass ratio is

$$\mathbf{b}' = \frac{m'}{\rho d^2} \tag{7-35}$$

in which m' is the mass per unit length of the footing and d is one-half its total width. Figure 7-13 shows the mass-ratio-dimensionless-frequency relationships for the two conditions of $\nu =$ [illegible] and $\nu =$ [illegible].

For a rectangular footing of finite length-to-width ratio, the resonant frequency should lie between the limits given by the theory for the rigid circular (or square) footing and that for the rigid strip of width 2[illegible] infinite length.

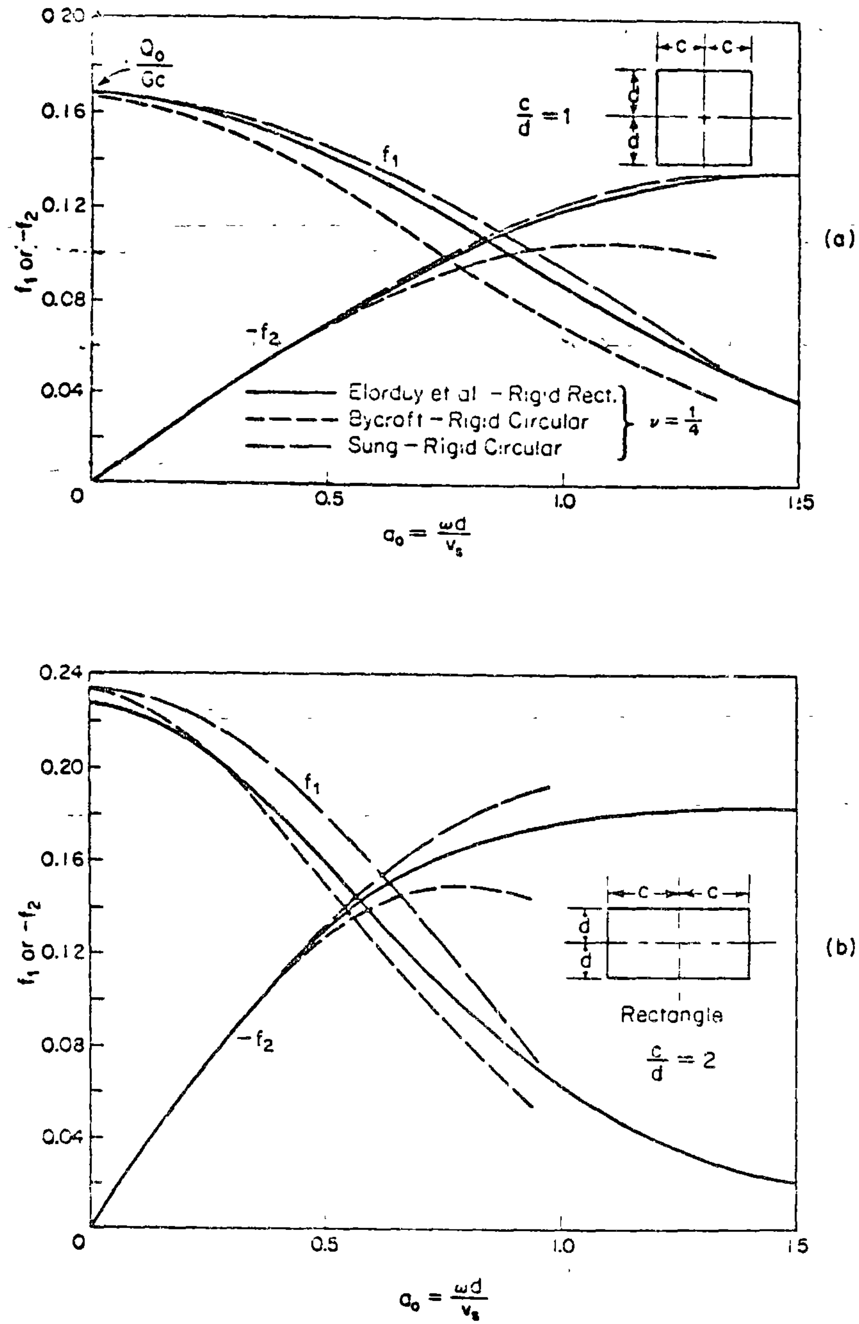


Figure 7-12 Displacement functions for vertical vibration of rigid rectangular and rigid circular footings.

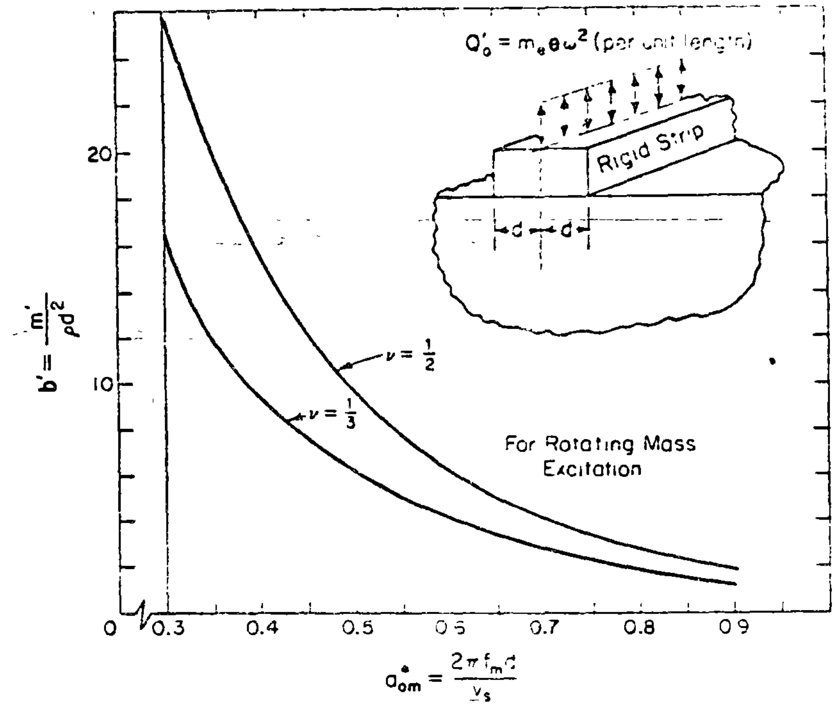


Figure 7-13. Mass ratio vs dimensionless frequency for vertical oscillation of rigid strip of infinite length (*after Quinlan, 1953*).

7.4 *Torsional Oscillation of Circular Footings on the Elastic Half-Space*

Reissner (1937) and Reissner and Sagoci (1944) presented analytical solutions for the torsional oscillation of a circular footing resting on the surface of the elastic half-space. In the first paper Reissner considered a linear variation of shearing stress varying from zero at the center of the circle to a maximum at the radius r, which bounded the loaded zone. Consequently, this represented a flexible footing. In the second approach, Reissner and Sagoci considered a linear variation in displacement from the center of the circle to the periphery. This represents the movement of a *rigid circular footing* oscillating about a vertical axis through the center of the contact area. Under static conditions the tangential shearing stress $\tau_{z\theta}$, which is developed by the applied torque T_θ, is given by the expression

$$\tau_{z\theta} = \frac{3}{4\pi} \frac{T_\theta r}{r_0^3 \sqrt{r_0^2 - r^2}} \qquad \text{for } 0 < r < r_0$$

Equation (7-36) demonstrates that the shearing stress is zero at the center of the footing and becomes infinite at the periphery. In practical cases the infinite shearing stresses cannot be developed by soils. Thus, it could be anticipated that calculated values of frequency would be higher and amplitudes of motion would be lower than those occurring for real footings. The relation between applied torque T_Θ and the resulting rotation Θ_s under static conditions determines the static spring constant

$$k_{\Theta s} = \frac{T_\Theta}{\Theta_s} = \frac{16}{3} G r_o^3 \tag{7-37}$$

In the dynamic solutions Reissner, and Reissner and Sagoci again employed the dimensionless frequency a_o (Eq. 7-2) and the "mass ratio." For torsional oscillation the mass ratio is

$$B_\Theta = \frac{I_\Theta}{\rho r_o^5} \tag{7-38}$$

in which I_Θ is the mass moment of inertia of the footing about the axis of rotation. Again, the analytical results can be presented in a simplified form by plotting the peak values of the amplitude-frequency response curves as relations between B_Θ, a_{om} (the dimensionless frequency at peak amplitude), and the dynamic magnification factor, which for constant torque excitation is

$$M_{\Theta m} = \frac{A_{\Theta m}}{\Theta_s} \tag{7-39}$$

For the case of excitation by a rotating-mass system, the exciting torque is

$$T_\Theta = m_e e x \omega^2 \tag{7-40}$$

in which x is the horizontal-moment arm of the unbalanced weights from the center of rotation. With this excitation the peak amplitude of motion is given by

$$A_{\Theta m} = \frac{m_e e x}{I_\Theta} M_{\Theta rm} \tag{7-41}$$

Values of $M_{\Theta m}$, $M_{\Theta rm}$, and a_{om} are given in Fig. 7-14 as functions of B_Θ.

The theoretical solution for the torsional oscillation disclosed several significant differences from the case for vertical oscillation: (1) this oscillation is not influenced by Poisson's ratio, (2) it is an uncoupled motion and may

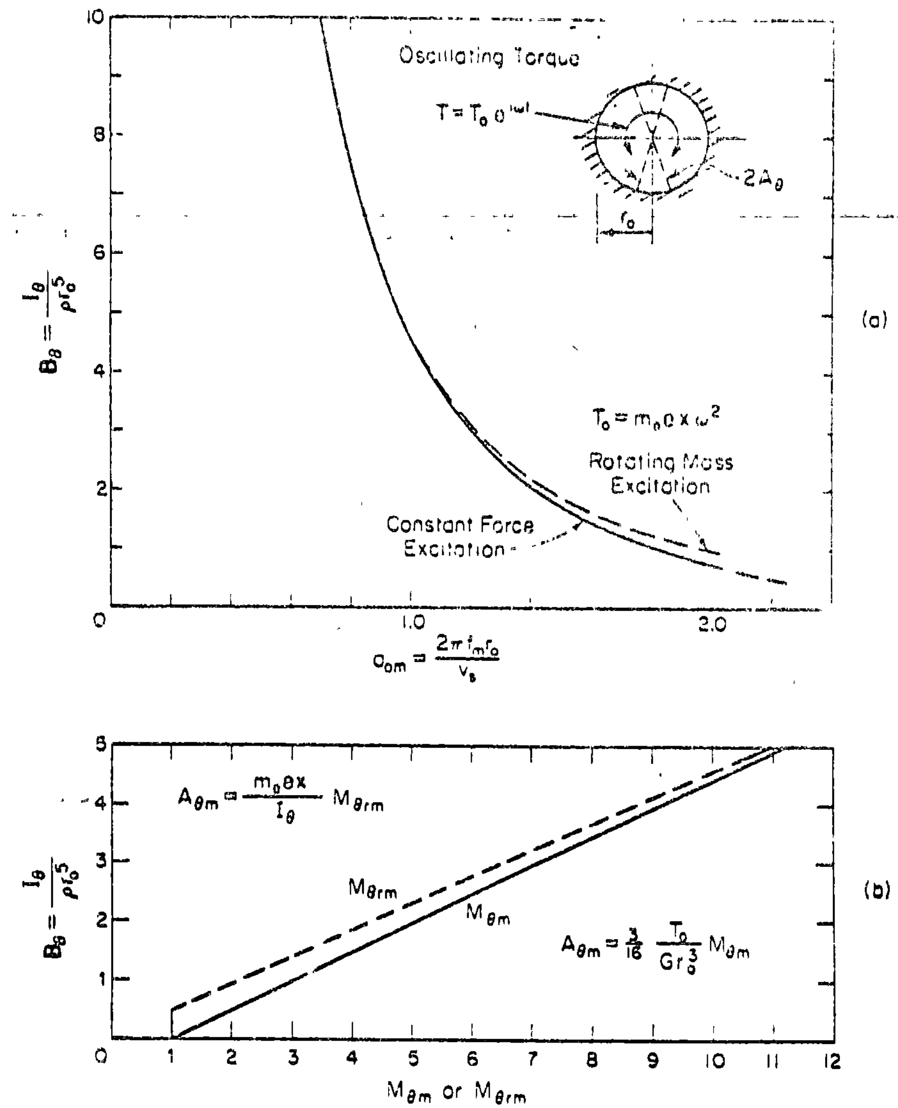


Figure 7-14. Torsional oscillation of rigid circular footing on elastic half-space. (a) Mass ratio vs. dimensionless frequency at resonance. (b) Mass ratio vs. magnification factor at resonance.

be treated independently of the possible vertical motion of the footing; and (3) energy is dissipated by propagation of elastic *shear waves*, only, from the footing; no compression or Rayleigh waves are developed. Furthermore, because this is a rotational-type oscillation, the geometric damping effect contributed by propagation of elastic waves is smaller vertical oscillation. Note in Fig. 7-14b that the magnification factors increase rapidly as B_υ increases.

7.5 Rocking Oscillation of Footings Resting on the Elastic Half-Space

Rigid Circular Footing

Analytical solutions for this problem were presented by Arnold, Bycroft, and Warburton (1955) and by Bycroft (1956). The distribution of vertical pressure on the circular zone of contact was assumed to vary as

$$\sigma_z = \frac{3T_\psi r \cos \theta}{2\pi r_o^3 \sqrt{r_o^2 - r^2}} \exp(i\omega t) \qquad \text{for } r \leq r_o \tag{7-42}$$

for rocking about the y-axis and with θ measured from the x-axis in the x-y plane. The orientation of rocking is indicated in Fig. 7-16, in which the y-axis is perpendicular to the plane of the page through point O. Rocking of the footing occurs about the y-axis (point O) with an angular rotation ψ. Under static application of the external moment T_ψ, the static rotation is (from Borowicka, 1943)

$$\psi_s = \frac{3(1-\nu)}{8} \frac{T_\psi}{G r_o^3} \tag{7-43}$$

Under dynamic conditions the amplitude of rocking is a function of the mass ratio, which now takes the form

$$B_\psi = \frac{3(1-\nu)}{8} \frac{I_\psi}{\rho r_o^5} \tag{7-44}$$

and of the dimensionless frequency a_o (Eq. 7-2). In Eq. (7-44) I_ψ denotes the mass moment of inertia of the footing in rotation about point O (see Eq. (7-49) for I_ψ of a circular footing). With constant amplitude of the exciting moment T_ψ, the response curves are shown as solid lines in Fig. 7-15 for several values of B_ψ. The ordinate of the graph in Fig. 7-15 is expressed as the dynamic magnification factor in rocking,

$$M_\psi = \frac{A_\psi}{\psi_s} \tag{7-45}$$

which is recorded on a logarithmic scale because of the large magnitudes

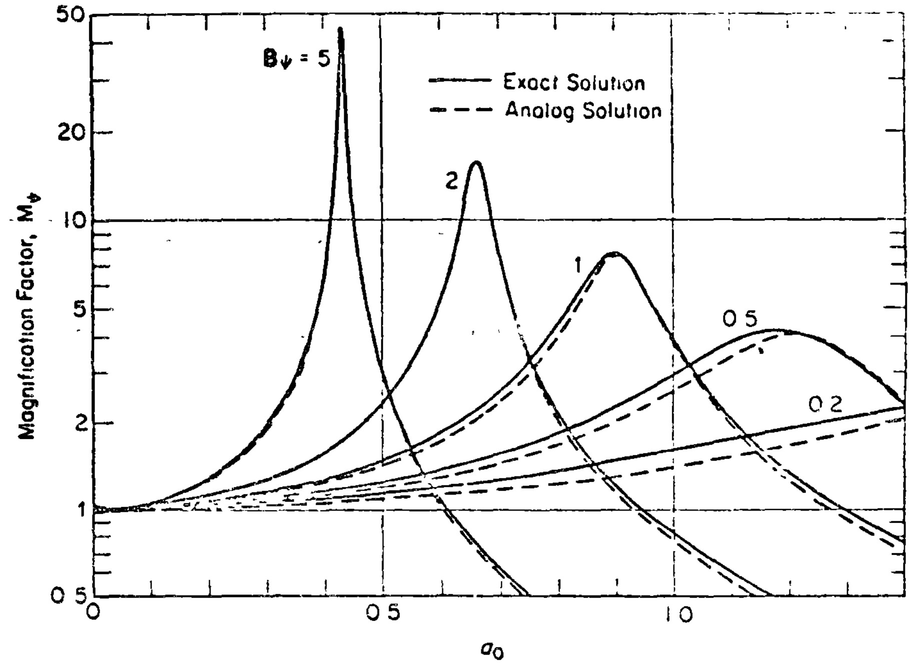


Figure 7-15. Magnification factor vs. dimensionless frequency relations for pure rocking of rigid circular footing on elastic half-space (*from Hall, 1967*)

involved. Figure 7-15 illustrates the sharp peaks for the response curves, even for small values of B_ψ. The shape of these curves as well as the high values of peak amplitudes are typical of a simple vibrating system which has *low damping*. From this we may infer that relatively little energy is dissipated into the elastic half-space by elastic waves but that the elastic strain energy in the supporting half-space is transferred back and forth beneath the two halves of the rocking circular footing.

The peak values of dimensionless frequency and magnification factor from Fig. 7-15 provide information for Fig. 7-16. In Fig. 7-16a the frequency at maximum amplitude is shown as a function of B_ψ. Because of the sharp peaks of the response curves in Fig. 7-15, the two curves in Fig. 7-16a—which denote the cases for constant moment and rotating-mass moment—are essentially identical. The peak values of the magnification factor M_ψ are shown in Fig. 7-16b as a function of B_ψ; the same diagram includes the curve of $M_{\psi rm}$ vs. B_ψ.

In order to develop a moment about the y-axis produced by a rotating mass, it is convenient to express the moment as

$$T_{\psi r} = m_e e z \omega^2$$

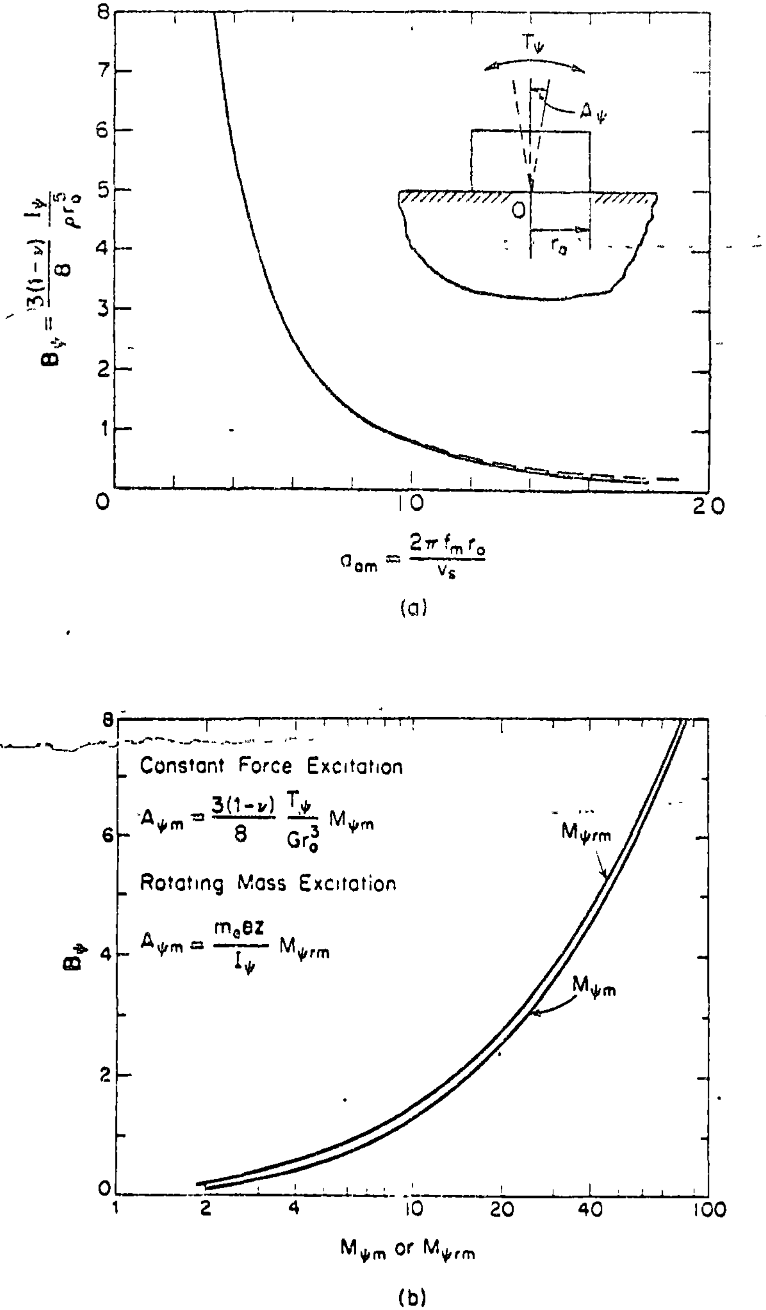


Figure 7-16. Rocking of rigid circular footings on elastic half-space. (a) Mass ratio vs. dimensionless frequency at resonance. (b) Mass ratio vs. magnification factor at resonance.

in which z represents the vertical distance above point O of a horizontally oscillating force $m_e e\omega^2$. By using $T_{\psi r}$ as the exciting moment, we are replacing a pure moment by a moment developed by an eccentric force. This substitution requires that there be a horizontal force applied to the base of the footing to maintain the center of rocking at point O. However, when we *start* by assuming that rocking occurs about point O, we automatically assume that the required restraints exist. Section 7.8 will treat the problem of coupled rocking and sliding, which happens in real situations when excitation is provided by a moment described by Eq. (7-46). With the exciting moment as indicated by Eq. (7-46), the amplitude of rotation A_ψ may be evaluated from

$$A_\psi = \frac{m_e e z}{I_\psi} M_{\psi r} \tag{7-47}$$

Rocking of the rigid circular footing on an elastic half-space develops an infinite vertical stress under the edge of the footing (see Eq. 7-42). Real soils cannot sustain this stress; therefore, a soil support is not as stiff as the ideal elastic medium having the same G. Thus, the actual maximum amplitude of rotation will be somewhat higher and the frequency at this maximum amplitude will be lower than the values calculated from Fig. 7-16.

Hall's Analog

Hsieh (1962) showed that all modes of vibration of a rigid circular footing resting on an elastic half-space could be represented in the form of Eq. (7-19), in which the damping and spring factors are functions of the frequency of oscillation. Because Lysmer had been successful in developing a mass–spring–dashpot analog to the vertical vibration of a footing resting on the half-space, Hall (1967) followed this approach to study the rocking problem. This required evaluations of the damping and spring constants for use in the equation of motion:

$$I_\psi \ddot{\psi} + c_\psi \dot{\psi} + k_\psi \psi = T_\psi \exp(i\omega t) \tag{7-48}$$

In Eq. (7-48) the mass moment of inertia of the footing about the center of rotation is designated I_ψ. For a cylindrical footing of radius r_0 and height h with uniformly distributed mass, the expression for I_ψ is

$$I_\psi = \frac{\pi r_0^2 h\gamma}{g}\left(\frac{r_0^2}{4} + \frac{h^2}{3}\right) \tag{7-49}$$

It was found convenient to introduce the static spring constant as $k_{\psi s}$ in Eq. (7-48), where $k_{\psi s}$ is obtained from Eq. (7-43) as

$$k_{\psi s} = \frac{T_\psi}{\psi_s} = \frac{8Gr_o^3}{3(1-\nu)} \tag{7-50}$$

Then it was necessary to provide a damping constant, which may be expressed as

$$c_\psi = \frac{0.80 r_o^4 \sqrt{G\rho}}{(1-\nu)(1+\mathrm{B}_\psi)} \tag{7-51}$$

The damping term described by Eq. (7-51) is adequate for establishing the maximum amplitude of rocking motion or the maximum dynamic magnification factor as given in Fig. 7-16b. This can be checked by introducing the expression for critical damping for the mass–spring–dashpot system—

$$c_{\psi c} = 2\sqrt{k_\psi I_\psi} \tag{7-52}$$

—into the calculation for the damping ratio

$$D_\psi = \frac{c_\psi}{c_{\psi c}} = \frac{0.15}{(1+\mathrm{B}_\psi)\sqrt{\mathrm{B}_\psi}} \tag{7-53}$$

and noting that for small damping the maximum magnification factor for rocking is

$$M_{\psi m} \approx \frac{1}{2D_\psi} \tag{7-54}$$

In the mass–spring–dashpot system the amplitude of motion at resonance is controlled by the damping, whereas the frequency at maximum amplitude—or "resonant frequency"—is established by the inertia term and spring constant. A discussion of the methods of varying these two quantities to account for the frequency-dependent effect is given in Chap. 10. However, for this particular case Hall found it simplest to consider that an additional mass moment of inertia be added to the real value for the rocking footing in order to force the resonant frequency to agree with the value obtained from the half-space theory. This may be expressed by a magnification factor by which B_ψ (or I_ψ) must be multiplied to give $\mathrm{B}_{\psi\text{eff}}$ (or $I_{\psi\text{eff}}$), or

$$\mathrm{B}_{\psi\text{eff}} = n_\psi \mathrm{B}_\psi \tag{7-55}$$

Values of n_ψ are indicated in the table below:

B_ψ	5	3	2	1	0.8	0.5	0.2
n_ψ	1.079	1.110	1.143	1.219	1.251	1.378	1.600

With this modified value of $\mathrm{B}_{\psi\text{eff}}$ (or $I_{\psi\text{eff}}$) and the damping term from Eq. (7-51), Hall found good agreement between the analog solution and the elastic-half-space solution, as indicated by the dashed curves in Fig. 7-15.

7.6 *Sliding Oscillation of a Circular Disk Resting on the Elastic Half-Space*

Rigid Circular Disk

This problem can exist only in a mathematical sense, for it requires that translation of the disk occur in the horizontal direction without rocking. Physically this requires that the mass of the disk be confined within an infinitely thin layer resting on top of the elastic half-space. Only by concentrating the mass of the disk in this thin layer can the center of gravity of the disk be on the line of action of the restraining force P developed by the half-space on the bottom face of the disk. The exciting force Q, the thin disk, and the restraining force P are shown in the sketch in Fig. 7-17.

In the mathematical treatment it is relatively easy to specify boundary conditions for oscillation of the disk, which demands only horizontal translation without rotation. The analytical solution for translation of the rigid circular disk was presented by Arnold, Bycroft, and Warburton (1955), and by Bycroft (1956), with results expressed in terms of the dimensionless frequency a_o (Eq. 7-2) and the mass ratio b (Eq. 7-3). Hall (1967) found that the modified mass ratio

$$\mathrm{B}_x = \frac{7-8\nu}{32(1-\nu)}\frac{m}{\rho r_o^3} \tag{7-56}$$

eliminated the effect of Poisson's ratio as a similar modified mass ratio had for the case of vertical oscillation. Consequently, the response parameter will be expressed here in terms of B_x. Figure 7-17 illustrates the response curves for the horizontal translation of the disk when excited by a horizontal force

$$Q = Q_o \exp(i\omega t) \tag{7-57}$$

for which Q_o is a constant. The abscissa of Fig. 7- [illegible]

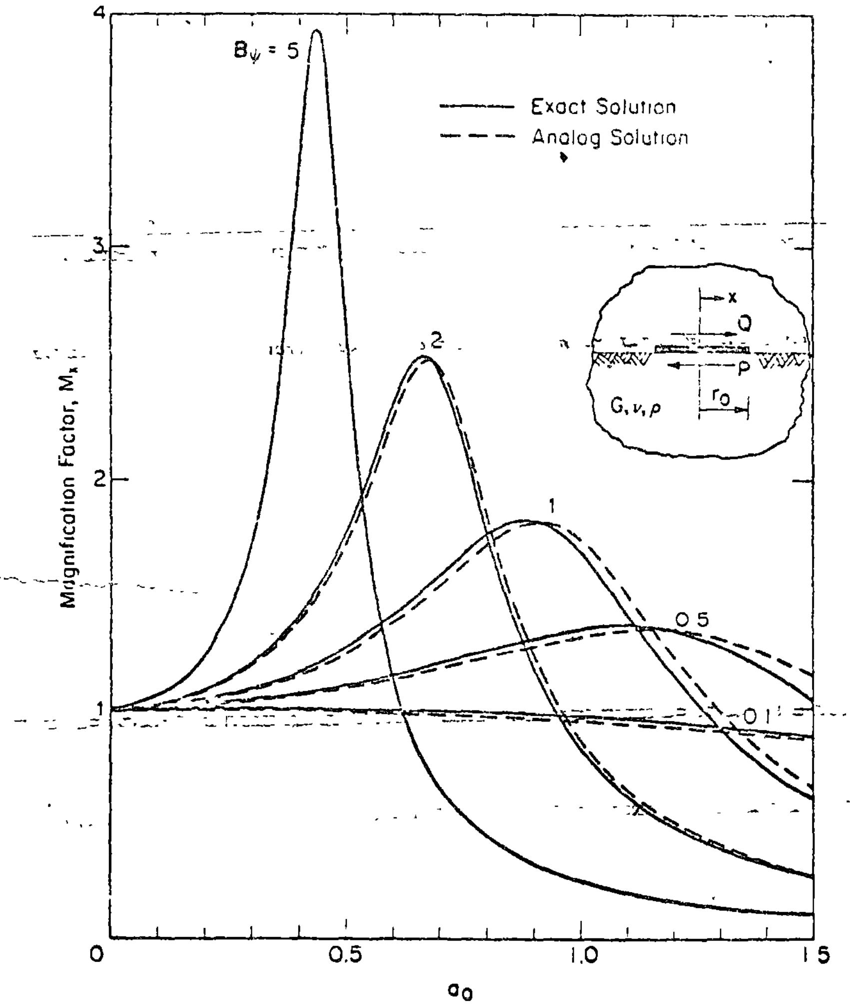


Figure 7-17. Magnification factor vs. dimensionless frequency relations for pure sliding of rigid circular footing on elastic half-space (*from Hall, 1967*).

frequency a_o, and the ordinate is the dynamic amplitude magnification factor

$$M_x = \frac{A_x}{x_s} \tag{7-58}$$

in which the static deflection (from Bycroft, 1956) is determined from

$$x_s = \frac{(7 - 8\nu)Q_o}{32(1 - \nu)Gr_o} \tag{7-59}$$

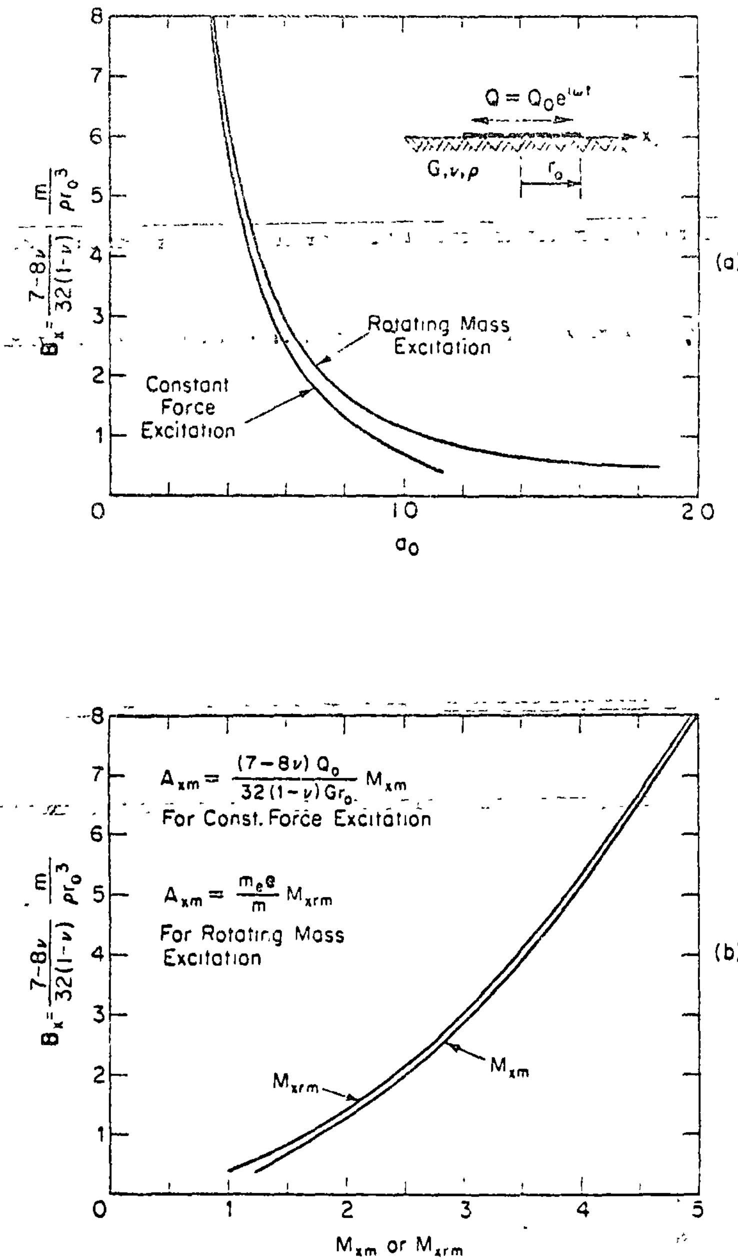


Figure 7-18. Sliding oscillation of rigid circular disk on elastic half-space. (a) Mass ratio vs. dimensionless frequency at resonance. (b) Mass ratio vs magnification factor at resonance.

Note in Fig. 7-17 that the magnitudes of M_x are relatively small and that the peaks of the response curves are relatively flat, particularly for the smaller values of $\mathbf{B}_x$. This indicates that the mode of vibration in horizontal translation is associated with relatively high damping, as was the case for vibration by translation in the vertical direction.

Figure 7-18a again shows the relations between the mass ratio and the value of a_o at the maximum amplitude of oscillation for both the constant-force and rotating-mass types of excitation. Figure 7-18b shows the magnitudes of the dynamic magnification factors as functions of $\mathbf{B}_x$. The fact that the frequency at peak amplitude (Fig. 7-18a) develops two distinct curves further demonstrates that a significant damping effect is associated with this mode of vibration.

Hall's Analog

For the sliding oscillation of a rigid circular disk it was again found possible (Hall, 1967) to describe this motion in terms of the mass–spring–dashpot analog. The mass for the analog was again taken equal to the mass resting on the half-space, and the spring constant was established as equal to the static response of the rigid disk to a horizontal load (see Eq. 7-59),

$$k_{xs} = \frac{32(1-\nu)}{7-8\nu} Gr_o \tag{7-60}$$

Then the damping constant required to provide satisfactory dynamic response for the model was found to be

$$c_x = \frac{18.4(1-\nu)}{7-8\nu} r_o^2\sqrt{\rho G} \tag{7-61}$$

The dashed curves in Fig. 7-17 illustrate how well the response curves for the analog agree with the response curves for the half-space model.

7.7 *Geometrical Damping Associated with Vibrations of Rigid Circular Footings on the Elastic Half-Space*

It is instructive to stop at this point and review the results so far described for the single-degrees-of-freedom vibration of the rigid circular footing supported by the elastic half-space. We have considered translation along the vertical and horizontal axes, rotation about the vertical axis through

the center, and rotation about a diameter through the base of the footing. All six degrees of freedom are represented by these four solutions because translations and rotations with respect to the x-axis are identical to similar motions with respect to the y-axis.

From the magnification-factor–frequency (M vs. a_o) response curves (Figs. 7-9, 7-15, 7-17), it has been demonstrated that the "resonance" condition is associated with a finite amplitude of motion, which indicates that damping is present in the system. However, the assumption of an ideal elastic half-space precludes loss of energy because of inelastic behavior of the material which constitutes the half-space. The indication of damping is evidence that energy is lost in the vibrating system, and in the case of the footing oscillating on the surface of the semi-infinite elastic body, or half-space, the loss of energy occurs through transmission of elastic-wave energy from the footing to infinity. This geometrical distribution of elastic-wave energy has been designated as *geometrical damping*.

From each solution for vibration of the footing on the half-space it is possible to establish a value of the equivalent damping ratio D, which can then be used in the lumped-parameter analysis. A convenient method for evaluating D is to equate the peak amplitude of motion from the half-space solution to the amplitude obtained from the mass–spring–dashpot system and then to solve for D. This procedure has been followed in preparing the curves shown in Fig. 7-19. Approximately the same results can be obtained by calculating D from the damping constants obtained in the analog solutions and the expression for critical damping,

$$c_c = 2\sqrt{km} \tag{2-31}$$

With this approach the damping ratio is

$$D = \frac{c}{c_c} \tag{2-32}$$

Expressions for the damping ratio are

For vertical oscillation—

$$D_z = \frac{0.425}{\sqrt{\mathbf{B}_z}} \tag{7-62}$$

For horizontal oscillation—

$$D_x = \frac{0.288}{\sqrt{\mathbf{B}_x}} \tag{7-63}$$

For rocking oscillations—

$$D_\psi = \frac{0.15}{(1+\mathbf{B}_\psi)\sqrt{\mathbf{B}_\psi}} \tag{7-64}$$

For torsional oscillations—

$$D_\theta = \frac{0.50}{1 + 2B_\theta} \tag{7-65}$$

The variations of the effective-damping ratio D with B for the various modes of vibration are shown in Fig. 7-19. The expressions for B_z, B_x, B_θ, and B_ψ are given by Eqs. 7-23, 7-56, 7-38, and 7-44, respectively; and it should be noted again that the effect of Poisson's ratio is incorporated into the computation for B.

From Fig. 7-19 it is evident that appreciable damping is associated with a wide range of B for the translational modes of vibration. On the other hand, damping is quite low for the rotational modes of vibration, particularly for values of $B_\theta > 2$ in torsional oscillation and for $B_\psi > 1$ in rocking. Because many machine foundations are subjected to some overturning forces, it is probable that some oscillation in the rocking mode will occur. Consequently, the results shown on Fig. 7-19 should indicate to the designer that he should provide the lowest possible value of B_ψ for his machine foundation in order to minimize the rocking motion.

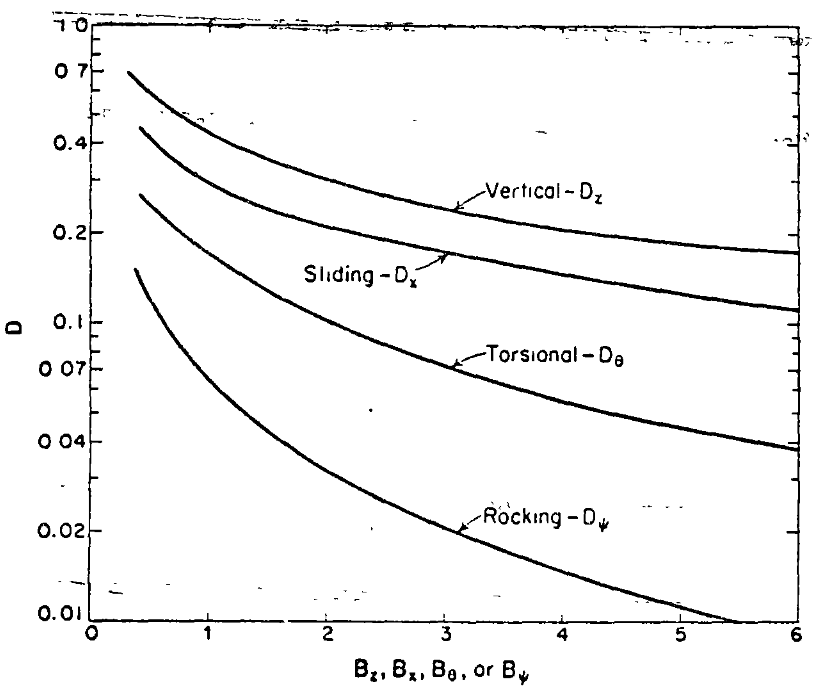


Figure 7-19 Equivalent damping ratio for oscillation of rigid circular footing

7.8 Coupled Rocking and Sliding of the Rigid Circular Footing on the Elastic Half-Space

As noted in the preceding article, there are six degrees of freedom possible for the motion of a rigid body: translation in the three coordinate directions—x, y, and z—and rotation about each of these axes. With the information presented in this chapter, it is possible to write the equations of motion for each degree of freedom, thereby establishing six equations of motion. Although this procedure is possible, we usually do not have enough information on the exciting forces or soil parameters to justify the effort involved. Furthermore, it is often found that the vertical mode of oscillation, the torsional mode, or both, occur as uncoupled motions. Coupled motion is most frequently encountered in the design of machine foundations as rocking and sliding. Therefore, the following discussion of coupled vibrations with damping will be restricted to the case of combined rocking and sliding.

Figure 7-20a shows a rigid circular footing which rests on the surface of the elastic half-space. Its center of gravity is assumed to lie on the vertical axis through the center of the circular base and is a distance h_0 above the surface of the half-space. We can express the motion of this rigid body in terms of the horizontal translation x_g, of its center of gravity (CG), and the rotation ψ of the body about the CG. The sign convention chosen is illustrated in Fig. 7-20b, which indicates that $+x$ and $+P$ act to the right and that $+\psi$ and $+R_\psi$ are clockwise. The force P'_x and the moment R_ψ are developed by the soil reaction on the base of the footing. From Fig. 7-20c it is seen that the resulting motion of the footing can be established by superposing the translation x_g of the CG and the rotation ψ about the CG. In this diagram both motions are + (inphase), which forces the center of rotation to lie *below* the CG; this is designated as the *first mode of vibration*. If the translation is + while the rotation is − (motions out of phase), then the center of rotation lies *above* the CG and the motion is designated as the *second mode of vibration*. These designations of first and second modes follow from the fact that resonance in the first mode of vibration occurs at a lower frequency than does resonance in the second mode.

In order to establish the equations of motion from which the amplitudes of motion and the frequencies at maximum amplitude (resonant frequencies) can be calculated, it is useful to designate the translation of the base of the footing as

$$x_b = x_g - h_0\psi \tag{7-66}$$

as noted in Fig. 7-20d. Then the horizontal force on the base of the footing is expressed in terms of this base displacement and velocity as

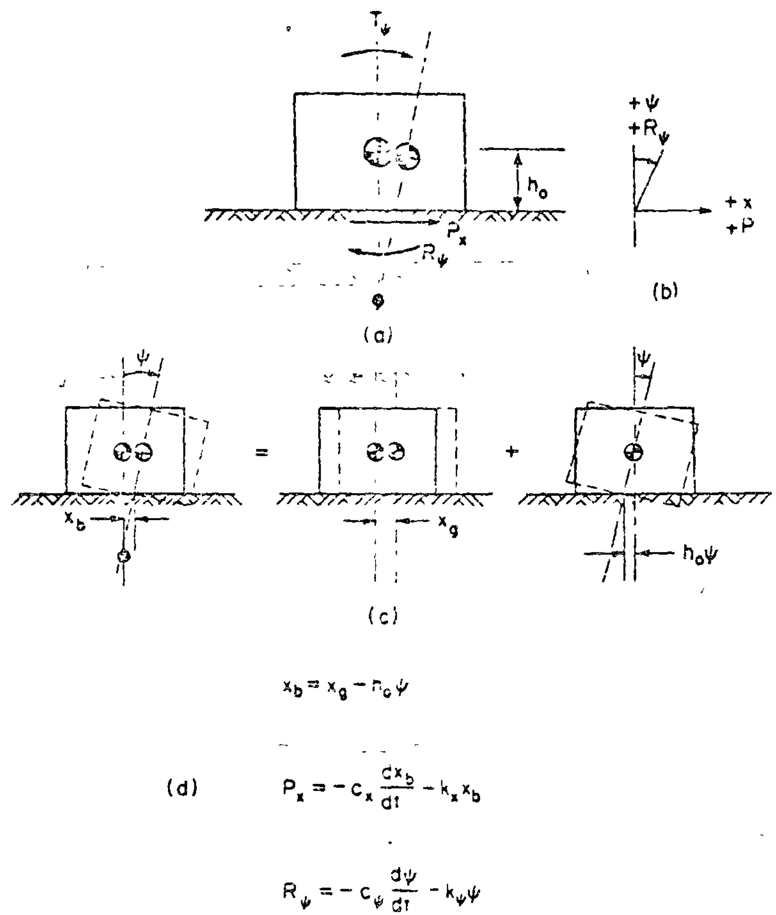


Figure 7-20 Notation for rocking and sliding mode of vibration.

The force described by Eq. (7-67) has the same form as that given in Eq. (7-15) for vertical vibration because they both involve translatory motions. The quantity c_x represents a damping coefficient and k_x represents a spring-reaction coefficient. Values of c_x and k_x obtained from the half-space theory by Hsieh (1962) are frequency-dependent. We have also seen previously that these quantities can be represented quite satisfactorily by the damping (Eq. 7-61) and spring constants (Eq. 7-60) for the analog. The expression for the resistance of the half-space to rocking of the footing—

$$R_\psi = -c_\psi \dot\psi - k_\psi \psi \tag{7-68}$$

—also includes the damping (c_ψ) and spring (k_ψ) terms which may be frequency-dependent or represented by the analog values (Eqs. 7-51 and 7-50).

The equation of motion for horizontal translation of the center of gravity of the footing is

$$m\ddot{x}_g = P_x = -c_x\dot{x}_b - k_x x_b \tag{7-69}$$

After substituting Eq. (7-66) and rearranging terms, Eq. (7-69) becomes

$$m\ddot{x}_g + c_x\dot{x}_g + k_x x_g - h_o c_x \dot\psi - h_o k_x \psi = 0 \tag{7-70}$$

The equation of motion for rotation about the CG is

$$I_g\ddot\psi = T_\psi + R_\psi - h_o P_x \tag{7-71}$$

in which I_g is the moment of inertia of the footing about the CG. Substitutions for R_ψ, P_x, and x_b change Eq. (7-71) to the form

$$I_g\ddot\psi + (c_\psi + h_o^2 c_x)\dot\psi + (k_\psi + h_o^2 k_x)\psi - h_o c_x \dot{x}_g - h_o k_x x_g = T_\psi \tag{7-72}$$

With the substitution of

$$x_g = A_{x1}\sin\omega t + A_{x2}\cos\omega t \tag{7-73}$$

$$\psi = A_{\psi 1}\sin\omega t + A_{\psi 2}\cos\omega t \tag{7-74}$$

$$T_\psi = T_{\psi o}\sin\omega t \tag{7-75}$$

into Eqs. (7-70) and (7-72), four equations in four unknowns are established. When the vibrating system is represented by the lumped-parameter analog, the spring and damping coefficients are constants and the solution of the four simultaneous equations at each value of the frequency provide for evaluation of the response. When the footing–soil system is represented by the footing on the elastic half-space, the damping and spring coefficients are frequency-dependent and the values of the coefficients must be calculated at any given frequency before the four simultaneous equations are solved for that value of frequency. In either case the calculations are most conveniently performed with the help of a high-speed digital computer. Hall (1967) has established that for a rigid circular footing the calculated response based on the lumped-parameter analog agrees very well with the "exact" solution.

Equations (7-70) and (7-72) demonstrate that coupling occurs [illegible] problem because the vertical location of the CG of the footing [illegible] line of action of the horizontal force P_x of the half-space on the bottom of the footing. If h_o is zero, no coupling is present, as demonstrated by the condition that in this case Eq. (7-70) includes only motion related to the coordinate x_o and Eq. (7-72) includes only motion related to the coordinate ψ.

7.9 Oscillation of the Rigid Circular Footing Supported by an Elastic Layer

For this problem the footing rests on the surface of a layer of thickness H of isotropic, homogeneous, elastic material which extends to infinity in the horizontal directions only. This layer is supported by a semi-infinite body which is infinitely rigid.

Reissner (1937) outlined the method of solution for the case of torsional oscillation of a circular footing on layered medium and Arnold, Bycroft, and Warburton (1955), and Bycroft (1956) have presented some solutions for this problem. The problem of vertical oscillation of the rigid circular footing on the elastic layer was treated by Arnold et al. (1955), Bycroft (1956), and Warburton (1957). The following discussions indicate only the general trends for the dynamic response of footings on a single layer under restricted conditions. The general problem of the dynamic behavior of footings on layered media or on elastic bodies with stiffness varying with depth needs further investigation, both theoretical and experimental.

Torsional Oscillation

In Reissner's (1937) discussion of the torsional oscillation of the circular footing on elastic layers, he established the basic equations for the solution and noted that the application of torsional oscillation at the surface provided one method of estimating the layer thickness. He did not establish the displacement functions f_1 and f_2 needed for evaluation of the dynamic response of the footing. These functions were presented by Arnold et al. (1955) and Bycroft (1956) for a few values of the *layer thickness ratio* H/r_o, in which H is the thickness of the elastic layer which is fixed to the rigid support and r_o is the footing radius (see Fig. 7-21). They computed values of f_1 and f_2 for H/r_o equal to 10, 1.0, and 0.5, and also indicated the agreement between the theoretical predictions and test results using a model footing resting on a layer of foam rubber. Figure 7-21 was prepared from information given in these two papers.

From both theory and tests they found that a resonant condition exists even when the mass of the footing is zero ($B_\theta = 0$). The frequency for this

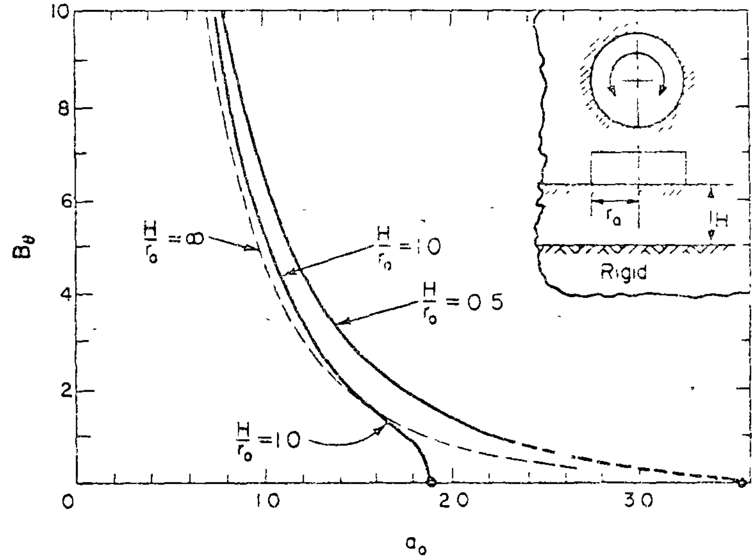


Figure 7-21 Mass ratio vs dimensionless frequency at resonance for torsional oscillation of rigid circular footing on an elastic layer (after *Bycroft, 1956*)

condition is shown in Fig. 7-21 as the a_o value for $B_\theta = 0$. Also note in Fig. 7-21 that for finite values of B_θ the frequency at maximum amplitude is higher than for the case of the semi-infinite elastic medium ($H/r_o = \infty$). This indicates that the presence of the rigid lower boundary introduces a stiffening effect, thereby increasing this frequency. Finally, the presence of the lower rigid boundary acts to reduce the geometrical damping of the system. This is illustrated by an increase in maximum amplitude of motion during vibration. For the footing on the semi-infinite medium the dynamic magnification factor M_θ is 1.0 for $B_\theta = 0$. The dynamic magnification factor was computed to be 1.6 for $H/r_o = 1.0$ and 2.6 for $H/r_o = 0.5$. Thus, as the layer becomes thinner with respect to the radius of the footing, the effective damping is decreased.

Bycroft (1956) made special note of the limiting condition for the frequency of torsional oscillation of a rigid circular footing on an elastic layer. He demonstrated that as H/r_o became smaller this frequency would approach the natural frequency for a rod of radius r_o, of length H, when oscillated as a torsional column fixed at the base and free at the top. The natural frequency of the resonant torsional column is

$$f_n = \frac{v_s}{4H} \qquad (7\text{-}7[illegible])$$

and the dimensionless frequencies (a_o) computed from this expression are $a_o = 1.57$ for $H/r_o = 1$, and $a_o = 3.14$ for $H/r_o = 0.5$. These values may be compared to $a_o = 1.87$ for $H/r_o = 1$, and $a_o = 3.55$ for $H/r_o = 0.5$ as computed from the displacement functions f_1 and f_2.

Vertical Oscillation

In the theoretical studies of vertical motion of a circular footing resting on the surface of the elastic layer, the pressure distribution corresponding to the rigid-base condition for the half-space (Eq. 7-8) was assumed. It was recognized that this does not correspond to the correct pressure distribution for a rigid footing on an elastic layer and Bycroft (1956) included a discussion of this point. In his study of the displacement of the footing, represented by the rigid-base pressure distribution, Bycroft computed the average static displacement for several values of the layer-thickness ratio H/r_o. Figure 7-22 shows the average displacement expressed in terms of the static displacement of the footing on an elastic half-space. This diagram shows clearly that the presence of the underlying rigid boundary provides a significant stiffening effect to the footing motion. The dashed curve in this same diagram illustrates the increase of the static spring constant as the layer thickness ratio H/r_o decreases. Note that this does not reach a factor of 2 until H/r_o is

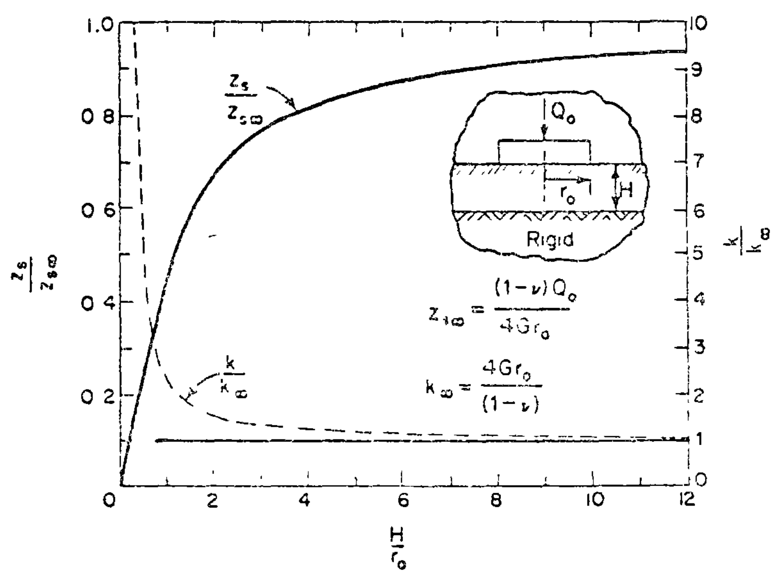


Figure 7-22 Static displacement and spring constant for vertical loading of rigid circular footing on elastic layer (after Bycroft, 1956).

reduced to about 1, but that the spring constant increases rapidly as H/r_o becomes smaller than 1.

In the studies of the dynamic behavior of the circular footing on the elastic layer, Bycroft (1956) considered only the case for $b = 0$, while Warburton (1957) presented solutions for $b > 0$. Both noted that true resonance, with *amplitudes of motion becoming infinite*, occurred for $b = 0$. This resonant vibration for $b = 0$ occurred when $f_1 \to \infty$ at values of a_o as indicated below:

$$\begin{aligned} &\text{for } \nu = 0, && f_1 \to \infty && \text{as} && a_o \frac{H}{r_o} \to 2.1582 \\ &\text{for } \nu = \tfrac{1}{4}, && f_1 \to \infty && \text{as} && a_o \frac{H}{r_o} \to 2.5742 \\ &\text{for } \tfrac{1}{3} < \nu < \tfrac{1}{2}, && f_1 \to \infty && \text{as} && a_o \frac{H}{r_o} \to \frac{\pi}{2}\sqrt{\frac{2(1-\nu)}{1-2\nu}} \end{aligned} \tag{7-77}$$

Bycroft noted that the case for $b = 0$ corresponds to the vibration of a rod of elastic material fixed at the base, free at the top, and constrained at the sides so that no lateral motion occurs. The resonant frequency of vibration of this rod is given by

$$f_o = \frac{(2n-1)v_P}{4H} = \frac{(2n-1)v_S}{4H}\sqrt{\frac{2(1-\nu)}{1-2\nu}} \tag{7-78}$$

which may be rearranged to

$$a_o = \frac{(2n-1)\pi r_o}{2H}\sqrt{\frac{2(1-\nu)}{1-2\nu}} \tag{7-79}$$

Note that higher modes of resonant frequency (i.e., $n = 1, 2, 3, \cdots$) are possible for this vertical motion of the weightless rigid plate on the elastic stratum. The vertical displacements of particles at different depths in the stratum are represented by the curves in Fig. 3-8 for the different modes of vibration.

For footings which have weight ($b > 0$), the amplitudes of motion are finite at the frequency we customarily describe as "resonance" (i.e., frequency for maximum amplitude of vibration). Warburton (1957) has presented curves for this resonant frequency, or frequency at maximum amplitude, as the usual mass ratio b-vs.-a_o plots for different values of H/r_o. He prepared one such diagram for $\nu = 0$ and a second for $\nu = \frac{1}{4}$, which is reproduced here as Fig. 7-23.

Warburton also evaluated the maximum dynamic displacements and expressed them in terms of a magnification of the static displacement

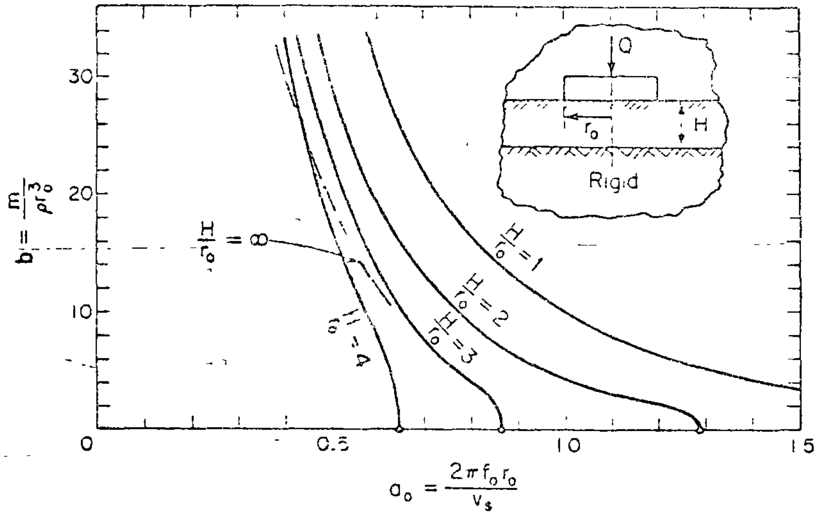


Figure 7-23 Variation of dimensionless frequency at resonance with mass ratio and thickness of the elastic layer for vertical oscillation of rigid circular footing (*after Warburton, 1957*)

Referring to Fig. 7-22, it is evident that the large magnification factors he presented are not quite as bad as they appear at first glance because they amplify a static displacement which is reduced in magnitude as the layer thickness decreases. The values indicated in Table 7-1 relate Warburton's values for dynamic motion to the static value for displacement of the rigid footing on the semi-infinite body, or

$$M_{Lm} = \frac{A_{zLm} 4 G r_o}{(1 - \nu) Q_o} \tag{7-80}$$

Table 7-1. Magnification Factors for Vertical Vibration of Rigid Circular Footing Supported by an Elastic Layer ($\nu = \frac{1}{4}$)

$\frac{H}{r_o}$	M_{Lm} for				
	b = 0	b = 5	b = 10	b = 20	b = 30
1	∞	5.3	11.4	20.5	23.9
2	∞	3.0	16.1	30.6	40.8
3	∞	4.7	9.5	23.7	36.0
4	∞	(3.4)	5.9	15.6	27.9
∞	1	1.21	1.60	2.22	2.72

in which M_{Lm} is the magnification factor for displacement at resonance for the layered system

Warburton's analysis treated the ideal elastic medium which has no internal damping. For a real footing-soil system, the relatively small amount of material or hysteresis damping will be important in reducing these high magnification factors which have been indicated by the theoretical treatment

7.10 *Vibrations of Rigid Foundations Supported by Piles or Caissons*

Piles will be effective in resisting vibratory loadings only if they can develop appreciable forces as their tops move through very small distances. As noted in Chap. 10 under *Design Criteria*, the permissible dynamic motions of machine foundations are often of the order of just a few thousandths of an inch. Consequently, the pile must contribute its resisting forces during this kind of movement or it is not effective Resistances to vertical motion may be provided by end bearing, skin friction, or by a combination of the two. The resistance developed by piles to horizontal forces is provided by horizontal bearing of the pile against the soil. In each of these cases the soil properties involved depend upon the magnitude of the local deformation developed by the pile acting against the soil and must be evaluated from tests involving the same order of magnitude of strain in the soil as occurs in the prototype situation. Fortunately, for these small strains many soils exhibit an approximately elastic response which may be evaluated by laboratory or field tests as described in Chap. 6. Thus, elastic solutions will again be used to estimate the response of pile-supported foundations.

Vertical Vibrations of a Foundation Supported by Point-bearing Piles to Rock

Point-bearing piles provide support for a foundation by transferring the vertical loads to a stronger soil stratum at some depth beneath the surface. Under stable dynamic conditions for which no further settlement occurs the dynamic loads are transferred through the elastic pile to the elastic contact zone at the tip of the pile where the loads are absorbed by the stratum. We can establish the *maximum* influence of piles for stiffening the support for a foundation if we first consider that the *stratum is rigid* and *no deformation* occurs at the pile tip when dynamic loads are transferred from the pile. This would be approximately true if piles were driven through soft soils to rock.

The theoretical procedure required for this study was discussed in Chap. 3 and is represented by the sketch in Fig. 3-10. It involves an e...

at the base and free at the top, with a mass m resting on the top. When no weight rests on top we have a solid resonant column with the fixed-free condition, which has a resonant frequency

$$f_n = \frac{v_C}{4\ell} = \frac{1}{4\ell}\sqrt{\frac{E}{\rho}} \tag{7-81}$$

in which

E = Young's modulus of elasticity of the pile,

$\rho\left(=\frac{\gamma}{g}\right)$ = mass density of the pile material, and

ℓ = length of the pile.

For the intermediate case in which the supported mass (i.e., the portion of the total load assigned to each pile) is of the same order of magnitude as the weight of the pile itself, the frequency equation has been given by Eq. (3-33). The solution for Eq. (3-33) is shown graphically by Fig. 7-24, from which f_n may be calculated. Of course, when the weight of the pile is negligible with respect to the supported weight, the natural frequency is given by Eq. (2-17b).

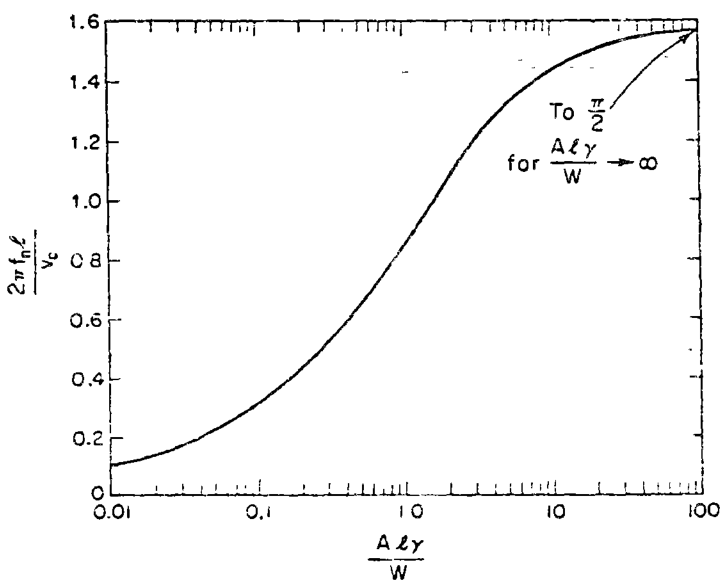


Figure 7-24. Graphical solution for Eq. (3-33)

Material	E, lb/in²	γ, lb/ft³
Steel	29.4 × 10⁶	480
Concrete	3.0 × 10⁶	150
Wood	1.2 × 10⁶	40

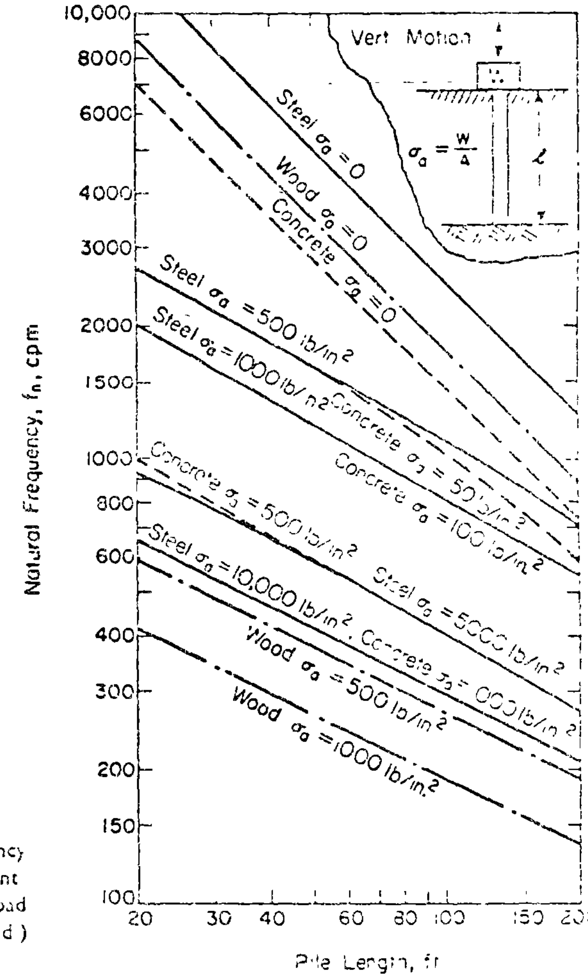


Figure 7-25 Resonant frequency of vertical oscillation for a point bearing pile carrying a static load W. (Loaded stratum is rigid.) (From Richart, 1962)

In order to illustrate the influence of axial loading on the resonant frequency of end-bearing piles to rock, Richart (1962) prepared a diagram which included the parameters of axial load, pile length, and pile material. This diagram is reproduced here as Fig. 7-25. The three curves at the top of the diagram illustrate the resonant frequencies of unloaded steel, concrete, and wooden piles, as computed from Eq. (7-81). As the axial load is increased on a pile of given length, the resonant frequency ...

In this analysis of a pile-supported foundation, only the resonant frequency was considered. From an elastic analysis of a closed system corresponding to this one, no geometrical damping occurs and the amplitude of motion at resonance are theoretically infinite. Actually, material or hysteresis damping will restrict the motions somewhat, but the resonant motion will still be of large magnitude relative to the static displacement.

Torsional Vibration of a Circular Foundation Supported by Piles

The problem of torsional vibration of a circular footing supported by piles is included to illustrate the effectiveness of piles in resisting lateral loads. For the following discussion it will be assumed that the base of the footing is in contact with the soil throughout the dynamic motions and that *no slippage* occurs between the base of the foundation and the soil. The piles are considered to be stub piles installed to increase torsional resistance only. The method of analysis involves (1) an estimate of the rotational motion of the soil beneath the footing as the footing twists and imparts shearing forces to the soil surface, (2) an estimate of the relative motion of a pile attached to the footing with respect to the soil motion, and (3) use of the theory of subgrade reaction to estimate the restraining torque provided by the lateral motions of the pile against the soil.

In their studies of torsional oscillations of rigid circular footings resting on the surface of the elastic half-space Reissner and Sagoci (1944) developed solutions for the static rotational displacements (referred to the vertical axis through the center of the footing) within the half-space. Their solution was developed using oblate-spheroidal coordinates. The tangential displacement s directly below the periphery of the disk has been evaluated and is illustrated in Fig. 7-26 as $s/r_o\Theta_s$ vs. z/r_o. The static rotation Θ_s represents the angular rotation of the disk at the surface, and the displacements vary linearly with the radius.

If a pile is attached to the footing, its point of attachment moves the same distance along a circumferential arc as does a point on the footing base or as does a point on the surface of the soil at the same radius r. Therefore, at the footing-soil contact zone the pile does not have any relative motion with respect to the soil and does not develop any lateral force. The only stiffening effect of the pile is through transfer of shear by bending from a deeper location where the pile moves against the soil. The amount of horizontal force developed on the pile at each elevation depends on the relative motion of the pile against the soil. For an infinitely rigid pile attached

to the footing, the motion of the pile against the soil would be

$$s_{rel} = r\Theta_s\left(1 - \frac{s}{r\Theta_s}\right) \quad (7\text{-}82)$$

where the quantity $s/r\Theta_s$ is given in Fig. 7-26 for each depth. However, a real pile will bend because of the forces developed along its length and the relative motion of pile against the soil will be reduced.

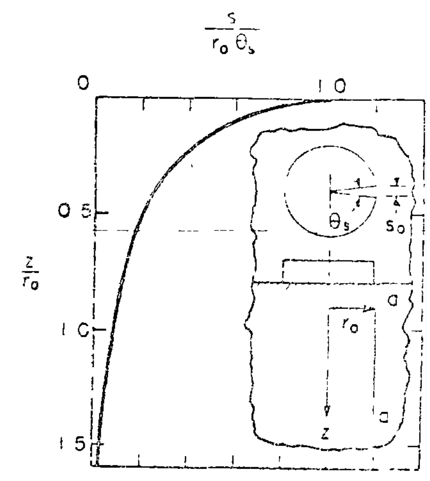


Figure 7-26 Tangential displacements along line a-a beneath the edge of a circular footing which has been rotated through an angle Θ_s

The force developed by the pile motion against the soil can be estimated by using the theory of horizontal subgrade reaction. From this approach the horizontal force P' per unit length of pile is given by

$$P' = K_h d \cdot s_{rel} \quad (7\text{-}83)$$

in which

K_h = coefficient of horizontal subgrade reaction, and
d = pile diameter.

The critical factor in this type of analysis is the proper selection of K_h, which relates the pressure developed as the surface of the pile moves a unit distance into the soil; that is,

$$K_h = \frac{p}{s} \quad (7\text{-}84)$$

The value of K_h *must be related to the order of magnitude of the motions involved*, and for vibration problems these motions are very small. Because methods are available for evaluating the "elastic" constants for soils for these small strains, it is useful to employ the theory of elasticity to establish K_h. If we consider first the behavior of a circular pile, we can approximate this by considering the pressure required to expand a cylindrical hole in an elastic medium. Westergaard (1952) gives the radial expansion of a hole of diameter d as

$$\frac{\Delta d}{2} = \frac{pd}{4G} \quad (7\text{-}85)$$

from which the coefficient of subgrade reaction is approximately

$$K_h = \frac{2p}{\Delta d} = \frac{4G}{d} \tag{7-86}$$

This approach has been followed because the inaccuracies introduced by the geometry of the problem were considered to be less important than the potential errors in G.

Now if we return to a consideration of the infinitely rigid cylindrical pile which extends downward from the base of a circular footing, this pile will move laterally a distance s_o as the footing rotates through an angle Θ_s, as shown in the sketch in Fig. 7-27a. At the same time, the soil moves laterally because of the shearing forces introduced by the footing on the surface (Fig. 7-27b). The net motion is illustrated in Fig. 7-27c. Then, according to Eq. (7-83), the lateral force developed at each elevation is

$$P' = \frac{4G}{d}\, r\Theta_s d\left(1 - \frac{s}{r\Theta_s}\right) \tag{7-87}$$

in which the displacement s is a function of the depth involved—or the length of the pile. By integrating Eq. (7-87) from $z = 0$ to $z = \ell$ we can evaluate the efficiency of *rigid* piles in developing resistance to lateral motion as they penetrate to greater depths. The efficiency factor is designated by

$$(\mathrm{EF})_M = \frac{\int_0^{\ell} P'\, dz}{K_h s_o \ell d} \tag{7-88}$$

and represents the total horizontal force developed as a fraction of the total force which *could* be developed if the soil did *not* move. Figure 7-28 illustrates the variation of this efficiency factor with length of *rigid pile* for the condition of K_h *constant with depth*.

For real piles the flexibility of the pile itself is of importance because pile bending will reduce the relative motion of pile against the soil. Thus, the flexibility introduces an efficiency factor which decreases as the pile length increases.

EXAMPLE. To illustrate the combined effects of soil motion and pile flexibility, consider 6-in.-diameter stub piles located along a 50-in.-diameter circle on the base of a 62-in.-diameter circular concrete footing. The footing is to be subjected to torsional oscillations and the stub piles are for the purpose of increasing the torsional [illegible] foundation.

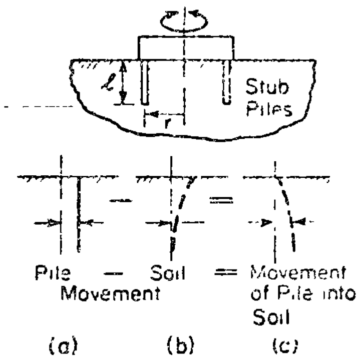


Figure 7-27. Action of stub piles

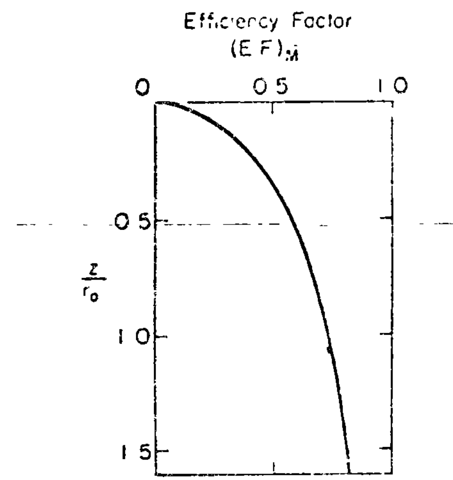


Figure 7-28 Efficiency factor for rigid stub piles attached to a circular footing twisted against the soil.

For this example the shear modulus G of the soil is taken as 4500 lb/in.² and the footing is not permitted to slip. Then the static torsional stiffness of the rigid footing on the soil is given by

$$\left(\frac{T_\Theta}{\Theta_s}\right)_F = \frac{16}{3} G r_o^3 = \frac{16}{3}(4500)(31)^3 = 7.15 \times 10^8 \quad \text{in.-lb/rad}$$

When the stub piles are introduced on a 50-in. diameter, the contribution to the torsional stiffness of the system by *each* pile is

$$\left(\frac{T_\Theta}{\Theta_s}\right)_p = \frac{P_o r^2}{s_o} \tag{7-89}$$

in which P_o and s_o are the horizontal force and displacement, respectively, at the top of the pile. Because we have already considered the effect of *soil motion* in reducing the pile efficiency below that for the infinitely rigid pile moving against still soil, we may express Eq. (7-89) as

$$\left(\frac{T_\Theta}{\Theta_s}\right)_p = \frac{K_h d s_o \ell r^2}{s_o}(\mathrm{EF})_M(\mathrm{EF})_F \tag{7-90}$$

in which $(\mathrm{EF})_M$ is the efficiency factor for a rigid pile including the effect [illegible] motion, and $(EF)_F$ is the efficiency factor which takes into account the pile flexibility.

In order to evaluate the effect of pile flexibility, we must first evaluate K_h from Eq. (7-86).

$$K_h = \frac{4G}{d} = \frac{4(4500)}{6} = 3000 \quad \text{lb/in}^3 \tag{7-91}$$

Then the stiffness for the infinitely rigid pile moving against still soil is

$$\left(\frac{T_\Theta}{\theta_s}\right)_R = K_h l r^2 d = 11.25 \times 10^6 \text{ in.-lb/rad} \tag{7-92}$$

Now, if the 6-in.-diameter stub pile is assumed to be made of reinforced concrete with $E = 5 \times 10^6$ psi, the effect of pile flexibility may be evaluated from the theory for beams on an elastic foundation (Hetenyi, 1946). For K_h constant with depth and a constant EI pile, we first calculate the value of λ as

$$\lambda = \sqrt[4]{\frac{K_h d}{4EI}} = 0.0613 \quad \left(\frac{1}{\text{in.}}\right) \tag{7-93}$$

For a pile of finite length, Hetenyi gives separate solutions for deflection and slope for end shear only, and deflection and slope for end moment only. Then, for a stub pile assumed fixed into the circular footing, it is only necessary to calculate the force required to produce unit translation of the point of fixity. The ratio of the stiffness of the flexible pile to the stiffness of a rigid pile, both moving against soil at rest, gives the efficiency factor for flexibility $(EF)_F$. The decrease of this quantity with length of the 6-in.-diameter stub pile is shown in Fig 7-29a.

The combined effect of soil motion and pile flexibility is shown by the curve in Fig 7-29b. Thus, the lateral resistance to rotation provided by a 6-in.-diameter stub pile 20-in. long is given by

$$\left(\frac{T_\Theta}{\theta_s}\right)_P = \left(\frac{T_\Theta}{\theta_s}\right)_R (EF)_M (EF)_F = 11.25 \times 10^6 \times 20 \times 0.44 = 99 \times 10^6 \text{ in.-lb/rad}$$

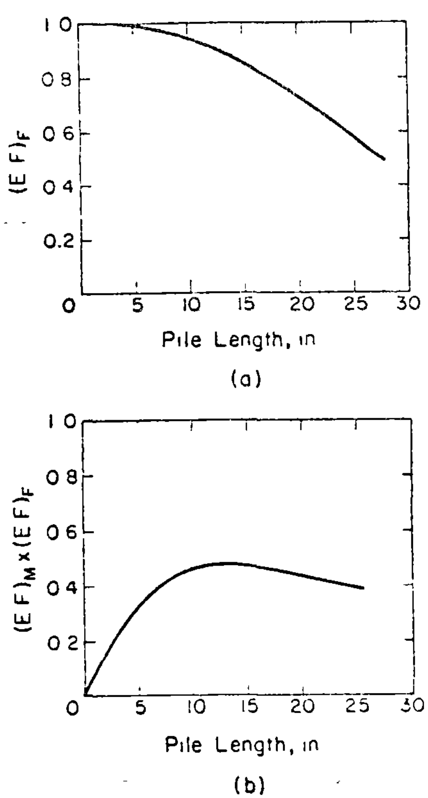


Figure 7-29 Efficiency factor for six-inch diameter concrete stub piles (a) Effect of pile flexibility (b) Combined effect of flexibility and soil motion

From this analysis it is seen that each 20-in.-long stub pile contributes a torsional stiffness equal to about 13.8 per cent of that due to the contact between the circular footing and the soil. Thus, the torsional stiffness of the foundation system could be doubled, theoretically, by adding seven of these stub piles.

The foregoing discussion was restricted to a consideration of stub piles which do not absorb any appreciable vertical load. However, it is possible for real piles to absorb the entire vertical load of the footing if the soil settles away from the footing. If this happens, the torsional resistance of the footing against the soil is lost and the *entire* torsional resistance must be provided by the piles. It should be evident from this example that the torsional restraint provided by the footing twisting against the soil is important and should be maintained if torsional oscillations of the footing are anticipated.

10

DESIGN PROCEDURES FOR DYNAMICALLY LOADED FOUNDATIONS

10.1 Introduction

A design procedure must deal with three major questions: (1) What is the condition for failure of the design function and how is it defined? (2) What are the loads or external conditions which produce the failure? (3) What is the analytical procedure to be followed for relating the applied loadings to the failure condition? Another factor, introduced after evaluating the uncertainties related to each of foregoing, is the factor of safety.

In the design of machine foundations or other dynamically loaded foundations, successive corrections are used to arrive at the final physical system. A set of physical parameters are assumed and then analyzed to determine if the design conditions are satisfied. If they are not satisfied, then some of the physical parameters are varied and the process is repeated. With this approach the *criteria for failure of the design function* and the *applied loadings* must be carefully determined, because these establish a control of the procedure. The analysis should describe clearly the influence of each of the major physical variables involved, in order that an intelligent choice can be made during successive corrections in design.

The design procedures to be discussed in this chapter relate primarily to the dynamic response of foundations subjected to steady-state vibrations or

transient loadings. Generally, the supporting soil is considered to be in a stable condition such that it does not compact or change geometry unless the design conditions are exceeded. The emphasis of the chapter lies in the procedure for carrying out the dynamic analyses after the design conditions have been established.

10.2 Design Criteria

The end product of the design procedure is the determination of a foundation-soil system which satisfactorily supports equipment or machinery. The supported unit may be the source of dynamic loads applied to the system or it may require isolation from external excitation. In each case the criteria for satisfactory operation of the unit dictate the design requirements.

In Table 10-1 are listed some criteria which may be considered during the design of the foundation system. This checklist is included only as a guide; all topics may not be applicable to a particular problem, and additional topics may be included to cover special installations.

The design criteria most often encountered relate to the dynamic response of the foundation. These are expressed in terms of the limiting amplitude of vibration at a particular frequency or a limiting value of peak velocity or peak acceleration. Figure 10-1 indicates the order of magnitudes which may be involved in the criteria for dynamic response. Five curves limit the zones for different sensitivities of response by persons, ranging from "not noticeable" to "severe." These categories are for persons standing and being subjected to vertical vibrations. The boundary between "not noticeable" and "barely noticeable" is defined by a line at a slope of −1 on the log–log plot which represents a peak *velocity* of about 0.01 in./sec. The line dividing the zones of "easily noticeable" and "troublesome" represents a peak *velocity* of 0.10 in./sec.

The envelope described by the shaded line in Fig. 10-1 as "limit for machines and machine foundations" indicates a limit for *safety* and *not a limit for satisfactory operation of machines*. Operating limits for machines are discussed in the next section. The shaded limit for machines in Fig. 10-1 is composed of two straight lines. Below about 2000 cycles/min this limit represents a peak velocity of 1.0 in./sec, and above 2000 cycles/min it corresponds to a peak acceleration of (0.5)g.

Two curves are also included in Fig. 10-1 to indicate limiting dynamic conditions associated with *blasting*. These magnitudes of motion correspond to effects applied once, or at most repeated a few times. They definitely do not apply for steady-state vibrations. The line at the lower limit of the zone "caution to structures" corresponds to a peak velocity of [illegible] in./sec

Table 10-1. Checklist for Design Criteria

I Functional Considerations of Installation
- A Modes of failure and the design objectives
- B Causes of failure
- C Total operational environment
- D Initial cost and its relation to item A
- E. Cost of maintenance
- F Cost of replacement

II. Design Considerations for Installations in Which the Equipment Produces Exciting Forces
- A. Static bearing capacity
- B. Static settlement
- C Bearing capacity. Static + Dynamic loads
- D Settlement Static + Repeated dynamic loads
- E. Limiting dynamic conditions
  - 1. Vibration amplitude at operating frequency
  - 2 Velocity
  - 3 Acceleration
- F. Possible modes of vibration—coupling effects
- G. Fatigue failures
  - 1 Machine components
  - 2. Connections
  - 3. Supporting structure
- H Environmental demands
  - 1. Physiological effects on persons
  - 2. Psychological effects on persons
  - 3. Sensitive equipment nearby
  - 4 Resonance of structural components

III Design Considerations for Installation of Sensitive Equipment
- A. Limiting displacement, velocity, or acceleration amplitudes
- B. Ambient vibrations
- C Possible changes in ambient vibrations
  - 1. by construction
  - 2. by new equipment
- D. Isolation of foundations
- E. Local isolation of individual machines

It is important to note from Fig. 10-1 that the magnitudes of vibration involved in these criteria are much smaller than the displacements usually considered in the designs of foundations for static loads. For example, at a frequency of 1000 cycles/min, an amplitude of 0.0001 in. may be noticed by persons, whereas it takes a motion of 0.01 in. at the same frequency to cause damage to machinery or machine foundations. Therefore, the order of magnitude of vibration amplitudes to be considered in this chapter will nearly always be less than 0.01 in. and will usually be of the order of 0.005 to 0.0001 in.

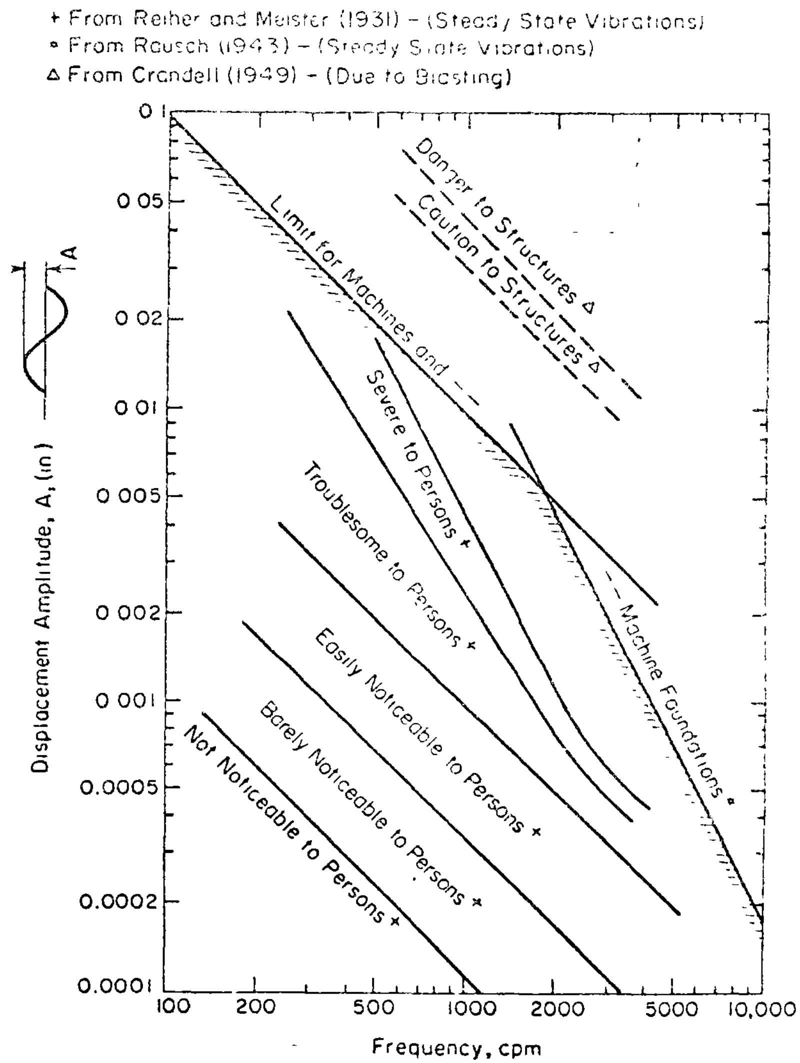


Figure 10-1. General limits of displacement amplitude for a particular frequency of vibration (*from Richart, 1962*)

Steady-State Vibrations of Machinery

The design criteria related to operation of machinery depend on the prime function of the entire installation and the importance of each ... unit to this function. Thus, the design criteria involve considerations of initial cost, cost of maintenance (which includes the ...

lown time"), and the cost of replacement of the unit. The foundation stem must be designed to accommodate the level of operation desired by e owner.

For rotating machinery the information presented by Blake (1964) may e used to establish the permissible amplitudes of motion at the operating eed. Figure 10-2 illustrates the categories of performance *A* through *E* n the amplitude-vs.-frequency diagram. Note that the amplitude of vibraon refers to horizontal motions measured on the bearing (not the shaft) the machine. Blake (1964) has also introduced the concept of *service ctor* (see Table 10-2), which indicates the importance of the particular achine to the prime function of the plant. The higher numerical values for e service factor indicate the more critical machines.

With the introduction of the service factor, Fig. 10-2 may be used to aluate the performance of a wide variety of machines. Several examples rom Blake, 1964) are included below to illustrate the use of the service ctor.

A. Measurements on an electric motor show 2 mils (0.002-in. single amplitude) at 3600 rpm From Table 10-2 the service factor is 1, and the effective vibration is 2 × 1 = 2 mils Enter Fig 10-2 at 3600 rpm and go up to 2 mils. This point falls in class *D* See "Explanation of Cases" at the bottom of Fig 10-2 for recommended action.

B. A stiff-shafted centrifuge shows 7 mils (0.007 in.) at 1000 rpm The service factor is 2. Thus, the effective vibration is 7 × 2 = 14 mils. From Fig. 10-2 at 1000 rpm and 14 mils (0.014 in), the point falls in Class *E*. See notes to Fig. 7-2 for recommended action.

Table 10-2. Service Factors*

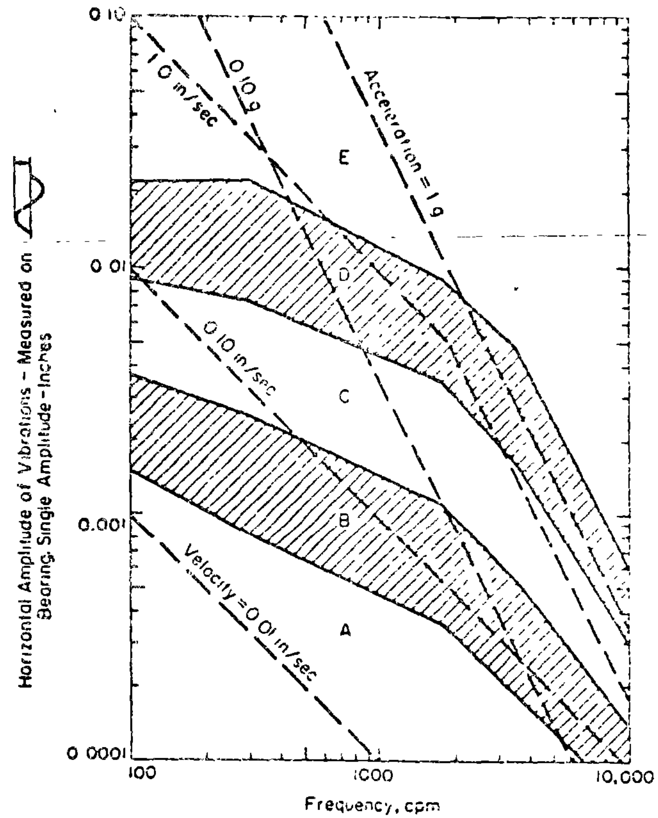
Single-stage centrifugal pump, electric motor, fan	1
Typical chemical processing equipment, noncritical	1
Turbine, turbogenerator, centrifugal compressor	1.6
Centrifuge, stiff-shaft†, multistage centrifugal pump	2
Miscellaneous equipment, characteristics unknown	2
Centrifuge, shaft-suspended, on shaft near basket	0.5
Centrifuge link-suspended, slung	0.3

Effective vibration = measured *single amplitude* vibration, inches multiplied by the *service factor*

Machine tools are excluded Values are for bolted-down equipment, when not bolted, multiply the service factor by 0 4 and use the product as a service factor

Caution Vibration is measured on the bearing housing, except as stated

*rom Blake (1964)
† ... displacement of basket housing



Explanation of cases
E Dangerous Shut it down now to avoid danger
D Failure is near Correct within two days to avoid breakdown
C Faulty Correct within 10 days to save maintenance dollars
B Minor faults Correction wastes dollars
A No faults Typical new equipment

Figure 10-2 Criteria for vibrations of rotating machinery (*after Blake, 1964*).

C. A link-suspended centrifuge operating at 950 rpm shows 2 5 mils, with the basket empty The service factor is 0 3 and the effective vibration is 0 75 mils (0.00075 in) The point at 950 rpm and 0 00075 in in Fig 7-2 falls in Class *B*.

For special types of machines, the organizations concerned with their manufacture, installation, and operation often develop ratings for different operating conditions. For example, Parvis and Appendino (1966) give values for vibrations at the bearings of turbo-alternator sets op[illegible] which have the ra[illegible] indicated in Table [illegible]

Table 10-3 Range of Values of Vibrations for Turboalternators Operating at 3000 rpm*

Rating of Turboalternator Operation	Vibration (Single-Amplitude) On Bearing Caps (in.)	On Shaft (in.)	On Turbine Table (in.)
Excellent	0.0002	0.0010	0.00002
Good	0.0004	0.0020	0.00004
Fair	0.0008	0.0040	0.00008
Bad	0.0016	0.0080	0.00016
Dangerous	0.0032	0.0160	0.00032

* After Parvis and Appendino (1966)

Additional information relating to the operation of rotating machinery in general is noted in Table 10-4 (from Baxter and Bernhard, 1967). These limits are based on peak-velocity criteria alone and would be represented by straight lines on plots similar to those of Figs. 10-1 and 10-2. Note the similarity in values of peak velocity for the lower limit of the range for machines as "smooth" (0.010 in./sec in Table 10-4) and the lower limit of the range "barely noticeable to persons" (0.01 in./sec in Fig. 10-1). Similarly, note the lower limits for "slightly rough" for machines (0.160 in./sec in Table 10-4) and "troublesome to persons" (0.10 in./sec in Fig. 10-1), and the danger limits of "very rough" (>0.63 in./sec in Table 10-4) and the Rausch limit for machines (1.0 in./sec in Fig. 10-1). The "dangerous" rating for turboalternators of 0.0032 in. at 3000 rpm (Table 10-3) also corresponds to 1.0 in./sec.

Baxter and Bernhard (1967) have also given a tentative guide to vibration tolerances for machine tools—this information is shown in Table 10-5. The motions indicated in this table represent only general magnitudes, actual operating tolerances must depend on the size and finish tolerances of the parts to be machined.

Table 10-4. General Machinery-Vibration-Severity Data*

Horizontal Peak Velocity (in./sec)	Machine Operation
< 0.005	Extremely smooth
0.005–0.010	Very smooth
0.010–0.020	Smooth
0.020–0.040	Very good
0.040–0.080	Good
0.080–0.160	Fair
0.160–0.315	Slightly rough
0.315–0.630	Rough
>0.630	Very rough

* After Baxter and Bernhard (1967)

Table 10-5 Tentative Guide to Vibration Tolerances for Machine Tools*

Type Machine	Displacement of Vibrations as Read with Pickup on Spindle-Bearing Housing in the Direction of Cut
Grinders	Tolerance Range (mils†)
Thread grinder	0.01–0.06
Profile or contour grinder	0.03–0.08
Cylindrical grinder	0.03–1.0
Surface grinder (vertical reading)	0.03–0.2
Gardner or Besly type	0.05–0.2
Centerless	0.04–0.1
Boring machines	0.06–0.1
Lathe	0.2–1.0

* These values came from the experience of personnel who have been trouble-shooting machine tools for over ten years. They merely indicate the range in which satisfactory parts have been produced and will vary depending on size and finish tolerance. After Baxter and Bernhard (1967)
† 1 mil = 0.001 in.

Vibrations of Structures

Although the topics of vibrations of structures and the allowable limits for such vibrations are beyond the scope of this book, it is useful to include a few comments on this subject. This is particularly important in relation to the problem of preventing damage to structures because of machine operations or construction operations in the immediate vicinity.

In Fig. 10-1 and in the text describing this figure, it was noted that limits have been established (Crandell, 1949) for motions of structures caused by blasting. Although the lower limit for the zone (in Fig. 10-1) denoted "caution to structures" represents a peak velocity of 3 in./sec, it is general practice to limit the peak velocity to 2 in./sec (see Wiss, 1968). The U.S. Bureau of Mines criteria for structural safety against damage from blasting involve both a limiting peak velocity and a limiting peak acceleration. Below 3 cycles/sec the limit is 2 in./sec peak velocity, and above 3 cycles/sec the limit is (0.10)g peak acceleration.

For failure conditions governed by limiting values of peak velocity or acceleration, it is sometimes more convenient to plot this information [illegible]

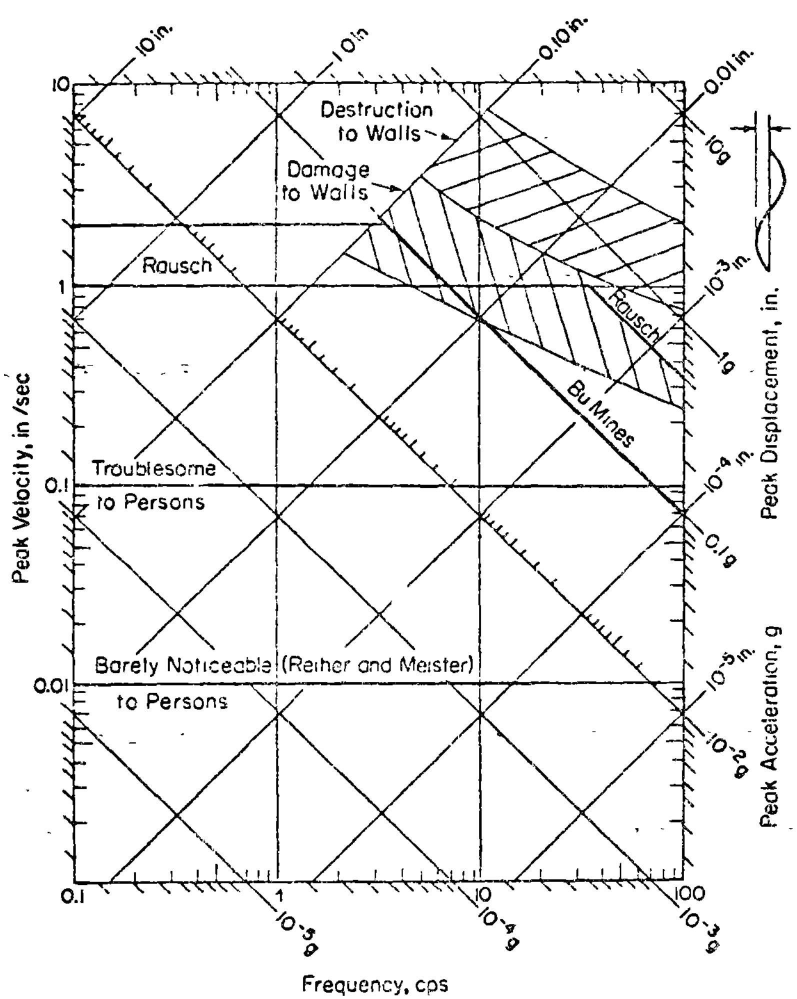


Figure 10-3. Response spectra for vibration limits

diagram similar to Fig. 10-3, which shows simultaneous values of displacement, velocity, and acceleration. The limiting condition for each of these three quantities forms an envelope on this diagram. Points falling above this envelope violate the "failure" conditions, while those points falling below the envelope represent satisfactory conditions. In Fig. 10-3 are shown the limiting conditions for (1) the "people" limits of "barely noticeable" and "troublesome to persons," (2) the Rausch limits for machines and machine foundations, and (3) the U.S. Bureau of Mines criteria. Also shown in Fig. 10-3 are two shaded zones which describe the possibility of structural damage, particularly to walls, which may be caused by steady-state vibrations (see [illegible]). Diagrams similar to Fig. 10-3 are quite

useful for evaluating the dynamic response of a vibrating system and are often designated as *shock or response-spectra diagrams*.

Effects of Vibrations on Persons

When the design of a foundation–soil system involves the consideration of people in the immediate vicinity, the problem may become complex. The first point to be established is the tolerable level of vibration for the area where persons are to be located. The next step is to ascertain if this level is possible within the ranges of vibration input, distances, and soil conditions anticipated.

In Fig. 10-1 several ranges of human tolerances to vibrations were noted which had been established experimentally by subjecting people to vertical vibrations as they stood on a shaking table (see Reiher and Meister, 1931). Generally, people are most susceptible to vibrations applied in the direction of the long axis of the body. The human tolerance limits of Reiher and Meister have been confirmed by subsequent investigations and are generally accepted as useful physiological vibration limits for people. For a comprehensive discussion of the effect of shock and vibrations on man, see Chap. 44, by Goldman and von Gierke, in the book edited by Harris and Crede (1961).

The physiological vibration limits represent only the first step in evaluating the effects of vibrations on persons. The next and often more important consideration is the psychological effect on persons. If the vibration is being generated "in his interest," then a person may accept the physiological vibration limit. However, if the vibration is generated "for someone else's benefit," a vibration which is "barely noticeable" may be effectively transformed into the "troublesome" category. Another example of the psychological effect occurs when a new foundation–soil system is designed for a dynamic unit. It is not satisfactory to provide the same level of vibration to persons in the neighborhood of the new unit as had existed from the old installation. Even though they had accepted the previous vibration levels and possibly even found them tolerable, people "expect" the new installation to perform better. Consequently, the reactions of people in the immediate vicinity of vibration-generating equipment may introduce a significant factor when establishing design criteria. It should be possible to make up a table similar to Table 10-2 (service factors for machinery) to indicate the variations in sensitivity of persons to vibrations, according to their psychological response.

Criteria for Transient Loadings

The report by Steffens (19[illegible]2) includes a [illegible] the methods proposed for [illegible]

Table 10-6. Some Typical Vibration Data*

Vibration from	Authority	Details	Observed: Amplitude (in.)	Observed: Frequency (cycles/sec)	Reiher & Meister Classification	Derived: Max. Velocity (microinches/sec)	Derived: Acceleration (g)
Traffic	Hyde and Lintern (1929)	Single-deck motor bus, 18 mph, 30 ft away	0.00012	26	Just perceptible	19,700	0.0082
Traffic	Hyde and Lintern (1929)	Light truck, 13.6 mph, 20 ft away; rough road	0.00012	20	Just perceptible	15,100	0.0049
Traffic	BRS (1934)	General traffic at Brentford	0.00012	19	Just perceptible	14,300	0.0044
Traffic	Tillman (1933)	Measurements in house 30–50 ft from traffic	0.00025	24	Clearly perceptible	37,700	0.0145
Traffic	BRS (1950)	Vibrations from London, traffic as measured inside a building	0.00014	25	Just perceptible	22,000	0.009
Traffic	BRS (1950)	Traffic measurements in Queens Street, London	0.00031	14	Just perceptible	27,000	0.0062
Traffic	BRS (1950)	Traffic measurements in Farrington Street, London	0.00036	10	Just perceptible	22,600	0.003
Railways	US	Measurements of vibration in Times Building (N.Y.), subway; floor vibrations	0.00078	15–20	Clearly perceptible	85,000	0.024
Railways	Mallock (1902)	Hyde Park area; building vibrations due to subway	0.001	10–15	Clearly perceptible	78,000	0.01
Railways	C. C. Williams	Freight train at 65 ft; passenger train at 25–30 ft	0.0009 0.0037		—	—	
Pile Driving	BRS	Close to occupied building	0.00053	30	Clearly perceptible—annoying	100,000	0.049
Blasting	BRS and RAE (1950)	Measurements in bomb-damaged tunnel, no damage caused by blasting vibration	0.0015 0.00007	6 80	Clearly perceptible Clearly perceptible	57,500 36,000	0.006 0.045
Blasting	G. Morris (1950)	Vibrations in villa 1100 ft away; firing 2000-lb explosive	0.0017	9.4	Clearly perceptible—annoying	100,000	0.015
Machinery	Tillman (1933)	Vibration from chocolate factory; measurements in nearby house	0.00056	42	Annoying	147,500	0.09
Machinery	Tillman (1933)	Vibration in houses (3rd storey), 400 ft from 120-hp diesel	0.0008	3.5	Just perceptible	17,500	0.01
Machinery	BRS	Vibrating table, measurements on table	0.005	25	Painful	780,000	0.32
Machinery	Tillman (1933)	Vibration in 70-year-old house adjacent to six lithographic presses	0.00031	64	Annoying	125,000	0.133

Notes

(a) From the foregoing results it would appear that the maximum velocities involved at the various stages of perceptibility are (approximately) in microinches per second

Just perceptible	10,000 to 30,000
Clearly perceptible	30,000 to 100,000
Annoying	Over 100,000

(b) Digby gives a nuisance-vibration velocity of 86,000 to 250,000 microinches per second and a faintly-perceptible-vibration of 25,000 to 63,000

* From Steffens (1952)

o the study of building vibrations. With relation to the effect of vibrations on people he concluded that the criteria presented by Reiher and Meister (1931) for any particular frequency cover reasonably well the values given by other investigators. The Reiher and Meister limits, established by tests, are shown in Fig. 10-1. Steffens also included tables which describe the ranges of intensity for the "Modified Mercalli Scale" of 1931 (abridged), Seiberg's "Mercalli–Cancani Scale" (abridged), the "Rossi–Forel Scale," and the 'Omori Scale" (abridged) for earthquakes and attempted to apply these to ndustrial vibrations. Generally, he found that the information provided by the earthquake scales was of little use for the industrial-vibration problem. He found the Reiher and Meister data useful and suggested that the Zeller cale, based on a unit called the "Pal," might have potential. The Pal is letermined by 10log 2X (see for example Zeller, 1933) in which X is equal to $16\pi^4 z^2 f^3$ or $16\pi^4$ (amplitude)^2 × (frequency)^3 of the exciting motion. The ibrations considered by Zeller ranged from 0 to 80 Pal; a vibration equivaent to 55 Pal causes seasickness and a vibration of 70 Pal causes a painful ensation to people. Finally, Steffens included a table containing typical ibration data from traffic, blasting, and machinery. These data are reproduced herein as Table 10-6.

The problem of vibrations produced by impact loads on soils or soilupported structures is of considerable importance. In Table 10-6 only one ase for pile driving and two cases for blasting are noted, but these are routine onstruction procedures, and many consulting firms, contractors, and inurance companies have extensive files relating to vibrations generated by pile driving and blasting. One criteria for evaluating the influence of impact or ibratory energy on soils and structures is the "energy ratio" (Crandell, 1949) given by

$$\text{E.R.} = \frac{(\text{Acceleration})^2}{(\text{Frequency})^2}\left(\frac{\text{ft}}{\text{sec}^2}\right)^2\left(\frac{\text{sec}}{\text{cycles}}\right)^2 \qquad (10\text{-}1)$$

The energy ratio decreases with distance from the source with the rate of ecay depending on the type of soil and local conditions; but Crandell ndicates a general trend of decrease of energy ratio according to (distance)$^{-2}$. From his study Crandell concluded that damage to structures did not occur when the energy ratio produced by blasting was less than 3. The concept of he energy ratio has also been selected as a criterion for evaluating the excitaon required to compact cohesionless soils (D'Appolonia, 1966). However a ower limit of energy ratio which does *not* affect soil structure does not seem o be well-defined at the present time (1969). Tschebotarioff (1965) has ointed out at least one instance for which energy ratios of 0.01 to 0.001 eveloped by repeated impacts from pile driving have caused serious settleent of soils in the vicinity of the construction. Consequently, the total nflu[illegible] vibratory loading on soil structure [illegible] be evaluated by a simple

limit indicated by the energy ratio because the *number of repetitions* is also important.

Several bits of information on the vibrations resulting from traffic are noted in Table 10-6. It is important to evaluate the influence of traffic and other background vibrations when considering the design criteria for a particular installation. The evaluation should include a range of the potential variables involved, such as the type and frequency of vehicles passing; the roughness of the road surface, including the effects of ice in cold climates, the influence of changing soil conditions, including frost and seasonal moisture variations; and the possible effects of the changes in soil geometry during construction, on the wave-energy transmission from the traffic source to the selected site. Several of these factors were investigated by Sutherland (1950), who found the effect of irregularities in the road to be the most significant. The ambient or background vibration level at a particular location is of particular significance when foundations for sensitive equipment are to be designed.

Foundations for Sensitive Equipment

Occasionally it is necessary to design a foundation which is "vibrationfree." This is impossible, of course, but it indicates an extremely low value of permissible motion. This requirement is often specified for sensitive equipment such as electron microscopes, calibration test stands, precisionmachining operations, and radar towers. For installations in which the equipment itself is not a significant source of vibration, it is necessary to evaluate the ambient vibrations at the site and then to provide isolation of the foundation and of the individual pieces of equipment by the methods indicated in Chap. 8. Generally, the design criteria should be established by the owner or equipment manufacturer because they must be satisfied with the eventual operation of the equipment. In the case of the electron microscope, a limiting criteria of $10^{-4}g$ at the machine has been established by tests (Sell 1963). In this particular instance, local isolation pads may provide about one order of magnitude of reduction in g values. Criteria for calibration test stands and similar facilities are often of the order of $10^{-4}g$ with some variations according to the frequency of input vibrations. The interesting part of the requirement of many of these "vibration-free" facilities is that they are often located relatively close to a major source of vibrations, which complicates the isolation problem.

Radar tracking towers are one type facility which require stable foundations and are at the same time often located near rocket-launching faci[illegible] For satisfactory operation of the equipment, Pschunder (196[illegible]) has [illegible] that the total allowable tilt of the tower is of the order of ¼ of the total pointing error. This includes the tilt induced by the flexibility of the tower structure in addition to the tilt introduced by rocking of the foundation mat. Maxwell (1965) has noted that typical values for the angular rotation in tilting of radar towers is often of the order of 0.02 mils (1020 mils = 1 radian, or 1 mil = 0.05617°); similar limits apply to the torsional motions. In addition, criteria for radar-tower foundations usually include a range of resonant frequencies to be avoided because of resonance in the structural or electrical systems. Each particular type radar tower has its own set of design criteria which must be satisfied by the foundation designer.

10.3 Dynamic Loads

Before a satisfactory design can be made for a machine foundation, it is necessary to obtain as much information as possible about the magnitude and characteristics of the dynamic loads involved. Often this is a relatively difficult task because manufacturers may not wish to admit that any unbalanced forces occur from operation of their equipment. However, there are certain basic types of equipment for which the unbalanced forces can be calculated, and a brief discussion of some of these are included in the following paragraphs. There is a definite need for reliable measurements of machine-induced forces which are transmitted to foundations, and the reader is encouraged to obtain this kind of information at every opportunity.

Rotating Machinery

Rotating machinery designed to operate at a constant speed for long periods of time includes turbines, axial compressors, centrifugal pumps, turbogenerator sets, and fans. In the case of each it is possible, theoretically, to balance the moving parts to produce no unbalanced forces during rotation. However, in practice, some unbalance always exists, and its magnitude includes factors introduced by design, manufacture, installation, and maintenance. These factors may include an axis of rotation which does not pass through the center of gravity of a rotating component; an axis of rotation which does not pass through the principal axis of inertia of a unit, thereby introducing longitudinal couples; gravitational deflection of the shaft; misalignment during installation, damage, corrosion, or wear of moving parts; improper tightening of components; or unbalances introduced by movement of materials being processed. The cumulative result of the unbalanced forces must not be great enough to cause vibrations of the machine–foundation system which exceed the design criteria. When excessive vibrations do occur, the obvious remedy is to reduce the unbalanced forces.

In certain types of machines, unbalanced forces are developed on purpose. Compaction machinery often contains unbalanced masses which rotate at a fixed eccentric radius about either a horizontal or vertical axis. The vibratory rollers for surface compaction have a horizontal axis of rotation, whereas the Vibroflot has a vertical axis. In either case the exciting-force amplitude can be evaluated from

$$Q_o = m_e e \omega^2 \qquad (10\text{-}2)$$

in which m_e is the total unbalanced mass and e is the eccentric radius to the center of gravity of the total unbalanced mass. The force transmitted to the soil by this type compacting machinery depends on the resulting motion of the contact faces of the machine against the soil. It is the purpose of this type device to produce inelastic deformations in the soil, and much of the input energy is absorbed in changing the soil structure.

Another type rotary mechanism which often develops unbalanced forces is the solid-waste shredder, rotary rock crusher, or hammermill. In this kind of machine a row of heavy steel weights or "hammers" are attached to disks or arms which rotate at relatively high speeds (See Fig. 10-4). When solid wastes, rocks, or automobile bodies are fed into this machine, the material is smashed against a slotted anvil by the rotating hammers, thereby fracturing or shredding the material because of the high shearing stresses involved. During this shredding operation the hammers are worn down. This changes the total rotating weight and possibly the eccentricity if the hammers wear unevenly or are not matched during replacement. Because of the need for frequent replacement of the hammers in this kind of machine, the operating unbalanced forces depend on the owner's maintenance procedure.

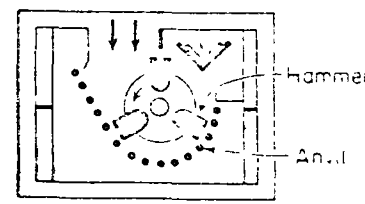


Figure 10-4 Elements of a hammermill

Multimass Vibrators

The centripetal force developed by a single rotating mass Eq. (10-2) is a vector force Q_o which acts outward from the center of rotation (Fig. 10-5a). By combining two rotating masses on parallel shafts within the same mechanism, it is possible to produce an oscillating force with a controlled direction. As shown in Fig. 10-5b, counterrotating masses can be so arranged that the horizontal-force components cancel but the vertical components are

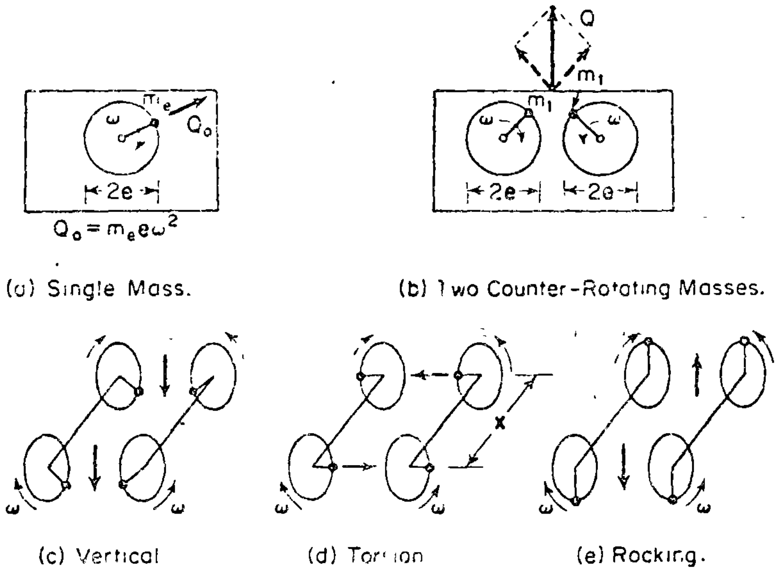


Figure 10-5 Forces from rotating mass exciters

added. If each mass m_1 has an eccentricity e, the *vertical* force produced is

$$Q = Q_o \sin \omega t = 2m_1 e\omega^2 \sin \omega t \tag{10-3}$$

In order to improve the flexibility of this type of vibrator, *four* masses may be arranged with one at each end of the two parallel shafts. A vertical oscillating force is developed by the arrangement of weights shown in Fig. 10-5c, a torsional couple about a vertical axis results from the arrangement of Fig. 10-5d, and a rocking couple is produced when the masses are located as in Fig. 10-5e. Vibrators of this design often include mechanisms which permit different settings of the eccentricities of the weights (see, for example, Bernhard and Spaeth, 1928; Hertwig, Früh, and Lorenz, 1933; or Fry, 1963). Note that for the rocking or torsional forces developed from the four-mass exciter, the torque or moment is given by

$$T = 4m_1 e \frac{x}{2} \omega^2 \sin \omega t \tag{10-4}$$

in which x represents the distance between the weights at the ends of each shaft.

Usually these multimass vibrators are designed such that adjustments of the size of the weights or of the throw of the eccentric are fixed before each

test and then maintained constant during the test. For this situation the exciting force increases as the square of the rotating speed.

Single-Cylinder Engines

Internal-combustion engines, piston-type compressors and pumps, steam engines, and other machinery involving a crank mechanism produce reciprocating forces. The crank mechanism transfers a reciprocating motion to a rotary motion, or vice versa. After the weight and center of gravity of each of the moving parts are determined, the forces resulting from operation of the machine can be evaluated.

The basic crank mechanism is shown in Fig. 10-6. It consists of a piston which moves vertically within a guiding cylinder, a crank of length r which rotates about point O, and a connecting rod of length L attached to the piston at point p and to the crank at point C. Thus, the crank pin C follows a circular path, while the wrist pin p oscillates along a linear path. Points on the connecting rod between C and p follow elliptical paths.

If the crank is assumed to rotate at a constant angular velocity ω, we may evaluate the acceleration of the piston along its axis of translation. In Fig. 10-6 the vertical displacement of the piston is z_p, measured from the top dead-center position at which ωt is taken as zero. The total motion of the piston is

$$z_p = \overline{DB} + L(1 - \cos \phi) \tag{10-5}$$

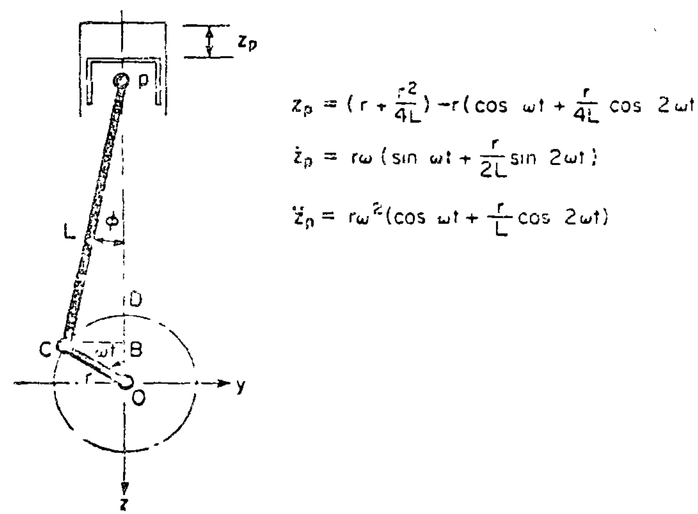


Figure 10-6 Crank mechanism

From the geometry of the diagram (Fig. 10-6),

$$\overline{DB} = r(1 - \cos \omega t) \tag{10-6}$$

and

$$\sin \phi = \frac{r}{L} \sin \omega t \tag{10-7}$$

With the introduction of expressions obtained from Eqs. (10-6) and (10-7) into Eq. (10-5), the displacement becomes

$$z_p = r(1 - \cos \omega t) + L\left(1 - \sqrt{1 - \frac{r^2}{L^2} \sin^2 \omega t}\right) \tag{10-8}$$

Because the ratio r/L is seldom greater than $\frac{1}{4}$, the expression beneath the radical in Eq. (10-8) can be replaced by the first two terms of the expansion into a power series by the binomial theorem, or

$$\sqrt{1 - \frac{r^2}{L^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2L^2} \sin^2 \omega t \tag{10-9}$$

and after substituting, Eq. (10-8) becomes

$$z_p = r(1 - \cos \omega t) + \frac{r^2}{2L} \sin^2 \omega t \tag{10-10}$$

The $\sin^2$ term in Eq. (10-10) can be represented by its equivalent expression for the double angle in order to simplify the differentiation of the displacement expression; thus,

$$z_p = \left(r + \frac{r^2}{4L}\right) - r\left(\cos \omega t + \frac{r}{4L} \cos 2\omega t\right) \tag{10-11a}$$

Then the velocity and acceleration are

$$\dot{z}_p = r\omega\left(\sin \omega t + \frac{r}{2L} \sin 2\omega t\right) \tag{10-11b}$$

and

$$\ddot{z}_p = r\omega^2\left(\cos \omega t + \frac{r}{L} \cos 2\omega t\right) \tag{10-11c}$$

The expressions for velocity and acceleration provide the momentum and inertia forces for the piston after multiplying by its mass. Note that one term

varies with the *same frequency* as the rotation; this is called the *primary term* The term which varies at *twice* the frequency of rotation is called the *secondary term* The importance of the secondary term is established by the ratio r/L. If the connecting rod is infinitely long the secondary term disappears and the piston executes harmonic motion For a connecting rod of finite length the motion of the piston is periodic but not harmonic Figure 10-7 illustrates the influence of the secondary term on the piston acceleration for a crank mechanism having $r/L = \frac{1}{4}$.

With Eqs. (10-11) to describe the dynamic characteristics of the piston, we may now consider the rotating parts of the crank. If there is any unbalance in the crankshaft, this may be replaced by a mass concentrated at the crank pin C, which produces the same inertia forces as the original system The vertical motion of point C is

$$z_C = r(1 - \cos \omega t) \tag{10-12a}$$

from which the velocity and acceleration are

$$\dot{z}_C = r\omega \sin \omega t \tag{10-12b}$$

and

$$\ddot{z}_C = r\omega^2 \cos \omega t \tag{10-12c}$$

The horizontal components are

$$y_C = -r \sin \omega t \tag{10-13a}$$

$$\dot{y}_C = -r\omega \cos \omega t \tag{10-13b}$$

$$\ddot{y}_C = r\omega^2 \sin \omega t \tag{10-13c}$$

The motions of the piston and the crank have now been established, leaving the characteristics of the connecting rod to be determined. Because the wrist pin p follows a linear path, the crank pin C a circular path, and all

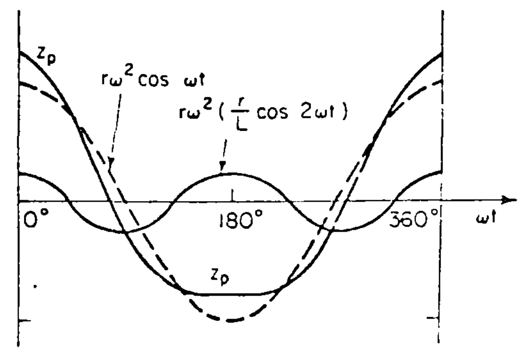


Figure 10-7 Piston acceleration as function of crank angle for $r/L = \frac{1}{4}$

points in between an elliptical path, the exact evaluation of the motions and resulting forces developed by the connecting rod is fairly complicated. However, it is satisfactory to replace the connecting rod by an equivalent structure consisting of a mass at the wrist pin and a mass at the crank pin which produce the same total mass and the same center of gravity. This procedure is correct for evaluating the inertia forces but is an approximation when establishing the inertia couple.

After adopting this procedure of dividing the connecting rod into two masses, one moving with the piston (reciprocating) and one moving with the crank pin (rotating), the total reciprocating and rotating masses may be designated by m_{rec} and m_{rot}, respectively. Then the total vertical-inertia force F_z and total horizontal-inertia force F_y are given by

$$F_z = (m_{rec} + m_{rot})r\omega^2 \cos \omega t + m_{rec}\frac{r^2}{L}\omega^2 \cos 2\omega t \qquad (10\text{-}14)$$

and

$$F_y = m_{rot}r\omega^2 \sin \omega t \qquad (10\text{-}15)$$

It should be noted that the vertical force has both a *primary* component acting at the frequency of rotation and a *secondary* component acting at twice that frequency. The horizontal force has only the *primary* component.

The torque of the inertia forces can also be evaluated from the arrangement of the masses and the geometry described in the preceding paragraphs. The torque is about the longitudinal axis O (perpendicular to plane of figure) in Fig. 10-6 and represents the torque acting on the shaft in the direction of rotation or the torque on the frame in the opposite direction. Its magnitude is given by

$$M = -m_{rec}\omega^2 r^2 \sin \omega t\left(\frac{r}{2L} + \cos \omega t + \frac{3r}{2L}\cos 2\omega t\right) \qquad (10\text{-}16)$$

By "counterbalancing," the inertia forces due to the rotating masses can be reduced or eliminated completely. Usually this is done in the design of internal-combustion engines or piston-type pumps or compressors. However, the reciprocating mass still produces unbalance in a simple system corresponding to that shown in Fig. 10-6. Thus, *a single-cylinder engine is inherently unbalanced*.

The following example illustrates the calculation for the primary and secondary unbalanced forces for a single-cylinder engine. Typical data for two single-cylinder engines are given in Table 10-7.

It will be assumed that the rotating mass is balanced. Then Eq. (10-14) is reduced to

$$F_z = m_{rec}r\omega^2 \cos \omega t + m_{rec}\frac{r^2}{L}\omega^2 \cos 2\omega t$$

Table 10-7. Data for Single-Cylinder Engines

Bore (in.)	Stroke (in.)	Crank r (in.)	Rod L (in.)	$\frac{r}{L}$	Piston Weight (lb)	Pin Weight (lb)	Rod Weight (lb)	Total Recip Weight (lb)	Total Engine Weight (lb)
5¾	8	4	15	0 267	10 6	2 9	18 0	19 0	2270
5⅛	6½	3¼	10¾	0 302	7 19	1 88	11 18	11 87	2270

For the 5¾-by-8-in. single-cylinder engine the amplitude of the primary force is

$$F' = \frac{19}{386}\,4\,\frac{4\pi^2}{(60)^2}\quad(\text{rpm})^2 = 0.00216\quad(\text{rpm})^2$$

and the amplitude of the secondary force is

$$F'' = \frac{r}{L}F' = (0\ 267)F' = 0.000576\quad(\text{rpm})^2$$

Then at an operating speed of 1000 rpm, these force amplitudes amount to

$$F' = 2160\quad\text{lb}$$

and

$$F'' = 576\quad\text{lb}$$

which are significant when compared with the machine weight of 2270 lb. Note that for this machine 30.6 per cent of the connecting-rod weight was considered to act as part of the reciprocating weight.

Multicylinder Machines

In multicylinder engines and compressors it is possible to arrange the cylinders in a manner which minimizes the unbalanced forces. Table 10-8 illustrates the forces developed by multicylinder machines for different crank arrangements and numbers of cylinders. For a particular machine the unbalanced primary and secondary forces as well as the torques should be available from the manufacturer because these quantities were required for the original design of the machine.

It should be noted that Table 10-8 illustrates the unbalanced forces developed for multicylinder engines having the same bore and stroke for each cylinder. For this condition the six-cylinder engine can be completely balanced. Consequently, a V-12 engine made up of two in-line 6's would also be balanced. However, if the bore and stroke of the cylinder [illegible]

Table 10-8. Unbalanced Forces and Couples for Different Crank Arrangements (after Newcomb, 1951)

Crank Arrangements	Forces		Couples	
	Primary	Secondary	Primary	Secondary
Single crank	F' without counterwts (0.5)F' with counterwts	F''	None	None
Two cranks at 180° In-line cylinders	0	2F''	$F'D$ without counterwts. $\frac{F'}{2}D$ with counterwts	None
Opposed cylinders	0	0	Nil	Nil
Two cranks at 90°	(1.41)F' without counterwts (0.707)F' with counterwts	0	(1.41)$F'D$ without counterwts (0.707)$F'D$ without counterwts	$F''D$
Two cylinders on one crank Cylinders at 90°	F' without counterwts 0 with counterwts	(1.41)F''	Nil	Nil
Two cylinders on one crank Opposed cylinders	2F' without counterwts F' with counterwts	0	None	Nil
Three cranks at 120°	0	0	(3.46)$F'D$ without counterwts (1.73)$F'D$ with counterwts	(3.46)$F''D$
Four cylinders Cranks at 180°	0	0	0	0
Cranks at 90°	0	0	(1.41)$F'D$ without counterwts. (0.707)$F'D$ with counterwts	4.0$F''D$
Six cylinders	0	0	0	0

r = crank radius (in)
L = connecting-rod length (in)
D = cylinder-center distance (in)
W = recip wt of one cylinder (lb)
F' = (0.0000284) r W (rpm)^2 = Primary
F'' = $\frac{r}{L}F'$ = Secondary

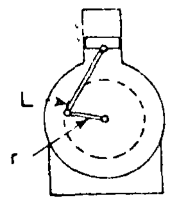
same, then Table 10-8 should not be used, but the unbalanced forces should be computed for each cylinder and the results superposed.

Table 10-9 illustrates the order of magnitude of forces which may be developed by one- and two-cylinder engines or compressors. For the single-cylinder engines oriented vertically, both the primary and secondary vertical forces are significant. An engine usually drives some additional machinery, and a compressor needs something to drive it. Consequently, these forces

Table 10-9. Order of Magnitude of Forces Developed by One- and Two-Cylinder Engines

Single-Cylinder Engine - Vertical

Bore = 5.125 in
Stroke = 6.5 in
r = 3.25 in
L = 10.75 in
Total Wt = 2270 lb
Reciprocating Weight = 11.87 lb
Operating Speed = 1800 rpm
Unbalanced Forces
Primary = 3450 lb at 1800 rpm
Secondary = 1075 lb at 1800 rpm



Single-Cylinder Compressor - Vertical

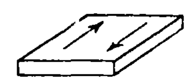
Bore = 14.5 in
Stroke = 9 in
Total Wt = 10,900 lb
Forces at Operating Speed of 450 rpm
Vert { Primary = 9180 lb; Secondary = 2210 lb }
Horiz { Primary = 310 lb; Secondary = 0 }

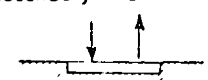
Horizontal Compressor - 2 Unequal Cylinders

Low Pressure Cyl Bore = 23", Stroke = 14", Unbal Wt = 1130 lb
High Pressure Cyl Bore = 14", Stroke = 14", Unbal Wt = 890 lb,
Wt Compressor = 22,400 lb

Unbalanced Forces at 277 rpm
Horiz { Primary = 6190 lb; Secondary = 730 lb } Vert { Primary = 5300 lb; Secondary = 0 }

Unbalanced Moment at 277 rpm
Horiz { Primary = 22,400 ft lb; Secondary = 11,300 ft lb } Vert { Primary = 19,300 ft lb; Secondary = 0 }





shown on Table 10-9 constitute only a part of the loads acting on a partic foundation. The response of the foundation will depend on the resul forces developed from the entire package of machinery and the locatio these forces.

The two-cylinder horizontal compressor described on Table 10-9 i trates the additional problems associated with the motion of two horizo in-line cylinders of different bore. Because of the large moving masse volved, the vertical components of the primary force and moment are no as large as the horizontal components. Furthermore, forces are devel which will cause horizontal and vertical transla[illegible] twisting of the foundation for this machine. It is also probable that pitching and lateral motions would occur because of coupling. Consequently, the forces developed by this two-cylinder compressor would excite vibrations of its foundation in all six degrees of freedom.

In general, multicylinder engines have smaller unbalanced forces than do the one- and two-cylinder engines and compressors. However, in each case it is necessary to evaluate the influence of the unbalanced forces and couples on the response of the machine foundation in *all six* modes of vibration.

Forces from Vibratory Conveyors

Vibratory conveyors are often used to transport masses of solid particles. Figure 10-8 illustrates the elements of this type machine, which consists essentially of the conveyor trough, supporting springs, a reaction block, and a motor–crank-drive mechanism. The springs may be either leaf springs, as shown, or coil springs Horizontal movements of particles in the conveyor trough are developed by oscillation of the trough along a path which produces a forward and upward acceleration of particles, then a backward and downward acceleration. This causes the particles to move forward in a small "hop" each oscillation The sketch in Fig. 10-8 shows leaf springs inclined at 60° from the horizontal, which forces the conveyor trough to move along a path inclined at 30° from the horizontal. A crank mechanism provides the input oscillation at a given amplitude of motion. Usually there is a speed-reduction system between the motor and crank-drive mechanism to provide a wide range of operating frequency of oscillation. If the operating frequency is "tuned" to the resonant frequency of the mass–spring system, then minimum input force is required to maintain the oscillation. The moving mass is primarily the weight of the conveyor trough but may include a portion of the transported material.

The foundation block is required to absorb an inclined oscillating force, indicated as Q_o in Fig. 10-8. There may also be a pitching moment developed if the line of action of the resultant force Q_o does not coincide with the center of the resistance developed by the foundation block. In any case, a horizontal

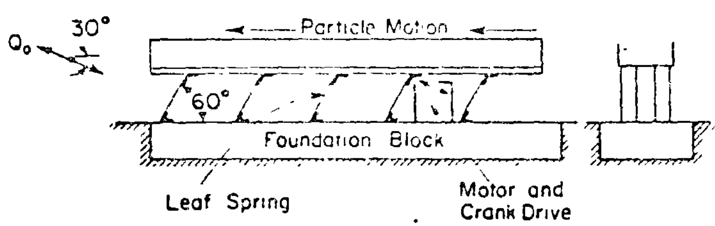


Figure 10-8 Elements of vibratory conveyor

and a vertical force will be transmitted to the supporting soil, and wave energy will be propagated from the foundation block outward. Because the horizontal component of the force applied to the soil is predominant, the elastic waves propagating outward from the foundation will have a greater intensity along the axis of conveyor vibration. This directional effect should be considered in the layout of vibrating conveyor systems if there is any vibration-isolation problem in the vicinity.

As indicated in Fig. 10-8, the oscillating force is inclined at an angle to the horizontal and has a magnitude

$$Q_o = me4\pi^2 f_o^2 \tag{10-17}$$

in which m is the total oscillating mass, e is the crank throw or eccentricity, and f_o is the operating frequency of vibration. This force is then broken down into its horizontal and vertical components in order to estimate the dynamic motions of the foundation block.

Loads Developed by Intermittent Machine Operation—Pulse Loads

Many types of machines produce intermittent loads which must be transmitted through the foundation block to the supporting soil The operation of punch presses forging hammers, drop tests, and stamping machines, for example, produce impulsive loads which may be considered as single pulses because the effect of one load dies out before the next load occurs

In order to evaluate the response of a foundation block to a pulse-type load, it is necessary to have reliable information about the force-time relation of the pulse. This information is often not readily available, and the reader is encouraged to obtain this information experimentally whenever possible Two pulse loads will be described to illustrate the needed information and to provide loads which will be employed later (in Sec. 10 6) in the evaluation of the dynamic response of footings. The pulse shown in Fig 10-9b was obtained experimentally from a load-sensitive column which supported a loading platen on a model footing (see Drnevich and Hall, 1966) Figure 10-9 illustrates the general test setup in which the load was applied by dropping a 5-lb sandbag a distance of 1 ft onto the loading platen The shape of the pulse and in particular the rise time were controlled by placing different thicknesses of foam-rubber sheets on the surface of the loading platen The solid curve in Fig. 10-9b represents the experimental load-time pulse for test Q-2 and the dashed rectangles constitute a step-type approximation

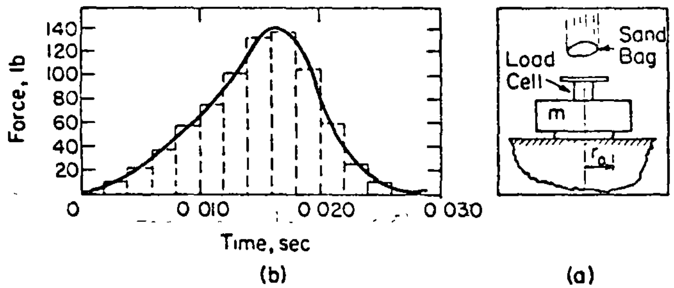
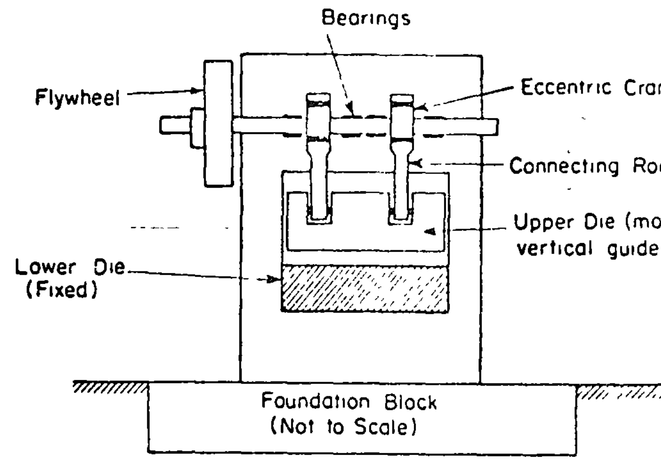


Figure 10-9. Impact load on model footing (a) Test setup showing sandbag dropping onto loading platen to force mass m to load soil through circular footing having $r_0 = 6$ in. (b) Load-time pulse measured in load-sensitive column for test Q-2 (*after Drnevich and Hall, 1966*).

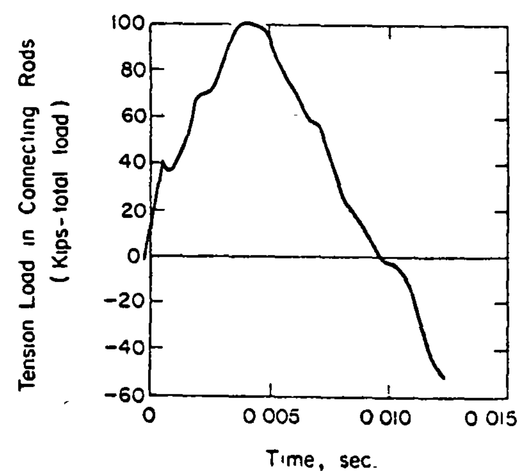
the loading pulse which will be used in the phase-plane solution for the footing response.

Figure 10-10a shows the elements of a punch press—a machine which punches shapes from sheet metal by forcing a moving upper die against a fixed lower die. The upper die moves vertically in guides and is actuated by an eccentric crank and connecting rods. As the upper die is forced downward against the metal to be processed, compression loads are built up in the connecting rods. These forces are resisted by developing tension in the structural frame; thus, the loads are contained within the machine. However, when the upper die punches through the sheet metal, there is a sudden release of this compressive energy in the connecting rods along with a loss of support for the weight of the upper die. These two forces accelerate the mass of the upper die in a downward direction. This results in a tension shock imparted to the connecting rods as they stop the motion of the die. Because this tension shock in the connecting rods is developed by inertia loads, it is an external load on the machine. The result is a downward force on the crankshaft which then causes motion of the entire machine and foundation block.

Figure 10-10b shows the tensile forces developed in the connecting rods of a punch press which has a capacity of 250 tons of compressive force between the dies. In this case, it is seen that the maximum transient tensile load in the connecting rods (or downward force on the machine and foundation) was of the order of 40 per cent of the rated capacity of the machine. This transient-load pulse reached a maximum of about 100,000 lb, but the pulse duration was only about 0.010 sec. This loading pulse, the weight of the machine and its moving parts, and soil data obtained by one of the methods described in Chap. 6 provide the information needed to design a foundation which will restrict the motion of the machine to acceptable limits.



(a) Elements of Punch Press Mechanism.



(b) Force-Time Pulse for Tension Load Developed in Connecting Rods after Upper Die Punches Through Metal Stock

Figure 10-10. Impulse developed by punch press operation.

Random Vibrations

Earthquakes, wind, and certain manmade forces have a random pattern which provides excitation to structures and foundations. In order to establish force or displacement patterns to be applied as design loads, *it is necessary to obtain reliable field data from previous excitations considered similar to the proposed design conditions.* For example, ground-motion records from previous earthquakes are often used to represent probable earthquake excitations when analyzing the dynamic response of a proposed structure.

10.4 Brief Review of Methods for Analyzing Dynamic Response of Machine Foundations

DEGEBO

An intensive study of the effect of vibrations on foundation response and upon soil properties was carried out by the Deutschen Forschungsgesellschaft für Bodenmechanik (DEGEBO) in Berlin primarily during the period 1928–1936. This group developed a rotating-mass mechanical oscillator with four eccentric masses to excite model footings into the vertical and torsional modes of vibration. The drive mechanism for the eccentric masses was arranged as shown in Fig. 10-11, with four shafts driven by bevel gears—instead of four masses on two parallel shafts as shown in Fig. 10-5c and d; but the principle is the same.

In the first major report on their investigations (DEGEBO No. 1) Hertwig, Früh, and Lorenz (1933) described the test equipment and included an extensive evaluation of the dynamic response of the oscillator and footing plate in vertical vibration. They attempted to fit the test results into the framework of the single-degree-of-freedom mass–spring–dashpot system and found it possible to do so for any particular test. However, the damping constant, in particular, was appreciably different for different tests. They noted that the dynamic response was nonlinear and that progressive settlements developed during vibration tests when the oscillator was supported on sand. The dynamic response was found to depend on the total weight of the oscillator and base plate, on the area of the base plate, and on the dynamic

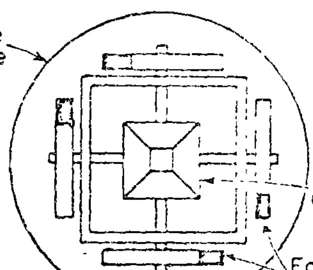


Figure 10-11. DEGEBO Oscillator (top view)

force applied as well as on the characteristics of the soil. At the end of the report is a table which indicates the "characteristic frequency" for a variety of soils. This table has been reproduced many times in the literature, to the point that many people believe that soil has a "natural frequency" and attempt to use this value in design. The table represents information obtained from a particular set of test conditions and should be considered only as interesting qualitative information.

In subsequent publications (for example, Lorenz, 1934) the effect of oscillator weight, base-plate area, and exciting force were studied for their influence on the resonant frequency. It was found that for the same base-plate area and exciting force, increasing the total weight *lowered* the resonant frequency. For a constant total weight and exciting force, an increase in base-plate area *raised* the resonant frequency, and for a constant weight and constant base-plate area, an increase in exciting force *lowered* the resonant frequency. Hertwig and Lorenz (1935) obtained similar results for both vertical and torsional tests of footings on sand and on clay.

The change in frequency with a change in exciting force indicated that the soil response was nonlinear. This is true and it is particularly important at the magnitude of motions involved in the DEGEBO tests, which often involved vertical accelerations of the oscillator of more than ±1g. Thus for many tests the oscillator was acting as a hammer. A discussion of the influence of range of strain on the effective modulus of elasticity of soils was presented in Chap. 6; this influence on the design of machine foundations will be discussed in Sec. 10.7.

Methods Based on the "In-Phase Mass"

From the DEGEBO tests and subsequent analyses there developed the concept that a mass of soil moved with the footing. This is illustrated by the zone labeled m_s beneath the footing in Fig. 10-12. By working backward from the equation for the resonant frequency—

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m + m_s}} \qquad (10\text{-}18)$$

—one is able to evaluate m_s for each test. However, it was found that m_s varied with the dead load, exciting force, base-plate area, mode of vibration, and type of soil on which the oscillator rested.

In spite of the difficulties in obtaining specific values of m_s, the mass of soil that

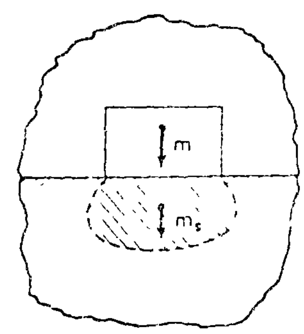


Figure 10-12 "In-phase mass" of soil

supposedly moves with the footing or is "in phase" with the footing, the concept has appeared periodically in the literature. Crockett and Hammond (1949) and Rao (1961), for example, have attempted to estimate a weight of soil within a "bulb of pressure" in order to force Eq. (10-18) to fit test results. These procedures are principally intuitive; reliable numbers are difficult to obtain for design purposes. Even if the "in-phase mass" could be determined satisfactorily, this information would not lead directly to an evaluation of the amplitude of vibration needed for design purposes. Consequently, at this stage of development of design procedures for dynamically loaded foundations, the "in-phase mass" is not a significant factor.

Tschebotarioff's "Reduced Natural Frequency"

In an attempt to improve the methods for evaluating the resonant frequency of machine foundations supported by different soils, Tschebotarioff and Ward (1948) and Tschebotarioff (1953) developed an expression for a "reduced natural frequency" of the system. Beginning with the DEGEBO expression (Eq. 10-18) for the resonant frequency of a foundation (including the effect of an "in-phase mass"), the spring constant k was replaced by $k'A$, where k' is the dynamic modulus of subgrade reaction (i.e., lb/ft³) and A is the contact area (ft²) of the foundation against the soil. With this substitution Eq. (10-18) takes the form

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k'A}{m + m_s}} \tag{10-19}$$

This equation was further rearranged to

$$f_n = \sqrt{\frac{A}{W}}\,\frac{1}{2\pi}\sqrt{\frac{k'g}{1 + \dfrac{m_s}{m}}} = \frac{1}{\sqrt{q_o}}f_{nr} \tag{10-20}$$

in which q_0 is the average vertical contact pressure between the base of the foundation and the soil, and the remaining terms are lumped together and called the "reduced natural frequency f_{nr}." Then from an evaluation of a limited number of case histories available at the time (1953), Tschebotarioff prepared curves which related f_{nr} to the base area of the foundation for several soils. These relations appear as straight lines on the log-log plot of Fig. 10-13a. In order to calculate the resonant frequency for a given size footing on a particular soil, Fig 10-13a gives a value of f_{nr} which then leads to f_n by the use of Eq. (10-20) and of the design value of q_o. It should be noted that

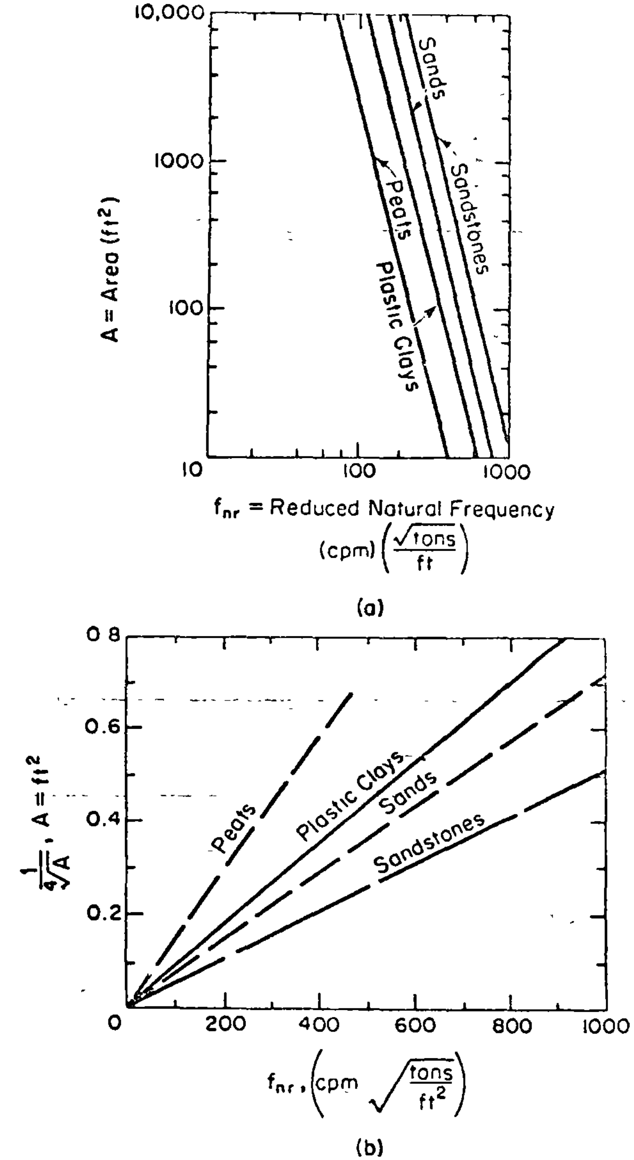


Figure 10-13. Tschebotarioff's "Reduced Natural Frequency"

this method gives *only* an estimate of the resonant *frequency* and te nothing about the amplitude of vibration.

Figure 10-13a may be replotted as f_{nr} vs. $A^{-1/4}$ to give the diagram sl in Fig. 10-13b. It is of interest to note that relations similar to the Fig. 10-13b can be predicted from the elastic-half-space theory describ

Chap. 7. For vertical vibrations the mass ratio **b** (Eq. 7-3) can be expressed as

$$\mathbf{b} = \frac{m}{\rho r_o^3} = \frac{W}{\gamma r_o^3} = \frac{W}{\frac{\gamma r_o}{\pi}\, \pi r_o^2} = \frac{q_o}{\frac{\gamma r_o}{\pi}} \tag{10-21}$$

and the dimensionless frequency a_{on} (Eq. 7-2) as

$$a_{on} = \omega r_o \sqrt{\frac{\rho}{G}} = 2\pi f_n r_o \sqrt{\frac{\rho}{G}} \tag{10-22a}$$

and

$$a_{on}^2 = 4\pi^2 f_n^2 r_o^2 \frac{\rho}{G} \tag{10-22b}$$

Then,

$$\mathbf{b}a_{on}^2 = \frac{4\pi^3 r_o}{gG} q_o f_n^2 = \frac{4\pi^{5/2}\sqrt{A}}{gG} q_o f_n^2 \tag{10-23}$$

if we substitute $A = \pi r_o^2$. Equation (10-23) represents the reduced natural frequency:

$$f_n\sqrt{q_o} = \sqrt{\frac{gG\mathbf{b}a_{on}^2}{4\pi^{5/2}}}\, \frac{1}{\sqrt[4]{A}} = K\frac{\sqrt{G}}{\sqrt[4]{A}} \tag{10-24}$$

For any particular **b**, a_{on} is fixed. Thus, Eq. (10-24) illustrates that for a constant value of G we get a linear relation between f_{nr} $(= f_n\sqrt{q_o})$ and $A^{-1/4}$. Therefore, the lines in Fig. 10-13b designated as peat, plastic clays, sands, and sandstones actually represent typical values of shear modulus G for these materials.

Method Based on the Dynamic Subgrade Reaction

One method for estimating the deflection of a loaded structure resting on soil involves replacing the soil by a set of independent elastic springs which produce an equivalent reactive force to the displacement developed. This concept has been designated as the theory of elastic subgrade reaction. It is discussed in the books by Hayashi (1921), Terzaghi (1943), and Hetenyi (1946), for example, and a comprehensive discussion of methods for evaluating the *coefficients* of elastic subgrade reaction was given by Terzaghi (1955).

Figure 10-14 illustrates the approximations involved in replacing the soil beneath a rigid foundation by a series of springs. Once the representative

values are chosen for the reaction springs, these values are fixed and there is no further modification of their behavior as a consequence of changing the total weight of the oscillating block (i.e., change in confining pressure in the soil) or of the amplitude of vibration (effect of strain). Furthermore, this elastic subgrade rests on a rigid base, and for the dynamic condition it represents a *closed* system. When the system shown in Fig. 10-14 is set into vertical vibration, it responds as an *elastic undamped* system with *amplitudes* of motion at resonance which *approach infinity*. Such a closed system does not include the damping of energy by radiation as does the elastic-half-space theory and gives *no* useful information on the amplitude of motion at frequencies near resonance. *This theory gives useful results only for the undamped natural frequency of vibration.*

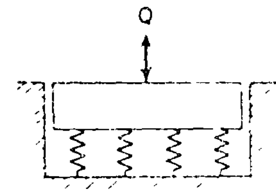


Figure 10-14 Springs replacing soil support to provide "dynamic subgrade reaction."

As indicated in the previous section, the coefficient of subgrade reaction is related to a spring constant for a given system by

$$k = k'A \tag{10-25}$$

in which the spring constant (lb/ft) is represented as the product of k' (lb/ft³ or pressure per unit displacement) multiplied by the foundation-contact area (A = ft²). Therefore, if we can obtain test information relating the applied load to the displacement we have evaluated k, and from Eq. (10-25) we can obtain k'.

Information from plate-bearing tests and field tests on foundations has been used to establish k for machine foundations. Barkan (1962) has cited numerous field tests which demonstrate that the spring constant applicable to dynamic motion is essentially equal to the ratio of increment of load to increment of deflection (or moment to rotation) during static repeated-loading tests. The resonant frequency observed during dynamic tests on a foundation block was compared with the undamped natural frequency computed using just the mass of the foundation block plus machinery and using the value of k measured during a static repeated-loading test on the same foundation block. From 15 data points from tests on foundations ranging from 5-ft² to 161-ft² base area resting on sand, clay, or loess, he found that the observed frequency averaged 97 per cent of the computed frequency and that the extremes ranged from 85 to 121 per cent. Therefore, it was considered that the procedure was satisfactory for estimating resonant frequencies.

The key to the procedure described by Barkan is the use of *repeated* loadings in the static tests. Furthermore, it is important that the magnitudes of the "dead load" and of the "live load" be similar to those anticipated under the actual foundation. Because of the small movements anticipated (note displacements allowed for prototype foundations given in Fig. 10-1), the process of obtaining reliable load-deformation data from model or prototype footings is not easy. Special instrumentation is usually required for these measurements and particular care is needed in carrying out the tests.

After field data are obtained from tests on model footings, the next problem involves extrapolating this information to prototype dimensions. The discussion by Terzaghi (1955) and others concerning the choice of a modulus of subgrade reaction for static loadings applies as well to the machine-foundation problem. Suggested methods for extrapolating test information are given by Terzaghi (1955) and are indicated below for vertical motions.

For cohesive soils:
$$k'_z = k'_{z_1}\frac{1}{2d} \tag{10-26a}$$

For cohesionless soils:
$$k'_z = k'_{z_1}\left(\frac{2d+1}{4d}\right)^2 \tag{10-26b}$$

in which

$2d$ = width of a beam, or least dimension of foundation base,

k'_z = coefficient of vertical subgrade reaction for base of least dimension of $2d$ (lb/ft³), and

k'_{z_1} = coefficient of vertical subgrade reaction for base of least dimension of 1 ft (lb/ft³).

Thus, the test data provide information for establishing the values of k'_{z_1} for the unit dimensions, then Eqs. (10-26a) and (10-26b) are used to adjust the subgrade coefficient to correspond to the prototype dimensions. This procedure is reasonable only when both the model footing and the prototype footing produce equivalent stresses in similar soils.

Barkan (1936, 1962) has utilized the concept of elastic-subgrade reaction extensively and has indicated spring constants for the various modes of vibration of rigid foundations in the following form:

For vertical motion:
$$k_z = k'_z A$$
For horizontal motion:
$$k_x = k'_x A$$
For rocking motion:
$$k_\psi = k'_\psi I' \tag{10-27}$$
For torsional motion:
$$k_\theta = k'_\theta I''$$

in which

A = contact area between foundation and soil,

I' = second moment of contact area about a horizontal axis normal to the plane of rocking through the centroid, and

I'' = second moment of the contact area about a vertical axis through the centroid.

The coefficients k'_z, k'_x, k'_ψ, and k'_θ are coefficients of subgrade reaction and are functions of soil type and of size and shape of the foundation. However, these are often assumed to be functions **only** of soil type. Barkan (1962) provided the data in Table 10-10 for k'_z and has suggested that the remaining coefficients can be evaluated as

$$k'_x \approx 0.5k'_z$$
$$k'_\psi \approx 2k'_z \tag{10-28}$$
$$k'_\theta \approx 1.5k'_z$$

The spring constants computed on the basis of Eq. (10-28) and Table 10-10 could be used for preliminary design when reliable soil information is not available. However, it is recommended that the procedures outlined in Secs. 10-6 and 10-7 be used for design purposes.

Table 10-10. Recommended Design Values for Subgrade Coefficient k'_z*

Soil Group	Allowable Static Bearing Stress (ton/ft²)	Coefficient k'_z (ton/ft³)
Weak soils (clay and silty clays with sand, in a plastic state, clayey and silty sands)	1.5	95
Soils of medium strength (clays and silty clays with sand, close to the plastic limit, sand)	1.5–3.5	95–155
Strong soils (clay and silty clays with sand, of hard consistency, gravels and gravelly sands, loess and loessial soils)	3.5–5	155–310
Rocks	5	310

* After Barkan (1962)

Elastic-Half-Space Theory

The representation of a foundation on soil by a footing resting on a semi-infinite elastic body was discussed extensively in Chap. 7. This elastic-half-space theory includes the dissipation of energy [illegible] by "geometrical damping." This theory permits calculation of finite amplitudes of vibration at the "resonant" frequency. The entire amplitude-frequency response curve may be obtained as well as the phase angle between the exciting force and footing motion and the input power required. Because the elastic-half-space theory is an analytical procedure, certain mathematical simplifications have been introduced which are not quite realistic. The footing is assumed to rest *on* the surface of the half-space and to have simple geometrical areas of contact, usually circular but occasionaly rectangular or a long strip. The half-space itself is assumed to consist of an ideal elastic, homogeneous, isotropic material. However, the analytical solutions serve as a useful guide for evaluation of the dynamic response of simple footings undergoing single modes of vibration. They also provide a rational means of evaluating the spring and damping constants which may then be incorporated into the lumped-parameter, mass-spring-dashpot vibrating system.

10.5 Lumped-Parameter Vibrating Systems

In a study by Richart and Whitman (1967) it was shown that the dynamic behavior of actual foundations could be predicted by the elastic-half-space theory. Furthermore, Lysmer (1965) had shown that vertical vibrations of a rigid circular footing on the elastic half-space could be represented quite satisfactorily by a mass-spring-dashpot system if the damping constant and spring constant were chosen correctly. Therefore, it followed that the lumped-parameter system represented by a mass, spring, and dashpot could be used to represent the motion of rigid foundations. The lumped-parameter system treats all the masses, springs, and damping components of the system as if they were lumped into a single mass, single spring, and single damping constant for each mode of vibration. A description of the lumped-parameter system equivalent to the half-space model for each mode of vibration was given in Chap. 7; Fig. 10-15 illustrates typical equivalent lumped systems for foundations subjected to vertical, horizontal, and torsional exciting forces. Note that in Fig. 10-15 the vertical and torsional excitations produce motion with a single degree of freedom but that the horizontal excitation produces a coupled motion involving both rocking and sliding

For a single degree-of-freedom system the lumped parameters lead to an equation of motion of the type

$$m\ddot{z} + c\dot{z} + kz = Q(t) \qquad (2\text{-}48)$$

in which

m = equivalent mass,

c = effective damping constant,

k = effective spring constant,

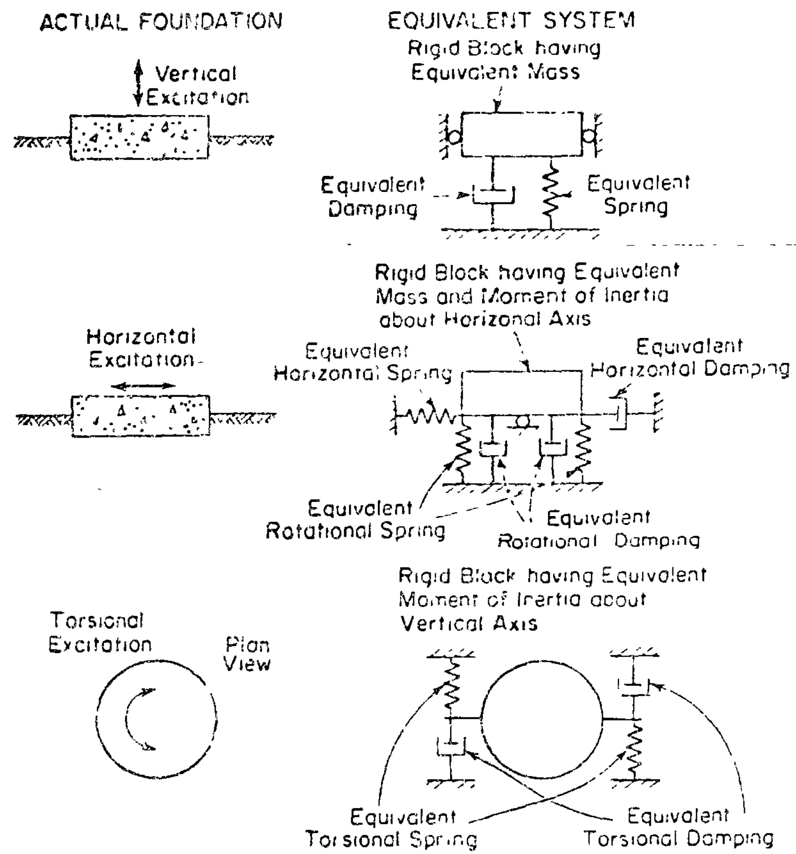


Figure 10-15. Typical equivalent lumped systems.

$Q(t)$ = time-dependent exciting force, and

$z, \dot{z}, \ddot{z}$ = displacement, velocity, and acceleration, respectively, of the mass in the direction of the chosen coordinate (in this example the vertical direction was chosen).

Analytical and graphical methods for treating the lumped-parameter systems were described in Chap. 2, and many books are available which include comprehensive discussions of this topic. Consequently, any procedure which permits a dynamically loaded foundation to be represented by lumped parameters simplifies our analysis of the foundation response.

In Chap. 2 it was noted that the exciting force $Q(t)$ in Eq. (2-48) can be expressed as ($Q_o \sin \omega t$), in which the force amplitude Q_o is either a constant or a function of the circular frequency ω. When the force amplitude is a function of the frequency it is evaluated from Eq. (10-2). The ex[illegible] which describe the response of a mass m to either t[illegible]

Table 10-11. Summary of Parameters Required for Dynamic Analysis

Analysis		Factors Required
Approximate estimate for resonant frequency		k and m
Approximate estimate for motion at frequencies well away from resonance	$\ll f_o$	k
	$\gg f_o$	m
Upper limit for motion at frequencies near resonant frequency		D and k or m

summarized in Table A-1 in the Appendix for the single-degree-of-freedom system. Note that the relations in Table A-1 apply to each of the six modes of vibration, but that the vertical coordinate z was chosen for the example From Table A-1 it can be seen that the lumped parameters influence different expressions relating to the response of the mass m; these effects are summarized in Table 10-11.

Choice of Mass for Equivalent Lumped Systems

The method recommended in this text for establishing the lumped parameters for the equivalent mass–spring–dashpot system is based on the elastic-half-space theory. *The lumped mass is chosen as the mass of the foundation and supported machinery.* Then the damping and spring constants are developed through the theory and have values as indicated in Table A-2 for the case of rigid circular footings.

The method based upon the concept of an "in-phase mass" of soil leads only to an estimate of the natural frequency of the system and gives no information relating to the amplitude of vibration at resonance. Consequently, this method is not satisfactory for determining the lumped parameters for a vibrating system which includes damping.

Choice of Damping for Equivalent Lumped Systems

The dashpots of the lumped system represent the damping of the soil in the foundation–soil system. There are two types of damping in the real system: one introduced by the loss of energy through propagation of elastic

waves away from the immediate vicinity of the footing, the other associated with internal energy losses within the soil due to hysteretic and viscous effects. The equivalent damping corresponding to the elastic-wave propagation has been designated as "geometrical damping" (Chap. 7) or is occasionally called "radiation damping" Expressions for the damping ratio D obtained through the half-space theory and corresponding analogs for *rigid circular footings* are summarized in Table A-2 This information is also shown in graphical form in Fig. 7-19.

The equations and diagrams for geometrical damping developed by vibrations of a rigid circular footing on the elastic half-space may also be used to provide estimates for the geometrical damping developed by footings with rectangular-plan form. This is accomplished by converting the rectangular base of dimensions $2c$-by-$2d$ into an equivalent circular base having a radius r_o, determined by the following:

$$\text{For translation:} \quad r_o = \sqrt{\frac{4cd}{\pi}} \tag{10-29a}$$

$$\text{For rocking:} \quad r_o = \sqrt[4]{\frac{16cd^3}{3\pi}} \tag{10-29b}$$

$$\text{For torsion:} \quad r_o = \sqrt[4]{\frac{16cd(c^2 + d^2)}{6\pi}} \tag{10-29c}$$

in which

$2c$ = width of the foundation (along axis of rotation for the case of rocking), and

$2d$ = length of the foundation (in the plane of rotation for rocking)

The internal damping in soils has been discussed in Chap. 6. Table 10-12 summarizes some of the available information relating to internal damping of soils at the level of stress changes occurring under machine foundations. (Where the test results are given as damping capacity or log decrement, they are expressed in terms of an equivalent damping ratio D.) From Table 10-12 it is evident that a typical value of D is on the order of 0.05 for internal damping in soils.

The lumped damping parameter for any particular foundation soil system will include both the effects of geometrical and internal damping. If we take the value of 0.05 to represent a typical internal-damping ratio, then by comparing this value with the geometrical damping from Fig. 7-19, we can estimate the contribution of each. It is evident from this examination that for vibrations in translatory modes the geometrical damping overshadows so the internal damping to the point where the latter may be disregarded

Table 10-12. Some Typical Values of Internal Damping in Soils

Type Soil	Equivalent D	Reference
Dry sand and gravel	0.03–0.07	Weissmann and Hart (1961)
Dry and saturated sand	0.01–0.03	Hall and Richart (1963)
Dry sand	0.03	Whitman (1963)
Dry and saturated sands and gravels	0.05–0.06	Barkan (1962)
Clay	0.02–0.05	Barkan (1962)
Silty sand	0.03–0.10	Stevens (1966)
Dry sand	0.01–0.03	Hardin (1965)

preliminary analyses. On the other hand, for the rotary modes of vibration—torsion and rocking—the geometrical damping is small and, for rocking in particular, these two damping terms may be of the same order of magnitude. In this case, the internal damping is important and should be included.

This comparison of the effectiveness of geometrical damping and internal damping illustrates the value of the elastic-half-space theory in establishing values for geometrical damping for motions of simple footings in each of the modes of vibration. The values of geometrical damping thus obtained should be considered as a first approximation, however, because the theory treats footings resting *on the surface* of the elastic half-space; whereas actual foundations are often partially embedded. Barkan (1962), Pauw (1952), and Fry (1963) have reported on tests of footings partially embedded as well as on footings resting on the surface of the soil. In general, partial embedment reduced the amplitude of motion at the resonant peaks and increased the value of the resonant frequency. This indicates an increase in the effective spring constant as well as a probable increase in the effective damping ratio. However, the effects on amplitude and frequency in the tests depended upon the mode of vibration and magnitude of the motion. For motions within the range of design criteria for machinery, it appears that this reduction in amplitude resulting from partial embedment is on the order of 10 to 25 per cent. Therefore, the design calculations will err on the conservative side if the footing is considered to rest on the surface. Further field tests are needed to establish the influence of partial embedment, particularly for the rocking mode.

A second major discrepancy between the assumptions made in the theoretical treatment and real conditions is the assumption that the soil is a homogeneous, isotropic, elastic body. Often a soil stratum is layered and may have a hard stratum of soil or rock at a shallow depth below the footing. This problem was discussed briefly in Sec. 7.9, in which it was noted that the amplitudes of vibration at resonance were increased by the presence of the underlying rigid layer. This indicates that radiation of energy from the footing was impeded by the presence of the rigid layer and that part of this elastic-wave energy was reflected back to the footing. Further studies should be directed toward evaluations of the geometrical damping related to vibrations of footings supported by layered media as well as of footings supported by soils which vary in stiffness with depth or confining pressure.

Choice of Spring Constant for Equivalent Lumped Systems

The spring constant k is the most critical factor in the lumped-parameter analysis. It governs the static displacement of the foundation which would be developed by application of a static force equal to the dynamic force Q_o; and this static displacement is multiplied by a magnification factor M to establish the maximum amplitude of dynamic motion. The magnification factor (Eq. 2-53) is influenced by k through its contribution to the critical-damping coefficient c_c (Eq. 2-31), and thus to the damping ratio D (Eq. 2-32). Finally, k is the significant unknown in establishing the resonant frequency (Eqs. 2-17, 2-55, or 2-60). Methods for establishing k include static field tests of prototype foundations, static or dynamic field tests of model foundations, or theoretical methods.

Tests on prototype foundations. Tests on the prototype foundations are, of course, preferable if the tests are carefully conducted to include ranges of load and deformations corresponding to acceptable operating conditions. Pile-loading tests and tests of foundations supported by pile groups have often been conducted, but usually these have been for the purpose of evaluating the load-carrying capacity rather than the spring constant. The same type of test can provide useful information about the k required for dynamic analysis if repeated static loadings or vibratory loadings are applied and realistic ratios of steady load to alternating load are maintained. Tests of prototypes are recommended if several foundations of similar characteristics are to be built at one construction site. However, if only one structure is planned, the test on the prototype may indicate either a satisfactory or unsatisfactory performance. An unsatisfactory performance may require costly repairs that could have been minimized by a more careful design in the first place.

Tests on model footings. Static or dynamic tests of model footings are useful for establishing relations between the applied loads and response of these footings for particular subsoil conditions. A comprehensive program of carefully controlled model tests, exemplified by the vibration tests reported by Fry (1963), provides not only information about the response of the individual footings but also permits evaluation of the best methods for extrapolating this information for use in the design of prototype foundations. It is the extrapolation procedure which governs the value of model-footing tests for design purposes.

Formulas for spring constants. The spring constant represents a linear relation between applied load and displacement of the foundation which implies a linear stress–strain relation for the soil. Therefore, it follows that theory of elasticity can provide useful formulas for the spring constants for footings of simple shapes. Tables 10-13 and 10-14 include spring constants obtained through the theory of elasticity for circular and rectangular footings resting *on the surface* of the elastic half-space. These expressions have been obtained for *rigid* footings except for the case of horizontal motion, for which the spring constant was obtained by assuming a uniform distribution of shearing stress on the contact area and computing the average horizontal displacement of this area. These formulas apply for situations corresponding to rigid block or mat foundations with shallow embedment.

Table 10-13. Spring Constants for Rigid Circular Footing Resting on Elastic Half-Space

Motion	Spring Constant	Reference
Vertical	$k_z = \dfrac{4Gr_o}{1-\nu}$	Timoshenko and Goodier (1951)
Horizontal	$k_x = \dfrac{32(1-\nu)Gr_o}{7-8\nu}$	Bycroft (1956)
Rocking	$k_\psi = \dfrac{8Gr_o^3}{3(1-\nu)}$	Borowicka (1943)
Torsion	$k_\Theta = \frac{16}{3}Gr_o^3$	Reissner and Sagoci (1944)

$\left(Note\text{: } G = \dfrac{E}{2(1+\nu)}\right)$

Table 10-14. Spring Constants for Rigid Rectangular Footing Resting on Elastic Half-Space

Motion	Spring Constant	Reference
Vertical	$k_z = \dfrac{G}{1-\nu}\beta_z\sqrt{4cd}$	Barkan (1962)
Horizontal	$k_x = 4(1+\nu)G\beta_x\sqrt{cd}$	Barkan (1962)
Rocking	$k_\psi = \dfrac{G}{1-\nu}\beta_\psi 8cd^2$	Gorbunov-Possadov (1961)

(*Note:* values for β_z, β_x, and β_ψ are given in Fig. 10-16 for various values of d/c)

The effect of embedment is to increase the soil resistance to motion of th foundation, thus, the effective spring constant is increased. Figure 10-1 illustrates the change in vertical spring constants for circular footings as th depth of embedment increases. Curve *a* represents a rigid footing whic adheres to the soil along the vertical surface, thereby developing skin-frictio resistance to vertical motion of the block as well as developing resistance b pressure on the base. Curve *b* corresponds approximately to the situation o an embedded foundation which is isolated from the soil along the vertica surfaces. It is included to point out the increase in spring constant develope only by base pressure applied at different depths. The spring constants corre sponding to curve *b* were obtained from the average settlement produced b a uniformly distributed load applied at the different depths of embedment By comparing the spring constants for curve *a* (with side adhesion) and curve *b* (without side adhesion), it is possible to separate the effects of end bearin and skin friction. The information shown in Fig. 10-17 was prepared b Kaldjian (1969) from a solution of the elasticity problem by the finite element method.

The depth of embedment should produce even more significant effect on the spring constants for rocking and sliding motions of the foundation However, by the end of the 1960s, satisfactory solutions for these problem were not known to the writers.

Another effect which provides a stiffening to the spring constant of th foundation is the presence of a rigid boundary beneath an elastic layer Tha is, a thin elastic layer supported by a rigid base permits a smaller displace ment of a footing for a given load than does the elastic half-space. This wa

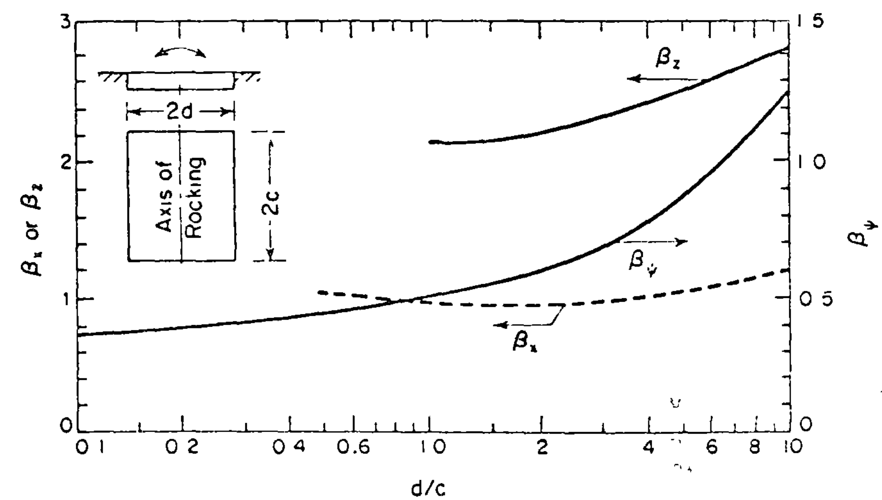


Figure 10-16 Coefficients β_x, β_z, and β_ψ for rectangular footings (after Whitman and Richart, 1967)

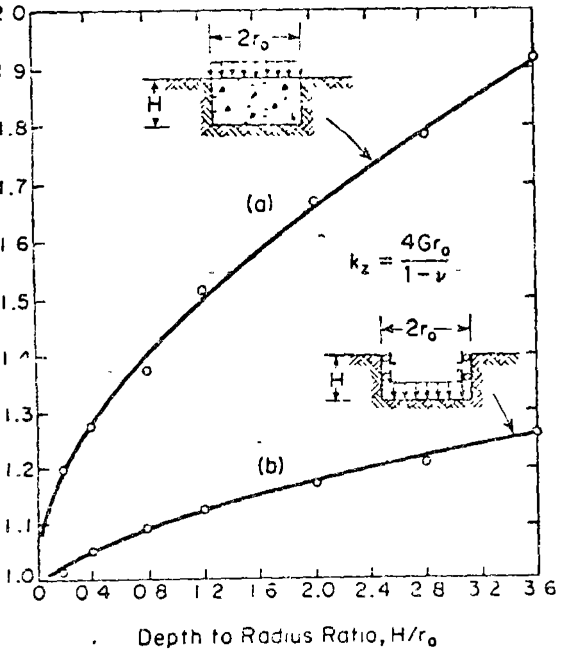


Figure 10-17. Effect of depth of embedment on the spring constant for vertically loaded circular footings (*from Kaldjian, 1969*).

illustrated by Fig. 7-22, in which the increase in k (for vertical loading) was shown to be significant as the value of H/r_0 (where H is the thickness of the elastic layer) decreased below about 2.

Elastic constants for soils. In the preceding section, which discussed formulas for spring constants, it was indicated that these were derived from solutions by the theory of elasticity. It should be emphasized that the elastic medium was assumed to be isotropic and homogeneous; therefore, only two elastic constants are required in the solution. Throughout this book the elastic constants chosen have been the modulus of elasticity in shear, G, and Poisson's ratio ν. Consequently, in order to evaluate spring constants for foundations from the formulas, we need reliable values for G and ν for the soil beneath the proposed foundation.

It is possible to compute Poisson's ratio for soils from measured values of the compression-wave and shear-wave velocities through the soil (see Chap. 3). However, these computations involve small differences of rather large numbers, and significant errors are possible. Generally, it has been found that Poisson's ratio varies from about 0.25 to 0.35 for cohesionless soils and from about 0.35 to 0.45 for cohesive soils which are capable of supporting block-type foundations. Consequently, for design purposes little error is introduced if Poisson's ratio is assumed as ⅓ for cohesionless soils and as 0.40 for cohesive soils.

Values of the shear modulus G may be evaluated in the field or from samples taken to the laboratory. Static plate-bearing tests in the field can establish an experimental value of the spring constant k, from which the shear modulus can be calculated (Table 10-13). *In-situ* steady-state-vibration tests may also be used to establish values of G at the construction site (see Sec. 4.3). In the laboratory the resonant-column test (see Sec. 9.6) is now a standard method for determining the effective G of soil samples for design purposes as well as being a research tool. Consequently, several methods are available for obtaining useful values of G by testing the actual soil which will support the proposed foundation. In the event the design study represents only a preliminary estimate or a feasibility study, reasonable values of G for soils can be estimated if we have some information on the void ratio of the soil and of the probable confining pressure $\bar{\sigma}_o$. Figure 6-8 illustrates the dependence of the shear-wave velocity of quartz sand on the void ratio and confining pressure. The shear modulus can be obtained from the shear-wave velocity given in Fig. 6-8 and the relation

$$G = \rho v_S^2 \tag{6-17}$$

For round-grained sands ($e < 0.80$) the shear modulus can be estimated from the empirical equation

$$G = \frac{2630(2.17 - e)^2}{1 + e} (\bar{\sigma}_o)^{0.5} \tag{6-19}$$

and, for angular-grained materials ($e > 0.6$), from

$$G = \frac{1230(2.97 - e)^2}{1 + e} (\bar{\sigma}_o)^{0.5} \tag{6-21}$$

in which both G and $\bar{\sigma}_o$ are expressed in lb/in.². Hardin and Black (1968) have indicated that Eq. (6-21) is also a reasonable approximation for the shear modulus of normally consolidated clays with low surface activity.

10.6 *Analysis and Design for Vertical Vibrations of Foundations*

In many cases machines which produce vertical forces can be located centrally on foundation blocks or mats with the result that only vertical vibrations of the machine-foundation system are important. This section includes examples of analyses of such systems based on both the elastic-half-space theory (Chap. 7) and on the lumped-parameter analog

Steady-State Vibrations of Model Footings

It is useful to begin this section with a comparison of the vibration response estimated by theory and that measured in carefully controlled field tests on model footings. Fry (1963) has reported on tests conducted on model footings from about 5-ft to 16-ft diameter constructed at the U.S. Army Waterways Experiment Station in Vicksburg, Mississippi, and at Eglin Field, Florida. The basic dimensions and weights of the footings are given in Table 10-15. All of these circular footings were constructed on the surface of the soil, except for base 5 at the Eglin Field site, which was embedded 25 in The soil at the WES site was a silty clay (CL), for which typical parameters needed for dynamic analysis are

$$\gamma = 117 \text{ lb/ft}^3$$

$$v_S = 460 \text{ ft/sec}$$

$$G = 5340 \text{ lb/in.}^2$$

$$\nu = 0.35$$

The water table was approximately 16 ft from the surface At the Eglin Field site the soil was a nonplastic uniform fine sand (SP) with uniform conditions indicated from borings to 25 ft below the surface as well as reasonably uniform conditions indicated throughout the test area. The water table was deep and was not encountered in any of the boreholes. For this material a typical void ratio was $e = 0.70$.

The footings were excited by a rotating-mass vibrator of the type illustrated in Fig. 10-5c. The four eccentric masses each had a weight of 339 lb; so the total eccentric weight was 1356 lb. The total static weight of the vibrator was 5600 lb. Four eccentric settings were used in the testing program: 0.105 in., 0.209 in., 0.314 in., and 0.418 in. Values of force output from

Table 10-15. Data on WES Test Bases

Base No.	r_o (in.)	Thickness (in)	Wt of Base + Vibrator (lb)
1—1st pour	31	14.3	12,820
1—2nd pour	31	29 7	25,640
1—3rd pour	31	36.0	30,970
2	43 81	20.5	18,465
3	54.0	24.0	24,315
4	62.0	24.5	30,970
5	43 81	25.0	18,465
:6	96 0	24 0	64,961

Table 10-16. Vertical Forces from Four-Mass WES Oscillator

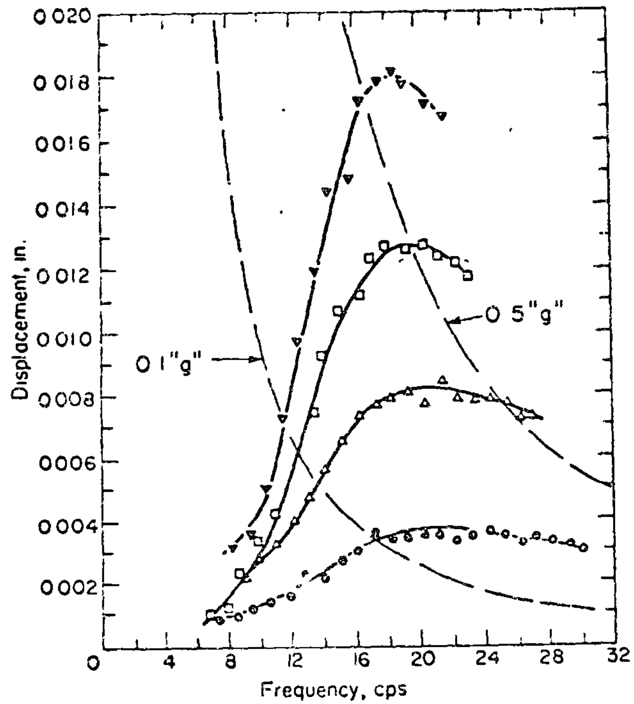
Rotating Frequency (cycles/sec)	15	20	25	30
Q_o (lb) for e = 0 105 in	3,285	5,840	9,125	13,140
Q_o (lb) for e = 0 209 in	6,540	11,620	18,160	26,160
Q_o (lb) for e = 0 314 in	9,825	17,460	27,290	39 300
Q_o (lb) for e = 0 418 in	13,020	23,140	36,160	52,0[illegible]0

this vibrator can be calculated from Eq. (10-2), and several values are indicated in Table 10-16 to establish the order of magnitude of forces involved

By comparing the static weights of the footings in Table 10-15 with the dynamic forces available from the vibrator at the higher speeds from Table 10-16, it is evident that the vibrator was capable of lifting some of the lighter footings free of the ground during vibration. Consequently, for some of the tests, additional weights were rigidly attached to the footing

For the purpose of comparing theoretical and test results, three tests at the WES site and three tests at the Eglin Field site were chosen. In each of these tests the vibrator had an eccentricity setting of 0 105 in., which produced the smallest set of exciting forces and the lowest accelerations in each pattern of tests. Figure 10-18 (after Fry, 1963) illustrates the effect on the amplitude-frequency response curve developed by changing the eccentric settings of the rotating weights for particular test conditions. Note that for WES test 3-6 in Fig. 10-18 a motion of about 0.0037 in. was developed at 20 cycles/sec. Dashed curves are shown in Fig. 10-18 which correspond to peak accelerations of $(0.1)g$ and $(0.5)g$. The response curves shown in Fig. 10-18 could also have been plotted in Fig. 10-3, which would permit an easier evaluation of the peak velocities and accelerations. From Fig. 10-18 it can be noted that the peak accelerations for tests 3-7, 3-8, and 3-9 were greater than $(0.5)g$. The largest value of peak acceleration for test 3-9 was $(0.7)g$.

The test results for the model footings can be evaluated better when they are presented on dimensionless plots For example, we may consider three tests at the WES site, tests 2-18, 3-6, and 4-5, for which the mass ratios were $\mathbf{b} = 3.12$, $\mathbf{b} = 2.83$, and $\mathbf{b} = 3.1$, respectively. Points representing test data are shown in Fig. 10-19a for comparison with the theoretical curves for $\mathbf{b} = 3$ and $\nu = \frac{1}{3}$ (or $\mathbf{B}_z = 0.5$). Theoretical curves for the rigid-base pressure (R) and the uniform-pressure (U) distribution (from Sung, 1953) are shown. From Fig. 10-19a it is evident that the agreement between the test results and the R-curve is reasonably good with respect to amplitude of vibration, but that the theoretical curve indicates a higher frequency at [illegible]



Legend

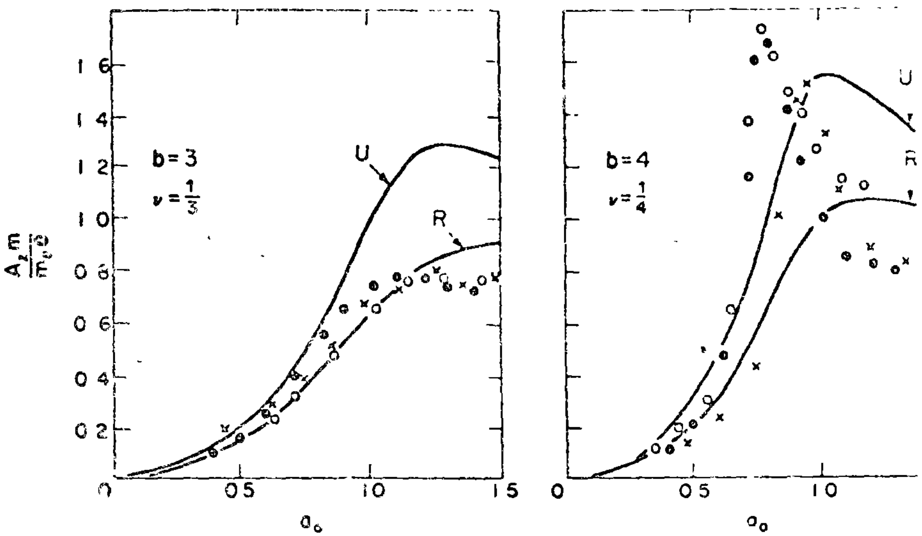
Symbol	Run No	ECC Setting—in
⊙	6	0 105
△	7	0 209
□	8	0 314
▽	9	0 418

Note 6655 lb ballast, 30,970 lb total weight including vibrator

Fig. 10-18 Typical amplitude-frequency response curves for tests on model footings (*after Fry, 1963*)

than that shown by test For these tests, in which the footings were supported by a cohesive soil, it appears that the rigid-base condition is approximated.

Tests 2-2, 5-1, and 3-5 run at the Eglin Field site provided the dimensionless test data shown in Fig. 10-19b. For these tests the b-values were 4.5, 4.5, and 3 61, respectively; so they were compared with the theoretical values for b = 4.



WES Test	W, lb	r_o ft
⊙ 2—18	18,465	3 67
× 3—6	30,970	4 50
o 4—5	51,280	5 17

(a)

Eglin Tests	W, lb	r_o ft
• 2—2	25,640	3 65
o 5—1	25,640	3 65
× 3—5	38,460	4 5

(b)

Figure 10-19 Comparison of test results with theory for vertical oscillation (a) Data from WES test site. (b) Data from Eglin Field test site

In order to interpret the test frequency in terms of the dimensionless frequency

$$a_o = \frac{2\pi f r_o}{v_S} \tag{7-2}$$

it is necessary to obtain a representative value of v_S. At the Eglin Field site the footings were poured on the sand surface (except for base 5, which was embedded). Thus, the sand beneath the footing was loaded by the weight of the footing and ballast as well as by its own weight. For an approximation to the pressure developed below the periphery of the footing, the theoretical solution obtained by Prange (1965) for a rigid circular footing on the isotropic, homogeneous, elastic half-space was used. These relations are given below for $\nu = \frac{1}{4}$. It is obvious that a bed of sand develops a different

$\frac{z}{r_o}$	0.1	0.2	0.4	0.6	0.8	1 0	1.5	2.0	3 0
$\frac{\bar{\sigma}_o}{Q/\pi r_o^2}$	0 642	0.440	0.293	0.222	0.178	0 147	0 096	0 067	0 037

distribution of pressure under loading than does the ideal half-space, and further information is needed on the effects of repeated loadings on the distribution of contact pressure at the footing base, as well as information on the vertical and horizontal pressures within the soil mass.

Figure 10-20a shows a rigid circular footing resting on the surface of a soil mass. The radius and weight of the footing correspond to that for base 3 used in the Eglin Field tests. The average contact pressure was $q = 604.6$ lb/ft²; Fig. 10-20b shows how the average confining pressure $\bar{\sigma}_{oq}$ caused by this surface pressure decreases with depth below the periphery of the footing according to Prange's solution.

The unit weight of the sand at the Eglin Field site was approximately 97 lb/ft³ and it was assumed that Poisson's ratio was $\frac{1}{4}$ for this material. Then the vertical and horizontal stresses at a depth in the soil mass were

$$\sigma_z = \gamma z \tag{10-34a}$$

and

$$\sigma_x = \sigma_y = \frac{\nu}{1-\nu}\,\sigma_z = \frac{\gamma z}{3} \tag{10-34b}$$

These stresses established $\bar{\sigma}_{os}$ as $(1.67/3)\gamma z$.

The total average confining pressure at any depth below the perimeter of the footing is the sum of $\bar{\sigma}_{oq}$ and $\bar{\sigma}_{os}$, as shown by Fig. 10-20d. A minimum value of $\bar{\sigma}_{o\,tot}$ usually occurs at a depth of $z/r_o \le 1$. This minimum value of $\bar{\sigma}_{o\,tot}$ and the void ratio of the sand ($e = 0.7$ at the Eglin Field site) were introduced into the equation for velocity of the shear wave

$$v_S = [170 - (78.2)e](\bar{\sigma}_o)^{0.25} \tag{6-18}$$

By this procedure, values of v_S of 460, 470, and 500 ft/sec were determined for the soil directly beneath Eglin Field bases 2, 3, and 5, respectively.

Note that errors in the calculation of v_S (or G) have an important influence on the value of a_o for maximum amplitude of vibration. Conversely, if the theoretical curves are to be used for predicting the frequency for design or analysis, this also depends on the value of v_S. In Fig. 10-19b the peak amplitudes are at a lower value of a_o than indicated by theory, and the shapes of the response curves indicate less damping than might be indicated from theory. Part of this difference could be assigned to a probable change in pressure distribution beneath the footing from the assumed rigid-base (R) condition to one more nearly uniform. The theoretical curve for the uniform (U) pressure distribution for $\mathbf{b} = 4$ and $\nu = \frac{1}{3}$ is shown in Fig. 10-19b for comparison. The amplitudes of oscillation for the test footings agree fairly well with those predicted from the uniform pressure distribution condition.

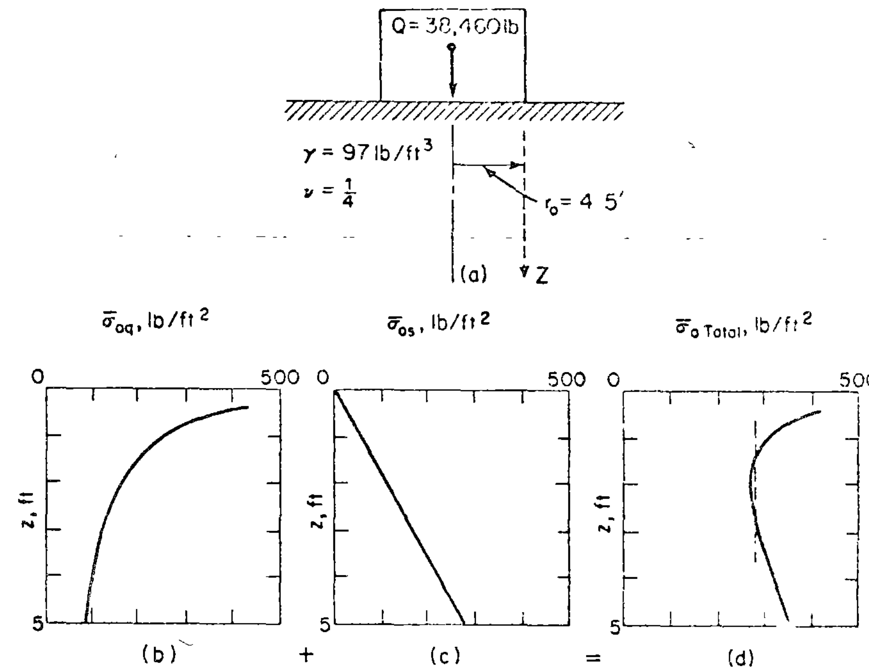


Figure 10-20 Distribution of average confining pressure, $\bar{\sigma}_o$, beneath periphery of rigid footing. (a) Vertical load on footing (b) $\bar{\sigma}_{oq}$ from footing load (c) $\bar{\sigma}_{os}$ from unit weight of soil (d) Total $\bar{\sigma}_o = \bar{\sigma}_{oq} + \bar{\sigma}_{os}$

The entire test program of vertical vibrations of the model footings included 94 tests. Figure 10-21 illustrates the relations between the maximum amplitudes of motion as computed from the half-space theory and those measured in the WES tests. The abscissa of Fig. 10-21 represents the maximum vertical acceleration of the footing as compared with the acceleration

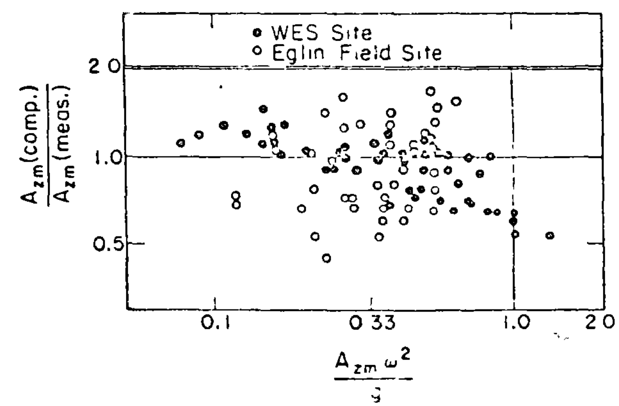


Figure 10-21 Vertical vibration of model footings, summary of 94 tests.

of gravity g. From Fig. 10-21 it is seen that the theoretical and test values for tests at the WES site agree closely for $A_{zm}\omega^2/g$ less than about $\frac{1}{4}$. As the acceleration ratio increases, these test and theoretical values differ more. The test results from the Eglin Field site show considerably more scatter throughout the entire range of acceleration ratios. However, the overall agreement between test and theory is within a factor of about 2, which is considered good for dynamic problems.

One item which should be noted with regard to the tests at the Eglin Field site is the matter of *total settlement* of the footings over the course of the vibration tests. For each pattern of tests, the response curves were obtained by increasing the eccentricity of the rotating mass successively through the four settings; then the dead load was changed and the next loading pattern was applied. By this procedure the soil beneath the footing had been subjected to a complex load-history by the time the second loading pattern was applied. Converse (1953) has shown that vibrating footings on sand tend to develop a "hard zone" beneath the center of the footings after sustained high-amplitude vibrations; so it would be anticipated that the pressure distribution beneath the footing would change as the loading history of the sand changed. Finally, settlement records were kept and the average total settlements at the end of the test program were about $4\frac{3}{8}$ in. for base 2, $1\frac{1}{2}$ in. for base 3, $\frac{5}{8}$ in. for base 4, and 3 in. for base 5. Consequently, these footings produced local failures and compaction of the supporting soil during some parts of the test program. For an actual machine foundation, a proper design would prevent this progressive settlement, and it could be anticipated that the soil would behave more nearly like the elastic half-space.

Vertical Single-Cylinder Compressor

This type of machine develops vertical periodic forces which can produce a vertical motion of the machine and its foundation block. This motion must be restricted to acceptable values, as noted in Sec. 10.2, to provide for satisfactory operation of the machine. The following discussion treats a method for establishing the foundation-block size for a vertical single-cylinder compressor having the following characteristics:

Bore =	14.5	in.
Stroke =	9	in.
Operating frequency =	450	rpm
<u>Unbalanced forces</u>		
Vertical: primary =	9,180	lb

secondary =	2,220	lb
Horizontal: primary =	310	lb
secondary =	0	lb
Weight of compressor + motor =	10,900	lb

The vertical primary and secondary forces produce a periodic vertical force as indicated in Fig. 10-22 by the heavy solid curve. However, for purposes of analysis, the reduced lower portion of the real-force-time curve will be ignored and the excitation will be considered to be developed by a sinusoidal force having an amplitude of $Q_o = 11,400$ lb at 450 rpm.

This compressor is to be supported by a foundation block resting directly upon the soil. From resonant-column tests of samples of the silty clay at the proposed site, the shear-wave velocity v_s was found to be 806 ft/sec. This value of v_s and the unit weight γ of 100 lb/ft^3 establish the shear modulus G as 14,000 lb/in.^{2}. Poisson's ratio ν was chosen as $\frac{1}{3}$ for the following calculations.

The first step in the design procedure is to establish the acceptable limits of motion (criterion of "failure"). For this example it is assumed that vertical motions equal to the horizontal motions noted as "case B" in Fig. 10-2 are acceptable. At 450 rpm the upper limit of case B corresponds to a

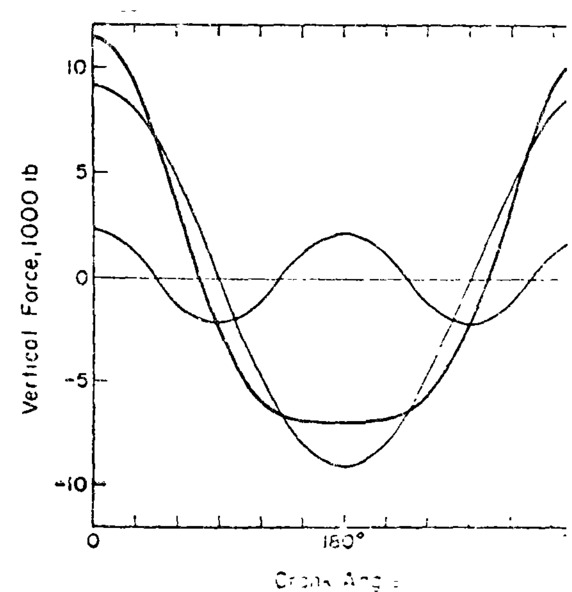


Figure 10-22. Unbalanced vertical force from vertical single-cylinder compressor.

single amplitude of 0.0021 in. Thus, the design criterion for the foundation–soil system requires the maximum amplitude of vertical motion to be less than 0.0021 in.

The first approximation for the foundation-plan dimensions may be obtained from the base area required to limit the static displacement, caused by $Q_o = 11{,}400$ lb, to a value of 0.002 in. The equivalent rigid circular footing will be used in both the static and dynamic analysis, although a rectangular foundation plan is needed. The static deflection

$$z_s = \frac{(1-\nu)Q_o}{4Gr_o} = 0.002 \quad \text{in.} = \frac{\frac{2}{3} \times 11{,}400}{4 \times 14{,}000 \times r_o}$$

leads to a required radius $r_o = 67.9$ in. $= 5.66$ ft. For convenience in further calculations, assume $r_o = 6$ ft, which determines a base area of 113 ft². With this assumed value of r_o, the corrected static displacement is now $z_s = 0.0019$ in.

The rigid circular footing of $r_o = 6$ ft will be used in further calculations to represent a rectangular foundation block 16-ft long and 7-ft wide. For a block 3-ft thick the total weight is

$$W_b = 16 \times 7 \times 3 \times 150 = 50{,}400 \quad \text{lb}$$

The total oscillating weight W, which includes the block and machinery supported upon it, is

$$W = 50{,}400 + 10{,}900 = 61{,}300 \quad \text{lb}$$

Then for the equivalent circular footing,

$$\mathbf{B}_z = \frac{(1-\nu)}{4}\frac{W}{\gamma r_o^3} = \frac{\frac{2}{3} \times 61{,}300}{4 \times 100(6)^3} = 0.473$$

From Fig. 7-19 the damping ratio D is 0.60 for this value of $\mathbf{B}_z$. The natural frequency f_n of the system depends upon the oscillating mass and the spring constant

$$k_z = \frac{4Gr_o}{1-\nu} = \frac{3}{2} \times 4 \times 14{,}000 \times 72 = 6.048 \times 10^6 \quad \text{lb/in.}$$

Then

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_z}{m}} = \frac{1}{2\pi}\sqrt{\frac{6.048 \times 10^6 \times 386}{61{,}300}} = 31.1 \quad \text{cycles/sec}$$

or

$$f_n = 1864 \quad \text{cycles/min}$$

Thus, the operating frequency of 450 rpm corresponds to $(0.24)f_n$. From Fig. A-1,

$$\frac{A_z}{z_s} = 1.02 \quad \text{at} \quad \frac{f}{f_n} = 0.24 \quad \text{for } D = 0.6$$

which permits calculation of the maximum amplitude of vertical motion

$$A_a = 1.02 \times z_s = 0.00194 \quad \text{in.}$$

This value of motion satisfies the design criterion.

For this example the 16-by-7-by-3-ft concrete block and the soil with $G = 14{,}000$ lb/in.² form a satisfactory foundation for vertical vibration. However, it has been assumed in the analysis that the input force, center of gravity of the oscillating mass, and the center of pressure of the soil on the base of the foundation block all lie along the same vertical line. In assembling the machinery on the foundation block, care should be taken to align these exciting and resisting forces as closely as possible to reduce coupling between the vertical and rocking or pitching modes of vibration.

The solution for the maximum amplitude of vertical vibration was obtained from the response curve (Fig. A-1) for constant amplitude of force excitation (Q_o = const. = 10,900 lb). It could have been obtained also from Fig. A-2, which corresponds to the frequency-dependent excitation ($Q_o = m_e e\omega^2$). In Chap. 2 it was noted that the ordinate of each curve on Fig. A-2 may be obtained at each frequency ratio from the ordinate of a similar curve on Fig. A-1 by

$$M\left(\frac{f}{f_n}\right)^2 = \frac{A_z k_z (2\pi)^2 \omega^2}{Q_0 (2\pi)^2 \frac{k_z}{m}} = \frac{A_z m\omega^2}{m_e e\omega^2} = \frac{A_z m}{m_e e}$$

The force of $Q_o = 11{,}400$ lb at 450 rpm is developed by

$$m_e e = \frac{Q_o}{\omega^2} = \frac{11{,}400}{4\pi^2\left(\frac{450}{60}\right)^2} = 5.13 \quad \text{lb-sec}^2$$

Then

$$1.02(0.24)^2 = 0.0587 = \frac{A_z \times 61{,}300}{386 \times 5.13}$$

or

$$A_z = 0.0019 \quad \text{in.}$$

as was obtained in the previous calculation. Note that the value of $A_z m/m_e e$

is usually obtained directly from Fig. A-2 at $f/f_n = 0.24$ on the curve for $D = 0.6$. However, in this low frequency range ($f/f_n < 0.3$) the curves are very steep on the semilog plot and more accurate values are obtained by calculation, as indicated above.

Response of Foundations to Transient Vertical Loads

In order to evaluate the motions of foundations responding to transient loadings, it is necessary to have reliable information on the load–time-pulse to be applied. This pulse is then applied to the lumped-parameter analog of the foundation–soil system and a solution can be obtained from the phase-plane method.

Drop test on model footing. The first example to be considered here was illustrated in Fig. 10-9a. A footing of 1-ft diameter rested on the surface of a bed of compacted Ottawa sand. This footing supported a dead weight and a loading platen onto which a 5-lb sandbag was dropped. For test Q-2 the force–time-pulse shown by the solid curve in Fig. 10-9b was developed, and the rectangular force–time-pulse approximations, also shown on this figure, were used in the phase-plane analysis of the footing response. The following quantities related to the footing–soil system entered into the computations.

r_o = Radius of the circular footing	= 6 in.
W = Weight of the footing	= 150 lb
γ = Unit weight of the sand	= 109 lb/ft^3
$\mathrm{b} = \dfrac{W}{\gamma r_o^3}$ = Mass ratio	= 11.0
$\mathrm{B}_z = \dfrac{1-\nu}{4}\mathrm{b}$ = Modified mass ratio	= 2.07
$D = \dfrac{0.425}{\sqrt{\mathrm{B}_z}}$ = Damping ratio	= 0.296
G = Shear modulus of soil	= 3400 lb/in.^{2}*
ν = Poisson's ratio of soil	= $\frac{1}{3}$
$k_z = \dfrac{4Gr_o}{1-\nu}$ = Static spring constant	= 108,800 lb/in.

* The effective G for the sand beneath the footing was established by the procedure described by Fig. 10-20

$m = \dfrac{W}{g}$ = Mass of footing	= 0.389 lb-sec^2/in.
$\omega_n = \sqrt{\dfrac{k_z}{m}}$ = Undamped natural frequency	= 529 rad/sec
$\omega_d = \sqrt{\dfrac{k_z}{m}}\sqrt{1-D^2}$ = Damped frequency	= 505 rad/sec
$T_d = \dfrac{2\pi}{\omega_d}$ = Natural period of footing	= 0.0124 sec
$\varphi_1 = \arcsin D$ = Angle of inclination of ordinate	= 17.2°

The rectangular force–time-pulse approximations to the pulse–loading curve indicate a constant-force amplitude over an interval of 0.002 sec (2 msec). This time interval corresponds to an angular movement $\omega_d\,\Delta t$ on the phase-plane of

$$\omega_d\,\Delta t = \frac{\Delta t}{T_d}\,360 = \frac{0.002}{0.0124}\,360 = 58°$$

The values of input force for each time interval establish the static displacements z_{s4}, z_{s6}, . . . shown in the phase-plane solution (Fig. 10-23a).

The phase-plane solution shown in Fig. 10-23a is constructed by the procedure described in Chap. 2. The circled numbers, ④, ⑥, . . . designate the points on the phase plane from which the displacement and velocity can be evaluated, corresponding to the time of 4, 6, . . . m sec. For example, to find the displacement at point ⑩ (end of 10 msec. of loading), a line is drawn from ⑩ parallel to the $\dot{z}/\omega_n$ axis until it intersects the z-axis at point ⑩'. This value of z is 0.0005 in. The circles in Fig. 10-23b represent the displacements at the end of the time intervals as obtained from the phase-plane solution shown in Fig. 10-23a. The length of the line ⑩–⑩' represents the value of $\dot{z}/\omega_n$ at the end of 10 msec of loading from which a velocity of 0.09 in./sec is calculated.

The acceleration at the end of each time interval could also be evaluated from Fig. 10-23a by the method described in Chap. 2, but because rather large instantaneous-force jumps are represented by the rectangles used to approximate the force pulse (Fig. 10-9b), the values of acceleration would be fairly crude. In order to improve the calculations of accelerations, it is preferable to use smaller time intervals for the force-pulse blocks. The problem described by the graphical phase-plane method can also be solved readily with a digital computer, which makes it easy to cut down the time duration on the force–time-pulse blocks. Figure 10-23c shows the acceleration–time diagrams obtained from the computer solution and the curve obtained from the test.

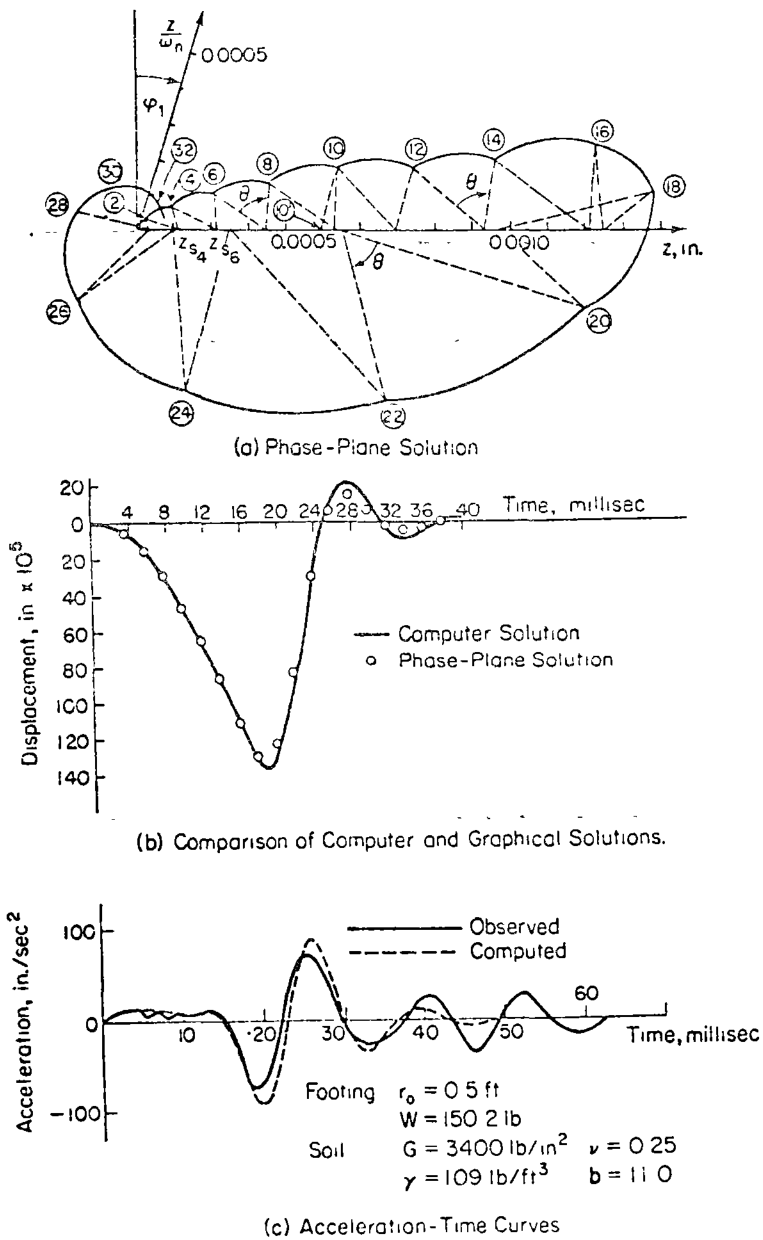


Figure 10-23. Dynamic response of model footings–transient loading test Q-2.

This good agreement between test and computed values is typical of the results reported by Drnevich and Hall (1966). The computed value of the displacement–time curve is shown as the solid curve on Fig. 10-23b, and it should be noted that the phase-plane solution produces a displacement–time curve which agrees closely with the computed solution.

From transient-loading tests on model footings similar to the one described for Test Q-2, it was demonstrated that theoretical methods may predict the displacement–time and acceleration–time behavior of the footing quite satisfactorily. The phase-plane solution provides approximate answers in a relatively short time, and the accuracy may be improved by taking smaller time intervals in the force–time-pulse approximations.

Impact on punch-press foundation. The loading pulse described in Fig. 10-10b resulted from a sudden release of elastic energy as the upper die of a punch press sheared through a metal blank. This loading pulse is reproduced in Fig. 10-24a along with the rectangular force-pulse approximations to this curve. This pulse is associated with a machine having a dead weight of 30,000 lb which rests on a concrete-block foundation. The block is supported directly on a soil for which $G = 10\,000$ psi and $\nu = \frac{1}{4}$.

Table 10-17 includes the significant quantities needed for a phase-plane solution for the response of a machine–foundation–soil system set into motion by this loading pulse. Three choices for the foundation block are noted in Table 10-17 having dimensions of 18 by 18 by 3 ft, 18 by 18 by 4 ft, and 15 by 15 by 3 ft. The phase-plane solution shown in Fig. 10-24b and the

Table 10-17. Data for Analysis of Response of Block Foundation to Transient Vertical Loading

Block Dimensions	18 × 18 × 3 ft	18 × 18 × 4 ft	15 × 15 × 3 ft
W*	175,800 (lb)	224,400 (lb)	131,250 (lb)
effective r_o	10.16 (ft)	10.16 (ft)	8.46 (ft)
b	1.40	1.78	1.81
B_z	0.26	0.334	0.339
D	0.83	0.735	0.730
k	6.5×10^6 (lb/in.)	6.5×10^6 (lb/in.)	5.41×10^6 (lb/in.)
φ_1	56.2	47.3°	46.9°
ω_n	119.5 (rad/sec)	105.8 (rad/sec)	126.2 (rad/sec)
ω_d	66.6 (rad/sec)	71.8 (rad/sec)	86.3 (rad/sec)
T_d	0.0943 (sec)	0.0876 (sec)	0.0729 (sec)
θ	3.82°	4.11°	4.94°
z_{max}	0.0042 (in.)	0.0041 (in.)	0.0048 (in.)
time for z_{max}	0.0125 (sec)	0.0130 (sec)	0.0126 (sec)

* W = wt. of foundation block + machine

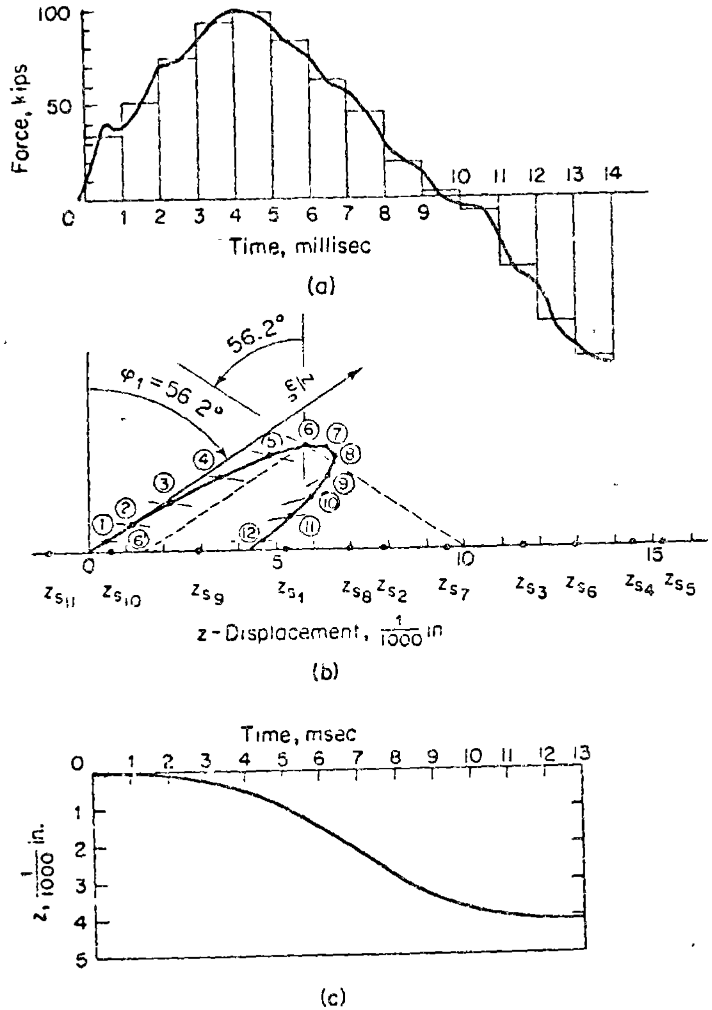


Figure 10-24 Transient loading of 18-ft-by-18-ft-by-3-ft foundation block. (a) Force-time diagram (b) Phase-plane solution. (c) Displacement-time diagram

displacement–time curve in Fig 10-24c are shown for the 18-by-18-by-3-ft foundation block to illustrate the method of analysis. Note from Table 10-17 that for vertical loading the weights and dimensions of each of these foundations lead to extremely high computed values for the geometrical damping. High damping ratios limit the maximum amplitude of motion developed during response to the impact.

The phase-plane solution shown in Fig 10-24b was constructed by the same procedure followed for the phase-plane solution shown in Fig. 10-23a.

However, the construction lines have been eliminated from Fig. 10-24b to simplify the drawing. This phase-plane solution was discontinued after the displacement had reached the maximum amplitude, again for convenience in illustration. As before, the displacement z is evaluated from Fig. 10-24b by following from a point on the curve—for example, point ⑥—down along a line parallel to the $\dot{z}/\omega_n$ axis to the abscissa. Thus, the displacement corresponding to point ⑥ (at end of 6 msec) is 0.0015 in. The velocity at this time is represented by the length of the line ⑥-⑥ (i.e., $\dot{z}/\omega_n$) multiplied by ω_n, which gives a value of $\dot{z} = 0.62$ in./sec at the end of 6 msec. The acceleration at this time is obtained by projecting a line at $-56.2°$ from the vertical through point ⑥ to the abscissa. This intersection at $z = 0.010$ in. is a distance of -0.0029 in. to the left of z_{s6}, from which the acceleration at the end of 6 msec is calculated to be

$$\ddot{z} = (0.0029)\omega_n^2 = 41.4 \quad \text{in./sec}^2$$

Data from phase-plane solutions of the three block foundations loaded by the same pulse are given in Table 10-17 for comparison. Note that the maximum displacement only increases from 0.0042 in. to 0.0048 in. by decreasing the side of the square block from 18 to 15 ft. The final choice of block size depends on the design criteria, which includes cost for a particular installation.

10.7 Analysis and Design for Rocking Vibrations of Foundations

In contrast to the high values of geometrical damping generally associated with vertical oscillations, rocking oscillations develop relatively low values of geometrical damping. This was illustrated in Fig. 7-19, where it was shown that for rigid circular footings a value of B_ψ of 0.75 or less was required in order to raise D above 0.10. The consequence of low damping is exhibited by large values for the magnification factor M_ψ, shown in Fig 7-16. Consequently, the dynamic response at the resonant frequency for rocking will result in large amplitudes of motion.

Design procedures for footings subjected to rocking motions must either provide such a low value for B_ψ (i.e., $B_\psi \leq 0.5$) that the magnification factors become small or assure that the resonant frequency for rocking is well above (by at least a factor of 2) the proposed operating frequency.

Rocking Tests of Model Footings

The series of tests on model footings described by Fry (1963) [illegible] excitation of the footings in the rocking mode of [illegible] vibrator to operate as indicated in Fig. 10-5c. Even though a pure couple was generated by the vibrator, the response of the footing involved both rocking and a horizontal translation because the center of gravity of the footing was above the center of sliding resistance (see Fig. 7-20). Therefore, a coupled motion resulted, and two modes of resonant vibration were possible, as indicated in Fig. 10-25. The lower-frequency mode is designated as mode I. Another resonant frequency, mode II, occurs at a higher frequency and corresponds to an out-of-phase relation between rotation and translation. In mode II the footing rotates clockwise about the center of gravity as the center of gravity moves to the left. Thus, the footing moves about some center of rotation which is above the center of gravity (Fig. 10-25b).

The design restrictions placed on the model footings by the limited range of frequencies available from the mechanical oscillator, as well as the desire to limit all resonant vibrations to relatively small amplitudes, affected the response of the footings in the rocking mode of vibration. Only base 1 (see Fig. 10-26b) had geometrical configurations which permitted mode-II rocking vibrations to develop within the range of available frequencies. The other footings developed only mode-I rocking vibrations.

For base 1—test 36 at the Vicksburg site, the test results for the rocking

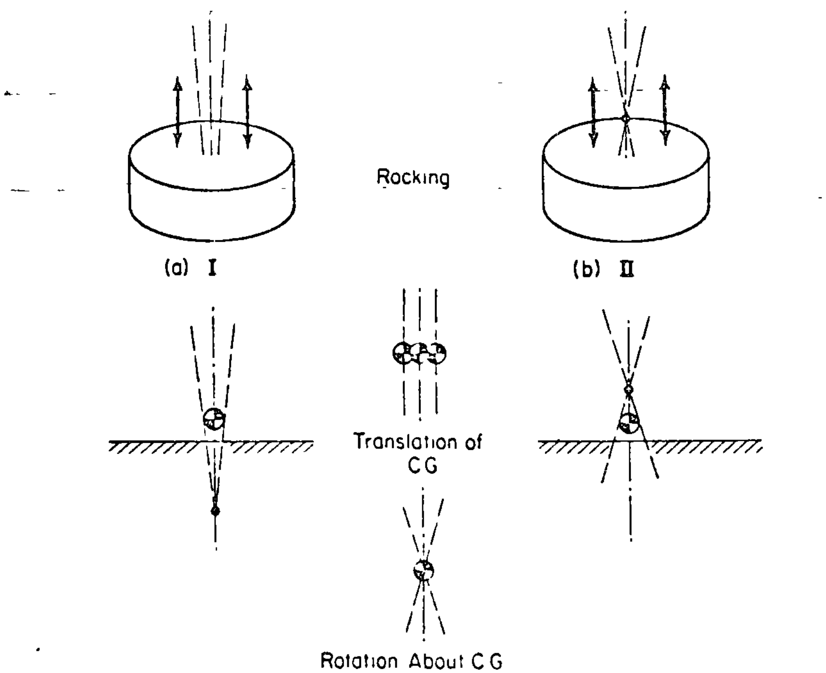


Figure 10-25 First (I) and second (II) coupled modes of rocking vibrations of model footings

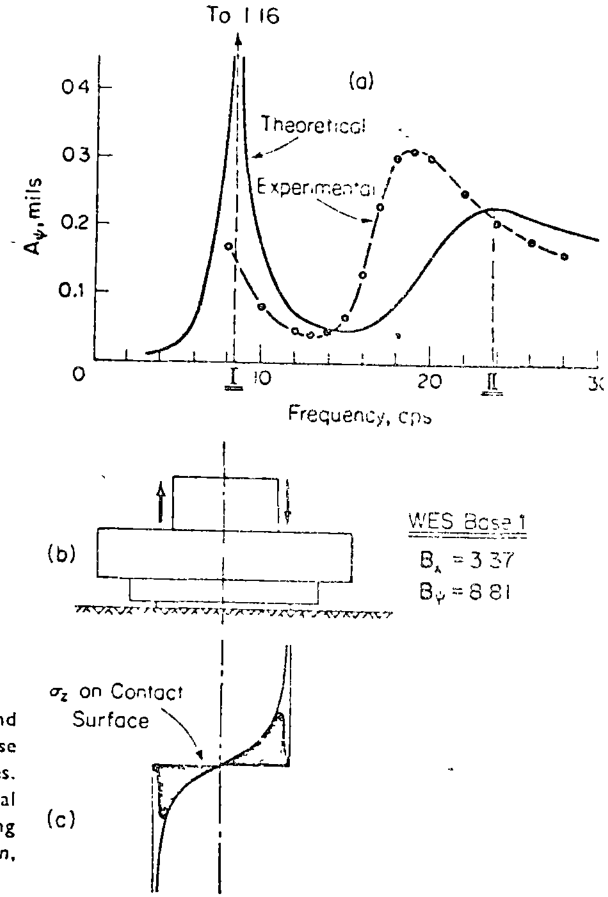
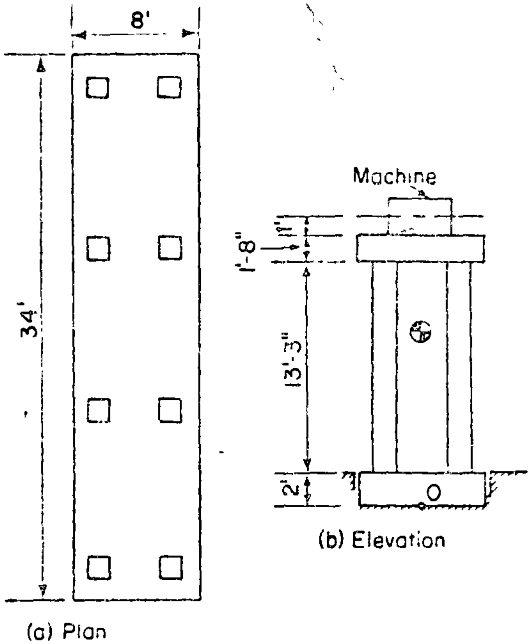


Figure 10-26 Coupled rocking and sliding vibrations of WES Base 1, Test 36 (a) Response curves. (b) Footing geometry. (c) Vertical pressure distribution on footing base (*After Richart and Whitman, 1967.*)

mode of vibration are shown by the dashed curve in Fig. 10-26a. These values were obtained from the vertical displacement measured 2 in. from the periphery of the upper surface of the concrete base on a diameter perpendicular to the axis of rocking. These vertical displacements were divided by the radius to the center of the circular top surface of the base to describe the amplitude of rotation A_ψ. A theoretical curve for the corresponding amplitude of rotation is also shown in Fig. 10-26a, as obtained by the analysis for coupled rocking and sliding described in Sec 7.8. Because the theoretical solution for rocking of the rigid disk was available only for the case of $\nu = 0$, the theoretical coupled solution applies only for $\nu = 0$.

Figure 10-26b shows the general configuration of WES base 1 which had a circular base of 62-in. diameter in contact with the soil and additional cylindrical concrete masses of 88-in. and 112-in. diameter added above. The 5600-lb mechanical vibrator was attached to the top of the footing [illegible]



Total Weight = 272,100 lb

Mass Moment of inertia in Rocking about Point O

$$I_\psi = \frac{4.81 \times 10^7}{32.2} \text{ ft lb sec}^2$$

Elevation of CG above Point O = 11.2 ft

Figure 10-27. Machine foundation.

top of the basement slab. For this installation the soil properties needed in the dynamic analysis of the foundation–soil system are: $G = 12{,}300$ psi, $v_S = 720$ ft/sec, $\nu = 0.25$, and $\gamma = 110$ lb/ft^3.

The problem is to evaluate the dynamic response of this foundation to the horizontal and vertical forces generated by rotating machinery. This will be carried out by analyses based on the elastic-half-space theory (Chap. 7) for separate single-degree-of-freedom responses of the foundation to the vertical and to the horizontal (or rocking) forces. For the vertical response the first step is to calculate the radius of the equivalent circular area (Eq. 10-29a)—

$$r_o = \sqrt{\frac{4cd}{\pi}} = \sqrt{\frac{34 \times 8}{\pi}} = 9.30 \quad \text{ft}$$

—to be used in the calculation for the modified mass ratio,

$$\mathbf{B}_z = \frac{(1-\nu)}{4}\frac{W}{\gamma r_o^3} = \frac{0.75 \times 272{,}100}{4 \times 110(9.3)^3} = 0.58$$

From Fig. 7-11 the dynamic magnification factor is about 1.1, and from Fig. 7-19 or Eq. (7-30) the damping ratio D is 0.56. This demonstrates that the vertical motion is highly damped and that the maximum amplitude of dynamic motion will be only slightly greater than the static displacement produced by the input force. Therefore, for the preliminary calculation, it appears that this foundation is satisfactory from the standpoint of vertical vibrations.

For rocking vibrations excited by the horizontal component of the machine forces, again we calculate the radius of an equivalent circular base, this time from Eq. (10-29b), as

$$r_o = \sqrt[4]{\frac{2c(2d)^3}{3\pi}} = \sqrt[4]{\frac{34 \times 8^3}{3\pi}} = 6.55 \quad \text{ft}$$

Then

$$\mathbf{B}_\psi = \frac{3(1-\nu)}{8}\frac{I_\psi}{\rho r_o^5} = \frac{2.25}{8}\frac{4.81 \times 10^7}{110(6.55)^5} = 10.2$$

Figure 7-16 indicates that the dynamic amplitude magnification factor for this value of $\mathbf{B}_\psi$ is greater than 100. From Eq. (7-64),

$$D_\psi = \frac{0.15}{(1 + \mathbf{B}_\psi)\sqrt{\mathbf{B}_\psi}} = 0.0042$$

from which the magnification factor can be calculated as

$$M_{\psi m} \approx \frac{1}{2D} = 119$$

With this low value of damping ratio, or high magnification factor, the peak of the amplitude–frequency response curve will occur at a frequency almost identical with the natural frequency. The dimensionless frequency a_{om} can be estimated from Fig. 7-16a as 0.30, from which the resonant frequency is

$$f_m = \frac{a_{om} v_S}{2\pi r_o} = \frac{0.30 \times 720}{2\pi \times 6.55} = 5.25 \quad \text{cycles/sec} = 315 \quad \text{cycles/min}$$

As a check, the resonant frequency for the lumped-mass system can be evaluated through Eq. (2-17). For this calculation the expression for the spring constant of the rectangular footing may be taken from Table 10-14 and with β_ψ from Fig. 10-16,

$$k_\psi = \frac{G}{1-\nu}\beta_\psi 8cd^2 = \frac{12{,}300 \times 144}{0.75} 0.40 \times 34 \times 8^2$$
$$= 2.055 \times 10^9 \quad \text{ft-lb/rad}$$

Then from Eq. (2-17),

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_\psi}{I_\psi}} = \frac{1}{2\pi}\sqrt{\frac{2.055 \times 10^9 \times 32.2}{4.81 \times 10^7}}$$

$$= 5.90 \quad \text{cycles/sec} = 354 \quad \text{cycles/min}$$

The difference between the resonant frequencies calculated by these two methods is due primarily to the differences in the k_ψ values computed from Table 10-13 and Table 10-14. However, these are reasonably close, and either is satisfactory to indicate the order of magnitude of the resonant frequency.

From this preliminary analysis, it is evident that the foundation will experience a severe rocking oscillation at a frequency in the range of 320 350 rpm. This particular foundation was scheduled to support several different combinations of rotating machinery at different times, and all of the machines operated around this range of frequencies or higher. When operating at the higher frequencies, there was always the necessity for passing through the resonant condition during starting up or stopping. Consequently, this configuration was considered unsuitable for the purpose of resisting rocking motions. In this particular case, several proposed foundations of this general type were located parallel and relatively close together. Therefore, it was expedient to tie these together with a shear wall at each end to develop a box-type foundation which was stable against rocking.

It should be fairly obvious that foundations needed to resist rocking forces induced by machines should be low and wide. This is demonstrated by the dependence of the dynamic response of foundations to the mass ratio B_ψ (Eq. 7-44) for rocking. Whenever possible, the best procedure for reducing the value of B_ψ is to increase the size of the footing, because r_o enters the computation as r_o^5.

Rocking of a Radar Tower

The supporting structure for a radar antenna must have dynamic responses which do not interfere with the operation of the electronic equipment. The radar disk itself must be rigid enough so that it does not distort unduly as the mechanism moves in azimuth and elevation, the rotating mechanisms must have close tolerances, the vertical tower must be stiff and must not develop resonant responses, and finally, the foundation which bears against the soil must not permit large motions of the entire tower.

As noted in the discussion of design criteria (Sec. 10 2), permissible rotations of radar-tower foundations are often of the order of 0 05 mil (or about 0.00005 rad) Figure 10-28 illustrates the rotation of the radar antenna

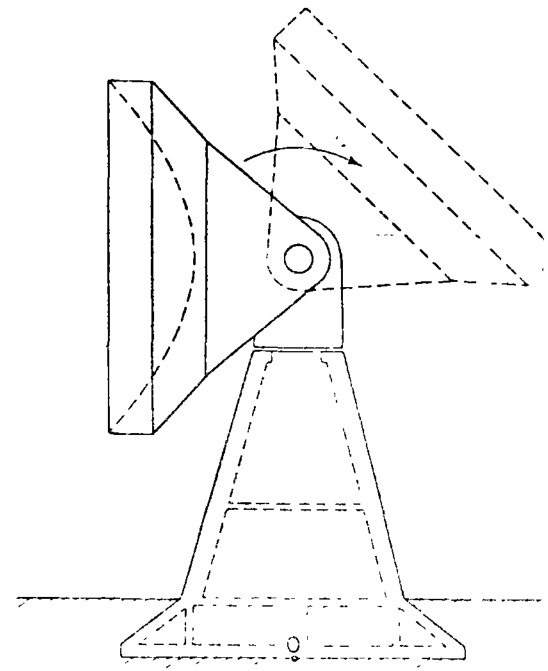


Figure 10-28. Rocking of radar tower.

in elevation about a horizontal axis. This motion introduces a transient rocking pulse into the tower which then may cause the tower to rock at its natural frequency because of the flexible connection between the foundation and the soil or because of the various flexibilities in the structural system. Obviously, the entire radar tower has many degrees of freedom in vibration, but a standard design can be prepared for the structural system to avoid the critical resonant frequencies. It is the foundation–soil flexibility which will vary from site to site and which must be evaluated for each tower installation. This section will consider only the rocking of the radar tower, considering the tower itself as a rigid mass and all the flexibility to be concentrated in the supporting soil.

The radar tower shown in Figure 10-28 is supported by a rigid circular concrete base 60 ft in diameter which rests directly on the soil. Field measurements were made of the dynamic soil properties (see W E S Misc. Paper No. 4-584, July 1963), and values of the shear modulus between 12,000 and 20,000 psi and a Poisson's ratio of 0 43 were determined by the steady-state-vibration method (see Sec 4.3). Because slightly higher confining pressures were to be developed in this soil under the completed structure, the limiting values of G = 14,000 psi and G = 20,000 psi were used in the original design computations.

The analysis for rocking of the tower as a rigid body is analyzed here by the lumped-parameter analog to the elastic theory for [illegible] circular foundation. For this tower the circular foundation was 60 ft in diameter, or $r_o = 30$ ft. The mass moment of inertia in rocking about a diameter through the base (point O, Fig. 10-28) was calculated to be

$$I_\psi = 80.545 \times 10^6 \quad \text{lb-ft-sec}^2$$

With the unit weight of the soil of approximately 100 lb/ft³, this leads to the calculation of the mass ratio in rocking as

$$B_\psi = \frac{3(1-\nu)}{8} \frac{80.545 \times 10^6}{\frac{100}{32.2}(30)^5} = (1-\nu)0.400$$

Calculations for the maximum amplitude of rotation and the frequency at which this occurs for an overturning moment of 212,000 ft lb, considered as a constant-moment excitation, are shown in Table 10-19.

Field tests were conducted on this tower after construction to evaluate the prototype performance (see Ballard and Fowler, 1967). A summary of the test results from excitation of the tower in the rocking mode are given in Table 10-20 along with the original design estimates, which were prepared with the aid of the elastic-half-space theory (with the assumption of $\nu = 0$). Note that there is reasonably good agreement between the calculated and measured amplitudes and frequencies.

Table 10-19. Calculations for Rocking of Radar Tower

Constant-Force Excitation, $T = T_o \sin \omega t$, $T_o = 212{,}000$ ft-lb			
	For $\nu = 0$	For $\nu = 0.4$	Eq. No.
B_ψ	0.40	0.24	7-44
D_ψ	0.169	0.247	7-53
$\sqrt{1 - D_\psi^2}$	0.986	0.969	
$\sqrt{1 - 2D_\psi^2}$	0.971	0.937	
$M_{\psi m}$	2.96	2.02	2-56
For $G = 14{,}000$ psi			
ψ_s (rad)	1.46×10^{-6}	0.876×10^{-6}	7-43
A_ψ (rad)	4.26×10^{-6}	1.72×10^{-6}	7-45
f_m (cycles/sec)	6.56	8.2	2-55
For $G = 20{,}000$ psi			
ψ_s (rad)	1.022×10^{-6}	0.613×10^{-6}	7-43
A_ψ (rad)	2.98×10^{-6}	1.20×10^{-6}	7-45
f_m (cycles/sec)	7.8	9.8	2-55

Table 10-20. Comparison of Measured and Calculated Values for Rocking of Radar Tower*

	Frequency Range (cycles/sec)	Rotation (10^{-6} rad)
Design	6.0-9.0	3.6-5.1
Measured	4.9-7.7	1.60

* From Ballard and Fowler (1967)

The design of the foundation for a radar tower should also include consideration of the torsional resistance of the foundation as well as rocking, and should include an evaluation of the coupling between the structural and foundation flexibilities. Additional design criteria may be found—in Fu and Jepson (1959), Horn (1964), and Pschunder (1966), for example—and useful data on foundation and tower stiffnesses are given by Weissmann and White (1961), Weissmann (1966) and Pschunder (1965, 1966).

10.8 Conclusions

This chapter has treated methods of analysis and design of dynamically loaded foundations. These methods depend on the design criteria, applied forces, soil response, and analytical procedures for relating these quantities.

The design criteria were based on a failure criterion of a limiting amplitude of motion, or a limiting velocity or acceleration of the foundation. In nearly all cases the motions involved were on the order of a few thousandths of an inch up to perhaps a few hundredths, and general guidelines have been established.

A critical part of the study of the dynamic response of a given system is to evaluate the type and magnitude of the input forces to be resisted. These may be calculated readily for certain types of machinery and can usually be evaluated experimentally for machines producing transient loads. It becomes more difficult to estimate the loads introduced by natural forces of wind, water waves, or earthquakes. Thus, it may be concluded that much more information is needed on the forces to be applied to foundations by machines or external sources.

Chapter 6 included a discussion of the response of soils to dynamic loads which produce small deformations. Because the design criteria for foundation behavior restrict motions to small values, it follows that the supporting soils will normally be subjected to small strains only. Consequently, the dynamic soil parameters described in Chap. 6 and the methods described in Chaps. 4 and 9 for determining these characteristics constitute a reasonably satisfactory part of the design procedure at the present (1969). However, it is anticipated that a considerable amount of effort will continue to be directed toward laboratory and field evaluation of soil behavior under dynamic loading.

The analytical procedures for establishing the dynamic behavior of a foundation relate the applied forces, soil properties, and foundation weights and geometry to the response. By successive corrections of the design parameters, the analytical procedures provide a method for developing a dynamic response of the foundation which falls within the design limits. Several simplified methods of analysis have been discussed in Chaps. 7 and 10; these have been found satisfactory when the prototype conditions correspond to the assumptions made in establishing the theory. Much more work is required in developing analytical procedures to cover the variables of shape of foundation, depth of embedment, variations of soil properties with depth, geometrical vs. hysteresis damping, coupling effects, and effects of adjacent footings. Very little information is available on the dynamic behavior of foundations supported by piles or caissons, or on flexible mats.

A final conclusion relates to the continuing need for field data from tests on prototype foundations: The only justification for using any design procedure is that it provides a reliable estimate for the behavior of the prototype. Thus, it is necessary to compare predicted and measured values at all opportunities in order to provide a realistic basis for subsequent efforts to improve the methods of design.

APPENDIX

Information that is needed frequently in design or analysis is included in the following two tables and four figures.

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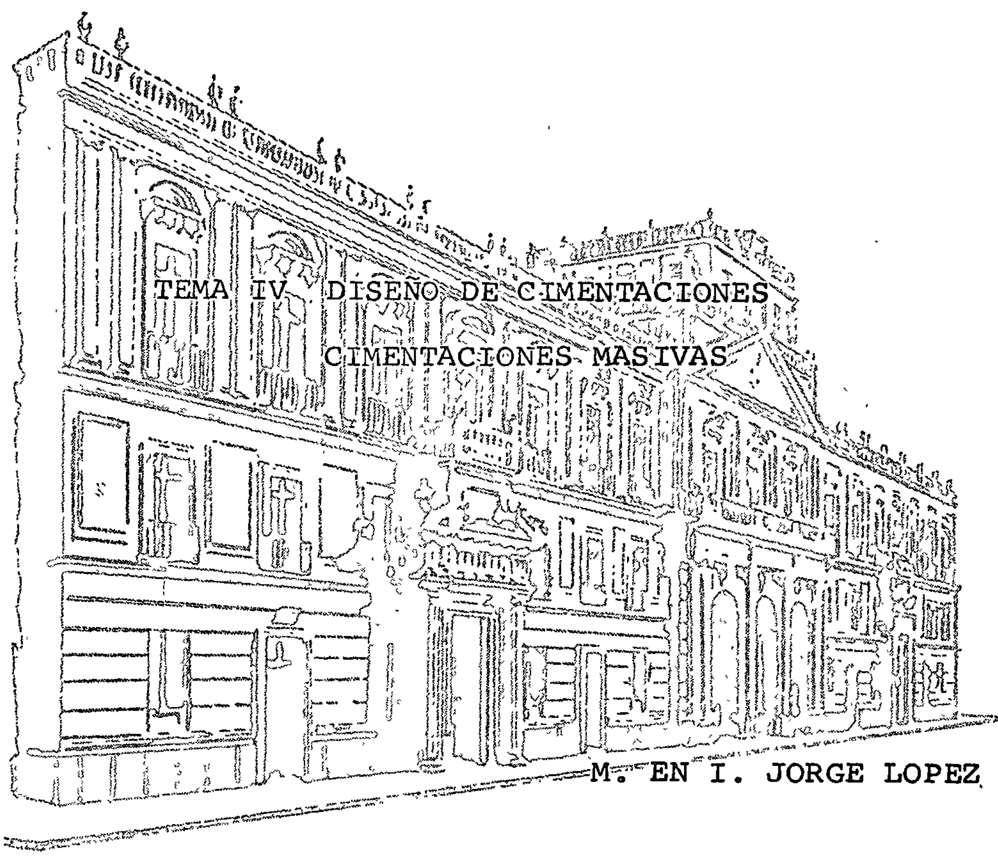

TEMA IV DISEÑO DE CIMENTACIONES

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induced by the impacts of a ram having a weight of 160 kg and dropped from a height of 1 to 1.1 m.

Similar investigations conducted under different soil conditions showed that a floating pile base had a very favorable effect on dynamic settlements.

The values of foundation settlements (in field experiments) after 1,500 blows on piles, presented in Table II-2, show that in all cases the rein-

TABLE II-2. DATA ON DECREASE OF DYNAMIC SETTLEMENTS BY THE USE OF PILES

Characteristics of the base	Foundation area in contact with soil, m^2	Settlement of the foundation after 1,500 blows, mm
Loess in natural state	2.0	8.9
Loess reinforced by seven soil piles 4.5 m long	2.0	3.6
Loess reinforced by seven wooden piles 4.5 m long	2.0	1.0
Medium-grained water-saturated yellow sand of medium density	1.5	45.7
Medium-grained water-saturated yellow sand, reinforced by five wooden piles 3 m long	1.6	9.0
Medium-grained water-saturated dense gray sand	1.5	19.0
Medium-grained water-saturated dense gray sand reinforced by four wooden piles 3 m long	1.5	0.5

forcement of the base under the foundation by means of short piles resulted in a considerable decrease of settlements induced by vibrations. When the foundation was erected on loess reinforced by wooden piles, the settlement of the foundation under the action of vibrations was reduced to one-ninth its value on natural soil. A foundation erected on short wooden piles driven into dense gray sands had settlements equal to approximately one-thirtieth of the settlement value on natural sand.

A decrease in residual settlements may also be achieved by decreasing the amplitude of vertical vibrations by selecting rational dimensions for the foundation.

III

THEORY OF VIBRATIONS OF MASSIVE MACHINE FOUNDATIONS

III-1. Vertical Vibrations of Foundations

a. Basic Assumptions. In general, the investigation of vibrations of a massive foundation placed on the soil surface can be reduced to the investigation of vibrations of a solid block resting on a semi-infinite elastic solid. To date no solution of this problem has been found. Therefore several simplifying assumptions concerning vibrations of solid blocks placed on soil are necessary.

First of all let us assume that there is a linear relation between the soil reacting on a vibrating foundation and the displacement of this foundation. Then the relation between the displacements and the reactions will be determined in terms of the coefficients of elastic uniform and nonuniform compression, as well as a coefficient of elastic shear. In Chap. I, the dependence of these coefficients on the elastic properties of the soil and on the size of the foundation was established; also, the numerical values of the coefficients for various soils were given. In addition, it is necessary to assume that the soil underlying the foundation does not have inertial properties, but only elastic properties as described by the coefficients. Thus, the foundation is considered to have only inertial properties and to lack elastic properties, while the soil is considered to have only elastic properties and to lack properties of inertia. These assumptions concerning foundation and soil make it possible, in the general case, to analyze foundation vibrations as a problem of a solid body resting on weightless springs, the latter serving as a model for the soil.

Frequently foundations under machinery are embedded into soil to a certain depth. In this case, the elastic reactions of the soil act not only along the horizontal contact surface between soil and foundation, but also along the side surfaces of the foundation. The reactions may have considerable effect on the frequencies of free vibration of the foundation and on the coefficient of damping. Therefore, reactions along the side surfaces of the foundation have considerable effect on amplitudes of free or forced vibration under conditions close to resonance.

It is difficult to evaluate in each case the effect of side reactions on foundation vibration. This effect is tentatively taken into account in design computations by increasing the values of the coefficients of elasticity of the base. For example, this method is applied in computations of foundations for forge hammers. If a foundation undergoes only forced vibrations (as, for example, foundations under reciprocating machinery), and the design values of frequencies of natural vibrations of this foundation are larger than the operational frequency of rotation of the machine, then the effect of the side reactions is relatively small and can be neglected. In these cases, disregarding the above soil reactions is conservative, since it results in a design dynamic stability lower than the actual stability.

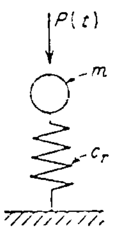


Fig. III-1. Vibration of a centered mass resting on a spring

b. Vertical Vibrations of Foundations Neglecting the Damping Effect of Soil Reactions. Let us consider vibrations of the foundation caused by a vertical exciting force $P(t)$ which changes with time. We assume that the center of mass of the foundation and machine and the centroid of the area of foundation in contact with soil lie on a vertical line which coincides with the direction of action of the exciting force $P(t)$. In this case, the foundation will undergo only vertical vibrations. Since the foundation is assumed to be an absolutely rigid body, its displacement is determined by the displacement of its center of gravity. As mentioned above, weightless springs serve as a model for the soil. Thus, the problem of vertical vibrations of a foundation is reduced to the investigation of vibrations of a centered mass resting on a spring (Fig. III-1). Let us denote by z the vertical displacement of the foundation computed with respect to the equilibrium position. We shall consider z to be positive in a downward direction. If the displacement of the center of gravity of the foundation equals z, then the reaction of the spring (i.e., the foundation base) will equal

$$R = W + c_r z \tag{III-1-1}$$

where W = weight of foundation and machine
c_r = coefficient of rigidity of the base

$$c_r = c_u A \tag{III-1-2}$$

c_u = coefficient of elastic uniform compression of soil
A = horizontal contact area of foundation with soil

If c_u has the dimensions tons per cubic meter and A is expressed in square meters, then c_r will evidently have the dimensions tons per meter. Using d'Alembert's principle, we may obtain the differential equation of vertical foundation vibrations. According to this principle, the equation of motion may be written in the same way as the equation of statics if one adds the inertial force to the external forces acting on a moving body. Then the equation of motion for the foundation will be

$$-mz + W + P(t) - R = 0$$

or, using Eq. (III-1-1), we obtain

$$mz + c_r z = P(t) \tag{III-1-3}$$

where m = mass of foundation and machine, $m = W/g$
g = acceleration of gravity

Dividing both parts of Eq. (III-1-3) by the mass m, we rewrite this equation as follows:

$$z + f_{nz}^2 z = p(t) \tag{III-1-4}$$

where

$$f_{nz}^2 = \frac{c_r}{m} = \frac{c_u A}{m} \tag{III-1-5}$$

$$p(t) = \frac{P(t)}{m}$$

Equation (III-1-4) describes the vertical vibrations of a foundation under the action of an exciting force.

Let us consider the case in which no exciting force acts on the foundation, but the motion results from an impact or from an initial displacement of the foundation. Setting $p(t) = 0$ in Eq. (III-1-4), we obtain

$$z + f_{nz}^2 z = 0 \tag{III-1-6}$$

This equation corresponds to the case in which the motion occurs only under the action of the inertial forces of the foundation and the elastic reaction of the base. Such vibrations are called natural or free vibrations. For example, foundations under forge hammers may be subjected to such vibrations.

The general solution of the homogeneous differential Eq. (III-1-6) may be written as follows:

$$z = A \sin f_{nz} t + B \cos f_{nz} t \tag{III-1-7}$$

Hence it is seen that free vibration under the action of elastic reactions and inertia forces is a harmonic motion with frequency f_{nz}, called the "natural circular frequency of vertical vibrations of the foundation." According to Eq. (III-1-5), this frequency is determined only by the foundation mass and the elasticity of the base and does not depend on the nature or condition of the exciting force.

Since the frequency of vibration is the number of oscillations per second, the period of the natural vibration, i.e., the time for one oscillation, is related to the circular frequency f_{nz} by the following equation:

$$T_{nz} = \frac{2\pi}{f_{nz}}$$

The numbers of oscillations per minute and per second are related to the circular frequency of vibrations by the following simple formulas:

$$N = \frac{60}{2\pi} f_{nz} \qquad n = \frac{f_{nz}}{2\pi}$$

One oscillation per second is called a hertz.

The coefficients A and B in Eq. (III-1-7) represent the amplitudes of natural vibrations of the foundation. Their values depend only on the initial conditions of motion, i.e., on the magnitudes of the velocity (or the displacement) of the foundation at a certain moment of time taken as the initial moment. Natural vibrations of foundations under machines are usually caused by an impact, i.e., the foundations experience a certain initial velocity. Therefore let us consider only this particular case.

Let us assume that at $t = 0$,

$$z = 0 \qquad \text{and} \qquad \dot{z} = v_0 \tag{III-1-8}$$

Differentiating both parts of the solution (III-1-7) with respect to time, we obtain the following expression for the velocity of the foundation:

$$\dot{z} = Af_{nz} \cos f_{nz}t - Bf_{nz} \sin f_{nz}t \tag{III-1-9}$$

Setting $t = 0$ in Eqs. (III-1-7) and (III-1-9), we obtain the following expressions for constants A and B:

$$A = \frac{v_0}{f_{nz}} \qquad B = 0$$

Thus, when vertical natural vibrations of the foundation are caused by an impact, the displacement is determined by the equation

$$z = \frac{v_0}{f_{nz}} \sin f_{nz}t \tag{III-1-10}$$

While the frequency of natural vibrations of a foundation depends only on the inertia and the elastic properties, the amplitude, i.e., the maximum deflection from the equilibrium position, depends also on the initial conditions of the motion, being proportional to initial velocity.

Returning to Eq. (III-1-2) for forced vertical vibrations of foundations, let us consider the case in which the exciting force $P(t)$ is a harmonic

function of the time (for example, $P(t) = p \sin \omega t$ (ω is exciter frequency and

$$p = \frac{P}{m}$$

where P is the exciting force).

An exciting force which changes with time according to $\sin \omega t$ or $\cos \omega t$ is of special interest in the study of forced vibrations of foundations, since in design work, exciting loads imposed by machines are usually harmonic functions of time. Substituting into the right-hand part of Eq. (III-1-1) the expression

$$p(t) = p \sin \omega t$$

we obtain the equation for forced vertical vibrations of foundations:

$$\ddot{z} + f_{nz}^2 z = p \sin \omega t \tag{III-1-11}$$

The general solution of this differential equation presents the sum of two solutions, corresponding to free and to forced vibrations caused by a given exciting force. Due to the action of damping soil reactions, free vibrations are damped out a short time after the beginning of the forced motion of foundations, and there remain only forced vibrations. The solution of Eq. (III-1-11), corresponding only to these steady-state vibrations, is as follows:

$$z = A_z \sin \omega t \tag{III-1-12}$$

We obtain the expression for the amplitude A_z of forced vibrations by substituting the formula for z [Eq. (III-1-12)] into differential Eq. (III-1-11); then we have

$$A_z = \frac{P}{m(f_{nz}^2 - \omega^2)} \tag{III-1-13}$$

The solution (III-1-12) shows that the frequency of forced vibrations is equal to the frequency of the exciting force. Thus, unlike the frequency of natural vibrations, the frequency of forced vibrations does not depend on the inertial and elastic properties of the foundation and its base. Since the exciting loads of machines are usually repeated periodically with every revolution of the machine, in many cases the frequency of the exciting force equals that of rotation of the machine.

In general, this conclusion is valid for all linear mechanical systems not capable of producing self-excited vibrations. Therefore, the identity of the frequency of forced vibrations and the frequency of exciting loads acting on the foundation holds so long as the relationship between elastic foundation displacements and soil reactions is linear. Numerous comparisons of the frequency of forced vibrations of machine foundations with the frequency of exciting forces developed by these machines con...

the coincidence of these frequencies. Thus it is clear that in forced vibrations of a foundation, there really exists a linear relationship between the foundation displacement and the soil reaction.

Figure III-2 presents graphs of the effect of the magnitude of the exciting force on the amplitude of forced vertical vibrations of a test foundation. These graphs substantiate the linear character of the relationship established by Eq. (III-1-13).

It follows from the same equation that the amplitude of forced vibrations depends also on the mass of the foundation and the difference between the frequencies of free and forced vibrations.

In order to better understand the influence of the mass and the natural frequency of the foundation, we transform expression (III-1-13) into

$$A_z = \frac{P}{mf_{nz}^2}\frac{1}{1 - \omega^2/f_{nz}^2} \qquad \text{(III-1-14)}$$

Since

$$\frac{P}{mf_{nz}^2} = A_{st}$$

where A_{st} is the displacement of the foundation under the action of force P if the latter were applied statically, expression (III-1-14) may be rewritten as

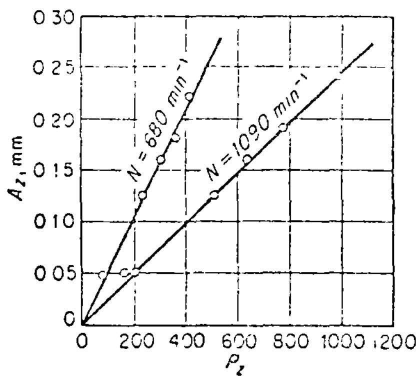


FIG. III-2. Relationship between the amplitude of vertical vibrations A_z and the exciting force P_z.

$$A_z = \eta A_{st} \qquad \text{(III-1-15)}$$

where η is a dynamic modulus (or magnification factor)

$$\eta = \frac{1}{1 - \xi^2} \qquad \text{(III-1-16)}$$

and $\xi = \omega/f_{nz}$ is the frequency ratio.

The value of the dynamic modulus depends only on the interrelationship between the frequency of the exciting force and the natural frequency of the foundation.

If the frequency of the excited vibrations is small in comparison with the natural frequency of the foundation, then the value of the dynamic modulus is close to unity and the amplitude of forced vibrations of the foundation does not differ much from A_{st}; the latter represents a static displacement of the foundation under the action of the exciting force P. With an increase in the frequency of the exciting force, ξ also increases; thus the denominator in expression (III-1-16) decreases, leading to an increase in the value of the dynamic modulus. When $\xi = 1$, i.e., when the frequency of the exciting force equals the natural frequency of the

foundation, the amplitude of vibrations of the foundation [illegible] equals infinity. This corresponds to resonance. With further increase in the frequency of the exciting force, ξ becomes larger than 1; the dynamic modulus continuously decreases with an increase of ξ and when $\xi = \sqrt{2}$, the dynamic modulus again equals 1. For ranges of frequency of exciting force corresponding to $\xi > \sqrt{2}$, the dynamic modulus uniformly decreases, asymptotically approaching zero. Hence it follows that an exciting force, the frequency of which is large in comparison with the natural frequency, may induce forced vibrations with an amplitude of infinitesimal value. This conclusion is used as the guiding principle for the design of various devices for insulation from vibrations, particularly for insulating machines and engines. When the exciting frequency is given, the design of a device for insulating a machine or an engine should be made in such a way that the frequency of natural vibrations of the device is as small as possible compared to the exciting frequency (for example, the frequency of oscillations caused by traffic).

Foundations under reciprocating machinery are usually designed in such a way that the natural frequency of the foundation is higher than the operating frequency of the machine, i.e., $\xi < 1$. If one increases the foundation mass without changing the foundation area, the frequency of natural vibrations decreases and the value of ξ increases. It follows that the denominator in expression (III-1-16) decreases, causing an increase in the dynamic modulus. Thus, other conditions being equal, an increase in the foundation height is accompanied by an increase in the amplitude of forced vibrations. This is the reason why modern foundations for machines (especially for reciprocating machinery) are designed as blocks with large bases and minimum height.

c. Vertical Vibrations of Foundations Considering the Damping Effect of Soil Reactions. As mentioned in the foregoing discussion, under conditions of resonance the amplitude of forced vibrations theoretically approaches infinity. However, this contradicts experimental data which show that under conditions of resonance, the amplitude of vibrations still remains finite. This contradiction between experience and theory is explained by the fact that amplitudes of vibrations with a frequency close to the natural frequency of the foundation are affected by deviations of the mechanical properties of the soil from those of an ideal elastic body. Like any real body, soil deviates from an idealized model represented by an ideally elastic solid. In reality, irreversible processes, characterized by energy dissipation, occur in soil. This deviation of soil properties from those of an ideally elastic solid may be taken into account if one assumes that the reaction of the soil depends not only on the displacement of the foundation, but also upon its velocity. Since the velocities of foundation vibrations are rather low, it can be taken as a first approximation that the damping reactions of soil are proportional to the first power of the velocity of vibration. Then the equation of free vibrations of foundations may be written as follows:

$$\ddot{z} + 2c\dot{z} + f_{nz}^2 z = 0 \qquad \text{(III-1-17)}$$

This expression differs from Eq. (III-1-6) by the presence of the term $2c\dot{z}$. Here

$$c = \frac{\alpha}{2m} \qquad \text{(III-1-17a)}$$

is called the damping constant; its double value equals the coefficient of resistance α per unit of foundation mass. Usually $c < f_{nz}$. The solution of Eq (III-1-17) corresponding to this case is as follows:

$$z = \exp(-ct)(A \sin f_{nd}t + B \cos f_{nd}t) \qquad \text{(III-1-18)}$$

where f_{nd} is the natural frequency of vertical vibrations of foundations in cases where the reaction of soil depends not only on the displacement, but also on the velocity. Substituting solution (III-1-18) into the left-hand part of differential Eq. (III-1-17), we find that the solution will satisfy this equation for any values of A and B if

$$f_{nd}^2 = f_{nz}^2 - c^2$$

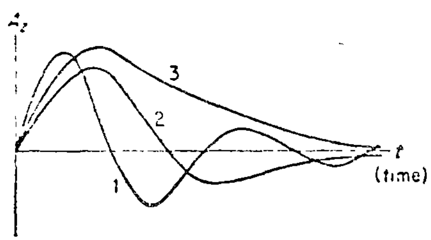


Fig. III-3. Effect of damping on vibration amplitude A_z: curve 1, damping coefficient c is small; curve 2, c approximately equals natural frequency f_{nz}; curve 3, c is larger than f_{nz}

Hence it follows that the damping properties of a soil decrease the natural frequency of vibration of the foundation. If the damping constant is small in comparison to the natural frequency of the foundation, then the influence of the damping properties of the soil on the natural frequency may be neglected.

However, the effect of the damping reactions of soil on the amplitudes of free vibrations of a foundation is rather considerable, even in cases of small values of c. It follows directly from Eq. (III-1-18) that amplitudes of vibrations decrease exponentially with time. This is illustrated by curve 1 of Fig. III-3.

If $c \cong f_{nz}$, then the character of free vibrations of foundations will correspond to curve 2. For large values of damping constant, when $c > f_{nz}$, free vibrations are not possible, and, under the action of an impact or an initial displacement, the foundation will undergo nonperiodic motion, as shown by curve 3 of Fig. III-3.

Thus damping reactions of the soil have considerable effect on free

vibrations of foundations and on amplitudes of forced vibrations, especially under conditions close to resonance.

We obtain an equation for forced vibrations of foundations, including the effect of damping reactions of the soil, if we insert into the right-hand part of Eq (III-1-17) the value of the exciting force; as before, we assume the latter to equal $p \sin \omega t = (P/m) \sin \omega t$; then we have

$$\ddot{z} + 2c\dot{z} + f_{nz}^2 z = p \sin \omega t \qquad \text{(III-1-19)}$$

The solution of this equation, corresponding only to steady forced vibrations of foundations, will be

$$z = A_z^* \sin(\omega t - \gamma) \qquad \text{(III-1-20)}$$

Here A_z^* is the amplitude of forced vibrations:

$$A_z^* = \frac{P}{m\sqrt{(f_{nz}^2 - \omega^2)^2 + 4c^2\omega^2}} \qquad \text{(III-1-21)}$$

The phase shift between the exciting force and the displacement induced by this force equals

$$\tan \gamma = \frac{2c\omega}{f_{nz}^2 - \omega^2} \qquad \text{(III-1-22)}$$

Similarly, Eq (III-1-21) may be reduced to the form

$$A_z^* = \eta^* A_{st}$$

The dynamic modulus η^* in this case will equal

$$\eta^* = \frac{1}{\sqrt{(1 - \xi^2)^2 + 4\Delta^2\xi^2}} \qquad \text{(III-1-23)}$$

where $\Delta = \frac{c}{f_{nz}}$

Figure III-4 presents curves of interrelationship between η^* and ξ; the latter is the ratio of the frequency of forced vibrations to the natural frequency of the foundation. These graphs are plotted for different magnitudes of Δ, proportional to the damping constant.

The particular case $\Delta = 0$ corresponds to the previously discussed case of foundation vibrations where the damping effects of soil reactions were not considered. Only here will the amplitude at resonance increase without limit. At all other times, when $\Delta \neq 0$, the amplitude remains finite. The larger the value of Δ, the smaller the amplitude. Under conditions of damping, the maximum value of the amplitude corresponds to

$$\xi = \sqrt{1 - 2\Delta^2}$$

Thus if damping occurs, the resonance frequency decreases somewhat, and the dynamic modulus at resonance equals

$$\eta^*_{max} = \frac{1}{2\Delta\sqrt{1-\Delta^2}} \qquad \text{(III-1-24)}$$

Hence it follows that the larger the damping constant, the smaller the dynamic modulus at resonance

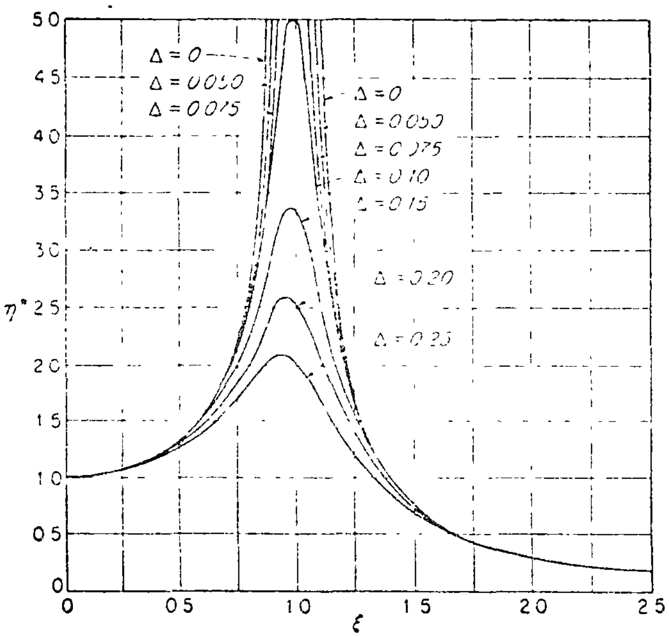


FIG III-1. Relationship between the dynamic modulus of damped vibrations η^* and the ratio ξ of actual to natural frequencies ω/f_{rz}, for varying values of the reduced damping coefficient Δ

The graphs presented in Fig. III-1 also show that the greatest effect of damping reactions of soil is observed in the zone of resonance, when the value of ξ is approximately equal to unity When the difference $1 - \xi^2$ increases, the influence of damping soil properties on amplitudes of forced vibrations decreases, when ξ is large in comparison to unity, this effect may be neglected Since foundations under machines with a steady regime of work are usually designed in such a way that there is a significant difference between ξ and unity, the effect of soil damping may be neglected in many computations of amplitudes of forced vibrations

d The Effect of Soil Inertia on Forced Vertical Vibrations of Foundations The foregoing theory of vertical vibrations of foundations is based on the assumption that soil reactions may be represented by weightless springs

characterized by the coefficient c_z This model [illegible] considerably [illegible] the real properties of soil Therefore, the results obtained should be considered as a first approximation only. As stated previously, an accurate solution of the problem of vertical vibrations of foundations resting on soil necessitates a consideration of the problem of vibrations of a solid resting on an elastic base, which in the simplest case presents a semi-infinite elastic solid. Limiting our analysis by considering only vertical forced vibrations induced by the force $P \sin \omega t$ which harmonically changes with time, we may write the equation for this case as follows

$$m\ddot{z} + R \exp(i\omega t) = P \exp[i(\omega t + \epsilon)] \qquad \text{(III-1-25)}$$

where R = magnitude of soil reaction against foundation

ϵ = phase shift between exciting force and soil reaction

In order to solve Eq. (III-1-25) it is necessary to determine the dependence of the value R upon the displacement, the characteristics of the foundation, and the soil properties.

E Reissner[7] gave an approximate solution of the problem of vibrations of a solid body resting on an elastic semi-infinite [illegible] the computation of R, he used the magnitude of the settlement of the soil under the center of a uniformly loaded absolutely [illegible] O Ya Shekhter[8] showed [illegible] involved in Reissner's solution [illegible] solved the same problem taking the magnitude of settlement of soil (needed for the computation of the value R) as an arithmetic mean between the magnitudes of settlement under the center of a flexible circular area and under its edge. As a result of rather complicated computations which are omitted here, the following simple relationship between R and z was established:

$$z = -\frac{R}{r_0 G}(f_1 + if_2)\exp(i\omega t) \qquad \text{(III-1-26)}$$

where G = modulus of elasticity in shear of soil

r_0 = radius of a circle = $\sqrt{A/\pi}$

A = contact area between foundation and soil

f_1, f_2 = functions depending on ratio between radius r_0 of circle and length of shear waves propagated by foundation under machine, and also depending on Poisson's ratio of soil

The following formulas give values of f_1 and f_2 corresponding to a Poisson ratio of $\nu = 0.5$, with a degree of precision sufficient for practical computations:

$$f_1 = -0.130 + 0.0536\kappa^2 - 0.0078\kappa^4 + \cdots$$
$$f_2 = 0.0545J_1(1.047\kappa)[1 + J_0(1.047\kappa)] + 0.0174\kappa - 0.0065\kappa^3 + \cdots$$

[7] This mistake was also noted in the paper by R. N. [illegible] N. Bycroft [illegible] W [illegible] J Appl Mech, vol 22, no [illegible]

where $x = 2\pi r_0/L_s$

L_s = length of shear waves propagated from foundation

J_1, J_0 = Bessel's functions of first kind, of order one, zero

The length of the shear waves is

$$L_s = 2\pi \frac{v_s}{\omega}$$

where v_s is the velocity of shear waves in soil. For conventional designs of machinery foundations the value x is considerably smaller than unity.

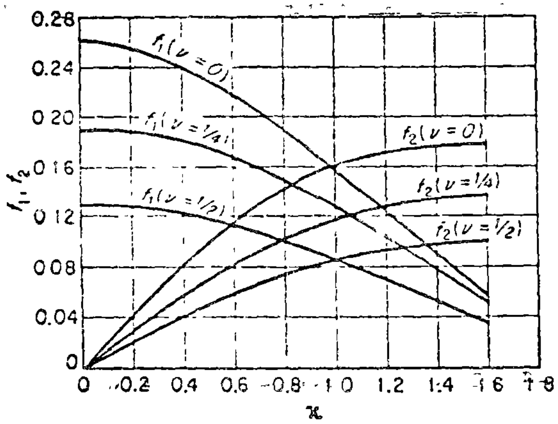


FIG. III-5. Auxiliary diagram for the solution of Eq. (III-1-26).

Figure III-5 presents graphs of f_1 and f_2 depending on magnitudes of the independent variable x, these graphs are plotted for Poisson's ratio equal to 0, 0.25, and 0.5 and make it possible to avoid computations when determining values of f_1 and f_2 corresponding to selected values of x and of ν.

Substituting the value of z determined by Eq. (III-1-26) into the left-hand part of Eq. (III-1-25), we obtain:

$$\frac{Rm\omega^2}{Gr_0}(f_1 + if_2)\exp(i\omega t) + R\exp(i\omega t) = P\exp[i(\omega t + \epsilon)]$$

From this we obtain two equations for determining R and ϵ:

$$\frac{m\omega^2 f_2}{Gr_0} R = P \sin \epsilon$$

$$\left(\frac{m\omega^2 f_1}{Gr_0} + 1\right) R = P \cos \epsilon$$

Solving this system of equations, we find

$$\tan \epsilon = \frac{(m\omega^2/Gr_0)f_2}{1 + (m\omega^2/Gr_0)f_1} \qquad \text{(III-1-27)}$$

$$R = \frac{P}{\sqrt{[1 + (m\omega^2/Gr_0)f_1]^2 + [(m\omega^2/Gr_0)f_2]^2}} \qquad \text{(III-1-28)}$$

The expression thus found for R is substituted into the right-hand part of Eq. (III-1-26); by neglecting the imaginary part, we obtain the following formula for the amplitude a of forced vibrations of the foundation:

$$a = \frac{P}{Gr_0}\sqrt{\frac{f_1^2 + f_2^2}{[1 + (m\omega^2/Gr_0)f_1]^2 + [(m\omega^2/Gr_0)f_2]^2}} \qquad \text{(III-1-29)}$$

The phase shift between the exciting force P and the displacement z equals $\varphi = \alpha + \epsilon$, where $\tan \alpha = -f_1/f_2$ (phase shift between displacement and reaction of soil). Using Eq. (III-1-27), we obtain

$$\tan \varphi = -\frac{f_2}{f_1 + (m\omega^2/Gr_0)(f_1^2 + f_2^2)} \qquad \text{(III-1-30)}$$

Let us introduce a dimensionless value b;

$$b = \frac{m}{\gamma r_0^3} \qquad \text{(III-1-31)}$$

where γ is the soil density. Then

$$\frac{m\omega^2}{Gr_0} = x^2 b$$

and formulas for the amplitude and phase of vibrations will be rewritten as follows.

$$\frac{aGr_0}{P} = \sqrt{\frac{f_1^2 + f_2^2}{(1 + x^2 bf_1)^2 + (x^2 bf_2)^2}} \qquad \text{(III-1-32)}$$

$$\tan \varphi = -\frac{f_2}{f_1 + bx^2(f_1^2 + f_2^2)} \qquad \text{(III-1-33)}$$

Figure III-6 presents graphs of changing aGr_0/P depending on changes of x, or, what is the same thing, changes in the frequency of excitement. These graphs have much in common with resonance curves for a system with one degree of freedom subjected to damping.

Thus, although the initial Eq. (III-1-25) does not take into account damping properties of soil, amplitudes of vibrations never reach infinity with changes in frequencies of excitement as is the case in an ideally elastic system with one degree of freedom. This means that even an ideally elastic soil has a damping effect on the amplitude of foundation vibrations. This is explained by the fact that the energy of a vibrating foundation, due to its propagation into the soil, is continuously dissipated; therefore the vibrations of a foundation, even of one resting on an ideal elastic solid representing a semi-infinite elastic mass, are damped with

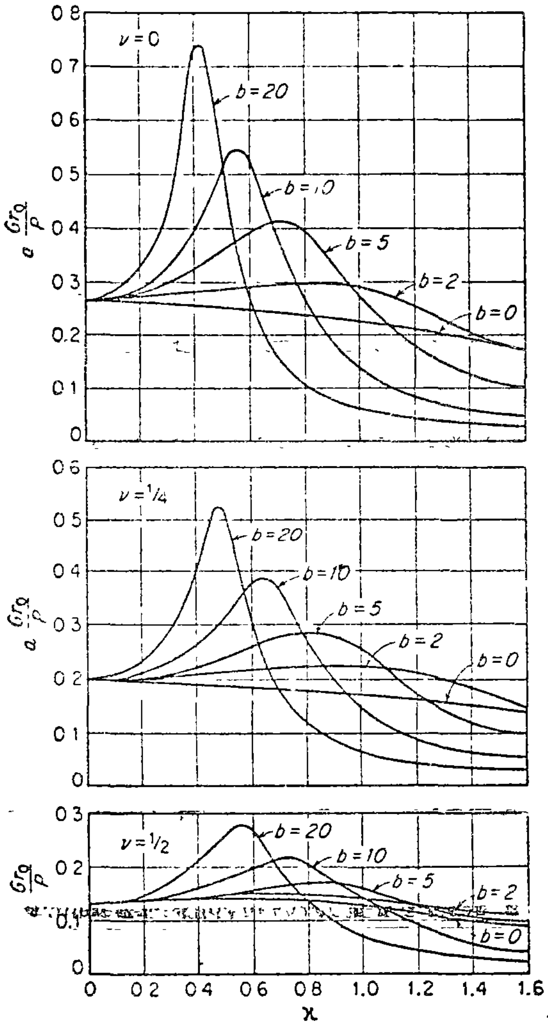


FIG. III-6. Auxiliary diagram for the solution of Eq. (III-1-32).

time. Since, other conditions being equal, amplitudes of maximum vibrations depend upon the value of Poisson's ratio, it follows that the damping of foundation vibrations also depends thereon. A comparison of graphs in Fig. III-6, plotted for several values of Poisson's ratio, shows

that an increase in this ratio leads to an increase in dissipation of energy from a vibrating foundation into the soil, and hence to a result equivalent to an increase in the damping properties of the soil.

The fact that amplitudes of vibrations depend considerably on the value of b shows that the damping properties of soil are determined not only by its characteristics (inertia and elastic properties) but also by the size and mass of the foundation.

The interrelationship between resonance values aGr_0/P and the value b is illustrated in Fig. III-7 by dashed lines. It is seen that this relationship is close to a linear one of the type

$$a_r \frac{Gr_0}{P} = k + pb \qquad \text{(III-1-34)}$$

where a_r is the amplitude under conditions of resonance.

The value b is determined by Eq. (III-1-31). Taking into account that

$$m = \frac{W}{g} \qquad r_0 = \sqrt{\frac{A}{\pi}}$$

where W is the weight of the foundation and machine and A is the contact area of the foundation with soil, Eq. (III-1-31) may be rewritten as follows:

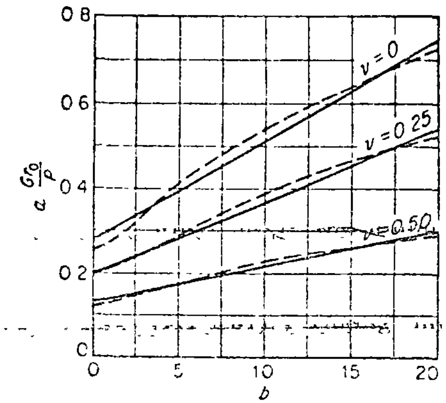


FIG. III-7. Dependence of peak (resonance) values of Fig. III-6 on Poisson's ratio ν.

$$b = \frac{\pi^{1/2}}{\gamma} \frac{p_{st}}{\sqrt{A}} \qquad \text{(III-1-35)}$$

where γ = unit weight of soil

p_{st} = normal static pressure

Substituting this value of b into Eq. (III-1-34), we obtain for the amplitude of vibrations under conditions of resonance (for $P = 1$)

$$a_r = \frac{k\pi^{1/2}}{GA^{1/2}} + \frac{l\pi^2}{G\gamma} \frac{p_{st}}{A} \qquad \text{(III-1-36)}$$

Since the unit weight of soils varies within a comparatively narrow range, the value of γ in Eq. (III-1-36) may be considered to be constant, equaling, for example, 1.70 tons/m³. Hence the conclusion is possible that the inertial characteristics of soil have small effect on amplitudes under conditions of resonance.

Equation (III-1-36) also shows that the resonance amplitude (reduced to unit of exciting force) increases with an increase in normal static pressure on the foundation base and decreases with an increase in the foundation contact area. Thus, under otherwise equal conditions, the damping of foundation vibrations by the soil decreases with an increase of the static pressure beneath the foundation. With an increase in the foundation contact area, resonance amplitudes decrease, i.e., damping increases. Consequently, damping properties of soil depend not only on its physicomechanical properties, but also on the foundation size and mass. Hence it is understandable why a damping constant established on the basis of the resonance curve of forced vibrations or on the basis of observation of natural vibrations of a foundation depends on the characteristics of the foundation itself.

In order to compare formulas for computations (III-1-32) and (III-1-33) with the results of the theory of vibrations, let us rewrite expression (III-1-21) for the amplitude of vibrations of a foundation on a base of zero inertia, considering the base to consist of weightless springs.

$$A_z^* = \frac{P}{\sqrt{(c_z - m_t\omega^2)^2 + (\alpha\omega)^2}} \qquad \text{(III-1-37)}$$

where m_t denotes the total mass of foundation and soil,

$$m_t = m + m_s$$

Assuming that the foundation contact area is a circular area with radius r_0, we transform Eq. (III-1-37) by introducing x and b as variables.

Multiplying (III-1-37) by Gr_0, we obtain

$$A_z^* \frac{Gr_0}{P} = \frac{1}{[(c_z/Gr_0) - (m_t\omega^2/Gr_0)]^2 + (\alpha\omega/Gr_0)^2} \qquad \text{(III-1-38)}$$

According to (III-1-5), if $m = m_t$ we will have

$$c_z = m_t f_{nz}^2 = c_u A$$

Further, using (I-1-9), (I-2-12), and (I-2-13), we obtain

$$c_z = c_e \frac{E\sqrt{A}}{(1 - \nu^2)} = c_e \frac{2\sqrt{\pi}\, Gr_0}{1 - \nu}$$

Here, as in (I-2-14), c_e is a coefficient which depends on the geometric form of the foundation contact area; for a circular area it equals 1.083 if one considers settlement as the arithmetic mean of settlements under the center of the circular area and the edge. We denote

$$\frac{\alpha}{2m_t f_{nz}} = \xi \qquad \text{(III-1-39)}$$

where α refers to Eq. (III-1-17a). Then

$$\left(\frac{\alpha\omega}{Gr_0}\right)^2 = \frac{4m_t^2\xi^2\omega^2 f_{nz}^2}{G^2r_0^2} = \frac{8\sqrt{\pi}}{1-\nu}\,\frac{\omega^2 m_t \xi^2}{Gr_0}\, c_e$$

Let $m_t = \beta m$, where β is the coefficient of increase in foundation mass due to participation of soil.

$$\frac{m_t\omega^2}{Gr_0} = x^2 b\beta$$

Consequently,

$$A_z^* \frac{Gr_0}{P} = \frac{1}{\sqrt{[3.84/(1 - \nu) - x^2b\beta]^2 + 15.35x^2b\xi^2\beta/(1 - \nu)}} \qquad \text{(III-1-40)}$$

Taking the derivative of the right-hand side of Eq. (III-1-40) and equating it to zero, we find that resonance corresponds to the following value of the independent variable x:

$$x = \sqrt{\frac{3.84(1 - 2\xi^2/\beta)}{b(1 - \nu)\beta}}$$

Let us assume that for selected values ν and b, the value of the amplitude of vibration computed from Eq. (III-1-32) will attain its maximum at $x = x_0$. Assuming that the maximums of amplitudes computed from Eqs. (III-1-40) and (III-1-32) correspond to the same value of the independent variable x, we obtain

$$\sqrt{\frac{3.84(1 - 2\xi^2/\beta)}{b(1 - \nu)\beta}} = x_0 \qquad \text{(III-1-41)}$$

Let us assume also that the maximum values of amplitudes of vibrations computed from Eqs. (III-1-40) and (III-1-32) coincide with each other. Then we obtain:

$$\frac{1}{\sqrt{[3.84/(1 - \nu) - x_0^2b\beta]^2 + 15.35x_0^2b\xi^2/(1 - \nu)}} = a_r \qquad \text{(III-1-42)}$$

Equations (III-1-40) and (III-1-42) may be transformed as follows:

$$\frac{\xi^2}{\beta} = \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{1 - \nu}{3.84a_r}\right)^2}\right] \qquad \text{(III-1-43)}$$

$$\beta = \frac{3.84(1 - 2\xi^2/\beta)}{bx_0^2(1 - \nu)} \qquad \text{(III-1-44)}$$

Figure III-8 presents graphs of ξ and β as functions of b for soils with different values of Poisson's ratio ν.

If we take from these graphs values of the coefficients ξ and β corresponding to various values of b and ν, and then, using Eq. (III-1-40), plot resonance curves of forced vibrations, we shall see that these curves coincide fairly well with graphs plotted on the basis of Eq. (III-1-32). Such comparisons were made for $b = 2.5$, 10, and 20 and $\nu = 0$, 0.25, and 0.50. The results of computations from Eq. (III-1-40) using values of coefficients ξ and β taken from Fig. III-8*a* and *b* coincide so well with the results of computations from Eq. (III-1-32) that the two curves merge completely.

Thus it is established that if numerical values of the reduced coefficient of damping ξ and the coefficient of increase in mass β are taken from the graphs of Fig. III-8*a* and *b*, then the results of computations of amplitudes

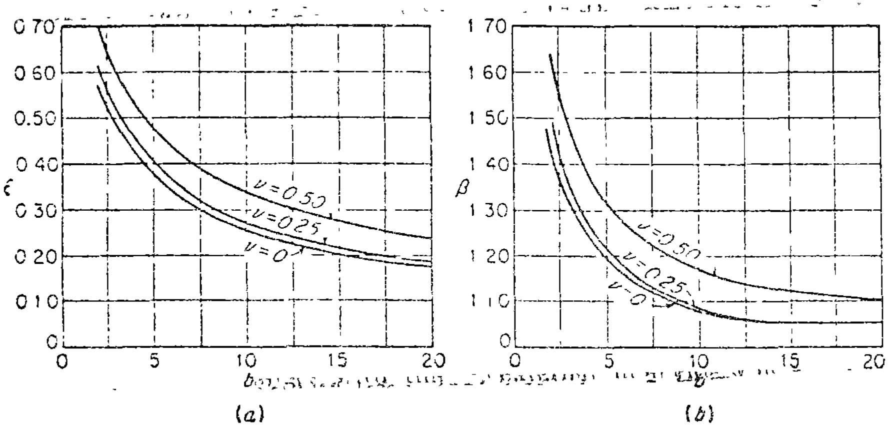


FIG. III-8 Auxiliary diagrams for the determination of the reduced damping coefficient ξ and the coefficient of mass increase β.

of forced vibrations from the approximate formula (III-1-40) (the latter obtained on the basis of an assumption that soil is represented by weightless springs) coincide fairly well with the results of a more precise theory which takes into account inertial properties of soil.

An analysis of Fig. III-8*a* and *b* leads to some general conclusions in respect to the effect of soil inertia on foundation vibrations. It follows directly from Fig. III-8*b* that, for all values of ν, with a decrease of b the coefficient of increase of mass β will increase, hence the effect of soil inertia on forced vertical vibrations of the foundation will grow.

According to Eq. (III-1-35) the value of b is proportional to the pressure p_{st} on the foundation contact area and inversely proportional to the square root of the area. Small values of b correspond to foundations with small height and large contact area. Such foundations are influenced more strongly by the effect of soil inertia than are foundations with considerable height and relatively small contact area. Thus foundations having the shape of thick slabs cause larger masses of soil to vibrate than do foundations having the shape of high blocks.

Numerical values of b for machine foundations attain 7 to 15. For these values, the coefficient of increase of mass β, even for $\nu = 0.5$, does not surpass the value 1.23. Hence, the soil mass participating in foundation vibrations does not exceed 23 per cent of the foundation mass. Since the period of natural vertical vibrations is inversely proportional to the square root of mass, it follows that the results of calculations of the period of natural vibrations of a foundation which include soil inertial effects are not more than 10 per cent smaller than those which neglect soil inertial properties. Thus a correction accounting for the influence of soil inertia will not be significant. Errors in computations of natural frequencies or amplitudes of forced foundation vibrations are usually not less than 10 to 15 per cent; therefore the effect of soil inertia on machinery foundations may be considered to be so slight that it may be neglected in many engineering calculations.

The foregoing discussion on the effect of soil inertia refers to a foundation resting on the soil surface. Where the foundation is embedded and the soil reacts not only on the horizontal foundation base area but also on the foundation sides, there can be a considerable soil-inertia effect.

III-2. Rocking Vibrations of Foundations

Let us consider vibrations of foundations due to the action of a rocking external moment which changes with time according to the function $M \sin \omega t$ and which lies in one of the principal vertical planes of the foundation (Fig. III-9). It is assumed that the center of inertia of the mass of the foundation and the centroid of its horizontal base area lie on a vertical line located in the plane of the rocking moment.

Let us further assume that the elastic resistance of the soil against sliding of the foundation is so large in comparison to the resistance of the foundation against rocking that it may be considered to be infinitely great.

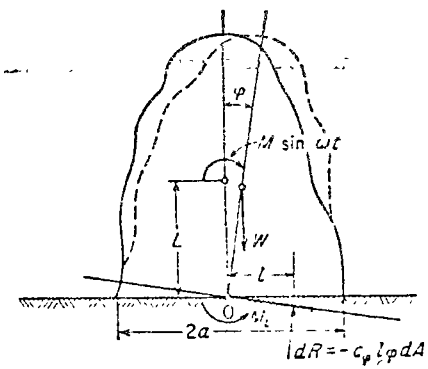


FIG. III-9 Analysis of rocking motion of a foundation.

In this case, the motion induced by an external moment $M \sin \omega t$ will be a rocking around the axis passing through the centroid of the area of foundation in contact with soil, perpendicular to the plane of vibrations. The position of the foundation is determined by one independent variable: the angle of rotation φ of the foundation around the axis passing through the point O (Fig. III-9).

Let us assume that at a certain instant the foundation has rotated a small angle φ around this axis.

The equation of its motion will be

$$-W_0\ddot{\varphi} + \Sigma M_i = 0 \qquad \text{(III-2-1)}$$

where W_0 = moment of inertia of mass of foundation and machine with respect to axis of rotation

ΣM_i = sum of all external moments with respect to same axis

In this case, the foundation weight and the soil reaction are external forces.

a. Foundation Weight. The moment of this force W in respect to the axis of rotation is

$$LW\varphi$$

where L is the distance between the axis of rotation and the center of gravity of the vibrating mass.

b. Soil Reaction. An element dA of the foundation area in contact with soil, located at a distance l from the axis of rotation, is acted upon by the soil reaction

$$dR = c_\varphi l\varphi\, dA$$

where c_φ is the coefficient of elastic nonuniform soil compression.

The moment of the elementary force dR with respect to the axis of rotation is

$$dM_r = -l\,dR = -c_\varphi l\varphi\, dA$$

If it is assumed that the foundation does not lose contact with the soil, then the total reactive moment against the foundation area in contact with soil is

$$M_r = -c_\varphi\varphi \int_A l^2\, dA = -c_\varphi I\varphi \qquad \text{(III-2-2)}$$

where I is the moment of inertia of the foundation area in contact with soil with respect to the axis of rotation of the foundation.

By adding the exciting moment $M \sin \omega t$ to the two other moments, we obtain the equation of forced vibrations of the foundation:

$$-W_0\ddot{\varphi} + WL\varphi - c_\varphi I\varphi + M \sin \omega t = 0$$
$$W_0\ddot{\varphi} + (c_\varphi I - WL)\varphi = M \sin \omega t \qquad \text{(III-2-3)}$$

By equating M to zero, we obtain the equation of free rocking vibrations with respect to the y axis:

$$W_0\ddot{\varphi} - (c_\varphi I - WL)\varphi = 0 \qquad \text{(III-2-4)}$$

The solution of this equation is

$$\varphi = C \sin (f_{n\varphi}t + \varphi_0) \qquad \text{(III-2-5)}$$

where

$$f_{n\varphi} = \frac{c_\varphi I - WL}{W_0} \qquad \text{(III-2-6)}$$

and $f_{n\varphi}$ = natural frequency of rocking vibrations of foundation

c, φ_0 = arbitrary constants determined from initial conditions of motion of foundation

The solution of Eq. (III-2-3) will have a form similar to that for forced vertical vibrations [Eq. (III-1-13)], but instead of P, m, and f_{nz}, the values of M, W_0, and $f_{n\varphi}$ should be inserted.

In the case under consideration, the following expression will be obtained for the amplitude of vibrations:

$$A_\varphi = \frac{M}{W_0(f_{n\varphi}^2 - \omega^2)} \qquad \text{(III-2-7)}$$

Since the product WL is usually small in comparison with $c_\varphi I$, that term may be neglected in Eq. (III-2-6); we then obtain for the frequency of natural rocking vibrations

$$f_{n\varphi}^2 = \frac{c_\varphi I}{W_0} \qquad \text{(III-2-8)}$$

If the foundation base area has a rectangular form with sides a and b, and a is the side perpendicular to the axis of rotation, we have

$$I = \frac{ba^3}{12}$$

$$f_{n\varphi}^2 = \frac{ba^3}{12}\frac{c_\varphi}{W_0}$$

It follows from the latter formula that the length of the side of the foundation area in contact with soil and perpendicular to the axis of rotation has considerable effect on the natural frequency of rocking vibrations of the foundation. Depending on the selected length, the natural frequency may change considerably; hence the amplitude of forced vibrations will also change. The length of the other side of the foundation area, i.e., the one parallel to the axis of rotation, does not much influence the values of $f_{n\varphi}$ and A_φ. This length is usually selected on the basis of design considerations.

The amplitude of the vertical component of vibrations of the edge b or the foundation area in contact with soil is

$$A = \frac{a}{2} A_\varphi$$

or

$$A = \frac{Ma}{2W_0(f_{n\varphi}^2 - \omega^2)}$$

Rocking vibrations occur mostly in high foundations under machines having unbalanced horizontal components of exciting forces and exciting moments. For example, such vibrations occur in foundations under sawmill log-sawing frames. These foundations are usually high and project above the first floor of the sawmill. Figure III-10 illustrates the distribution of amplitudes of horizontal forced vibrations along the height of the foundation under a log frame. It is seen from the graph that the centroid of the foundation area in contact with soil is subjected to forced rocking vibrations.

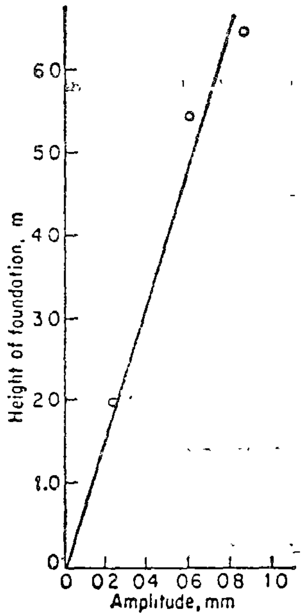


FIG. III-10. Variation of the horizontal amplitude of rocking forced vibrations along the height of a foundation.

Therefore Eq. (III-2-7) may be used for the computation of the amplitude of forced vibrations of foundations under log frames induced by the horizontal component of exciting forces developed in these frames.

III-3. Vibrations of Pure Shear

If the resistance of soil to compression is large in comparison with the resistance to shear, then displacement of the foundation under the action of horizontal forces will occur mainly in the direction of the action of horizontal exciting forces.

Let us assume that a horizontal exciting force $P_T \sin \omega t$ acts on the foundation. The equations of forced and free vibrations will be analogous to the equations of vertical vibrations of a foundation, e.g., Eq. (III-1-11), in which c_r should be inserted instead of c_u; thus the equation of forced horizontal vibrations will be

$$\ddot{x} + f_{nx}^2 x = p_T \sin \omega t \qquad \text{(III-3-1)}$$

where x is the horizontal displacement of the center of gravity of the foundation and

$$f_{nx}^2 = \frac{c_r A}{m} \qquad \text{(III-3-2)}$$

f_{nx} is the natural frequency of vibrations in shear

The solution of Eq. (III-3-1) is

$$A_x = \frac{P_T}{m(f_{nx}^2 - \omega^2)} \qquad \text{(III-3-3)}$$

All conclusions and formulas obtained while considering vertical vibrations of foundations apply also to vibrations in shear.

In addition to the foregoing type of vibrations of foundations, characterized by horizontal displacement of the center of gravity of the vibrating mass, vibrations in shear may have a form of rotational vibrations with respect to the vertical axis passing through the center of gravity of the foundation and the centroid of the area of its base. Letting

W_z = moment of inertia of vibrating mass with respect to above axis
J_z = polar moment of foundation base area
ψ = angle of torsion of foundation
$M_z \sin \omega t$ = exciting moment acting in horizontal plane
c_ψ = coefficient of elastic nonuniform shear

we obtain the following equation of forced vibrations of a foundation induced by an exciting moment.

$$W_z \ddot{\psi} + c_\psi J_z \psi = M_z \sin \omega t \qquad \text{(III-3-4)}$$

A particular solution of this equation may be presented in the form

$$\psi = \frac{M_z}{W_z(f_{n\psi}^2 - \omega^2)} \qquad \text{(III-3-5)}$$

where $f_{n\psi}^2 = c_\psi J_z / W_z$ is the square of the natural frequency of vibrations of a foundation for the vibrations under consideration.

III-4. Vibrations of Foundations Accompanied by Simultaneous Rotation, Sliding, and Vertical Displacement

a. Equations of Vibration. The foregoing discussion concerned cases of vibrations of massive foundations in which the soil was characterized by infinite rigidity with respect either to compression or to shear. Let us consider now the simplest case of vibrations of foundations, where the soil is able to offer elastic resistance both to compression and to shear. As before, it is assumed that the center of gravity of the foundation and machine and the centroid of the foundation base area are located on a vertical line which lies in the main central plane of the foundation. External exciting forces induced by the machine also lie in this plane; if O (Fig. III-11) is the center of gravity of the vibrating mass, then these forces may be reduced to the exciting force $P(t)$ applied at O and a couple with moment $M(t)$. Let us ...

the center of gravity of the foundation and machine mass at an instant when the foundation is motionless; the direction of coordinate axes is shown in Fig. III-11.

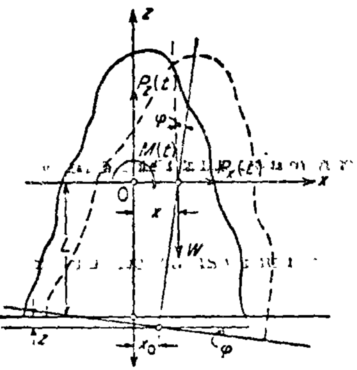


FIG III-11. Analysis of combined types of foundation motion

Under the action of loads $P(t)$ and $M(t)$, the foundation will undergo a two-dimensional motion determined by the values of three independent parameters: the projections x and z of displacement of the foundation center of gravity on the coordinate axes and the angle φ of rotation of the foundation with respect to the y axis which passes through the center of gravity of the foundation and machine, perpendicular to the plane of vibrations.

By projecting all forces acting on the foundation at time t on the x and z axes and adding to them the projections on the same axes of the inertia forces, we obtain, according to d'Alembert's principle,

$$-m\ddot{x} + \Sigma X_i = 0$$
$$-m\ddot{z} + \Sigma Z_i = 0 \qquad \text{(III-4-1)}$$

where m = foundation mass

X_i, Z_i = projections on x, z axes of all external forces acting on foundation

The equation of the moments with respect to the y axis should also be added to Eqs. (III-4-1).

$$-M_m\ddot{\psi} + \Sigma M_i = 0 \qquad \text{(III-4-2)}$$

where M_m is the moment of inertia of the mass with respect to the y axis.

The following forces are acting on the foundation at time t:

1. Weight W of foundation and machine. A projection of the force W on the x axis equals zero, that on the z axis equals

$$Z_1 = -W$$

2. Soil reaction caused by settlement of the foundation under the action of weight. A projection of this force on the x axis also equals zero; the projection of this force on the z axis equals

$$Z_2 = c_u A z_{st}$$

where c_u = coefficient of elastic uniform compression of soil

A = area of foundation in contact with soil

z_{st} = elastic settlement caused by action of weight of foundation and machine

The soil reaction is applied to the centroid of the contact area of foundation and soil. It produces a moment with respect to the y axis:

$$M_1 = WL\varphi$$

where L is the distance from the center of gravity of the mass to the foundation base.

If at a given instant of time t, the foundation has a displacement z measured from the equilibrium position, then the soil reaction induced by this displacement is

$$Z_3 = c_u A z$$

3. Horizontal reaction of elastic resistance of soil. Its projection on the x axis is

$$X_1 = -c_\tau A x_0$$

where c_τ = coefficient of elastic uniform shear of soil

x_0 = displacement centroid of contact area of foundation

$$x_0 = x - L\varphi$$

where x is horizontal displacement of the common center of gravity of foundation and machine.

Substituting this value of x_0 into the expression for X_1, we obtain

$$X_1 = -c_\tau A(x - L\varphi)$$

The moment of this force with respect to the y axis is

$$M_2 = c_\tau AL(x - L\varphi)$$

4. Reactive resistance of soil induced by rotation of foundation base area. In order to compute the moment caused by this resistance, let us single out an infinitely small element dA of the foundation area in contact with soil. The reaction dR of soil on this element is

$$dR = c_\varphi l\varphi\, dA$$

where c_φ = coefficient of elastic nonuniform compression of soil

l = distance between area element dA and axis of rotation

The moment of this elementary reaction with respect to the y axis is

$$dM_3 = -c_\varphi l^2\varphi\, dA$$

By integrating over the whole foundation area in contact with soil, we obtain the total reactive moment of soil developed when the foundation base contact area turns an angle φ. This moment is

$$M_3 = -c_\varphi I\varphi$$

where I is the moment of inertia of foundation contact area with respect to the axis passing through the centroid of this area, perpendicular to the plane of vibrations.

To these four forces there should be added the projections $P_z(t)$ and $P_x(t)$ of the external exciting force $P(t)$ on the coordinate axes and the moment $M(t)$.

Substituting into Eqs. (III-4-1) and (III-4-2) the established values of the projections of forces on the x and z axes, as well as the values of moments with respect to the y axis, we obtain, after some elementary transformations, a system of three differential equations of forced vibrations of a foundation:

$$m\ddot{z} + c_u A z = P_z(t) \qquad \text{(III-4-3)}$$

$$\begin{aligned} m\ddot{x} + c_\tau A x - c_\tau A L\varphi &= P_x(t) \\ M_m\ddot{\varphi} - c_\tau A L x + (c_\varphi I - WL + c_\tau A L^2)\varphi &= M(t) \end{aligned} \qquad \text{(III-4-4)}$$

N. P. Pavliuk[32] was the first to give these equations of foundation vibration

Equations (III-4-4) are interdependent because each of them includes x and φ. Equation (III-4-3) in no way depends upon Eqs (III-4-4) Hence it follows that vertical vibrations of foundation occur independently of vibrations associated with the other two coordinates. If a foundation is acted upon by exciting loads having no vertical components, then no vertical vibrations of the foundation develop In this case the foundation will undergo rotation around the y axis and horizontal displacement in the direction of the x axis If a foundation is acted upon by an exciting load producing only a vertically centered force, then the foundation will undergo only vertical vibrations.

In the same way, if the equilibrium of a foundation at a certain instant of time is disturbed only by a vertical displacement of its center of gravity, or if at this moment the foundation is given a velocity in the vertical direction, then the foundation will undergo only vertical natural vibrations. If the equilibrium of a foundation is disturbed by a displacement of its center of gravity in the horizontal direction or if it is given a velocity in the horizontal direction, then no vertical vibrations appear. In this case the foundation motion will be characterized by changes in two coordinates: x and φ. The same coordinates characterize the foundation motion if its equilibrium is disturbed by changes in either x or φ.

The fact that vertical vibrations of foundations are independent of vibrations in the directions x and φ gives us a chance to consider each type of vibration separately. An investigation of vertical vibrations has already been made in Art. III-1. Therefore it remains to investigate vibrations corresponding to the system of Eqs. (III-4-4).

b Free Vibrations If the equilibrium of a foundation is disturbed by subjecting it at the initial instant of time to certain changes in the

coordinates x and φ and the velocities $\dot{x}$ and $\dot{\varphi}$, then during the time which follows the foundation will be subjected to elastic soil reactions and inertial forces and will undergo free vibrations. The equations of these vibrations are as follows:

$$\begin{aligned} m\ddot{x} + c_\tau A x - c_\tau A L\varphi &= 0 \\ M_m\ddot{\varphi} - c_\tau A L x + (c_\varphi I - WL + c_\tau A L^2)\varphi &= 0 \end{aligned} \qquad \text{(III-4-5)}$$

Particular solutions of these equations may be written in the form

$$x = A_a \sin(f_n t + \alpha) \qquad \varphi = B_a \sin(f_n t + \alpha)$$

where A_a, B_a, and α are arbitrary constants

Substituting these solutions into (III-4-5) and reducing all terms by eliminating $\sin(f_n t + \alpha)$, we obtain two homogeneous equations:

$$\begin{aligned} (c_\tau A - m f_n^2)A_a - c_\tau A L B_a &= 0 \\ -c_\tau A L A_a + (c_\varphi I - WL + c_\tau A L^2 - M_m f_n^2)B_a &= 0 \end{aligned} \qquad \text{(III-4-6)}$$

The constants A_a, B_a, and f_n should satisfy these equations if the particular solutions are to satisfy the system of differential equations of free vibrations of the foundation

System (III-4-6) does not permit the determination of values of all three constants A_a, B_a, and f_n In order to do this, it is necessary to know the initial conditions of the foundation However, if we consider that in Eqs (III-4-6) only A_a and B_a (that is, only the amplitudes of vibrations) are unknown, then we obtain from the first equation

$$A_a = \frac{c_\tau A L}{c_\tau A - m f_n^2} B_a$$

Substituting this expression for A_a into the second equation, we obtain

$$B_a[-c_\tau^2 A^2 L^2 + (c_\varphi I - WL + c_\tau A L^2 - M_m f_n^2)(c_\tau A - m f_n^2)] = 0$$

If B_a does not equal zero, then, in order to satisfy the above equation, it is necessary to assume that the factor in brackets equals zero. Then we obtain the frequency equation

$$\Delta(f_n)^2 = -c_\tau^2 A^2 L^2 + (c_\varphi I - WL + c_\tau A L^2 - M_m f_n^2)(c_\tau A - m f_m^2) = 0 \qquad \text{(III-4-7)}$$

This equation contains only one unknown constant f_n, the natural frequency of vibrations of the foundation.

Let us transform Eq. (III-4-7) by opening brackets and grouping members containing the same powers of f_n Then we obtain a second-degree equation for f_n^2.

After dividing all members of the new equation by mM_m, it may be rewritten as follows:

$$f_n^4 - \left(\frac{c_\varphi I - WL}{M_m} + \frac{c_\tau A}{m}\frac{L^2m + M_m}{M_m}\right)f_n^2 + \frac{c_\varphi I - WL}{M_m}\frac{c_\tau A}{m} = 0$$

Let us denote by M_{m0} the moment of inertia of the total vibrating mass (the foundation and machine) with respect to the axis passing through the centroid of the base contact area and perpendicular to the plane of vibrations; this moment equals

$$M_{m0} = M_m + mL^2$$

Let

$$\frac{M_m}{M_{m0}} = \gamma$$

where $1 > \gamma > 0$.

Substituting $M_m = \gamma M_{m0}$ into the equation for frequencies, we rewrite it as follows:

$$f_n^4 - \left(\frac{c_\varphi I - WL}{M_{m0}} + \frac{c_\tau A}{m}\right)\frac{f_n^2}{\gamma} + \frac{c_\varphi I - WL}{\gamma M_{m0}}\frac{c_\tau A}{m} = 0$$

But according to (III-2-6) and (III-3-2) the expressions

$$\frac{c_\varphi I - WL}{M_{m0}} = f_{n\varphi}^2 \quad \text{and} \quad \frac{c_\tau A}{m} = f_{nx}^2$$

represent limiting frequencies of the foundation when the resistance of soil to shear is very large in comparison to its resistance to rotational vibrations or vice versa.

Using these two expressions we obtain the final equation of frequencies in the following form:

$$\Delta(f_n^2) = f_n^4 - \frac{f_{n\varphi}^2 + f_{nx}^2}{\gamma}f_n^2 + \frac{f_{n\varphi}^2 f_{nx}^2}{\gamma} = 0 \qquad \text{(III-4-8)}$$

This equation will have two positive roots f_{n1} and f_{n2} corresponding to the two principal natural frequencies of the foundation.

It can be proved that the natural frequencies which are the roots of Eq. (III-4-8) have the following interrelationship with the limiting frequencies $f_{n\varphi}$ and f_{nx}: the smaller of the two natural frequencies (for example, f_{n2}) is smaller than the smallest of the two limiting frequencies; the larger natural frequency is always larger than $f_{n\varphi}$ and f_{nx}.

In the case under consideration, involving a foundation with two degrees of freedom, specific forms of vibrations correspond to the frequencies f_{n1} and f_{n2} of the foundation; these vibrations are characterized by a certain interrelationship between the amplitudes A_a and B_a which depends on the foundation size and the soil properties, but does not depend on the initial conditions of foundation motion.

Let us determine from the first equation of system (III-4-6) the ratio A_a/B_a:

$$\rho = \frac{A_a}{B_a} = \frac{f_{nx}^2 L}{f_{nx}^2 - f_n^2} \qquad \text{(III-4-9)}$$

If the foundation vibrates at the lower frequency f_{n2}, then, according to the above statement,

$$f_{nx}^2 - f_2^2 > 0$$

and ρ also is larger than zero; consequently, the amplitudes A_a and B_a have the same sign. It means that during vibration at frequency f_{n2}, when the center of gravity deviates from the equilibrium position,

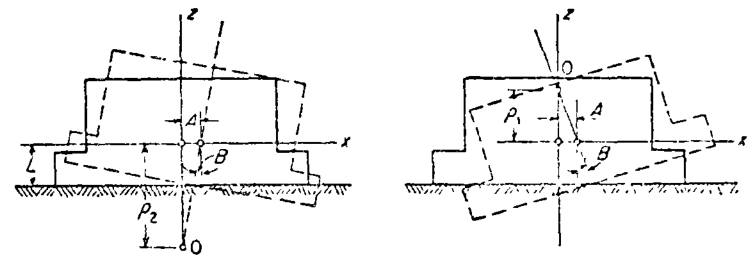


FIG. III-12. Two types of foundation vibrations which correspond to Eq. (III-4-9).

for example, in the positive direction of the x axis, the rotation of the foundation will be also positive, and changes of amplitudes A_a and B_a will be in phase. The form of vibrations in this case will be analogous to that shown in Fig. III-12a; i.e., the foundation will undergo rocking vibrations with respect to a point situated at a distance ρ_2 from the center of gravity of the foundation. The value of ρ_2 is determined by the absolute value of expression (III-4-9) if f_{n2} is substituted for f_n. However, if a foundation vibrates at the higher frequency f_{n1}, then, since $f_{nx}^2 - f_{n1}^2 < 0$, ρ will be negative, and A_a and B_a will be 180° out of phase. Figure III-12b illustrates the form of vibrations corresponding to this case. Here the foundation vibrates around a point which lies higher than the center of gravity and at a distance ρ_1 determined from expression (III-4-9) if f_{n1} is substituted for f.

There is a simple relationship between ρ_2 and ρ_1:

$$\rho_1\rho_2 = i^2$$

where

$$i^2 = \frac{M_m}{m}$$

i is the radius of gyration of the masses of foundation and machine

If the main dimensions of a foundation which determine its mass, base area, and moments of inertia are selected, then the limiting natural frequencies $f_{n\varphi}$ and f_{nx} will depend only on the coefficients of elastic nonuniform compression and shear c_φ and c_τ of the soil.

Often the exact values of these coefficients are not known and only the range of the most probable values of c_φ and c_τ may be assumed. Then the computation of the natural frequencies f_{n1} and f_{n2} should be performed for the whole range of values of these coefficients.

The natural frequencies f_{n1} and f_{n2} of the foundation are determined as the roots of Eq. (III-4-8):

$$f_{n1,2}^2 = \frac{1}{2\gamma}\left[f_{n\varphi}^2 + f_{nx}^2 \pm \sqrt{(f_{n\varphi}^2 + f_{nx}^2)^2 - 4\gamma f_{n\varphi}^2 f_{nx}^2}\right]$$

If it is necessary to compute a range of possible values of frequencies corresponding to the most probable values of elastic coefficients of the soil, then it is more convenient to transform the latter expression as follows:

$$\beta_{1,2} = \frac{f_{n1,2}^2}{f_{n\varphi}^2} = \frac{1}{2\gamma}\left[1 + \mu \pm \sqrt{(1 + \mu)^2 - 4\gamma\mu}\right]$$

where μ is the ratio of the squares of the limiting natural frequencies:

$$\mu = \frac{f_{nx}^2}{f_{n\varphi}^2}$$

If the dimensions of a foundation are selected, then the value β depends only on the assumed values of the coefficients of elastic compression and shear. After the selection of a range of values for these coefficients, it is easy to calculate all possible values of frequency.

Figures III-13*a* and *b* present graphs of $\beta_{1,2}$ as functions of μ. The values of β_1 and β_2 are plotted along the y axis, the values of μ along the x axis. Curves are plotted for different values of γ, from $\gamma = 0.4$ (high foundations) up to $\gamma = 0.9$ (low foundations).

With these graphs it is easy to determine a possible range of changes in f_{n1} and f_{n2}, using a given range of values of c_φ and c_τ. The frequencies are determined from the formula

$$f_{n1,2}^2 = f_{n\varphi}^2 \beta_{1,2}$$

Since the values of $f_{n1,2}$ depend not only on $\beta_{1,2}$, but also on $f_{n\varphi}$, in order to determine the smallest and the largest value of f_{n1} and f_{n2}, it is necessary to compute the minimum and maximum values of the right-hand part of the above expression.

c. *Forced Vibrations* Returning to Eqs (III-4-1), describing forced vibrations of foundations, let us consider separately several particular cases of the action of exciting loads.

Assume that a horizontal force of magnitude $P \sin \omega t$ is applied at the center of gravity of the foundation and machine. This case is of the greatest interest in engineering practice. Equations (III-4-1) (forced vibrations of a foundation accompanied by changes with time in the

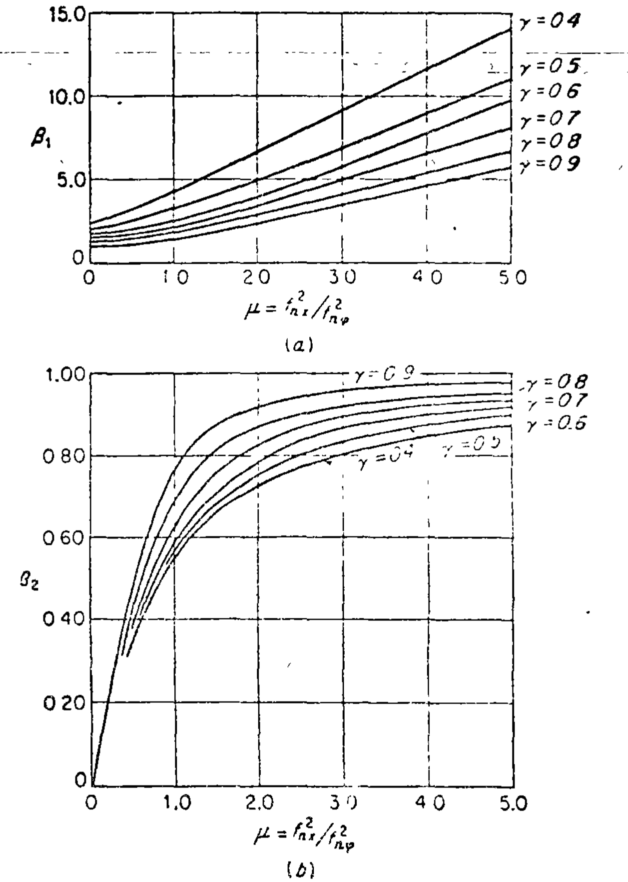


FIG. III-13 Variation of the coefficients β_1 and β_2 (which govern the two main natural frequencies f_{n1} and f_{n2} of the foundation shown in Fig. III-12) with the ratio μ of natural horizontal and rocking frequencies and the ratio γ of moments of inertia which inversely reflects the height of a foundation.

angle of rotation φ and the horizontal component x of the coordinates of the center of gravity) will be rewritten as follows:

$$\begin{aligned} m\ddot{x} + c_\tau A x - c_\tau A L \varphi &= P_x \sin \omega t \\ M_m \ddot{\varphi} - c_\tau A L x + (c_\varphi I - WL + c_\tau A L^2)\varphi &= 0 \end{aligned} \qquad \text{(III-4-10)}$$

We shall seek particular solutions of this system, corresponding only to the forced vibrations of a foundation, in the form

$$x = A_x \sin \omega t$$
$$\varphi = A_\varphi \sin \omega t$$

Substituting these expressions for x and φ into Eqs. (III-4-10), we find that selected particular solutions will satisfy this system if the coefficients A_x and A_φ are to be the roots of the following system of equations:

$$(c_\tau A - m\omega^2)A_x - c_\tau ALA_\varphi = P_T$$
$$-c_\tau ALA_x + (c_\varphi I - WL + c_\tau AL^2 - M_m\omega^2)A_\varphi = 0$$

Solving this system, we find the following expressions for the amplitudes of forced vibrations:

$$A_x = \frac{c_\varphi I - WL + c_\tau AL^2 - M_m\omega^2}{\Delta(\omega^2)} P_T$$
$$A_\varphi = \frac{c_\tau AL}{\Delta(\omega^2)} P_T \qquad \text{(III-4-11)}$$

where $\Delta(\omega^2) = mM_m(f_{n1}^2 - \omega^2)(f_{n2}^2 - \omega^2)$

If vibrations are caused by an exciting moment M_t, the equations of forced vibrations of the foundation will be

$$m\ddot{x} + c_\tau A_x - c_\tau AL\varphi = 0$$
$$M_m\ddot{\varphi} - c_\tau ALx + (c_\varphi I - WL + c_\tau AL^2)\varphi = M_t \sin \omega t$$

For the amplitudes of forced vibrations we obtain

$$A_x = \frac{c_\tau AL}{\Delta(\omega^2)} M_t$$
$$A_\varphi = \frac{c_\tau A - m\omega^2}{\Delta(\omega^2)} M_t \qquad \text{(III-4-12)}$$

Changes in the frequency of exciting forces lead to changes in the amplitudes of vibrations, even when magnitudes of exciting forces remain the same. The phenomenon of resonance is observed when one of the natural frequencies coincides with the frequency of exciting forces. Since in the case under consideration the foundation has two natural frequencies, two resonances are possible when amplitudes grow rapidly. Figure III-14 illustrates the general character of resonance curves for the forced vibrations of foundations under discussion; here an experimental resonance curve is plotted on the basis of data obtained by the author during investigations of a test foundation with a 4-m² base area in contact with soil.

To every frequency of forced vibrations of a foundation there corresponds a particular form of vibrations which is characterized by the magnitude and sign of the radius vector ρ connecting the center of

gravity of the foundation and the point O (Fig. III-12a and b), around which the foundation rotates. The magnitude and sign of ρ are determined from the equation

$$\rho = \frac{A_x}{A_\varphi} \qquad \text{(III-4-13)}$$

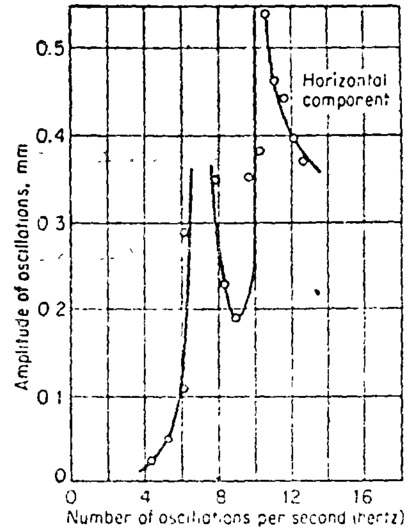


FIG. III-14 General character of resonance curves corresponding to Eq. (III-4-12).

The character of the exciting loads causing forced vibrations of a foundation also has an effect on the dependence of the form of vibrations upon changes in the frequency of the exciting force.

If a foundation is subjected only to the action of the exciting moment M_t, then according to (III-4-12),

$$\rho = \frac{f_{nx}^2}{f_{nx}^2 - \omega^2} L \qquad \text{(III-4-14)}$$

When ω is small in comparison with f_{nx}, ρ does not differ much from L, i.e., at low exciting frequencies, the foundation will vibrate with respect to the axis passing through the center of gravity of the base contact area, perpendicular to the plane of vibrations. With an increase in exciting frequency, the denominator of (III-4-14) will decrease rapidly and ρ will grow, i.e., the foundation vibrations will be accompanied not only by changes in ρ, but also by changes in x_0; in other words, the foundation contact area will undergo sliding. If $\omega = f_{nx}$, then ρ will be indefinitely large. In this case the foundation will undergo only vibrations of shear (sliding) with a certain amplitude. With a further increase in the exciting frequency, ρ changes its sign, with an increase in ω, ρ continuously decreases, approaching zero as ω becomes infinitely large.

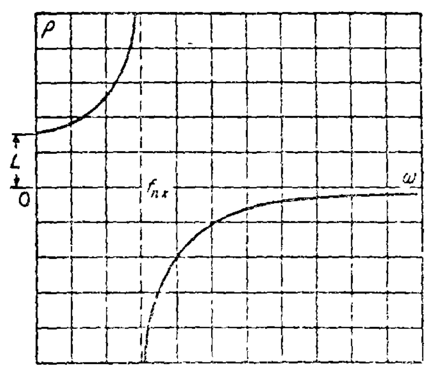


FIG. III-15 General character of changes in the radius vector ρ with the frequency of vibrations ω.

This means that at exciting frequencies considerably larger than the limiting frequency f_{nx}, a foundation will principally undergo rotation around the axis passing through its center of gravity. Figure III-15 illustrates the general character of changes in ρ, depending on ω

Figure III-16 gives position of the main vertical axis versus frequency ω, plotted for a test foundation and computed from Eq. (III-1-14) on the basis of an experimental value of f_{nz} and a selected value of ω. Circles plotted on the same figure show amplitudes of the horizontal component of vibrations corresponding to some magnitudes of exciting frequency. These amplitudes were measured at various foundation heights. The experimental points agree well with values established on the basis of the foregoing theory.

d. The Effect on the Natural Frequencies of Eccentric Distribution of the Foundation and Machine Mass. An eccentric distribution of the machine

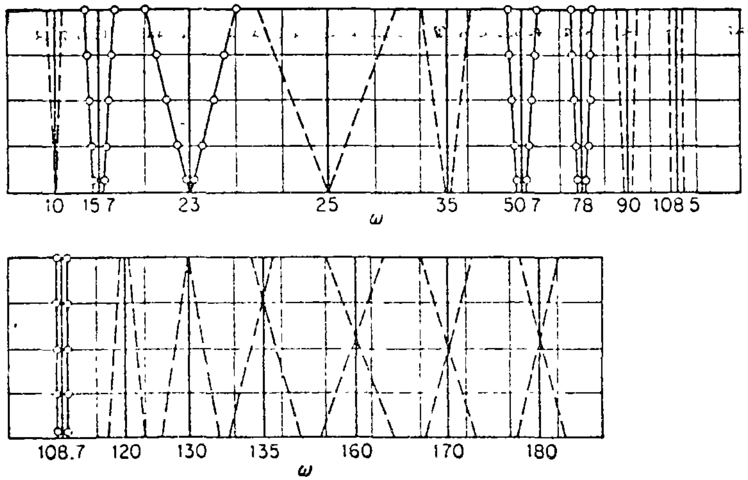


FIG. III-16. Variations in the position of the main vertical axis (Fig. III-12) with changes in the frequency ω.

mass may occur when a machine and a generator or a motor are coupled on the same shaft. Sometimes an eccentricity in the mass distribution is caused by asymmetry of the foundation resulting from various cavities, channels, etc. The asymmetry can often be eliminated by moving the centroid of the foundation area in contact with the soil. Sometimes this cannot be done; then foundation vibrations should be computed with the asymmetric distribution of mass taken into account.

Let us consider the simplest case of asymmetry of foundation mass, that in which the center of gravity of the foundation and machine mass and the centroid of the foundation contact area lie in one of the main foundation planes, but not on the same vertical line. We shall investigate foundation vibrations in the main plane, in which both the centers of gravity lie.

The foundation motion is again determined by three parameters: the projections x and z of the displacement of the foundation center of

gravity on the corresponding coordinate axes and angle φ of rotation of the foundation around the y axis, passing through the center of gravity and perpendicular to the plane of vibrations.

Let us assume that foundation motion has been caused by an initial disturbance (for example, an impact) and examine the forces acting on the foundation during its motion. Then we shall set up the differential equations for this motion taking into consideration the soil reactions produced by foundation displacements only.

As before, we match the origin of the coordinate system used for the study of the foundation with the center of gravity, when the foundation is at rest.

We assume that at a certain instant of time, the projections of the displacement of the center of gravity of the foundation will equal x and z, and the projection of the rotation vector on the y axis will equal φ. We measure these values from the equilibrium position of the foundation when it is subjected to the action of weight and to static soil reactions.

The horizontal displacement of the centroid of the foundation area in contact with soil equals

$$x_0 = x - L\varphi$$

where L, as before, is a distance between the center of gravity of the foundation and the foundation base, hence, the soil reaction caused by horizontal displacement of the centroid of the contact area equals

$$R_x = -c_\tau A(x - L\varphi)$$

The vertical component of the displacement of the centroid of the contact area equals

$$z_0 = z - \epsilon\varphi$$

where ϵ is the eccentricity of the foundation and machine mass.

The soil reaction caused by this displacement of the centroid is

$$R_z = -c_z A(z - \epsilon\varphi)$$

Let us compute the moments of all forces with respect to the y axis.

The moment of the force of gravity equals zero.

The moment of the soil reaction caused by the static action of weight equals

$$M_1 = WL\varphi$$

where W is the total weight of the foundation and machine.

The horizontal component of the soil reaction produces the moment

$$M_2 = c_\tau A(x - L\varphi)L$$

The moment due to the vertical component of the soil reaction is

$$M_3 = c_u A(z - \epsilon\varphi)\epsilon$$

The reactive moment produced by soil due to rotation of the foundation by the angle φ equals

$$M_4 = -c_\varphi I \varphi$$

where I is the moment of inertia of the foundation contact area with respect to the axis passing through its centroid, perpendicular to the plane of vibrations.

Substituting the above expressions for forces and moments into Eqs. (III-4-1) and (III-4-2), we obtain the following differential equations for the eccentric distribution of masses now under consideration:

$$\begin{aligned} m\ddot{x} + c_\tau A x - c_\tau A L \varphi &= 0 \\ m\ddot{z} + c_u A z - c_u A \epsilon \varphi &= 0 \\ M_m \ddot{\varphi} - c_\tau A L x + (c_\varphi I - WL + c_u A \epsilon^2 + c_\tau A L^2)\varphi - c_u A \epsilon z &= 0 \end{aligned} \qquad \text{(III-4-15)}$$

Equations (III-4-15) show that—unlike the previous case, in which the center of inertia of mass and the center of gravity of the foundation contact area lay on the same vertical line—here the three differential equations of motion are interrelated. Therefore, if at the initial moment a change took place in only one parameter related to the motion, then as a consequence there would be changes in all three parameters determining the position of the foundation. Thus, if at the initial instant the foundation is subjected to the action of a disturbance inducing only a horizontal displacement of its center of gravity, then it will move not only in this direction, but also in the vertical direction, and will undergo rotational vibrations around the y axis as well.

If the asymmetry in the distribution of masses is very small ($\epsilon \cong 0$), then Eqs. (III-4-15) at this limit value of ϵ are transformed into the systems (III-4-3) and (III-4-4).

We shall proceed as before in order to obtain the frequency equation for the asymmetrical case under consideration.

First of all, let us simplify the differential equations of motion by denoting

$$\begin{aligned} c_\tau A &= c_x \\ c_u A &= c_z \\ c_\varphi I - WL + (c_u \epsilon^2 + c_\tau L^2)A &= c \end{aligned}$$

Inserting these notations into Eqs. (III-4-15), we obtain

$$\begin{aligned} m\ddot{x} + c_x x - c_x L \varphi &= 0 \\ m\ddot{z} + c_z z - c_z \epsilon \varphi &= 0 \\ M_m \ddot{\varphi} - c_x L x + c\varphi - c_z \epsilon z &= 0 \end{aligned} \qquad \text{(III-4-16)}$$

We will then seek a particular solution of Eqs. (III-4-16) in the form

$$x = A_a \sin f_n t \qquad z = B_a \sin f_n t \qquad \varphi = C_a \sin f_n t$$

here A_a, B_a, and C_a are arbitrary constants

Substituting these solutions into Eqs (III-4-16), we obtain three homogeneous equations:

$$\begin{aligned} (c_x - mf_n^2)A_a - c_x L C_a &= 0 \\ (c_z - mf_n^2)B_a - c_z \epsilon C_a &= 0 \\ (c - M_m f_n^2)C_a - c_x L A_a - c_z \epsilon B_a &= 0 \end{aligned} \qquad \text{(III-4-17)}$$

The constants A_a, B_a, C_a, and f_n should satisfy these equations if our selected particular solutions are to satisfy the system (III-4-16) of differential equations. Equations (III-4-17) are linear and homogeneous. In order for these equations to have solutions other than zero for A_a, B_a, and C_a, their determinant should be identically reduced to zero:

$$(c_x - mf_n^2)(c_z - mf_n^2)(c - M_m f_n^2) - (c_x - mf_n^2)c_z^2\epsilon^2 - (c_z - mf_n^2)c_x^2 L^2 = 0 \qquad \text{(III-4-18)}$$

If the eccentricity in distribution of mass equals zero, then this equation of frequencies is reduced to the equation

$$(c_z - mf_n^2)[(c_x - mf_n^2)(c_1 - M_m f_n^2) - c_x^2 L^2] = 0$$

Thus, when $\epsilon \rightarrow 0$,

$$f_n^2 = f_{nz}^2 = \frac{c_z}{m}$$

$$\Delta(f_n^2) \equiv (c_x - mf_n^2)(c_1 - M_m f_n^2) - c_x^2 L^2 = 0 \qquad \text{(III-4-19)}$$

where $\quad c_1 = c_\varphi I - WL + c_\tau L^2 A$

Equation (III-4-19) is identical with Eq. (III-4-8). Denoting the natural frequencies of vibrations of the foundation which correspond to the limiting case $\epsilon = 0$ by $\bar{f}_{n1}$ and $\bar{f}_{n2}$, and assuming that $\bar{f}_{n1} > \bar{f}_{n2}$, we may rewrite Eq (III-4-19) in the form

$$\Delta(f_n^2) = mM_m(f_n^2 - \bar{f}_{n1}^2)(f_n^2 - \bar{f}_{n2}^2) = 0 \qquad \text{(III-4-20)}$$

On the basis of the general characteristics of the interrelationships between frequencies of systems with a limited number of degrees of freedom, we may state that the following dependence exists between the frequencies f_{n1}, f_{n2} and f_{n3}, corresponding to $\epsilon \neq 0$, and the frequencies $\bar{f}_{n1}$, $\bar{f}_{n2}$ when $\epsilon = 0$.

$$f_{n3} < \bar{f}_{n2} < f_{nz} < \bar{f}_{n1} < f_{n1}$$

We rewrite Eq. (III-4-18) in the form

$$(c_z - mf_n^2)(c_x - mf_n^2)(c_1 + c_z\epsilon^2 - M_m f_n^2) - (c_x - mf_n^2)c_z^2\epsilon^2 - (c_z - mf_n^2)c_x^2 L^2 = 0$$

Dividing by m, we obtain

$$i(f_{nz}^2 - f_n^2)(\bar{f}_{n1}^2 - f_n^2)(\bar{f}_{n2}^2 - f_n^2) - \epsilon^2(f_{nx}^2 - f_n^2)f_{nz}^2 = 0$$

from which we find

$$\epsilon^2 f_n^2 = \frac{i(f_{nz}^2 - f_n^2)(\bar{f}_{n1}^2 - f_n^2)(\bar{f}_{n2}^2 - f_n^2)}{(f_{nz}^2 - f_n^2) f_{nz}^2} \qquad \text{(III-4-21)}$$

where $$i = \frac{M_m}{m}$$

The right-hand part of Eq. (III-4-21) does not depend on ϵ. Let us draw a graph of this member, taking f_n^2 as an independent variable (Fig. III-17). This graph will be formed by two separate curves with an asymptote corresponding to the value $f_n^2 = f_{nz}^2$. Branch A meets the x axis at point $f_n^2 = \bar{f}_{n2}^2$. Branch B crosses the same axis at point $f_n^2 = \bar{f}_1^2$ and point $f_n^2 = f_{nz}^2$ (Fig. III-17 is plotted on the assumption that $f_{nz} > \bar{f}_{n1}$). Straight line C corresponds to the left member of Eq. (III-4-21). The abscissas of points at which the curves cross this straight line give the unknown roots f_{n3}^2, f_{n2}^2, and f_{n1}^2 of Eq. (III-4-21). The graph clearly illustrates the effect of eccentricity ϵ on the natural frequencies of foundation vibration. It is seen that in the case of an eccentric distribution of mass, the two smaller frequencies f_{n3} and f_{n2} become somewhat lower, while the largest frequency becomes higher. If the eccentricity ϵ is small, the frequencies f_{n3}, f_{n2}, and f_{n1} do not differ much from $\bar{f}_{n1}$, $\bar{f}_{n2}$, and $\bar{f}_{nz}$, and at the limit, when $\epsilon = 0$, they become equal.

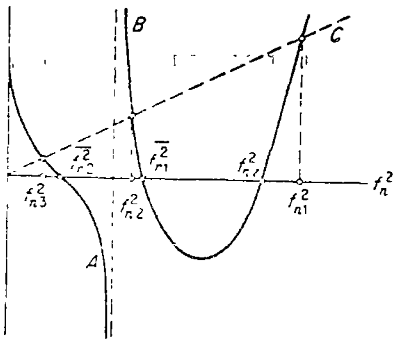


FIG. III-17. Graph illustrating Eq. (III-4-21).

For foundations having a relatively small eccentricity, say 5 per cent of the length of a side of the foundation contact area, its effect may be neglected and computations may be based on formulas derived for $\epsilon = 0$.

III-5. Experimental Investigations of Vibrations of Massive Foundations

a. Verification of the Theory of Vertical Vibrations. The first field investigations of vibrations of foundations were performed by the author together with A. Mikhalchuk on a porous water-saturated silty clay with some sand. On the site of investigations the clay had a thickness of 4.5 m and was underlaid by a sand bed having a thickness of about 4 m. The sand rested on a thick layer of clay. The ground-water level was 20 to 30 cm higher than the base of the test foundations, which were all placed in the same excavation at a depth of about 2.5 m. For dynamic investigations three test foundations were employed with areas in contact with soil of 2.0, 1.0, and 8 m² and weights up to 30 tons.

In addition to dynamic investigations of forced and free vibrations,

static investigations were also performed in order to determine the coefficient of elastic uniform compression of soil.

Figure III-18 presents resonance curves of vertical vibrations of a foundation with an area in contact with soil of 8 m² for different eccentricities ϵ of unbalanced mass of a vibromachine. Analogous curves were obtained for other foundations. Table III-1 presents data on computed

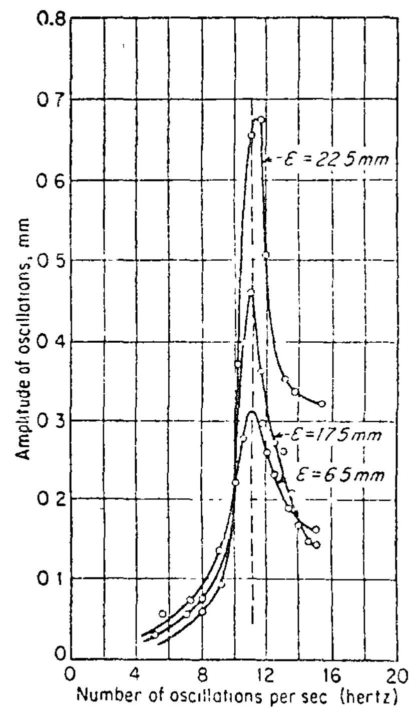


FIG. III-18. Resonance curves of vertical vibrations for three different eccentricities of unbalanced mass of a vibromachine.

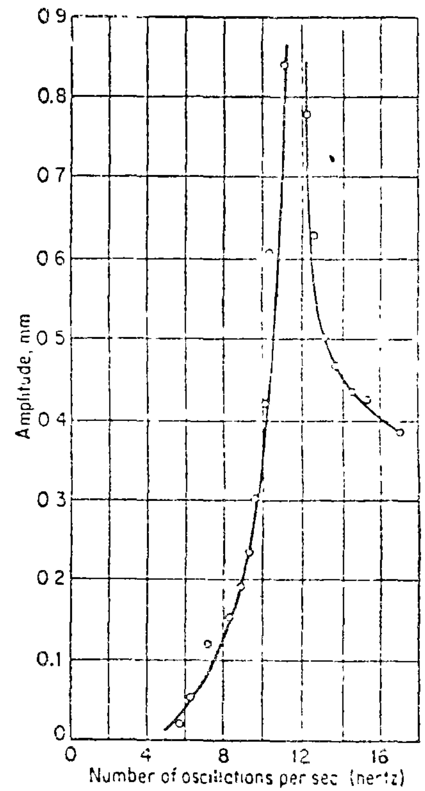


FIG. III-19. Resonance curve of 1.5-m² test of Table III-2.

and experimentally established values of natural frequencies of foundation vibrations. The computation was performed on the basis of values of c_u established by static investigations.

The author and Ya. N. Smolikov performed analogous investigations on a water-saturated soft gray clay with an admixture of organic silt. Investigations were performed on test foundations with base areas of 0.5, 1.0, and 1.5 m².

Figure III-19 presents one of the resonance curves of forced vibrations obtained for a foundation with a contact base area of 1.5 m². In addition

Table III-1. Comparison of Natural Frequencies Computed and Observed during Tests on Water-saturated Clay with a 4-m-thick Sand Layer at 4.5 m Depth

Foundation contact area, m^2	Mass of system, tons × sec^2/m	c_u from static investigations, kg/cm^3	Frequency of natural vertical vibrations, sec^{-1}	
			Computed	Established from observations of forced vibrations
2	1.66	4.40	72.8	88.0
4	1 92	2 45	71.4	60.0
8	3.05	2 05	73 2	69 0

Table III-2. Comparison of Natural Frequencies Computed and Observed during Tests on Soft Saturated Silty Clay

Foundation contact area, m^2	Mass of system, tons × sec^2/m	c_u from static investigations, kg/cm^3	Frequency of natural vertical vibrations, sec^{-1}	
			Computed	Established from observations of forced vibrations
0.5	0 332	3.5	72.5	72.8
1.0	0.520	2.52	69.5	69.0
1.5	0.685	2.1	67.8	70.2

Table III-3. Comparison of Natural Frequencies Computed and Observed during Tests on Loess

Foundation contact area, m^2	Mass of system, tons × sec^2/m	c_u from static investigations, kg/cm^3	Frequency of natural vertical vibrations, sec^{-1}		
			Computed	Established from observations of natural vibrations	Established from observations of forced vibrations
0 81	0 44	14 2	162	158	159
1 40	1 08	10 8	118	113	107
2 00	1 10	10 3	137	117	117
4.00	[illegible]	8.2	[illegible]	[illegible]	121

to dynamic investigations, the coefficient c_u was also determined by the static method. Results are presented in Table III-2.

Similar investigations were performed by the author, Ya. N. Smolikov, and P. A. Saichev on loessial loams and on loess.

Table III-3 gives values of c_u secured from static investigations and natural frequencies of vertical vibrations established by two different dynamic methods.

Table III-4 gives results of similar investigations performed on water-saturated gray fine dense sands containing, in places, peat and organic silt. The first two foundations listed in Table III-4 were placed on sand with an admixture of peat and organic silt; the remaining three rested on pure sand.

Table III-4. Comparison of Natural Frequencies Computed and Observed during Tests on Fine Saturated Sands

Foundation contact area, m^2	Mass of the system, tons × sec^2/m	c_u from static investigations, kg/cm^3	Frequency of natural vertical vibrations, sec^{-1}		
			Computed	Established from observations of natural vibrations	Established from observations of forced vibrations
1.0	0 38	3 96	102	103	95
4.0	0 84	4 45	145	136	143
4 0	0.84	7 54	189	155	181
8 0	2 76	5.55	126	126	130
15 0	3 60	4 00	127	121	124

In all tests the foundations were placed either directly on the soil surface, or at the bottoms of excavations. Therefore the soil reacted only along the foundation base area. The free vertical vibrations were excited by impacts; the forced vibrations by special vibromachines.

Frequencies of natural vertical vibrations of test foundations were computed from Eq. (III-1-5), which does not take into account the inertial properties of soil.

Errors in evaluation of results of experiments are about 10 per cent. The analysis of Tables III-2 to III-4 leads to the conclusion that there is only a small difference between the values of natural frequencies of vertical vibrations as computed by the two methods, i.e., from the data of static investigations and from those established on the basis of observation of free and forced vibrations. Hence, Eq. (III-1-5) is in rather good agreement with experimental data.

The conclusion is also possible that [illegible] foundations studied [illegible] inertia had small effect on the natural frequencies of vertical vibrations of foundations. Apparently this is explained by the fact that for these foundations the value b, found from Eq. (III-1-35), was comparatively large, reaching 20 in some cases. Therefore the effect of soil inertia on the natural frequency of vertical foundation vibrations was comparatively small, being within the range of errors involved in the experiments and the evaluation of results.

b. Experimental Investigations of the Coefficient of Damping. Table III-5 summarizes the results of the determination of the coefficient of damping of vibrations ξ. The values of ξ were determined from the measured amplitudes of forced vibrations at resonance. This was done as follows: at resonance $\omega = f_{nz}$, and Eq. (III-1-21) becomes:

$$A_z^* = A_{res} = \frac{P}{m2cf_{nz}} = \frac{P}{2m\xi f_{nz}^2}$$

since from Eq. (III-1-17a), $c = \alpha/2m$ and from Eq. (III-1-39),

$$\xi = \frac{\alpha}{(2m_t f_{nz})} = \frac{\alpha}{(2mf_{nz})}$$

if one sets $m_t = m$ (i.e., if one considers only the mass m of the foundation and neglects the mass m_s of the soil). Then:

$$\xi = \frac{P}{A_r \, 2mf_{nz}^2}$$

The values of b were established from Eq. (III-1-35).

The analysis of Table III-5 leads to the conclusion that the coefficient of damping is much smaller for soft gray silty clays with some sand than

TABLE III-5. SUMMARY OF TEST RESULTS FOR THE DETERMINATION OF THE REDUCED COEFFICIENT OF DAMPING ξ

Soil description	Foundation contact area, m²	Weight of system, tons	b	ξ
Water-saturated brown silty clay with some sand	2	16.3	16.5	0.145
	4	18.8	6.7	0.133
	8	30.0	4.0	0.181
Water-saturated soft gray clay with sand and organic silt	0.5	3.25	26.0	0.071
	1.0	5.10	14.5	0.058
	1.5	6.72	9.0	0.051
Water-saturated fine dense gray sand	1	6.76	19.0	0.132
	1	6.76	20	0.190
	1	6.76	20	0.175
	1	16.4	55	0.08[illegible]

for brown silty clay with some sand and for fine gray sands. In foundations placed on sands, much higher values of b were observed than in foundations placed on brown silty clay with some sand. Hence it follows that for similar values of b, the coefficient of damping will be larger in sands than in brown clays.

The value of the coefficient of damping is strongly influenced by several factors very difficult to take into account (for example, the backfilling of foundation excavations). Figure III-20 presents two resonance curves of vertical forced vibrations of a foundation with a contact area of 1.0 m²; curve 1 corresponds to the situation in which the foundation is exposed along all its height; curve 2 characterizes the same foundation, but back-filled. The depth of backfilling was around 2 m. The foundation was placed on silty clays interbedded with sands; the ground-water level was considerably below the foundation base.

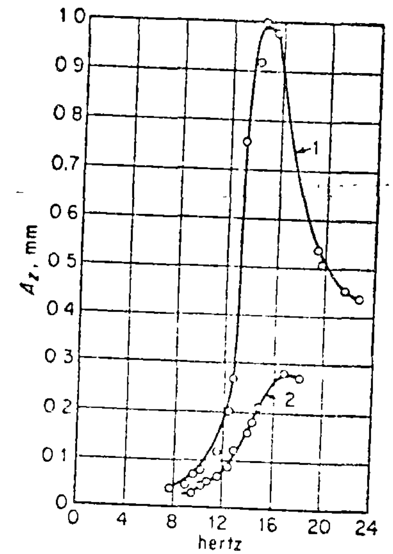


FIG. III-20. Resonance curves of a 1.0-m² test foundation: curve 1, sides of footing free; curve 2, sides of foundation backfilled.

It follows from the comparison of graphs 1 and 2 of Fig. III-20 that amplitudes of vibrations of a backfilled foundation at resonance are about 3.5 times smaller than those of an exposed foundation. Since the coefficient of damping is inversely proportional to the amplitude of vibrations at resonance, it follows that in the case under consideration, the value of the coefficient of damping ξ for a backfilled foundation will be approximately 3.5 times larger than that for an exposed foundation.

A considerable effect of backfilling on the value of ξ was also observed in investigations of foundations placed on gray sands. For example, ξ increased from 0.19 to 0.32 when a foundation was backfilled to the height of 1 m.

Even when foundation sides are not subjected to the influence of soil reactions, but are flooded by ground water, the value ξ grows. For example, investigations on the same fine gray sands established that submerging of the foundation to a height of 1 to 1.5 m is accompanied by an increase in the coefficient of damping by 1.5 to 2 times its value.

The increase in the coefficient of damping when foundation sides are not free is explained by the increase in the total foundation surface dissipating energy into the soil. There is an increase in the dissipation of energy of foundation vibrations, and therefore an increase in the coefficient of damping. Besides, the value of ξ is affected by the forces of friction, whose magnitude increases with an increase of the area of backfilling of the foundation.

Experimentally established values of ξ were in most cases smaller than computed values taken from graphs of Fig. III-8*a*, corresponding to selected values of the coefficient *b*. For foundations characterized by data presented in Table III-5, no experimental value of ξ was greater than 0.20. However, according to the graphs of Fig. III-8*a*, the values of this coefficient (corresponding to the value of *b* for the test foundations) should almost always be larger than 0.20. Apparently the absolute values of the computed coefficients are close to the values typical for backfilled foundations.

TABLE III-6. DATA ON THE VIBRATION OF PILE FOUNDATIONS

Foundation contact area, m^2	Mass of system, tons × sec^2/m	Data on pile foundation: Number of piles	Distance between piles, m	Length of piles, m	K_z from static investigations, kg/cm	Frequency of natural forced vibrations, sec^{-1}: Computed	Established from observation of forced vibrations
10.8	3.0	16	0.81	5.4	153×10^4	227	201
8.6	2.3	12	0.81	5.6	104×10^4	215	186
8.3	1.7	12	0.81	5.4	105×10^4	247	235
6.5	2.0	9	0.81	5.6	55×10^4	166	138

If one plots a resonance curve of forced vertical vibrations of a foundation on the basis of computed values of the coefficient ξ at resonance, it will turn out that the computed amplitudes of forced vibrations are not in full agreement with experimentally established amplitudes. This divergence is partly explained by errors involved in the operation of frequency-measuring devices. In addition, values of ξ apparently depend on the frequency of vibrations and, possibly, on the amplitude; therefore the values of ξ are different for different sections of the resonance curve.

Consequently, amplitudes computed on the basis of an assumption that ξ remains constant only approximately correspond to the true values.

Available experimental data permit the assertion that if the coefficients of elasticity of the soil are correctly selected, then the divergence between computed and test values of amplitude outside the resonance zone will not exceed 10 to 20 per cent.

c. Investigations of Vibrations of Pile Foundations. Investigations of vertical vibrations were performed on test pile foundations. Results are presented in Table III-6.

Piles were driven into water-saturated fine dense sands. The natural frequency of vertical vibrations of the foundation was computed from Eq. (III-1-5); the foundation mass was considered to be the only vibrating mass. It is natural, therefore, that the natural frequencies of vertical vibrations established by static investigations turned out to be somewhat higher than the frequencies obtained from investigations of forced vibrations. For all foundations investigated, a magnitude of vibrating mass was computed on the basis of values of the coefficient of elastic uniform compression, established by static investigations, and on the basis of the resonance frequency of vertical forced vibrations. This vibrating mass was, on the average, 30 per cent larger than the mass restricted to the foundation only.

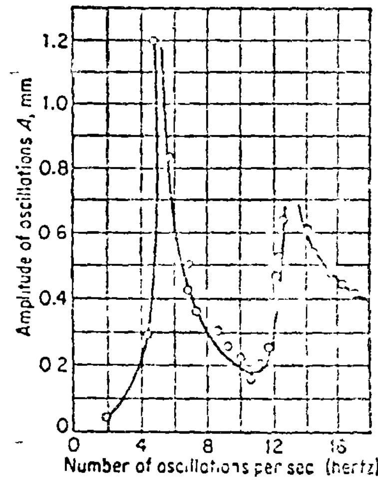


FIG. III-21. Resonance curves of a 1.5-m^2 test foundation of silty clay with sand subjected to forced horizontal vibrations

d. Experimental Investigations of Complicated Forms of Foundation Vibrations. Other forms of vibrations of foundations were also investigated. For example, forced vibrations in one of the principal planes, induced by a horizontal force, were studied. Figure III-14 gives a resonance curve of forced horizontal vibrations of a foundation with a contact area of 4 m^2 placed on brown silty clay with sand, Fig. III-21 shows a similar curve for a foundation with a contact area of 1.5 m^2 placed on water-saturated soft silty clays with sand. Similar investigations were performed on loess and on gray sands. In the first two cases, for all foundations investigated, two frequencies f_{n1} and f_{n2} were obtained which correspond to a sharp rise in the amplitudes of vibrations.

Thus, there is experimental corroboration of a theoretical conclusion concerning two maximums in the resonance curves of forced vibrations of foundations, corresponding to the two frequencies f_{n1} and f_{n2}. No static tests for determining c_φ were performed during the investigations of foundations on water-saturated silty clays with some sand. Therefore, in this case there is no way to verify directly how far the frequencies f_{n1} and f_{n2}, computed from values of c_φ and c_τ (established by means of static investigations), coincide with those which were established experimentally. Static tests for determining c_z and c_τ were performed on foundations resting on loess. Using the established values, it is possible to compute the frequencies f_{n1} and f_{n2} and to verify how close the computed values of these frequencies are to those established experimentally. Results of this analysis are presented in Table III-7.

TABLE III-7. DATA ON THE TWO FUNDAMENTAL FREQUENCIES OF A FOUNDATION SUBJECTED TO FORCED HORIZONTAL VIBRATION

Foundation contact area, m²	Frequencies computed from static investigations		Experimentally established values of f_{n2}	
	f_{n1}, sec⁻¹	f_{n2}, sec⁻¹	From free vibrations	From forced vibrations
0 81	174 0	58 4	65 3	48 3
1 40	181 0	73 5	73 5	64 0
2 60	140 0	65 2	69 1	50 2
4 00	167 0	89 2	77 8	54 0

Static investigations for determining c_u and c_τ were performed on gray silty clays with some sand, also f_{n1} and f_{n2} were experimentally determined. Since these frequencies are connected with the limiting frequencies $f_{n\varphi}$ and f_{nx}, it was possible to establish the latter analytically. Then, on the basis of the values of $f_{n\varphi}$ and f_{nx}, the values of c_τ were determined. Results of this processing of data from dynamic investigations on gray silty clay with some sand are presented in Table III-8.

TABLE III-8. DATA ON THE COEFFICIENT c_τ OF ELASTIC UNIFORM SHEAR OF SOIL

Foundation contact area, m²	c_τ, kg/cm²	
	From experimental frequencies f_{n1} and f_{n2}	From static investigations
0 5	1 88	1 90
1 0	1 64	1 58
1 5	1 27	1 40

It is seen from the tables that the values of c_τ computed on the basis of static investigations and those obtained as a result of the investigations of vibrations coincide with a satisfactory degree of accuracy.

Available data, including those cited above, lead to the conclusion that the theory of vibrations of foundations for several-degrees-of-freedom systems, as presented in this chapter, is supported by experiments.

IV

FOUNDATIONS UNDER RECIPROCATING ENGINES

IV-1. General Directives for the Design of Foundations

a Design Values of Permissible Amplitudes of Foundation Vibrations. Many types of reciprocating engines belong to the group of unbalanced machines which are dangerous in respect to vibrations. The fact that these engines usually operate at comparatively low speed increases the probability that vibrations may develop in adjoining buildings or structures. Therefore a thorough analysis of vibrations in such foundations is of the utmost importance.

The greater the amplitude of vibrations of the foundation, the more danger there is to adjoining structures. In addition, if the amplitude of foundation vibrations is large, the foundation may lose its stability and undergo a nonuniform settlement endangering the normal work of the engine. Finally, vibrations of large amplitude may lead to the destruction of the foundation and to damage of the engine, for example, it has been observed that sometimes crankshafts of log-sawing frames fail as a result of vibrations of their foundations and frames, to which the crankshafts are rigidly tied. Similarly, foundation vibrations often cause dangerous vibrations of machine connections.

It is extremely difficult to establish a limit for the permissible value of amplitude of foundation vibrations on the basis of general principles. There are some cases in which vibrations with an amplitude up to 0.4 to 0.5 mm did not have any harmful effects. However, many cases have been observed in which foundations under engines with low-frequency vibrations underwent vibrations at smaller amplitudes than those cited above, but induced strong vibrations of structures located at a distance of several tens of meters. It may happen that even when the amplitude of vibration of a machine foundation is smaller than an accepted permissible limit, the adjoining structure will vibrate due to resonance.

On the strength of data gained by experience it is possible to state that if no resonance is to occur in adjoining buildings and structures, then the amplitude of vibrations of a foundation should not exceed 0.20 to 0.25 mm. This range of amplitude values may serve as a basis for evaluation of the adequacy of foundation-design computations.

b. Machine Data Required for Foundation Design Data supplied by the maker of the engine or motor is the basic information for a foundation designer, together with data on soil conditions and on the exciting loads imposed by the engine. The following information should be given:

1. The normal speed and power of the engine
2. The character, magnitude, and point of application of dynamic loads which will develop in the process of operation of the engine. If this data cannot be supplied, the designer of the machine foundation should be given all the data needed for the computation of exciting forces
3. The distribution of static loads imposed by the engine over the foundation surface
4. The size and shape of the engine supporting plate
5. The location of openings and grooves in the foundation provided for anchor bolts, pipe lines, the flywheel, etc.

c. Foundation Material. The following loads are imposed on a machine foundation:

1. The weight of the machine and equipment
2. The dynamic loads which develop in the process of machine operation

For diesel engines, the total load is such that the reduced pressure on the upper surface of the foundation usually does not exceed 3 to 5 kg/cm².

For horizontal piston engines this value is still smaller. In any case, the permissible bearing values of concrete and masonry are considerably higher. It is natural that since pressures imposed on foundations are small, the stress analysis should be performed only for cross sections weakened by large openings or grooves.

Thus the question of selection of a material for the foundation—concrete or masonry—is first of all a question of cost and of availability of material on the site Concrete type 100† is usually employed for foundations under reciprocating machinery.

† TRANSLATION EDITOR'S NOTE: The figure 100 indicates the 28-day compressive cube strength in kilograms per square centimeter of the concrete mixture used (100 kg/cm² equals 1,420 psi)

d. Comments in Regard to Design Foundations under reciprocating engines are usually built as massive blocks provided with grooves and channels for machine details and openings for anchor bolts

Due to the massive shapes of such foundations, when studying their vibrations it is possible to consider them as absolutely rigid bodies and to use in the computations of frequencies and amplitudes the theory of vibrations of a solid resting on an elastic base, as presented in Chap III

The main condition to be observed when designing a machine foundation is as follows. the minimum dimensions of the foundation should be selected in such a way that the amplitudes of its forced vibrations will not exceed the permissible value

If a foundation is erected on a natural soil base, its depth should not be less than that of frost penetration There is an established opinion among practicing engineers that in order to decrease the transmission of vibrations the depth of a machine foundation should be no less than the depth of the footings of adjoining walls and columns Theoretical and experimental data on wave propagation in soils, presented in Arts VII-2 and VII-5, lead to the conclusion that provision for machine foundations deeper than footings for walls has no effect on the transmission of vibrations to walls Therefore the depth of a machine foundation may be selected without taking into account the transmission of vibrations

In order to obtain uniform settlement of the foundation, it is recommended to place the common center of gravity of the system (i e , of the foundation and machine) on the same vertical line with the centroid of the foundation area in contact with the soil. In any case the eccentricity in the distribution of masses should not exceed 5 per cent of the length of the side of the contact area. Satisfying this condition makes it possible to simplify the computation of foundation vibrations. When the common center of gravity does not lie on the same vertical line as the centroid of the foundation contact area, it is necessary, as stated above, to solve at least three interrelated differential equations of vibrations. In order to simplify the computations, an attempt should be made to achieve a condition such that the plane of action of the exciting forces imposed by the machine coincides with one of the principal planes of inertia of the foundation

In order to decrease the transmission of vibrations to adjoining parts of buildings, it is necessary to leave a gap between the foundation of an unbalanced machine and the adjoining structures (footings, walls, floors, and so on). As a rule, the machine foundation is not allowed to serve as a support for other parts of the building or for mechanisms not related to a given machine. If it is not possible to avoid placing unimportant parts of a building on the machine foundation, measures should be taken to soften the connection by providing gaskets made of rubber, cork, felt, or other insulating materials.

If several machines are to be installed in the same shop and if the distances between these machines are comparable to the foundation dimensions, then, in soft soils, it is recommended to place the foundations under similar machines on one common mat of sufficient thickness. The rigidity of this mat should be selected so that its possible deformations remain small in comparison with the amplitudes of vibrations. Only then may a group of machine foundations installed on the same mat be regarded as an absolutely solid block resting on an elastic base. The computation of vibrations of such groups of foundations is very difficult. If an asymmetry and an alternating phase shift are observed, then computations are practically impossible.

Therefore if several foundations are erected on the same mat, it is conditionally broken up into sections corresponding to separate foundations; the computations of vibrations proceed as if each foundation were installed separately. Then the design value for the permissible amplitude of vibrations may be increased somewhat (by 25 to 30 per cent).

To avoid a distorted tilt of the master shaft of the machine, its external bearing should be placed on the same machine foundation. This directive refers also to the installation of motors coupled directly to reciprocating engines such as electromotors and generators.

In any case, the foundation area in contact with soil should be selected in such a way that pressure on the soil does not exceed permissible values.

The larger the foundation contact area, the smaller the reduced pressure on the soil and the higher the natural frequencies of the foundation. This is of considerable importance for low-frequency machines, including most of the reciprocating engines. The easiest way to change the natural frequencies of a foundation is to increase or decrease the dimensions of the foundation contact area and change its configuration in plan. Therefore a final selection of the contact area should be based on requirements obtained as a result of design computations regarding vibrational loads on the foundation.

Foundations under low-frequency machines should be designed so that their natural frequencies are much higher than the operational frequencies of the machines.

The natural frequencies of foundations are affected by the absolute value of the foundation mass and by its distribution in space. The designer should try to distribute the mass so that the smallest possible value of its moment of inertia is obtained with respect to the principal axis passing through the centroid of the foundation contact area. To meet this requirement, the minimum foundation height should be selected

IV-2. Unbalanced Inertial Forces in Reciprocating Engines

Forced vibrations in foundations under piston engines are largely caused by the unbalanced inertia forces in the moving parts of crank mechanisms.

a. Single-line Machines. Figure IV-1 illustrates the main features of a reciprocating mechanism. The piston A and the piston rod B execute an alternating motion; the connection rod C executes a complicated periodic motion; all points of the crank D execute a rotational motion around the main axis O. Any of these parts may have unbalanced inertial forces which independently may cause foundation vibrations.

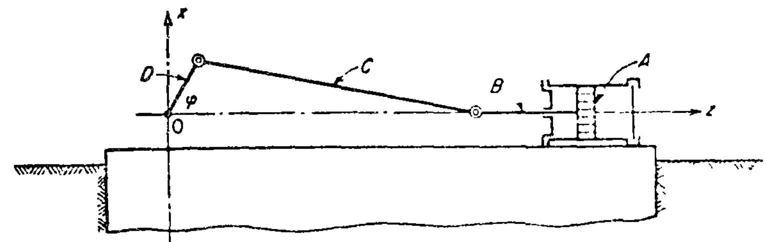


FIG. IV-1 The main parts of a reciprocating engine.

According to the laws of statics, we may replace all these forces by forces acting at point O and place at this point the origin of a coordinate system x, y, z.

Let us place the z axis in the direction of piston movement, the x axis perpendicular to this direction, and the y axis perpendicular to the plane of the drawing. Since all points of the crank mechanism move in the plane xz, the y ordinates will remain constant for all these points. As a result of replacing all unbalanced inertia forces by forces acting at point O, we obtain one force and one couple. Resolving these into their components along the coordinate axes, we obtain the force components P_x and P_z and the moment M_y.

The position of any point of the crank mechanism is determined by one independent variable: the angle of rotation φ of the crankshaft. Therefore the coordinates x_i, y_i, and z_i of any point i of the moving machine parts are functions of the angle φ, the latter being a function of time.

Let us denote by $\ddot{x}_i$ and $\ddot{z}_i$ the projections of the acceleration of an element i of the crank mechanism on the x and z axes; then the projections of the inertial forces acting on this element will be as follows:

$$P_{xi} = m_i \ddot{x}_i \qquad P_{zi} = m_i \ddot{z}_i$$

The projections of the inertial forces acting on all elements of the crank mechanism will be

$$P_x = \Sigma m_i \ddot{x}_i \qquad P_z = \Sigma m_i \ddot{z}_i \tag{IV-2-1}$$

Considering the x_i and z_i coordinates as functions of t, we obtain

$$\frac{dx_i}{dt} = \frac{dx_i}{d\varphi}\frac{d\varphi}{dt}$$

but

$$\frac{d\varphi}{dt} = \omega$$

where ω is the angular velocity of the machine rotation, hereafter assumed constant Therefore,

$$\frac{dx_i}{dt} = \omega \frac{dx_i}{d\varphi}$$

Taking the derivative with respect to t, we find

$$\ddot{x}_i = \omega^2 \frac{d^2x_i}{d\varphi^2}$$

similarly,

$$\ddot{z}_i = \omega^2 \frac{d^2z_i}{d\varphi^2}$$

Substituting the values for $\ddot{x}_i$ and $\ddot{z}_i$ into Eqs. (IV-2-1), we obtain

$$P_x = \omega^2 \sum m_i \frac{d^2x_i}{d\varphi^2} \qquad P_z = \omega^2 \sum m_i \frac{d^2z_i}{d\varphi^2} \tag{IV-2-2}$$

The total resultant inertial force of the crank mechanism evidently will equal the sum of the inertial forces of its moving parts: the crank, the piston, and the connecting rod Consequently the component of the resultant of the inertial forces acting in the direction of the piston motion can be described by the formula

$$P_z = P_{z1} + P_{z2} + P_{z3}$$

where P_{z1} = projection of inertial forces of crank on the z axis
P_{z2} = projection of inertial forces of rod, crosshead, and piston
P_{z3} = projection of inertial forces of connecting rod

Further,

$$P_{z1} = \omega^2 \sum m_{1i} \frac{d^2z_{1i}}{d\varphi^2}$$
$$P_{z2} = \omega^2 \sum m_{2i} \frac{d^2z_{2i}}{d\varphi^2}$$
$$P_{z3} = \omega^2 \sum m_{3i} \frac{d^2z_{3i}}{d\varphi^2}$$

The first moment, for example, of the mass of the crank with respect to the rotation axis will equal $\Sigma m_{1i}z_{1i}$. On the other hand, if M_1 is the mass of the crank and x_1 and z_1 are the coordinates of its center of gravity, then

$$\Sigma m_{1i}z_{1i} = M_1 z_1$$

Differentiating this equation twice with respect to φ, we obtain

$$\sum m_{1i} \frac{d^2z_1}{d\varphi^2} = M_1 \frac{d^2z_1}{d\varphi^2}$$

Using this relationship, we find

$$P_{z1} = M_1\omega^2 \frac{d^2z_1}{d\varphi^2}$$
$$P_{z2} = M_2\omega^2 \frac{d^2z_2}{d\varphi^2}$$
$$P_{z3} = M_3\omega^2 \frac{d^2z_3}{d\varphi^2}$$

similarly,

$$P_{x1} = M_1\omega^2 \frac{d^2x_1}{d\varphi^2}$$
$$P_{x2} = M_2\omega^2 \frac{d^2x_2}{d\varphi^2}$$
$$P_{x3} = M_3\omega^2 \frac{d^2x_3}{d\varphi^2}$$

The expressions for the projections of the resultant inertial force will be

$$P_x = \omega^2\left(M_1 \frac{d^2x_1}{d\varphi^2} + M_2 \frac{d^2x_2}{d\varphi^2} + M_3 \frac{d^2x_3}{d\varphi^2}\right)$$
$$P_z = \omega^2\left(M_1 \frac{d^2z_1}{d\varphi^2} + M_2 \frac{d^2z_2}{d\varphi^2} + M_3 \frac{d^2z_3}{d\varphi^2}\right) \tag{IV-2-3}$$

Without limiting the general validity of the solution, and without involving any significant error, it is possible to concentrate the entire mass of the crank mechanism not at three points, as has been done above, but at two points. This will simplify the expressions obtained for P_x and P_z.

It has been assumed that the crank executes its motion at a uniform rate. Therefore the magnitude of its inertial force will be the magnitude of the centrifugal force; i.e.,

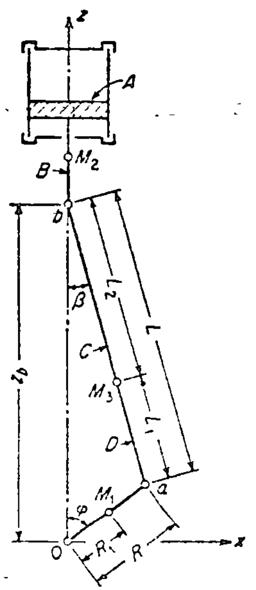


FIG. IV-2. Reciprocating engine, illustrating the derivation of Eq. (IV-2-4).

$$P_c = R_1 M_1 \omega^2$$

where R_1 is the distance between the center of gravity of the crank and the axis of rotation.

This force will be directed along the radius of rotation. Let us consider that it is applied not to the center of gravity of the crank, but to point a, i.e., to the crankpin (Fig. IV-2). In order to obtain a force equal to P_c and applied at point a, it is necessary to assume that mass M_{11} is concentrated at this point and is smaller than M_1 in the same proportion as R_1 is smaller than R; thus we should set

$$M_{11} = \frac{R_1}{R} M_1$$

Since point b (the crosshead) executes a reciprocating motion which does not differ from the motion of the center of gravity of mass M_2, we may consider that mass M_2 is concentrated at point b; this will not change the magnitude of its induced inertial force.

Finally the mass M_3 of the connecting rod may be replaced by masses M_{31} and M_{32} concentrated at points a and b. However, this distribution of mass M_3 should be made so as not to change the magnitude of the inertial force of the connecting rod computed on the assumption that its mass is concentrated at the center of gravity (x_3,z_3).

Denoting the coordinates of points a and b respectively by (x_a,z_a) and (x_b,z_b), we obtain

$$P_{z3} = \omega^2 \left(M_{31} \frac{d^2 z_a}{d\varphi^2} + M_{32} \frac{d^2 z_b}{d\varphi^2} \right)$$

On the other hand,

$$P_{z3} = \omega^2 M_3 \frac{d^2 z_3}{d\varphi^2}$$

Equating the right-hand parts of these expressions, we obtain

$$M_{31} z_a + M_{32} z_b = M_3 z_3$$

Similarly, deriving the expression for the projection of the inertial force of the connecting rod on the x axis, we obtain the second equation:

$$M_{31} x_a + M_{32} x_b = M_3 x_3$$

It follows from the equations obtained for M_{31} and M_{32} that the values of these masses should be selected so that their center of gravity lies at point (x_3,z_3), i.e., the masses should be distributed in an inverse relationship to their distances from mass M_3. Denoting these distances by L_1 and L_2, we obtain

$$\frac{M_{31}}{M_{32}} = \frac{L_2}{L_1}$$

On the other hand,

$$M_{31} + M_{32} = M_3$$

Solving these equations for M_{31} and M_{32}, we find

$$M_{31} = \frac{L_2}{L} M_3 \qquad M_{32} = \frac{L_1}{L} M_3 \qquad L = L_1 + L_2$$

where L is the length of the crank.

Thus the three masses, concentrated at the centers of gravity corresponding to the parts of the crank mechanism, may be replaced, without changing the magnitudes of the inertial forces of the mechanism, by two masses:

1. Mass M_a, concentrated at the crankpin:

$$M_a = M_{11} + M_{31} = \frac{R_1}{R} M_1 + \frac{L_2}{L} M_3 \qquad \text{(IV-2-4)}$$

2. Mass M_b, concentrated at the crosshead:

$$M_b = M_2 + M_{32} = M_2 + \frac{L_1}{L} M_3 \qquad \text{(IV-2-5)}$$

The projections of the inertial force of mass M_a on the coordinate axes will be

$$P_{xa} + M_a \omega^2 R \sin \varphi \qquad P_{za} = M_a \omega^2 R \cos \varphi$$

The inertial force of mass M_b will only have a projection on the z axis; this will be equal to

$$P_{zb} = -M_b \omega^2 z_b$$

The projections of the resultant inertial force of the whole mechanism will equal

$$P_x = M_a \omega^2 R \sin \varphi$$
$$P_z = M_a \omega^2 R \cos \varphi - M_b \omega^2 z_b \qquad \text{(IV-2-6)}$$

It follows directly from Fig. IV-2 that

$$z_b = R \cos \varphi + L \cos \varphi$$

We have, from the triangle *Oab*,

$$\sin \beta = \frac{R}{L} \sin \varphi = \alpha \sin \varphi$$

from which $\cos \beta = \sqrt{1 - \alpha^2 \sin^2 \varphi}$

Expanding $\cos \beta$ into a series according to Newton's binomial theorem, we obtain

$$\cos \beta = (1 - \alpha^2 \sin^2 \varphi)^{1/2} = 1 - \tfrac{1}{2}\alpha^2 \sin^2 \varphi - \tfrac{1}{8}\alpha^4 \sin^4 \varphi - \tfrac{1}{16}\alpha^6 \sin^6 \varphi - \cdots$$

Using the formula converting an even exponential trigonometric function into a linear one,

$$2^{n-1}(-1)^{n/2} \sin^n \varphi = \cos n\varphi - n \cos (n-2)\varphi + \frac{n(n-1)}{1.2} \cos (n-4)\varphi - \cdots + (-1) \frac{(n/2)n(n-1) \cdots (n/2+1)}{2 \cdot 1 \cdot 2 \cdots n/2}$$

we replace powers of sines by the cosines of multiples of 2φ. Then

$$\cos \beta = A_0 + A_2 \cos 2\varphi + A_4 \cos 4\varphi + \cdots$$

where $A_0, A_2, A_4, \ldots$ are constants depending only on the characteristic number α of the crank mechanism:

$$A_0 = 1 - \tfrac{1}{4}\alpha^2 - \tfrac{3}{64}\alpha^4 - \tfrac{5}{256}\alpha^6 - \cdots$$
$$A_2 = \tfrac{1}{4}\alpha^2 + \tfrac{1}{16}\alpha^4 + \tfrac{15}{512}\alpha^6 + \cdots$$
$$A_4 = -(\tfrac{1}{64}\alpha^4 + \tfrac{3}{256}\alpha^6 + \cdots)$$

Substituting the expression established for $\cos \beta$ into Eq. (IV-2-6), we obtain

$$z_b = R\left[\cos \varphi + \frac{1}{\alpha}(A_0 + A_2 \cos 2\varphi + A_4 \cos 4\varphi + \cdots)\right]$$

$$\frac{d^2 z_b}{d\varphi^2} = -R(\cos \varphi + B_2 \cos 2\varphi + B_4 \cos 4\varphi + \cdots)$$

where $B_2 = \dfrac{4A_2}{\alpha}$ $\quad B_4 = \dfrac{16A_4}{\alpha}$

Substituting into these formulas the expressions for A_2 and A_4, and, in view of the small value of α, disregarding all terms containing its

fourth or higher powers, we obtain

$$B_2 = \alpha\left(1 + \frac{\alpha^2}{4}\right) \qquad B_4 = -\frac{\alpha^3}{4}$$

Therefore,

$$\frac{d^2 z_b}{d\varphi^2} = -R\left[\cos \varphi + \alpha\left(1 + \frac{\alpha^2}{4}\right)\cos 2\varphi - \frac{\alpha^3}{4}\cos 4\varphi\right]$$

Substituting the expression established for $d^2z_b/d\varphi^2$ into the formula for P_{zb}, we find that

$$P_{zb} = M_b R\omega^2\left[\cos \varphi + \alpha\left(1 + \frac{\alpha^2}{4}\right)\cos 2\varphi - \frac{\alpha^3}{4}\cos 4\varphi\right]$$

Substituting this expression for P_{zb} into Eq. (IV-2-6) describing the projection on the z axis of the resultant inertial force of the machine and replacing φ by ωt, where ω is the angular velocity of machine rotation, we finally obtain

$$P_x = R\omega^2 M_a \sin \omega t$$

$$P_z = R\omega^2\left[(M_a + M_b)\cos \omega t + \alpha M_b\left(1 + \frac{\alpha^2}{4}\right)\cos 2\omega t - \frac{M_b \alpha^3}{4}\cos 4\omega t\right] \qquad \text{(IV-2-7)}$$

Thus the formula describing the exciting loads causing the forced vibrations of a foundation contains terms depending not only on the frequency ω of machine rotation, but also on the double, quadruple, etc. of this frequency. However, the coefficients preceding $\cos 2\omega t$ and $\cos 4\omega t$ decrease very quickly, and these terms may be disregarded in engineering calculations.

The terms containing $\cos \omega t$ are called primary inertial forces (the first harmonics); those containing $\cos 2\omega t$, secondary forces (secondary harmonics); and so on.

The foregoing discussion leads to the conclusion that rotating machinery masses produce primary inertial forces, reciprocating masses produce both primary inertial forces and forces of even higher orders.

By installing counterweights of mass M'_a on a shaft, it is possible to balance inertial forces induced by mass M_a. If M'_a is fixed on the shaft so that the angle between the radius vectors of masses M_a and M'_a equals π, then, in order to have the inertial forces of the rotating parts balanced, one of the two following conditions should be satisfied. Either

$$M_a - \frac{l}{R} M'_a = 0$$

or

$$\frac{R'}{R} M_1 + \frac{L_2}{L} M_3 - \frac{l}{R} M'_a = 0$$

In order to balance fully the projections of primary inertial forces in the direction of piston motion it is necessary to select mass M'_a and distance l (between its center of gravity and the axis of rotation) so that either

$$M_a + M_b - \frac{l}{R} M'_a = 0$$

or

$$\frac{R_1}{R} M_1 + M_2 + M_3 - \frac{l}{R} M'_a = 0 \qquad \text{(IV-2-8)}$$

If the selected values of l and M'_a satisfy one of the above equations, then in the expression for P_z there will remain terms depending only on $\cos 2\omega t$ and $\cos 4\omega t$, while the expression for P_x will be as follows:

$$P_x = R\omega^2 M_b \sin \omega t$$

Usually M_b is larger than M_a; therefore the selection of a counterweight mass satisfying Eq. (IV-2-8) leads to an enlargement of inertial forces in the direction perpendicular to the sliding of the piston

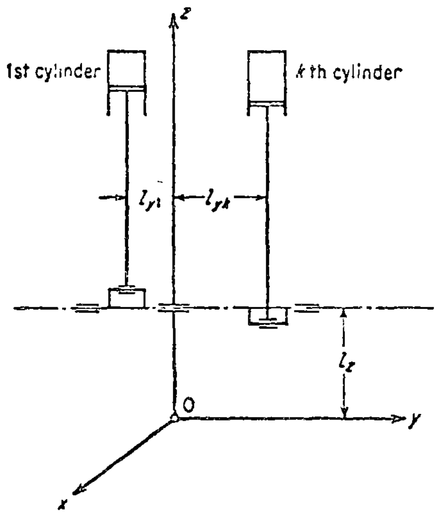


FIG. IV-3 Multicylinder engine, illustrating the derivation of Eq (IV-2-9).

b. Multicylinder Engines The method of determination of exciting loads in multicylinder engines is in principle the same as in single-cylinder engines.

Consider a vertical engine in which the cylinders are situated in the same plane, parallel to each other (the so-called linear arrangement of cylinders). Usually the number n of cylinders does not exceed 10. Unbalanced inertial forces are calculated similarly for vertical and horizontal reciprocating engines.

Let us direct the y axis (Fig. IV-3) along the crankshaft of the engine, the x axis perpendicular to the shaft and horizontal, the z axis upward, along the axis of sliding of the pistons Let us place the origin at the mass center of the foundation and engine and let us assume that the yz plane passes through the principal axis of engine rotation.

We confine ourselves to the case in which the engine has only main cylinders (no auxiliaries such as compressor and exhaust cylinder). We denote by β_k the angle between the crank of the kth cylinder and the first crank (the wedging angle) By the reasoning of Art. IV-2-a we obtain the following expressions for the component exciting force along the x and z axes for the kth cylinder:

$$P_{xk} = R_k\omega^2 M_{ak} \sin(\omega t + \beta_k)$$
$$P_{zk} = R_k\omega^2[(M_{ak} + M_{bk}) \cos(\omega t + \beta_k) + M_{bk}\alpha_k \cos(\omega t + \beta_k)]$$

The terms
$$M_{bk}\frac{\alpha k^3}{4} \cos 2(\omega t + \beta_k)$$
and
$$\frac{M_{bk}\alpha k^3}{4} \cos 4(\omega t + \beta_k)$$
have been neglected.

In order to obtain the resultant exciting force transmitted to the foundation from all engine cylinders, it suffices to sum the above expressions for all n cylinders. Then we have:

$$P_x = \omega^2 \sum_{k=1}^{n} R_k M_{ak} \sin(\omega t + \beta_k)$$
$$P_z = \omega^2 \sum_{k=1}^{n} R_k[(M_{ak} + M_{bk}) \cos(\omega t + \beta_k) + M_{bk}\alpha_k \cos 2(\omega t + \beta_k)] \qquad \text{(IV-2-9)}$$

In addition to exciting forces, there are exciting moments; their magnitudes equal

$$M_x = \sum_{k=1}^{n} P_{zk} l_{yk} \qquad M_y = \sum_{k=1}^{n} P_{xk} l_{zk} \qquad M_z = \sum_{k=1}^{n} P_{xk} l_{yk}$$

If the crank mechanisms are identical in all cylinders, then the equations for the exciting force will be simplified:

$$P_x = R\omega^2 M_a \sum_{k=1}^{n} \sin(\omega t + \beta_k)$$
$$P_z = R\omega^2 \left[(M_a + M_b) \sum_{k=1}^{n} \cos(\omega t + \beta_k) + M_b\alpha \sum_{k=1}^{n} \cos 2(\omega t + \beta_k) \right]$$

Hence it follows that to balance the first harmonics of the exciting forces, the following equations should be satisfied:

$$\sum_{k=1}^{n} \cos(\omega t + \beta_k) = 0 \qquad \sum_{k=1}^{n} \sin(\omega t + \beta_k) = 0$$

The second harmonics will be satisfied if

$$\sum_{k=1}^{n} \cos 2(\omega t + \beta_k) = 0$$

To balance the exciting moments of the first harmonics, the following equations should be satisfied:

$$\sum_{k=1}^{n} l_{zk} \cos(\omega t + \beta_k) = 0$$

$$\sum_{k=1}^{n} l_{zk} \sin(\omega t + \beta_k) = 0$$

$$\sum_{k=1}^{n} l_{yk} \sin(\omega t + \beta_k) = 0$$

Similar conditions hold for the second harmonics.

Let us consider several particular computations of exciting loads imposed by multicylinder engines, assuming all cylinders are identical and neglecting all higher harmonics of exciting loads.

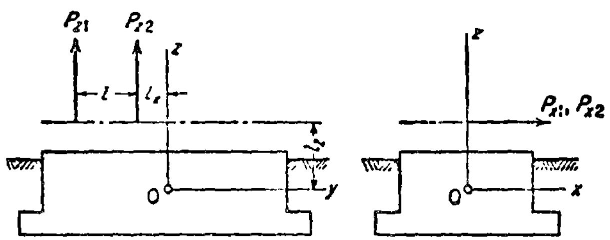


FIG. IV-4. Derivation of equations for a vertical two-cylinder engine.

c. Vertical Two-cylinder Engines Let us assume that the engine is mounted asymmetrically on the foundation (Fig. IV-4). Both cylinders are identical.

CASE 1. CRANKS IN SAME DIRECTION. Here,

$$\beta_1 = 0 \qquad \beta_2 = 2\pi$$

Assuming in Eq. (IV-2-9) that $k = 1.2$, we obtain

$$P_{x1} = P_{x2} = R\omega^2 M_a \sin \omega t$$
$$P_{z1} = P_{z2} = R\omega^2 (M_a + M_b) \cos \omega t$$

The resultant components of the exciting forces will be

$$P_x = 2R\omega^2 M_a \sin \omega t$$
$$P_z = 2R\omega^2 (M_a + M_b) \cos \omega t$$

The components of the exciting moment equal

$$M_x = P_{z1}(l + 2l_x)$$
$$M_y = 2P_{z1} l_z$$
$$M_z = P_{x1}(l + 2l_x)$$

The values of l, l_x and l_z are shown in Fig. IV-4.

The engine under consideration belongs to the class of highly unbalanced engines, dangerous with respect to vibrations.

CASE 2. TWO-CYLINDER ENGINE WITH 90° CRANK ANGLE. On the basis of Eqs. (IV-2-9), we have

$$P_{x1} = R\omega^2 M_a \sin \omega t$$
$$P_{x2} = R\omega^2 M_a \sin\left(\omega t + \frac{\pi}{2}\right) = R\omega^2 M_a \cos \omega t$$
$$P_{z1} = R\omega^2 (M_a + M_b) \cos \omega t$$
$$P_{z2} = R\omega^2 (M_a + M_b) \cos\left(\omega t + \frac{\pi}{2}\right) = -R\omega^2 (M_a + M_b) \sin \omega t$$

The resultant components of exciting forces are

$$P_x = R\omega^2 M_a (\sin \omega t + \cos \omega t) = \sqrt{2}\, R\omega^2 M_a \sin\left(\omega t + \frac{\pi}{4}\right)$$
$$P_z = R\omega^2 (M_a + M_b)(\cos \omega t - \sin \omega t)$$
$$= \sqrt{2}\, R\omega^2 (M_a + M_b) \cos\left(\omega t + \frac{\pi}{4}\right)$$

Hence it follows that the resultant components of exciting forces are 1.41 times the resultant forces in each cylinder.

Let us determine the components of the exciting moment:

$$M_x = P_{z1}(l + l_y) + P_{z2} l_y$$

Analogously,

$$M_y = (P_{x1} + P_{x2}) l_z$$
$$M_z = P_{x1}(l + l_y) + P_{x2} l_y$$

This case, the 90° crank angle, is the most characteristic for two-cylinder engines.

CASE 3. TWO-CYLINDER ENGINE WITH 180° CRANK ANGLE. Here $\beta_1 = 0$, $\beta_2 = \pi$. According to Eq. (IV-2-9), the resultant components of the exciting forces will equal

$$P_x = 0 \qquad P_z = 0$$

The components of the exciting moment equal

$$M_x = P_{z1} l \qquad M_y = 0 \qquad M_z = P_{x1} l$$

d. Reciprocating Horizontal Compressors These engines usually have two cylinders with 90° crank angles. The expressions for the exciting forces here are the same as for a vertical engine; the only difference is that in the equations x should be changed to z and vice versa.

The exciting moments will equal (Fig IV-5)

$$M_x = P_{z1}l_{y1} - P_{z2}l_{y2}$$
$$M_y = P_zl_x + P_xl_z$$
$$M_z = P_{x1}l_{y1} - P_{x2}l_{y2}$$

e. Vertical Three-cylinder Engine. These engines usually have 120° crank angles; i.e., $\beta_1 = 0$, $\beta_2 = 120°$, and $\beta_3 = 240°$. Since

$$\cos 0 + \cos 120° + \cos 240° = 0$$
$$\sin 0 + \sin 120° + \sin 240° = 0$$

the first harmonics of the exciting forces are balanced:

$$P_x = 0 \qquad P_z = 0$$

If all three cylinders are spaced alike, then the exciting moments of the engine are

$$M_x = P_{z1}(2l + l_y) + P_{z2}(l + l_y) + P_{z3}l_y$$
$$M_y = 0$$
$$M_z = P_{x1}(2l + l_y) + P_{x2}(l + l_y) + P_{x3}l_y$$

f. Vertical Four-cylinder Engine. This engine is so designed that $\beta_1 = 0°$, $\beta_2 = 180°$, $\beta_3 = 180°$, and $\beta_4 = 360°$. All components of exciting forces and moments are balanced as a result of this arrangement.

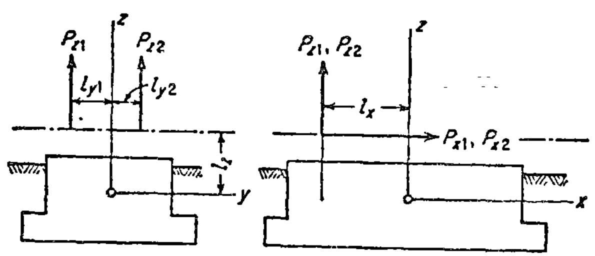


FIG. IV-5. Derivation of equations for horizontal piston compressors.

g. Vertical Six-cylinder Engine. In this engine crank angles are usually as follows:

$$\beta_1 = 0 \qquad \beta_2 = \frac{2\pi}{3} \qquad \beta_3 = \frac{4\pi}{3} \qquad \beta_4 = \frac{4\pi}{3} \qquad \beta_5 = 2\pi \qquad \beta_6 = \frac{8}{3}\pi$$

For such crank positions, the first and second harmonics of disturbing forces are balanced. The exciting moments equal

$$M_x = \sqrt{3}\,P_zl \qquad M_y = 0 \qquad M_z = \sqrt{3}\,P_xl$$

Their absolute values are comparatively small and they cannot cause vibrations with an amplitude exceeding the permissible value. Therefore

in design computations of foundations under six-cylinder engines there is no need to compute forced vibrations.

If the engines have auxiliary cylinders (a compressor and an exhaust cylinder in addition to the main cylinders), then in the computations of exciting forces a load imposed by the auxiliaries should be added to the loads produced by the main cylinders. However, the exciting loads caused by auxiliary cylinders are small in comparison to loads caused by the main cylinders, and they may often be neglected in computations of foundation vibrations.

IV-3. Stresses Imposed by Belt Pull

In many cases reciprocating engines set into rotary motion some types of operating machines, usually electric generators, by means of a belt

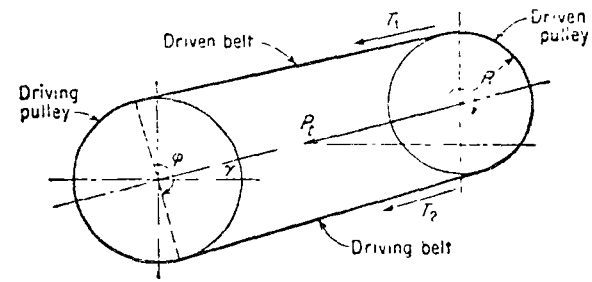


FIG. IV-6. Diagram of forces transmitted to pulleys by belt.

drive. On the other hand, some reciprocating engines are given rotary motion by means of a belt drive from an electromotor. Saw frames and compressors belong to this group.

When a belt drive is in operation, the force of the belt pull acts on the engine bearings and consequently on the foundation.

Let us consider a reciprocating engine set in rotation by means of a belt drive and examine the stresses transmitted to the foundation.

If T_1 is the magnitude of belt tension in the slack side of a belt (Fig IV-6) and T_2 is tension in the driving side, the resultant force of pull transmitted to the bearings of the engine, and consequently to the foundation, equals

$$T_1 + T_2 = P_t \qquad \text{(IV-3-1)}$$

The peripheral tension transmitted by the belt to the driven pulley is the difference between tensions in the driving and driven belts:

$$P_r = T_1 - T_2 \qquad \text{(IV-3-2)}$$

If W is the engine power and v is the peripheral speed of the belt, then it is known that

$$P_r = \frac{W}{v}$$

Since
$$v = R\omega = \frac{2\pi}{60} NR$$

where N = speed of engine, rpm
R = radius of driven pulley

then
$$P_r = \frac{60}{2\pi} \frac{W}{NR} \cong 9.55 \frac{W}{NR} \qquad \text{(IV-3-3)}$$

The interrelationship between the pull values in the driven and driving belts is approximately expressed by the formula

$$T_2 = T_1 e^{\varphi\mu} \qquad \text{(IV-3-4)}$$

where φ = smallest angle of arc of belt contact
μ = coefficient of friction between belt and pulley

The magnitude of μ depends on the type of flexible connection used.

From Eqs. (IV-3-2) and (IV-3-3) we have:

$$T_1 - T_2 = 9.55 \frac{W}{RN}$$

Substituting here the expression for T_2 from Eq. (IV-3-4), we obtain

$$T_1 = 9.55 \frac{W}{RN} \frac{1}{1 - e^{\varphi\mu}}$$

and consequently,

$$T_2 = 9.55 \frac{W}{RN} \frac{e^{\varphi\mu}}{1 - e^{\varphi\mu}} \qquad \text{(IV-3-5)}$$

Substituting these expressions for T_1 and T_2 into the right-hand part of Eq. (IV-3-1), we obtain the following expression for the force transmitted to the foundation by the belt pull:

$$P_r = 9.55 \frac{W}{RN} \frac{1 + e^{\varphi\mu}}{1 - e^{\varphi\mu}} \qquad \text{(IV-3-6)}$$

The direction of this force depends on the respective locations of the axes of the driving and driven pulleys.

If the straight line passing through the axes of rotation of the driving and driven pulleys forms an angle γ with the horizontal (Fig. IV-6), then the horizontal component of belt pull equals

$$P_{tx} = P_t \cos \gamma$$

The vertical component of pull tension usually may be neglected since it is small in comparison with the engine weight. In cases in which the engine is driving, the expression for P_t remains the same but the direction sign changes.

Therefore if the driven and driving pulleys are mounted on the same foundation, the forces imposed by a driving gear represent internal forces and do not influence the displacement of the foundation.

IV-4. Examples of Dynamic Analyses of Foundations for Reciprocating Engines

Example 1. *Dynamic computations of foundation for a vertical compressor coupled on a shaft with an electromotor*

1. Design Data A two-cylinder compressor has the following characteristics: crank angles, $\beta_1 = 0$, $\beta_2 = \pi/2$, compressor weight, 12 tons, electromotor weight, 4 tons; operational speed, 180 rpm.

The first harmonics of the exciting forces equal (in tons), in the direction of sliding of the piston,

$$P_{z1} = 3.0 \cos \omega t \qquad P_{z2} = -3.0 \sin \omega t$$

and in the horizontal direction perpendicular to the shaft axis,

$$P_{x1} = 0.4 \sin \omega t \qquad P_{x2} = 0.1 \cos \omega t$$

where ω is the angular velocity of rotation of the compressor, equaling

$$\omega = 0.101 \times 180 = 50 \text{ sec}^{-1}$$

The base of the foundation consists of silty clays with some sand characterized by the following design elastic coefficients:

$$c_z = 5.0 \times 10^3 \text{ tons/m}^3 \qquad c_\varphi = 10.0 \times 10^3 \text{ tons/m}^3 \qquad c_\tau = 2.5 \times 10^3 \text{ tons/m}^3$$

2. Design Diagram of Foundation To simplify computations, it is advisable to shape the foundation in plan as simply as possible, avoiding all small grooves, projections, asymmetry, and so on. Figure IV-7 gives a design diagram for the foundation under consideration, selected on the basis of the foregoing reasoning. Somewhat larger dimensions were selected for the projection of the foundation slab on the left-hand side, due to the eccentric distribution of the equipment on the foundation.

3. Centering of the Foundation Area in Contact with Soil and Determination of Static Pressure on Soil. Let us determine the coor[illegible]

the common center of gravity of the system (the foundation and compressor with the electromotor) with respect to the axes shown in Fig. IV-7:

$$x_0 = \frac{\Sigma m_i x_i}{m} \qquad y_0 = \frac{\Sigma m_i y_i}{m} \qquad z_0 = \frac{\Sigma m_i z_i}{m}$$

where m_i = masses of single elements of system
x_i, y_i, z_i = coordinates of centers of gravity of single elements with respect to axes
m = mass of system

We will consider that the masses of the compressor and electromotor are concentrated at the height of the level of the master-shaft axis (at a distance of 0.8 m from the foundation surface).

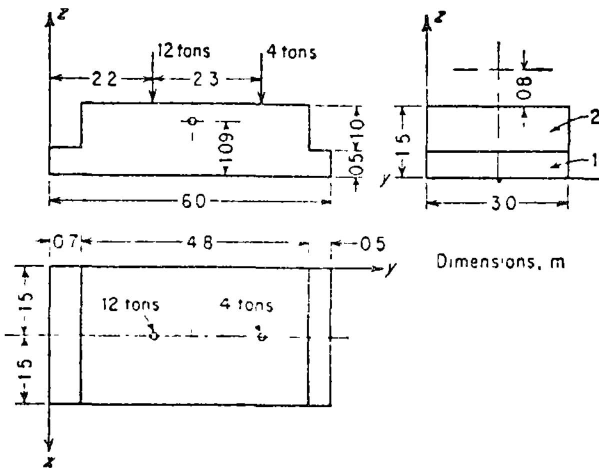


FIG. IV-7. Design diagram of foundation, example 1: (1) foundation slab; (2) upper part of foundation.

The results of computations of static moments of single elements of the system are given in Table IV-1. Using those data, we obtain the coordinates of the common center of gravity of the system:

$$x_0 = \frac{10.35}{6.91} = 1.5 \text{ m} \qquad y_0 = \frac{20.55}{6.91} = 2.98 \text{ m} \qquad z_0 = \frac{7.52}{6.91} = 1.09 \text{ m}$$

The relative values of the eccentricity in the directions of the x and y axes equal, in per cent:

$$\epsilon_x = 0 \qquad \epsilon_y = \frac{3.0 - 2.98}{3.0} \times 100 = 0.7$$

These values of eccentricity in the distribution of the masses are so small that they may be neglected in further computations of the foundation Thus we obtain: the weight of the whole system,

$$W = mg = 6.91 \times 9.81 = 67.5 \text{ tons}$$

the foundation area in contact with soil,

$$A = 6.0 \times 3.0 = 18 \text{ m}^2$$

and the static pressure on soil,

$$p_{st} = \frac{W}{A} = \frac{67.5}{18} = 3.8 \text{ tons/m}^2 = 0.38 \text{ kg/cm}^2$$

4. POSSIBLE FORMS OF FOUNDATION VIBRATIONS AND DESIGN VALUES OF EXCITING LOADS. The foregoing data lead to the conclusion that horizontal components of the disturbing forces of the compressor are small in comparison with vertical components Therefore the dynamic analysis of the foundation may be confined merely to determining the amplitudes of forced vibrations caused by vertical components of the exciting forces and of their moments.

TABLE IV-1. SUMMARY OF DATA FOR THE SOLUTION OF EXAMPLE IV-4-1

Elements of system	Dimensions of elements, m			Mass of element, tons × sec²/m	Coordinates of center of gravity of element, m			Static moments of mass of elements, tons × sec²		
	a_x	a_y	a_z		x_i	y_i	z_i	$m_i x_i$	$m_i y_i$	$m_i z_i$
Compressor				1.23	1.5	2.2	2.3	1.85	2.70	2.82
Electromotor				0.41	1.5	4.5	2.3	0.62	1.84	0.90
Foundation slab .	3	6	0.5	2.02	1.5	3.0	0.25	3.03	6.06	0.55
Upper part of foundation .	3	4.8	1.0	3.25	1.5	3.0	1.0	4.85	9.75	3.25
Total .				6.91				10.35	20.55	7.52

The resultant vertical component of the disturbing forces equals (see Art. IV-2)

$$P_z = P_{z1} \cos \omega t - P_{z2} \sin \omega t = 3.0(\cos \omega t - \sin \omega t)$$
$$= 4.2 \cos \left(\omega t + \frac{\pi}{4}\right)$$

The design value of the vertical component of the exciting loads will be

$$P_z = 4.2 \text{ tons}$$

This load will induce vertical forced vibrations of the foundation.

Due to the asymmetric position of the compressor, the foundation will be subjected to the action of the disturbing moment M_x with respect to the x axis. The magnitude of this moment is

$$M_x = P_z(l + l_y) + P_z l_y$$

where l = distance between cylinder axes; in the case under consideration $l = 1.3$ m
l_y = distance between second cylinder and vertical axis passing through center of gravity of complete system, in the case under consideration $l_y = 0.2$ m

Thus

$$M_x = 3.0(1.3 + 0.2)\cos \omega t - 3.0 \times 0.2 \sin \omega t$$
$$= 4.6 \cos\left(\omega t + \frac{\pi}{4}\right)$$

The design value of the disturbing moment should equal its greatest magnitude:

$$M_x = 4.6 \text{ tons} \times \text{m}$$

Under the action of this moment, vibrations will develop in the plane parallel to yz; they will be accompanied by a simultaneous sliding of the foundation in the direction of the y axis and a rotation of the foundation with respect to an axis parallel to the x axis and passing through the common center of gravity.

5. COMPUTATIONS OF THE AMPLITUDE OF FORCED VERTICAL VIBRATIONS OF THE FOUNDATION. From Eq (III-1-5) we determine the frequency of vertical natural vibrations of the foundation:

$$f_{nz}^2 = \frac{5.0 \times 10^3 \times 18}{6.91} = 13.0 \times 10^3 \text{ sec}^{-2}$$

The amplitude of forced vertical vibrations is found from Eq. (III-1-13):

$$A_z = \frac{4.2}{6.91(13.0 - 2.5) \times 10^3} = 0.058 \times 10^{-3} \cong 0.06 \text{ mm}$$

Hence it follows that the amplitude of vertical vibrations of the foundation will be much smaller than permissible (0.15 mm).

6. DETERMINATION OF THE MOMENTS OF INERTIA OF THE FOUNDATION AREA IN CONTACT WITH SOIL AND OF THE MASS OF THE WHOLE SYSTEM. The moment of inertia I of the foundation contact area with respect to the axis passing through its center of gravity perpendicular to the plane of vibrations is

$$I_a = \frac{3 \times 6^3}{12} = 51.0 \text{ m}^4$$

The moments of inertia of the masses of separate elements of the system with respect to the same axis are for the compressor, whose mass is considered to be concentrated at the height of the shaft axis,

$$I_{01} = m_1(0.8^2 + 2.3^2) = 1.23 \times 5.94 = 7.3 \text{ tons} \times \text{m} \times \text{sec}^2$$

for the electromotor,

$$I_{02} = m_2(1.5^2 + 2.3^2) = 0.41 \times 7.55 = 3.1 \text{ tons} \times \text{m} \times \text{sec}^2$$

for the foundation slab,

$$I_{03} = \frac{m_3}{12}(a_{3y}^2 + a_{3z}^2) + m_3 h_3^2 = \frac{2.02}{12}(6.0^2 + 0.5^2)$$
$$+ 2.02 \times 0.25^2 = 6.1 \text{ tons} \times \text{m} \times \text{sec}^2$$

(h_3 is the distance between the center of gravity of the mat and the foundation contact area).

For the upper part of the foundation, located above the mat, the moment of inertia, from an analogous formula, is

$$I_{04} = \frac{3.25}{12}(4.8^2 + 1.0^2) + 3.25 \times 1.0^2 = 9.8 \text{ tons} \times \text{m} \times \text{sec}^2$$

The total moment of inertia of the mass of the whole system with respect to this axis is

$$W_0 = \sum_{i=1}^{4} I_{0i} = 7.3 + 3.1 + 6.1 + 9.8 = 26.3 \text{ tons} \times \text{m} \times \text{sec}^2$$

The moment of inertia of the whole system with respect to the axis passing through the center of gravity of the whole system perpendicular to the plane of vibrations is

$$I = I_0 - mh^2 = 26.3 - 6.91 \times 1.09^2 = 18.2 \text{ tons} \times \text{m} \times \text{sec}^2$$

since $h = z_0 = 1.09$ m.

The ratio between the moments of inertia is

$$\gamma = 18.2/26.3 = 0.69$$

7. COMPUTATION OF AMPLITUDES OF FORCED VIBRATIONS OF A FOUNDATION ACCOMPANIED SIMULTANEOUSLY BY SLIDING AND ROCKING. The limiting natural frequency of rocking vibrations of the foundation, according to Eq. (III-2-6), is

$$f_{n\varphi}^2 = \frac{10 \times 10^3 \times 51 - 67.5 \times 1.09}{26.3} = 20.5 \times 10^3 \text{ sec}^{-2}$$

The limiting frequency of vibrations in shear, from Eq (III-3-2), is

$$f_{nx}^2 = \frac{2.5 \times 10^3 \times 18}{6.91} = 6.5 \times 10^3 \text{ sec}^{-2}$$

The frequency equation for the foundation [Eq. (III-1-8)] is

$$f_n^4 - \frac{(20.5 + 6.5) \times 10^3}{0.69} f_n^2 + \frac{20.5 \times 6.5}{0.69} 10^6 = 0$$

or

$$f_n^4 - 39.2 \times 10^3 f_n^2 + 193.0 \times 10^6 = 0$$

By solving this equation we find the natural frequencies of vibrations of the system:

$$f_{n1}^2 = 33.4 \times 10^3 \text{ sec}^{-2} \qquad f_{n2}^2 = 5.8 \times 10^3 \text{ sec}^{-2}$$

We compute the coefficient $\Delta(\omega^2)$:

$$\Delta(\omega^2) = mI(f_{n1}^2 - \omega^2)(f_{n2}^2 - \omega^2)$$
$$= 6.91 \times 18.2(33.4 - 2.5)(5.8 - 2.5) \times 10^6$$
$$= 13.8 \times 10^9$$

From Eqs (III-4-12) we determine the amplitudes of sliding shear and rotation of the foundation. The amplitude of sliding shear of the center of gravity of the whole system is

$$A_y = \frac{2.5 \times 10^3 \times 18 \times 1.09}{13.8 \times 10^9} 4.6 = 0.016 \times 10^{-3}\ \text{m} = 0.016\ \text{mm}$$

The amplitude of rotation is

$$A_\varphi = \frac{2.5 \times 10^3 \times 18 - 6.91 \times 2.5 \times 10^3}{13.8 \times 10^9} 4.6 = 0.009 \times 10^{-3}\ \text{radians}$$

The maximum horizontal displacement of the foundation surface in the plane yz is

$$A = A_y + h_1 A_\varphi = (0.016 + 0.41 \times 0.009) \times 10^{-3} = 0.020 \times 10^{-3}\ \text{m} \cong 0.02\ \text{mm}$$

The foregoing computations show that the amplitude of horizontal vibrations, as well as the amplitude of vertical vibrations, lies within the range of permissible values. Hence the conclusion is possible that the dimensions of the foundation for the machine under consideration were selected properly.

It is clear that in the case under review an increase in foundation height would lead to greater amplitudes of vibrations, hence an increase in height would not only raise the cost of the construction, but would also have a negative effect on the dynamic condition of the foundation.

This conclusion holds for all cases in which the natural frequencies of a foundation supported on soil are higher than the operational frequency of the engine mounted on the foundation. This occurs in the overwhelming majority of reciprocating engines.

Example 2. Dynamic analysis of a foundation for a reciprocating horizontal compressor

1. **Design Data.** The reciprocating horizontal compressor has two cylinders. The distance between the axis of the engine master shaft and the foundation surface is 0.9 m. The operational speed is 167 rpm.

Maximum values of unbalanced inertia forces of the engine are: horizontal component in the direction of piston motion P_x = 12.8 tons, vertical component P_z = 0.73 tons.

The foundation rests on a soil of medium strength having a permissible bearing value of 2 kg/cm². The design values of the coefficients of elasticity of the soil may be selected according to Table I-8 as follows:

Coefficient of elastic uniform compression:

$$c_u = 4.0 \times 10^3\ \text{tons/m}^3$$

Coefficient of elastic nonuniform compression

$$c_\varphi = 8.0 \times 10^3\ \text{tons/m}^3$$

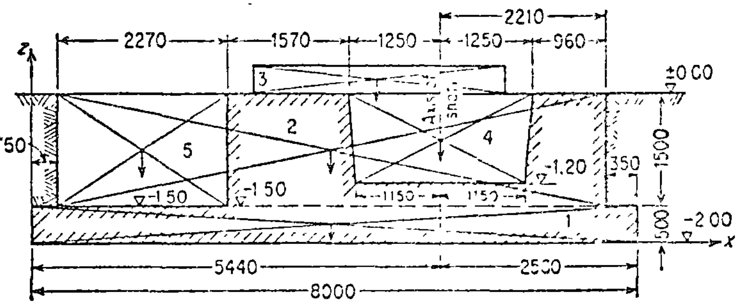
Coefficient of elastic uniform shear:

$$c_\tau = 2.0 \times 10^3\ \text{tons/m}^3$$

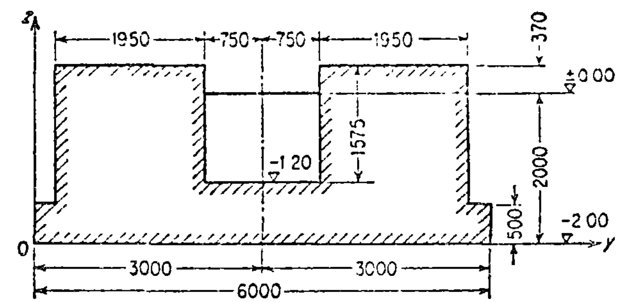
The dimensions of the foundation are not limited by structures, communication lines, or plant equipment

2. **Selection of a Design Diagram for the Foundation.** The dimensions of the foundation are to be selected according to design considerations based on the requirements of the plant management. Figure IV-8 gives the design diagram selected on the basis of these considerations. Concrete type 100† is to be employed for the foundation.

3. **Centering of the Foundation Area in Contact with Soil and Determination of Pressure on the Soil.** Let us determine the coordinates x_0, y_0, and z_0 of



Section along longitudinal axis of the foundation



Section along axis of main shaft

Fig. IV-8. Design diagram of foundation, example 2.

the center of gravity of the whole system (the foundation and engine) with respect to the axes shown in Fig. IV-8:

$$x_0 = \frac{\Sigma m_i x_i}{m} \qquad y_0 = \frac{\Sigma m_i y_i}{m} \qquad z_0 = \frac{\Sigma m_i z_i}{m}$$

where m_i = masses of separate elements of system

x_i, y_i, z_i = coordinates of centers of gravity of these elements with respect to x, y, z axes

m = mass of complete system

Separate elements of the foundation are marked in Fig IV-8 by the numbers 1, 2, and so on.

The foundation should be divided into elements of such shape that the data on magnitudes of masses and coordinates of centers of gravity of separate elements may

† See footnote in Art IV-1-c, p 132

be used later, when the moment of inertia of the mass of the whole system will be computed.

The data for the computation of coordinates of the center of gravity of the system are given in Table IV-2. Masses corresponding to cavities in the foundation are shown with minus signs. From the data of Table IV-2 we find the coordinates of the center of gravity of the system:

$$x_0 = \frac{89.14}{21.5} = 4.16 \text{ m} \qquad y_0 = \frac{64.52}{21.5} = 3.05 \text{ m} \qquad z_0 = \frac{28.55}{21.5} = 1.33 \text{ m}$$

The displacement of the center of gravity of the mass of the system with respect to the center of the foundation area in contact with soil is:

In the direction of the x axis:

$$4.16 - 4.00 = 0.16 \text{ m}$$

In the direction of the y axis:

$$3.05 - 3.00 = 0.05 \text{ m}$$

The relative magnitude of the eccentricity in the direction of the x axis is

$$\frac{0.16}{8.0} 100 = 2 \text{ per cent}$$

which is less than 5 per cent. The eccentricity in the direction of the y axis is even smaller.

Since the eccentricity in the mass distribution is small, its influence on the amplitudes of forced vibrations will be insignificant. Therefore we neglect hereafter the eccentricity and consider that the center of gravity of the mass of the system and the centroid of the foundation contact area are located on the same vertical line.

The pressure on the soil imposed by the static load is assumed to be uniformly distributed over the foundation contact area and equals

$$p_{st} = \frac{mg}{A} = \frac{21.5 \times 9.81}{48} = 4.4 \text{ tons/m}^2 = 0.44 \text{ kg/cm}^2$$

Thus the static pressure on the soil is considerably smaller than the permissible pressure.

4. Computation of Amplitudes of Forced Vibrations of the Foundation. Since the horizontal component of the unbalanced inertial forces of the engine in the direction perpendicular to the motion of the piston is zero, and since the vertical component of the above forces is insignificant, we compute the amplitudes of forced vibrations only for foundation vibrations caused by the horizontal component of unbalanced inertial forces in the direction of piston motion (the system will be subjected to vibrations in the xz plane). We also neglect the action of exciting moments tending to produce rocking vibrations of the foundation.

We begin by establishing the data needed for the computation of amplitudes of foundation vibrations. The frequency of machine rotation equals

$$\omega = 0.105N = 0.105 \times 167 = 17.3 \text{ sec}^{-1}$$
$$\omega^2 = 300 \text{ sec}^{-2}$$

TABLE IV-2. SUMMARY OF DATA FOR THE SOLUTION OF EXAMPLE IV-4-2

Elements of system (engine and foundation)	Dimensions of elements, m			Mass, tons × sec²/m	Coordinates of center of gravity of element with respect to x, y, z axes, m			Static moment of mass of element with respect to x, y, z axes, tons × sec²			Moment of inertia of mass of element with respect to axes passing through center of gravity of element, tons × sec² × m	Distance between center of gravity of element and common center of gravity, m		r
	a_{xi}	a_{yi}	a_{zi}	m_i	x_i	y_i	z_i	m_ix_i	m_iy_i	m_iz_i	$\frac{m_i}{12}(a_{xi}^2 + a_{zi}^2)$	x_{0i}	z_{0i}	$m_i(x_{0i}^2 + z_{0i}^2)$
Compressor				2.23	3.61	2.84	2.90	8.05	6.35	6.47	..	0.39	1.57	5.98
Motor...				1.47	5.44	3.18	2.90	8.00	4.67	4.26		1.44	1.57	6.76
1 ..	8.00	6.00	0.50	5.38	4.00	3.00	0.25	21.52	16.14	1.34	28.80	0	0.81	6.05
2 ..	7.30	5.40	1.50	13.27	4.00	3.00	1.25	53.08	39.81	16.59	61.00	0	0.03	0.01
3(2) ...	3.36	1.95	0.37	1.10	4.66	3.00	2.19	5.13	3.30	2.41	1.05	0.66	0.86	1.33
-4 ..	2.40	1.50	1.20	−0.97	5.44	3.00	1.40	−5.27	−2.91	−1.35	−0.58	1.44	0.07	−2.02
5 ..	2.27	1.25	1.50	−0.95	1.44	3.07	1.25	−1.37	−2.92	−1.19	−0.58	2.56	0.08	−6.03
Total ..				21.5				89.14	64.52	28.55	89.69			12.03

The distance from the axis of the master shaft of the engine to the common center of gravity of the mass of the system is $h_1 = 1.53$ m. The exciting moment of the engine is then

$$M = P_x h_1 = 12.8 \times 1.53 = 19.6 \text{ tons} \times \text{m}$$

The moment of inertia of the foundation contact area with respect to the axis passing through its center of gravity perpendicular to the plane of vibrations equals

$$I = \frac{6.00 \times 8.00^3}{12} = 256 \text{ m}^4$$

The weight of the whole system is

$$W = mg = 21.5 \times 9.81 = 211 \text{ tons}$$

The moment of inertia of the mass of the whole system with respect to the axis passing through the common center of gravity perpendicular to the plane of vibrations equals

$$W_m = \tfrac{1}{12}\Sigma m_i(a_{xi}^2 + a_{zi}^2) + m\Sigma(x_{0i}^2 + z_{0i}^2)$$
$$= 89.69 + 12.09 = 101.78 \cong 102 \text{ tons} \times \text{m} \times \text{sec}^2$$

The moment of inertia of the mass of the whole system with respect to the axis passing through the centroid of the foundation contact area perpendicular to the plane of vibrations equals

$$W_0 + I_m = mh^2 = 102 + 21.5 \times 1.33^2 = 140 \text{ tons} \times \text{m} \times \text{sec}^2$$

The ratio between the moments of inertia of the masses is

$$\gamma = {}^{102}\!/_{140} = 0.73$$

The limit value of natural frequency of rocking vibrations of the foundation is determined from Eq. (III-2-6):

$$f_{n\varphi}^2 = \frac{8 \times 10^3 \times 256 - 1.33 \times 211}{140} = 14.6 \times 10^3 \text{ sec}^{-2}$$

The limit value of the natural frequency of sliding shear vibrations, from Eq. (III-3-2), is

$$f_{nx}^2 = \frac{2 \times 10^3 \times 48}{21.5} = 4.46 \times 10^3 \text{ sec}^{-2}$$

We set up the frequency equation of the foundation according to Eq. (III-4-8):

$$f_n^4 - \frac{14.6 \times 10^3 + 4.46 \times 10^3}{0.73} f_n^2 + \frac{14.6 \times 10^3 \times 4.46 \times 10^3}{0.73} = 0$$
$$f_n^4 - 26.0 \times 10^3 f_n^2 + 89.0 \times 10^6 = 0$$

Solving this equation,

$$f_{n1,2}^2 = [13.0 \pm \sqrt{(13.0^2 - 89.0)}]10^3$$

Hence

$$f_{n1,2}^2 = (13.0 \pm 8.9)10^3$$

Thus the natural frequencies of the foundation will be

$$f_{n1} = 21.9 \times 10^3 \text{ sec}^{-2} \qquad f_{n2}^2 = 4.1 \times 10^3 \text{ sec}^{-2}$$

We compute the coefficient:

$$\Delta(\omega^2) = mW_m(f_{n1}^2 - \omega^2)(f_{n2}^2 - \omega^2)$$
$$= 21.5 \times 102(21.9 - 0.30)(4.1 - 0.30) \times 10^6 = 18.0 \times 10^{10}$$

We then compute the amplitudes of forced vibrations induced by the horizontal force P_x and by the moment $M = P_x h_1$, according to Eqs. (III-4-11) and (III-4-12) the horizontal displacement of the common center of gravity of the foundation and the engine is

$$A_x = \frac{(8 \times 10^3 \times 256 - 211 \times 1.33 + 2 \times 10^3 \times 48 \times 1.33 - 102 \times 17.3^2)12.8}{18.0 \times 10^{10}}$$
$$+ \frac{2 \times 10^3 \times 48 \times 1.33 \times 19.6}{18.0 \times 10^{10}} = 0.17 \times 10^{-3} \text{ m} = 0.17 \text{ mm}$$

From the same equations we find the amplitude of rocking vibrations of the foundation about the horizontal axis passing through the center of gravity of the foundation perpendicular to the plane of vibrations.

$$A_\varphi = \frac{(2 \times 10^3 \times 48 \times 1.33 \times 12.8 + (2 \times 10^3 \times 48 - 21.5 \times 17.3^2)19.6}{18.0 \times 10^{10}}$$
$$= 0.019 \times 10^{-3} \text{ radians}$$

Thus the amplitude of forced vibrations of the upper edge of the foundation equals

$$A = 0.17 \times 10^{-3} + 1.04 \times 0.019 \times 10^{-3} = 0.19\ 10^{-3} \text{ m} < 0.2 \text{ mm}$$

The design value of the amplitude of vibrations does not exceed the permissible value, hence the dimensions of the foundation are selected correctly.

The foregoing computations show that vibrations of the foundation are produced mainly by its horizontal displacement in the direction of the action of the horizontal component of the disturbing force of the compressor. This is explained by the fact that the dimensions of the foundation in the direction of the action of this force is large in comparison with the height of the foundation. Therefore rocking results only in small dynamic displacements.

Hence the rocking vibrations of a foundation may be neglected when computing the amplitude of forced vibrations if the foundation is elongated in the direction of action of the horizontal exciting force, in this case the vibrations of the foundation may be considered to be vibrations of sliding shear. This assumption greatly simplifies dynamic computations.

However, this simplification of computations should be very cautiously applied. For example, if such a simplification were made in the case of the foundation under consideration, then from Eq. (III-3-3) we would obtain for the amplitude of horizontal displacements of the foundation

$$A_x = \frac{12.8}{21.5(4.46 - 0.30) \times 10^3} = 0.14 \times 10^{-3} \text{ m} = 0.14 \text{ mm}$$

The computed amplitude is 26 per cent smaller than that obtained by means of the foregoing computations (0.19 mm). This cannot be admitted as a good approximation. The results of computations will be more accurate if we add to the [illegible] vibrations of sliding shear a displacement produced by rocking vibrations of the foundation, computed from the formula

$$A_{x\varphi} = A_\varphi h$$

where A_φ is the amplitude of rocking vibrations when no shear is present, determined from Eq. (III-2-7), and h is the full height of the foundation.

Assuming that the horizontal exciting force acts at height h_1 from the base of the foundation, and assuming that, in Eq. (III-2-7), $M = P_x h_1$, we obtain

$$A_x = \frac{P_x h h_1}{W_0(f_n^2 - \omega^2)}$$

For the foundation under consideration, $h = 2.0$ m and $h_1 = 2.9$ m; consequently,

$$A_{x\varphi} = \frac{12.8 \times 2.0 \times 2.9}{140(14.6 - 0.30) \times 10^3} = 0.038 \times 10^{-3} \text{ m} = 0.04 \text{ mm}$$

Thus the total amplitude of the horizontal displacement of the foundation will equal

$$A = A_x + A_{x\varphi} = 0.14 + 0.04 = 0.18 \text{ mm}$$

An amplitude computed by means of the above approximate method will not differ much from the value obtained as a result of computations taking into account vibrations of the foundation accompanied by simultaneous sliding shear and rocking.

IV-5. Methods for Decreasing Vibrations of Existing Foundations

a. Counterbalancing of Exciting Loads Imposed by Engines. As stated in Art. IV-2, there are different methods of balancing primary inertial forces by means of counterweights.

It is possible to counterbalance completely a component in the direction perpendicular to piston motion and partly a component in the direction of piston motion. Or, the dimensions of counterweights and their distances from axes of rotation may be selected to counterbalance completely the first harmonic of the component exciting forces in the direction of piston motion. Then the component in the perpendicular direction will increase.

Usually the first method is employed for the counterbalancing of engines because stresses in the engine itself are smaller than those occurring when the other method is used. Another advantage of the first method is that it requires a smaller counterweight mass.

The efficiency of a certain method of counterbalancing the exciting forces induced by an engine for the purpose of decreasing foundation vibrations depends on the type of engine and on special features of the foundation.

For a horizontal reciprocating engine, the most dangerous foundation vibrations are those which are accompanied simultaneously by rocking and sliding. In this case, a decrease in the vibrations of the foundation may be achieved by counterbalancing the inertial forces of the engine

by the second method, even if this leads to some increase in vertical vibrations. Therefore, if such an engine was counterbalanced by means of the first method but impermissible horizontal vibrations were observed after the construction of the foundation, then counterbalancing by means of the second method (i.e., by changing the character of counterbalancing) may be recommended as one of the simplest measures to decrease these vibrations.

In cases in which vertical vibrations of an impermissible amplitude are present in systems with horizontal motors, the second method is unsuitable, and the first method should be applied.

Similarly, for a vertical motor, the method of counterbalancing selected depends on the type of foundation vibrations—vertical, horizontal, or rocking.

The installation of counterweights for balancing a motor does not require dismantling or prolonged interruption of operation. The interruption is only for the time needed to attach the counterweight to the sides of the crank.

b. Chemical Stabilization of Soils. If a foundation rests on sandy soil, then, in order to decrease vibrations, chemical or cement stabilization of the soil under the foundation may be used. Such soil stabilization will result in an increase in the rigidity of the base and consequently in an increase in the natural frequencies of the foundation. Therefore this method is very effective when natural frequencies of the foundation on a nonstabilized soil are higher than the operational frequency of the engine—which is usually the case. An increase in rigidity will increase still further the difference between the frequency of natural vibrations and the frequency of the engine; consequently the amplitudes of foundation vibrations will decrease. When a foundation resting on a natural soil has natural frequencies smaller than the operational frequency of the engine, then soil stabilization may cause an increase in the amplitudes of vibrations. This may be undesirable if a soil is stabilized to such a degree that frequencies of natural vibrations of the foundation merely approach the operational frequency. But if a soil is thoroughly stabilized and natural frequencies of the foundation became much higher than the operational frequency of the engine, then such soil stabilization may result in a considerable decrease in amplitudes of vibrations.

Chemical and cement stabilization of soils is economically advantageous, since its costs are low in comparison, for example, with structural measures. The principal advantage of this method lies in the fact that it can be applied without a prolonged interruption in the work of the engine. The interruption is only for the period of direct work connected with soil stabilization and then for 2 to 3 days more. Thus the over-all result is that the engine will be inactive only for a few days.

The limits of the stabilized zones of soil and their shape are determined by the character of the vibrations. If, for example, a foundation is subjected mainly to rocking vibrations about an axis passing through the centroid of the base contact area, then it suffices to stabilize the soil near the foundation edges, perpendicular to the plane of vibrations, and it is not necessary to stabilize the soil under the entire foundation. The depth of the stabilized zone should be no less than 1 to 2 m.

This method of decreasing vibrations was applied at one of the Soviet plants when it became necessary to decrease the amplitudes of vibrations of an operating horizontal compressor without a long interruption in its work. Soil was stabilized to a depth of about 1.0 m; the zone extended horizontally 30 cm beyond the foundation edges. The results of foundation vibration measurements before and after stabilization showed that the amplitudes of vibrations, on the average, decreased by 50 per cent. The work of the compressor was stopped only for the period of injection of the silicates; the engine was set in motion immediately after silicatization was completed. It can therefore be assumed that when the compressor renewed its motion, the stabilized soil had not as yet formed a sufficiently rigid base, and it is possible that foundation vibrations acted unfavorably on the stabilized zone of soil, which had not fully hardened.

c. Structural Measures The use of structural measures for decreasing foundation vibrations often requires a long interruption in the engine's operation and considerable expense of funds and materials Therefore the use of this method may be suggested only in cases in which for some reason no other methods may be applied. At the same time, it should be noted that the correct change in foundation design may prove very effective in decreasing the amplitude of vibrations.

Structural measures are applied with the purpose of changing the natural frequencies of a foundation in such a way as to achieve the largest possible difference between them and the operational frequency of the engine. The choice of structural measure depends on the nature of the vibrations and the interrelationships between the frequencies of natural and forced vibrations. The operational frequencies of reciprocating engines are usually lower than the fundamental frequencies of foundations; therefore most of the structural measures are directed towards increasing still further the natural frequencies of the foundation. This is achieved by increasing the foundation contact area and its moments of inertia, as well as by increasing the rigidity of its base by means of piles

In addition, it is possible to increase the foundation mass without inducing changes in the frequency of foundation vibrations. This results in a decre... in the amplitudes of vertical vibrations.

When check calculations of the natural frequencies of a vibrating foundation show that they are lower than the operational frequencies of

the engine, an enlargement of the foundation contact area or an increase in the soil rigidity not only may not decrease the amplitudes of vibrations, but may even increase them. In this case, it is better to decrease still more the natural frequency of the foundation. This may be achieved by enlarging the foundation mass without an increase in its area in contact with soil.

The selection of particular structural measures depends on local conditions. For example, if a vibrating foundation lies close to another foundation, it can be attached to the latter. As an illustration we will

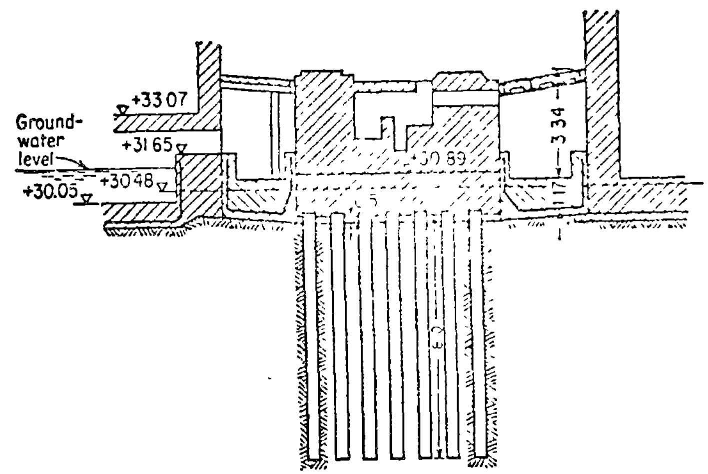


FIG IV-9. Machine foundation which developed horizontal vibrations of high amplitude.

describe here the structural measures which were applied to a foundation under a horizontal compressor in order to decrease its vibrations

The horizontal component of the exciting forces induced by the whole system was 30 tons. The foundation rested on a medium-grained sand with clay laminae.

The foundation consisted of a block about 4.6 m high, with a base area 7 by 8 m², placed on 55 situ-cast piles The length of the piles was about 8 m Figure IV-9 shows a cross section of the foundation

Horizontal vibrations of extremely large amplitude (around 0.9 mm) were observed while the engine was in operation At the same time, there occurred settlement of the basement of an adjoining structure, under which no piles were provided. On the side nearest the foundation under discussion, settlement of the basement reached 70 mm. It appears that this considerable settlement of the basement was caused by vibrations transmitted from the foundation These vibrations further... loosening of the soil and the carrying away of soil particles from beneath the foundation and basement by ground water. When reinforcement of the foundation was started, it was found that soil under it was washed out or had subsided to a depth of about 0.5 m. No damage was found in the foundation block.

The ground-water level was approximately 1.5 m above the level of the foundation base.

Reinforcement of the foundation was undertaken with the purpose of decreasing its vibrations. After the ground-water level had been arti-

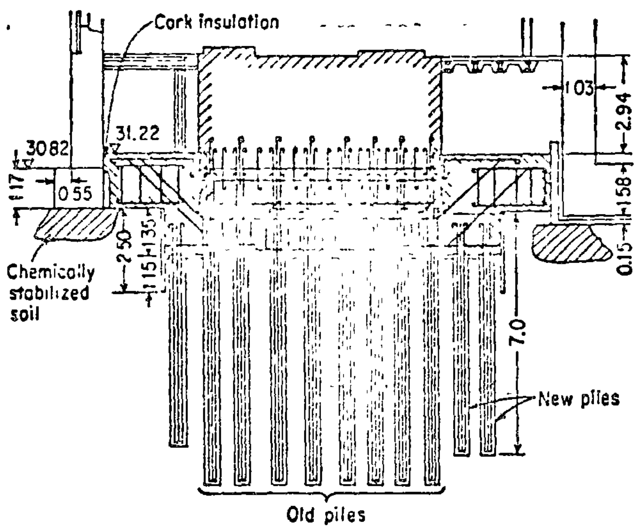


FIG. IV-10. Reinforcement of foundation shown in Fig. IV-9.

ficially lowered, soil was removed from beneath the foundation to a depth of 0.75 m. Thus an excavation of a total depth of about 1.25 m was formed. This excavation was filled with concrete. The foundation area in contact with the soil was extended on all sides and 33 new situ-cast piles were installed. Figure IV-10 illustrates the measures recommended for reinforcement. Reinforcement of the new part of the foundation provided a good connection with the old part. To avoid settlement of the footings under the building walls due to excavation of the soil, sheetpiling and chemical stabilization of soil were used beneath the footings under the walls.

This reinforcement of the foundation was very effective. The amplitudes of foundation vibrations decreased from 0.9 to 0.05 mm, i.e., 18 times.

This decrease in amplitudes was caused by considerable increase in the natural frequencies of foundation vibrations due to an increase in the

foundation area in contact with the soil, as well as to an increase in the moment of inertia of the contact area. In addition, a considerable effect was produced by the extra foundation mass and the increase in the rigidity of the base due to the installation of supplementary piles.

To facilitate the application of various structural measures for decreasing foundation vibrations, it is recommended in doubtful cases to leave projecting reinforcement which may be used, if necessary, for the attachment of an additional mass to the foundation or for the extension of its area in contact with soil. These measures, of course, should be applied only after recognizing the fact that the foundation is undergoing vibrations of an impermissible magnitude.

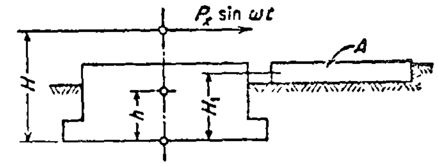


FIG. IV-11. Analysis of the effect of attaching a slab A to an engine foundation.

The use of special slabs, first proposed by Professor N. P. Pavliuk and Engineer A. D. Kondin,[43] may be considered as a structural measure to decrease foundation vibrations. By means of these slabs, it is possible in some cases to decrease the amplitudes of rocking and horizontal vibrations of foundations.

Let us assume (Fig. IV-11) that slab A, resting on soil, is attached to a foundation undergoing rocking vibrations around the axis passing through the centroid of the foundation area in contact with soil. Let us set up the equation of forced rocking vibrations of the foundation. The following symbols will be used:

W_0 = moment of inertia of foundation mass and of mass of engine with respect to axis of vibrations

I = moment of inertia of foundation area in contact with soil, with respect to same axis

$P_T \sin \omega t$ = magnitude of horizontal exciting force induced by engine and transmitted to foundation, where ω = frequency of engine rotation

H = distance between line of action of exciting force and foundation contact area

h = distance between center of mass of foundation and engine, and foundation contact area

m_1 = mass of attached slab

A_1 = contact area of attached slab

H_1 = distance between place of connection of foundation with attached slab, and foundation contact area

W = foundation weight

c_ϕ, c_τ = coefficients of elastic nonuniform compression, shear of soil

The differential equation of forced vibrations of the foundation together with the attached slab will be as follows:

$$(W_0 + m_1H_1^2)\ddot{\varphi} + (c_\psi I - Wh + H_1^2c_rA_1)\varphi = P_TH \sin \omega t \quad \text{(IV-5-1)}$$

From this we obtain in the usual way the expression for the natural frequency of rocking vibrations of the foundation with attached slab:

$$f_{n\varphi 1}^2 = \frac{c_\varphi I - Wh + H_1^2c_rA_1}{W_0 + m_1H_1^2} \quad \text{(IV-5-2)}$$

The amplitudes of rocking vibrations of the foundation will be found from the equation

$$A_{\varphi 1} = \frac{P_TH}{(W_0 + m_1H_1^2)(f_{n\varphi 1}^2 - \omega^2)} \quad \text{(IV-5-3)}$$

Equation (IV-5-2) shows that under certain conditions an attached slab may have no effect on the natural frequency $f_{n\varphi}$ of rocking vibrations of the foundation. To determine these conditions, let us use the following expression:

$$f_{n\varphi 1}^2 \geq f_{n\varphi}^2 \quad \text{(IV-5-4)}$$

or since

$$f_{n\varphi}^2 = \frac{c_\varphi I - Wh}{W_0}$$

we substitute into the left-hand part of (IV-5-4) the expression for $f_{n\varphi 1}^2$ from Eq. (IV-5-2); then, neglecting the term containing Wh because of its smallness, we obtain

$$\frac{c_\varphi I + H_1^2c_rA_1}{W_0 + m_1H_1^2} \geq \frac{c_\varphi I}{W_0} \quad \text{(IV-5-5)}$$

Hence

$$\frac{c_rA_1}{m_1} \geq \frac{c_\varphi I}{W_0}$$

or

$$f_{nx1}^2 \geq f_{n\varphi}^2 \quad \text{(IV-5-6)}$$

Thus if one selects the attached slab so that the frequency of its natural vibrations of pure shear equals the frequency of rocking vibrations of the foundation, then the attached slab will have no effect on the magnitude of the frequency of natural vibrations of the foundation. Besides, the amplitude of forced vibrations of the foundation will decrease according to the ratio

$$a = \frac{A_{\varphi 1}}{A_\varphi} = \frac{1}{1 + \delta^2} \quad \text{(IV-5-7)}$$

where

$$\delta = \frac{m_1H_1^2}{W_0}$$

Usually in foundations under engines, $f_{n\varphi 1}$ is considerably larger than ω; therefore, approximately,

$$A_{\varphi 1} \approx \frac{P_TH}{c_\varphi I + H_1^2c_rA_1}$$

To determine the amplitude of foundation vibrations before the slab is attached, if $f_{n\varphi} \gg \omega$ and the value of WH in comparison with $c_\varphi I$ is small, we have from Eq. (IV-5-3):

$$A_\varphi = \frac{P_TH}{c_\varphi I}$$

Hence

$$a = \frac{A_{\varphi 1}}{A_\varphi} = \frac{c_\varphi I}{c_\varphi I + H_1^2c_rA_1}$$

or

$$a = \frac{1}{1 + H_1^2c_rA_1/c_\varphi I} \quad \text{(IV-5-8)}$$

It follows from Eq. (IV-5-8) that the effect of the attached slab on the decrease in foundation vibrations will be proportional to the frequency f_{nx} of natural vibrations of shear of the slab and proportional to the height H_1 of the slab above the foundation base. The foundation area in contact with soil should always be as large as possible; the contact area of the slab A_1 is limited by local conditions and economic considerations. Since the value of f_{nx1} depends also on the coefficient of the elastic shear of soil c_r, a pile foundation may be installed under the slab to increase f_{nx1} as much as possible. Short frictional reinforced-concrete piles should be used.

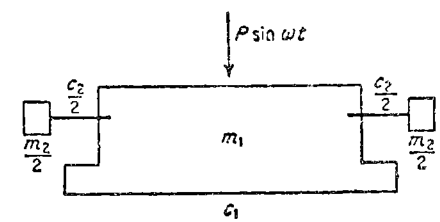


FIG. IV-12. Analysis of the effect of attaching dampers $m_2/2$ to an engine foundation m_1.

Professor N. P. Pavliuk and A. D. Kondin describe a case in which a reinforced-concrete slab was used to decrease the vibrations of a foundation under a compressor. By installation of the slab, foundation vibrations were practically reduced to zero.

d. Dynamic Vibration Dampers. Consider a foundation with mass m_1 (Fig. IV-12) subjected to the action of external exciting forces induced by an engine and producing only vertical vibrations. Let us assume that two masses, each equaling $m_2/2$, are attached to this foundation by means of elastic ties (elastic rods, springs, etc.). Then the foundation with attached masses will have not one but two degrees of freedom. Formulas from the theory of vibrations of a system with two degrees of freedom may be directly applied when considering the forced vibrations of the system.

For example, for the amplitudes A_1 and A_2 of forced vibrations of the foundation and the attached masses, respectively, we have:

$$A_1 = \frac{f_{22}^2 - \omega^2}{\Delta(\omega^2)} p$$

$$A_2 = \frac{f_{22}^2}{\Delta(\omega^2)} p \qquad \text{(IV-5-9)}$$

Also,

$$f_{na}^2 = \frac{c_2}{m_2}$$

where f_{na} = natural frequency of vibrations of attached masses
c_2 = coefficient of elastic rigidity of tie between these masses and foundation
p = magnitude of exciting force per unit of foundation mass

$$\Delta(\omega)^2 = (f_{n1}^2 - \omega^2)(f_{n2}^2 - \omega^2) \qquad \text{(IV-5-10)}$$

where f_{n1} and f_{n2} are natural fundamental frequencies of the foundation with dampers.

It follows from the first of Eqs. (IV-5-9) that the amplitude of foundation vibrations becomes zero when

$$\omega^2 = f_{na}^2 \qquad \text{(IV-5-11)}$$

i.e., when the frequency of natural vibrations of the attached masses equals the frequency of the exciting force. In order to determine the amplitude of vibrations of the damper masses, we substitute f_{2} into the right-hand part of Eq. (IV-5-10) in place of ω; then

$$\Delta(\omega^2) = \frac{c_2}{m_1}$$

Substituting this expression for $\Delta(\omega^2)$ into the second of Eqs. (IV-5-9), we obtain the following expression for the amplitude of vibrations of the damper:

$$A_2 = \frac{m_1}{c_2} p$$

Since

$$p = \frac{P(t)}{m_1}$$

it follows that

$$A_2 = \frac{P(t)}{c_2} \qquad \text{(IV-5-12)}$$

Thus, the amplitude of vibrations of the damper equals its static deflection as produced by a force of magnitude equal to the maximum value of the exciting force $P(t)$.

Equations (IV-5-11) and (IV-5-12) determine the selection of the damper. It should be noted that neither the frequencies of the damper nor the amplitudes of its vibrations depend on the properties of the soil base or the mass of the foundation.

It follows from Eq. (IV-5-9) that theoretically it is possible to damp vibrations of infinitely large foundations subjected to the action of periodic exciting forces by attaching dampers to these foundations, even dampers with small masses. However, the smaller the mass of the damper, the smaller should be the rigidity c_2 of its elastic tie with the foundation, and consequently the larger will be the amplitude of its vibrations.

At values of c_2 smaller than a certain limit, amplitudes of vibrations of the damper may attain magnitudes endangering its strength. Therefore the minimum value of the damper mass is limited by permissible values of stresses in the elastic tie between the damper and the foundation.

It has already been mentioned that when dampers are used, the foundation has not one, but two natural frequencies of vibrations, determined as roots of the equation

$$f_n^4 - [f_{nz}^2 + (1 + \mu)f_{na}^2]f_n^2 + f_{nz}^2 f_{na}^2 \qquad \text{(IV-5-13)}$$

where

$$f_{nz}^2 = \frac{c_1}{m_1} \qquad c_1 = c_u A$$

The roots of Eq. (IV-5-13) are.

$$f_{n1,2}^2 = \tfrac{1}{2}(f_{nz}^2 + f_{na}^2(1 + \mu) \pm \sqrt{[f_{nz}^2 + f_{na}^2(1 + \mu)^2]^2 - 4f_{nz}^2 f_{na}^2})$$

where f_{nz} = frequency of natural vertical vibrations of foundation
$f_{na} = \omega$ = average operational machine rotation
$\mu = m_2/m_1$ = ratio between masses of dampers and foundation mass

When a damper is installed, f_{n1} will be larger than both f_{nz} and ω, and f_{n2} will be smaller than these frequencies. Besides, either f_{n1} or f_{n2} will lie close to ω, and the other will be close to f_{nz}.

Let us assume that $f_{nz} > \omega$. Then the lower fundamental frequency f_{n2} will be close to ω; the higher one, f_{n1}, will be close to f_{nz}. If the engine has varying angular frequency, then, with the installation of the damper, the danger arises that one of the values of ω will coincide with f_{n2}, i.e., that resonance will occur with the lower frequency of the system "foundation and damper." In this case, the amplitudes of foundation vibrations will be sharply increased and the masses attached to the foundation will work not as dampers, but as intensifiers of the vibrations.

In order to avoid such an intensification of vibrations, m_2 should be selected so that the maximum decrease in operational frequency of the engine, as compared with the average value of this frequency, is smaller than the difference between the average operational frequency of the engine ω_{av} (equaling the natural frequency f_{22} of the damper) and the lower frequency f_{n2} of the foundation. Hence, the following condition should be satisfied when the damper sizes are selected:

$$f_{n2}^2 < \omega_{min}^2$$

or $$\tfrac{1}{2}\left(f_{nz}^2 + f_{na}^2(1 + \mu) - \sqrt{[f_{nz}^2 + f_{na}^2(1 + \mu)]^2 - 4f_{nz}^2 f_{na}^2}\right) < \omega_{min}^2$$

Solving this inequality for μ and noting that

$$f_{na}^2 = \omega_{av}^2$$

we obtain

$$\mu > \frac{f_{nz}^2(\omega_{av}^2 - \omega_{min}^2) + \omega_{min}^4}{\omega_{min}^2\omega_{av}^2} - 1$$

Let us assume that the nonuniformity in engine speed is as follows:

$$\epsilon = 1 - \frac{\omega_{min}}{\omega_{av}}$$

Hence $$\omega_{min} = \omega_{av}(1 - \epsilon)$$

Substituting this expression for ω_{min} into the right-hand part of the inequality obtained for μ,

$$\mu > \frac{\beta^2 - (1 - \epsilon)^2}{(1 - \epsilon)^2}\epsilon(2 - \epsilon)$$

where $$\beta = \frac{f_{nz}}{\omega_{av}} > 1$$

If $0 < \beta < 1$, then $\omega_{max} = \omega_{av}(1 + \epsilon)$ and we obtain:

$$\mu > \frac{(1 + \epsilon)^2 - \beta^2}{(1 + \epsilon)^2}\epsilon(2 + \epsilon) \qquad \text{(IV-5-14)}$$

The inequalities obtained show that the selection of a proper interrelationship between the damper mass and the foundation mass depends not only on the values of the irregularity in the engine speed, but also on the interrelationship between the natural frequency of the foundation and the average operational speed of the engine.

If the frequency of natural vibrations is higher than the operational frequency of the engine (i.e., $\beta > 1$), then the damper mass should be selected in proportion to the value of β. If $\beta < 1$, the value of μ decreases with an increase in β. If $\beta = 1$, μ should have its lowest value.

The nonuniformity in the speed of reciprocating engines lies in the range from 0.01 to 0.10. The most uniform speed is observed in multicylinder diesels with flywheel, where ϵ is about 0.01 to 0.02; in saw frames ϵ is 0.05 to 0.10.

The frequencies of natural vertical vibrations of foundations are usually higher than the operational frequencies of low-speed reciprocating engines, i.e., usually $\beta > 1$. With variations in the angular speed of the engine, the foundation may develop resonance with the lower frequency of the "foundation-damper" system. If one assumes that the average smallest irregularity in engine speed is around 0.075, then in order to avoid resonance, it is necessary that the lower frequency of the system differ by at least 3 per cent from the average operational speed of the engine. Thus, in calculations of the smallest value of μ, the design value of ϵ should be taken not less than 0.03. For this value of ϵ, and with $\beta = 1.3$, the value $\mu = 0.05$, if $\beta = 1.6$, then the damper mass should be about one-tenth the foundation mass. If the natural frequency of vertical vibrations is two times larger than the operational frequency of the engine, the damper mass should not be less than 20 per cent of the foundation mass.

The weights of foundations under reciprocating engines may reach several hundred tons. For the previously mentioned values of ϵ and β the damper weight will equal several tens of tons. In practice it is difficult to attach a mass of this size elastically to the foundation so that the frequency of natural vibrations of this mass corresponds exactly to the average operational speed of the engine. For example, for a foundation weighing 300 tons, for $\epsilon = 0.03$ and $\beta = 1.5$, the damper should weigh not less than 27 tons, for $\epsilon = 0.05$ it should weigh 45 tons. The difficulty in attaching such blocks to the foundation limits the use of dynamic dampers even for machines with uniform speed. It is out of the question for such machines as saw frames, in which the irregularity in speed attains 0.1.

For high-frequency engines, such as turbogenerators and electromotors with small irregularity in speed, the employment of dynamic dampers may be effective because of the low value of ϵ and because usually $\beta < 1$ for their foundations.

By introducing damping into the system of the damper, it is possible to increase the difference $\omega_{av} - f_{n2}$ and decrease the amplitude of foundation vibrations when ω approaches f_{n2}. However, these effects will take place only for some optimum values of damping. In order to achieve in practice these optimum values, the damper should be thoroughly tuned up. The maintenance of constant damping is especially difficult under working conditions. Temperature, moisture, and contamination may affect a damper's natural period and its damping, thus upsetting its tuning.

When foundations undergo vibrations close to those of pure sliding shear, all the above interrelationships will be valid, except that f_{nx} should be inserted everywhere in the equations instead of f_{nz}. They are valid also when foundation vibrations are close to being pure rotational vibrations around an axis passing through the centroid of the foundation area in contact with soil. In this case, μ designates the ratio between the

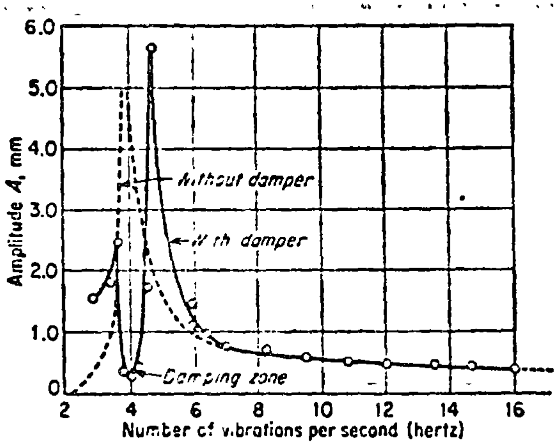


Fig. IV-13. Experimentally determined shift of resonance peak as a result of the use of a vibration damper.

moment of inertia of the damper masses with respect to the axis passing through the center of gravity of the foundation perpendicular to the plane of vibrations and the moment of inertia of both the foundation and engine mass with respect to the same axis.

Let us note in conclusion that the author and his associates investigated experimentally the effects of dampers on model foundations. These experiments confirmed the fundamental theoretical conclusions. For example, the experiments verified that after the damper is installed, one of the resonances of the newly formed "foundation-damper" system (Fig. IV-13) appears close to the operational frequency of the engine. This resonance is dangerous even at negligible changes in frequency of engine rotation.

IV-6. Analysis and Design of Foundations with Vibration Absorbers

In some cases much lower than usual permissible amplitude values of machine foundation vibrations are necessary. It is very difficult to

decrease these amplitudes by means of proper selection of the mass or the foundation contact area or by increasing the rigidity of the base.

However, the amplitudes of foundation vibrations under reciprocating engines may be considerably decreased by means of special spring absorbers.

These absorbers are comparatively inexpensive, reliable in operation, and effective in decreasing the amplitudes of forced vibrations of foundations. Absorbers considerably decrease vibrations produced not only by the main (first) harmonics, but also by higher harmonics of exciting loads, as well as by loads developing as a result of various factors not taken into account by design computations. Therefore spring absorbers are sometimes used to decrease vibrations of machine foundations having unbalanced second harmonics. This is done in order to eliminate completely the transmission of the inevitable vibrations to adjacent structures and especially to equipment and precision measuring instruments. Human beings feel vibrations of even very small amplitudes. Sometimes small vibrations interfere with the work of precise devices or are the reason for undesirable distortions in various technological processes (for example, in the operation of precision devices, in molding, etc.).

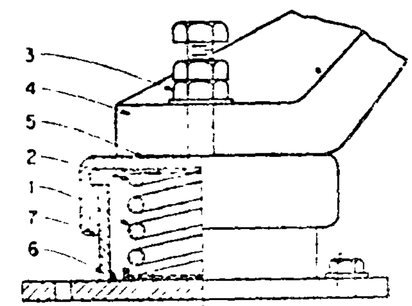


Fig. IV-14. Small one-spring vibration absorber.

There are various types of absorbers employed, depending on the type of machine to which they will be attached, on the static load transmitted to the absorber, and on special requirements in regard to assembling and adjusting.

Figure IV-14 shows a sketch of a small one-spring absorber used for small engines producing no considerable unbalanced exciting loads. This absorber consists of a coil spring 1 which fits into the adjusting slab 2. The regulating bolt 3 rests on this slab. The frame 4 is placed on the lid 5 of the absorber. The position of the frame is adjusted by turning the regulating bolt 3. To eliminate harmful external effects on the spring, the latter is enclosed in the housing 6, having insulating pads 7 (made of rubber or cork) which protect the spring from water and dirt.

Such light absorbers are used for vibroisolation of small diesel engines, ventilation units, presses, pumps, and other machines.

For the vibroisolation of reciprocating engines of medium and high capacities, absorbers containing several springs are used. An absorber of this type with four springs is shown schematically in Fig. IV-15. The housing and parts of the absorber are made of steel plates and several other metals.

The main characteristics of the springs (i.e., the diameters and number of coils) are selected according to the results of dynamic computations.

In addition to spring absorbers, rubber absorbers may also be employed for vibroisolation of light engines and devices. In comparison with spring absorbers, rubber absorbers are simpler and less expensive. Besides, they are characterized by a larger coefficient of resistance to vibrations, useful when they are used for vibroisolating machines of irregular performance. A disadvantage of rubber absorbers is the variation in their modulus of elasticity, which depends on the load. Computations related to vibroisolation always involve relatively large errors if rubber is used.

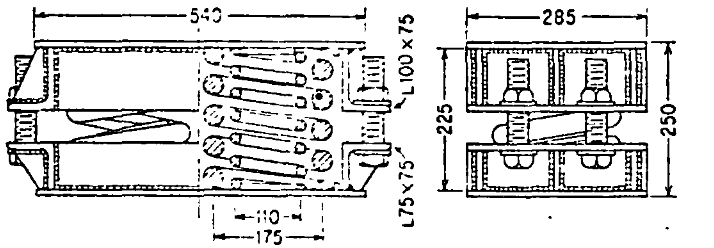


Fig. IV-15. Four-spring vibration absorber.

Depending on the balance of the engine and its operational speed, different arrangements may be used for the vibroisolation of foundations by spring absorbers. Fundamentally these arrangements can be reduced to two types: supporting and suspension springs.

When designing the vibroisolation of a foundation for a high-speed engine (more than 300 rpm) which is relatively well balanced, i.e., no first harmonics of exciting loads are present, it is not necessary to provide a heavy foundation above the springs. It may be designed as a reinforced-concrete slab of comparatively small thickness. In this case, a "supported" type of vibroisolation is employed, in which the absorbers are installed directly under the mass above the springs. Figures IV-16 and IV-17 illustrate vibroisolations of this type. Figure IV-16 shows the arrangement of absorbers employed for the vibroisolation of a six-cylinder diesel engine operating on one shaft with a generator. Here the absorbers are installed directly under a metal frame made of rolled steel shapes, used instead of the cast-iron frame of the diesel and generator. The mass above the springs consists here of the mass of the motor with the generator and the supporting frame; no portion of the foundation is above the springs. This arrangement of absorbers can

be used only for high-speed engines. The supporting frame should be very rigid to avoid the harmful effects of its deformations on the connector of the shaft.

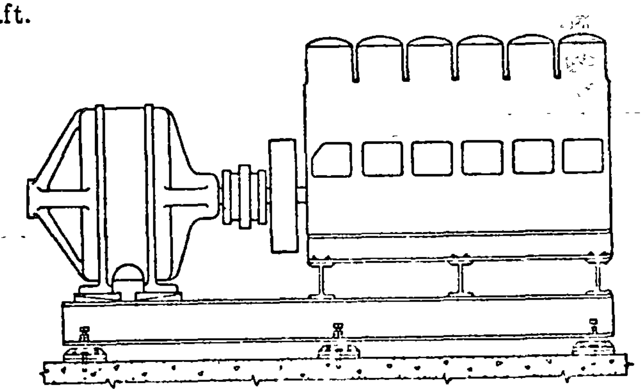


Fig. IV-16. Supporting-spring type of vibroisolation of a six-cylinder diesel with generator on the same shaft.

Figure IV-17 shows the "supported" vibroisolation of a high-speed two-cylinder diesel engine having unbalanced first harmonics of exciting loads. In this case, the mass above the springs was increased by means of a special thick reinforced-concrete slab under which the absorbers were placed. They rest on supporting slabs which also support rolled steel beams embedded in the lower part of the foundation above the springs.

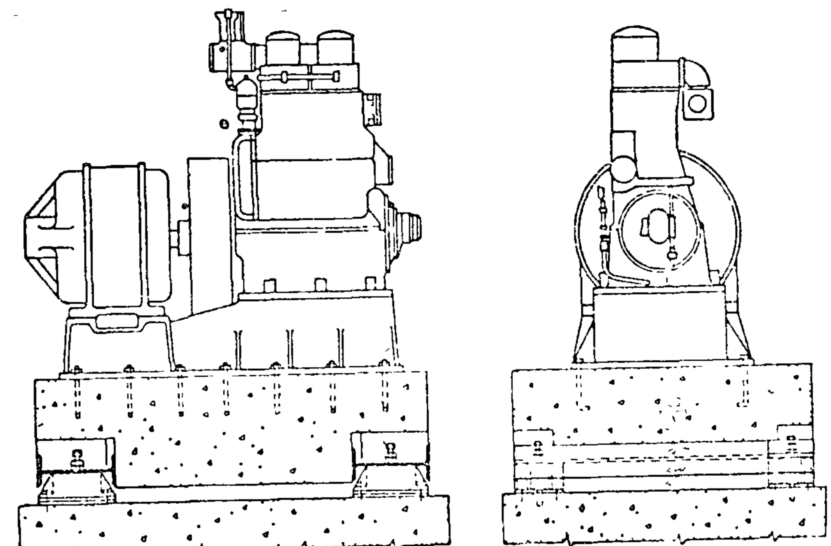


Fig. IV-17. Vibroisolation of high-speed two-cylinder diesel with generator on the same shaft. System has unbalanced first harmonics of exciting loads.

A rigid coupling between the absorbers and the beams is formed by bolting a beam to the cover of each absorber. Special cavities are left in the concrete above the springs to permit access to the absorbers.

The installation of vibroisolation of the supported type proceeds successively as follows: first the foundation beneath the springs is concreted. Usually it consists of a reinforced-concrete slab with a thickness of 0.20 to 1.00 m, depending on the type and size of the engine and on soil properties. After the concrete has hardened, the surface of the slab is covered with Ruberoid, tar paper, or plywood, on which the lower slabs of the absorbers are placed in the proper order. Above these slabs a prefabricated frame of rolled steel beams is installed. Then the formwork is prepared for the concrete of the foundation above the springs; cavities should be left for each absorber. Then the concrete is poured. Due to the presence of the layer of Ruberoid (or tar paper or plywood), the concrete of the upper part of the foundation will not bind to the concrete beneath the springs.

After the concrete of the upper part of the foundation has hardened, the absorbers are mounted. The springs are placed on the lower slabs of the absorbers and are covered by the upper supporting slabs, which are bolted to girders. Finally, the restraining anchor bolt is installed to permit lifting of the mass above the springs. The lifting is carefully regulated by means of a level. If the absorbers are installed correctly, the lifting and regulation of the mass above the springs do not take much time and do not involve any difficulties.

The vibroisolation of low-frequency engines by means of absorbers leads to the necessity for providing a heavy foundation above the springs. Therefore, if the foundation is placed directly on the absorbers, the latter should be installed at a level considerably lower than the floor of the shop. This hinders access to the absorbers and their mounting, regulation, and maintenance.

In such cases, absorbers of the "suspended" type are used. It is seen from the sketch of an absorber of this type in Fig. IV-18 that it differs from the previously described "supported" type only by the considerable length of the restraining anchor bolt passing through the absorber. Projections cantilevered from the body of the foundation above the springs are attached by girders to the lower end of the restraining anchor bolt. The absorbers are placed on the upper edges of the foundation mass below the springs. This mass is designed in the shape of a box in which the mass located above the springs is inserted.

The procedure for mounting and regulating absorbers of the suspended type does not differ much from that used for the supported type.

Generally the dynamic computations of a foundation with absorbers are reduced to an investigation of vibrations of a system having up to

12 degrees of freedom. However, since absorbers are mostly used for the vibroisolation of engines with vertical cylinders, the analysis may in many cases be limited to an investigation of vertical vibrations only. Then the problem of foundation vibrations is reduced to an investigation of a system with 2 degrees of freedom.

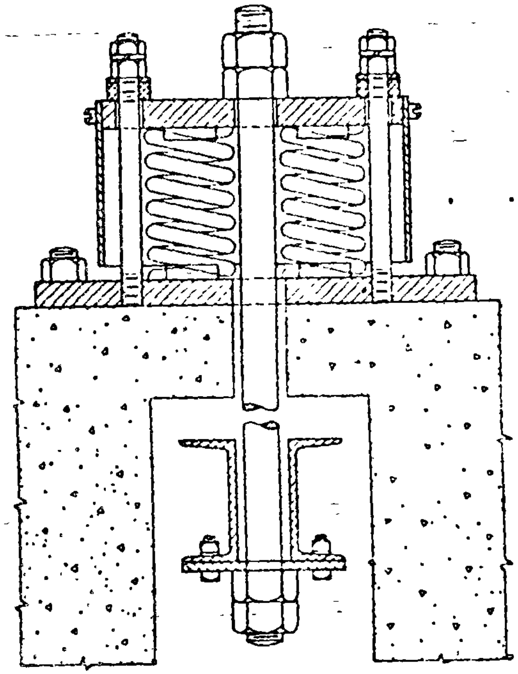


FIG. IV-18. Suspended-type absorber.

Let us assume that the masses of the foundation above and beneath the springs are concentrated in their centers of gravity, located on the same vertical line. Let us further assume that an exciting force $P(t) \sin \omega t$ acts on the mass m_2 above the springs (Fig. IV-19). The differential equations of forced vertical vibrations of the system under consideration will be as follows:

$$m_1\ddot{z}_1 + c_1 z_1 - c_2(z_2 - z_1) = 0$$
$$m_2\ddot{z}_2 + c_2(z_2 - z_1) = P(t) \sin \omega t \qquad \text{(IV-6-1)}$$

where $P(t)$ = magnitude of exciting force

ω = frequency of exciting force

z_1, z_2 = vertical displacements of centers of gravity of masses below and above springs m_1, m_2

c_1 = coefficient of elastic rigidity of base under foundation beneath springs

$$c_1 = c_u A \qquad \text{(IV-6-2)}$$

c_u = coefficient of elastic uniform compression of base
A = area of foundation beneath springs, in contact with soil
c_2 = total coefficient of rigidity of all springs

$$c_2 = \frac{n_1 n_2 d^4}{8nD^3} G \qquad \text{(IV-6-3)}$$

n_1 = number of springs in each absorber
n_2 = number of absorbers
n = number of coils in each spring
d = diameter of spring
D = diameter of coil
G = modulus of elasticity of material of springs

Each spring should be designed so that stresses developed therein under the action of static and dynamic loads will not exceed a permissible value.

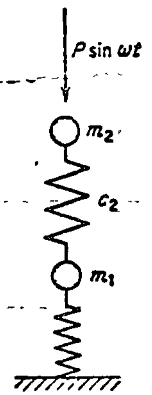


FIG. IV-19. Derivation of Eq. (IV-6-1) concerning the vibration of foundation masses above and below absorber springs.

Limiting our discussion to forced vibrations only, we take the solution of the system (IV-6-1) in the form

$$z_1 = A_1 \sin \omega t \qquad z_2 = A_2 \sin \omega t$$

where the amplitudes A_1 and A_2 of forced vibrations of the foundation beneath and above the springs are

$$A_1 = \frac{f_{nl}^2}{m_1 \Delta(\omega^2)} P(t) \qquad \text{(IV-6-4)}$$

$$A_2 = \frac{(1 + \mu) f_{nlz}^2 + \mu f_{nl}^2 - \omega^2}{m_2 \Delta(\omega^2)} P(t) \qquad \text{(IV-6-5)}$$

where f_{nl} is the limiting frequency of natural vibrations of the foundation above the springs, computed on the basis of the assumption that the foundation beneath the spring is infinitely large; the value of f_{nl} is determined by the equation

$$f_{nl}^2 = \frac{c_2}{m_2} \qquad \text{(IV-6-6)}$$

f_{nlz} is the limiting frequency of natural vibrations of the complete system when it is assumed that no absorbers are used:

$$f_{nlz}^2 = \frac{c_1}{m_1 + m_2} \qquad \text{(IV-6-7)}$$

Finally,

$$\mu = \frac{m_2}{m_1}$$

The coefficient $\Delta(\omega^2)$ is determined by the expression

$$\Delta(\omega^2) = \omega^4 - (1 + \mu)(f_{nl}^2 + f_{nlz}^2)\omega^2 + (1 + \mu) f_{nl}^2 f_{nlz}^2 \qquad \text{(IV-6-8)}$$

Returning to Eq. (IV-6-4), let us investigate the dependence of the amplitude of forced vibrations of the foundation beneath the springs on f_{nl}, which is proportional to the rigidity c_2 of the absorbers. The value of the exciting force is proportional to the square of the frequency of engine rotation; therefore

$$P(t) = \gamma\omega^2$$

where γ is a coefficient which depends on parameters of the engine.

Substituting this expression for P into the right-hand part of Eq. (IV-6-4) and dividing the numerator and denominator by ω^2, we obtain the following expression for the vibration amplitude of the foundation beneath the springs:

$$A_1 = \frac{\gamma}{m_1} \frac{\xi_l^2}{1 - (1 + \mu)(\xi_l^2 + \xi_{lz}^2 - \xi_l^2 \xi_{lz}^2)} \qquad \text{(IV-6-9)}$$

where

$$\xi_l = \frac{f_{nl}}{\omega} \qquad \xi_{lz} = \frac{f_{nlz}}{\omega}$$

If no absorbers are used and the upper and lower parts of the foundation are rigidly connected, then according to Eq. (III-1-13) the amplitude of vertical forced vibrations will equal

$$A_z = \frac{\gamma}{m_1} \frac{1}{(1 + \mu)(\xi_{lz}^2 - 1)} \qquad \text{(IV-6-10)}$$

The degree of absorption of vibrations will be

$$\eta = \frac{A_z}{A_1} = \frac{1 - (1 + \mu)(\xi_l^2 + \xi_{lz}^2 - \xi_l^2 \xi_{lz}^2)}{(1 + \mu)(\xi_{lz}^2 - 1)\xi_l^2} \qquad \text{(IV-6-11)}$$

Let us investigate the effect of changes in the value of ξ_l on the value of η. We assume that f_{nl} is very small in comparison with ω; i.e., the value ξ_l is also very small. It follows directly from Eq. (IV-6-11) that

$$\text{If } \xi_l \to 0, \text{ then } \eta \to \infty$$

hence it follows that if the natural frequency of foundation vibrations above the springs is small in comparison with the frequency of engine rotation, then the amplitude A_1 of the foundation with absorbers is small in comparison with the amplitude of vibrations of the same foundation without absorbers.

Suppose $\xi_l \to \infty$; this corresponds to a very large value of f_{nl}. One can see from Eq. (IV-6-11) that in this case $\eta \to 1$; i.e., absorbers will not have any influence on the amplitudes of foundation vibrations.

Figure IV-20 gives a graph of changes in η depending on changes in ξ_l. It is evident that the absorbers cause a decrease in the amplitudes of vibrations only when $\eta > 1$. It is seen from the graph that the zone of usefulness of absorbers is limited to a very narrow range of values of ξ_l lying between

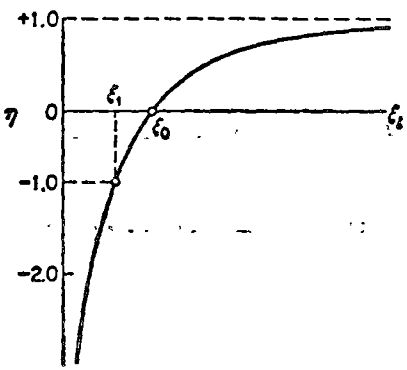


FIG. IV-20 Diagram illustrating Eq. (IV-6-12) and the limits of usefulness of vibration absorbers.

$$\xi_l = 0$$

and

$$\xi_l = \xi_1 = \sqrt{\frac{(1 + \mu)\xi_{l_2}^2 - 1}{2(1 + \mu)(\xi_{l_2}^2 - 1)}} \qquad \text{(IV-6-12)}$$

When $\xi_l > \xi_1$, the following inequality exists:

$$|\eta| < 1$$

Consequently, the use of absorbers does not bring any advantage, but on the contrary is harmful, because when $|\eta| < 1$ the amplitude of vibrations of the foundation with absorbers is larger than that of the same foundation without absorbers.

The foregoing theory of vibrations of foundations with absorbers leads to the conclusion that in order for absorbers to have a favorable effect on the amplitudes of foundation vibrations, the following condition is necessary: the frequency of natural vibrations of the mass above the springs should be as small as possible in comparison with the frequency of engine rotation. A decrease in the frequency of natural vibrations of the foundation above the springs may be achieved by the use of absorbers of a suitable stiffness and by an increase in the mass above the springs. For high-frequency engines the required relationship between ω and f_{nl} can be easily achieved without a considerable increase of the weight of the foundation above the springs. For low-frequency engines the relationship is usually difficult to achieve by just decreasing the rigidity of the absorber because, due to strength requirements, this decrease cannot extend below a certain limit determined by the strength of the springs. In such cases a decrease in f_{nl} is achieved by providing a massive foundation above the springs.

If the degree of absorption of foundation vibrations η is specified, then from Eq. (IV-6-11) we obtain for ξ_l:

$$\xi_l^2 = \frac{1 - (1 + \mu)\xi_{l_2}^2}{(1 + \mu)(\eta - 1)(\xi_{l_2}^2 - 1)} \qquad \text{(IV-6-13)}$$

Example. Design computations for a foundation with absorbers under a vertical compressor

1. DATA. A foundation for a 120-kw vertical compressor is to be designed. Foundation vibrations are objectionable, since a precision apparatus is adjacent to the foundation. The base of the foundation is formed by a soil characterized by a coefficient of elastic uniform compression c_u of 2×10^3 tons/m³.

In order to avoid harmful influences of the foundation on the apparatus, the amplitude of foundation vibrations should not exceed 0.03 mm.

The main exciting force imposed by the compressor is the vertical exciting force $P_z = 2.6$ tons; the compressor speed is 480 rpm.

2. COMPUTATIONS. In order to have an amplitude of foundation vibrations under the compressor smaller than 0.03 mm, the foundation should be very heavy and should have a large area in contact with the soil.

For example, assuming that the ratio between the frequency of natural vertical vibrations of the foundation and the engine frequency equals 2, for $A_z = 0.03 \times 10^{-3}$ m and $P_z = 2.6$ tons, we obtain from Eq. (III-1-13) the following foundation weight:

$$W = \frac{2.6 \times 9.81}{0.03 \times 10^{-3} \times 3 \times 2.5 \times 10^3} = 115 \text{ tons}$$

Then the foundation area in contact with soil should equal

$$A = \frac{f_{nz}^2 W}{c_u g} = \frac{4 \times 2.5 \times 10^3 \times 115}{2 \times 10^3 \times 9.81} = 57.5 \text{ m}^2$$

For such a low-power engine as the compressor under consideration it would be unreasonable to construct a foundation with weight and dimensions as large as those obtained above. In order to meet the requirements in regard to the amplitude of foundation vibrations, it is better to use spring absorbers. The selected dimensions of the foundation with absorbers are shown in Fig. IV-21.

The data for the computations are as follows: the foundation area in contact with soil $A = 12.5$ m²; the weight of the foundation beneath the springs is 21.0 tons; the weight of the foundation above the springs (together with the engine) is 35.0 tons. The coefficient of rigidity of the base is

$$c_1 = c_u A = 2 \times 10^3 \times 12.5 = 25.0 \times 10^3 \text{ tons/m}$$

The mass of the foundation beneath the springs is

$$m_1 = 21.0/9.81 = 2.15 \text{ tons} \times \text{sec}^2/\text{m}$$

The mass of the foundation above the springs is

$$m_2 = 35.0/9.81 = 3.56 \text{ tons} \times \text{sec}^2/\text{m}$$

The limit natural frequency of vertical vibrations of the whole system (assuming that no absorbers are used) equals

$$f_{na}^2 = \frac{c_1}{m_1 + m_2} = \frac{25.0 \times 10^3}{2.15 + 3.56} = 4.38 \times 10^3 \text{ sec}^{-2}$$

The coefficient ξ_{la} is computed to be

$$\xi_{la} = \frac{f_{na}^2}{2} = \frac{4.38 \times 10^3}{2.5 \times 10^3} = 1.75$$

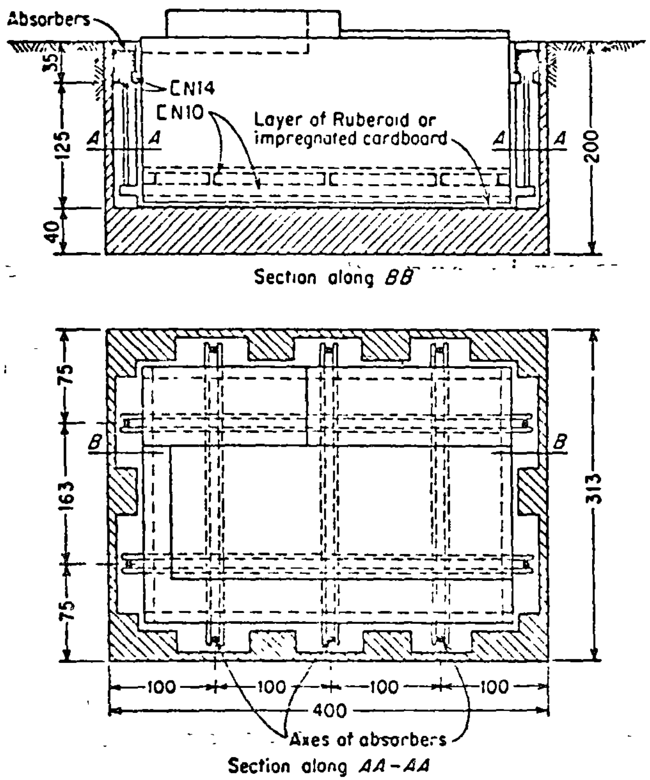


Fig. IV-21. Example of design computations for foundation with absorbers.

The ratio between values of the masses is

$$\mu = \frac{m_2}{m_1} = \frac{3.56}{2.15} = 1.65$$

If no absorbers were present, then for the selected dimensions of the foundation the amplitude of vertical vibrations would equal

$$A_s = \frac{2.6}{(2.15 + 3.56)(4.38 - 2.5)10^3} = 0.25 \times 10^{-3} = 0.25 \text{ mm}$$

In order that the permissible amplitude of vibrations of the foundation with absorbers is not exceeded, the degree of absorption of vibrations should equal

$$\eta = \frac{A_s}{A_1} = \frac{0.25}{0.03} = 8.4$$

Let us assume that the design value of η equals -10. From Eq. (IV-6-13) we determine the required value of the coefficient ξ_l:

$$\xi_l^2 = \frac{1 - (1 + 1.65)1.75}{(1 + 1.65)(-10 - 1)(1.75 - 1)} = 0.165$$

We determine the required value of the limiting frequency f_{nl_2} of natural vertical vibrations of the mass of the foundation above the springs:

$$f_{nl_2}^2 = \xi_l^2\omega^2 = 0.165 \times 2.5 \times 10^3 = 414 \text{ sec}^{-2}$$

The required total rigidity of all the absorbers will be

$$c_2 = f_{nl_2}^2 m_2 = 414 \times 3.56 = 1{,}480 \text{ tons/m}$$

If c_{sp} is the rigidity of one spring, then

$$c_{sp} = \frac{c_2}{n_1 n_2}$$

where n_1 = number of springs in each absorber
n_2 = number of absorbers

Assume $n_1 = 2$ and $n_2 = 8$; then the required rigidity of one spring will be

$$c_{sp} = \frac{1{,}480}{2 \times 8} = 92.0 \text{ tons/m}$$

On the other hand, using Eq. (IV-6-3), we obtain

$$c_{sp} = \frac{G}{8}\frac{d^4}{D^3}\frac{1}{n} \qquad \text{(IV-6-14)}$$

where G is the modulus of elasticity in shear of the spring material; its value may be assumed to be 7.5×10^6 tons/m².

We assume there are five coils in a spring. Substituting values of n and G into formula (IV-6-14) we obtain

$$92.0 = \frac{7.5 \times 10^6}{8}\frac{d^4}{D^3}\frac{1}{5} = 1.88 \times 10^5 \frac{d^4}{D^3}$$

Hence,

$$\frac{d^4}{D^3} = \frac{92.0}{1.88 \times 10^5} = 4.9 \times 10^{-4}$$

Let us assume the diameter of the spring $d = 2.5 \times 10^{-2}$ m; then

$$D^3 = \frac{d^4}{4.9 \times 10^{-4}} = \frac{39.0 \times 10^{-8}}{4.9 \times 10^{-4}} = 8.0 \times 10^{-4} \text{ m}^3$$
$$D = 9.3 \text{ cm}$$

3. Stress Analysis of the Spring. The permissible load on the spring equals

$$P_p = \frac{\pi d^3 R_p}{8D}. \qquad \text{(IV-6-15)}$$

where R_p is the permissible torsional stress for the spring material; we assume its value is 40×10^3 tons/m².

Assuming in accordance with the foregoing computations that $d = 2.5 \times 10^{-2}$ m and $D = 9.3 \times 10^{-2}$ m, we obtain for the permissible load on the spring

$$P_p = \frac{3.14 \times 15.6 \times 10^{-6} \times 40 \times 10^3}{8 \times 9.3 \times 10^{-2}} = 2.64 \text{ tons}$$

In order to find the actual load on each spring, it is necessary to determine the amplitude of forced vibrations of the foundation above the spring. From Eq. (IV-6-8), we determine the value of $\Delta(\omega^2)$:

$$\Delta(\omega^2) = 6.25 \times 10^6 - (1 + 1.65)(0.414 \times 10^3 + 4.38 \times 10^3) \times 2.5 \times 10^3 + (1 + 1.65) \times 0.414 \times 10^3 \times 4.38 \times 10^3 = 21.0 \times 10^6$$

According to (IV-6-5),

$$A_2 = \frac{(1 + 1.65) \times 4.38 \times 10^3 + 1.65 \times 0.414 \times 10^3 - 2.5 \times 10^3}{3.56 \times 21.0 \times 10^6} 2.64 = 0.34 \times 10^{-3} \text{ m}$$

The dynamic force imposed on the springs as a result of vibrations is

$$W_r = 0.34 \times 10^{-3} \times 1.48 \times 10^3 \times 3.56 \doteq 0.5 \text{ tons}$$

Thus, the actual load on each spring equals

$$P_{act} = \frac{0.5 + 35.0}{16} = 2.22 \text{ tons}$$

which is smaller than the permissible load.

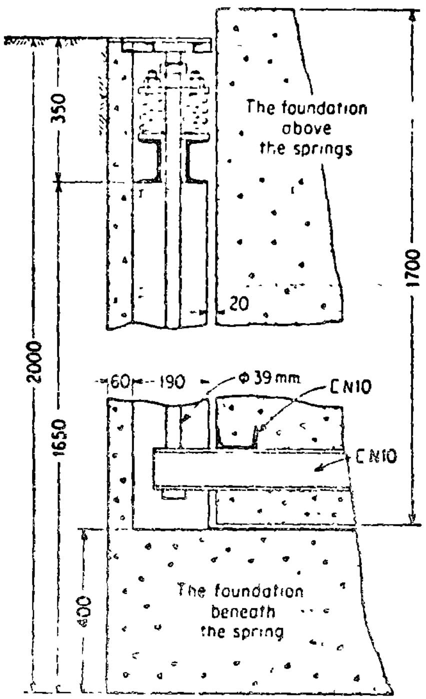


FIG. IV-22. Detail of absorbers with suspension system.

A schematic diagram of the main part of the arrangement of the foundation with suspended absorbers is shown in Fig. IV-22.

The construction of a foundation with absorbers proceeds analogously to the procedure described in Art. V-7.

V

FOUNDATIONS FOR MACHINES PRODUCING IMPACT LOADS

V-1. General Directives for the Design of Foundations for Forge Hammers

a. Classification of Forge Hammers. Forge hammers are divided into two groups: drop hammers for die stamping and forge hammers proper.

The side frame of a drop hammer is mounted on an anvil (Fig. V-1), thus giving rigidity to the system. The side frame, together with guides for the ram, contributes to the precision of blows required in forging. This peculiarity in the design of drop hammers predetermines to a certain degree the design of their foundations, since the foundation block under the anvil serves as a support for the whole hammer.

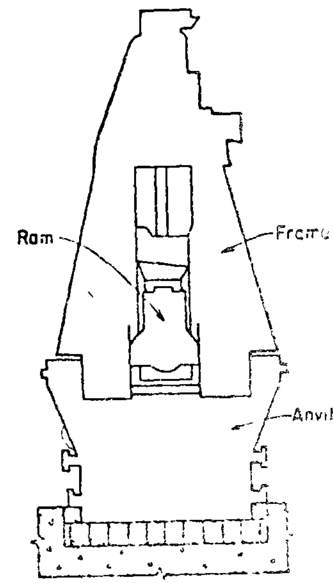


FIG. V-1. Drop hammer with frame mounted on anvil.

Free forging operations are usually performed by forge hammers proper. The anvil and the side frame, as a rule, are mounted separately. Forge hammers are built as single-support frames (Fig. V-2) and as double-support frames (Fig. V-3). The latter can be of the arch or bridge types. Pneumatic hammers with air compressors are single-frame hammers.

b. Design Data for a Hammer Foundation. The following technological data are required for the design of a hammer foundation:

1. The nominal weight of dropping parts, which usually characterizes the power of the hammer. In drop hammers the real weight of dropping parts, in addition to the weight of the ram, piston, and rod, includes also the upper half of the die. Therefore in these hammers the real weight of dropping parts is greater than the nominal weight or that shown in catalogues.

The design of a hammer foundation is made on the basis of the real weight of dropping parts. The total average height of the upper and

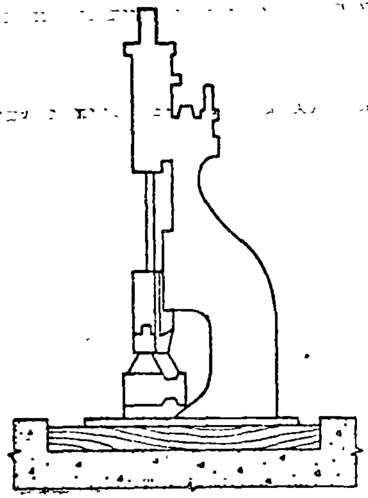


Fig. V-2. Hammer with single support frame.

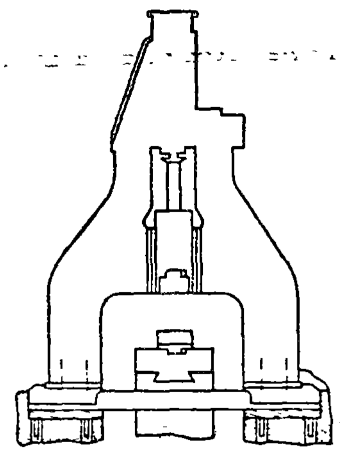


Fig. V-3. Hammer with double support frame.

lower halves of the die lies within the range 250 to 500 mm. It should be noted that some dies for long units (axes, shafts, etc.) are provided with heavy upper parts, whose weight reaches 100 per cent of the nominal weight of dropping parts. Such cases should be mentioned in the technological assignment.

2. The weight of the anvil and the dimensions of its base area.

3. The weight of the hammer, including side frames, ram cylinders with the anchor plate, etc., but excluding the anvil weight. When no data are available on the weight of the anvil and frames, it is permissible to assume the anvil weight to be 15 to 20 times the weight of the ram; the total weight of the hammer and anvil is taken to be 25 to 30 times the weight of the ram.

4. The maximum height of the ram drop (or the maximum piston stroke).

5. The upper piston area.

6. The average working pressure on the piston.

For the design and analysis of the foundation, either the machine assembly drawing or the following data should also be made available:

7. Dimensions in plan, the thickness and elevation of the top of the pad under the frame and anvil of the forge hammer.

8. The position, diameter, and length of anchor bolts.

9. The elevation of the anvil base with respect to the floor level of the shop.

10. Dimensions in plan and the thickness of the wooden pad under the anvil.

11. The location of the hammer in the shop with respect to adjacent foundations under engines, machinery, and supporting structures of the building; the dimensions, elevations, and depths of these foundations.

c. Material of the Foundation and Pad under the Anvil. Foundations under hammers with a weight of dropping parts in excess of 1 ton are made of concrete type 150,† with coarse aggregate of hard rocks with a compressive strength not less than 250 kg/cm². Normal portland cement, of a type not below 300, is used for concrete. The latter is reinforced according to design data or according to instructions given on the job.

The pad under the anvil is usually made of oak. Experience in the operation of hammers under war conditions showed that for hammers with a weight of dropping parts up to 2 tons, pine and larch timber may be used as material for under-anvil pads. Timbers of the best quality, having a moisture content below 15 to 18 per cent, should be used.

d. Directives for Design. Until recently there was a tendency to design foundations under hammers as massive blocks embedded to a considerable depth in the soil.[44] The purpose was to provide such dimensions of the foundation that its static elastic settlement would be larger than the amplitude of its vertical vibrations. Since design values of amplitudes of foundation vibrations under hammers were selected within the range 2.0 to 2.5 mm, the height of the foundation had to be increased considerably to obtain the desired static settlement. Figure V-1*a* shows a typical design of a foundation under a hammer, as used until recently. Here for each 1 ton of dropping parts weight, there correspond 80 to 100 and often 120 tons of foundation weight.

The discussion which follows will show that the use of such heavy foundations is not necessary, especially since they involve a considerable increase in the cost of construction.

Foundations under hammers should be designed as blocks or slabs loaded from above by backfill. Figure V-4*b* is a typical design of such a block foundation. The ratio between the weight of the foundation and that of the dropping parts is about 40.

† See footnote, Art. IV-1-c, p. 132.

In block-type foundations the thickness of the part below the anvil should be selected as follows: for a ram weight up to 0.75 tons, the thickness should be not less than 0.75 m; for a weight of 0.75 to 2.5 tons, the thickness should be 1.5 m; for a weight of 2.5 tons and more, the thickness must be 1.25 to 2.50 m, depending on the power of the hammer.

Previously, foundations under forge hammers were built under the anvils separated from the footings under the frames. This decreased the stresses in the hammer frames during eccentric forging. However, the separation of the foundation elements results in their tilting with respect to each other and in considerable nonuniform settlement of the foundation under the anvil. In recent years foundations under forge hammers have been designed by the method shown in Fig. V-4c; i.e., the footings

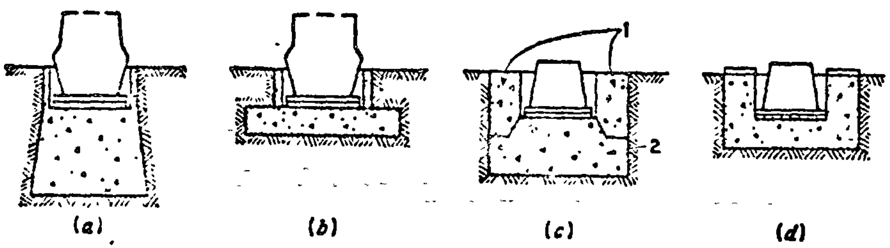


FIG. V-4. Types of hammer foundations: (a) deep block foundation; (b) slab block foundation weighted by backfill; (c) anvil block foundation 2 supports frame 1; (d) single block foundation for anvil and frame.

under the frame were not placed directly on the soil, but on the block under the anvil, and boards 2 to 3 cm thick (or several layers of Ruberoid) were placed between these two footings. Thus tilting of the anvil with respect to the frame was prevented. However, due to insufficient elasticity of the pad, this foundation design was not very effective in decreasing the stresses which developed as a result of nonuniform forging. These stresses may be decreased much more efficiently by means of spring washers in foundation bolts and by oak timbers installed under the anvil and under the frame of the hammer. In such cases the foundation under the forge hammer should be designed as a single block (Fig. V-4d).

Reinforcement is placed as directed on the job. The reinforcement used for the foundation under the anvil consists of 2 to 4 horizontal steel grillages formed by 8- to 12-mm bars spaced at 10 to 20 cm; the upper grillage is placed at a distance of 2 to 3 cm from the foundation surface.

Near the foundation surface in contact with soil, the reinforcement consists of 1 or 2 horizontal grillages formed by 12- to 20-mm bars and spaced 15 to 30 cm apart. Distances between the grillages are 10 to 15 cm in the part of the foundation under the anvil and 15 to 30 cm near the foundation contact surface.

The number of grillages is determined by the power of the hammer. It should be kept in mind that double-acting hammers belong to the group of heaviest hammers with respect to their impact effect on the foundation.

Pads under the anvil are made of square timbers from 10 by 10 cm to 20 by 20 cm in cross section. Timbers are laid flat in one or several rows, one over the other. Each row is braced by transverse bolts every 0.5 to 1.0 m and forms a separate mat.

If several rows of timbers are employed, then in order to decrease wear and tear and make the pad more rigid, the timbers are placed in the form of grillages. The upper row of timbers is laid along the short side of the

TABLE V-1. THICKNESS OF TIMBER PADS UNDER THE ANVIL

No.	Type of hammer	Thickness of pad, m, if weight of dropping parts of hammer is:		
		Up to 1 ton	1–3 tons	Over 3 tons
1	Double-action drop hammer	Up to 0.20	0.20–0.60	0.60–1.20
2	Single-action drop hammer	Up to 0.10	0.10–0.30	0.30–0.90
3	Forge hammer	Up to 0.20	0.20–0.60	0.60–1.00

anvil base. The mats must be strictly horizontal, smoothly planed, and easily fitted into the excavation. They should be checked by means of a water level. To prevent decay resulting from moisture, it is advisable to impregnate timbers with wood-preserving solutions.

The space between the pad and the excavation walls may be filled with petroleum asphalt. In order to prevent a horizontal displacement of the anvil along the pad, four timbers are placed around it near the base.

The pad thickness is shown on the assembly drawings of the hammer or indicated in the technical assignment. Tentative values of pad thickness under the anvil are given in Table V-1.

The thickness of the pad should be selected so that the stresses therein do not exceed permissible values, which are as follows:

Oak: 300 to 350 kg/cm²
Pine: 200 to 250 kg/cm²
Larch: 150 to 200 kg/cm²

e. Remarks on Construction Procedures. Concrete for the footings should be placed using vibrators. In the presence of ground water containing chemicals which may produce deterioration of concrete, pozzolan cement should be used or special measures should be provided to protect concrete from the action of water, the velocity of wa...

and possible fluctuations of ground-water level should be taken into account.

In the process of foundation construction, special care should be taken to provide accurate location of holes for anchor bolts (if the latter are foreseen by the design) and the excavation for the anvil or frame.

The lower part of the excavation for the anvil should be strictly horizontal; no additional corrective pouring of concrete is permitted. If supplementary cement grout has to be placed under the frame of the forge hammer, then the foundation surface in contact with the cement grout should be roughened, cleaned, and washed. The underlying soil should be compacted by tamping in of broken stone. In moist soils a working mat of concrete type 50† is placed under the foundation.

Concrete should be placed in horizontal layers and, as a rule, without interruption in the work. In case of an emergency interruption, the following measures should be taken to secure the monolithic character of the foundation:

1. Dowels of 12 to 16 mm diameter should be embedded on both sides of the joint to a depth of 30 cm at a distance of 60 cm from each other.

2. Prior to placing a new layer of concrete, the previously laid surface should be roughened, thoroughly cleaned, washed by a jet of water, and covered by a layer of a rich 1:2 cement grout, 20 mm thick. The placement of concrete should be started not later than 2 hr after this mixture is laid on the surface.

The anvil may be mounted on the foundation only after hardening of the concrete, i.e., not less than a week after its placement. The foundation may be put in operation as soon as the concrete attains the design strength value.

Vibrations of the forge-hammer anvil may result in some soil falling into the space between the anvil base and the upper row of the timber pad. This may lead to tilting of the anvil or damage to the pad. To avoid this, after the anvil is built a protective wall is usually installed around it, extending from the top of the foundation to floor level. Such a protective box permits an easy and rapid inspection of the anvil and pad; it also simplifies the mounting and dismounting of the anvil.

V-2. Initial Conditions of Foundation Motion under Impact Action

a. The Velocity of Dropping Parts at the Beginning of Impact. Large hammers may be divided into two groups: those with an unrestricted drop of the ram, and those with a restricted ram movement.

The first group includes hammers with frictional hoisting of the ram and hammers in which the ram, rigidly tied to the piston, is lifted by

† See footnote to Art. IV-1-c, p. 132.

steam pressure from underneath. In frictional hammers, the ram is connected to a plate which moves between two frictional disks pressed against this plate. When the ram is lifted to the height desired, one of the disks is moved away from the plate and the ram falls, moving along its guides.

In single-acting hammers, the ram, which is rigidly tied to the piston by means of a rod, is lifted by the pressure of steam released through a valve located under the piston and opened when the latter is in its extreme low position. After the piston is raised to the height desired, the access of steam into the cylinder under the piston is stopped, the valve opens, and the steam or compressed air escapes. The piston, together with the ram, drops at increasing speed.

After the access of steam is discontinued and the exhaust valve opens, steam cannot escape at once from the space in the cylinder under the piston. Therefore a counterpressure against the ram drop is created, resulting in a loss both in the ram's velocity and in the kinetic energy of its drop.

The velocity v of the ram drop under the condition of unrestricted motion equals

$$v = \eta\sqrt{2gh} \qquad \text{(V-2-1)}$$

where g = acceleration of gravity

h = height of ram drop

η = coefficient which takes into account counterpressure and frictional forces

The numerical value of η depends on the design of the hammer, its working order, the regulation of valves, etc.

Modern forging practice mostly employs the large double-acting hammers. In these hammers, steam or compressed air acts on the ram not only while it is being lifted, but also during its drop; therefore the velocity and kinetic energy are considerably larger at the moment of impact of the ram against the workpiece.

Changes in steam pressure during the drop of the ram, both under the piston (counterpressure) and above the piston (active pressure), depend on many varying factors such as the proper operation of the valves, the tightness of the piston in the cylinder, and the working order of the stuffing box. It is impossible to take into account all these factors with sufficient accuracy. Therefore in computations of ram velocity, one usually works with the average pressure of steam and air in the feed pipe. Then the velocity of the forced motion of the ram under the action of its own weight and the steady pressure will equal

$$v = \eta\sqrt{\frac{2g(\text{[illegible]})h}{\text{[illegible]}}} \qquad \text{(V-2-2)}$$

where A = piston area

p = total pressure on piston

W = total weight of dropping parts

η = correction coefficient

b. Experimental Determination of the Correction Coefficient η. The values of the correction coefficient in Eqs. (V-2-1) and (V-2-2) may be determined only empirically by comparing the velocities obtained from the equations with the corresponding values obtained experimentally.

TABLE V-2. RESULTS OF HAMMER EFFICIENCY MEASUREMENTS

Type of hammer	Nominal weight of dropping parts, tons	Velocity at beginning of impact, m/sec		Ratio η between measured and computed velocities
		Measured	Computed from Eq. (V-2-2) or (V-2-1) for $\eta = 1$	
Double-acting hammer	5 4	6 2	9 0	0 69
	3.6	6 0	8.4	0 71
	2 25	5 4	8.6	0 63
	1.8	4 5	8 1	0 56
	1 125	6.2	8.6	0 72
	1 0	6 8	8.5	0.80
	1.0	5.8	9.8	0.59
	0.635	5 5	9.0	0.61
Hammer with unrestricted action	0 54	3 3	3 56	0 96
	1 125	3 5	3 93	0.89

The author performed such measurements for ten drop hammers of different powers and makes. The measurements were made under working conditions in shops, without any special adjustment. Therefore the results of these measurements may be considered to be characteristic for working conditions.

The results of these measurements are presented in Table V-2, which also gives velocities computed from Eqs. (V-2-1) and (V-2-2) for $\eta = 1$.

This table shows that the measured velocity of dropping parts at the moment of impact is much lower in double-acting hammers than the values computed from Eq. (V-2-2). For these hammers, the ratio between the values of measured and computed velocities lies within the range 0 45 to 0 80; the average value of η in Eq. (V-2-2) may be taken to equal 0 65.

In addition, it follows from Table V-2 that the absolute velocity does not depend much on the power of the hammer. This is explained in part by the fact that usually the height of drop of the ram and the steam or air pressure vary for different hammers only within comparatively narrow ranges. Therefore in many cases the design velocity value may be taken to equal approximately the average value of velocities measured in hammers with different powers. This value for double-acting die-stamping hammers equals 6.0 to 6.5 m/sec.

In hammers with unrestricted drop, especially frictional hammers, no counterpressure of steam is encountered, and a decrease in the velocity of dropping parts is mostly caused by friction in the guides. When a hammer is properly adjusted, the effect of friction will be negligible; therefore the correction coefficient in Eq. (V-2-1) will be close to unity. This conclusion is confirmed by data for hammers with unrestricted drop, presented in Table V-2.

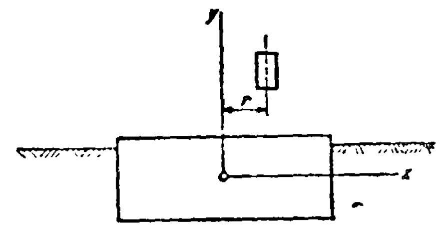


FIG. V-5. Derivation of Eq. (V-2-1).

c. Initial Velocities of Foundation Motion. Let us investigate the values of velocity characteristic for the foundation at the end of a vertical eccentric impact. Such an impact occurs in drop hammers when edge grooves are stamped.

We assume that no pad is present under the anvil and that the anvil and foundation form one single body whose elasticity may be neglected in comparison with the elasticity of the soil. Let us also assume that the foundation can be represented by the body shown in Fig V-5, that the eccentric impact produced by the falling mass occurs in a plane which we shall consider to be one of the principal planes of the foundation, and that the center of gravity of the foundation and the centroid of its contact area with the soil lie on the same vertical line. Due to the impact, the foundation undergoes vibrations which occur in the aforementioned principal plane.

If the ram and foundation are considered as one closed system, it may be assumed that linear momentum is conserved during the impact. The foundation is motionless before the impact: at that time linear momentum equals the momentum of the ram, i.e., m_0v, where v is the velocity of the falling ram of mass m_0 at the moment it touches the foundation (the beginning of impact).

After impact, i.e., during the period following the instant when the ram detaches itself from the foundation, the momentum of the ram and foundation is

$$m_. \quad - m v_0$$

where v_1 = velocity at which ram rebounds from foundation
m = foundation mass
v_0 = initial velocity of forward motion of center of mass of foundation

The momentum of the system before the impact equals the momentum after the impact; therefore

$$m_0v = m_0v_1 + mv_0 \qquad \text{(V-2-3)}$$

In addition to progressive downward motion under the action of an eccentric impact, the foundation undergoes a rotational movement around the axis passing through its center of gravity, perpendicular to the plane in which impact occurs. The moment of momentum will be

$$m_0vr = m_0v_1r + I\varphi_0 \qquad \text{(V-2-4)}$$

where I = moment of inertia of mass of foundation and hammer in regard to axis of rotation
r = eccentricity of impact
φ_0 = initial velocity of rotation of foundation

Equations (V-2-3) and (V-2-4) include three unknown values. In order to derive a third equation, let us use Newton's hypothesis concerning the restitution of impact. According to this hypothesis, if there occurs an impact between two bodies moving in relation to each other, the relative velocity after the impact is proportional to the relative velocity before the impact. The ratio between these two depends only on the material of the bodies which underwent the impact. The foundation was motionless before impact; therefore the relative velocity of the ram equals v. After impact, the absolute velocity of ram motion equals v_1, but the point of the foundation which was subjected to impact acquired a velocity whose vertical component equals $v_0 + r\varphi_0$; it follows that the relative velocity of the ram after the impact equals $v_0 + r\varphi_0 - v_1$. According to Newton's hypothesis,

$$e = \frac{v_0 + r\varphi_0 - v_1}{v} \qquad \text{(V-2-5)}$$

where e is the coefficient of restitution.

From Eqs. (V-2-3) to (V-2-5) we obtain expressions for initial velocities of the foundation motion:

$$v_0 = (1 + e)\frac{\rho}{(1 + \mu)(r^2 + \rho^2) - r^2}v \qquad \text{(V-2-6)}$$

$$\varphi_0 = \frac{\mu r(1 + e)}{(1 + \mu)(r^2 + \rho^2) - r^2}v \qquad \text{(V-2-7)}$$

where

$$\rho^2 = \frac{I}{m_0} \qquad \mu = \frac{m}{m_0}$$

If the impact was at the center of the foundation, then $r = 0$ and

$$v_0 = \frac{1 + e}{1 + \mu}v \qquad \varphi_0 = 0 \qquad \text{(V-2-8)}$$

When the elasticity of the pad cannot be neglected, it should be considered that the initial velocity of motion is acquired only by the anvil (for forge hammers) or by the anvil and frame (for drop hammers). Then Eqs. (V-2-6) and (V-2-7) remain the same, but the symbols m and I denote the mass and the moment of inertia either of the anvil or of both the anvil and the frame, without taking into account the foundation mass.

d. Coefficient of Restitution e. It follows from the foregoing formulas that the initial velocities of motion of the foundation or anvil depend considerably on the coefficient of restitution e. If the impact was perfectly elastic, then $v = v_1$, and consequently $e = 1$. For the impact of a rigid body against a plastic one, $v_1 = 0$, and consequently $e = 0$. For real bodies, the numerical values of e lie within the range $0 < e < 1$.

In forge hammers, e depends on many factors, the most important of which are: the temperature of a forged piece, the dimensions and forms of grooves (in stamping hammers), and the elastic properties of materials of the ram, head, and anvil.

Since the design values of the amplitude of hammer foundation vibrations depend on the selected values of the coefficient of restitution e, the designer of a foundation naturally has a practical interest in knowing its real values. However, the answer to this question is poorly elucidated in special publications on heat treatment of metals. In this connection the author carried out special measurements to determine numerical values of the coefficient of restitution of hammers.[11] Measurements were performed under working conditions with both single- and double-acting hammers. The computation of e was made from measured values of the heights of fall and rebound of the ram after impact or from the interval of time between two rebounds of the ram.

The results showed that the values of the coefficient of restitution depend to a great extent on the state of a forged piece. Figure V-6 presents a graph of changes in e as a function of the number of blows on a forged piece under a hammer having a weight of dropping parts equaling 5.3 tons. Analogous graphs were obtained for other hammers. It follows from these plots that during the first blows against the forged piece, when its temperature is high and it is in a plastic state, the coefficient of impact velocity restitution is very small, equaling approximately 0.10. As the number of blows increases, the temperature of the forged piece decreases, the impact rigidity increases, and consequently the value of e increases. For the last blows, when a comparatively cooler piece is being forged, the coefficient of restitution approaches 0.5. Measurements of this coefficient during idle blows and under conditions of cold forging showed that its value does not exceed 0.5.

Since computations of hammer foundation vibrations should be performed for the most unfavorable conditions of operation, the design value of the coefficient of restitution for hammers forging steel parts should be taken as 0.5.

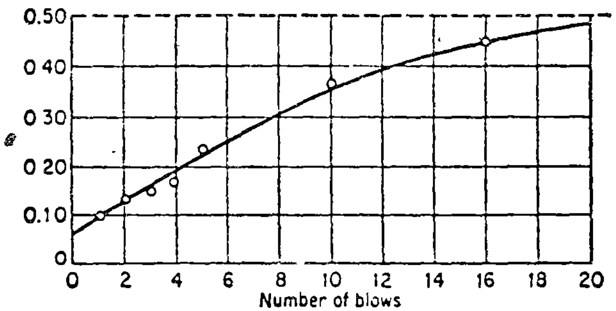


FIG. V-6. Variation of the coefficient of impact velocity restitution e with the number of hammer blows on the forge piece.

Values of e for forge hammers proper are much smaller than those for stamping hammers, and corresponding design values may be taken to equal 0.25.

Finally, for hammers forging nonferrous metals, this coefficient is considerably smaller than for hammers working on steel parts and may be considered to equal zero.

V-3. Natural Vibrations of a Hammer and Its Foundation as a Result of a Centered Impact

a. The Main Assumptions Involved in Design Computations. The foundation and hammer present a system which includes at least seven bodies: the frame, the dropping parts, the forged piece, the anvil, the elastic pad under the anvil, the foundation block, and, finally, the soil. From the point of view of mechanics, the phenomena which develop as a result of the impact of the ram against a forged piece lying on the anvil are extremely complicated and may be analyzed only with a high degree of approximation.

The main problems in computations for a hammer foundation are to determine the amplitude of foundation vibrations and to establish the values of stresses in the pad under the anvil.

The solution of these problems is usually based on the assumption that the hammer frame, the forged piece, the anvil, the elastic pad under the anvil, and the foundation block form one solid body. Such an assumption in regard to the foundation, the anvil, and the frame is justified by the fact that the deformation (due to impact) of each of these bodies is small in comparison with soil settlement under the foundation and therefore may be neglected.

However, deformation of the pad under the anvil may be much larger than soil settlement. Therefore the assumption that the pad has an infinitely large rigidity may lead in some cases to large errors in computation. This assumption is permissible only when the masses of both the anvil and the frame, if the latter is placed directly on the anvil, are comparatively small in relation to the foundation mass. Only in this case will the pad have no considerable effect on the amplitude of foundation vibrations. Otherwise, the elasticity of the pad cannot be neglected. In the case under consideration, the computation setup will be reduced to a system of three bodies: the ram, which is the striking body; the anvil, which is separated from the foundation by an elastic connection, and the foundation on an elastic base. The anvil and the foundation are the impact-receiving bodies.

In determining the amplitudes of foundation vibrations, it is possible to assume that the time of actual impact is small in comparison with the period of natural vibrations of the system, therefore, during the impact, there is no time for the foundation and anvil to undergo displacements comparable to their displacements during the vibrations which follow the impact. Since the reactions of the pad and the soil depend only on the displacements of the anvil and the foundation (we neglect damping reactions), it is possible to assume that during impact no additional reactions occur from the pad and soil. Thus only static reactions develop, imposed by the weight of the foundation, hammer, and anvil. These reactions existed before the impact and balanced the weight of the installation.

Therefore during impact, the foundation (with anvil and frame) and the dropping ram, in the first approximation, may be considered to be free bodies. Then an analysis of the impact of the system may be reduced to the analysis of a free impact of two or more absolutely solid bodies moving with given initial velocities.

The striking body (the ram) in all computations is assumed to be absolutely rigid.

b. Equations of the Vibrations of Foundation and Anvil. We begin by considering the simplest conditions: those in which pad elasticity may be neglected and the vibrations of the foundation, anvil, and frame occur as vibrations of a body with only one degree of freedom

In this case the equation of vertical free vibrations of the foundation will be (Art. III-1)

$$\ddot{z} + f_{nz}^2 z = 0 \tag{V-3-1}$$

where z = vertical displacement of center of mass of foundation and anvil, measured from equilibrium position

f_{nz}^2 = square of frequency of natural vibrations of foundation:

$$f_{nz}^2 = \frac{c_u A}{m}$$

A = foundation area in contact with soil

m = total vibrating mass

c_u = coefficient of elastic uniform compression of soil

Equation (V-3-1) is the equation of free vibrations of the foundation without damping. The general solution of this equation is

$$z = A \sin f_{nz}t + B \cos f_{nz}t \tag{V-3-2}$$

The constants A and B, as usual, are determined from the initial conditions of motion. Taking as the beginning of readings the instant when the impact of the ram against the anvil ends, we obtain, for $t = 0$,

$$z = 0 \qquad \dot{z} = v_0$$

Using these initial conditions, we obtain

$$A = \frac{v_0}{f_{nz}} \qquad B = 0$$

Equation (V-3-2) will take the form

$$z = \frac{v_0}{f_{nz}} \sin f_{nz}t \tag{V-3-3}$$

The maximum deflection of the foundation will occur after time t_1:

$$t_1 = \frac{\pi}{2f_{nz}}$$

Its value will be

$$A_z = \frac{v_0}{f_{nz}} \tag{V-3-4}$$

If one is to take into account the soil reactions which are proportional to the velocity of foundation displacement, then the amplitude of real vibrations will be smaller than the one computed without considering the damping forces. However, it is a difficult task to evaluate the influence

of damping forces by means of computations. As [illegible] in Chap. [illegible] these forces depend on many factors (for example, the foundation area in contact with soil, the foundation mass and the period of its free vibration and the foundation depth).

The pad under the anvil is fairly elastic in comparison with the anvil and foundation; therefore the anvil and frame (if the latter rests on the anvil) will not only participate in vibrations of the foundation on soil but will undergo some vibration with respect to the foundation.

In order to evaluate the amplitude of vibrations of the anvil in relation to the foundation, it is necessary to consider vibrations of a system with two degrees of freedom. Free vibrations of such a system are determined by the following differential equations:

$$\begin{aligned} m_1\ddot{z}_1 + c_1 z_1 - c_2(z_2 - z_1) &= 0 \\ m_2\ddot{z}_2 + c_2(z_2 - z_1) &= 0 \end{aligned} \tag{V-3-5}$$

where m_1, m_2 = masses of foundation, anvil (with frame, if latter is mounted on anvil)

$c_1 = c_u A$ = coefficient of rigidity of soil base under foundation

$c_2 = (E/b)A_2$ = coefficient of rigidity of pad under anvil

A_2 = base area of pad

b = thickness of pad

E = Young's modulus of material of pad

z_1, z_2 = displacements of foundation, anvil measured from equilibrium position

We denote by f_{n1} and f_{n2} the natural frequencies of the system whose motion is determined by Eqs. (V-3-5); by

$$f_{na}^2 = \frac{c_2}{m_2}$$

we denote the frequency of natural vibrations of the anvil with the frame (or for forge hammers proper that of the anvil on a motionless foundation); then we obtain a general solution of the system of Eqs. (V-3-5)

$$\begin{aligned} z_1 &= C_1(f_{na}^2 - f_{n1}^2) \sin (f_{n1}t + \alpha_1) + C_2(f_{na}^2 - f_{n2}^2) \sin (f_{n2}t + \alpha_2) \\ z_2 &= C_1 f_{na}^2 \sin (f_{n1}t + \alpha_1) + C_2 f_{na}^2 \sin (f_{n2} + \alpha_2) \end{aligned} \tag{V-3-6}$$

Setting

$$C^{(1)} = C_1 \cos \alpha_1 \qquad C^{(2)} = C_1 \sin \alpha_1$$
$$C^{(3)} = C_2 \cos \alpha_2 \qquad C^{(4)} = C_2 \sin \alpha_2$$

we obtain

$$\begin{aligned} z_1 &= C^{(1)}(f_{na}^2 - f_{n1}^2) \sin f_{n1}t + C^{(2)}(f_{na}^2 - f_{n1}^2) \cos f_{n1}t \\ &\quad + C^{(3)}(f_{na}^2 - f_{n2}^2) \sin f_{n2}t - C^{(4)}(f_{na}^2 - f_{n2}^2) \cos f_{n2}t \\ z_2 &= C^{(1)}f_{na}^2 \sin f_{n1}t + C^{(2)}f_{na}^2 \cos f_{n1}t + C^{(3)}f_{na}^2 \sin f_{n2}t \\ &\quad + C^{(4)}f_{na}^2 \cos f_{n2}t \end{aligned} \tag{V-3-7}$$

The natural frequencies f_{n1} and f_{n2} are determined as roots of the equation

$$f_n^4 - (f_l^2 + f_{na}^2)(1 + \mu_1)f_n^2 + (1 + \mu)f_l^2 f_{na}^2 = 0 \qquad \text{(V-3-8)}$$

where

$$\mu = \frac{m_2}{m_1} \qquad f_l^2 = \frac{c_1}{m_1 + m_2}$$

f_l^2 is the limiting frequency of the foundation together with the hammer placed on soil (for the condition that the pad is infinitely rigid).

The initial conditions of motion in the case under consideration (at $t = 0$) are as follows:

$$z_1 = z_2 = 0 \qquad \dot{z}_1 = 0 \qquad \dot{z}_2 = v_a$$

where v_a is the initial velocity of motion of the anvil,

$$v_a = \frac{1 + e}{1 + \mu_a} v$$

and

$$\mu_a = \frac{m_2}{m_0}$$

Particular solutions of system (V-3-5) which correspond to these initial conditions are as follows:

$$\begin{aligned} z_1 &= \frac{(f_{na}^2 - f_{n2}^2)(f_{na}^2 - f_{n1}^2)}{f_{na}^2(f_{n1}^2 - f_{n2}^2)} v_a \left(\frac{\sin f_{n1}t}{f_{n1}} - \frac{\sin f_{n2}t}{f_{n2}}\right) \\ z_2 &= \frac{v_a}{f_{n1}^2 - f_{n2}^2}\left(\frac{f_{na}^2 - f_{n2}^2}{f_{n1}} \sin f_{n1}t - \frac{f_{na}^2 - f_{n1}^2}{f_{n2}} \sin f_{n2}t\right) \end{aligned} \qquad \text{(V-3-9)}$$

With these expressions it is possible to compute stresses which develop in the pad as a result of combined vibrations of the anvil and foundation.

The maximum stress σ in the pad evidently will equal

$$\sigma = \frac{c_2}{A_2}(z_2 - z_1) \qquad \text{(V-3-10)}$$

V-4. Experimental Studies of Vibrations of Foundations under Forge Hammers

a. Introduction. The theory of vertical vibrations of hammer foundations, presented in Art. V-3, is based on some assumptions which may be verified only by comparing the results of computations with experimental data. This refers primarily to the negligibility of the mass and damping properties of soil. As stated in Chap. IV, due to the fact that forced vibrations of foundations under reciprocating engines are usually characterized by a frequency different from the natural frequency of foundation vibrations, the influence of damping soil reactions in such cases is small and may be ignored.

Under an impact, the foundation below a hammer undergoes free vertical vibrations. Therefore the damping soil reactions will have considerable influence on the amplitudes of foundation vibrations. The introduction of damping-reaction values into the computations of these vibrations will somewhat complicate the formula used, but the calculations will still be practicable from a mathematical point of view. However, in order that this complication, caused by the introduction of damping reactions, should be of some practical value, it is necessary to know the constants characterizing the dissipative properties of the soil. Great difficulties are involved in establishing these constants, because their values depend not only on the soil, but also on the design of the foundation (in particular, on the depth of the foundation, the ratio between the length and width of the foundation, the foundation height, and the material and density of the backfill). It is very difficult to take into account the influence of all these factors on the value of the damping constant of a soil

The inertial properties of the soil, which were not considered by the theory of vertical vibrations presented in Art. V-3, also may have great effect. In addition, the results of computations may be influenced by values of the coefficient of elastic rigidity c_2 of the pad under the anvil. This coefficient depends not only on the properties of the material of the pad under the anvil, but also on its design and on other special features which cannot always be taken into account by computations.

The pad under the anvil in hammer foundations of conventional design usually consists of several shields made of timber beams bolted together. The horizontal surfaces of these shields, just as the base of the anvil and the surface of the foundation under the anvil, are not ideally smooth surfaces and consequently do not come into contact with each other at all points Because of this, some sections of the surfaces of the pad, the anvil, and the foundation are subjected to considerable stresses while others are not loaded at all. As a result, the elastic properties of the whole pad depend not only on its material, but also on the conditions of its contact with the surfaces of the foundation and anvil

Only by means of measurements of vibrations occurring in a sufficiently large number of operating hammer foundations is it possible to elucidate the influence of all the above factors on vertical foundation vibrations. It is obvious that measurements do not give us an opportunity to establish separately the influence of each of these factors, for example, that of damping and inertial properties of soil However, the measurement data do make it possible to introduce corrective coefficients into the formulas of Art. V-3, which then permit the adjustment of the results of computations performed on the basis of these formulas to the results of vibration measurements

As early as 1939, the author carried out a large-scale investigation of foundation vibrations.[5] He studied 47 foundations under hammers located at six different plants.

That study had a threefold purpose: the verification of formulas for computations of hammer foundation vibrations, the collection of data on the design of normally operating foundations, and the determination of values of vibration amplitudes which could be accepted by designers as permissible and on which the main dimensions of the foundation would depend.

Table V-3. Data on Plant-Sites where Hammer-Foundation Measurements Were Made

Plant no.	*Geological and hydrogeological description of site*
1	A brown sandy clay with yellow inclusions comes to the surface everywhere on the site of the plant. This clay has a thickness of 0.1–2.5 m and is underlaid by a fine quartz sand alternating with lenses of clayey sand and of clay with some sand and silt. The thickness of these layers is not uniform over the area of the plant, varying from 0.3–6 m. With increasing depths, sands free of clay admixture predominate. Ground-water level is at a depth of 7 m, i.e., below all the hammer foundations.
2	Yellow medium-grained dense sands at a natural moisture content
3	Yellow medium-grained dense sands at a natural moisture content
4	Fine dense sand, ground-water level at a depth of 2.2–2.8 m, i.e., above the base of the hammer foundations
5	Medium-grained sands of medium density reaching to a depth of 9.4 m
6	Heavy brown clays with some sand and silt

Foundations of various designs were studied. Slab-shaped foundations predominated at one plant only. At other plants, only deeply embedded block-type foundations were present. This design of foundations was very popular at the time of the investigation (1939); slab-shaped foundations embedded to a small depth were seldom used then.

b. Description of Bases and Foundations. The greatest part of the forge hammers mounted on the foundations studied (35 out of 47) were double-acting steam or air stamping hammers. Only 6 foundations were under drop hammers of unrestricted action. The remaining 6 foundations were under forge hammers proper. The foundations investigated were located at six different plants. The geological conditions for each plant are given in Table V-3.

c. Results of Measurements of Foundation and Anvil Vibrations. Preliminary measurements of foundation vibrations showed that, in addition to vertical vibrations, hammer foundations also undergo rocking vibrations. However, the latter are less important because their amplitudes are much smaller than those of vertical vibrations.

The vibration amplitudes of the foundation, the anvil, and the frame are strongly affected by the state of the forged piece. For example, during the first impacts of the hammer against the piece, the energy of impact is largely consumed in plastic deformation of the metal, and the coefficient of restitution is small, as are the vibration amplitudes. The amplitudes of vibrations of the anvil and foundation grow with each subsequent impact. The largest amplitudes come with the last few impacts, when the forged piece is already deformed to such a degree that the greater part of the impact is taken not by the forged piece, but by the lower die, which transfers the impact energy to the anvil and foundation. Since the last few impacts induce the most unfavorable dynamic conditions for the foundation and anvil, their vibrations were measured during these impacts.

Figure V-7 shows samples of vibrograms obtained for some of the hammers investigated. It is seen from these vibrograms that vibrations of the hammer foundation and anvil, in most cases, differ considerably from damped sinusoids, which could be assumed on the basis of theoretical considerations. This shows that the foundation together with the anvil presents a much more complicated vibrating system than was assumed in Art. V-3, wherein vibration equations were derived.

In addition, as was to be expected, the vibrograms reveal considerable influence of the damping reactions. In some cases this influence is so large that the motion is almost aperiodic.

Finally it was found that identical foundations built under the same geologic conditions and subjected to the action of identical impacts underwent vibrations of varying amplitudes, sometimes sharply differing from one another. For example, two identical foundations under 3 6-ton hammers were investigated; one of them had 0.48 mm amplitude of vibrations, the other 0.78 mm. In the same way, two identical foundations under 2.25-ton hammers had 0.80- and 1.80-mm amplitudes of vibrations.

These data on vibrations of existing foundations permit the assumption that they are greatly affected by factors not considered by theory. Thus the observed differences in the amplitudes of foundation vibrations under hammers operating under the same conditions are apparently explained by the influence of the following factors (1) the state of the timber pad under the anvil; (2) the contacts between this pad and both the anvil and the foundation; (3) the backfill of the foundation, there may also be influences of other factors which are difficult to include in design computations.

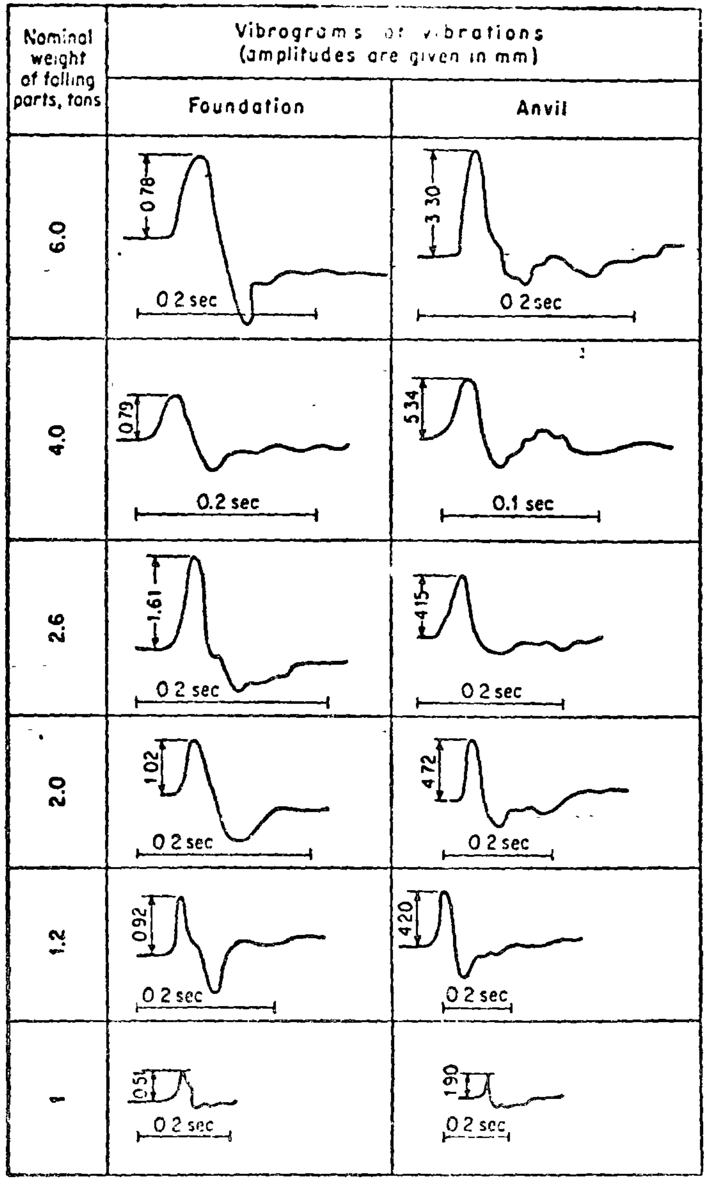


FIG. V-7. Typical vibrograms of operating hammer foundations

Therefore the results of computations carried out on the basis of the formulas of Art V-3 should be considered as tentative values only, showing the order of magnitude of vibration amplitudes, but not their absolute values.

A comparison of computed and measured vibration amplitudes of

hammer foundations leads to the conclusion that if data required for calculation are selected correctly, then an average error in computation will be around ±30 per cent.

d. Amplitudes of Foundation Vibrations. On the basis of investigations of 47 foundations, points are plotted in Fig. V-8 giving measured vibration amplitudes versus actual weights of dropping parts of hammers. It is seen that the amplitudes of hammer vibrations never attain the value of 2.0 mm, i.e., the value which in the past was taken as permissible in computations.[44] With a decrease in power of the hammer, a

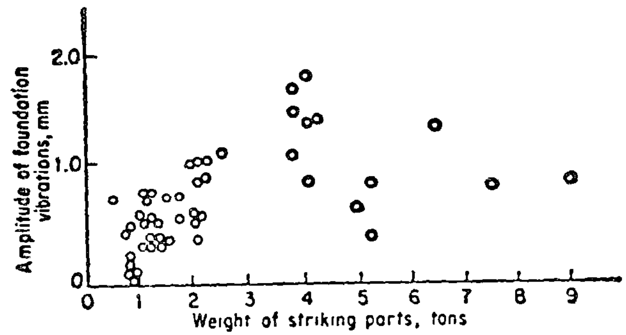


FIG. V-8. Measured vibration amplitudes of 47 hammer foundations plotted against the weight of the striking parts.

decrease in the amplitudes of foundation vibrations is observed. The overwhelming majority of foundations studied had vibration amplitudes of about 1.0 to 1.3 mm or less. Foundation vibrations characterized by these amplitudes did not exercise any noticeable harmful influence on the structures of forge shops. Similarly, no considerable settlements of foundations were observed where amplitudes of vibrations were of the order of 1.0 to 1.3 mm.

However, foundations having vibration amplitudes greatly exceeding 1.0 mm underwent considerable settlements. For example, a foundation with an amplitude of around 1.8 mm underwent a settlement reaching 0.3 m.

The above discussion leads to the conclusion that a design value of vertical vibrations of hammer foundations may be taken in the range 1 to 1.2 mm.

e. Amplitudes of Anvil Vibrations. Amplitudes of vibrations of anvils and frames of stamping hammers are much larger than amplitudes of foundation vibrations. For powerful hammers having thick pads under the anvil, absolute values of anvil vibrations reach 5 mm, although most of the hammers studied had amplitudes in the range 2 to 4 mm. With an increase in power of the hammer, the amplitude of anvil vibrations increases. For hammers with a weight of dropping parts up to 1 ton, a typical amplitude reaches 1 mm; for 2-ton hammers this amplitude reaches 2 mm; for hammers in which the weight of dropping parts exceeds 3 tons, the amplitude of anvil vibrations is usually 3 to 4 mm. These values of amplitudes of anvil and hammer vibrations may be taken as permissible.

The above values show that when stamping hammers are in operation, the anvil rebounds on the pad. Shocks of the anvil and frame (if the latter is attached to an anvil whose amplitude of vibrations reaches considerable values) have a harmful effect on the condition of the hammer. In addition, the larger the amplitude of vibrations of the anvil, the more kinetic energy of impact is consumed by these vibrations and consequently the smaller the hammer's efficiency. Loss of impact energy due to vibrations reaches 10 per cent of the work of the hammer's dropping parts.

The large vibration amplitudes of anvils are explained by insufficient rigidity of the pad under the anvil, which in some hammers had a thickness of 1.5 m. There is no reason to use such thick pads either from the point of view of the forging process or from a structural point of view. The pad thickness is usually assigned by the hammer supplier on the basis of traditional recommendations of the manufacturer and is not substantiated by any design data. Therefore different plants producing hammers of the same power recommend pads of different thicknesses.

The thickness of the pad should be selected so that the vibration amplitudes of the anvil do not exceed a particular value; in addition, stresses in the pad should not be greater than is permissible. Table V-1 [illegible] on the basis of these considerations. Thicknesses of pads, as recommended in that table, are somewhat smaller than those which usually have been employed up to the present time. The decrease in pad thickness as compared with usually accepted thicknesses is based on considerations concerning the harmful effects of anvil vibrations of large amplitude.

f. The Determination of Elastic Constants of the "Anvil-Foundation" System. If one is to consider a foundation together with the anvil mounted thereon as a system with two degrees of freedom, as was done in Art. V-3, then vibrograms of the anvil and foundation vibrations will show two sinusoids of different periods superimposed on each other. It follows from theory that the amplitudes of these sinusoids will be inversely proportional to their frequencies.

As stated before, the shapes of the measured foundation vibrograms in many cases approach aperiodic curves. In no case was it possible to determine from vibrograms both natural frequencies of the combined

vibrations of the anvil and foundation. Vibrograms usually reveal only the vibrations at the lower principal frequency. Therefore, it is possible to consider (with a precision sufficient for practical purposes) that in Eqs. (V-3-9) the amplitude of vibrations for $\sin f_{n1}t$ (where $f_{n1} > f_{n2}$) equals zero. Then approximate expressions for dynamic displacement of the foundation and anvil will be as follows:

$$z_1 = -\frac{(f_{na}^2 - f_{n2}^2)(f_{na}^2 - f_{n1}^2)}{f_{na}^2(f_{n1}^2 - f_{n2}^2)f_{n2}^2} v_a \sin f_{n2}t$$
$$z_2 = -\frac{f_{na}^2 - f_{n1}^2}{(f_{n1}^2 - f_{n2}^2)f_{n2}} v_a \sin f_{n2}t \qquad \text{(V-4-1)}$$

Hence
$$\beta = \left|\frac{z_2}{z_1}\right| = \frac{1}{1 - \gamma^2} \qquad \text{(V-4-2)}$$

where
$$\gamma^2 = \frac{f_{n2}^2}{f_{na}^2} \qquad \text{(V-4-3)}$$

Thus, having found, from vibrograms obtained for the anvil and foundation, the value β and the lower natural frequency of vibrations, one can establish from Eq. (V-4-3) the limiting frequency f_{na} of vibrations of the anvil on the pad. From the formula

$$f_{na}^2 = \frac{EA}{bm_2}$$

one can establish the value of the modulus of elasticity E of the pad under the anvil.

Then no difficulties are involved in establishing the value of the second higher frequency f_{n1}, as well as the limiting frequency f_l of the natural vertical vibrations of the entire installation on the soil. After some transformations (not shown here), we obtain

$$f_{n1}^2 = \frac{1 + \mu - \gamma^2}{(1 - \gamma^2)\gamma^2} f_{n2}^2$$
$$f_l^2 = \frac{1 + \mu - \gamma^2}{(1 + \mu)(1 - \gamma^2)} f_{n2}^2 \qquad \text{(V-4-4)}$$

Here, as before, μ is the ratio between the anvil mass (in drop hammers the frame mass is also included) and the foundation mass. Knowing f_l^2 from the formula

$$f_l^2 = \frac{c_u A}{m_1 + m_2}$$

one can establish the real value of the coefficient of elastic uniform compression c_u of the base under the hammer foundation.

The results of computations of moduli of elasticity for pads and coefficients of uniform compression c_u for bases, performed in accordance with the above methods for the several foundations studied, lead to the following conclusions:

As was to be expected, the computed moduli of elasticity E for pads under the anvils of different hammers vary within a comparatively wide range of values. The probable reasons for this have already been discussed. The moisture content and working life of the pad should also be mentioned, as their influence may be very noticeable.

The average value of E was established from computations to equal 4.7×10^4 tons/m², i.e., approximately two times smaller than the value customarily used in stress analyses of oak beams when the latter are compressed across their fibers. On the basis of these results, it is recommended that a design value of 5×10^4 tons/m² be used for the modulus of elasticity of timber pads.

In most hammers the dynamic stress in the pad under the anvil does not exceed 200 tons/m², i.e., it is much lower than the permissible value of about 300 to 350 tons/m² for oak timbers compressed across their fibers. This attests to the fact that pads which were employed up to this time have had a considerable safety factor. As stated before, this is because the thickness of the pad has usually been taken much larger than was necessary from the point of view of dynamic computations.

In low-power hammers, dynamic stresses in pads do not exceed 100 tons/m². Therefore it is possible to employ in these hammers pads made of pine or larch instead of oak.

Special investigations showed that the design value of the coefficient c_u of elastic uniform compression of the soil base of hammer foundations was about 4.0 kg/cm³. However, the average value of this coefficient obtained from measurements of hammer vibrations was around 25 kg/cm³, i.e., approximately six times larger. Such a large divergence between these values attests to the fact that the amplitudes of foundation vibrations under hammers are greatly affected by factors not considered by the theory presented in Art. V-3. In particular, this theory, as stated above, does not consider the influence of the damping and inertial properties of the soil, but this influence may be considerable, just as in the case of natural foundation vibrations. In addition, the value of c_u may be affected by the backfill of the foundation.

Investigations of a test foundation showed that with backfilling the value of the coefficient c_u of elastic uniform compression increases approximately two times as compared with the value established from tests on an exposed foundation. Consequently, in computations of natural vertical vibrations of foundations under hammers, the value of the coefficient of elastic uniform compression c_u should not be taken equal

to that used in computations of vibrations of other machines whose foundations do not undergo natural vertical vibrations. According to the above data, in design computations of hammer foundations the coefficient c'_u should be used instead, where

$$c'_u = kc_u \tag{V-4-5}$$

With allowance for some safety reserve for dynamic stability of the foundation, the value of the correction coefficient k may be taken as equal to 3.

g. Comparison of Different Formulas for the Computation of Amplitudes. To simplify the practical computations of vibrations of the foundation and anvil, the vibrations are considered to be independent of each other. This is equivalent to the assumptions that the presence of the pad under the anvil has no influence on the amplitudes of vibrations of the foundation and that the vibrations of the anvil are not affected by elastic properties of the soil base or the mass of the foundation. This assumption leads to considerable simplification of formulas for the computation of vibration amplitudes of the foundation and anvil. According to Eqs. (V-3-4), the amplitude of vibrations of the foundation can be established from

$$A_z = \frac{(1 + e)W_0 v}{(W_1 + W_2)f_{nz}} \tag{V-4-6}$$

and the amplitude of vibrations of the anvil from

$$A_a = \frac{(1 + e)W_0 v}{W_2 f_{na}} \tag{V-4-7}$$

It is interesting to establish and compare the computational errors involved in these equations and the more accurate Eqs. (V-3-9). Taking into account the fact that in Eqs. (V-3-9) the amplitudes for $\sin f_{n1}t$ are much smaller than the amplitudes for $\sin f_{n2}t$, we may neglect the terms containing $\sin f_{n1}t$ in these formulas. Then the vibration amplitudes of the anvil and foundation will be determined by Eqs. (V-4-1).

Table V-4 gives the results of computations of vibration amplitudes of the foundation and anvil, performed for several foundations studied; both methods of computation were employed. The same values of the modulus of elasticity of the pad under the anvil and the coefficient of elastic uniform compression of the base under the foundation were taken for all hammers (respectively, 5×10^4 tons/m² and 20×10^3 tons/m³). The same velocity of dropping parts (6.5 m/sec) was used for all hammers.

It is seen from Table V-4 that there is considerable difference between the amplitude values of foundation vibrations as computed by Eqs. (V-4-1) and (V-4-6). Hence it follows that a control computation of

TABLE V-4. COMPARISON OF VIBRATION AMPLITUDES COMPUTED BY DIFFERENT PROCEDURES

W, tons[a]	m_1, tons × sec²/m	m_2, tons × sec²/m	c_1, tons/m	c_2, tons/m	A_1, mm[b]	A_2, mm[c]	A_f, mm[d]	A_a, mm[e]
7.50	51.2	19.8	136.2 × 10⁴	42.2 × 10⁴	0.915	4.13	0.74	4.10
4.10	17.2	9.17	56.8 × 10⁴	39.6 × 10⁴	1.32	2.81	0.85	2.80
5.30	29.4	9.54	62.4 × 10⁴	23.1 × 10⁴	0.55	4.5	0.55	4.31
3.25	17.4	4.92	60.0 × 10⁴	21.9 × 10⁴	1.61	6.7	1.25	7.36
2.10	9.37	3.44	40.0 × 10⁴	22.0 × 10⁴	1.74	4.46	1.04	4.90
1.20	5.05	2.85	29.4 × 10⁴	28.6 × 10⁴	0.77	2.00	0.58	2.28

[a] These are the actual weights of the dropping parts; they differ from the nominal values.
[b] $A_1 = z_1$ = amplitude of foundation vibration computed from Eqs. (V-4-1).
[c] $A_2 = z_2$ = amplitude of anvil vibration computed from Eqs (V-4-1).
[d] A_f = amplitude of foundation vibration computed from Eq. (V-4-6).
[e] A_a = amplitude of anvil vibration computed from Eq. (V-4-7).

vibrations of a foundation under a hammer should be performed using Eqs. (V-4-1), considering the effect of the pad under the anvil on these vibrations.

However, Table V-4 shows that there is only a small difference between computations of anvil vibration amplitudes by Eq. (V-4-7) and by the more precise Eqs. (V-4-1). Therefore it is permissible to use the simplified formulas in computations of anvil vibration amplitudes and in stress analysis of the pad under the anvil.

V-5. Selection of the Weight and Base Area of a Hammer Foundation

It was formerly held that the weight of the foundation for a hammer and the size of its area in contact with the soil should be selected in such a way as to meet the following requirements: the total pressure on the soil should not exceed the bearing capacity of this soil; and the foundation should not bounce on the soil. These conditions may be written as follows:

$$p_{st} + p_{dy} \leq \alpha p_0 \qquad \text{(V-5-1)}$$

$$p_{st} > p_{dy} \qquad \text{(V-5-2)}$$

where p_{st}, p_{dy} = static, dynamic pressures on soil
p_0 = permissible bearing value under condition that only static load is acting
α = coefficient of required reduction

The condition expressed by Eq. (V-5-1) is based on an assumption that either the static and dynamic pressures are equivalent or that the "coefficient of required reduction" is not a constant for a given soil and foundation. As stated in Chap. II, dynamic pressure transmitted to soils (especially to granular soils) may induce settlements and deformations which are tens and hundreds of times larger than those caused by static pressure of the same magnitude. Therefore if one of the items of the left-hand part of expression (V-5-1) changes, but the sum of these items remains constant, the total settlements and deformations will change. Hence it follows that under the assumption that α has a constant value for a given soil and foundation, the condition expressed by Eq. (V-5-1) cannot be accepted, because it is contrary to the physical nature of the phenomenon.

The condition expressed by Eq. (V-5-2) has no practical significance, because the bouncing of a foundation is of no essential importance and cannot be observed under working conditions. However, an observance of this condition led to carrying foundations down to a considerable depth. Since the design value of vibration amplitude was taken to equal 2 mm and more, in order to obtain a static settlement of this value it was necessary to increase considerably the heights of foundations. The result was that all hammer foundations erected with designs complying with the conditions of Eqs. (V-5-1) and (V-5-2) represented massive blocks carried down to a considerable depth. Figure V-4*a* shows a typical foundation of this kind.

Limitation of the vibration amplitude of a hammer foundation is the most important condition which must be satisfied by the design of such a foundation. The smaller the vibration amplitude, the smaller the influence of vibrations on adjacent structures and buildings, on the foundation, on the soil, and on the hammer. Another significant condition is the limiting of the value of static pressure on soil; as shown in Art. II-3, the smaller the static pressure, the smaller the settlement of the foundation (other conditions remaining equal).

Thus instead of the conditions of Eqs (V-5-1) and (V-5-2), the foundation design should satisfy the following two conditions:

$$A_s < A_0 \qquad \text{(V-5-3)}$$

$$p_{st} \leq \alpha p_0 \qquad \text{(V-5-4)}$$

As indicated in Art. V-4, an average value of vibration amplitudes of hammer foundations, obtained from the results of numerous investigations of operating hammers, is approximately 1 mm. This may be taken as a design value for the permissible amplitude. Therefore the condition of Eq. (V-5-3) may be rewritten as follows:

$$A_s < 10^{-3} \text{ m} \qquad \text{(V-5-5)}$$

In the simplest case, under the assumption that the foundation together with the anvil presents a system with one degree of freedom, the value of the vibration amplitude is determined by Eq. (V-4-6), and the condition of Eq. (V-5-5) may be written in the form

$$\frac{(1+e)W_0 v}{\sqrt{kc_u W A g}} < 10^{-3} \qquad \text{(V-5-6)}$$

where all dimensions are in tons, meters, and seconds.

From Eqs. (V-5-4) and (V-5-6) values of foundation contact area and weight can be found for which the amplitude of foundation vibrations will not exceed 1 mm and the static pressure on the soil will not exceed the value αp_0; thus we obtain

$$A \gg \frac{(1+e)W_0 v}{\sqrt{kc_u \alpha p_0 g}} 10^3 \quad \text{meters} \qquad \text{(V-5-7)}$$

$$W_f = \frac{(1+e)W_0 v\sqrt{\alpha p_0}}{\sqrt{kc_u g}} 10^{-3} - W_a \quad \text{tons} \qquad \text{(V-5-8)}$$

where W_f = weight of foundation together with backfill, if present
W_a = weight of anvil and frame

Dividing both parts of Eq. (V-5-8) by W_0, we obtain a formula for determining the reduced foundation weight corresponding to a unit of actual weight of dropping parts of the hammer:

$$n_f = \frac{(1+e)\sqrt{\alpha p_0}}{\sqrt{kc_u g}} v \times 10^3 - n_a \qquad \text{(V-5-9)}$$

where $n_f = \dfrac{W_f}{W_0}$ $\quad n_a = \dfrac{W_a}{W_0}$

In accordance with the values of p_0 and c_u for different soils given in Art. I-2, the following approximate relationship can be used:

$$\frac{p_0}{c_u} = 0.07$$

The corrective coefficient k, in accordance with data presented in Art. V-4, we set equal to 3.0; the coefficient of reduction of bearing capacity α, we set equal to 0.4.

TABLE V-5. VALUES OF SOME HAMMER COEFFICIENTS

Type of hammer	v, m/sec	e	n_a	n_f
Stamping hammers:				
Double-acting hammers (stamping of steel pieces)	6.5	0.5	30	48
Unrestricted hammers:				
Stamping of steel pieces	4.5	0 5	20	34
Stamping of nonferrous metals	4 5	0 0	...	16
Forge hammers proper:				
Double-acting	6 5	0 25	30	35
Unrestricted	4 5	0 25	20	25

Substituting the values of these coefficients into (V-5-9), we obtain a simple formula for the tentative determination of the foundation weight depending on the velocity of dropping parts of the hammer and the coefficient of restitution:

$$n_f = 8\,0(1+e)v - n_a \qquad \text{(V-5-10)}$$

According to data of machine-building plants, one can take approximately:

For double-acting hammers: $n_f = 30$
For unrestricted hammers: $n_a = 20$

Numerical values of n_f for different hammers are given in Table V-5.

In order to compare the computed values of n_f with data secured from experience, Table V-6 shows values of n_f for some of the drop-hammer slab foundations investigated at one of the plants.

The average experimental value of n_f for the hammers of Table V-6 is 46. It should be mentioned, however, that the actual weight of dropping parts of the hammers studied is much larger than their nominal weight. In most other cases, the operating dies are lighter and consequently the difference between the actual and the nominal weights of the dropping parts is smaller. Therefore such hammers have a larger value of n_f,

TABLE V-6. COMPARISON OF ACTUAL AND COMPUTED REDUCED FOUNDATION WEIGHTS n_f [EQ. (V-5-10)] AND REDUCED FOUNDATION AREAS a_f [EQ. (V-5-11)] OF SEVERAL HAMMER SLAB FOUNDATIONS

Nominal weight of dropping parts, tons	W_0, tons	W_f, tons	A, m²	Actual value of n_f	Actual value of a_f	Computed value of n_f	$\frac{W_b}{W_0}$†	Computed value of a_f
1.0	1.2	49.7	14.7	41.1	12.2	48.0	32.6	12.2
1.25	2.10	92.0	20.0	43.8	9.6	48.0	31.3	9.5
5.4	7.5	502	65.6	67.0	8.8	48.0	30.8	8.8
2.25	4.10	217	38.2	53.0	9.4	48.0	32.4	9.3
1.35	2.0	73.5	16.5	36.8	8.2	48.0	36.8	8.2
0.54	0.75	28.4	9.9	38.0	13.0	48.0	31.0	13.2
2.25	4.10	168	40	41.1	10.0	48.0	34.0	9.8

† W_b = weight of concrete foundation without consideration of the backfill.

reaching 50 to 55 for the same foundation weights. For massive foundations, which until recently were accepted by all design organizations, this ratio is much larger, lying in the range 70 to 80 and often reaching 100 to 120.

If in slab foundations one takes into account only the weight of concrete, neglecting the weight of backfill above the slab, then the value of n_f will be around 30 to 35. In massive foundations no such backfill is present; therefore the value of n_f of 70 to 120 represents the ratio between the weight of the foundation and the actual weight of dropping parts. Thus the expenditure of material for slab foundations is two times smaller than for massive foundations. And, as stated in Art. V-4, the results of instrumental investigations of hammer slab foundations show that the amplitudes of their vibrations lie within the range of permissible values (around 1 mm and less).

Equation (V-5-7) for the selection of the foundation area in contact with soil may be simplified on the basis of the following considerations:

According to available data on the values of c_u and p_0 for different soils,

it can be stated that

$$\frac{p_0}{c_u} \cong 0.5 \times 10^{-2}\,\mathrm{m}$$

Hence

$$c_u = 2 \times 10^{-2}\,p_0$$

where all dimensions are tons and meters.

Dividing both parts of Eq. (V-5-7) by W_0, we obtain an expression for the reduced contact area of foundation per unit of the actual weight of dropping parts:

$$a_f = \frac{A}{W_0} = \frac{(1+e)v \times 10^3}{p_0\sqrt{\alpha k g \times 2 \times 10^{-2}}}$$

Setting as before $k = 3.0$ and $\alpha = 0.4$, we obtain a simple formula for the tentative determination of a_f and consequently of the entire contact area of the foundation:

$$a_f = \frac{20(1+e)v}{p_0} \qquad \text{(V-5-11)}$$

Equation (V-5-11) establishes the dependence of the foundation contact area not only on the hammer characteristics, but also on soil properties; the required dimensions of the foundation contact area increase in an inverse proportion to the bearing capacity of soil. Table V-7 presents values of a_f computed for different types of soils.

TABLE V-7. VALUES OF REDUCED FOUNDATION CONTACT AREAS a_f REQUIRED FOR DIFFERENT SOILS

Type of hammer	Values of a_f for following groups of soil:		
	Weak soils, $p_0 \leq$ 1.5 kg/cm²	Soils of medium strength, p_0 = 1.5–3.5 kg/cm²	Soils of high strength, p_0 = 3.5–6 kg/cm²
Stamping hammers:			
Double-acting hammers (stamping of steel pieces)	13	13–5.5	5.5–3.3
Unrestricted hammers			
Stamping of steel pieces	9	9–4	4–2.5
Stamping of nonferrous metals	6	6–2.5	2.5–1.5
Forge hammers Proper			
Double-acting	11	11–5	5–3
Unrestricted	7.5	7.5–3	3–2

For example, it is seen from this table that double-acting drop hammers used for stamping steel pieces, i.e., the type most frequently employed in forge shops, are characterized by the following ratio when the soil is of medium strength, on the average about 9 m² of foundation contact area will be required per unit of actual weight of dropping parts.

Table V-6 gave values of a_f which were used for hammer slab foundations when $p_0 = 2.5$ kg/cm². It follows from the data of this table that values of a_f established from Eq. (V-5-11) are close to those accepted for the design of hammer slab foundations. Massive hammer foundations designed as blocks are characterized by values of a_f of 7 to 8, i.e., by somewhat smaller values than those used for slab foundations.

V-6. Design of a Hammer Foundation

Example. Dynamic computations for the design of a hammer foundation

1. Design Data. A double-acting stamping hammer has the following specifications:

Nominal weight of dropping parts:

3.0 tons

Actual weight of dropping parts:

$W_0 = 3.5$ tons

Height of drop:

$h = 1.0$ m

Piston area from above:

$A = 0.15$ m²

Steam pressure:

$p = 8$ atm

Weight of the anvil and frame:

$W_2 = 90$ tons

Base area of the anvil:

$A_2 = 1.75$ m²

Thickness of pad under anvil:

$b = 0.60$ m

Soils on the site of the foundation consist of brown clays with some sand and silt, with a permissible pressure $p_0 = 2$ kg/cm² if only static pressure is acting.

2. Velocity of Dropping Parts at the Beginning of Impact. From Eq. (V-2-2),

$$v = 0.65\sqrt{\frac{2 \times 9.81 \times 1.0(3.5 + 80 \times 0.15)}{3.5}} = 6.1 \text{ m/sec}$$

3. Preliminary Computation of the Required Values of Foundation Weight and Soil Contact Area. Design computations for determining the required weight of the foundation corresponding to a unit weight of dropping parts are made from Eq. (V-5-10). The coefficient of restitution is taken as $e = 0.5$. The weight of the anvil and frame corresponding to a unit weight of dropping parts will be:

$$n_a = \frac{90}{3.5} = 25.7$$

According to Eq. (V-5-10), the weight of the foundation corresponding to a unit weight of dropping parts equals

$$n_f = 8.0(1 + 0.5)6.1 - 25.7 = 47.3$$

The required weight of the foundation (together with backfill) equals

$$W = 3.5 \times 47.3 = 166 \text{ tons}$$

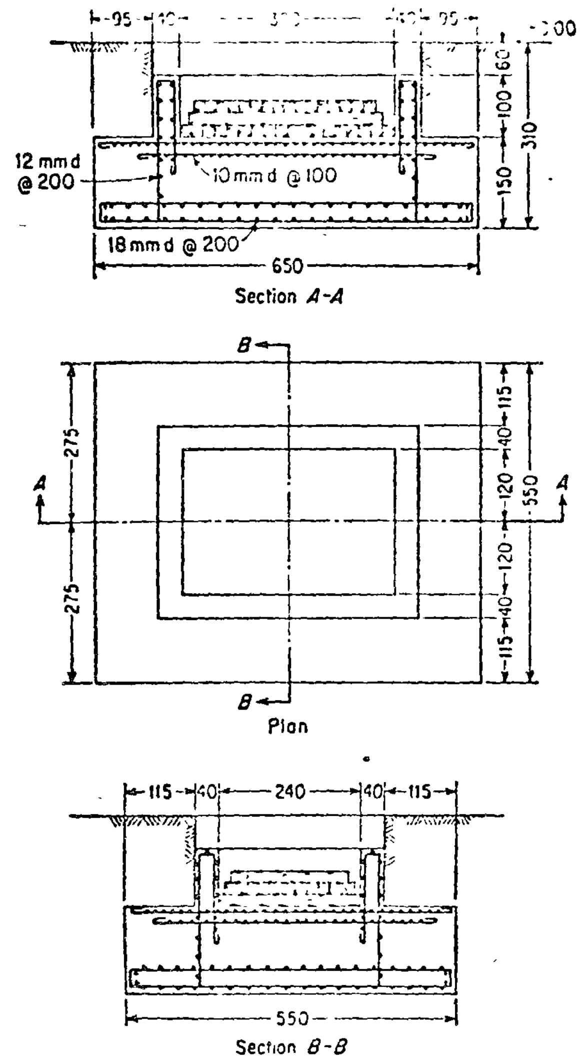


Fig. V-9. Design of foundation for example of Art. V-6.

The foundation contact area corresponding to a unit weight of dropping parts is determined from Eq. (V-5-11).

$$a_f = \frac{20(1 + 0.5)6.1}{20} = 9.2 \text{ cm}^2$$

The required foundation area in contact with soil equals

$$A = 9.2 \times 3.5 = 32.2 \text{ m}^2$$

4. Design of Foundation. On the basis of the values determined above for the required foundation weight and area in contact with soil, we design the foundation in the form of a slab. The outline of the foundation design is shown in Fig. V-9.

The actual volume of concrete will be as follows:

$$V_c = 6.5 \times 5.5 \times 1.5 + 2 \times 3.2 \times 0.10 \times 1.0 + 2 \times 3.8 \times 0.10 \times 1 = 58.0 \text{ m}^3$$

The volume of backfill is

$$V_b = 2 \times 0.95 \times 5.5 + 2 \times 4.60 \times 1.15 = 20.9 \text{ m}^3$$

The total weight of the foundation and backfill is

$$W_1 = 58.0 \times 2.2 + 20.9 \times 1.6 = 161.4 \text{ tons}$$

The foundation area in contact with soil is

$$A = 6.50 \times 5.5 = 35.7 \text{ m}^2$$

5. AMPLITUDE OF FOUNDATION VIBRATIONS. We take the modulus of elasticity of the pad under the anvil to equal

$$E_2 = 50 \times 10^3 \text{ tons/m}^2$$

We take the thickness of the pad under the anvil from the design data:

$$b = 0.60 \text{ m}$$

The coefficient of rigidity of the pad under the anvil will equal

$$c_2 = \frac{50 \times 10^3 \times 4.75}{0.60} = 39.5 \times 10^4 \text{ tons/m}$$

The mass of the hammer is

$$m_2 = 90/9.81 = 9.18 \text{ tons} \times \text{sec}^2/\text{m}$$

The limiting frequency of natural vibrations of the anvil on the oak timber pad is

$$f_{na} = \frac{39.5 \times 10^4}{9.18} = 43 \times 10^3 \text{ sec}^{-2}$$

We set the coefficient of elastic uniform compression to equal

$$c_u = 4 \times 10^3 \text{ tons/m}^3$$

and the value of the correction coefficient $k = 3$. Then

$$c_u' = 3 \times 4 \times 10^3 = 12 \times 10^3 \text{ tons/m}^3$$

The coefficient of rigidity of the base under the foundation equals

$$c_1 = 12 \times 10^3 \times 35.7 = 43.8 \times 10^4 \text{ tons/m}$$

The mass of the foundation together with the backfill is

$$m_1 = \frac{161.4}{9.81} = 16.5 \text{ tons} \times \text{sec}^2/\text{m}$$

The square of the limiting frequency of natural vibrations of the whole system is then

$$f_l^2 = \frac{42.8 \times 10^4}{16.5 + 9.18} = 16.7 \times 10^3 \text{ sec}^{-2}$$

The ratio between the mass of the hammer and the mass of the foundation together with the backfill is

$$\mu_1 = \frac{9.18}{16.5} = 0.557$$

Using Eq. (V-3-8), we set up the equation for determining the frequencies of natural vibrations of the foundation-hammer system:

$$f_n^4 - (1 + 0.557)(43.0 \times 10^3 + 16.7 \times 10^3)f_n^2 + (1 + 0.0557) \times 43.0 \times 10^3 \times 16.7 \times 10^3 = 0$$

Or

$$f_n^4 - 92.5 \times 10^3 f_n^2 + 1115 \times 10^6 = 0$$

Solving this equation, we obtain

$$f_{n1,2}^2 = [46.25 \pm \sqrt{(46.25)^2 - 1115}]10^3 = (46.25 \pm 32.2)10^3$$

Hence we have the frequencies

$$f_{n1}^2 = 78.5 \times 10^3 \text{ sec}^{-2} \qquad f_{n1} = 288 \text{ sec}^{-1}$$
$$f_{n2}^2 = 14.1 \times 10^3 \text{ sec}^{-2} \qquad f_n^2 = 199 \text{ sec}^{-1}$$

We determine the initial velocity of the motion of the anvil together with the frame:

$$v_a = \frac{(1 + 0.5)3.5 \times 6.1}{3.5 + 90.0} = 0.342 \text{ m/sec}$$

From Eqs. (V-4-1) we establish the amplitudes of vibration of the foundation and anvil. The amplitude of vibration of the foundation is

$$A_z = -\frac{(43.0 \times 10^3 - 14.1 \times 10^3)(43.0 \times 10^3 - 78.5 \times 10^3)}{43.0 \times 10^3(78.5 \times 10^3 - 14.1 \times 10^3)119} 0.342$$
$$= 1.07 \text{ mm}$$

The amplitude of vibrations of the anvil together with the frame is

$$A_a = -\frac{(43.0 \times 10^3 - 78.5 \times 10^3)\sqrt{.342}}{(78.5 \times 10^3 - 14.1 \times 10^3)119} = 1.6 \times 10^{-3} \text{ m} = 1.6 \text{ mm}$$

Thus the results of computations show that the amplitude of vibrations of the foundation will not exceed the permissible value of 1.0 to 1.2 mm.

The dynamic stresses in the pad under the anvil approximately equal

$$\sigma = \frac{c_2(A_a - A_z)}{A_2} = \frac{39.5 \times 10^4(1.6 \times 10^{-3} + 1.07 \times 10^{-3})}{4.75} = 222 \text{ tons/m}^2$$

which is much smaller than the permissible value of 300 to 350 tons/m².

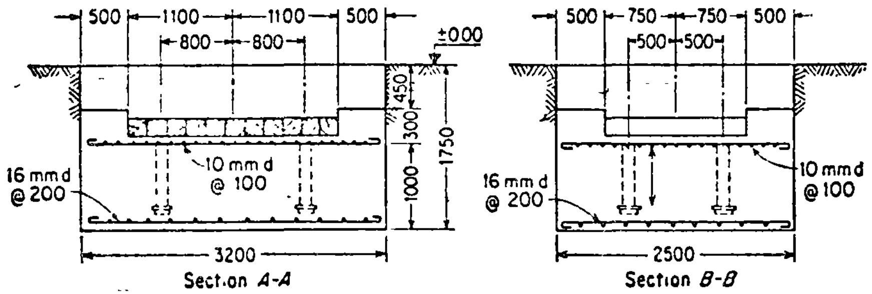
6. REINFORCEMENT OF THE FOUNDATION. The foundation is reinforced as shown in Fig. V-9 according to practical requirements pointed out in Art. V-1. Concrete type 150† is used for the foundation.

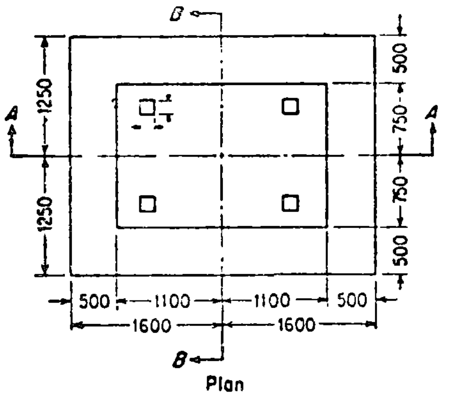
Standard Illustrative Designs of Hammer Foundations. Computation and design of foundations for stamping hammers of different p... dr...as

† See footnote, Art. IV-I-c, p. 132

well as foundation design for forge hammers proper, may be performed in a manner similar to the preceding numerical example.

Figures V-10 to V-17 show several standard illustrative foundation designs for stamping hammers and forge hammers of several systems. In the preparation of these examples, actual data on hammers were borrowed from instructive design manuals which had been compiled with the author's participation as a consultant.[18]





Concrete type 110

Actual weight of falling parts, tons	0.75
Velocity at beginning of impact, m/sec	5.5
Weight of anvil and frame, tons	26.6
Thickness of pad under anvil, m	0.20
Volume of foundation concrete, m^3	8.7
Weight of foundation and backfill, tons	22.6
Foundation contact area, m^2	8

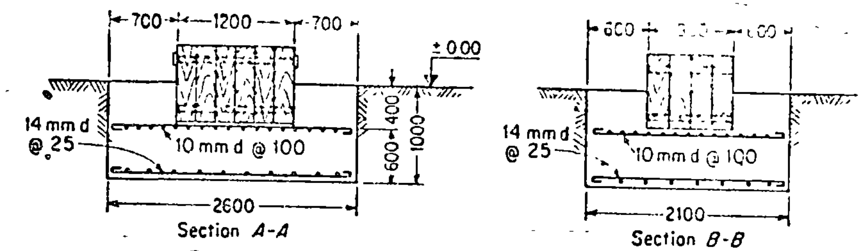
FIG. V-10. Foundation for 750-kg frictional hammer of "KLF" plant.

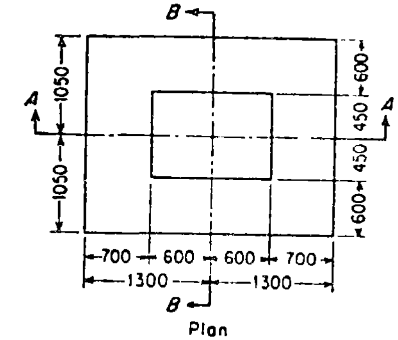
Design computations for these foundations were performed for a soil of medium strength with the coefficient of elastic uniform compression of soil c_u equal to approximately 4 kg/cm³.

V-7. Computation and Design of Hammer Foundations with Vibration Absorbers

a. General Directives on Computation and Design. Sometimes the decrease in vibration amplitudes of hammer foundations is of great practical importance.

It follows from the approximate Eqs. (V-4-6) and (V-4-7) that vibration amplitudes of the foundation and anvil are inversely proportional to the square roots of the products of the base rigidity and mass. Conse-

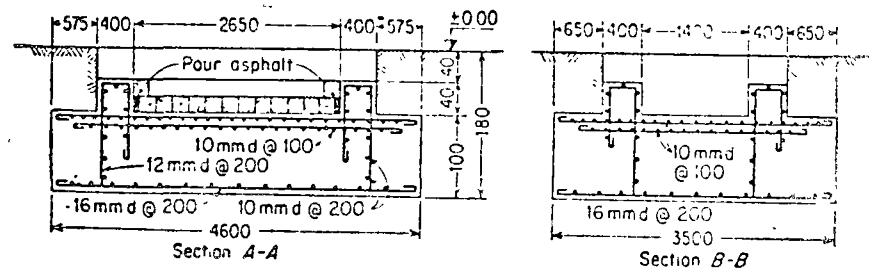


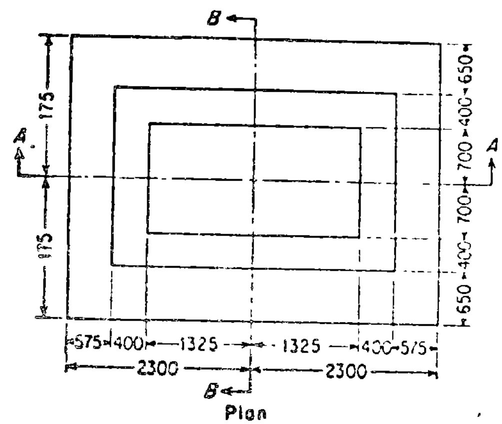


Concrete type 110

Actual weight of falling parts, tons	1.0
Velocity at beginning of impact, m/sec	4.2
Weight of anvil and frame, tons	6.0
Volume of foundation concrete, m^3	5.0
Weight of foundation and backfill, tons	11.0
Foundation contact area, m^2	5.5

FIG. V-11. Foundation for 1.0-ton free-falling cable hammer

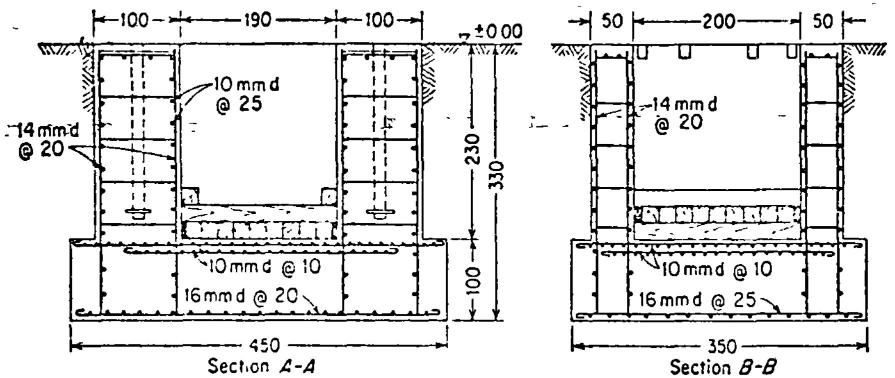


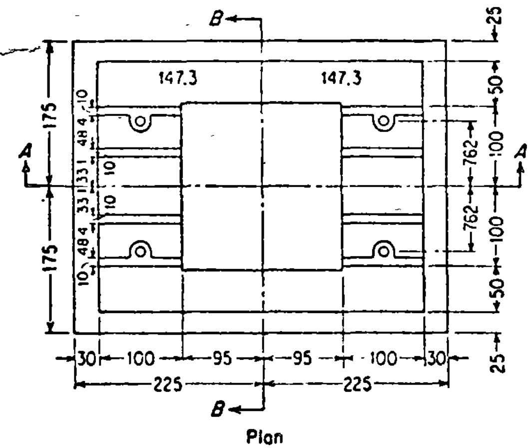


Concrete type 110

Actual weight of falling parts, tons	1.2
Velocity at beginning of impact, m/sec	6.5
Weight of anvil and frame, tons	3.0
Thickness of pad under anvil, m	0.2
Volume of foundation concrete, m^3	17.7
Weight of foundation and backfill, tons	50
Foundation contact area, m^2	16

FIG. V-12. Foundation for 1.2-ton steam or compressed-air stamping hammer quently, if one is to try to decrease the vibration amplitude of a foundation by increasing its mass, difficulties will arise; for example, if one wishes to make the amplitude of vibrations three times smaller (i.e., to use a design value of 0.3 to 0.4 mm instead of 1.0 to 1.2 mm) it will be necessary to increase the weight of the foundation at least nine times. It is clear that this method is impracticable. Similarly, it is very difficult to





Concrete type 110	
Actual weight of falling parts, tons	2.0
Velocity at beginning of impact, m/sec	6.0
Thickness of pad under anvil, m	0.40
Volume of foundation concrete, m³	34.0
Weight of foundation and backfill, tons	80.0
Foundation contact area, m²	15.8

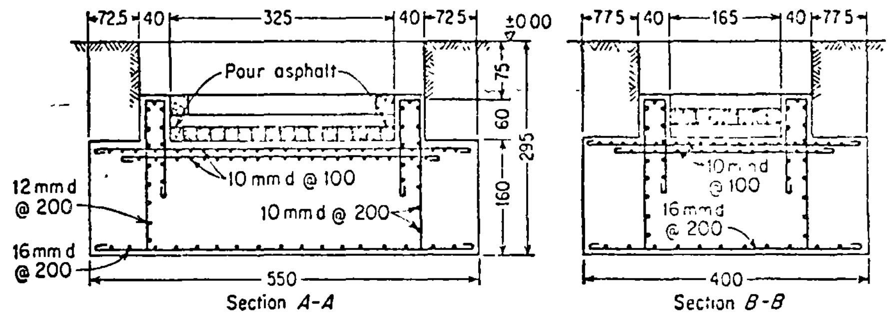
FIG. V-13. Foundation for 2-ton steam or compressed-air forging hammer.

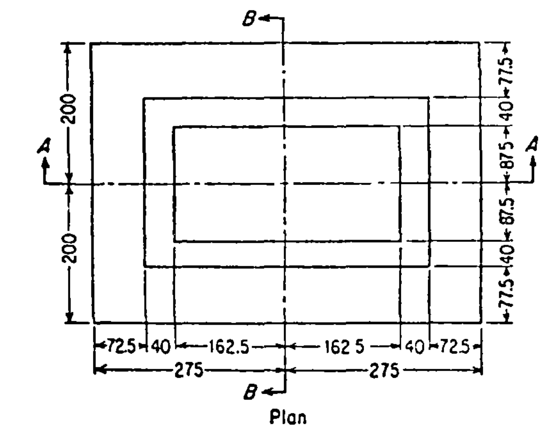
decrease the amplitude of foundation vibrations by increasing the foundation contact area or the rigidity of the base.

Since foundation vibrations, if they are considered together with vibrations of the anvil, depend not only on the parameters of the foundation c and m_1 (i.e., on the rigidity of the base and the mass of the foundation beneath the springs), but also on the parameters c_2 and m_2 (i.e., on the rigidity of the absorbers and the mass of the foundation above the springs), theoretically it is possible to decrease the amplitudes of foundation vibration by selecting suitable values of c_2 and m_2. These parame-

ters should be selected so as not to increase sharply the vibration amplitude of the anvil in comparison with values customary in forging practice.

Thus the problem is reduced to the following: values of c_2 and m_2 should be found for which the corresponding vibration amplitudes of the foundation and anvil do not exceed selected values. Hammer characteristics (the weight of dropping parts, their velocity at the beginning of





Concrete type 110	
Actual weight of falling parts, tons	2.0
Velocity at beginning of impact, m/sec	6.5
Weight of anvil and frame, tons	5.5
Thickness of pad under anvil, m	0.40
Volume of foundation concrete, m³	33
Weight of foundation and backfill, tons	111
Foundation contact area, m²	22

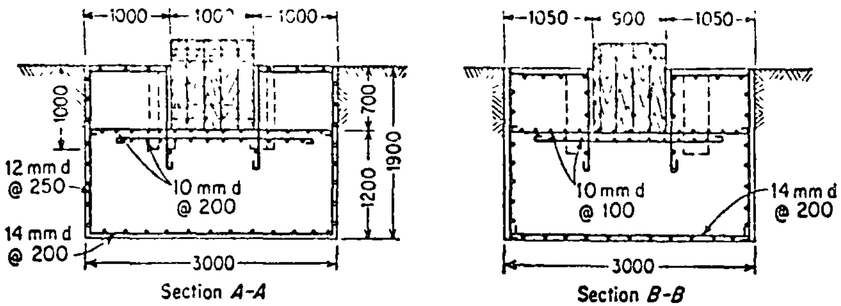
FIG. V-14. Foundation for 2-ton steam or compressed-air stamping hammer.

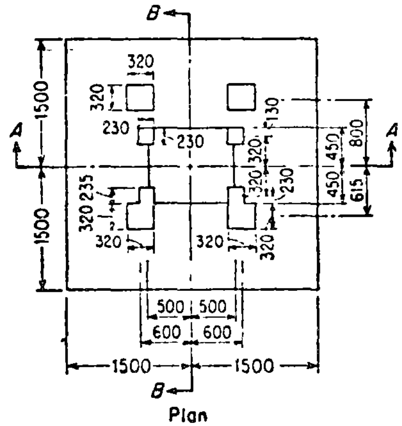
impact, and the coefficient of restitution) are considered to be assigned and fixed.

Let us consider vibrations of the foundation above the springs (the anvil), as a first approximation, to be a system with one degree of freedom. Then, according to Eq. (V-4-7), the amplitude of vibrations of this portion of the foundation equals

$$A_a = \frac{\alpha}{\sqrt{c_2 W_2}} \tag{V-7-1}$$

Equation (V-7-1) was derived from Eq. (V-4-7) as follows, assuming the foundation does not move: the natural frequency f_{na} of the hammer ...





Concrete type 110

Actual weight of falling parts, tons	1.5
Velocity at beginning of impact, m/sec	4.7
Weight of anvil and frame, tons	7.0
Thickness of pad under anvil, m	1.0
Volume of foundation concrete, m³	16 5
Weight of foundation and backfill, tons	36 3
Foundation contact area, m²	9.0

FIG. V-15. Foundation for 1 5-ton cable hammer.

the pad under the anvil is

$$f_{na} = \sqrt{\frac{c_2 g}{W_2}}$$

where c_2 = rigidity of pad
W_2 = weight of hammer

Substituting the above value of f_{na} into Eq. (V-4-7), we obtain

$$A_a = \frac{(1+e)W_0 v}{W_2\sqrt{c_2 g/W_2}} = \frac{(1+e)W_0 v}{\sqrt{g}\sqrt{c_2 W_2}} = \frac{\alpha}{\sqrt{c_2 W_2}}$$

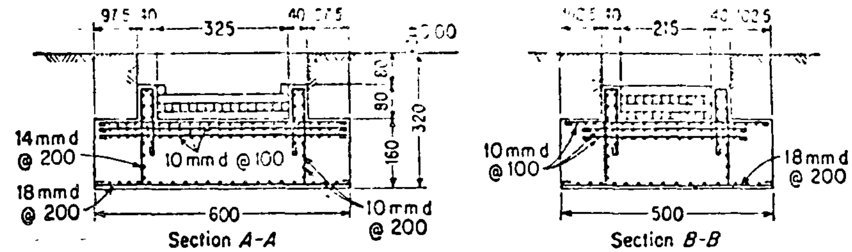
where α is a coefficient depending only on the characteristics of the hammer and equaling

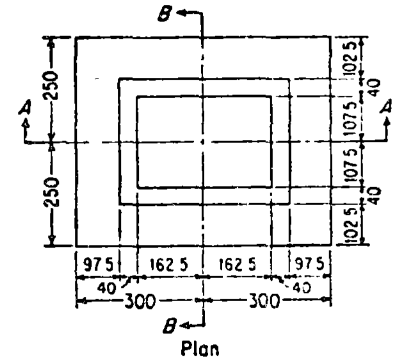
$$\alpha = \frac{(1+e)W_0 v}{\sqrt{g}} \tag{V-7-2}$$

We denote by z_2 the static settlement of the foundation above the springs on absorbers; then

$$z_2 = \frac{W_2}{c_2} \tag{V-7-3}$$

We consider z_2 to be fixed.

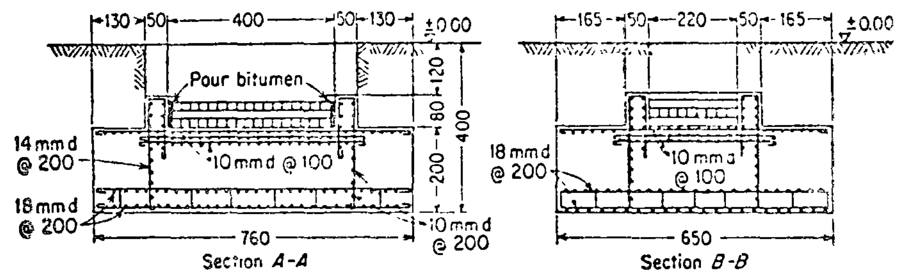


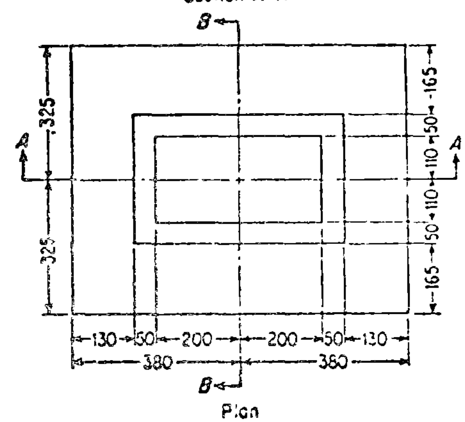


Concrete type 110

Actual weight of falling parts, tons	3 0
Velocity at beginning of impact, m/sec	6.5
Weight of anvil and frame, tons	92 5
Thickness of pad under anvil, m	0.60
Volume of foundation concrete, m³	52.5
Weight of foundation and backfill, tons	201
Foundation contact area, m²	30

FIG. V-16. Foundation for 3-ton steam or compressed-air stamping hammer





Concrete type 110

Actual weight of falling parts, tons	4 2
Velocity at beginning of impact, m/sec	6 5
Weight of anvil and frame, tons	107 5
Thickness of pad under anvil, m	0.60
Volume of foundation concrete, m³	107
Foundation contact area, m²	

FIG. V-17. Foundation for 4 2-ton steam or [illegible]-air stamping hammer

From Eqs. (V-7-1) and (V-7-3) we determine approximate values for the weight of the foundation above the springs and the total coefficient of rigidity of all absorbers:

$$W_2 = \frac{\alpha}{A_a}\sqrt{z_2} \tag{V-7-4}$$

$$c_2 = \frac{W_2}{z_2} \tag{V-7-5}$$

The mass and the area of the foundation under the springs in contact with the soil are selected on the basis of design considerations.

From the selected parameters of the system, the amplitudes of its vibrations are computed taking into account the fact that the system has not one but two degrees of freedom. The design values of vibration amplitudes computed from Eqs. (V-3-9) should not exceed permissible values.

The installation of absorbers should not decrease the efficiency of the hammer, which is

$$\eta = 1 - \frac{W_1 + K_{in}}{W}$$

where W is the work done by the dropping parts of the hammer, equaling

$$W = \frac{W_0 v^2}{2g}$$

W_1 is the energy lost on the rebound of dropping parts:

$$W_1 = \frac{W_0 v_1^2}{2g}$$

where, approximately,

$$v_1 = ev$$

K_{in} is the maximum value of the kinetic energy of vibrations of the anvil on the timber pad or of the foundation above the springs on the absorbers:

$$K_{in} = \frac{A_a^2 c_2}{2}$$

The values W and W_1 do not depend on the characteristics of the anvil and its base; therefore the following condition must be satisfied so that the efficiency of the hammer with absorbers is not less than that of the hammer without absorbers:

$$A_{a0}^2 c_{20} > A_a^2 c_2$$

The subscript 0 refers to the design without absorbers.

Since the amplitudes of foundation vibrations above the springs are about the same whether or not absorbers are used, the last condition may be rewritten as follows:

$$c_2 < c_{20}$$

Example. Computations for a stamping-hammer foundation with absorbers

1. Data. The following specifications are given:
Weight of dropping parts of the hammer:

$$W_0 = 2.0 \text{ tons}$$

Weight of anvil together with frame:

$$W_a = 33.7 \text{ tons}$$

Coefficient of restitution:

$$e = 0.5$$

Velocity of dropping parts:

$$v = 6.0 \text{ m/sec}$$

Design values of permissible amplitudes are as follows:

For the anvil $A_a = 3$ mm
For the foundation: $A_z = 0.2$ mm

2. Computations. The soil is of medium strength, with a coefficient of elastic uniform compression c_u equaling 3.3 kg/cm³. According to data of Art. V-4, the value of the coefficient of rigidity of the base under the hammer foundation will be

$$c_u' = kc_u = 3 \times 3.3 = 10 \text{ kg/cm}^3$$

Let us assume that the static settlement of the mass above the springs equals 0.01 m. We determine the hammer coefficient from Eq. (V-7-2):

$$\alpha = \frac{(1 + 0.5)2 \times 6.0}{\sqrt{9.81}} = 5.75$$

From Eq. (V-7-4) we determine the tentative value of the weight of the mass above the springs:

$$W_2 = \frac{5.75}{3 \times 10^{-3}}\sqrt{10^{-2}} = 191 \text{ tons}$$

The weight of the concrete block of the foundation above the springs, which is added to the weight of the hammer, is.

$$W_{f2} = 191 - 33.7 = 157.3 \text{ tons}$$

From Eq. (V-7-5) we determine the required rigidity of the absorbers:

$$c_2 = 191/10^{-2} = 19{,}100 \text{ tons/m}$$

The foundation above the springs is designed as a block of height 2.3 m and 6.0 by 5.0 m² in plan. It has a depression for the anvil, which is placed not on timber beams but on a pad made from steel wool.

The foundation under the springs is designed in the shape of a box. [illegible] thickness of the protecting walls will be 0.3 m. The cross-sectional dimensions of the columns which support the absorbers will be 0.15 m, the thickness of the supporting slab will be 1.0 m. Thus the area of the foundation beneath the springs in contact with soil equals

$$A = 7.76 \times 6.80 = 52.7 \text{ m}^2$$

Figure V-18 gives a sketch of the foundation with absorbers. The weight of the mass under the springs equals $W_1 = 180$ tons.

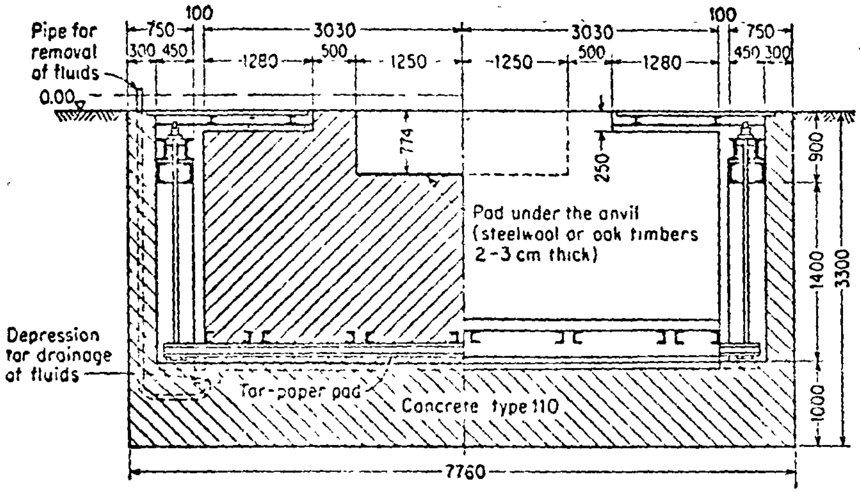


FIG. V-18. Design of foundation for example of Art. V-7.

The foundation both above and beneath the springs is built of properly reinforced concrete type 110.†

Vibrations are computed from Eqs. (V-3-9). The coefficient of rigidity of absorbers is

$$c_2 = 1.9 \times 10^4 \text{ tons/m}$$

The mass above the springs is

$$m_2 = 191/9.81 = 19.5 \text{ tons} \times \text{sec}^2/\text{m}$$

The coefficient of rigidity of the base under the foundation equals

$$c_1 = c_u' A = 10 \times 10^3 \times 52.7 = 52.7 \times 10^4 \text{ tons/m}$$

The mass of the foundation beneath the springs is

$$m_1 = 189/9.81 = 18.1 \text{ tons} \times \text{sec}^2/\text{m}$$

The square of the frequency of natural vertical vibrations of the foundation above the springs is

$$f_a^2 = \frac{c_2}{m_2} = \frac{1.9 \times 10^4}{19.5} = 0.975 \times 10^3 \text{ sec}^{-2}$$

†See footnote, Art. IV-1-c, p 13

The square of the frequency of vibrations of the foundation beneath the springs is

$$f_b^2 = \frac{c_1}{m_1 + m_2} = \frac{52.7 \times 10^4}{18.1 + 19.5} = 13.4 \times 10^3 \text{ sec}^{-2}$$

The ratio between masses is

$$\mu_1 = \frac{19.5}{18.4} = 1.06$$

We then set up the frequency Eq (V-3-8):

$$f_n^4 - (0.975 \times 10^3 + 13.4 \times 10^3)(1 + 1.06)f_n^2 + (1 + 1.06) \times 13.4 \times 10^3 \times 0.975 \times 10^3 = 0$$

or

$$f_n^4 - 29.7 \times 10^3 f_n^2 + 27.0 \times 10^6 = 0$$

Solving this equation, we find the natural frequencies of the system.

$$f_{n1}^2 = 29.7 \times 10^3 \text{ sec}^{-2}$$
$$f_{n2}^2 = 0.946 \times 10^3 \text{ sec}^{-2}$$

We then determine the initial velocity of motion of the foundation above the springs

$$v_a = \frac{(1 + e)W_0 v}{W_0 + W_2} = \frac{(1 + 0.5) \times 2.0 \times 6.0}{(2.0 + 191)} = 0.093 \text{ m/sec}$$

The displacements of separate parts of the foundation are found from Eqs. (V-3-9). The displacement of the foundation beneath the springs is

$$z_1 = \frac{(0.975 \times 10^3 - 0.946 \times 10^3)(0.975 \times 10^3 - 29.7 \times 10^3)}{0.975 \times 10^3(29.7 \times 10^3 - 0.942 \times 10^3)} \times 0.093 \left(\frac{\sin f_{n1}t}{172} - \frac{\sin f_{n2}t}{30.8}\right) = -0.0199 \sin f_{n1}t + 0.109 \sin f_{n2}t \quad \text{mm}$$

The displacement of the foundation above the springs is

$$z_2 = \frac{0.093}{29.7 \times 10^3 - 0.946 \times 10^3}\left(\frac{0.975 \times 10^3 - 0.946 \times 10^3}{172} \sin f_{n1}t - \frac{0.975 \times 10^3 - 29.7 \times 10^3}{30.8} \sin f_{n2}t\right) = 0.0007 \sin f_{n1}t + 3.03 \sin f_{n2}t \quad \text{mm}$$

Neglecting terms containing $\sin f_{n1}t$, we obtain for the amplitudes of vibrations

$$A_1 = 0.101 \text{ mm} \qquad A_2 = 3.03 \text{ mm}$$

Thus the selected dimensions of the foundation mass above the springs and the selected value of the coefficient of rigidity of the absorbers lead to an amplitude of vibrations of the foundation above the springs approximately one-half the design value (0.2 mm). The foundation will be practically motionless and no harmful influence will be exercised by the vibrating foundation on structures or on technological processes.

Absorbers are made of cylindrical standard springs used in railway rolling stock. The dimensions of the springs are chosen as follows.

Diameter of the coil D:	80 mm
Diameter of the spring d:	30 mm
Number of coils n.	5.5

If the number of absorbers is n_1 and the number of springs in each absorber is n_2, then the required rigidity of each spring will be

$$c_{sp} = \frac{c_2}{n_1 n_2}$$

On the other hand, as mentioned in Art. IV-6,

$$c_{sp} = 940 \frac{d^4}{D^3 n} 10^3 \quad \text{tons/m}$$

Equating the left-hand parts of the two latter expressions, we obtain

$$n_1 n_2 = \frac{c_2 D^3 n}{940 d^4} 10^{-3}$$

Substituting here the corresponding numerical values as given above,

$$n_1 n_2 = \frac{1.91 \times 10^4 \times 8^3 \times 10^{-6} \times 5.5}{940 \times 3^4 \times 10^{-8}} \times 10^{-3} = 70$$

If each absorber is made of four springs, then the required number of absorbers will be

$$n_1 = \frac{70}{n_2} = \frac{70}{4} = 18$$

Thus the actual number of springs will be 72.

The permissible torsional stress for the spring is

$$T_0 = 40 \times 10^3 \text{ tons/m}^2$$

Then the permissible load on each spring will be

$$P_p = \frac{\pi d^3 T_0}{8D} = \frac{3.14 \times 3^3 \times 10^{-6} \times 40 \times 10^3}{8.8 \times 8 \times 10^{-2}} = 5.3 \text{ tons}$$

The rigidity of one spring equals

$$c_{sp} = \frac{1.91 \times 10^4}{72} = 264 \text{ tons/m}$$

The permissible deflection of one spring is

$$z_0 = \frac{P_0}{c_{sp}} = \frac{5.3}{264} = 0.020 \text{ m} = 20 \text{ mm}$$

The actual deflection will be smaller, namely,

$$z = z_2 + A_2 = 10 + 3.03 = 13.03 \text{ mm}$$

We design the absorber to be of the suspension type (see Art. IV-6). Each absorber, consisting of four springs, is placed in a case made of steel channels welded together. A general view of the absorber as designed is shown in Fig. V-19. The inside dimensions of the case 5, within which the springs are placed, are 248 by 248 mm. The case is fastened by bolts 14 to the lower supporting plate 2. The lower guide disks 4, for springs 10, are fastened by screws 11 to the same plate. The upper pressure plate 1 is placed on the springs and is also provided with guide disks for the springs. The

mass below the springs may be lifted by tightening the regulating bolt 6 by means of nuts 9; the anchor cap of this bolt fits between two edges of girders built into the lower part of the foundation above the springs.

The walls of case 5 containing the absorbers are fastened on shelves formed by girder pieces embedded into projections of the walls of the foundation below the springs.

The cantilevers of girders embedded in the lower part of the foundation above the springs should be designed in such a way that bending stresses produced in them by the action of the weight and inertia forces of the foundation above the springs do not exceed a maximum value.

b. Construction Procedure for a Foundation with Absorbers. A foundation with absorbers should be constructed as follows:

1. Place the concrete of the foundation under the springs, walls, and projections.
2. Place two or three layers of Ruberoid or tar paper on the surface of the foundation slab.
3. Install girders on the Ruberoid or tar paper, thus forming a lower frame; projecting sills of this frame serve as a support for the anchor plates of regulating bolts of the absorbers. Thoroughly check all required dimensions and positions of girders, then weld the frame. Install the frame in a position corresponding to the design location of absorbers.
4. Install absorbers without tightening the springs; insert caps of regulating bolts between each pair of girders forming the frame.

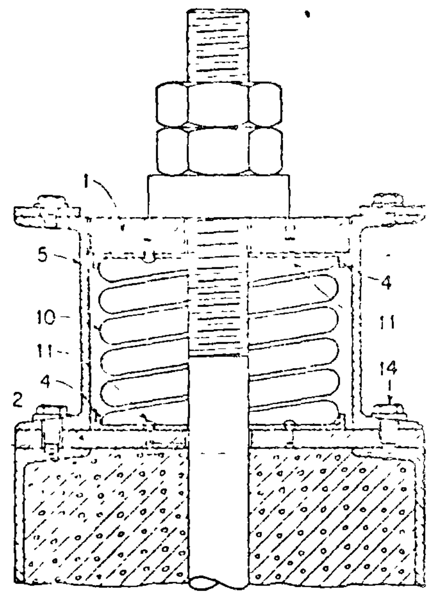


FIG. V-19. Design of vibration absorber for example of Art. V-7.

5. Place the concrete of the foundation above the springs.
6. After the required period of time has elapsed (not less than 10 days), erect the hammer.
7. When the hammer is mounted, lift the foundation above the springs. This is done by gradually tightening the regulating bolts so that the mass above the springs is lifted 1 to 1.5 cm without tilting. A level is used to check that no tilting has occurred.
8. Cover the absorbers and the foundation above the springs with a demountable metal plate.

When the foregoing procedure has been completed, the hammer is ready for operation.

V-8. Pressures in the Base under a Foundation Subjected to Horizontal Impacts

If horizontal impacts are transferred from an operating engine to a massive foundation, natural vibrations of this foundation will develop. When a horizontal impact occurs in one of the vertical principal planes of inertia of the foundation, the equations of this vibration do not differ from Eqs. (III-4-5):

$$m\ddot{x} + c_\tau A(x - L\varphi) = 0$$
$$M_m\ddot{\varphi} - c_\tau ALx + (c_\varphi I - WL + c_\tau AL^2)\varphi = 0 \qquad \text{(V-8-1)}$$

where x = projection on a horizontal axis of displacement of center of mass of foundation
φ = angle of rotation of foundation with respect to axis passing through foundation mass center perpendicular to axis of vibrations
m = mass of foundation and engine
W = weight of foundation and engine
c_τ, c_φ = coefficients of elastic shear, elastic nonuniform compression
A = foundation area in contact with soil
I = moment of inertia of foundation area in contact with soil, with respect to axis passing through its centroid and perpendicular to plane of vibrations
M_m = moment of inertia of mass of foundation and engine with respect to axis passing through center of mass
L = distance between center of mass and foundation base

Solutions of Eq. (V-8-1) should satisfy the initial conditions; when $t = 0$,

$$x = \varphi = 0 \qquad \dot{x} = \dot{x}_0 \qquad \dot{\varphi} = \dot{\varphi}_0$$

where $\dot{x}_0$ and $\dot{\varphi}_0$ are respectively the initial velocities of forward motion in the horizontal direction and of rotation around a horizontal axis passing through the center of mass of the system. They are established from Eqs. (V-2-6) and (V-2-7).

Solutions of Eq. (V-8-1) which correspond to these initial conditions are as follows:

$$x = \frac{1}{f_{n1}^2 - f_{n2}^2}\left(\frac{f_{nx}^2\dot{x}_{0c} - f_{n2}^2\dot{x}_0}{f_{n1}}\sin f_{n1}t - \frac{f_{nx}^2\dot{x}_{0c} - f_{n1}^2\dot{x}_0}{f_{n2}}\sin f_{n2}t\right)$$
$$\varphi = \frac{1}{f_{nx}^2(f_{n1}^2 - f_{n2}^2)L}\left[\frac{(f_{nx}^2 - f_{n1}^2)(f_{nx}^2\dot{x}_{0c} - f_{n2}^2\dot{x}_0)}{f_{n1}}\sin f_{n1}t - \frac{(f_{nx}^2 - f_{n2}^2)(f_{nx}^2\dot{x}_{0c} - f_{n1}^2\dot{x}_0)}{f_{n2}}\sin f_{n2}t\right] \qquad \text{(V-8-2)}$$

where f_{nx} = frequency of natural vibrations in shear, accompan[ied by] sliding of foundation
f_{n1}, f_{n2} = frequencies of foundation determined from solution of I (III-4-8)
$\dot{x}_{0c}$ = initial velocity of centroid of foundation area in cont[act] with soil

$$\dot{x}_{0c} = \dot{x}_0 - L\dot{\varphi}$$

From these solutions for x and φ it is possible to find the dynam[ic] stresses which develop in the base of a foundation as a result of impact. Dynamic compressive stress near the edge of the foundati[on] contact area will be

$$\sigma_\varphi = -c_\varphi b\varphi \qquad \text{(V-8-}$$

where $2b$ is the length of the foundation contact area.

V-9. Foundations (Bases) for Drop Hammers to Break Scrap Iron

a. Location of Drop Hammer within the Steelworks. Special dr[op] hammers are installed at metallurgical works for breaking up scrap ... pigs, and large blocks. These hammers are distinguished by the gr[eat] kinetic energy of ram impact required to break the scrap. While double-acting 5-ton forge hammers the kinetic energy at the moment [of] impact against the forged piece does not exceed 10 to 12 tons × m, [in] modern powerful drop hammers used in scrap yards the kinetic energy [of] the dropping ram attains 150 tons × m. Therefore these hammers m[ay] become a powerful source of elastic waves spreading through soil; som[e]times they may also have a harmful influence on various technologic[al] processes.

Because of this, scrapyards with drop hammers should be located far as possible from other structures. It will be shown in Chap. VI that the propagation of waves through soil is greatly affected by s[oil] properties; therefore the minimum permissible distance from a shop w[ith] drop hammers will depend on soil conditions. In addition, it is cle[ar] that the greater the kinetic energy of the hammer, the greater the ener[gy] of the waves propagated through soil, and consequently, other conditio[ns] being equal, the larger should be the values of minimum permissib[le] distances between the drop hammer and other structures.

The location of a scrapyard with drop hammers also depends on th[e] character of technological operations in certain structures and on th[e] vibration amplitudes which are permissible in connection with thes[e] operations. It is clear that the distance between a scrapyard with dro[p] hammers and a warehouse can be much smaller than that between scrapyard and a laboratory with precision instruments or a shop w[here] precision machines are operating.

Generally it is not possible to establish by means of computations the dependence of the minimum safe distance from a scrapyard with drop hammers on the three factors indicated above. In each case this problem should be solved on the basis of the following data: (1) results of experimental investigations of wave propagation at the construction site under study; (2) values of permissible vibration amplitudes for local technological processes; (3) data on construction characteristics of structures.

Table V-8 gives data on tentative values of minimum distances depending on power of drop hammers and soil conditions.

TABLE V-8. DATA ON MINIMUM DISTANCES BETWEEN DROP-HAMMER INSTALLATIONS USED TO BREAK UP SCRAP IRON AND OTHER STRUCTURES

Soil conditions	Minimum distances to the drop hammers, m, for ram weights of:		
	Up to 3 tons	3–7 tons	Over 7 tons
Plastic clays, clays with some sand and silt, moist sands	30	50	Over 70
Sands, clays, clays with some sand and silt below ground-water level	30	50	Over 70
Swamp soils	50	80	Over 100
Dry sandy soils, hard clays, clays with sand and silt, loess, loessial soils	30	40	Over 60
Rocks	20	30	Over 50

b. Design of Crushing Platforms under Drop Hammers. Up to the present time, bases under crushing platforms have been designed according to methods which have much in common with methods of design and computation of forge-hammer foundations. However, the energy of the dropping part (ram) of a crushing drop hammer is many times greater than the energy of extremely powerful forge hammers; therefore if forge-hammer foundation requirements are applied to foundations for crushing platforms, the latter turn out to be extremely heavy blocks sometimes weighing more than a thousand tons.

Figure V-20 shows a massive foundation designed for a breaking hammer with a 10-ton ram weight and a drop height of 30 m. The foundation for the crushing platform was designed similarly to foundations and anvils under forge hammers; the only difference is that sand and crushed rock were used as a pad under the metal anvil instead of the oak timber girders generally used in hammer foundations. To reduce the cost of construction, the lower part was made of cyclopean concrete; the upper part is of heavily reinforced concrete type 130.† The total

† See footnote IV-1-c, p. 132

weight of the whole structure reaches 900 tons, the depth of the foundation 9 m, the foundation area in contact with soil 85 m².

Technically and economically, such massive foundations cannot be considered rational. The ram velocity of the hammer just described at the moment of impact is 24.2 m/sec, and the kinetic energy is 295 ton × m. If one considers that the impact occurs not against the scrap lying on the anvil, but directly against the anvil, so that the coefficient of impact velocity restitution is of the order of 0.5, then the foundation should undergo vibrations of an amplitude within the range 5 to 15 mm,

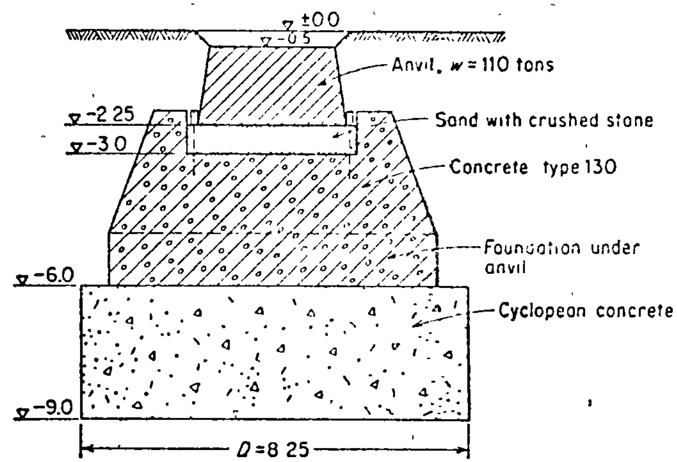


FIG. V-20. A heavy foundation for a scrap-crushing hammer installation

depending on soil conditions. For a soil with a coefficient of elastic uniform compression equaling 5 kg/cm³, the amplitude of foundation vibrations will be 7.5 mm, and the dynamic pressure on soil will be of the order of 4 kg/cm². The impact of a ram weighing several tons dropping from a height of 20 to 30 m will induce large stresses in the anvil and foundation. Therefore the foundation under the anvil should be made of concrete of better quality and should be thoroughly reinforced. In spite of this, cases have been recorded in which the anvil and the portion of the foundation under the anvil were destroyed in the operation of drop hammers breaking up scrap.

Foundations for crushing platforms can also be designed as hollow cylinders made of reinforced concrete and filled with sand and small scrap. Figure V-21 shows a sketch of this type of base. In order to increase the efficiency of the whole installation, the largest possible degree of compaction should be achieved in filling up the cylinder. Sand may be used for filling; compaction can be accomplished by means of vibrators applied to successive layers, each layer about 0.5 m thick. The sand is covered by a layer of broken scrap to a thickness of 1.5 to 2 m and mixed with the sand which has been subjected to vibration. To protect against flying chips, joists are suspended on hinges from a metallic ring installed above the cylinder and are tied to each other by a rope.

If the walls of the cylinder are sufficiently high, they may screen waves propagated inside of the cylinder and thus may prevent the propagation

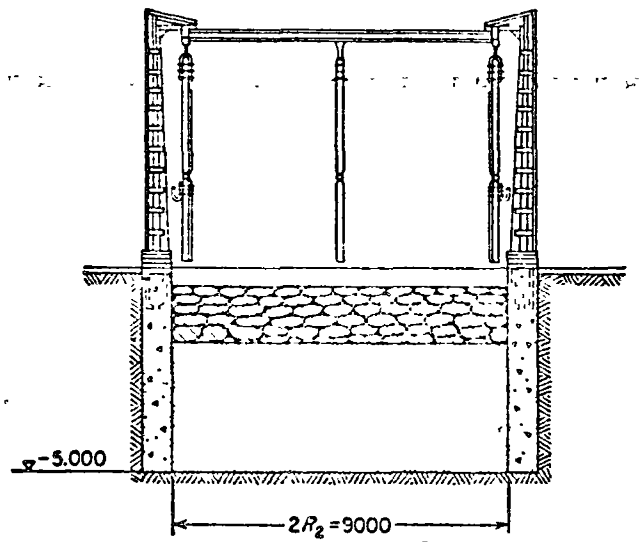


FIG. V-21. Design of foundation for hammer to break up pig-iron scrap.

of waves in the soil beyond it. Therefore a cylindrical foundation may be especially useful if the soils are dangerous in regard to the spreading of vibrations and settlement under the action of vibratory loads.

The larger the depth of the cylinder, the larger its effect on the screening of waves. The frequency of waves induced by the ram impact is smaller in loose soils than in dense soils; therefore, other conditions being equal, cylinder depth should be larger in poor soils than in strong soils. Waves propagated in soil under the action of an impact may be classified as of high frequency, since the number of such waves is of the order of 1,000 min^{-1} and more. As we shall see in Chap. VIII, the dimensions of a screening device should be selected according to the frequency of the propagating waves. For waves of 1,000 cycles/min and more, the depth of the screen in soils of medium strength should not be less than 5 to 6 m.

There is no known accurate stress analysis of a hollow cylinder, filled with a material whose strength properties are other than its own, subjected to the action of elastic waves propagated inside of the cylinder.

Therefore the stress analysis of the cylinder is necessarily limited by a very rough approximation of the real distribution of stresses in its walls.

Let us determine approximately the amplitude of vibrations of the ram which is dropped on the broken scrap; the equation of its vibrations will be

$$\ddot{z} + 2c\dot{z} + f_{nz}^2 z = 0 \qquad \text{(V-9-1)}$$

where c = damping constant; its value for the case under consideration may be approximately taken to equal 0.5 to 0.7 f_{nz}

f_{nz} = frequency of natural vertical vibrations of ram, equaling

$$f_{nz} = \frac{c_u A g}{W}$$

c_u = coefficient of elastic uniform compression of base subjected to impacts; its value may be approximately taken to be of the order of 3 to 5 $\times$ 10^3 kg/cm^3

A = base area of ram

W = weight of ram

After impact, for a certain time (equal to one-fourth the period of its natural vibrations) the ram will be pressed into the scrap. Its velocity at the beginning of impact will be

$$v = \sqrt{2gh} \qquad \text{(V-9-2)}$$

where h is drop height.

Let us take as the start of readings the instant at which the ram touches the scrap. The solution of Eq. (V-9-1) for the time $0 < t < T/4$ (where T is the period of natural vibrations of the ram on the scrap) will be

$$z = A_z \sin f_{nz1} t \qquad \text{(V-9-3)}$$

where A_z is the maximum penetration of the ram into the scrap.

$$A_z = \frac{v}{f_{nz}} \exp\left(\frac{-\pi c}{2f_{nz}}\right) \qquad \text{(V-9-4)}$$

$$f_{nz1} = \sqrt{f_{nz}^2 - c^2} \cong 0.7 f_{nz}$$

Impact of the ram against the scrap will induce an elastic wave spreading from the point of impact over the volume contained by the cylinder. In the first approximation this wave may be considered to be a spherical three-dimensional wave. After the wave reaches the cylinder walls, it will exert a pressure on them, inducing stresses therein.

Neglecting the absorption of wave energy by the medium filling the cylinder, it can be approximately estimated that the wave amplitudes decrease in an inverse proportion to distance from their source.

If A_s is the amplitude of the source acting on the surface in an area of radius R_0, then the amplitude A_r of soil at a distance r from the source may be approximately taken as

$$A_r = A_s \frac{R_0}{r} \tag{V-9-5}$$

The amplitude of the wave component perpendicular to the cylinder wall is

$$A' = A_r \frac{R_2}{\sqrt{R_2^2 + \xi^2}} \tag{V-9-6}$$

where R_2 = inside radius of hollow cylinder
ξ = depth of element of cylinder wall under consideration

Using Eqs. (V-9-4) and (V-9-5) and substituting $\sqrt{R_2^2 + \xi^2}$ for r, we obtain

$$A = \frac{v_0}{f_{nz}} \frac{R_0 R_2}{R_2^2 + \xi^2} \exp\left(-\frac{\pi c}{2 f_{nz}}\right) \tag{V-9-7}$$

When computing A we neglected the absorption of vibrations by the mass inside of the cylinder. Besides, it was assumed that all the impact energy is consumed only by the formation of elastic waves. As a matter of fact, a considerable part of the energy is spent in breaking up the iron blocks and on vibrations of the cylinder, together with the mass it contains, acting as a solid body on an elastic base. Therefore the assumption is possible that the values of A established by Eq. (V-9-7) are larger than actual. However, taking into account that the dynamic wave propagating in the mass contained by the cylinder exerts a dynamic pressure on the cylinder walls, it is possible, in static computations of strength, to evaluate stresses with sufficient accuracy by using the amplitudes established from Eq. (V-9-7).

It follows from the condition stipulating continuous contact between the mass included in the cylinder and the cylinder walls that the amplitudes A computed for the soil may be taken as equaling the amplitudes of elastic expansion of the cylinder walls. Stresses in the material of the cylinder should be established from these amplitudes.

The elastic impact wave is not propagated instantaneously in the mass inside the cylinder, but with a certain finite velocity of the order of 2,000 to 2,500 m/sec. First it exerts a pressure on the elements of the wall situated at the level of the impact. As the wave travels in a downward direction, it exerts a pressure on the lower elements of the cylinder. Consequently, under the action of the moving wave, the cylinder is subjected to a nonuniform pressure along its height which results in a

bending of the cylinder walls as shown by the dashed line in Fig. V-22. It is very difficult to take into account stresses which develop in the cylinder walls as a result of this bending, but it is necessary to provide longitudinal reinforcement along the inner and outer faces of the cylinder (for example, rods of 16 mm diameter spaced every 0.30 to 0.40 m).

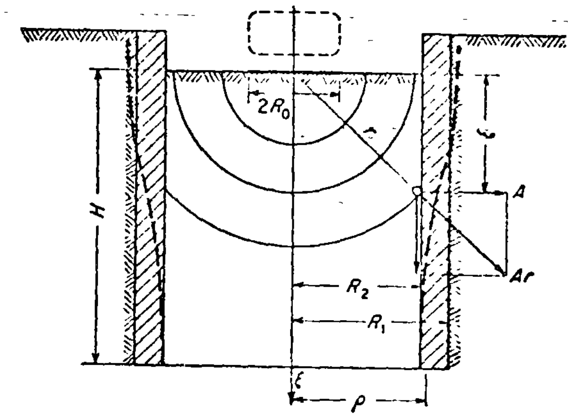


FIG. V-22. Estimation of stresses in cylindrical walls of foundation in Fig. V-21.

For computation of radial and tangential stresses in the cylinder walls, formulas for thick-walled cylinders may be used.† For radial stresses:

$$\sigma_r = \frac{R_2^2}{R_1^2 - R_2^2}\left(1 - \frac{R_1^2}{\rho^2}\right) q \tag{V-9-8}$$

For tangential stresses:

$$\sigma_\varphi = -\frac{R_2^2}{R_1^2 - R_2^2}\left(1 + \frac{R_1^2}{\rho^2}\right) q \tag{V-9-9}$$

where R_1 is the outside radius of the cylinder and

$$R_2 < \rho < R_1$$

The magnitude of internal pressure acting on a cylinder ring at a depth ξ below the level of impact is determined by the formula

$$q = -\frac{EA}{R_2[(R_1^2 + R_2^2)/(R_1^2 - R_2^2) + \nu]} \tag{V-9-10}$$

where ν and E are the Poisson ratio and the modulus of elasticity.

† Cf., for example, S. P. Timoshenko and J. M. Lessels, *Applied Elasticity*, Westinghouse Technical Night School Press, East Pittsburgh, Pa., 1925 (Translated into German and Russian.)

Example *Dynamic computations for a cylindrical base under a crushing platform*

1. GIVEN DATA. A scrap hammer has the following dimensions:

Weight of ram breaking up scrap:

$$W = 5 \text{ tons}$$

Area of the ram base:

$$A = 1 \text{ m}^2$$

Reduced radius of ram base area:

$$R_0 = 0.56 \text{ m}$$

Radii of the cylinder: internal:

$$R_2 = 4.5 \text{ m}$$

external:

$$R_1 = 4.8 \text{ m}$$

2. ASSUMED DATA

Coefficient of elastic uniform compression of scrap:

$$c_u = 3 \times 10^3 \text{ kg/cm}^3 = 3 \times 10^6 \text{ tons/m}^3$$

Modulus of elasticity of concrete:

$$E = 3 \times 10^6 \text{ tons/m}^2$$

The Poisson ratio for concrete:

$$\nu = 0.35$$

Damping constant for vibrations of the ram on scrap:

$$c = 0.7 f_{nz}$$

3. CALCULATIONS. We determine the frequency of natural vibrations of the ram:

$$f_{nz} = \sqrt{\frac{3 \times 10^6 \times 1 \times 9.81}{5}} = 2.44 \times 10^3 \text{ sec}^{-1}$$

From Eq. (V-9-2) we determine the velocity of ram motion at the instant of impact against scrap:

$$v_0 = 2 \times 9.81 \times 15 = 17.1 \text{ m/sec}$$

From Eq (V-9-7) we find the amplitude of the normal component of displacement caused by the wave propagating in scrap as a result of impact. The computation is performed for the highest stressed upper zone of the cylinder, i.e., where $\xi = 0$.

$$A = \frac{17.1}{2.44 \times 10^3} \frac{0.56}{4.5} \exp\left(-\frac{0.7\pi}{2}\right) = 0.29 \times 10^{-3} = 0.29 \text{ mm}$$

From Eq. (V-9-10) we find the value of dynamic pressure of the wave on the upper zone of the cylinder.

$$q = \frac{-3 \times 10^6 \times 0.29 \times 10^{-3}}{4.5(4.8^2 + 4.5^2)/(4.8^2 - 4.5^2) + 0.35} = -12.9 \text{ tons/m}^2 = -1.3 \text{ kg/cm}^2$$

Assuming in Eq. (V-9-9) that $\rho = R_2$, we find the maximum value of the tangential stresses in the cylinder wall:

$$\sigma_\theta = \frac{[illegible] \times 2}{[illegible] - 4.5^2} 12.9 = 172 \text{ tons/m}^2 = 17.2 \text{ kg/cm}^2$$

An adequate reinforcement should be installed to resist these tensile stresses. Since the latter decrease along the depth of the cylinder, the lower zones must be reinforced less intensively than the upper zones. From computations performed on the basis of the above formulas, it is easy to see that in a cylinder having, for example, a depth of 5 m, the lower zone will be under the action of tensile stresses which are approximately two times smaller than those acting on the upper zone.

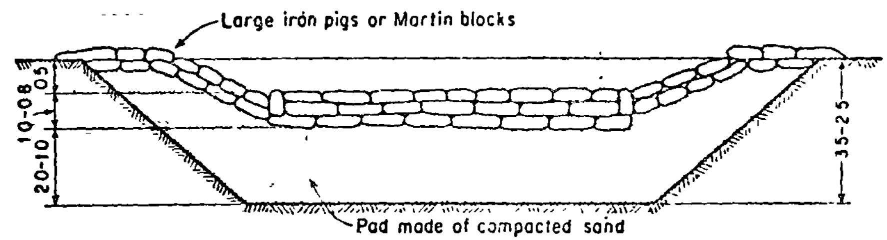


FIG. V-23. Crushing platform on good soil at a considerable distance from building

If soils are relatively strong and no buildings or shops with technological processes which may be affected by vibrations are located nearby, the base of the crushing platform may be made "without a foundation." It may be formed by iron blocks and scrap placed directly on soil or on a layer of a compacted sand, as shown in Fig. V-23.

Criterios de Diseño para Cimentaciones de Maquinaria

José A. NIETO y Daniel RESENDIZ

RESUMEN

Se propone un método racional para el diseño de cimentaciones de máquinas de baja velocidad de operación. El método propuesto se basa en el estudio del comportamiento dinámico de un modelo matemático de un grado de libertad amortiguado linealmente. Se detallan procedimientos para determinar los parámetros del modelo mediante pruebas de campo y de laboratorio y se dan los lineamientos para determinar las solicitaciones que actúan sobre el modelo. Finalmente se presenta un ejemplo de aplicación del método propuesto.

SYNOPSIS

A rational method for the design of foundations of low-speed machines is presented. The proposed method is based on the study of the dynamic behavior of a single-degree of dreedom, linearly damped mathematical model. Procedures for determining the parameters of the model are detailed These include field and laboratory tests Guidelines for determining the disturbances acting on the model are given Finally, an example of application of the method is worked out.

1. INTRODUCCION

1.1. *ANTECEDENTES*

El diseño de cimentaciones de maquinaria pesada de cualquier tipo es un problema sumamente complejo. Aun en los países más industrializados se acostumbra diseñar estas cimentaciones por medio de *recetas* más o menos empíricas desarrolladas localmente y, por tanto, aplicables solamente a las características de la maquinaria y del suelo para las que fueron deducidas.

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Daniel Reséndiz, se tituló de Ingeniero Civil en 1959 en la Facultad de Ingeniería, UNAM, y de Doctor en Ingeniería en 1965 en la División de Estudios Superiores de la propia Facultad, en la actualidad es Consultor para el Programa Latinoamericano de la Fundación Ford e Investigador del Instituto de Ingeniería.

Está por demás decir que estas recetas (del tipo: *tantos metros cúbicos de concreto para cada kilowatt de potencia de la maquina*) además de antieconómicas pueden llevar y han llevado a fracasos lamentables En el Instituto de Ingeniería se han efectuado con anterioridad estudios bibliográficos amplios relacionados con dinámica de los suelos y con normas para cimentaciones de maquinaria: sin embargo estos estudios no son directamente aplicables al establecimiento de criterios de diseño.

La complejidad del problema que nos ocupa se debe a tres factores fundamentales·

a) Se desconocen las perturbaciones a que va a estar sujeto el sistema cimentación-suelo. Aunque los fabricantes de maquinaria suministran datos referentes a fuerzas y momentos de desbalanceo en sus maquinas, estos datos son puramente analíticos En la practica los valores reales de esas fuerzas y momentos de desbalanceo son mucho más elevados debido a excentricidades accidentales y a la imprecisión propia del acabado de los elementos de la máquina No es raro encontrar momentos de desbalanceo en compresoras de gas que sean de 10 a 30 veces mayores que los teóricos.[4]

b) Se desconocen las características dinámicas y de amortiguamiento de los suelos. Aun con la hipótesis simplificatoria de comportamiento linealmente elástico del suelo hasta determinado nivel de esfuerzos, queda la incertidumbre del módulo de elasticidad y la relación de Poisson aplicables en condiciones dinámicas.[5] Si a esto se añaden los efectos de dispersión de energía vibratoria y amortiguamiento interno del suelo el problema se complica aun mas.

c) Hasta fecha reciente no se tenía una solución analítica del problema de vibración de un bloque rígido de base rectangular desplantado en la superficie de un semiespacio elástico. La solución analítica a este problema[6] da fundamento a la formulacion de un modelo matemático simple, semejante al propuesto para el caso de bases circulares,[3] que permite analizar la cimentación de maquinaria pesada utilizando principios elementales de dinámica.

En nuestro medio no ha dejado de utilizarse lo que podríamos llamar *metodo estático* de diseño de cimentaciones para maquinaria, consistente en incrementar el peso propio de la máquina con un factor de impacto, y diseñar la cimentación sujeta únicamente a la carga estatica incrementada. Sin embargo, aunque de esta manera se logre un diseño que cumpla las condiciones de capacidad de carga y asentamientos permisibles, no es difícil imaginar la posibilidad de que la frecuencia de vibracion correspondiente a la velocidad de operación de la máquina o alguna de sus componentes armónicas coincida con la frecuencia fundamental de vibración del sistema suelo-cimentación produciéndose un fenómeno de resonancia en que las amplitudes de vibración resultante pueden ser intolerables. También se puede visualizar el caso de que la vibración inducida por la máquina produzca modificaciones inadmisibles en el suelo sobre el que descansa la cimentación, tales como densificación de arenas sueltas o remoldeo de arcillas sensitivas. Estas razones, entre otras, resaltan la necesidad de recurrir a un método de diseño de cimentaciones de maquinaria que tome en cuenta la naturaleza eminentemente dinámica del fenómeno. El metodo estático podria utilizarse entonces como un primer tanteo.[7]

Los criterios dinámicos que se utilizan en la actualidad para el diseño de cimentaciones de maquinaria pueden clasificarse en dos grandes grupos dependiendo de que consideren al suelo como una cama de resortes linealmente elasticos y sin masa o como un medio elástico, homogéneo, isótropo y seminfinito. En realidad ninguna de las dos idealizaciones del suelo es rigurosamente correcta. Al considerar al suelo como una cama de resortes, se estarán despreciando fenómenos muy importantes, entre ellos la dispersión de energía en el terreno y la propagación de ondas en su superficie, pero se tiene la ventaja de que una vez determinados los parametros del suelo idealizado, el análisis dinámico del sistema maquina-cimentacion-suelo resulta muy sencillo. Por otra parte, la idealización del suelo como un semiespacio elastico permite considerar los fenómenos mencionados anteriormente pero complica el analisis dinámico. Además se sabe que el suelo no es perfectamente elastico y que, debido a la estratificación, en muchas ocasiones no puede considerarse como un medio seminfinito.

1.2 OBJETO Y ALCANCE

El objetivo principal de este trabajo es presentar los resultados de una investigación realizada por el Instituto de Ingeniería de la UNAM bajo el patrocinio de Petróleos Mexicanos encaminada a establecer lineamientos de diseño para cimentaciones de compresoras de gas natural. Los resultados de esta investigación son también aplicables a otras máquinas cuya cimentación consista esencialmente en un bloque masivo de concreto o mampostería. De acuerdo con el convenio celebrado, el Instituto llevó a cabo pruebas de laboratorio y de campo encaminadas a determinar los valores de los parámetros que intervienen en el problema y estudiar con base en los parametros citados, el comportamiento dinámico de cimentaciones de máquinas instaladas con anterioridad. Estas pruebas, cuyos resultados se dan por separado,[8] permitieron establecer un modelo matemático simple para el diseño de cimentaciones sujetas a cargas dinámicas. El modelo matematico constituye entonces el metodo de aplicación de los criterios de diseño.

En este artículo se presentan unicamente el modelo matemático propuesto y la manera de determinar los parámetros que intervienen en él. La aplicación del modelo a casos particulares se ilustra mediante ejemplos.

1.2.1 *Formulación del problema.*

El diseño de la cimentación de una maquina debe satisfacer los requisitos generales siguientes:

a) Los esfuerzos dinámicos inducidos en la cimentación por la operacion de la maquina en combinación con los esfuerzos debidos a otras fuentes, no deben exceder los límites permisibles para el material que constituye la cimentación.

b) El suelo debe ser capaz de soportar las fuerzas periódicas que se trasmiten a traves de la superficie de contacto o a traves de pilotes en cimentaciones piloteadas sin sufrir asentamientos importantes.

c) El movimiento de la cimentación y del terreno en que descansa para cualquier modo de vibración y cualquier combinación de cargas y velocidades de operacion no debe ser objetable para la máquina misma ni para máquinas, conexiones o estructuras vecinas, ni para las personas que se encuentren en lugares inmediatos.

1.2.2 *Vibraciones objetables.* El cuerpo humano es sumamente sensible a movimientos vibratorios. Las amplitudes de vibración perceptibles por el hombre son solo una fracción de las amplitudes que interfieren con la operación de una máquina o que son objetables para las estructuras civiles.[6] Por consiguiente, en este trabajo se considerará que las vibraciones que no sean perjudiciales a estructuras o a maquinaria en operación son tolerables para las personas aun cuando rebasen los niveles de percepción humana.

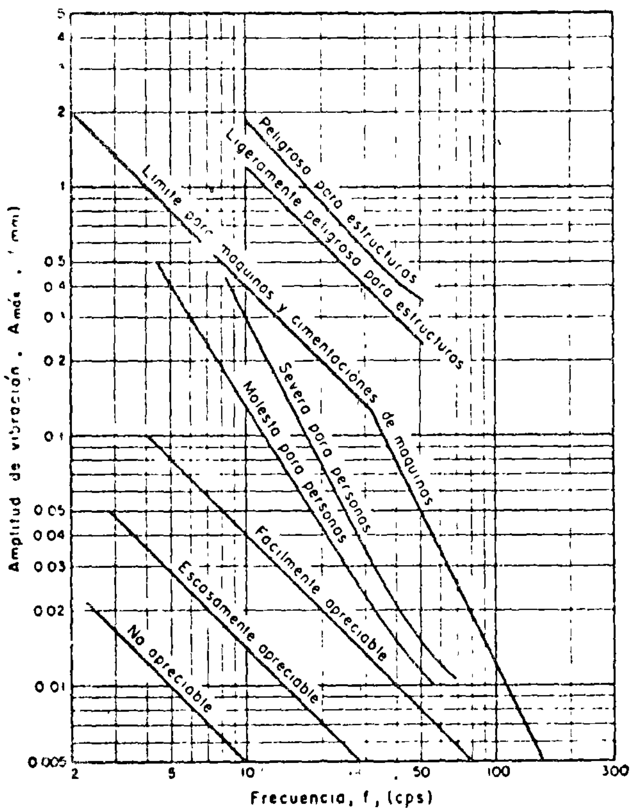

FIG. 1 *Amplitud permisible de vibración vertical en función de la frecuencia.*[10]

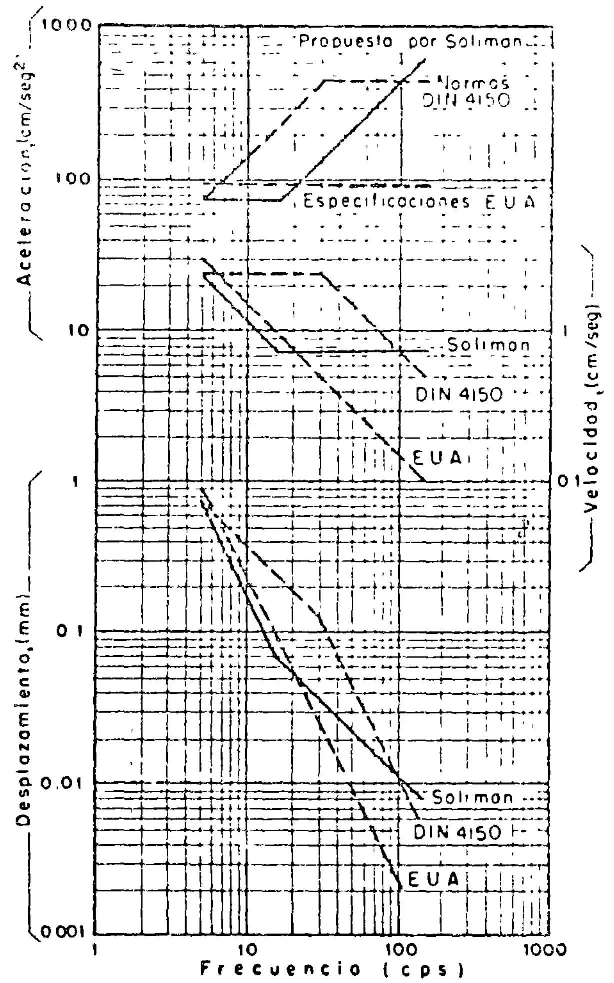

FIG. 2 *Amplitudes permisibles de aceleración, velocidad y desplazamiento vertical en función de la frecuencia.*

En la fig. 1 se establecen los límites de amplitud de desplazamiento vertical admisible, en función de la frecuencia, según la ref. 10. En la fig. 2 se presentan datos semejantes para las amplitudes de aceleración y de velocidad, y se establece una comparación entre normas de diferentes países.[7] Estas dos figuras servirán de base para determinar si el diseño de una cimentación es aceptable o no.

1.2.3 *Método de solución propuesto.* Como consecuencia de los estudios de campo y laboratorio efectuados y para satisfacer los requisitos del subinciso 1.2.1, se propone un método de solución consistente en el análisis dinámico de un modelo matemático que se describe en detalle en la sección 2. Este modelo se basa en la consideración del suelo como un medio elástico homogéneo, isótropo y semiinfinito, pero incluye las ventajas inherentes a la idealización del suelo como una cama de resortes lineales sin masa. La bondad del modelo propuesto, para representar el comportamiento dinámico de cimentaciones de maquinaria existente, se pudo comprobar en la interpretación de las vibraciones registradas en pruebas de campo.[3]

1.3 *DISEÑO PRELIMINAR*

Para poder aplicar el modelo matemático al diseño de una cimentación sujeta a solicitaciones dinámicas es preciso partir de un diseño preliminar. Para el diseño preliminar puede procederse por tanteos, o pueden utilizarse las recomendaciones del fabricante de la máquina, pero en cualquier caso deben satisfacerse los requisitos básicos de la estática relativos a momentos de volteo y los requisitos de la mecánica de suelos referentes a la capacidad de carga en condiciones estáticas y a los asentamientos producidos por cargas estáticas. A menos que se tomen medidas especiales para aumentar la compacidad, se debe evitar cimentar maquinaria en suelos granulares cuya compacidad relativa sea inferior a 90 por ciento.

En general, la cimentación de máquinas reciprocantes y máquinas rotatorias de baja velocidad consiste en un bloque masivo de concreto que para el estudio de las vibraciones puede considerarse infinitamente rígido. Para evitar asentamientos diferenciales y vibración torsional de la cimentación hay que procurar que el centro de gravedad común de la máquina y el bloque de cimentación se encuentre en la vertical del centroide del área de contacto entre el bloque de cimentación y el suelo

El principal requisito que debe satisfacer el diseño preliminar es que la frecuencia natural de vibración vertical del sistema máquina-cimentación-suelo no coincida con la frecuencia de operación de la máquina. En máquinas de baja velocidad (compresoras, generadores diesel, etc.), se recomienda que la frecuencia natural del sistema máquina-cimentación-suelo exceda de una a dos veces la velocidad de operación. La frecuencia natural de vibración vertical del sistema puede determinarse utilizando los datos de la fig 3 En esta figura la frecuencia natural reducida del sistema f_{nr} se determina en función del área de contacto de la cimentación para diferentes tipos de suelo.[11] De la frecuencia natural reducida puede obtenerse la frecuencia natural de vibración mediante la ecuación

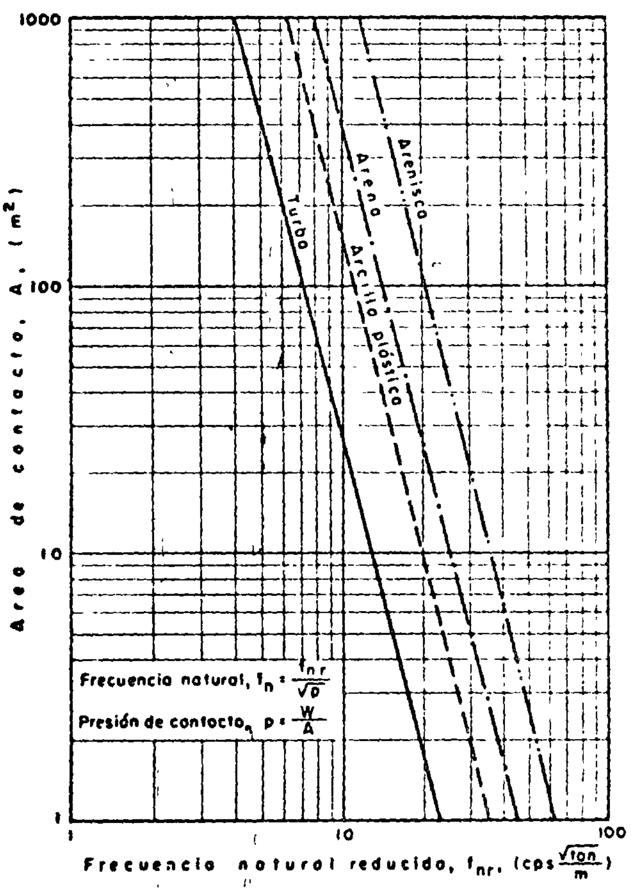

Fig. 3 *Frecuencia natural reducida f_{nr}, en función del área de contacto de la cimentación.*[11]

$$f_n = \frac{f_{nr}}{\sqrt{p}} \qquad (1.1)$$

donde

$$p = \frac{W}{A} = \frac{\text{Peso de la maquina y de la cimentacion (ton)}}{\text{Area de contacto } (m^2)}$$

Como no existen gráficas semejantes para las frecuencias naturales correspondientes a otros modos de vibración, en particular para vibración de cabeceo, se recomienda únicamente minimizar el momento de inercia de masas del bloque de cimentación respecto a los ejes de simetría del área de desplante

La frecuencia natural de vibración vertical de una cimentación apoyada en pilotes puede estimarse a partir de los datos de la fig. 4 tomada de la ref 10.

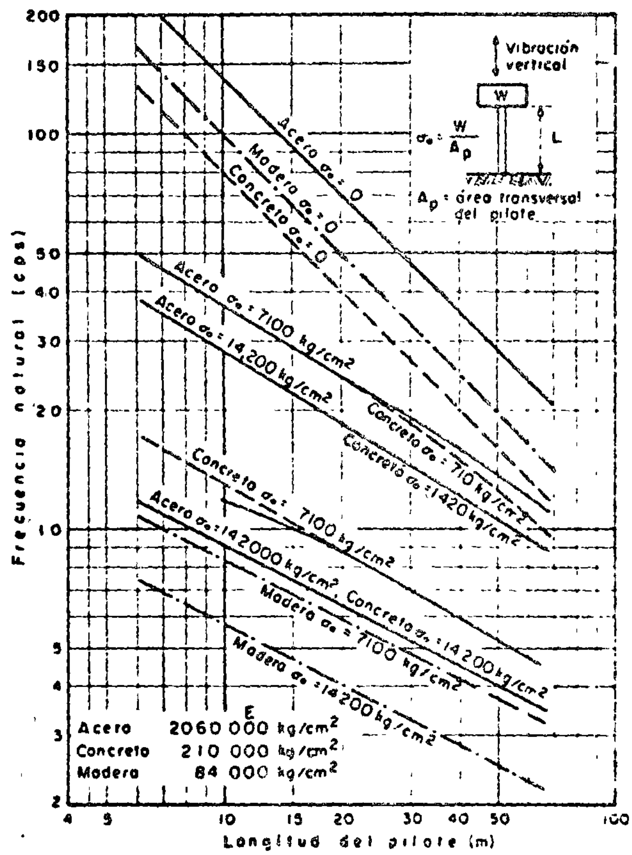

Fig. 4 *Frecuencia natural de vibración vertical en pilotes trabajando por punta sujetos a una carga estática W*[10]

Una vez desarrollado un diseño preliminar se puede proceder a afinarlo utilizando el modelo matemático que se presenta a continuación.

2. MODELO MATEMATICO

2.1 *DESCRIPCION Y JUSTIFICACION DEL MODELO*

El modelo matemático que se propone para analizar el comportamiento dinámico del conjunto maquina-cimentación-suelo es un sistema de un grado de libertad amortiguado linealmente. Consiste en una masa rígida constituida por el conjunto maquina-cimentación y un prisma virtual de suelo, cuya base es idéntica a la de la cimentación, pero cuya altura depende del grado de libertad considerado. La masa rígida está soportada por un elemento flexible, linealmente elástico, sin peso. La forma y colocación del elemento flexible se muestran en la fig. 5 para cada modo de vibración considerado.

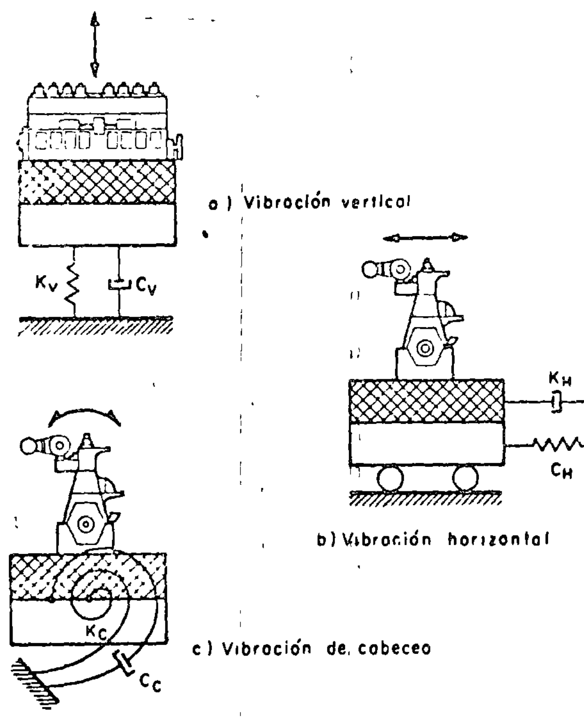

FIG. 5 *Modelos matemáticos propuestos para los tres modos de vibración considerados.*

Aunque el bloque rígido admite seis modos de vibración independientes, en la fig 5 se consideran únicamente tres: vibración vertical, vibración horizontal y vibración de cabeceo respecto a un eje centroidal principal del área de contacto del bloque. Como constante elástica del elemento flexible se utiliza la que suministra la teoría de la elasticidad, al considerar para cada grado de libertad la acción estática de la solicitación aplicada al bloque rígido desplantado en la superficie del semiespacio elástico que representa al suelo. De esta manera se asegura que la solución es correcta cuando la frecuencia de excitación tiende a cero. En la constante elástica se incluye el efecto de la relación de Poisson. En la Tabla 1 se presentan los valores de la constante elástica del elemento flexible del modelo matemático correspondiente a cada modo de vibración. La constante elástica se representa con la letra K y el subíndice v, h o c, según se trate de vibración vertical, horizontal o de cabeceo, respectivamente.

En la Tabla 1, A denota el área de contacto de la base, I_o el momento de inercia del área de contacto respecto al eje de cabeceo, E el módulo de elasticidad y ν la relación de Poisson del medio sobre el que descansa la base.

TABLA 1

CONSTANTES ELÁSTICAS PARA BASES RECTANGULARES

Modo de vibración	*Constante elástica**
Vertical	$K_v = \frac{E}{1-\nu^2} k_v \sqrt{A}$
Horizontal	$K_h = \frac{E}{1-\nu^2} k_h \sqrt{A}$
Cabeceo	$K_h = \frac{E}{1-\nu^2} k_c \frac{I_o}{\sqrt{A}}$

* Los valores de k_v, k_h y k_c se presentan en las Tablas 2, 3 y 4, respectivamente, para algunos valores de la relación largo/ancho de la base. Estos datos fueron tomados de la ref. 12.

TABLA 2

VALORES DEL COEFICIENTE k_v

Relacion largo/ancho	k_v
1.0	1.08
2.0	1.10
3.0	1.15
5.0	1.24
10.0	1.41

TABLA 3

VALORES DEL COEFICIENTE k_h

Desplazamiento horizontal en dirección paralela al lado a

ν	*Relación a/b*						
	0.5	1.0	1.5	2 0	3 0	5.0	10 0
0.1	1.040	1.000	1.010	1 020	1.050	1.150	1 250
0.2	0.990	0 938	0 942	0 945	0 975	1.050	1.160
0.3	0.926	0.868	0.864	0 870	0 906	0.950	1 040
0.4	0 844	0.792	0 770	0 784	0.806	0 850	0 940
0.5	0.770	0 704	0.692	0 686	0.700	0.732	0 940

TABLA 4

VALORES DEL COEFICIENTE k_c

Cabeceo respecto al eje paralelo al lado largo

Relación largo/ancho	k_c
1.0	1.984
1.5	2.254
2.0	2.510
3.0	2.955
5.0	3.700
10.0	4.981

La altura del prisma virtual de suelo y la constante de amortiguamiento para cada grado de libertad se presentan en la Tabla 5. En esta tabla, ρ denota la densidad de masa del suelo.

TABLA 5

PARÁMETROS DEL MODELO

Modo de vibración	*Altura del prisma virtual de suelo*	*Constante de amortiguamiento lineal*
Vertical	$h_v = 0.26\sqrt{A}$	$C_v = 6.7\sqrt{K_v \rho h_v^3}$
Horizontal	$h_h = 0.05\sqrt{A}$	$C_h = 41.1\sqrt{K_h \rho h_h^3}$
Cabeceo	$h_c = 0.35\sqrt{A}$	$C_c = 0.97\sqrt{K_c \rho h_c^3}$

Los valores de los parámetros del modelo propuesto son tentativos. Se obtuvieron igualando la frecuencia y amplitud de resonancia del mismo con las correspondientes a un bloque rígido desplantado en un semiespacio elástico. Como no existe solución cerrada para el caso de un bloque de base rectangular, se utilizaron los resultados obtenidos mediante la discretización mencionada en la ref. 6. A partir de ellos se elaboraron los modelos matemáticos correspondientes a vibración vertical[13] y a vibración de cabeceo[14] de bases rectangulares. Los estudios con modelos físicos que actualmente se efectúan en el Instituto de Ingeniería, pueden conducir a valores mejorados de estos parametros.

Se propone este modelo por la facilidad con que se puede analizar con él el comportamiento de un diseño propuesto sujeto a diferentes tipos de perturbación. Su eficacia se ha verificado al interpretar los resultados de pruebas de campo.

2.2 *RESPUESTA DEL MODELO A PERTURBACIONES EXTERNAS*

No se pretende establecer la solución a las ecuaciones de movimiento de sistemas de un grado de libertad sometidos a excitaciones periodicas. Se presentan únicamente los resultados principales.

El sistema constituido por la maquina y el bloque de cimentación está sujeto esencialmente a las solicitaciones que se muestran en la fig. 6, a saber:

a) Una fuerza vertical, P_z que pasa por el centro del área de contacto con el suelo.

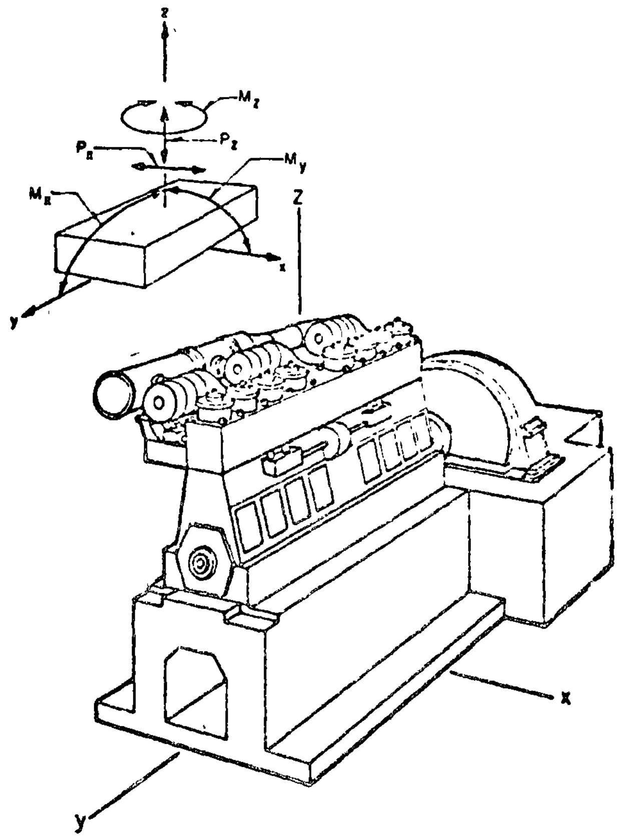

FIG. 6 *Solicitaciones que actúan sobre el sistema máquina-cimentación.*

b) Una fuerza horizontal, P_x, perpendicular a la flecha de la máquina.

c) Un par M_x, contenido en el plano vertical que contiene la flecha de la maquina.

d) Un par M_y, en un plano vertical perpendicular a la flecha de la máquina.

e) Un par M_z, en el plano horizontal.

La determinación de las magnitudes y frecuencias de estas solicitaciones se presentan en el inciso 3.3. Se ha observado que en las maquinas reciprocantes de baja velocidad el par M_z y el par M_y son despreciables, por lo que en el análisis de la respuesta de una cimentación para este tipo de maquinas basta considerar independientemente los grados de libertad siguientes:

1. Vibración vertical
2. Vibración horizontal acoplada con cabeceo alrededor del eje centroidal principal ($y - y$) del área de contacto de la base. (ver fig. 6)

En la mayoria de los problemas de interés práctico solamente las fuerzas y pares primarios cuya frecuencia corresponde a la velocidad de operación de la maquina son suficientemente importantes para justificar su empleo en el analisis de la respuesta de una cimentación. Sin embargo, se recomienda revisar también los efectos de los pares y fuerzas secundarias, cuya frecuencia es el doble de la de operación.

Una vez analizados por separado estos modos de vibración sus efectos se pueden combinar fácilmente como se indica en el subinciso 2 2 2

2 2.1 *Respuesta del modelo a perturbaciones independientes* La amplitud de vibración vertical que se produce en el modelo por la aplicacion de una carga vertical periodica P_z sen ωt está dada por la ecuación

$$A_z = \frac{P_z}{K_v \sqrt{\left[1 - \frac{\omega^2}{\omega_v^2}\right]^2 + \left[2\xi_v \frac{\omega}{\omega_v}\right]^2}} \quad (2.1)$$

donde

$$\omega_v = \sqrt{\frac{K_v}{M_v + M}} \quad (2.2)$$

y

$$\xi_v \doteq \frac{C_v}{2\sqrt{K_v(M_v + M)}} \quad (2.3)$$

En estas expresiones M_v denota la masa del prisma de suelo que se considera vibra verticalmente junto con la cimentación Su valor resulta de multiplicar el área de contacto por la altura h_v y por la densidad de masa del suelo en cuestión.

Cuando en el modelo actua únicamente un par periódico M_y sen ωt contenido en el plano vertical *xz*, la amplitud del desplazamiento angular producido está dada por

$$A_\phi = \frac{M_y}{K_c \sqrt{\left[1 - \frac{\omega^2}{\omega_c^2}\right]^2 + \left[2\xi_c \frac{\omega}{\omega_c}\right]^2}} \quad (2.4)$$

en la que

$$\omega_c = \sqrt{\frac{K_c}{I_c + I}} \quad (2.5)$$

y

$$\xi_c = \frac{C_c}{2\sqrt{K_c(I_c + I)}} \quad (2.6)$$

En estas expresiones I denota el momento de inercia de masa de la cimentación y la máquina respecto al eje de cabeceo ($y - y$), e I_c el momento de inercia de masa del prisma virtual de suelo correspondiente a este modo de vibración respecto al mismo eje. El momento de inercia de masa de este prisma es $^1/_{12}$ del producto de la masa del mismo por la suma de $4h^2_c$ más el cuadrado de la dimensión de la base, perpendicular al eje de cabeceo.

Las amplitudes de los desplazamientos vertical y horizontal en una esquina del bloque de cimentación debidos al cabeceo están dados por

$$A_{v\phi} = \frac{1}{2} c \cdot A_\phi \quad (2.7)$$

y

$$A_{h\phi} = h \cdot A_\phi \quad (2.8)$$

donde c es la dimensión de la base perpendicular al eje de cabeceo, y h la altura del bloque

La amplitud de vibración horizontal A_x, debida a una fuerza periódica P_x sen ωt se determina mediante las ecs 2 1, 2 2 y 2 3 y sustituyendo los indices z, v, por x, h, respectivamente.

2 2.2 *Frecuencias naturales de vibraciones acopladas horizontales y de cabeceo* Como se indicó anteriormente en máquinas en operacion la vibración horizontal se encuentra siempre acoplada con la vibración de cabeceo, puesto que la fuerza horizontal de desbalanceo no está aplicada al nivel de la superficie de contacto entre suelo y cimentación sino a la altura de la flecha de la máquina. En esas condiciones la vibración acoplada tiene dos frecuencias naturales de vibración dadas aproximadamente por

$$\omega_{1,2}^2 = \frac{1}{2\gamma}\left[\omega_c^2 + \omega_h^2 \pm \sqrt{(\omega_c^2 + \omega_h^2)^2 - 4\gamma\omega_c^2\omega_h^2}\right] \quad (2.9)$$

en la que

$$\gamma = \frac{\overline{I}_c + \overline{I}}{I_c + I} \quad (2.10)$$

En la ec 2 10 $\overline{I}_c$ e $\overline{I}$ denotan los momentos de inercia de masa del prisma virtual de suelo y del conjunto máquina-cimentación, respectivamente, respecto al eje paralelo al de cabeceo que pasa por el centro de gravedad del sistema máquina-cimentación-suelo.

El cálculo de los desplazamientos angular y horizontal del bloque de cimentación en condiciones de acoplamiento es bastante complicado por lo que se recurre al procedimiento que se indica a continuación.

2.2 3 *Amplitudes de vibración resultante*. Las amplitudes de desplazamiento vertical y horizontal de una arista de la cara superior del bloque de cimentación paralela al eje de cabeceo se pueden obtener con bastante aproximación mediante el procedimiento siguiente:[15]

a) *Desplazamiento vertical*. Calcúlense independientemente las amplitudes de desplazamiento vertical dadas por las ecs. 2.1 y 2 7. La amplitud resultante estará dada por:

$$A_v = \sqrt{A_z^2 + A_{v\phi}^2} \quad (2.11)$$

b) *Desplazamiento horizontal* Una vez calculadas A_x y $A_{h\phi}$, se tiene:

$$A_h = \sqrt{A_x^2 + A_{h\phi}^2} \quad (2.12)$$

En cualquier caso, ninguna de las frecuencias naturales calculadas mediante las ecs. 2 2 ó 2.9

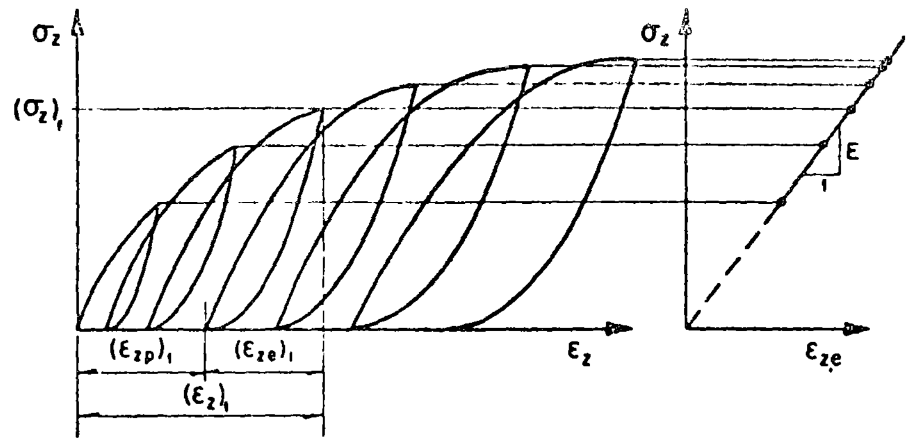

FIG. 7 *Determinación del módulo de deformación recuperable de un suelo*

deberá ser menor que una vez y media la frecuencia correspondiente a la velocidad de operación de la máquina.

3. DETERMINACION DE LOS PARAMETROS QUE INTERVIENEN EN EL MODELO

3.1 *PARAMETROS RELACIONADOS CON EL SUELO*

3.1.1 *Densidad de masa*. Se entiende por densidad de masa de un suelo, ρ, el cociente de su peso volumétrico en estado natural (determinado por cualquiera de los procedimientos usuales), entre la aceleración de la gravedad. Es importante hacer notar que la densidad de masa en problemas de dinámica de suelos no se debe confundir con la densidad de sólidos del suelo que es siempre mayor. En la densidad de masa se toman en consideración los huecos del suelo que pueden contener la fase liquida y la fase gaseosa del mismo.

3.1.2 *Relación de Poisson*. Para la determinación de la relación de Poisson, ν, existen varios procedimientos.[4] Para los fines de este trabajo se recomienda el siguiente:

a) Determinar el módulo de elasticidad, *E*, como se indica en el subinciso siguiente.

b) Realizar una prueba de compresion confinada (bajo condiciones de deformacion lateral nula) con lo que se determina el modulo de deformación confinada $M_c = (\sigma_z / \varepsilon_z)$ donde σ_z denota el esfuerzo vertical y ε_z la deformación unitaria vertical inmediata.

c) Calcular la relación de Poisson mediante la expresión

$$\nu = -\frac{M_c - E}{4M_c} - \sqrt{\left(\frac{M_c - E}{4M_c}\right)^2 + \frac{M_c - E}{2M_c}} \tag{3.1}$$

Si no se dispone de los resultados de una prueba de compresión confinada se pueden utilizar valores de la relación de Poisson comprendidos entre 0.45 y 0.50 para arcillas saturadas y entre 0.30 y 0.35 para arenas.[12b] Mientras más densa es la arena, mayor su relación de Poisson.

3.3.1 *Módulos de elasticidad y rigidez*. Como se indica en la ref. 4, existen por lo menos cuatro definiciones aplicables a la determinación de un módulo de deformación relacionado con la respuesta elástica de los suelos a cargas repetidas. Para fines de diseño de cimentaciones de maquinaria se obtienen buenos resultados utilizando el módulo de deformación recuperable. Este módulo es la pendiente *E* de la curva esfuerzo axial (σ_z) contra deformación axial recuperable (ε_{ze}) determinada como se indica en la fig. 7. El módulo *E* puede obtenerse en el laboratorio a partir de pruebas de compresión triaxial con carga repetida. La deformación axial recuperable resulta de sustraer a la deformación total la deformación remanente en cada ciclo. Se recomienda utilizar el valor medio de *E* determinado mediante pruebas triaxiales con presión confinante similar a la del suelo *in situ*, en probetas inalteradas* del material que se encuentra desde el nivel de desplante de la cimentación hasta una profundidad de una y media veces la dimensión máxima de la base. En suelos con permeabilidad mayor que aproximadamente 10^{-4} cm/seg estas pruebas deben ser no drenadas con presión confinante efectiva al principiar el incremento de ($\sigma_1 - \sigma_3$) igual a la presión confinante *in situ*.

Sin gran error se pueden utilizar los valores de la relación de Poisson recomendados en el subinciso anterior y calcular el modulo de deformación recuperable a partir del coeficiente de compresión elástica uniforme, c_u, determinado mediante pruebas de campo consistentes en carga y descarga de placas rigidas colocadas al nivel de desplante de la

* Tratándose de materiales granulares limpios (gravas y arenas sin finos) en que no es practicable el muestreo inalterado, las pruebas pueden realizarse en especimenes preparados con la relacion de vacios *in situ* a partir de muestras representativas. Si la relacion de vacios in situ no se conoce con precision deben usarse valores extremos e interpolación.

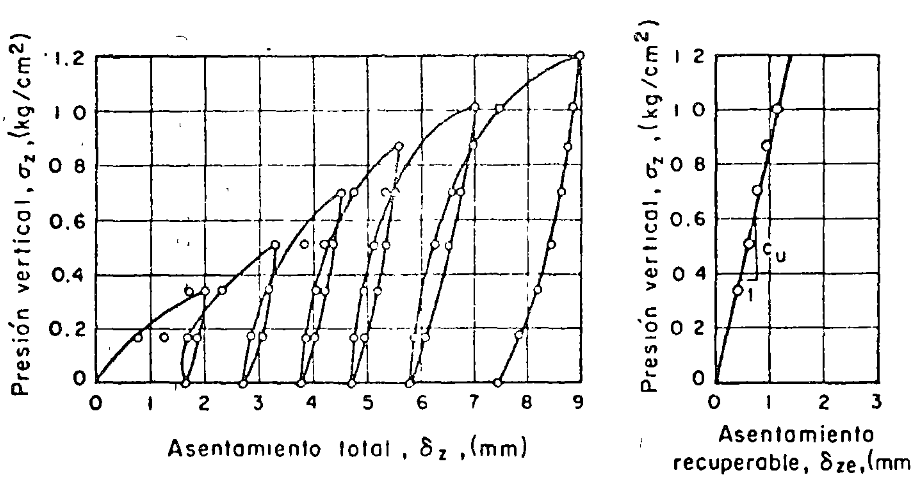

FIG. 8 *Determinación del coeficiente de compresión elástica uniforme," c_u.*

cimentación. Como se indica en la fig. 8 el coeficiente de compresión elástica uniforme es la pendiente de la curva esfuerzo vertical (σ_z) contra asentamiento vertical recuperable (δ_{ze}) Para placas cuadradas de área A, el módulo de deformación recuperable estará dado por

$$E = \frac{(1-\nu^2)\sqrt{A}}{1.13} c_u$$

Como hay cierta discrepancia relativa a la dependencia de c_u del área de la placa, es conveniente utilizar cuando menos dos dimensiones diferentes de placas cuadradas, (por ejemplo 60 × 60 cm y 1.20 × 1 20 m), y extrapolar los resultados.

Tanto en las pruebas de laboratorio como en las pruebas de campo se recomienda que el esfuerzo vertical máximo sea del orden de una y media veces la presión estática que se presentará bajo la cimentación real. Se debe llegar a este esfuerzo en unos diez incrementos de carga y se recomienda reducir los efectos de histéresis aplicando cuando menos cuatro ciclos de carga y descarga total en cada incremento del esfuerzo.

El módulo de rigidez se puede determinar a partir de la expresión

$$G = \frac{E}{2(1+\nu)} \tag{3.3}$$

3.1.4 *Propagación de vibraciones en el suelo* Aunque no está directamente relacionado con el diseño de cimentaciones de maquinaria, es importante tener una idea del amortiguamiento de las ondas vibratorias con la distancia. Esto permite estimar, en forma aproximada los efectos que las vibraciones inducidas por una maquina pueden tener en instalaciones cercanas Segun la ref. 12 si A_o es la amplitud de vibración vertical a la distancia r_o del foco de perturbación, la amplitud A_r a la distancia r está dada por

$$A_r = A_o \sqrt{\frac{r_o}{r}} e^{-\alpha(r-r_o)} \tag{3.4}$$

donde α es el coeficiente de absorción de energia de las ondas y sus unidades son m^{-1} o cm^{-1} En la Tabla 6 se presentan valores de α para algunos tipos de suelo

TABLA 6

VALORES DEL COEFICIENTE DE ABSORCIÓN DE ENERGÍA, α

(Según Barkan[12])

Suelo	α, m^{-1}
Arena fina, saturada	0 100
Arena saturada con capas de turba y limo orgánico	0 040
Arena arcillosa no saturada, interestratificada con arcilla	0.040
Arcilla saturada, con algo de arena y limo	0.040 — 0 120
Caliza marmórea	0.100
Loess	0.100

Como las instalaciones de maquinaria pesada generalmente están lejos de instalaciones en las que puedan producirse daños serios por vibraciones transmitidas a través del suelo, puede afirmarse que si las vibraciones que se producen en la cimentación son tolerables para la máquina misma, no hay que preocuparse por la propagacion de las vibraciones en el terreno.

3 2 *PARAMETROS RELACIONADOS CON LA CIMENTACION*

3 2 1 *Forma y dimensiones.* Como se indicó anteriormente las máquinas de baja velocidad deben ser cimentadas en bloques rigidos de concreto. Estos bloques son generalmente rectangulares. El fabricante de la maquina especifica la colocación en el bloque de los pernos de anclaje y los espacios necesarios para lubricación, paso de ductos y conexiones, etc. Se debe procurar que el centro de gravedad común de la maquina y el bloque de

cimentación se encuentre en la vertical del centroide de la base de contacto entre cimentación y suelo Para reducir la magnitud del par M_v es conveniente que la altura del bloque de cimentación sea la menor posible. Como primer tanteo se puede utilizar una relación largo ancho: alto del orden de 9:3:1. Si después de un diseño preliminar la frecuencia natural de vibración del sistema máquina-cimentación-suelo no es suficientemente grande en comparación con la frecuencia operacional de la máquina, es preciso incrementar dicha frecuencia natural. La forma más sencilla de lograrlo consiste en aumentar el área de contacto de la cimentación y/o reducir la masa de la misma.

La profundidad de desplante del bloque de cimentación carece de importancia por lo que a vibraciones se refiere. En general se acostumbra desplantar a una profundidad del orden de 0 7 a 0.8 veces la altura del bloque. El efecto del confinamiento lateral de las paredes del bloque es despreciable cuando la frecuencia natural de vibración del sistema es mayor que la frecuencia de operación de la máquina.[12]

3.2.2 *Masa e inercia de la cimentación.* Una vez definidas la forma y dimensiones del bloque se puede proceder a determinar su peso, masa y posición de su centro de gravedad Otros datos de interés en el diseño son el momento de inercia de masa del bloque respecto al eje que pasa por su centro de gravedad y es paralelo al eje de cabeceo.

3.3 *PARAMETROS RELACIONADOS CON LA MAQUINA*

3.3.1 *Forma, dimensiones, peso, masas móviles, velocidad de operación.* Todos estos datos son suministrados por el fabricante o pueden determinarse fácilmente con los planos de la máquina Es importante conocer las masas giratorias para determinar las fuerzas y pares de desbalanceo debidos a excentricidades accidentales. También es necesario conocer la velocidad máxima de operación a que puede llegar la máquina en circunstancias especiales, pues se ha observado que en ocasiones las máquinas operan durante lapsos considerables a velocidades superiores a las de diseño

3.3.2 *Excentricidades accidentales tolerables.* Aun en máquinas del mismo tipo y modelo se observan marcadas diferencias en la magnitud de las vibraciones que producen. Esto se debe a variaciones individuales en el acabado de las piezas móviles y de sus apoyos, que originan excentricidades accidentales. A partir de las dimensiones de la máquina es posible calcular las componentes primarias teóricas de las fuerzas y momentos de desbalanceo y sus componentes armónicas Estos valores son suministrados generalmente por el fabricante, pero no se tienen datos relativos a fuerzas y momentos de desbalanceo reales debidos a excentricidades accidentales Para conocer con exactitud las fuerzas y momentos de desbalanceo que existen en determinada máquina sería preciso hacerla funcionar colocada sobre resortes calibrados y medir cuidadosamente las vibraciones producidas.

Como se menciona en la ref 8, se llevó a cabo un estudio comparativo de las vibraciones inducidas por diferentes máquinas desplantadas en diversas condiciones En las máquinas observadas no existían en teoría fuerzas primarias ni pares primarios de desbalanceo Sin embargo, se pudieron medir vibraciones con frecuencia igual a la frecuencia operacional de la máquina, lo que indica la existencia de fuerzas y pares primarios Se observó también que la variación en las amplitudes de vibración medidas en máquinas del mismo tipo desplantadas en el mismo tipo de suelo es de igual orden que la variación observada en máquinas del mismo tipo desplantadas en suelos diferentes Dada la imprecisión en la determinación de las propiedades elásticas del suelo de cada sitio resulta difícil establecer si la causa de la variación en las amplitudes reside en el suelo o en la máquina misma. Más difícil aún resulta establecer valores razonables de las excentricidades *normales*, ya que según se indicó, las fuerzas y momentos de desbalanceo reales pueden exceder de 10 a 30 veces sus valores teóricos. Sin embargo, es lógico suponer que excentricidades excesivas que pudieran presentarse durante la vida útil de la máquina serían prontamente corregidas mediante el balanceo dinámico de la misma. En la ref. 12 se propone la siguiente expresión para la excentricidad probable en condiciones normales de operación.

$$e = \frac{500}{N^2} \tag{3.5}$$

en la que e denota la excentricidad en metros y N la velocidad de operación en revoluciones por minuto. Esta ecuación fue obtenida para turbinas y máquinas de alta velocidad, por lo que su aplicación a máquinas lentas conduce a valores excesivos de las fuerzas y momentos de desbalanceo.

3.3.3 *Fuerzas y momentos de desbalanceo.* Por lo mencionado en el subinciso anterior, se recomienda utilizar valores de las fuerzas y pares de desbalanceo diez veces *mayores* que los valores teóricos suministrados por el fabricante o valores diez veces *menores* que los calculados mediante la excentricidad accidental dada por la ec 3 5.

En la segunda alternativa utilizando como excentricidad $50/N^2$, las fuerzas y momentos de desbalanceo se calculan de la manera siguiente:

a) *Fuerza horizontal* Es el producto de la masa giratoria por la excentricidad y por el cuadrado de la velocidad de operación expresada en radianes por unidad de tiempo.

b) *Fuerza vertical.* Al valor de la fuerza horizontal se suma el producto de la masa giratoria por 1/10 de la aceleración de la gravedad

c) *Par vertical M_v.* Resulta de multiplicar la fuerza horizontal por la altura de la flecha de la máquina respecto a la superficie de desplante de la cimentación

Como se indicó anteriormente los otros momentos debidos a excentricidad accidental pueden considerarse despreciables

3.4 CIMENTACIONES PILOTEADAS

Cuando la capacidad de carga del terreno no permita cimentaciones por superficie o cuando exista el peligro de densificación de suelos granulares por vibración o perdida de resistencia por remoldeo en suelos cohesivos es preciso recurrir a cimentaciones piloteadas. La frecuencia natural de una cimentación piloteada se puede estimar en el diseño preliminar como se indicó en el inciso 1.3

Como en general se desconoce el comportamiento dinámico de cimentaciones piloteadas, para el diseño definitivo de una cimentación de este tipo se parte de las hipótesis siguientes

a) No se transmite ninguna carga por superficie. Es decir, se supone que el bloque de cimentación está desligado del suelo y soportado integramente por los pilotes

b) Para efectos de cargas dinámicas longitudinales en los pilotes (esto es, para vibración vertical y/o vibración de cabeceo) se sustituye cada pilote por un pilote ideal que trabaja únicamente por punta apoyado a la profundidad L_t. Esta profundidad puede determinarse mediante una prueba de cargas repetidas usando la expresión

$$L_t = \frac{AE}{k} \cdot \frac{1}{\mu} \qquad (3.6)$$

donde E es el módulo de elasticidad del material del pilote, A el área de su sección, transformada a un mismo material k la pendiente de la curva carga-deformación total recuperable y μ un coeficiente de corrección cuyos valores se presentan en la Tabla 7 y es función del espaciamiento medio entre los pilotes que soportan la cimentación. La prueba de cargas repetidas debe efectuarse en las condiciones descritas en el inciso 3.1.3 para pruebas de cargas repetidas en placas rigidas. Si no se dispone de los resultados de una prueba de carga se puede sin gran error tomar como longitud efectiva ¾ de la longitud total de pilotes que trabajan por fricción o la longitud de los pilotes si trabajan principalmente por punta.

c) Para cargas dinamicas transversales a los pilotes (esto es para vibracion horizontal), estos se pueden suponer empotrados en ambos extremos con una longitud efectiva L_h. La longitud efectiva puede determinarse mediante una prueba de cargas repetidas horizontales sin permitir giro de la cabeza del pilote, a partir de la expresion

$$L_h = \sqrt[3]{\frac{12AEr^2}{k}} \cdot \frac{1}{\mu} \qquad (3.7)$$

donde E, A y k han sido definidas previamente, y r es el radio de giro de la seccion transformada del pilote respecto a un eje centroidal perpendicular a la dirección del desplazamiento. Si no se dispone de los resultados de una prueba de cargas repetidas, se puede obtener una aproximacion razonable usando para L_h el valor de L_t

TABLA 7

COEFICIENTE DE CORRECCIÓN μ EN FUNCIÓN DEL ESPACIAMIENTO MEDIO Y DEL DIAMETRO DE LOS PILOTES (Según Barkan)[12]

Espaciamiento medio / Diámetro de pilotes	μ
> 6	1.00
6	0.65
4.5	0.64
3	0.41

3.4.1 *Modelo matematico para cimentaciones piloteadas.* Para cimentaciones piloteadas no se requiere considerar masa virtual de suelo vibrando con la cimentacion. Basta agregar a la masa de la cimentación la masa de pilotes correspondiente a la mitad de su longitud efectiva para el tipo de excitación considerado. Como constante de amortiguamiento para cada modo de vibración se debe utilizar la misma que para cimentaciones de superficie. La constante elástica del elemento flexible se determina a partir de la longitud efectiva y de las propiedades y distribución de los pilotes en la forma siguiente

a) Vibración vertical

$$K_r = n\frac{AE}{L_r} \cdot \mu \qquad (3.8)$$

donde n denota el número de pilotes que soportan el bloque de cimentación.

b) Vibración horizontal

$$K_h = n\frac{12}{(L_h/r)^2}\frac{AE}{L_h} \cdot \mu \qquad (3.9)$$

c) Vibración de cabeceo alrededor del eje y

$$K_c = \sum_{i=1}^{n} \frac{AE}{L_i} x_i^2 \cdot \mu \qquad (2.11)$$

en la que x_i es la distancia del i-ésimo pilote al eje de cabeceo.

Utilizando estos valores de los parámetros del modelo la respuesta del mismo a diferentes tipos de perturbación se puede calcular como se indicó en el inciso 2 2.

3.5 *DISEÑO ESTRUCTURAL DE LA CIMENTACION*

En la actualidad no existe un procedimiento riguroso para diseñar un bloque masivo de concreto sujeto al estado de esfuerzos que se presenta en cimentaciones de maquinaria. Sin embargo, los esfuerzos son generalmente pequeños por lo que basta proporcionar al bloque un refuerzo nominal por temperatura y colocar parrillas de acero de refuerzo bajo las concentraciones de carga Para el refuerzo por temperatura se recomienda lo especificado en el inciso XII del articulo 230 del Reglamento de Construcciones para el Distrito Federal.[16] Se aconseja también revisar el diseño por aplastamiento segun lo indicado en el inciso V del articulo 226 del mismo reglamento

En el diseño estructural de los pilotes es preciso tomar en cuenta los efectos de fricción negativa que pudieran presentarse por asentamiento de los estratos. En general se recomienda diseñar estos pilotes como columnas cortas.

4. EJEMPLO DE APLICACION DEL METODO PROPUESTO*

4.1 *DISEÑO PRELIMINAR*

El suelo sobre el que se va a cimentar es una arcilla medianamente compacta Sus caracteristicas pertinentes determinadas mediante pruebas de rutina en el laboratorio y/o pruebas de carga y descarga en placas rígidas son:

$$\gamma_h = 1.5 \text{ ton/m}^3$$
$$\rho = 0.153 \text{ ton-seg}^2\text{/m}^4$$
$$q_u = 10.34 \text{ kg/cm}^2 = 103.4 \text{ ton/m}^2$$
$$E = 1{,}000 \text{ kg/cm}^2 = 10{,}000 \text{ ton/m}^2$$
$$\nu = 0.45$$

Las características de la máquina según datos suministrados por el fabricante son:

Peso total	90 toneladas
Masa total	9.18 ton-seg²/m
Peso de elementos giratorios	16 toneladas
Masa de elementos giratorios	1.63 ton-seg²/m
Velocidad de operación	300 rpm = 5 cps
Altura de la flecha	1 20 m
Dimensiones en planta	7 00 m × 2.50 m
Altura del centro de masa	0.90 m
Momento de inercia de masa respecto a eje centroidal paralelo al eje de cabeceo	3.0 ton-m-seg²

Fuerzas y pares teóricos de desbalanceo

	Componente primaria	Componente secundaria
Fuerza horizontal	0 ton	0 ton
Par horizontal	1 80 ton-m	3 74 ton-m
Fuerza vertical	0 30 ton	0 ton
Par vertical	3.60 ton-m	2 35 ton-m

Las características del bloque de cimentacion propuesto son:

Dimensiones	9 00 m × 3 00 m × 1.00 m
Peso	65 toneladas
Masa	6 63 ton-seg²/m
Momento de inercia minimo del área de la base	20.2 m⁴
Momento de inercia de la masa respecto al eje de cabeceo	7 2 ton-m-seg²

De acuerdo con los datos anteriores se tiene:

Presión estática de contacto	5 74 ton/m²
Presión dinámica de contacto afectando a la máquina de un factor de imparto de 1 5	7 20 ton/m²
Momento de inercia de masa de la máquina respecto al eje de cabeceo	36.1 ton-m-seg²

4.1.1 *Revisión del diseño preliminar*. Afectando la capacidad de carga del suelo con un coeficiente de seguridad de 3 se obtiene una presion de contacto admisible de 34 5 ton/m² bastante mayor que la presión dinámica de contacto, por lo que el diseño es adecuado en lo referente a esfuerzos permisibles. Los asentamientos que se producirán bajo el bloque de cimentación pueden estimarse conociendo los espesores de los estratos compresibles mediante los nomogramas de Newmark.[17]

En la fig 3, para una superficie de contacto de 27 m² y cimentación en arcilla se obtiene una frecuencia natural reducida del orden de 25, así que la frecuencia resonante de vibración vertical del conjunto es 10 4 cps Esta frecuencia es superior al doble de la frecuencia de operación de la máquina, por lo que el diseño preliminar es aceptable en lo referente a frecuencia de resonancia

4 2 *RESPUESTA DEL SISTEMA A PERTURBACIONES EXTERNAS*

4.2.1 *Modelo matematico* De la Tabla 1 para las condiciones del ejemplo se obtienen los siguientes valores de las constantes elasticas del elemento flexible:

$$K_r = 74{,}000 \text{ ton m}$$
$$K_h = 48{,}600 \text{ ton m}$$
$$K_v = 144{,}000 \text{ ton-m}$$

* Los datos utilizados en este ejemplo son imaginarios.

De la Tabla 5 se obtienen los siguientes valores de la altura del prisma virtual de suelo y de la constante de amortiguamiento lineal para cada modo de vibración

$$h_v = 1.35 \text{ m}$$
$$h_h = 0.26 \text{ m}$$
$$h_c = 1.82 \text{ m}$$

$$C_v = 1.126 \text{ ton-seg/m}$$
$$C_h = 473 \text{ ton-seg/m}$$
$$C_c = 643 \text{ ton-m-seg}$$

De la altura del prisma virtual de suelo para cada modo de vibración se obtienen los siguientes valores:

$$M_v = 5.88 \text{ ton-seg}^2/\text{m}$$
$$M_h = 1.07 \text{ ton-seg}^2/\text{m}$$
$$I_\phi = 13.9 \text{ ton-m-seg}^2$$

4.2.2 *Frecuencias naturales del modelo.* Utilizando las ecs. 2.2, 2.3, 2.5 y 2.6 se tiene:

$$\omega_v = 59.2 \text{ rad/seg} = 9.4 \text{ cps}$$
$$\xi_v = 0.445$$
$$\omega_h = 53.7 \text{ rad/seg} = 8.5 \text{ cps}$$
$$\xi_h = 0.259$$
$$\omega_c = 50.5 \text{ rad/seg} = 8.0 \text{ cps}$$
$$\xi_c = 0.112$$

Para determinar con la ec 2.9 las frecuencias de vibración acoplada es preciso conocer la altura del centro de gravedad del sistema máquina-cimentación-suelo. Como el prisma virtual de suelo tiene diferente altura según se trate de vibraciones horizontales o de cabeceo, se recomienda tomar una altura media, que en este ejemplo es 1.04 m. La masa del prisma será 4.29 ton-seg²/m. Entonces, la altura del centro de gravedad general respecto a la superficie de contacto es:

$$\frac{9.18 \times 1.90 + 6.63 \times 0.50 - 4.29 \times 0.52}{9.18 + 6.63 + 4.29} = 0.92 \text{m}$$

Los momentos de inercia de masa del prisma de suelo y del conjunto máquina-cimentación respecto al eje que pasa por el centro de gravedad general y es paralelo al eje de cabeceo son:

$$\bar{I}_c = 12.5 \text{ ton-m-seg}^2$$
$$\bar{I} = 12.2 + 6.7 = 18.7 \text{ ton-m-seg}^2$$

El factor γ de la ec. 2.10 será entonces

$$\gamma = \frac{12.5 + 18.7}{13.9 + 36.1 + 7.2} = \frac{31.2}{57.2} = 0.544$$

y las frecuencias de vibración acoplada serán

$$\omega^2_{1,2} = \frac{5{,}425 \mp 3{,}644}{1.09} = \begin{cases} 1{,}637 \\ 8{,}336 \end{cases}$$

$$\therefore \omega_1 = 40.2 \frac{\text{rad}}{\text{seg}} = 6.4 \text{ cps}$$

$$\omega_2 = 91.3 \frac{\text{rad}}{\text{seg}} = 14.5 \text{ cps}$$

Aunque el diseño preliminar resultó aceptable por lo que toca a resonancia en vibración vertical, para vibraciones acopladas horizontales y de cabeceo la frecuencia resonante mínima es solo ligeramente mayor que la correspondiente a la velocidad de operación de la máquina. En estas circunstancias convendría modificar de inmediato el diseño, aumentando el área de contacto y reduciendo la altura del bloque de cimentación. Sin embargo, para fines de ilustración se calcularan mediante las ecs 2.1, 2.4, 2.7 y 2.8 las amplitudes de desplazamiento resultantes. Si están dentro de los límites tolerables se podrá aceptar el diseño

4.2.3. *Perturbaciones externas* De la ec 3.5 se obtiene una excentricidad probable $e = 5.5$ mm, que es obviamente absurda. Sin embargo, como se recomienda en el inciso 3.3.3, se pueden calcular las fuerzas y momentos de desbalanceo debidos a excentricidades accidentales utilizando un valor de e diez veces menor al obtenido de la ec 3.5. Entonces, las amplitudes de las fuerzas y momentos de desbalanceo son:

$$P_x = 4.98 \text{ ton}$$
$$P_z = 13.98 \text{ ton}$$
$$M_y = 14.94 \text{ ton-m}$$

Los valores de las fuerzas y momentos de desbalanceo que resultan de multiplicar por diez los datos suministrados por el fabricante son:

$$P_x = 0 \text{ ton}$$
$$P_z = 3.00 \text{ ton}$$
$$M_y = 36.00 \text{ ton-m}$$

Obsérvese que las fuerzas y los pares de desbalanceo calculados por ambos metodos son del mismo orden de magnitud. Dada la incertidumbre de los datos tal vez convenga utilizar el promedio de ambos resultados. Las amplitudes de desplazamiento correspondientes a cada modo de vibración independiente son:

$$A_z = 0.13 \text{ mm}$$
$$A_x = 0.07 \text{ mm}$$
$$A_{1\phi} = 0.39 \text{ mm}$$
$$A_{2\phi} = 0.26 \text{ mm}$$

Aplicando las ecs 2.11 y 2.12 se obtienen las amplitudes resultantes:

$$A_v = 0.41 \text{ mm}$$
$$A_h = 0.29 \text{ mm}$$

En la fig 1 puede observarse que estas amplitudes de vibración están dentro de los límites permisibles, por lo que el diseño es satisfactorio

5. RECONOCIMIENTO

Se agradece el patrocinio de Petroleos Mexicanos para llevar a cabo la investigación que condujo a los criterios de diseño objeto de este informe El señor B Martinez Romero efectuó los cálculos incluidos en el ejemplo de aplicación

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presoras de gas que sean de 10 a 30 veces mayores que los teóricos.[4]

b) Se desconocen las características dinámicas y de amortiguamiento de los suelos. Aun con la hipótesis simplificatoria de comportamiento linealmente elástico del suelo hasta determinado nivel de esfuerzos, queda la incertidumbre del módulo de elasticidad y la relación de Poisson aplicables en condiciones dinámicas.[5] Si a esto se añaden los efectos de dispersión de energía vibratoria y amortiguamiento interno del suelo, el problema se complica aun mas.

c) Hasta fecha reciente, no se tenía una solución analítica del problema de vibración de un bloque rígido de base rectangular desplantado en la superficie de un semiespacio elástico. La solución analítica a este problema[5] da fundamento a la formulacion de un modelo matemático simple, semejante al propuesto para el caso de bases circulares,[6] que permite analizar la cimentación de maquinaria pesada utilizando principios elementales de dinámica.

En nuestro medio no ha dejado de utilizarse lo que podríamos llamar *método estático* de diseño de cimentaciones para maquinaria, consistente en incrementar el peso propio de la máquina con un factor de impacto, y diseñar la cimentación sujeta únicamente a la carga estática incrementada. Sin embargo, aunque de esta manera se logre un diseño que cumpla las condiciones de capacidad de carga y asentamientos permisibles, no es difícil imaginar la posibilidad de que la frecuencia de vibracion correspondiente a la velocidad de operación de la máquina o alguna de sus componentes armonicas coincida con la frecuencia fundamental de vibración del sistema suelo-cimentación produciéndose un fenómeno de resonancia en que las amplitudes de vibración resultante pueden ser intolerables. También se puede visualizar el caso de que la vibración inducida por la máquina produzca modificaciones inadmisibles en el suelo sobre el que descansa la cimentación tales como densificacion de arenas sueltas o remoldeo de arcillas sensitivas. Estas razones, entre otras, resaltan la necesidad de recurrir a un método de diseño de cimentaciones de maquinaria que tome en cuenta la naturaleza eminentemente dinámica del fenómeno. El metodo estático podría utilizarse entonces como un primer tanteo.[7]

Los criterios dinámicos que se utilizan en la actualidad para el diseño de cimentaciones de maquinaria pueden clasificarse en dos grandes grupos dependiendo de que consideren al suelo como una cama de resortes linealmente elasticos y sin masa o como un medio elástico, homogeneo, isótropo y seminfinito. En realidad, ninguna de las dos idealizaciones del suelo es rigurosamente correcta. Al considerar al suelo como una cama de resortes se estarán despreciando fenómenos muy importantes, entre ellos la dispersión de energía en el terreno y la propagación de ondas en su superficie, pero se tiene la ventaja de que una vez determinados los parametros del suelo idealizado el analisis dinamico del sistema maquina-cimentacion-suelo resulta muy sencillo. Por otra parte, la idealizacion del suelo como un semiespacio elástico permite considerar los fenómenos mencionados anteriormente pero complica el analisis dinámico. Ademas, se sabe que el suelo no es perfectamente elastico y que, debido a la estratificación, en muchas ocasiones no puede considerarse como un medio seminfinito.

1.2 OBJETO Y ALCANCE

El objetivo principal de este trabajo es presentar los resultados de una investigación realizada por el Instituto de Ingeniería de la UNAM bajo el patrocinio de Petroleos Mexicanos, encaminada a establecer lineamientos de diseño para cimentaciones de compresoras de gas natural. Los resultados de esta investigación son tambien aplicables a otras máquinas cuya cimentación consista esencialmente en un bloque masivo de concreto o mampostería. De acuerdo con el convenio celebrado, el Instituto llevó a cabo pruebas de laboratorio y de campo encaminadas a determinar los valores de los parámetros que intervienen en el problema y estudiar, con base en los parámetros citados, el comportamiento dinámico de cimentaciones de máquinas instaladas con anterioridad. Estas pruebas, cuyos resultados se dan por separado,[8] permitieron establecer un modelo matemático simple para el diseño de cimentaciones sujetas a cargas dinámicas. El modelo matemático constituye entonces el método de aplicación de los criterios de diseño.

En este artículo se presentan únicamente el modelo matemático propuesto y la manera de determinar los parametros que intervienen en el. La aplicación del modelo a casos particulares se ilustra mediante ejemplos.

1.2.1 *Formulación del problema.* El diseño de la cimentación de una maquina debe satisfacer los requisitos generales siguientes:

a) Los esfuerzos dinámicos inducidos en la cimentación por la operacion de la maquina en combinación con los esfuerzos debidos a otras fuentes, no deben exceder los límites permisibles para el material que constituye la cimentación.

b) El suelo debe ser capaz de soportar las fuerzas periódicas que se trasmiten a traves de la superficie de contacto o a traves de pilotes en cimentaciones piloteadas sin sufrir asentamientos importantes.

c) El movimiento de la cimentación y del terreno en que descansa, para cualquier modo de vibración y cualquier combinacion de cargas y velocidades de operacion no debe ser objetable para la máquina misma ni para máquinas, conexiones o estructuras vecinas, ni para las personas que se encuentren en lugares inmediatos.

1.2.2 *Vibraciones objetables*. El cuerpo humano es sumamente sensible a movimientos vibratorios; las amplitudes de vibración perceptibles por el hombre son sólo una fracción de las amplitudes que interfieren con la operación de una máquina o que son objetables para las estructuras civiles.[6] Por consiguiente en este trabajo se considerará que las vibraciones que no sean perjudiciales a estructuras o a maquinaria en operación son tolerables para las personas aun cuando rebasen los niveles de percepción humana.

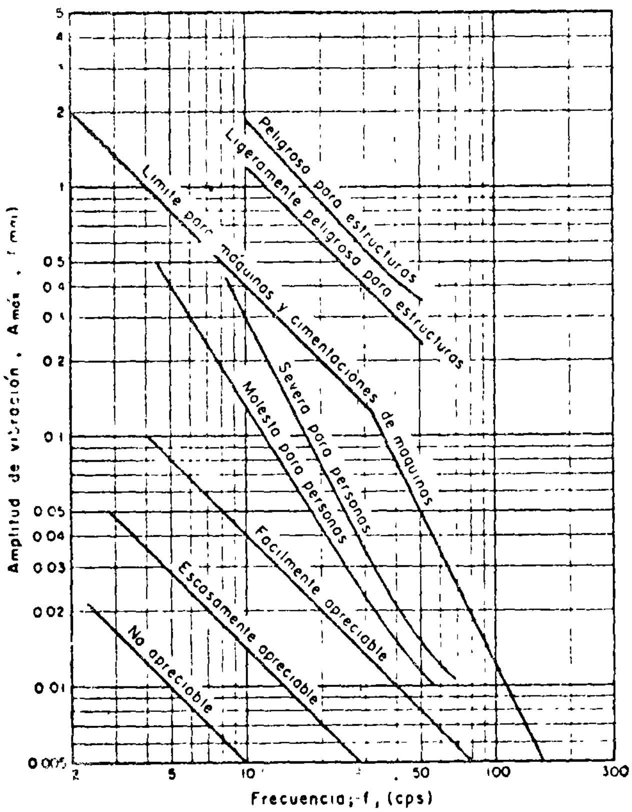

FIG. 1 *Amplitud permisible de vibración vertical en función de la frecuencia.*[10]

En la fig. 1 se establecen los límites de amplitud de desplazamiento vertical admisible, en función de la frecuencia, según la ref. 10. En la fig. 2 se presentan datos semejantes para las amplitudes de aceleración y de velocidad, y se establece una comparación entre normas de diferentes países.[1] Estas dos figuras servirán de base para determinar si el diseño de una cimentación es aceptable o no.

1.2.3 *Método de solución propuesto*. Como consecuencia de los estudios de campo y laboratorio efectuados y para satisfacer los requisitos del subinciso 1.2.1, se propone un método de solución consistente en el análisis dinámico de un modelo matemático que se describe en detalle en la sección 2. Este modelo se basa en la consideración del suelo como un medio elástico, homogéneo, isótropo, y semiinfinito, pero incluye las ventajas inherentes a la idealización del suelo como una cama de resortes lineales sin masa. La bondad del modelo propuesto, para representar el comportamiento dinámico de cimentaciones de maquinaria existente, se pudo comprobar en la interpretación de las vibraciones registradas en pruebas de campo.[5]

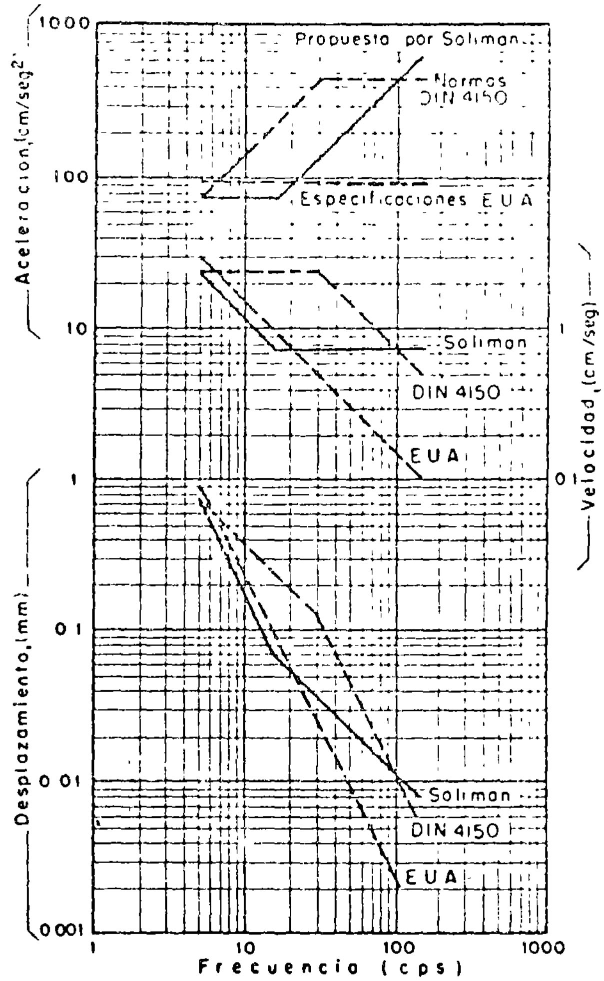

FIG. 2 *Amplitudes permisibles de aceleración, velocidad y desplazamiento vertical en función de la frecuencia.*[1]

1.3 *DISEÑO PRELIMINAR*

Para poder aplicar el modelo matemático al diseño de una cimentación sujeta a solicitaciones dinámicas es preciso partir de un diseño preliminar. Para el diseño preliminar puede procederse por tanteos, o pueden utilizarse las recomendaciones del fabricante de la máquina, pero en cualquier caso deben satisfacerse los requisitos básicos de la estática relativos a momentos de volteo y los requisitos de la mecánica de suelos referentes a la capacidad de carga en condiciones estáticas y a los asentamientos producidos por cargas estáticas. A menos que se tomen medidas especiales para aumentar la compacidad, se debe evitar cimentar maquinaria en suelos granulares cuya compacidad relativa sea inferior a 90 por ciento.

En general, la cimentación de máquinas reciprocantes y máquinas rotatorias de baja velocidad consiste en un bloque masivo de concreto que para el estudio de las vibraciones puede considerarse infinitamente rígido. Para evitar asentamientos diferenciales y vibración torsional de la cimentación hay que procurar que el centro de gravedad común de la máquina y el bloque de cimentación se encuentre en la vertical del centroide del área de contacto entre el bloque de cimentación y el suelo.

El principal requisito que debe satisfacer el diseño preliminar es que la frecuencia natural de vibración vertical del sistema máquina-cimentación-suelo no coincida con la frecuencia de operación de la máquina. En máquinas de baja velocidad (compresoras, generadores diesel, etc.), se recomienda que la frecuencia natural del sistema máquina-cimentación-suelo exceda de una a dos veces la velocidad de operación. La frecuencia natural de

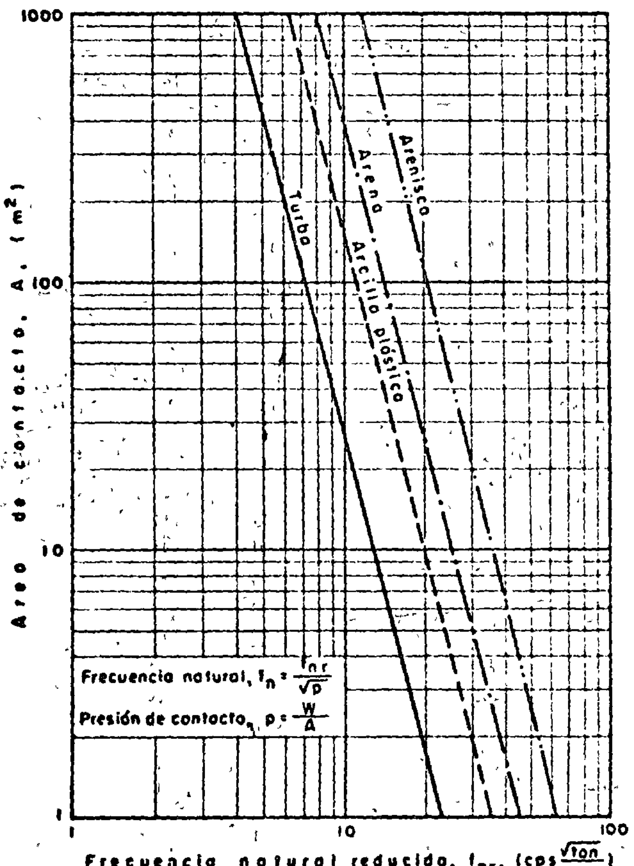

Fig. 3 *Frecuencia natural reducida f_{nr}, en función del área de contacto de la cimentación.*[11]

vibración vertical del sistema puede determinarse utilizando los datos de la fig 3. En esta figura la frecuencia natural reducida del sistema f_{nr}, se determina en función del área de contacto de la cimentación para diferentes tipos de suelo.[11] De la frecuencia natural reducida puede obtenerse la frecuencia natural de vibración mediante la ecuación

$$f_n = \frac{f_{nr}}{\sqrt{p}} \qquad (1.1)$$

donde

$$p = \frac{W}{A} = \frac{\text{Peso de la máquina y de la cimentación (ton)}}{\text{Area de contacto } (m^2)}$$

Como no existen gráficas semejantes para las frecuencias naturales correspondientes a otros modos de vibración, en particular para vibración de cabeceo, se recomienda únicamente minimizar el momento de inercia de masas del bloque de cimentación respecto a los ejes de simetría del área de desplante.

La frecuencia natural de vibración vertical de una cimentación apoyada en pilotes puede estimarse a partir de los datos de la fig. 4 tomada de la ref. 10.

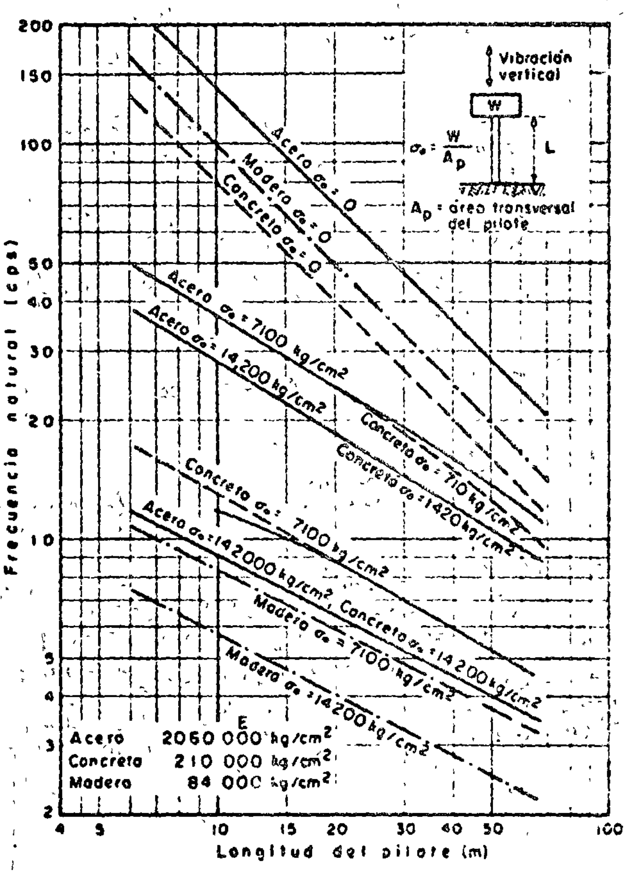

Fig. 4 *Frecuencia natural de vibración vertical en pilotes trabajando por punta sujetos a una carga estática W*[10]

Una vez desarrollado un diseño preliminar se puede proceder a afinarlo utilizando el modelo matemático que se presenta a continuación.

2. MODELO MATEMATICO

2.1 *DESCRIPCION Y JUSTIFICACION DEL MODELO*

El modelo matemático que se propone para analizar el comportamiento dinámico del conjunto maquina-cimentación-suelo es un sistema de un grado de libertad amortiguado linealmente. Consiste en una masa rígida constituida por el conjunto máquina-cimentación y un prisma virtual de suelo, cuya base es idéntica a la de la cimentación, pero cuya altura depende del grado de libertad considerado. La masa rígida está soportada por un elemento flexible, linealmente elástico, sin peso. La forma y colocación del elemento flexible se muestran en la fig. 5 para cada modo de vibración considerado

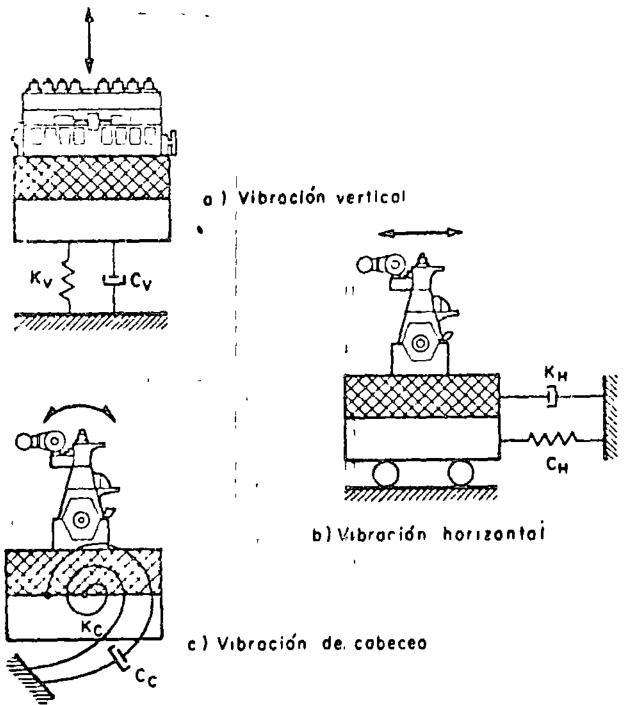

FIG. 5 *Modelos matemáticos propuestos para los tres modos de vibración considerados.*

Aunque el bloque rígido admite seis modos de vibración independientes, en la fig 5 se consideran únicamente tres: vibración vertical, vibración horizontal y vibración de cabeceo respecto a un eje centroidal principal del área de contacto del bloque Como constante elástica del elemento flexible se utiliza la que suministra la teoría de la elasticidad, al considerar para cada grado de libertad la acción estática de la solicitación aplicada al bloque rígido desplantado en la superficie del semiespacio elástico que representa al suelo De esta manera se asegura que la solución es correcta cuando la frecuencia de excitación tiende a cero En la constante elástica se incluye el efecto de la relación de Poisson En la Tabla 1 se presentan los valores de la constante elástica del elemento flexible del modelo matemático correspondiente a cada modo de vibración La constante elástica se representa con la letra K y el subíndice v, h o c, según se trate de vibración vertical, horizontal o de cabeceo, respectivamente.

En la Tabla 1, A denota el área de contacto de la base, I_o el momento de inercia del área de contacto respecto al eje de cabeceo, E el módulo de elasticidad y ν la relación de Poisson del medio sobre el que descansa la base.

TABLA 1

CONSTANTES ELÁSTICAS PARA BASES RECTANGULARES

Modo de vibración	*Constante elástica**
Vertical	$K_v = \frac{E}{1-\nu^2} k_v \sqrt{A}$
Horizontal	$K_h = \frac{E}{1-\nu^2} k_h \sqrt{A}$
Cabeceo	$K_h = \frac{E}{1-\nu^2} k_c \frac{I_o}{\sqrt{A}}$

* Los valores de k_v, k_h y k_c se presentan en las Tablas 2, 3 y 4, respectivamente, para algunos valores de la relación largo/ancho de la base Estos datos fueron tomados de la ref. 12.

TABLA 2

VALORES DEL COEFICIENTE k_v

Relación largo/ancho	k_v
1.0	1.08
2.0	1.10
3.0	1.15
5.0	1.24
10 0	1.41

TABLA 3

VALORES DEL COEFICIENTE k_h

Desplazamiento horizontal en dirección paralela al lado *a*

	Relación a/b						
ν	0.5	1.0	1.5	2.0	3 0	5 0	10 0
0.1	1.040	1.000	1.010	1 020	1 050	1.150	1.250
0.2	0.990	0 938	0 942	0 945	0 975	1.050	1 160
0.3	0.926	0.868	0.864	0 870	0 906	0.950	1 040
0.4	0 844	0.792	0.770	0.784	0.806	0 850	0 940
0.5	0.770	0 704	0.692	0.686	0.700	0.732	0.940

TABLA 4

VALORES DEL COEFICIENTE k_c

Cabeceo respecto al eje paralelo al lado largo

Relación largo/ancho	k_c
1.0	1.984
1.5	2.254
2.0	2.510
3.0	2.955
5.0	3.700
10.0	4.981

La altura del prisma virtual de suelo y la constante de amortiguamiento para cada grado de libertad se presentan en la Tabla 5. En esta tabla, ρ denota la densidad de masa del suelo.

TABLA 5

PARÁMETROS DEL MODELO

Modo de vibración	*Altura del prisma virtual de suelo*	*Constante de amortiguamiento lineal*
Vertical	$h_v = 0.26\sqrt{A}$	$C_v = 6.7\sqrt{K_v \rho h_v^3}$
Horizontal	$h_h = 0.05\sqrt{A}$	$C_h = 41.1\sqrt{K_h \rho h_h^3}$
Cabeceo	$h_c = 0.35\sqrt{A}$	$C_c = 0.97\sqrt{K_c \rho h_c^3}$

Los valores de los parámetros del modelo propuesto son tentativos. Se obtuvieron igualando la frecuencia y amplitud de resonancia del mismo con las correspondientes a un bloque rígido desplantado en un semiespacio elástico. Como no existe solución cerrada para el caso de un bloque de base rectangular, se utilizaron los resultados obtenidos mediante la discretización mencionada en la ref. 6. A partir de ellos se elaboraron los modelos matemáticos correspondientes a vibración vertical[13] y a vibración de cabeceo[11] de bases rectangulares. Los estudios con modelos físicos que actualmente se efectúan en el Instituto de Ingeniería, pueden conducir a valores mejorados de estos parámetros.

Se propone este modelo por la facilidad con que se puede analizar con él el comportamiento de un diseño propuesto sujeto a diferentes tipos de perturbación. Su eficacia se ha verificado al interpretar los resultados de pruebas de campo.

2.2 *RESPUESTA DEL MODELO A PERTURBACIONES EXTERNAS*

No se pretende establecer la solución a las ecuaciones de movimiento de sistemas de un grado de libertad sometidos a excitaciones periódicas. Se presentan únicamente los resultados principales.

El sistema constituido por la maquina y el bloque de cimentación está sujeto esencialmente a las solicitaciones que se muestran en la fig. 6, a saber:

a) Una fuerza vertical, P_z que pasa por el centro del área de contacto con el suelo.

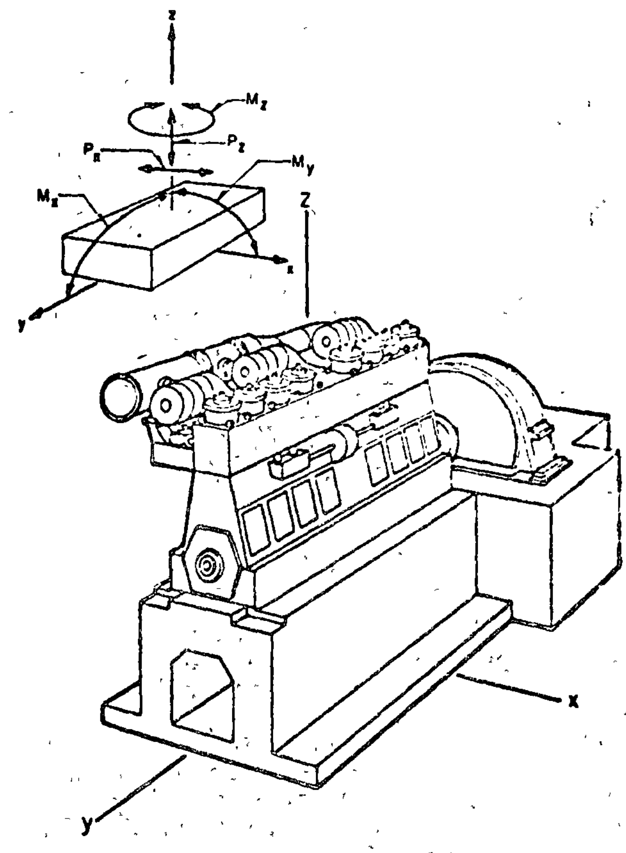

FIG. 6 *Solicitaciones que actúan sobre el sistema máquina-cimentación.*

b) Una fuerza horizontal, P_x, perpendicular a la flecha de la máquina.

c) Un par M_x, contenido en el plano vertical que contiene la flecha de la máquina.

d) Un par M_y, en un plano vertical perpendicular a la flecha de la máquina.

e) Un par M_z, en el plano horizontal.

La determinación de las magnitudes y frecuencias de estas solicitaciones se presentan en el inciso 3.3. Se ha observado que en las máquinas reciprocantes de baja velocidad el par M_z y el par M_x son despreciables, por lo que en el análisis de la respuesta de una cimentación para este tipo de maquinas basta considerar independientemente los grados de libertad siguientes:

1. Vibración vertical
2. Vibración horizontal acoplada con cabeceo alrededor del eje centroidal principal ($y - y$) del área de contacto de la base. (ver fig. 6)

En la mayoría de los problemas de interés práctico solamente las fuerzas y pares primarios cuya frecuencia corresponde a la velocidad de operación de la máquina son suficientemente importantes para justificar su empleo en el análisis de la respuesta de una cimentación. Sin embargo, se recomienda revisar también los efectos de los pares y fuerzas secundarias, cuya frecuencia es el doble de la de operación.

Una vez analizados por separado estos modos de vibración sus efectos se pueden combinar fácilmente como se indica en el subinciso 2 2 2

2 2 1 *Respuesta del modelo a perturbaciones independientes.* La amplitud de vibración vertical que se produce en el modelo por la aplicacion de una carga vertical periódica P_z sen ωt esta dada por la ecuación

$$A_z = \frac{P_z}{K_v \sqrt{\left[1 - \frac{\omega^2}{\omega_v^2}\right]^2 + \left[2\xi_v \frac{\omega}{\omega_v}\right]^2}} \quad (2.1)$$

donde

$$\omega_v = \sqrt{\frac{K_v}{M_v + M}} \quad (2.2)$$

y

$$\xi_v \doteq \frac{C_v}{2\sqrt{K_v (M_v + M)}} \quad (2.3)$$

En estas expresiones M_v denota la masa del prisma de suelo que se considera vibra verticalmente junto con la cimentación Su valor resulta de multiplicar el área de contacto por la altura h_v y por la densidad de masa del suelo en cuestion.

Cuando en el modelo actua únicamente un par periódico M_y sen ωt contenido en el plano vertical xz, la amplitud del desplazamiento angular producido está dada por

$$A_\phi = \frac{M_y}{K_c \sqrt{\left[1 - \frac{\omega^2}{\omega_c^2}\right]^2 + \left[2\xi_c \frac{\omega}{\omega_c}\right]^2}} \quad (2.4)$$

en la que

$$\omega_c = \sqrt{\frac{K_c}{I_c + I}} \quad (2.5)$$

y

$$\xi_c = \frac{C_c}{2\sqrt{K_c (I_c + I)}} \quad (2.6)$$

En estas expresiones I denota el momento de inercia de masa de la cimentación y la maquina respecto al eje de cabeceo $(y - y)$, e I_c el momento de inercia de masa del prisma virtual de suelo correspondiente a este modo de vibración respecto al mismo eje El momento de inercia de masa de este prisma es $1/12$ del producto de la masa del mismo por la suma de $4h_c^2$, más el cuadrado de la dimensión de la base, perpendicular al eje de cabeceo.

Las amplitudes de los desplazamientos vertical y horizontal en una esquina del bloque de cimentación debidos al cabeceo están dados por

$$A_{v\phi} = \frac{1}{2} c \cdot A_\phi \quad (2.7)$$

y

$$A_{h\phi} = h \cdot A_\phi \quad (2.8)$$

donde c es la dimensión de la base perpendicular al eje de cabeceo, y h la altura del bloque

La amplitud de vibración horizontal, A_x, debida a una fuerza periodica P_x sen ωt se determina mediante las ecs 2 1, 2 2 y 2.3 y sustituyendo los indices z, v, por x, h, respectivamente

2 2 2 *Frecuencias naturales de vibraciones acopladas horizontales y de cabeceo* Como se indicó anteriormente en maquinas en operacion la vibración horizontal se encuentra siempre acoplada con la vibración de cabeceo, puesto que la fuerza horizontal de desbalanceo no está aplicada al nivel de la superficie de contacto entre suelo y cimentación sino a la altura de la flecha de la máquina. En esas condiciones la vibracion acoplada tiene dos frecuencias naturales de vibración dadas aproximadamente por

$$\omega_{1,2}^2 = \frac{1}{2\gamma}\left[\omega_c^2 + \omega_h^2 \pm \sqrt{(\omega_c^2 + \omega_h^2)^2 - 4\gamma\omega_c^2\omega_h^2}\right] \quad (2.9)$$

en la que

$$\gamma = \frac{\bar{I}_c + \bar{I}}{I_c + I} \quad (2.10)$$

En la ec. 2 10 $\bar{I}_c$ e $\bar{I}$ denotan los momentos de inercia de masa del prisma virtual de suelo y del conjunto máquina-cimentación, respectivamente, respecto al eje paralelo al de cabeceo que pasa por el centro de gravedad del sistema máquina-cimentación-suelo.

El cálculo de los desplazamientos angular y horizontal del bloque de cimentación en condiciones de acoplamiento es bastante complicado por lo que se recurre al procedimiento que se indica a continuación.

2.2 3 *Amplitudes de vibración resultante.* Las amplitudes de desplazamiento vertical y horizontal de una arista de la cara superior del bloque de cimentación paralela al eje de cabeceo, se pueden obtener con bastante aproximación mediante el procedimiento siguiente:[13]

a) *Desplazamiento vertical.* Calcúlense independientemente las amplitudes de desplazamiento vertical dadas por las ecs. 2.1 y 2 7. La amplitud resultante estará dada por:

$$A_r = \sqrt{A_z^2 + A_{r\phi}^2} \quad (2.11)$$

b) *Desplazamiento horizontal* Una vez calculadas A_x y $A_{h\phi}$, se tiene:

$$A_h = \sqrt{A_x^2 + A_{h\phi}^2} \quad (2.12)$$

En cualquier caso, ninguna de las frecuencias naturales calculadas mediante las ecs. 2 2 ó 2.9

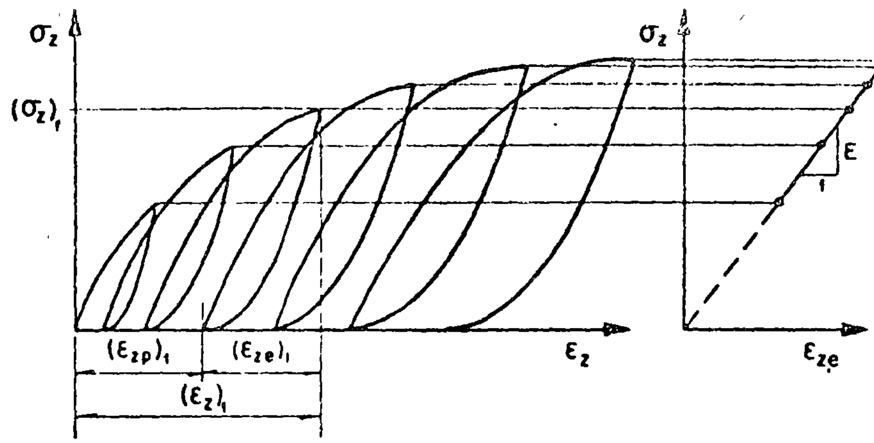

FIG. 7 *Determinación del módulo de deformación recuperable de un suelo.*

deberá ser menor que una vez y media la frecuencia correspondiente a la velocidad de operación de la máquina.

3. DETERMINACION DE LOS PARAMETROS QUE INTERVIENEN EN EL MODELO

3.1 *PARAMETROS RELACIONADOS CON EL SUELO*

3.1.1 *Densidad de masa*. Se entiende por densidad de masa de un suelo, ρ, el cociente de su peso volumétrico en estado natural (determinado por cualquiera de los procedimientos usuales), entre la aceleración de la gravedad. Es importante hacer notar que la densidad de masa en problemas de dinámica de suelos no se debe confundir con la densidad de sólidos del suelo que es siempre mayor. En la densidad de masa se toman en consideración los huecos del suelo que pueden contener la fase liquida y la fase gaseosa del mismo.

3.1.2 *Relación de Poisson*. Para la determinación de la relación de Poisson, ν, existen varios procedimientos.[4] Para los fines de este trabajo se recomienda el siguiente:

a) Determinar el módulo de elasticidad, *E*, como se indica en el subinciso siguiente.

b) Realizar una prueba de compresion confinada (bajo condiciones de deformacion lateral nula) con lo que se determina el modulo de deformación confinada $M_c = (\sigma_z/\varepsilon_z)$ donde σ_z denota el esfuerzo vertical y ε_z la deformación unitaria vertical inmediata

c) Calcular la relación de Poisson mediante la expresión

$$\nu = -\frac{M_c - E}{4M_c} - \sqrt{\left(\frac{M_c - E}{4M_c}\right)^2 + \frac{M_c - E}{2M_c}} \tag{3.1}$$

Si no se dispone de los resultados de una prueba de compresión confinada se pueden utilizar valores de la relación de Poisson comprendidos entre 0.45 y 0.50 para arcillas saturadas y entre 0.30 y 0.35 para arenas.[12b] Mientras más densa es la arena, mayor su relación de Poisson.

3.3.1 *Módulos de elasticidad y rigidez*. Como se indica en la ref. 4, existen por lo menos cuatro definiciones aplicables a la determinación de un módulo de deformación relacionado con la respuesta elástica de los suelos a cargas repetidas. Para fines de diseño de cimentaciones de maquinaria se obtienen buenos resultados utilizando el módulo de deformación recuperable. Este módulo es la pendiente *E* de la curva esfuerzo axial (σ_z) contra deformación axial recuperable (ε_{ze}) determinada como se indica en la fig. 7. El módulo *E* puede obtenerse en el laboratorio a partir de pruebas de compresión triaxial con carga repetida La deformación axial recuperable resulta de sustraer a la deformación total la deformación remanente en cada ciclo. Se recomienda utilizar el valor medio de *E* determinado mediante pruebas triaxiales con presión confinante similar a la del suelo *in situ*, en probetas inalteradas* del material que se encuentra desde el nivel de desplante de la cimentación hasta una profundidad de una y media veces la dimensión máxima de la base En suelos con permeabilidad mayor que aproximadamente 10^{-4} cm/seg estas pruebas deben ser no drenadas con presión confinante efectiva al principiar el incremento de $(\sigma_1 - \sigma_3)$ igual a la presión confinante *in situ*.

Sin gran error se pueden utilizar los valores de la relación de Poisson recomendados en el subinciso anterior y calcular el modulo de deformación recuperable a partir del coeficiente de compresión elástica uniforme, c_u, determinado mediante pruebas de campo consistentes en carga y descarga de placas rígidas colocadas al nivel de desplante de la

* Tratándose de materiales granulares limpios (gravas y arenas sin finos) en que no es practicable el muestreo inalterado, las pruebas pueden realizarse en especímenes preparados con la relación de vacíos *in situ* a partir de muestras representativas Si la relación de vacíos in situ no se conoce con precisión deben usarse valores extremos e interpolación.

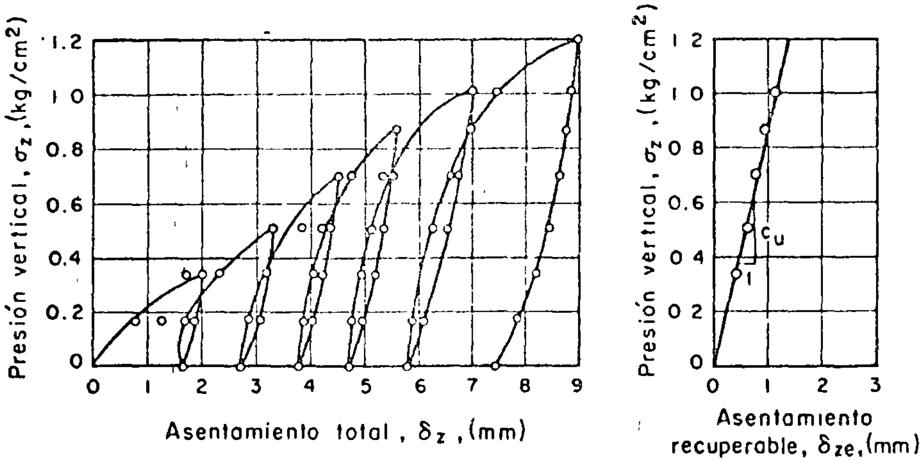

FIG. 8 *Determinación del coeficiente de compresión elástica uniforme,"* c_u.

cimentación. Como se indica en la fig. 8 el coeficiente de compresión elástica uniforme es la pendiente de la curva esfuerzo vertical (σ_z) contra asentamiento vertical recuperable (δ_{ze}). Para placas cuadradas de área A, el módulo de deformación recuperable estará dado por

$$E = \frac{(1 - \nu^2)\sqrt{A}}{1.13} c_u$$

Como hay cierta discrepancia relativa a la dependencia de c_u del área de la placa, es conveniente utilizar cuando menos dos dimensiones diferentes de placas cuadradas, (por ejemplo 60 × 60 cm y 1.20 × 1 20 m), y extrapolar los resultados

Tanto en las pruebas de laboratorio como en las pruebas de campo se recomienda que el esfuerzo vertical máximo sea del orden de una y media veces la presión estatica que se presentará bajo la cimentación real. Se debe llegar a este esfuerzo en unos diez incrementos de carga y se recomienda reducir los efectos de histéresis aplicando cuando menos cuatro ciclos de carga y descarga total en cada incremento del esfuerzo.

El módulo de rigidez se puede determinar a partir de la expresión

$$G = \frac{E}{2(1 + \nu)} \qquad (3.3)$$

3.1.4 *Propagación de vibraciones en el suelo* Aunque no está directamente relacionado con el diseño de cimentaciones de maquinaria es importante tener una idea del amortiguamiento de las ondas vibratorias con la distancia Esto permite estimar, en forma aproximada los efectos que las vibraciones inducidas por una maquina pueden tener en instalaciones cercanas Segun la ref. 12 si A_o es la amplitud de vibración vertical a la distancia r_o del foco de perturbación, la amplitud A_r a la distancia r está dada por

$$A_r = A_o \sqrt{\frac{r_o}{r}}\, e^{-\alpha(r - r_o)} \qquad (3.4)$$

donde α es el coeficiente de absorción de energia de las ondas y sus unidades son m^{-1} o cm^{-1} En la Tabla 6 se presentan valores de α para algunos tipos de suelo.

TABLA 6

VALORES DEL COEFICIENTE DE ABSORCIÓN DE ENERGÍA, α

(Según Barkan[12])

Suelo	α, m^{-1}
Arena fina, saturada	0.100
Arena saturada con capas de turba y limo orgánico	0.040
Arena arcillosa no saturada, interestratificada con arcilla	0.040
Arcilla saturada, con algo de arena y limo	0.040 — 0 120
Caliza marmórea	0.100
Loess	0.100

Como las instalaciones de maquinaria pesada generalmente están lejos de instalaciones en las que puedan producirse daños serios por vibraciones transmitidas a través del suelo, puede afirmarse que si las vibraciones que se producen en la cimentación son tolerables para la máquina misma, no hay que preocuparse por la propagacion de las vibraciones en el terreno.

3 2 *PARAMETROS RELACIONADOS CON LA CIMENTACION*

3 2 1 *Forma y dimensiones.* Como se indicó anteriormente las maquinas de baja velocidad deben ser cimentadas en bloques rigidos de concreto. Estos bloques son generalmente rectangulares. El fabricante de la maquina especifica la colocacion en el bloque de los pernos de anclaje y los espacios necesarios para lubricación, paso de ductos y conexiones, etc. Se debe procurar que el centro de gravedad común de la máquina y el bloque de

cimentación se encuentre en la vertical del centroide de la base de contacto entre cimentación y suelo Para reducir la magnitud del par M_v es conveniente que la altura del bloque de cimentación sea la menor posible Como primer tanteo se puede utilizar una relación largo ancho: alto del orden de 9:3:1. Si después de un diseño preliminar la frecuencia natural de vibración del sistema máquina-cimentación-suelo no es suficientemente grande en comparación con la frecuencia operacional de la máquina, es preciso incrementar dicha frecuencia natural. La forma más sencilla de lograrlo consiste en aumentar el área de contacto de la cimentación y/o reducir la masa de la misma.

La profundidad de desplante del bloque de cimentación carece de importancia por lo que a vibraciones se refiere. En general se acostumbra desplantar a una profundidad del orden de 0 7 a 0.8 veces la altura del bloque El efecto del confinamiento lateral de las paredes del bloque es despreciable cuando la frecuencia natural de vibración del sistema es mayor que la frecuencia de operacion de la máquina.[12]

3.2.2 *Masa e inercia de la cimentación.* Una vez definidas la forma y dimensiones del bloque se puede proceder a determinar su peso, masa y posición de su centro de gravedad Otros datos de interés en el diseño son el momento de inercia de masa del bloque respecto al eje que pasa por su centro de gravedad y es paralelo al eje de cabeceo.

3.3 *PARAMETROS RELACIONADOS CON LA MAQUINA*

3.3.1 *Forma, dimensiones, peso, masas móviles, velocidad de operación* Todos estos datos son suministrados por el fabricante o pueden determinarse fácilmente con los planos de la máquina. Es importante conocer las masas giratorias para determinar las fuerzas y pares de desbalanceo debidos a excentricidades accidentales Tambien es necesario conocer la velocidad maxima de operación a que puede llegar la máquina en circunstancias especiales, pues se ha observado que en ocasiones las máquinas operan durante lapsos considerables a velocidades superiores a las de diseño

3.3.2 *Excentricidades accidentales tolerables.* Aun en máquinas del mismo tipo y modelo se observan marcadas diferencias en la magnitud de las vibraciones que producen. Esto se debe a variaciones individuales en el acabado de las piezas móviles y de sus apoyos, que originan excentricidades accidentales. A partir de las dimensiones de la máquina es posible calcular las componentes primarias teóricas de las fuerzas y momentos de desbalanceo y sus componentes armónicas Estos valores son suministrados generalmente por el fabricante, pero no se tienen datos relativos a fuerzas y momentos de desbalanceo reales debidos a excentricidades accidentales. Para conocer con exactitud las fuerzas y momentos de desbalanceo que existen en determinada máquina sería preciso hacerla funcionar colocada sobre resortes calibrados y medir cuidadosamente las vibraciones producidas.

Como se menciona en la ref 8 se llevó a cabo un estudio comparativo de las vibraciones inducidas por diferentes máquinas desplantadas en diversas condiciones. En las máquinas observadas no existían en teoría fuerzas primarias ni pares primarios de desbalanceo Sin embargo se pudieron medir vibraciones con frecuencia igual a la frecuencia operacional de la máquina lo que indica la existencia de fuerzas y pares primarios Se observó también que la variación en las amplitudes de vibración medidas en máquinas del mismo tipo desplantadas en el mismo tipo de suelo es de igual orden que la variación observada en máquinas del mismo tipo desplantadas en suelos diferentes Dada la imprecisión en la determinación de las propiedades elásticas del suelo de cada sitio resulta difícil establecer si la causa de la variación en las amplitudes reside en el suelo o en la máquina misma. Mas difícil aun resulta establecer valores razonables de las excentricidades *normales*, ya que según se indicó, las fuerzas y momentos de desbalanceo reales pueden exceder de 10 a 30 veces sus valores teóricos. Sin embargo, es lógico suponer que excentricidades excesivas que pudieran presentarse durante la vida util de la máquina serán prontamente corregidas mediante el balanceo dinámico de la misma En la ref 12 se propone la siguiente expresión para la excentricidad probable en condiciones normales de operación.

$$e = \frac{500}{N^2} \tag{3.5}$$

en la que e denota la excentricidad en metros y N la velocidad de operación en revoluciones por minuto. Esta ecuación fue obtenida para turbinas y máquinas de alta velocidad, por lo que su aplicación a máquinas lentas conduce a valores excesivos de las fuerzas y momentos de desbalanceo.

3.3.3 *Fuerzas y momentos de desbalanceo.* Por lo mencionado en el subinciso anterior, se recomienda utilizar valores de las fuerzas y pares de desbalanceo diez veces *mayores* que los valores teóricos suministrados por el fabricante o valores diez veces *menores* que los calculados mediante la excentricidad accidental dada por la ec 3 5

En la segunda alternativa utilizando como excentricidad $50/N^2$, las fuerzas y momentos de desbalanceo se calculan de la manera siguiente:

a) *Fuerza horizontal* Es el producto de la masa giratoria por la excentricidad y por el cuadrado de la velocidad de operacion expresada en radianes por unidad de tiempo.

b) *Fuerza vertical.* Al valor de la fuerza horizontal se suma el producto de la masa giratoria por 1/10 de la aceleración de la gravedad

c) *Par vertical M_y.* Resulta de multiplicar la fuerza horizontal por la altura de la flecha de la máquina respecto a la superficie de desplante de la cimentación.

Como se indicó anteriormente los otros momentos debidos a excentricidad accidental pueden considerarse despreciables.

3.4 CIMENTACIONES PILOTEADAS

Cuando la capacidad de carga del terreno no permita cimentaciones por superficie o cuando exista el peligro de densificación de suelos granulares por vibración o pérdida de resistencia por remoldeo en suelos cohesivos, es preciso recurrir a cimentaciones piloteadas. La frecuencia natural de una cimentación piloteada se puede estimar en el diseño preliminar como se indicó en el inciso 1.3.

Como en general se desconoce el comportamiento dinámico de cimentaciones piloteadas para el diseño definitivo de una cimentación de este tipo se parte de las hipótesis siguientes:

a) No se transmite ninguna carga por superficie. Es decir, se supone que el bloque de cimentación está desligado del suelo y soportado íntegramente por los pilotes.

b) Para efectos de cargas dinámicas longitudinales en los pilotes (esto es, para vibración vertical y/o vibración de cabeceo) se sustituye cada pilote por un pilote ideal que trabaja únicamente por punta apoyado a la profundidad L_v. Esta profundidad puede determinarse mediante una prueba de cargas repetidas usando la expresión

$$L_v = \frac{AE}{k} \cdot \frac{1}{\mu} \qquad (3.6)$$

donde E es el módulo de elasticidad del material del pilote, A el área de su sección transformada a un mismo material, k la pendiente de la curva carga-deformación total recuperable y μ un coeficiente de corrección cuyos valores se presentan en la Tabla 7 y es función del espaciamiento medio entre los pilotes que soportan la cimentación. La prueba de cargas repetidas debe efectuarse en las condiciones descritas en el inciso 3.1.3 para pruebas de cargas repetidas en placas rígidas. Si no se dispone de los resultados de una prueba de carga se puede sin gran error tomar como longitud efectiva 2/3 de la longitud total de pilotes que trabajan por fricción o la longitud de los pilotes si trabajan principalmente por punta.

c) Para cargas dinámicas transversales a los pilotes (esto es para vibración horizontal), estos se pueden suponer empotrados en ambos extremos con una longitud efectiva L_h. La longitud efectiva puede determinarse mediante una prueba de cargas repetidas horizontales sin permitir giro de la cabeza del pilote, a partir de la expresión

$$L_h = \sqrt[3]{\frac{12AEr^2}{k}} \cdot \frac{1}{\mu} \qquad (3.7)$$

donde E, A y k han sido definidas previamente, y r es el radio de giro de la sección transformada del pilote respecto a un eje centroidal perpendicular a la dirección del desplazamiento. Si no se dispone de los resultados de una prueba de cargas repetidas, se puede obtener una aproximación razonable usando para L_h el valor de L_v.

TABLA 7

COEFICIENTE DE CORRECCIÓN μ EN FUNCIÓN DEL ESPACIAMIENTO MEDIO Y DEL DIAMETRO DE LOS PILOTES (Según Barkan)[12]

Espaciamiento medio / Diámetro de pilotes	μ
> 6	1.00
6	0.65
4.5	0.64
3	0.41

3.4.1 *Modelo matemático para cimentaciones piloteadas.* Para cimentaciones piloteadas no se requiere considerar masa virtual de suelo vibrando con la cimentación. Basta agregar a la masa de la cimentación la masa de pilotes correspondiente a la mitad de su longitud efectiva para el tipo de excitación considerado. Como constante de amortiguamiento para cada modo de vibración se debe utilizar la misma que para cimentaciones de superficie. La constante elástica del elemento flexible se determina a partir de la longitud efectiva y de las propiedades y distribución de los pilotes en la forma siguiente:

a) Vibración vertical

$$K_v = n\frac{AE}{L_v} \cdot \mu \qquad (3.8)$$

donde n denota el número de pilotes que soportan el bloque de cimentación.

b) Vibración horizontal

$$K_h = n\frac{12}{(L_h/r)^2}\frac{AE}{L_h} \cdot \mu \qquad (3.9)$$

c) Vibración de cabeceo alrededor del eje y

$$K_c = \sum_{i=1}^{n} \frac{AE}{L_i} x_i^2 \cdot \mu \qquad (2.11)$$

en la que x_i es la distancia del i-ésimo pilote al eje de cabeceo.

Utilizando estos valores de los parámetros del modelo la respuesta del mismo a diferentes tipos de perturbación se puede calcular como se indicó en el inciso 2.2.

3.5 *DISEÑO ESTRUCTURAL DE LA CIMENTACION*

En la actualidad no existe un procedimiento riguroso para diseñar un bloque masivo de concreto sujeto al estado de esfuerzos que se presenta en cimentaciones de maquinaria. Sin embargo, los esfuerzos son generalmente pequeños por lo que basta proporcionar al bloque un refuerzo nominal por temperatura y colocar parrillas de acero de refuerzo bajo las concentraciones de carga. Para el refuerzo por temperatura se recomienda lo especificado en el inciso XII del artículo 230 del Reglamento de Construcciones para el Distrito Federal.[16] Se aconseja también revisar el diseño por aplastamiento según lo indicado en el inciso V del artículo 226 del mismo reglamento.

En el diseño estructural de los pilotes es preciso tomar en cuenta los efectos de fricción negativa que pudieran presentarse por asentamiento de los estratos. En general se recomienda diseñar estos pilotes como columnas cortas.

4. EJEMPLO DE APLICACION DEL METODO PROPUESTO*

4.1 *DISEÑO PRELIMINAR*

El suelo sobre el que se va a cimentar es una arcilla medianamente compacta. Sus características pertinentes determinadas mediante pruebas de rutina en el laboratorio y/o pruebas de carga y descarga en placas rígidas son:

$$\gamma_h = 1.5 \text{ ton/m}^3$$
$$\rho = 0.153 \text{ ton-seg}^2/\text{m}^4$$
$$q_u = 10.34 \text{ kg/cm}^2 = 103.4 \text{ ton/m}^2$$
$$E = 1{,}000 \text{ kg/cm}^2 = 10{,}000 \text{ ton/m}^2$$
$$\nu = 0.45$$

Las características de la máquina según datos suministrados por el fabricante son:

Peso total	90 toneladas
Masa total	9.18 ton-seg²/m
Peso de elementos giratorios	16 toneladas
Masa de elementos giratorios	1.63 ton-seg²/m
Velocidad de operación	300 rpm = 5 cps
Altura de la flecha	1.20 m
Dimensiones en planta	7.00 m × 2.50 m
Altura del centro de masa	0.90 m
Momento de inercia de masa respecto a eje centroidal paralelo al eje de cabeceo	3.0 ton-m-seg²

* Los datos utilizados en este ejemplo son imaginarios.

Fuerzas y pares teóricos de desbalanceo

	Componente primaria	*Componente secundaria*
Fuerza horizontal	0 ton	0 ton
Par horizontal	1.80 ton-m	3.74 ton-m
Fuerza vertical	0.30 ton	0 ton
Par vertical	3.60 ton-m	2.35 ton-m

Las características del bloque de cimentación propuesto son:

Dimensiones	9.00 m × 3.00 m × 1.00 m
Peso	65 toneladas
Masa	6.63 ton-seg²/m
Momento de inercia mínimo del área de la base	20.2 m⁴
Momento de inercia de la masa respecto al eje de cabeceo	7.2 ton-m-seg²

De acuerdo con los datos anteriores se tiene:

Presión estática de contacto	5.74 ton/m²
Presión dinámica de contacto afectando a la maquina de un factor de imparto de 1.5	7.40 ton/m²
Momento de inercia de masa de la máquina respecto al eje de cabeceo	36.1 ton-m-seg

4.1.1 *Revisión del diseño preliminar.* Afectando la capacidad de carga del suelo con un coeficiente de seguridad de 3 se obtiene una presión de contacto admisible de 34.5 ton/m², bastante mayor que la presión dinámica de contacto por lo que el diseño es adecuado en lo referente a esfuerzos permisibles. Los asentamientos que se producirán bajo el bloque de cimentación pueden estimarse conociendo los espesores de los estratos compresibles mediante los nomogramas de Newmark.[17]

En la fig. 3, para una superficie de contacto de 27 m² y cimentación en arcilla se obtiene una frecuencia natural reducida del orden de 25, así que la frecuencia resonante de vibración vertical del conjunto es 10.4 cps. Esta frecuencia es superior al doble de la frecuencia de operacion de la máquina, por lo que el diseño preliminar es aceptable en lo referente a frecuencia de resonancia.

4.2 *RESPUESTA DEL SISTEMA A PERTURBACIONES EXTERNAS*

4.2.1 *Modelo matemático.* De la Tabla 1 para las condiciones del ejemplo se obtienen los siguientes valores de las constantes elásticas del elemento flexible:

$$K_v = 74{,}000 \text{ ton/m}$$
$$K_h = 48{,}600 \text{ ton/m}$$
$$K_r = 144{,}000 \text{ ton-m}$$

De la Tabla 5 se obtienen los siguientes valores de la altura del prisma virtual de suelo y de la constante de amortiguamiento lineal para cada modo de vibración

$$h_v = 1.35 \text{ m}$$
$$h_h = 0.26 \text{ m}$$
$$h_c = 1.82 \text{ m}$$

$$C_v = 1{,}126 \text{ ton-seg/m}$$
$$C_h = 473 \text{ ton-seg/m}$$
$$C_c = 643 \text{ ton-m-seg}$$

De la altura del prisma virtual de suelo para cada modo de vibración se obtienen los siguientes valores:

$$M_v = 5.88 \text{ ton-seg}^2/\text{m}$$
$$M_h = 1.07 \text{ ton-seg}^2/\text{m}$$
$$I_c = 13.9 \text{ ton-m-seg}^2$$

4.2.2 *Frecuencias naturales del modelo*. Utilizando las ecs. 2.2, 2.3, 2.5 y 2.6 se tiene:

$$\omega_v = 59.2 \text{ rad/seg} = 9.4 \text{ cps}$$
$$\xi_v = 0.445$$
$$\omega_h = 53.7 \text{ rad/seg} = 8.5 \text{ cps}$$
$$\xi_h = 0.259$$
$$\omega_c = 50.5 \text{ rad/seg} = 8.0 \text{ cps}$$
$$\xi_c = 0.112$$

Para determinar con la ec. 2.9 las frecuencias de vibración acoplada es preciso conocer la altura del centro de gravedad del sistema máquina-cimentación-suelo. Como el prisma virtual de suelo tiene diferente altura según se trate de vibraciones horizontales o de cabeceo, se recomienda tomar una altura media, que en este ejemplo es 1.04 m. La masa del prisma será 4.29 ton-seg²/m. Entonces, la altura del centro de gravedad general respecto a la superficie de contacto es:

$$\frac{9.18 \times 1.90 + 6.63 \times 0.50 - 4.29 \times 0.52}{9.18 + 6.63 + 4.29} = 0.92 \text{m}$$

Los momentos de inercia de masa del prisma de suelo y del conjunto máquina-cimentación respecto al eje que pasa por el centro de gravedad general y es paralelo al eje de cabeceo son:

$$\bar{I}_c = 12.5 \text{ ton-m-seg}^2$$

$$\bar{I} = 12.2 + 6.7 = 18.7 \text{ ton-m-seg}^2$$

El factor γ de la ec. 2.10 será entonces

$$\gamma = \frac{12.5 + 18.7}{13.9 + 36.1 + 7.2} = \frac{31.2}{57.2} = 0.544$$

y las frecuencias de vibración acoplada serán

$$\omega^2_{1,2} = \frac{5.425 \mp 3.644}{1.09} = \begin{cases} 1{,}637 \\ 8{,}336 \end{cases}$$

$$\therefore \omega_1 = 40.2 \frac{\text{rad}}{\text{seg}} = 6.4 \text{ cps}$$

$$\omega_2 = 91.3 \frac{\text{rad}}{\text{seg}} = 14.5 \text{ cps}$$

Aunque el diseño preliminar resultó aceptable por lo que toca a resonancia en vibración vertical, para vibraciones acopladas horizontales y de cabeceo la frecuencia resonante mínima es solo ligeramente mayor que la correspondiente a la velocidad de operación de la máquina. En estas circunstancias convendría modificar de inmediato el diseño, aumentando el área de contacto y reduciendo la altura del bloque de cimentación. Sin embargo, para fines de ilustración se calcularán mediante las ecs 2.1, 2.4, 2.7 y 2.8, las amplitudes de desplazamiento resultantes. Si están dentro de los límites tolerables se podrá aceptar el diseño.

4.2.3. *Perturbaciones externas*. De la ec. 3.5 se obtiene una excentricidad probable $e = 5.5$ mm, que es obviamente absurda. Sin embargo, como se recomienda en el inciso 3.3.3, se pueden calcular las fuerzas y momentos de desbalanceo debidos a excentricidades accidentales utilizando un valor de e diez veces menor al obtenido de la ec. 3.5. Entonces, las amplitudes de las fuerzas y momentos de desbalanceo son:

$$P_x = 4.98 \text{ ton}$$
$$P_z = 13.98 \text{ ton}$$
$$M_y = 14.94 \text{ ton-m}$$

Los valores de las fuerzas y momentos de desbalanceo que resultan de multiplicar por diez los datos suministrados por el fabricante son:

$$P_x = 0 \text{ ton}$$
$$P_z = 3.00 \text{ ton}$$
$$M_y = 36.00 \text{ ton-m}$$

Obsérvese que las fuerzas y los pares de desbalanceo calculados por ambos métodos son del mismo orden de magnitud. Dada la incertidumbre de los datos tal vez convenga utilizar el promedio de ambos resultados. Las amplitudes de desplazamiento correspondientes a cada modo de vibración independiente son:

$$A_z = 0.13 \text{ mm}$$
$$A_x = 0.07 \text{ mm}$$
$$A_{x\phi} = 0.39 \text{ mm}$$
$$A_{h\phi} = 0.26 \text{ mm}$$

Aplicando las ecs 2.11 y 2.12 se obtienen las amplitudes resultantes:

$$A_v = 0.41 \text{ mm}$$
$$A_h = 0.29 \text{ mm}$$

En la fig 1 puede observarse que estas amplitudes de vibración están dentro de los límites permisibles, por lo que el diseño es satisfactorio

5. RECONOCIMIENTO

Se agradece el patrocinio de Petroleos Mexicanos para llevar a cabo la investigacion que condujo a los criterios de diseño objeto de este informe El señor B Martinez Romero efectuó los cálculos incluidos en el ejemplo de aplicación

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DISEÑO DE CIMENTACIONES SUJETAS A VIBRACION

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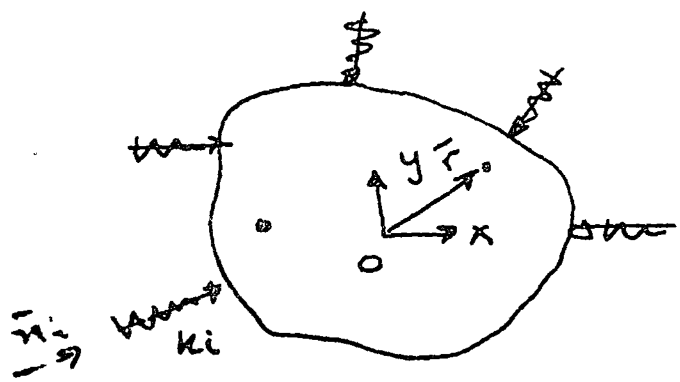
ACOPLAMIENTO DE RESORTES Y AMORTIGUADORES DISCRETOS CON MOVIMIENTO DE CUERPO RIGIDO

1.- Planteamiento en el plano



Supongamos que existe un cuerpo rígido que acopla una serie de resortes discretos. que trabajan únicamente en el sentido y dirección de la $\bar{n}_i$ que define su posición vectorial en el plano xy

El movimiento de cuerpo rígido se define como:

$$\bar{\delta} = \bar{r}_0 + \bar{\theta} \times \bar{r}_i$$

en el cual $\bar{r}_0$ es el movimiento de translación del origen. $\bar{\theta}$ es el angulo que gira el cuerpo rígido y $\bar{r}_i$ es el vector de posición de un punto cualquiera del cuerpo rígido así como $\bar{\delta}$ el vector de movimiento de ese punto cualquiera.

Ahora si los puntos considerados son los puntos de aplicación de los resortes luego la deformación en su di-

Sección Serie

$$\delta_o = \bar{\delta} \cdot \bar{n}_i$$

y si se multiplica por k_i se obtiene la fuerza que se origina

$f_o = \bar{\delta} \cdot \bar{n}_i k_i$, ahora vectorialmente se obtiene:

$\bar{f}_i = (\bar{\delta} \cdot \bar{n}_i) k_i \bar{n}$ esto es:

$$\bar{f}_i = (\bar{r}_o + \bar{\theta} \times \bar{r}_i) \cdot \bar{n}_i k_i \bar{n}$$

$$= (\bar{r}_o \cdot \bar{n}_i + [\bar{\theta}, \bar{r}_i, \bar{n}_i]) k_i \bar{n}$$

que desglozando se tiene

$$\bar{r}_o \cdot \bar{n}_i = (x_o \bar{i} + y_o \bar{j}) \cdot (\alpha_i \bar{i} + \beta_i \bar{j}) = \alpha_i x_o + \beta_i y_o$$

$$[\bar{\theta}\ \bar{r}_i\ \bar{n}_i] = \begin{bmatrix} 0 & 0 & \theta_z \\ x_i & y_i & 0 \\ \alpha_i & \beta_i & 0 \end{bmatrix} = -\theta_z y_i \alpha_i \bar{i} + \theta_z x_i \beta_i$$

luego

$$\bar{f}_i = (\alpha_i x_o + \beta_i y_o + \theta_z x_i \beta_i - \theta_z y_i \alpha_i) k_i (\alpha_i \bar{i} + \beta_i \bar{j})$$

$$\bar{f}_i = \begin{bmatrix} \alpha_i^2 k_i x_o + \alpha_i \beta_i k_i y_o + (x_i \alpha_i \beta_i k_i - y_i \alpha_i^2 k_i)\theta_z \\ \alpha_i \beta_i k_i x_o + \beta_i^2 k_i y_o + (x_i \beta_i^2 k_i - y_i \alpha_i \beta_i k_i)\theta_z \end{bmatrix}$$

que matricialmente puede representarse como:

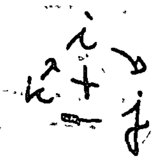
$$\bar{f}_i = \begin{bmatrix} \alpha_i^2 k_i & \alpha_i \beta_i k_i & (x_i \alpha_i \beta_i k_i - y_i \alpha_i^2 k_i) \\ \alpha_i \beta_i k_i & \beta_i^2 k_i & (x_i \beta_i^2 k_i - y_i \alpha_i \beta_i k_i) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ \theta_z \end{bmatrix}$$

Sacando momentos respecto al origen.

$$\bar{m}_i = \bar{r}_i \times \bar{f}_i = \bar{r}_i \times \left(\bar{r}_0 \cdot \bar{n}_i + [\theta, \bar{r}_i, \bar{n}_i]\right) k_i \bar{n}_i$$

$$= k_i \left(\bar{r}_0 \cdot \bar{n}_i + [\theta, \bar{r}_i, \bar{n}_i]\right) \bar{r}_i \times \bar{n}_i$$

$$\bar{r}_i \times \bar{n}_i = (\bar{i} x_i + \bar{j} y_i) \times (\alpha_i \bar{i} + \beta_i \bar{j})$$



$$= (x_i \beta_i - \alpha_i y_i) \bar{k}$$

O sea que únicamente hay componente en el eje z.

$$\bar{m}_i = (\alpha_i x_0 + \beta_i y_0 + \theta_z x_i \beta_i - \theta_z y_i \alpha_i) k_i (x_i \beta_i - \alpha_i y_i) \bar{k}_i$$

$$= \alpha_i x_0 x_i \beta_i k_i - \alpha_i^2 y_i x_0 k_i + x_i \beta_i^2 k_i y_0 - y_i \alpha_i \beta_i k_i y_0$$
$$+ (x_i \beta_i - y_i \alpha_i)(x_i \beta_i - \alpha_i y_i) k_i \theta_z$$

$$= (\alpha_i x_i \beta_i k_i - \alpha_i^2 y_i k_i) x_0 + (\beta_i^2 x_i k_i - y_i \alpha_i \beta_i k_i) y_0$$
$$+ (x_i^2 \beta_i^2 - 2 x_i y_i \alpha_i \beta_i + y_i^2 \alpha_i^2) k_i \theta_z$$

Esto es matricialmente:

$$m_i = \left[(\alpha_i x_i \beta_i k_i - \alpha_i^2 y_i k_i),\ (\beta_i^2 x_i k_i - y_i \alpha_i \beta_i k_i),\ (x_i^2 \beta_i^2 - 2 x_i y_i \alpha_i \beta_i + y_i^2 \alpha_i^2) k_i\right] \begin{bmatrix} x_0 \\ y_0 \\ \theta_z \end{bmatrix}$$

Si se efectúa la sumatoria en todo el cuerpo rígido se obtiene $\bar{F} = \sum_{n} \bar{f}_i$ y $\bar{M} = \sum_{n} \bar{m}_i$ y organizando las fuerzas generalizadas se obtiene el siguiente arreglo matricial

$$
\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} =
\begin{bmatrix}
\sum \alpha_i^2 k_i & \sum \alpha_i \beta_i k_i & \sum (x_i \alpha_i \beta_i k_i - y_i \alpha_i^2 k_i) \\
\sum \alpha_i \beta_i k_i & \sum \beta_i^2 k_i & \sum (x_i \beta_i^2 k_i - y_i \alpha_i \beta_i k_i) \\
\sum (x_i \alpha_i \beta_i k_i - y_i \alpha_i^2 k_i) & \sum (x_i \beta_i k_i - y_i \alpha_i \beta_i k_i) & \sum (x_i^2 \beta_i^2 - 2 x_i y_i \alpha_i \beta_i + y_i^2 \alpha_i^2) k_i
\end{bmatrix}
\begin{bmatrix} x_o \\ y_o \\ \theta_z \end{bmatrix}
$$

Si existen dos direcciones preferentes x y y. y si se les asigna a estos ejes sus nombres de Kh y Kv respectivamente. Se obtiene que para Kh $[\alpha = 1 \quad \beta = 0]$ y para Kv $[\alpha = 0, \beta = 1]$ por lo tanto.

$$\sum \alpha_i^2 k_i = \sum k_{hi} \quad ; \quad \sum \alpha_i \beta_i k_i = 0$$
$$\sum \alpha_i \beta_i k_i = 0 \quad ; \quad \sum \beta_i^2 k_i = \sum k_{vi}$$

$$\sum (x_i \alpha_i \beta_i k_i - y_i \alpha_i^2 k_i) = -\sum y_i k_{hi}$$
$$\sum (x_i \beta_i^2 k_i - y_i \alpha_i \beta k_i) = \sum x_i k_{vi}$$ y finalmente.

$$\sum (x_i^2 \beta_i^2 - 2 x_i y_i \alpha_i y_i + y_i^2 \alpha_i^2) k_i = \sum x_i^2 k_{vi} + \sum y_i^2 k_{hi}$$

La matriz nos queda:

$$\begin{bmatrix} \sum K_{hi} & 0 & -\sum y_i K_{hi} \\ 0 & \sum K_{vi} & \sum x_i K_{vi} \\ -\sum y_i k_{vi} & \sum x_i k_{vi} & \sum x_i^2 k_{vi} + \sum y_i^2 K_{hi} \end{bmatrix}$$

Si algun resorte de estos es continuo luego se convierte la sumatoria en una integral. siendo en ese caso el Kho Kv un resorte uniformemente repartido o distribuido con alguna ley. a los cuales llamaremos Kho y Kvo

Si K_{ho} y K_{vo} fueran resortes uniformemente distribuidos.

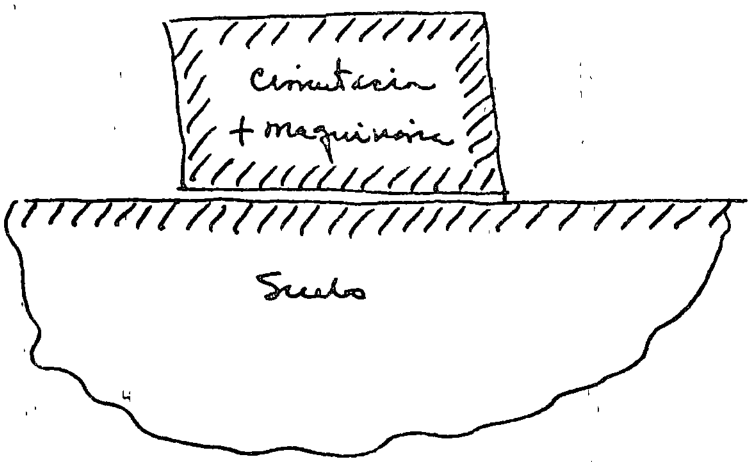
luego

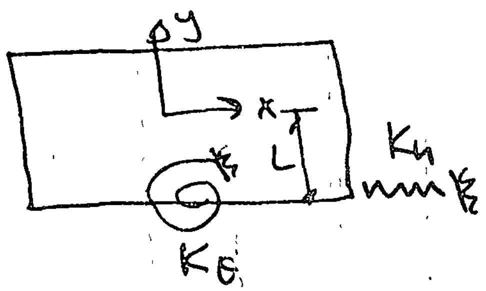
$$\sum x_i^2 K_{vi} = \int_A x_i^2 K_{v_0}\, dA = K_{v_0} J$$

en el que J es el momento de inercia del area de contacto.

El requisito para que el movimiento vertical este desacoplado respecto al resto es necesario que $\sum x_i K_{vi} = 0$

Si esto se cumple entonces el modelo de cimentación es el siguiente





y los terminos de la matriz se transforman en:

$\sum K_{hi} = K_h$

$-\sum K_{hi} y_i = K_h L$

y $\sum x_i^2 K_{vi} + \sum y_i^2 K_{hi} = K_\theta + K_h L^2$

Si se plantea el equilibrio respecto al centro de masa. la matriz de masas resulta desacoplada como sigue:

$$M = \begin{bmatrix} m & 0 \\ 0 & I_\theta \end{bmatrix}$$

en la que m es la masa total de cimentación + maquina y I_θ es el momento de inercia de la masa respecto al centro de masa. y el sistema de ecuaciones diferenciales es:

$$\begin{bmatrix} m & 0 \\ 0 & I_\theta \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} K_h & K_h L \\ K_h L & K_\theta + K_h L \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Suponiendo una solución $\begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} e^{i\omega t} = \begin{bmatrix} u \\ \theta \end{bmatrix}$ se induce la solución

$$-\omega^2 \begin{bmatrix} m & 0 \\ 0 & I_\theta \end{bmatrix} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} + \begin{bmatrix} K_h & K_h L \\ K_h L & K_\theta + K_h L \end{bmatrix} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} K_h - \omega^2 m & K_h L \\ K_h L & K_\theta + K_h L^2 - \omega^2 I_\theta \end{bmatrix} \begin{bmatrix} u_0 \\ \theta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

y el determinante será.

$$(K_h - \omega^2 m)(K_\theta + K_h L^2 - \omega^2 I_\theta) - K_h^2 L^2 = 0$$

$K_h K_\theta + K_h^2 L^2 - \omega^2 K_h I_\theta - \omega^2 m K_\theta - \omega^2 m K_h L^2$

$+ \omega^4 m I_\theta - K_h^2 L^2 = 0$

organizando la ecuación:

$$m I_\theta \omega^4 - (K_h I_\theta + m K_\theta + m K_h L^2)\omega^2 + K_h K_\theta = 0$$

Por otro lado $I_\theta = I_0 - mL^2$ siendo I_0 el momento de inercia de la mesa respecto a un eje que pasa por el suelo. Substituyendo este valor en la ecuación anterior se tiene

$$m I_\theta \omega^4 - [K_h(I_0 - mL^2) + m K_\theta + m K_h L^2]\omega^2 + K_h K_\theta = 0$$

$$m I_\theta \omega^4 - [K_h I_0 - K_h m L^2 + m K_\theta + m K_h L^2]\omega^2 + K_h K_\theta = 0$$

$$m I_\theta \omega^4 - [K_h I_0 + m K_\theta]\omega^2 + K_h K_\theta = 0$$

dividiendo ahora toda la ec por $m I_0$ y llamando

$I_\theta / I_0 = \gamma$ ~~se tiene~~ y $\frac{K_h}{m} = \omega_1^2$ $\frac{K_\theta}{I_0} = \omega_2^2$ luego

$$\gamma \omega^4 - [\omega_1^2 \gamma + \omega_2^2]\omega^2 + \omega_1^2 \omega_2^2 = 0$$

$$\omega_{1,2} = \frac{1}{2\gamma}\left[(\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2)^2 - 4\gamma\omega_1^2\omega_2^2}\right]$$

Modelo matemático para representar la interacción dinámica de suelo y cimentación*

José A. NIETO, Emilio ROSENBLUETH y Octavio A. RASCON **

1. INTRODUCCION

1.1. Descripción del problema

La influencia del suelo de cimentación en la respuesta dinámica de una estructura no ha sido suficientemente estudiada. Aunque es posible un tratamiento riguroso del problema[1] este no es de fácil aplicación en la práctica. Conviene contar con un modelo matemático sencillo que permita determinar con bastante precisión la respuesta de un sistema suelo estructura a perturbaciones dinámicas. El modelo debe proporcionar resultados razonablemente correctos en el intervalo de frecuencias de mayor interés. El tratamiento se simplifica considerando primeramente la cimentación de una estructura como un bloque rígido. El problema consiste en determinar el comportamiento dinámico del sistema constituido por el cuerpo rígido y el suelo: este último se puede idealizar, en una extensa gama de condiciones prácticas, como un medio elástico, homogéneo, isótropo y semiinfinito.

Una vez determinada la respuesta dinámica del bloque rígido de cimentación tomando en cuenta su interacción con el suelo, es fácil incorporar al sistema la estructura flexible que se levanta sobre dicha cimentación.

1.2. Antecedentes

Las principales soluciones disponibles hasta la fecha para el problema de interacción dinámica entre un cuerpo rígido y un semiespacio elástico pueden clasificarse en cuatro grupos.

1) Soluciones exactas clásicas, que suponen que la distribución de esfuerzos de contacto entre bloque y suelo no depende de la frecuencia de vibración[2] y generalmente toman dicha distribución igual a la que corresponde a carga estática. El cálculo de las respuestas dinámicas exige trabajar independientemente con cada frecuencia de excitación, pues los parámetros del modelo matemático correspondiente son funciones de esta variable: tal situación no es objetable cuando se trata de calcular el efecto de vibraciones debidas a maquinaria, pues entonces sólo interesan una o unas cuantas frecuencias de excitación, pero introduce complicaciones injustificadas cuando se desea calcular la respuesta a una perturbación que contiene componentes significativas en un extenso intervalo de frecuencias, como lo es un sismo, y prácticamente imposibilita el cálculo de respuestas cuando sobre la cimentación existe una estructura de comportamiento no lineal. Además a muy altas frecuencias esta solución da errores inadmisibles provenientes de la hipótesis referente a la distribución de esfuerzos de contacto.

2) Para una placa circular sujeta a vibración se dispone de la solución exacta que toma en cuenta la distribución correcta de esfuerzos de contacto[3]. Por lo demás esta solución adolece de las mismas limitaciones que las del grupo que antecede cuando se trata de análisis para diseño sísmico.

3) Se han propuesto diversos modelos matemáticos que incluyen masas virtuales, elementos flexibles y amortiguadores lineales para representar al suelo[4]. Al tomar estos parámetros independientes de la frecuencia se elimina la dificultad mencionada a propósito de la aplicación de las soluciones de los grupos anteriores a diseño sísmico, si bien se introducen errores inadmisibles a muy altas frecuencias. (Se ha demostrado que la masa virtual de suelo debe ser nula para que el orden de magnitud de las respuestas quede correctamente predicho cuando la frecuencia de excitación tiende a infinito[5]). Las masas virtuales de estos modelos matemáticos se basan en consideraciones de carácter intuitivo mientras las constantes de los amortiguadores se han determinado a partir de un número limitado de pruebas de laboratorio en modelos físicos de pequeñas dimensiones usando o simulando suelos en forma tal que se antoja peligroso extrapolar a partir de ellos.

4) Para eliminar la objeción mencionada respecto al empleo de una masa virtual cuando interesan frecuencias sumamente elevadas se ha propuesto un modelo matemático que comprende sólo un elemento flexible y un amortiguador lineal en representación del suelo, ambos elementos carentes de masa[5]. Dicho modelo suministra resultados excelentes en los intervalos de frecuencias sumamente bajas o excepcionalmente altas, pero introduce errores hasta de un 30 por ciento en el intervalo intermedio (si bien el error generalmente no excede 20 por ciento en el intervalo de mayor interés). Esta solución

* Ponencia presentada en el Primer Congreso Nacional de Ingeniería Sísmica, Guadalajara, Jal., nov. 1965.

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sólo se ha formulado, aparentemente para las oscilaciones verticales de una placa circular.

De las soluciones mencionadas las del último grupo son las únicas que pudieran considerarse satisfactorias para diseño sísmico. Sin embargo, tienen el inconveniente de perder precisión a frecuencias bajas y medias, que son las de mayor interés en el diseño de cimentaciones de maquinaria lenta y en diseño sísmico de estructuras.

El modelo matemático que se propone en este trabajo está basado en las soluciones exactas clásicas. Presenta la ventaja de mayor precisión en el intervalo de interés práctico de las frecuencias. Está limitado por lo pronto a bases de cimentación de forma circular, aunque su extensión a otras formas de base es inmediata. Se limita también a la consideración de modos desacoplados de vibración. Para los grados de libertad elegidos, estos modos no pueden existir en general cuando se trata de una base rígida por lo que respecta a cabeceo y traslación horizontal, salvo cuando la relación de Poisson, ν, vale 0.5. En la práctica los grados de libertad horizontal y de cabeceo se encuentran acoplados. En el modelo propuesto es factible tomar en consideración este acoplamiento variando la colocación de los elementos flexibles para los modos en cuestión.

1.3. *Fundamentos del modelo matemático propuesto*

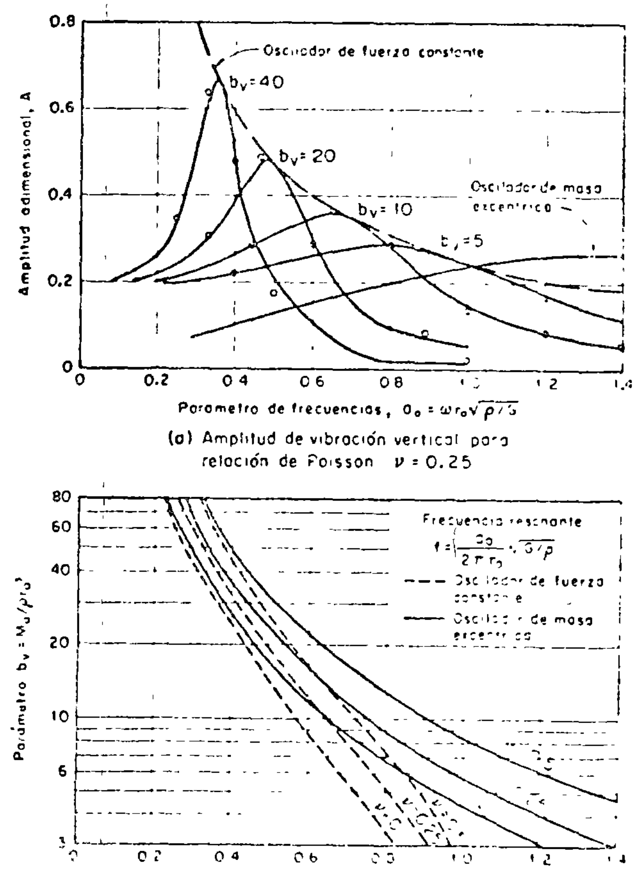
1.3.1. Soluciones "exactas" para bases circulares

La respuesta dinámica de un cilindro circular de masa M_o y radio r_o desplantado en la superficie de un semiespacio elástico homogéneo e isótropo cuyas constantes elásticas son G y ν y cuya densidad de masa es ρ ha sido estudiada por varios investigadores [?] para los seis grados de libertad del sistema, a saber: vertical, dos horizontales iguales entre sí, cabeceo respecto a dos diámetros ortogonales de la base y torsión respecto al eje vertical que pasa por el control de la base. Se han considerado en la mayoría de los casos dos tipos de perturbación actuando sobre el cilindro: solicitación armónica cuya amplitud es independiente de la frecuencia ω y solicitación armónica cuya amplitud es proporcional a ω^2 (Este último tipo es el que produciría un excitador mecánico de masa excéntrica). Como se desconoce, excepto para vibración vertical [?], la distribución real de esfuerzos dinámicos en la superficie de contacto entre el cilindro y el semiespacio, en los estudios referidos se ha supuesto una distribución de esfuerzos dinámicos análoga a la distribución estática existente bajo alguna de las dos condiciones extremas siguientes: *a*) cilindro infinitamente rígido y *b*) cilindro infinitamente flexible. Con la primera suposición se prescriben los desplazamientos en la superficie de contacto y se determina la condición de esfuerzos correspondientes. Con la segunda suposición la distribución de esfuerzos se prescribe de antemano para cada tipo de movimiento. Estas suposiciones conducirán sólo a valores aproximados de la respuesta dinámica del cilindro, pero se ha comprobado [?] que las aproximaciones son adecuadas para la mayor parte de los fines prácticos.

Richart [?] ofrece una excelente presentación de los resultados obtenidos por él y por otros investigadores para la determinación de las frecuencias resonantes y amplitudes de vibración del cilindro circular infinitamente rígido. Sus gráficas se reproducen en las figs 1 a 4, correspondientes a cada uno de los grados de libertad. En la mitad inferior de estas figuras se presenta la frecuencia resonante en función de la masa o la inercia del cilindro rígido para los dos tipos de perturbación considerados y para varios valores de la relación de Poisson, ν. Tanto la frecuencia resonante como la masa o momento de inercia se grafican en forma adimensional, aquella como el parámetro de frecuencias $a_o = \omega r_o \sqrt{\rho/G}$, y esta como el parámetro b, que se define en la forma siguiente:

i. Para movimiento vertical $b_v = M_o/\rho r_o^3$

ii. Para movimiento horizontal $b_H = M_o/\rho r_o^3$



(a) Amplitud de vibración vertical para relación de Poisson ν = 0.25

(b) Frecuencia resonante en función de b_v para tres valores de la relación de Poisson

Fig 1 *Curvas de respuesta y frecuencias resonantes de vibración vertical* (Según Richart, ref 5)

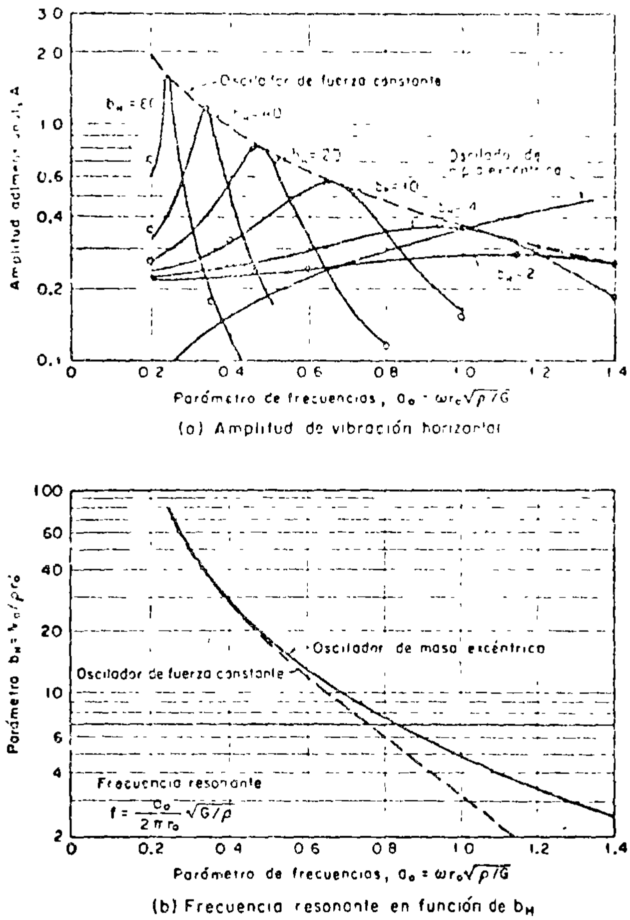


FIG. 2. *Curvas de respuesta y frecuencias resonantes de vibración horizontal.* Relación de Poisson, $\nu = 0$ (Según Richart, ref 5)

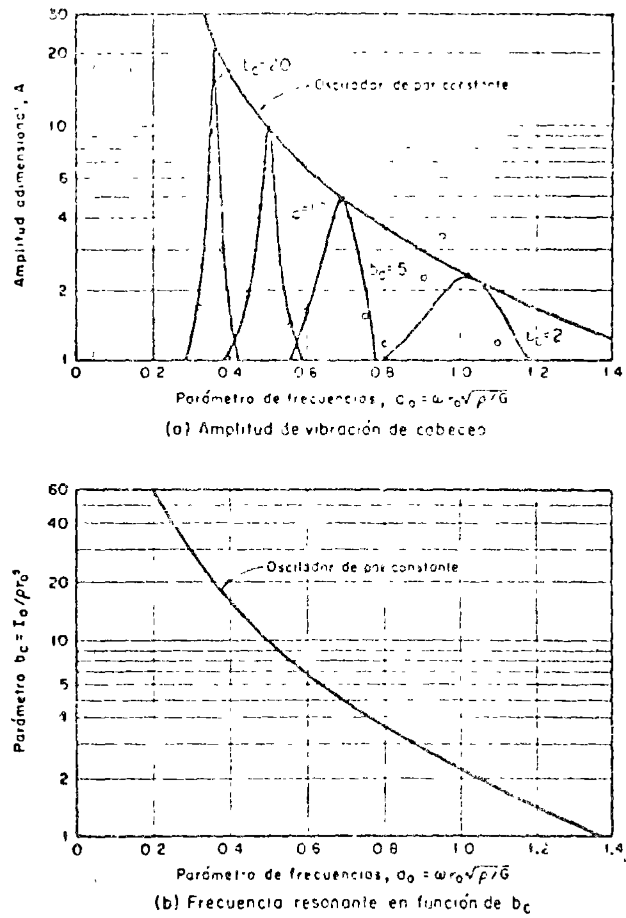


FIG. 3. *Curvas de respuesta y frecuencias resonantes de vibración de cabeceo.* Relación de Poisson, $\nu = 0$ (Según Richart, ref 5)

iii. Para cabeceo. $b_a = I_o/\rho r_o^5$

iv. Para movimiento torsional. $b_t = J_o/\rho r_o^5$

En estas expresiones I_o es el momento de inercia de la masa de la base respecto al eje de cabeceo y J_o su momento de inercia respecto al eje de torsión. En la mitad superior de las figs 1 a 4 Richart presenta la amplitud de vibración en función del parámetro de frecuencias para varios valores de b y para ambos tipos de perturbación. La amplitud A, que corresponde a cada grado de libertad se presenta adimensionalmente en la forma siguiente:

i. Para vibración vertical $A_1 G r_o/Z$ si la amplitud de la fuerza perturbadora Z es independiente de la frecuencia y $A_1 \rho r_o^4/2m_1 l$ si la perturbación es debida a un excitador mecánico con masa excentrica m_1 y brazo giratorio de longitud l

ii. Para vibración horizontal $A_H G r_o/Q_H$, o $A_H \rho r_o^3/2m_1 l$, en que Q_H es la amplitud de la fuerza perturbadora independiente de la frecuencia.

iii. Para vibración de cabeceo $A_a G r_o^3/M$, en que M es la amplitud del momento de cabeceo.

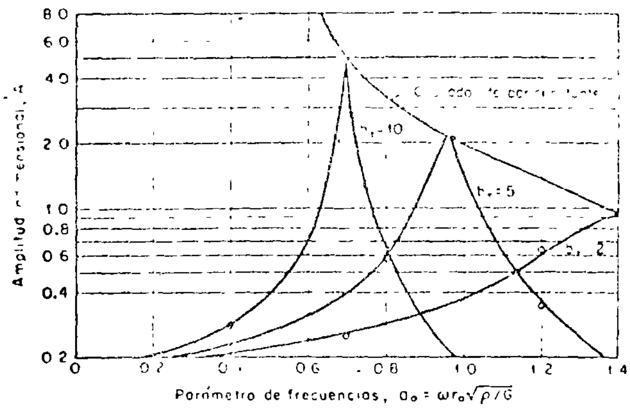
iv. Para vibración torsional $A_t G r_o^3/T$, en que T es la amplitud del par torsional

(Para estos dos últimos casos no se presentan curvas correspondientes a la perturbación producida por un excitador mecanico).

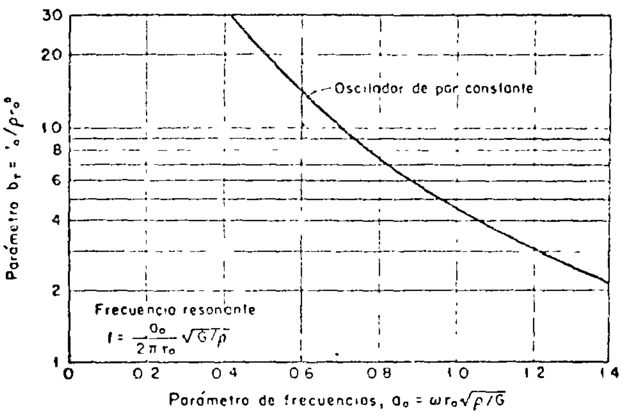
De la similitud entre las curvas de amplitudes presentadas por Richart y los espectros de desplazamiento de un sistema amortiguado con un grado de libertad sujeto a los mismos tipos de perturbación surgió la idea de desarrollar el modelo matemático que se describe en la siguiente sección.

1.3.2 RESPUESTA DINÁMICA DE SISTEMAS AMORTIGUADOS DE UN GRADO DE LIBERTAD

Se sabe que la frecuencia circular resonante de un sistema amortiguado con un grado de libertad



(a) Amplitud de vibración torsional



(b) Frecuencia resonante en función de b_T

FIG. 4. *Curvas de respuesta y frecuencias resonantes de vibración torsional.* Independiente de la relación de Poisson (Según Richart, ref 5)

con masa, M, constante elástica K y constante de amortiguamiento C, está dada por

$$\omega_1 = \sqrt{\frac{K(1-2\xi^2)}{M}} \quad (1)$$

cuando la amplitud máxima de la solicitación dinámica es constante, y por

$$\omega_2 = \sqrt{\frac{K}{M(1-2\xi^2)}} \quad (2)$$

cuando la amplitud máxima de la solicitación es proporcional al cuadrado de la frecuencia perturbadora. En estas expresiones $\xi = C/2\sqrt{KM}$ es el porcentaje de amortiguamiento crítico del sistema.

Si se conoce la constante elástica de un sistema amortiguado y se determinan experimentalmente sus frecuencias resonantes ω_1 y ω_2 cuando la excitación es, respectivamente, independiente de ω y proporcional al cuadrado de la frecuencia perturbadora, es posible, mediante las ecs 1 y 2, determinar el valor de la masa M y de la constante de amortiguamiento C del sistema. En esta idea se basa el modelo matemático propuesto en este trabajo con las modificaciones que se indican más adelante.

Se sabe también que la frecuencia resonante del sistema amortiguado cuando la amplitud máxima de la excitación es proporcional a la frecuencia perturbadora es igual a la frecuencia natural del sistema sin amortiguamiento. La amplitud resonante del sistema amortiguado sujeto a este último tipo de excitación es $M/2\xi K$.

2. *MODELO MATEMATICO PROPUESTO*

El modelo que se propone es un sistema de un grado de libertad con amortiguamiento lineal. Consiste en una masa rígida constituida por la cimentación y por un prisma virtual de suelo cuya base es idéntica a la de la cimentación pero cuya altura va a depender del grado de libertad considerado. En este trabajo se estudian exclusivamente bases de cimentación de forma circular, por lo que el prisma virtual es un cilindro. La masa rígida está soportada por un elemento flexible, linealmente elástico sin peso. La forma del elemento flexible y su colocación respecto a la masa rígida dependen del modo de vibración considerado y se presentan en la fig 5. Como constante elástica del elemento flexible se utiliza la obtenida por medio de la teoría de la elasticidad al considerar para cada grado de libertad la acción estática de la solicitación aplicada al cilindro rígido de radio r_o desplantado en la superficie del semiespacio elástico que representa al suelo. En esta constante elástica se incluye el efecto de la relación de Poisson. De esta manera se asegura

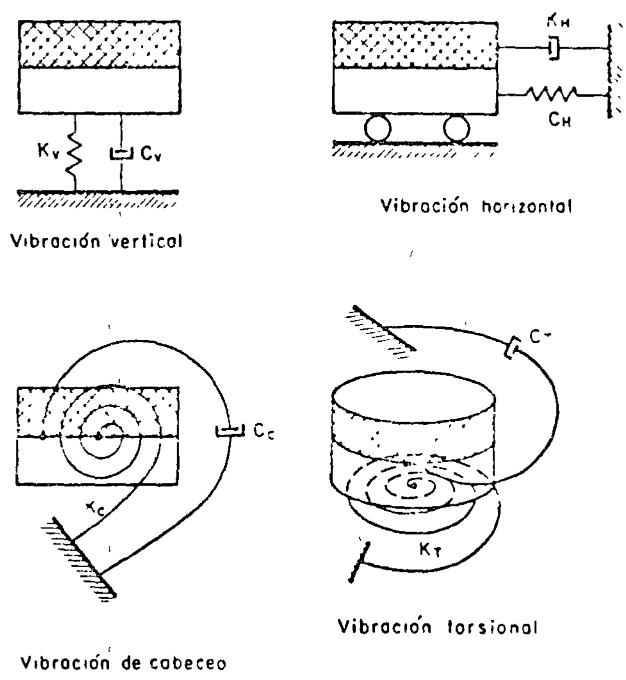


FIG 5. *Modelos matemáticos propuestos para los diferentes grados de libertad*

que la solución es exacta cuando la frecuencia de excitación tiende a cero

Debe notarse que al hablar de frecuencia de excitación nula y carga estática se entiende una velocidad de carga suficientemente lenta para evitar la aparición de fuerzas de inercia apreciables en el suelo, pero no tan lenta que se induzcan los fenómenos de consolidación y de deformaciones diferidas en gran escala. Este concepto es importante cuando se trata de aplicación a cimentaciones que se apoyan en suelos reales

Resta determinar la altura de la masa virtual de suelo y la constante de amortiguamiento para cada grado de libertad. Se desea que la respuesta del modelo matemático se ajuste a los resultados de la teoría clásica, dado que en la mitad inferior de las figs. 1 a 4 se dispone del valor de la frecuencia resonante para cada modo de vibración y para los dos tipos de perturbación considerados es sencillo determinar los valores requeridos utilizando estas figuras y las ecs. 1 y 2. Sin embargo, como en diseño sísmico la gama de frecuencias de interés es amplia, no interesa especialmente que el modelo tenga exactamente la misma frecuencia resonante que se determinó con la teoría clásica. Por otra parte, en los espectros de diseño sísmico más comunes, el intervalo de mayor interés de las frecuencias corresponde a amplitudes máximas de excitación proporcionales a la frecuencia perturbadora. Por estas dos razones se optó por utilizar el procedimiento descrito arriba para determinar solamente la altura de la masa virtual de suelo y encontrar el valor de la constante de amortiguamiento ajustando la ordenada máxima del espectro de seudovelocidades correspondiente a la solución clásica con la amplitud resonante del modelo indicado en la subsección 1.3.2. El espectro de seudovelocidades se puede determinar utilizando las curvas de las figs. 1 a 4.

Mediante este procedimiento se llega a los valores que se consignan en la Tabla 1 en la cual la constante elástica del elemento flexible está identificado por K con el subíndice correspondiente al grado de libertad considerado y A representa el área de cimentación. Los resultados se dan en función de A y de su raíz cuadrada para facilitar una primera estimación de los parámetros que corresponden a cimentaciones no circulares, a reserva de que estudios ulteriores permitan afirmar tales estimaciones

En las figs. 1 a 4 se incluyen algunos puntos representativos de las respuestas de los modelos desarrollados utilizando los valores de la Tabla, para diversos tipos de perturbación. El modelo proporciona una aproximación excelente a la respuesta *exacta* para los modos de vibración vertical y horizontal, aunque para los modos torsional y de cabeceo la aproximación no es tan buena. Hay que hacer notar también que en las estructuras ordinarias en las que la influencia del segundo modo de vibración es importante, este tiene una frecuencia de 2 a 2.5 veces mayor que la frecuencia fundamental, por lo que bastará que el modelo dé una buena aproximación en el intervalo comprendido entre la frecuencia resonante y 2.5 veces el valor de dicha frecuencia

La precisión del modelo puede mejorarse drásticamente si se hace la constante del amortiguador variable con la frecuencia; se ha preferido no proceder en tal forma para preservar la simplicidad del tratamiento.

El modelo propuesto permite incorporar fácilmente en el análisis la influencia del comportamiento inelástico de la estructura. Está limitado por ahora a bases circulares y a la consideración de modos de vibración desacoplados

Se trabaja actualmente en la preparación de modelos semejantes para bases de cimentación rectangulares con diversas relaciones de largo a ancho. El estudio de estas se basa en los resultados clásicos presentados en la ref. 15. Para otras formas de cimentación se pueden aplicar los resultados de este trabajo utilizando los coeficientes elásticos del suelo determinados mediante los monogramas de Newmark[16] e integrando en toda el área de la base la solución clásica de Cerruti[1]. Aunque suele suponerse[13] que la vibración horizontal y la de cabeceo de una cimentación están desacopladas si el centro de masa de la cimentación se halla a la altura de la base de contacto entre cimentación y suelo, esto no acontece en la práctica. Por otra parte, soluciones exactas basadas en la teoría de la elasticidad[11] demuestran que aun para el caso de esfuerzos tangenciales

TABLA 1

PARÁMETROS DEL MODELO PROPUESTO

TIPO DE VIBRACION	*ALTURA DEL PRISMA DE SUELO*		*CONSTANTE DE AMORTIGUAMIENTO*	
	En función del radio de la base	*En función del área de la base*	*En función del radio de la base*	*En función de la altura del prisma*
Vertical	0.48 r_0	0.27 $\sqrt{A}$	1.8 $\sqrt{K_v \rho r_0^3}$	5.42 $\sqrt{K_v \rho h^3}$
Horizontal	0.10 r_0	0.05 $\sqrt{A}$	1.3 $\sqrt{K_H \rho r_0^3}$	41.1 $\sqrt{K_H \rho h^3}$
Cabeceo	0.63 r_0	0.35 $\sqrt{A}$	0.30 $\sqrt{K_c \rho r_0}$	0.731 $\sqrt{K_c \rho h}$
Torsión	0.44 r_0	0.25 $\sqrt{A}$	0.50 $\sqrt{K_T \rho r_0^3}$	6.90 $\sqrt{K_T \rho h}$

TABLA 2

ALGUNOS VALORES DE LAS CONSTANTES ELÁSTICAS

FORMA DE LA BASE	MODO DE VIBRACION: vertical	horizontal	cabeceo	torsión
Circular	$\frac{4}{1-\nu} G r_0$	$5.8\pi \frac{(1-\nu)}{(2-\nu)^2} G r_0$	$2.7\, G r_0^3$ *	$\frac{16}{3} G r_0^3$ **
Rectangular [18]	$\frac{E}{1-\nu^2} c_z \sqrt{A}$ +	$\frac{E}{1-\nu^2} k_1 \sqrt{A}$ +	$\frac{E}{1-\nu^2} k_\phi \frac{I}{\sqrt{A}}$ +	$\frac{1.5\, E}{1-\nu} k_1 \sqrt{A}$

*Solo se presenta el valor de K correspondiente a $\nu = 0$

** El valor de K_T para base circular es independiente de ν

+ Los valores de c_z, k_1 y k_ϕ se presentan en las tablas 3, 4 y 5 respectivamente para algunos valores de la relacion largo/ancho

aplicados sobre la superficie del terreno se producen rotaciones de la misma así que el acoplamiento no debe ignorarse

Finalmente es de notarse que en un análisis modal que tome en cuenta la interseccion con el terreno los modos naturales de vibración no son rigurosamente ortogonales Mientras la participación del terreno no gobierne en forma pronunciada el comportamiento la falta de ortogonalidad no afecta seriamente las respuestas sismicas para fines de diseño En cambio si debe tenerse en cuenta en el cálculo de los modos naturales

TABLA 3

VALORES DEL COEFICIENTE c_z

(Según Barkan ref 18)

Relación largo/ancho	c_z
1.0	1.06
1.5	1.07
2.0	1.09
3.0	1.13
5.0	1.22
10.0	1.41

TABLA 4

VALORES DEL COEFICIENTE k_1

(Según Barkan ref 18)•
(Desplazamiento horizontal en la direccion paralela al lado a)

ν	Relacion a/b: 0.5	1.0	1.5	2.0	3.0	5.0	10.0
0.1	1.040	1.000	1.010	1.020	1.050	1.150	1.250
0.2	0.990	0.938	0.942	0.915	0.975	1.050	1.160
0.3	0.926	0.868	0.864	0.870	0.906	0.950	1.040
0.4	0.844	0.792	0.770	0.784	0.806	0.850	0.940
0.5	0.770	0.704	0.692	0.686	0.700	0.732	0.940

de vibración sobre todo cuando se acude a métodos numericos que hacen uso de esta propiedad

Algunos valores de las constantes elasticas de los elementos flexibles obtenidas a partir de la teoría de la elasticidad para los modos de vibración considerados y diferentes formas de la base se consignan en la Tabla 2.

TABLA 5

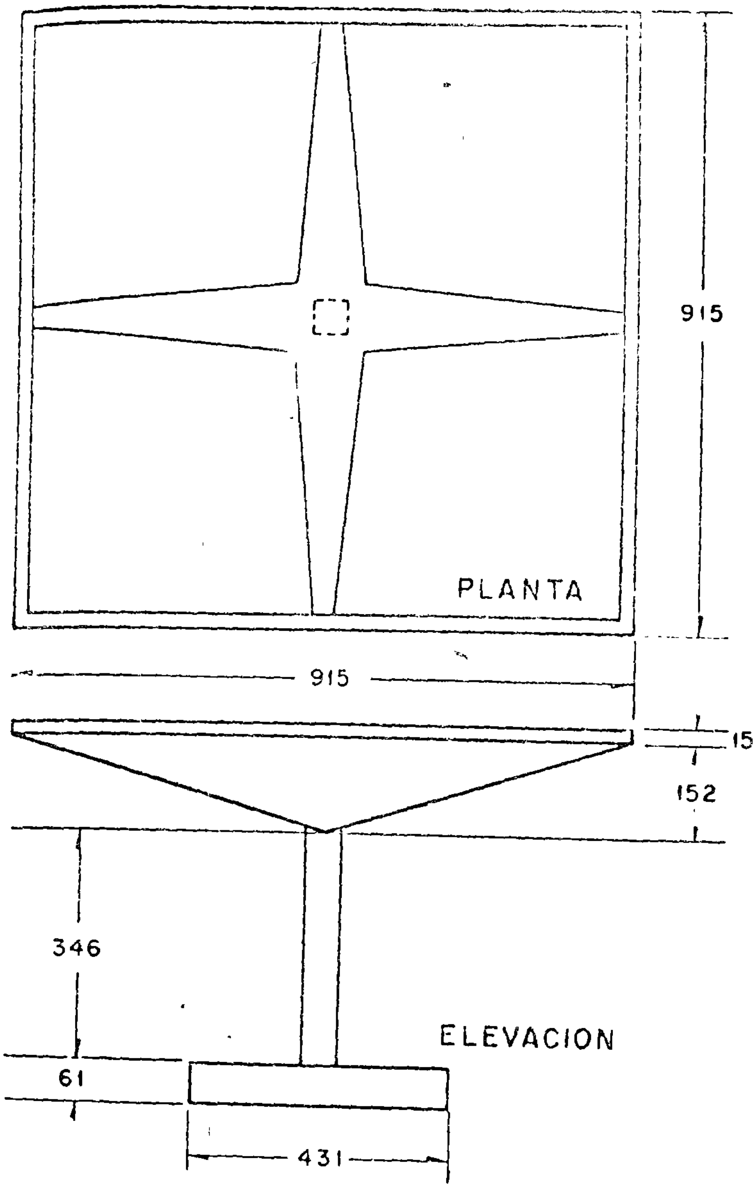
VALORES DEL COEFICIENTE k_ϕ

(Según Barkan ref 18)
(Cabeceo respecto al eje paralelo al lado largo)

largo/ancho	k_ϕ
1.0	1.984
1.5	2.254
2.0	2.510
3.0	2.955
5.0	3.700
10.0	4.981

3. *EJEMPLO DE APLICACION*

Para ilustrar la forma de aplicación de los resultados obtenidos con el modelo propuesto se resolverá el problema presentado en la ref 20 Dicho problema consiste en calcular las frecuencias, modos de vibración y respuestas sismicas de un péndulo invertido (fig 6) tomando en cuenta la intersección dinamica suelo-estructura y la inercia rotacional de la cubierta La solución difiere de la presentada en la ref 20 principalmente en que se introducirán dos grados de libertad adicionales al tomar en cuenta la masa y el momento de inercia de la losa de cimentacion Los parametros del suelo de cimentacion son $\rho = 0.112$ ton seg²/m⁴, $\nu = 0.5$ y $G = 166$ ton m² Utilizando las expresiones propuestas se obtiene $K = 2097$ ton/m $C_h = 67.80$ ton seg m $K = 8040$ ton m $C_r = 40.86$ ton seg m $M = 0.34$ ton seg²/m e $I_1 = 4.67$ ton seg² m. Los parametros de la estructura son distancia del centro de



$L = 4.19$ m
$L' = 4.80$ m
$I = 13.86$ ton seg^2 m
$W = 20.45$ ton
$m = 2.08$ ton seg^2/m
$W' = 43.6$ ton
$M_0 = 2.2$ ton seg^2/m
$I_0 = 3.03$ ton seg^2 m
$I_A = 1.065 \times 10^{-2}$ m^4
$k = 1266$ ton/m
$k_r = 7410$ ton m/rad
$\theta = 0.358$ rad/m
$\delta = 2.08$ m

$\rho = 0.112$ ton seg^2/m^4
$\nu = 0.5$
$G = 166$ ton/m^2
$K_H = 2097$ ton/m
$C_H = 67.80$ ton seg/m
$K_C = 8040$ ton m
$C_C = 40.86$ ton seg/m

FIG. 6 *Cascaron utilizado para ejemplo* (Despues de R McLean)

gravedad de la cubierta a la base de la columna $L = 4.19$ m; distancia de dicho centro a la base de la cimentación, $L' = 4.80$ m, momento de inercia de la masa de la cubierta respecto al eje de cabeceo, $I_m = 13.86$ ton seg^2/m; peso de la cubierta, $W = 20.45$ ton, masa de la cubierta $m = 2.08$ ton seg^2/m, peso de la estructura $W' = 43.6$ ton; momento de inercia centroidal principal de la sección transversal de la columna $I_A = 1.065 \times 10^{-2}$ m^4, rigidez por traslación de la columna $k = 1266$ ton/m rigidez por flexión de la columna, $k_r = 7410$ ton m/rad; rotación al nivel del centro de gravedad debida a una fuerza horizontal de valor k, $\theta = 0.358$ rad/m desplazamiento lateral del centro de gravedad debido a un momento de valor k, aplicado en dicho punto $\delta = 2.08$ m; $M_0 = 2.2$ ton seg^2/m, $I_0 = 3.03$ ton seg^2 m. (Para la obtencion de k, k_r, θ y δ vease la ref. 20.)

Para el cálculo de los modos y frecuencias de vibración se empleó una extensión del metodo propuesto en ref. 20. La extension consiste en tomar en cuenta los dos grados de libertad adicionales debidos a la masa y a la inercia de la losa de cimentación La solucion se llevó a cabo mediante una tabulacion en la cual x es el desplazamiento del centro de gravedad ε su rotación, x_0 el desplazamiento de la cimentación ε_0 su giro: x_1 y ε_1 desplazamiento y rotación del centro de gravedad debido a la flexibilidad de la columna y x_2 es el desplazamiento de dicho centro debido a la rotación de la cimentacion como cuerpo rigido

Después de varios ciclos se llego a $\omega_1 = 9.07$ rad seg. $T_1 = 2\pi/\omega_1 = 0.692$ seg $\overline{X}_1^t$ = vector modal traspuesto = {4.36 1 0.39 0.64}

Para el cálculo del segundo modo utilizando los conceptos antes mencionados se obtuvo

$\omega_2 = 25.3$ rad/seg, $T_2 = 0.248$ seg

$\overline{X}^t_2 = [1.26 \quad 1 \quad 3.73 \quad -0.26]$

La respuesta sísmica se calculó tomando en cuenta solamente los dos primeros modos. La manera de introducir los modos restantes es obvia mas cabe suponer que el efecto de estos será despreciable.

La respuesta sísmica se obtuvo utilizando el espectro de respuesta propuesto en el Reglamento del Distrito Federal[21] para la zona de alta compresibilidad. Dicho espectro lleva implícito un amortiguamiento total de la estructura. En el caso particular considerado el porcentaje de amortiguamiento se encuentra comprendido entre 2 por ciento correspondiente a la estructura y 34 por ciento obtenido como si esta fuese un cuerpo rígido y el amortiguador tuviese la constante C_H.

Los coeficientes de participación para la respuesta sísmica[11] son

$$r_n = \frac{\overline{X}_n^t \overline{M}\, \overline{\iota}}{\overline{X}_n^T \overline{M}\, \overline{X}_n}$$

en la que $\overline{\iota}$ es un vector que representa los desplazamientos estáticos de cada grado de libertad de la estructura inducidos por un desplazamiento estático unitario de la base.

En nuestro caso,

$$\overline{\iota}^T = [1 \quad 0 \quad 1 \quad 0]$$

La respuesta máxima en cada uno de los modos será[14, 20]

$$\overline{R}_n = \begin{Bmatrix} V_{cn} \\ M_{cn} \\ V_{bn} \\ M_{bn} \end{Bmatrix} = r_n \overline{M}\, \overline{X}_n A_n$$

donde $\overline{R}_n$ es el vector de las respuestas en el modo enésimo cuyas componentes son la fuerza y el momento en la cubierta V_{cn} y M_{cn}, respectivamente y la fuerza y el momento en la losa de cimentación V_{bn} y M_{bn} respectivamente y A_n es la ordenada del espectro de aceleraciones afectada del coeficiente sísmico correspondiente[21]. En nuestro caso dicho coeficiente vale 0.15

Los resultados obtenidos fueron

$$\overline{R}_1^T = [1.995 \text{ ton} \quad 3.049 \text{ ton m} \quad 1.792 \text{ ton} \quad 1.084 \text{ ton m}]$$

$$\overline{R}_2^T = [0.548 \text{ ton} \quad -2.897 \text{ ton m} \quad 1.980 \text{ ton} \quad -0.418 \text{ ton m}]$$

La respuesta total se obtiene utilizando el criterio propuesto en la ref. 22, según el cual

$$V_c = \sqrt{V_{c1}^2 + V_{c2}^2} \quad M_c = \sqrt{M_{c1}^2 + M_{c2}^2} \quad \text{etc.}$$

Los resultados son

$$V_c = 2.30 \text{ ton.} \quad M_c = 4.21 \text{ ton m.}$$
$$V_b = 2.67 \text{ ton.} \quad M_b = 1.16 \text{ ton m}$$

En la fig. 7a se resumen los resultados y se comparan con los de la fig. 7b obtenidos en la ref. 20. Se observa que las respuestas son muy parecidas y es seguro que al tomar en cuenta los dos modos faltantes la respuesta del caso (a) será un poco mayor.

Los resultados de este ejemplo en el que se consideraron masas y momentos de inercia virtuales, difieren poco de los obtenidos en la ref. 20 en la cual no se tomaron en cuenta dichos parámetros. Esto se debe a que la gran flexibilidad de la estructura juega un papel importante en la respuesta. Es fácil imaginar casos en los cuales ello no sucede.

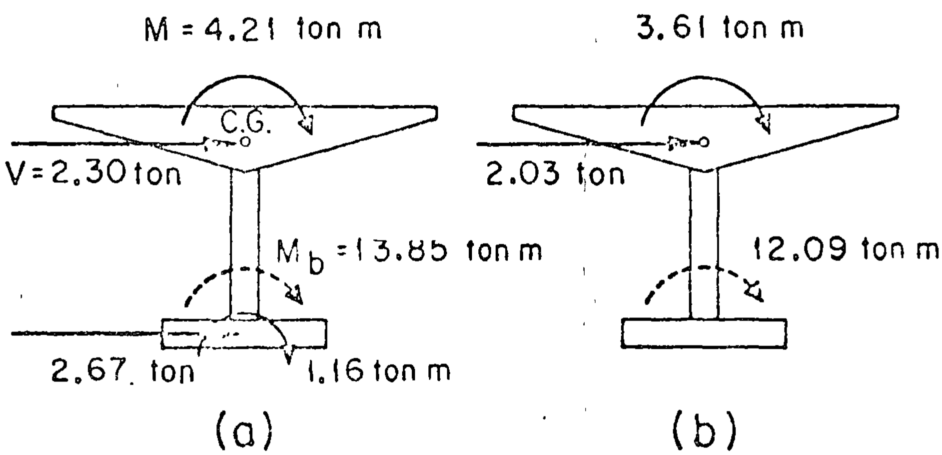


FIG. 7. *Comparación de las respuestas sísmicas a) Incluyendo masa virtual del suelo b) Despreciando la masa virtual del suelo*

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DISEÑO DE CIMENTACIONES SUJETAS A VIBRACION

Cimentaciones Reticuladas

Reglamentos

M. en I. Juan Dyer de León

Mayo de 1976.

Palacio de Minería
Tacuba 5, primer piso. México 1, D. F.
Tels.: 521-40-23 521-73-35 5123-123

1. INTRODUCCION

Aquí se presenta un estudio crítico de algunos aspectos del análisis y diseño de las estructuras y cimentaciones en que se apoyan los turbogeneradores y de las excitaciones que estos producen.

En la actualidad se puede disponer de turbogeneradores cuya potencia puede ser hasta 775 Mw; esto da lugar a cimentaciones y estructuras de apoyo muy masivas que, además de ser demasiado costosas y presentar serios problemas en su construcción, pueden sufrir asentamientos diferenciales que ocasionan funcionamiento inadecuado del equipo por desalineaciones del rotor del generador y la turbina.

El propósito que se persigue al construir así estas estructuras es evitar desplazamientos grandes de las mismas que pondrían en peligro el buen funcionamiento y la duración del equipo. Se pretende lograr esto mediante miembros muy robustos que conduzcan a que la frecuencia fundamental de resonancia de la estructura sea mayor que la de operación de la máquina, ya que si se emplearan miembros flexibles podría suceder que su frecuencia de resonancia quedara por abajo de la de operación, lo cual originaría amplitudes grandes de vibración cuando la velocidad de rotación de la flecha de la máquina pasara por la de resonancia de la estructura, durante el proceso de arranque o frenado.

Pero, si se menciona que el volumen de concreto empleado tan solo para la superestructura de un turbogenerador de 150 Mw es de 760 m^3 y que en la losa de cimentación se emplea casi igual cantidad, se aprecia la necesidad de estudiar con mayor detalle el proceso de análisis y diseño de esas estructuras, así como de conocer con precisión las excitaciones que producen los turbogeneradores.

De los libros y publicaciones consultados (algunos de los cuales aparecen al final en las referencias y bibliografía) destacan las refs 1 a 3, que de manera general presentan los criterios más usuales y las tendencias básicas actuales seguidos en Europa.

Actualmente existen normas como las DIN[4] y algunos libros dedicados exclusivamente al análisis y diseño de este tipo de estructuras[1 a 3] en los cuales se analizan, empleando los modelos que a continuación se mencionarán, algunas estructuras para turbogeneradores de potencia menor de 50 Mw.

En las dos últimas décadas se han logrado adelantos en el análisis y diseño de estructuras de ingeniería civil; sin embargo, parece que no se ha logrado un avance notable en el análisis de las estructuras de apoyo y cimentaciones de los turbogeneradores, a pesar de que se han realizado esfuerzos para analizar el comportamiento dinámico de dichas estructuras, sobre todo en Alemania y en la URSS.

Esta situación probablemente se debe a que la complejidad del problema ha obligado a formular modelos matemáticos con hipótesis que simplifican grandemente el modelo que se estudia, en relación con el prototipo (restringiendo la geometría y distribución de los miembros de la estructura), y a idealizar el comportamiento dinámico de la misma reduciendo en el modelo el problema tridimensional a uno bidimensional, en términos de sistemas discretos de masas y resortes que representan al conjunto máquina-estructura-cimentación-suelo. Para estudiar este problema se han realizado pruebas de campo, para conocer el comportamiento del suelo en términos de sus características dinámicas, se han medido amplitudes de vibración y aceleraciones en distintos puntos de las estructuras estando los turbogeneradores funcionando, y actualmente se está intentando determinar las magnitudes de fuerzas, desplazamientos y aceleraciones producidas por las máquinas mismas.[1, 2 y 5]

De la revisión de la literatura existente se puede concluir que los modelos matemáticos propuestos son muy burdos para representar a las estructuras prototipo, pues con frecuencia se recurre a modelos dinámicos de uno o dos grados de libertad (figs 1 y 2), sin considerar que las estructuras son tridimensionales y están ligadas a una cimentación que interactúa con el suelo. Para estudiarlas así habría que seguir aceptando hipótesis que las simplifiquen, pero que conduzcan a una mejor aproximación del comportamiento de los prototipos que las que actualmente se tienen. (Conviene mencionar que los modelos actuales evidentemente representan una mejor aproximación del comportamiento de las estructuras que las reglas empíricas que antes se aplicaban, como proporcionar determinada cantidad de concreto por cada unidad de potencia de la máquina.)

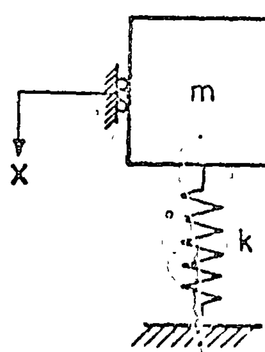


Fig 1. *Modelo matemático de un sistema de un grado de libertad, no amortiguado*

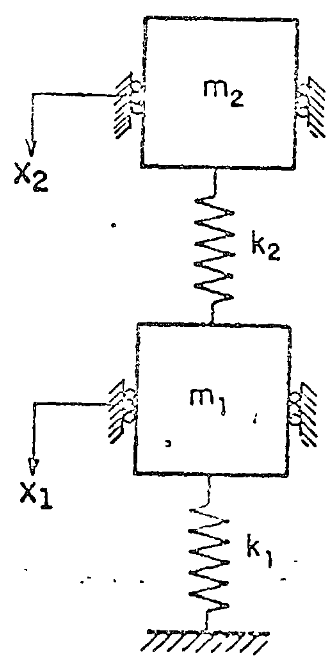


Fig 2. *Modelo matemático de un sistema de dos grados de libertad, no amortiguado*

En lo que respecta a los modelos matemáticos, mucho se puede hacer para mejorarlos; por ejemplo, se podría considerar que la masa de los miembros estructurales se encuentra repartida en ellos, y no concentrada en tres o menos puntos como suele suponerse; o se podrían formular modelos tridimensionales que tomen en cuenta el comportamiento dinámico del suelo bajo la cimentación, etc. Es justificable realizar un esfuerzo en este sentido, especialmente si se piensa que los cálculos, tal vez laboriosos, a que darían lugar los modelos más complejos que se llegaran a proponer, se podrían realizar empleando computadoras digitales.

La calibración de los distintos modelos se podría realizar mediante mediciones de campo, lo cual permitiría evaluar el grado con que estos se aproximarían al comportamiento del prototipo, o mediante modelos físicos de laboratorio.

Para el primer tipo de prueba se pueden emplear criterios como el propuesto en la ref 6, en términos de las funciones que caracterizan el comportamiento dinámico de las estructuras (funciones de transferencia) que se determinen empíricamente.

2. SOLICITACIONES

Aunque el mayor énfasis en el diseño de la estructura de apoyo de un turbogenerador se hace para que tenga un buen comportamiento bajo condiciones dinámicas, también debe resistir las cargas estáticas que sobre ella actúan. Como se verá más adelante, con frecuencia las cargas dinámicas se toman en cuenta como si fueran estáticas, empleando factores de amplificación para hacerlas equivalentes.

Las solicitaciones que usualmente se consideran son: cargas estáticas, dinámicas, par de torsión, solicitaciones sísmicas, cambios de temperatura y contracción. A continuación se describen los distintos criterios empleados en su evaluación, con lo que podrá apreciarse la precisión con que estas se conocen.

2.1 Cargas estáticas

Son las que se conocen con mayor precisión, ya que los propios fabricantes pueden pesar las diversas partes del equipo y, por consiguiente, indicar las magnitudes y puntos de aplicación de las cargas estáticas correspondientes.

2.1.1 Criterios norteamericanos

En la ref 7 (fabricante norteamericano de turbogeneradores) se presentan el valor y distribución de las cargas estáticas debidas al equipo del turbogenerador, correspondientes a cada tipo de máquina.

2.1.2 Criterio alemán

En la ref 4 se menciona que el cálculo por cargas estáticas deberá incluir los efectos del peso propio de la construcción, cargas de la máquina, incluyendo el peso del rotor, aspiración del vacío, etc., y que estas cargas se tomarán sin incrementar el valor dado por el fabricante. Se añade que las partes de construcción que no hayan de soportar cargas de la máquina se calcularán para cargas de montaje.

2.1.3 Criterio ruso

En la ref 1 se dice que para analizar una cimentación para un turbogenerador es necesario contar con información acerca de la localización de las cargas estáticas que obran sobre la cimentación, debidas tanto a partes estacionarias como rotatorias.

2.2 Cargas dinámicas

2.2.1 Criterios norteamericanos

Para efectos de análisis y diseño, un fabricante norteamericano[7] establece factores de carga por los cuales hay que multiplicar el peso de la máquina para tomar en cuenta variaciones de magnitud, distribución y punto de aplicación, y el efecto dinámico debido a desbalanceo. El incremento de cargas propuesto, para el caso de máquinas cuya velocidad de rotación sea de 3 600 rpm, es de 50 por ciento del peso de la máquina (aunque en dicha referencia no explican ni justifican el aumento propuesto).

Los componentes verticales de las cargas se localizan en los puntos de los pernos de anclaje del turbogenerador, y los horizontales a la altura del eje de la máquina o bien en la cara superior de las trabes longitudinales y transversales (fig 3).

Dependiendo de la forma como se instale el condensador (ya sea que se ligue a la turbina en forma rígida o con juntas de dilatación), se considerará determinada porción de su peso actuando sobre la estructura.

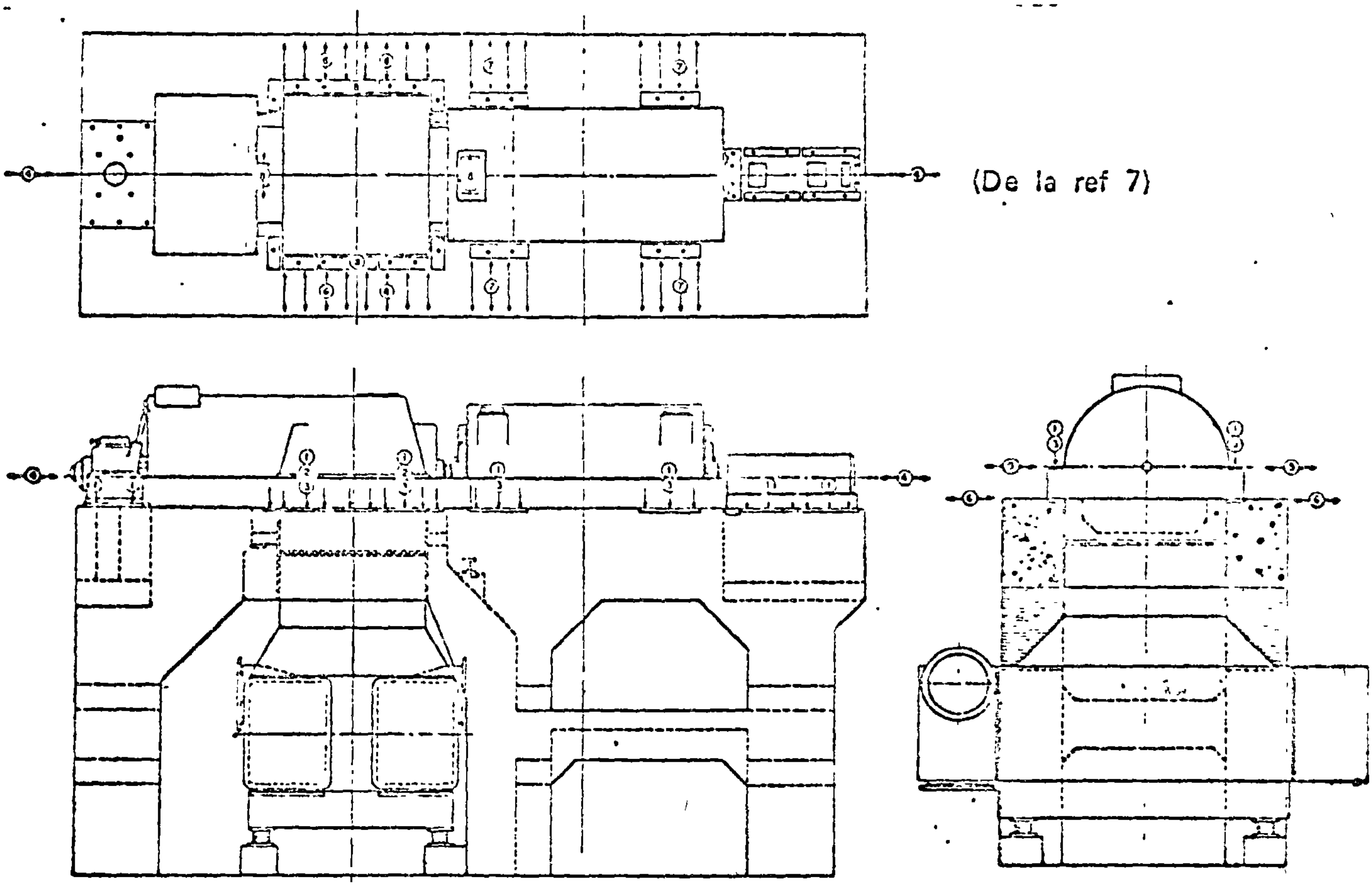


Fig 3. *Localización de las cargas que actúan sobre la cimentación de un turbogenerador*

2.2.2 Criterio alemán

En la literatura europea se aprecia que, en general, 1 a 5 se ha dedicado mayor atención al problema de cargas dinámicas, aunque también hay desconocimiento de los valores precisos que pueden adquirir. Respecto a cargas dinámicas, en la ref 4 se menciona que se deberá calcular la influencia dinámica de una fuerza centrífuga que puede actuar en todas las direcciones perpendiculares al eje de la máquina. Para ello, introducen el concepto de una fuerza "supletoria (o equivalente) estática" que es proporcional, en los mismos puntos a las cargas estáticas de la máquina, esto es, la fuerza supletoria se pondrá de modo que en cada punto que haya una carga de máquina, actúe vertical u horizontalmente una fuerza supletoria que sea proporcional a esa carga. Solo se considerarán las cargas de la máquina que se apoyen directamente sobre la estructura. Para simplificar el problema se considera que las fuerzas supletorias horizontales actúan en los ejes de las vigas.

Además, en las partes de la construcción que no estén cargadas directamente, para tomar en cuenta su vibración como conjunto, se debe emplear una fuerza supletoria de 50 por ciento del peso propio, a menos que las cargas de montaje produzcan esfuerzos más desfavorables.

2.2.3 Criterio ruso

En la ref 1, que forma la base de las especificaciones de la URSS, en relación con las fuerzas de excitación dinámica debidas a desbalanceo, se menciona que aunque teóricamente debieran ser nulas, en realidad no es posible hacer coincidir el centro de gravedad de las partes rotatorias con el eje de rotación, pudiendo llegar a ser muy grande la magnitud de estas fuerzas. Añade, además, que durante mucho tiempo la magnitud de estas fuerzas fue desconocida y, por lo tanto, en el cálculo de las cimentaciones de turbogeneradores solo se consideraban fuerzas "temporales", suponiendo que la acción estática de estas cargas era equivalente a la acción dinámica de las fuerzas de excitación reales, creadas por el desbalanceo de la máquina, con lo cual el análisis de las cimentaciones se reducía al cálculo de los esfuerzos estáticos producidos por la acción de estas cargas escogidas arbitrariamente. También se menciona que recientemente se ha reunido en la URSS bastante información sobre las vibraciones de estas máquinas, lo que ha hecho posible establecer valores para diseño de las fuerzas de excitación con un grado de exactitud suficiente para los propósitos prácticos, por lo que ya no es necesario introducir en los cálculos las cargas estáticas equivalentes antes mencionadas. Así, el método de análisis de cimentaciones ha cambiado, de manera que en lugar de emplear cargas estáticamente equivalentes, se calculan las vibraciones forzadas de las cimentaciones, producidas por las fuerzas y momentos de excitación. En consecuencia, la cimentación se puede diseñar de manera que las amplitudes de las vibraciones forzadas no excedan de los valores permisibles.

2.2.4 Comentarios

A continuación se harán algunos comentarios en relación con los conceptos expresados sobre las cargas dinámicas:

- Señalar un incremento fijo de cargas de 50 porciento del peso de la máquina,[7] sin que, en todo caso, dependa de la curva de amplificación dinámica de la estructura que se tenga, con toda seguridad implica un profundo desconocimiento de la magnitud de las fuerzas dinámicas y, por lo tanto, se les asignan valores arbitrariamente incrementados, los que pueden ser mayores o menores que los reales.

- En la ref 4 no se aclara o justifica el alcance y limitaciones de suponer una fuerza supletoria (equivalente) estática para tomar en cuenta los efectos dinámicos producidos por la máquina (este punto se volverá a discutir en el próximo capítulo).

- Tampoco se aclara por qué o cómo se eligió una fuerza supletoria del 50 por ciento de la carga propia para elementos estructurales que no estén cargados de manera directa; esto quizá se deba al efecto dinámico de las excitaciones que le trasmiten otros miembros estructurales que sí están cargados.

- Aunque se conozca la existencia de fuerzas de desbalanceo debidas a que el centro de gravedad de las partes rotatorias no coincide con el eje de rotación,[1] sus magnitudes no se conocen por anticipado, ya que dependen de características individuales de cada rotor, tales como marca de fábrica, diferencias aleatorias en la fabricación, diferentes formas de desgaste del eje en las chumaceras, etc.

- Es necesario hacer mediciones rutinarias en prototipos de las excitaciones que producen los turbogeneradores de diversas potencias, para contar con fuerzas y momentos dinámicos más realistas para el análisis dinámico de la estructura y cimentación.

2.3 Par de torsión

2.3.1 Criterios norteamericanos

En la ref 7 se presenta una expresión para valuar el par de torsión del turbogenerador. Se recomienda que las vigas y columnas se diseñen para un par de torsión cinco veces mayor que el normal (en condiciones accidentales, tales como un cortocircuito o falla mecánica de alguna parte de la máquina, pueden presentarse pares de torsión de 50 veces el normal).

2.3.2 Criterio alemán

El momento de cortocircuito se tomará como un par de fuerzas verticales trasmitido por la máquina en ambos sentidos de rotación. El valor para usar en el análisis es del doble del que proporciona la fábrica de la maquinaria (ref 4).

2.3.3 Criterio ruso

No fue posible obtener la información correspondien diente para el criterio ruso.

2.3.4 Comentarios

En la ref 7 no se aclara por qué se debe tomar un par de torsión cinco veces mayor que el normal, si se contempla la posibilidad de que se tenga uno de hasta 50 veces el normal. Tal vez lo hagan considerando que se cuenta con sistemas automáticos que paran la máquina de inmediato, con lo cual la duración del par accidental es tan corto que con los incrementos indicados la estructura queda en condiciones de soportar el efecto de tales pares. Puesto que no aclaran tal diferencia, se puede suponer que hay desconocimiento del efecto dinámico que produce el par de torsión, y de cómo trabaja la estructura para tomarlo.

En la ref 4 no se aclara el motivo del aumento de un cien por ciento sobre los valores proporcionados por la fábrica de la maquinaria. Además, como se puede apreciar, los valores propuestos difieren grandemente de los de la ref 7, lo cual apoya al comentario anterior.

2.4 Solicitaciones sísmicas

2.4.1 Criterios norteamericanos

Se sugiere[7] que, debido a la posibilidad de efectos destructivos producidos por sismos, se preste particular atención a la continuidad de las juntas, al anclaje de los extremos de las columnas, y a otros detalles que contribuyen a la rigidez de la estructura.

En las zonas de alta sismicidad se deberán emplear fuerzas horizontales adicionales, de acuerdo con la información local de que se disponga.

2.4.2 Criterio alemán

2.4.3 Criterio ruso

No fue posible obtener la información correspondiente a estos criterios.

2.4.4 Comentarios

- Tanto la continuidad de las juntas como el anclaje en los extremos de las columnas más bien contribuyen a la ductilidad que a la rigidez de la estructura.

2.5 Solicitaciones por temperatura y contracción

2.5.1 Criterios norteamericanos

Se recomienda[7] que se minimicen las fuerzas producidas por cambios de temperatura mediante refuerzo de acero en las caras de los elementos de concreto, pudiendo recurrir en casos extremos a aislantes térmicos o a ventilación especial. También se menciona que deben tomarse en cuenta las cargas producidas por expansión o contracción de tuberías y líneas sometidas a presión o vacío que tengan juntas de expansión. Respecto a la contracción del concreto indican que gran parte de esta tiene lugar en pocas horas después del colado y que continúa, aunque a menor velocidad, durante algún tiempo, siendo el tiempo necesario para llegar a la contracción total, una función de la temperatura, riqueza de la mezcla, cantidad de agua, volumen del concreto y velocidad del colado. Además, se menciona como ventaja del concreto reforzado que, por su baja conductividad térmica, no permite una rápida distorsión debido a calentamiento local, y que puede resistir altas temperaturas resintiendo pocos daños.

2.5.2 Criterio alemán

Los efectos de temperatura y contracción, para estructuras de concreto, se toman en cuenta como sigue:[4] Para la contracción de la losa de apoyo de la turbina, respecto a la losa de cimentación, se debe considerar una caída equivalente de temperatura de 10°C; si entre la construcción de ambas losas transcurren más de dos meses, la diferencia de temperatura será de 15°C. Para considerar el calentamiento uniforme entre las losas de cimentación de apoyo del turbogenerador, cuando se haya previsto una protección contra el calor mediante aislamiento térmico, debido a que no se pueden proporcionar datos más precisos, se supone un aumento de temperatura de 20°C, pudiendo deducirse la cantidad debida a contracción; por lo tanto, la contracción y el calentamiento uniforme se han de considerar bajo la hipótesis de que ocurre un cambio de temperatura desde −10°C hasta −15°C o desde 10° hasta 5°C, respectivamente.[4]

Además, la parte de la estructura correspondiente a la turbina se debe calcular imponiendo sobre los miembros un gradiente de temperatura de ± 10°C (en la parte interior 20°C más caliente que afuera), para proteger el concreto contra el calor.

Recomiendan que, para evitar grietas en las estructuras de concreto, el refuerzo de cada elemento constructivo, excepto la losa de cimentación, sea como mínimo de 50 kg/m^3 de concreto, y deberá colocarse siguiendo tres ejes ortogonales, aun cuando la cantidad requerida por el cálculo fuera menor; además, para evitar la tendencia a la contracción y aumentar la resistencia a la tracción, indican que es conveniente usar concreto elaborado con poca agua.

2.5.3 Criterio ruso

Se dice[1] que, debido a que los turbogeneradores trabajan con vapor a altas temperaturas, se deberá procurar un aislamiento térmico adecuado a las tuberías que conduzcan vapor o aire caliente y que deberán aislarse, por lo menos, hasta que se encuentren fuera de la estructura. El aislante deberá ser tal, que la temperatura de la superficie exterior del aislamiento no exceda de 40 a 50°C, ya que de otra forma se desarrollarán en la estructura esfuerzos considerables por temperatura, por lo que es objetable la instalación de tuberías de vapor o aire caliente directamente dentro de la estructura.

2.5.4 Comentarios

Respecto al tema del subcapítulo 2.5, se pueden hacer los siguientes comentarios:

En general se aprecia que se dispone de poca información en lo que a temperatura se refiere, pues solo en una de las referencias se menciona de manera directa la existencia de gradientes de temperatura, y aun en este caso se suponen fijos; en lo que se refiere a contracción, son aún más vagas las recomendaciones.

En la ref 7 se menciona como ventaja del concreto lo que en realidad representa una desventaja, ya que debido a su baja conductividad cualquier gradiente de temperatura puede originar distorsiones locales y aun altos niveles de esfuerzos.[8]

En la ref 4 no explican cómo se eligieron las diferencias de temperatura para tomar en cuenta los efectos de contracción entre las losas de la estructura, ni cómo se decidió fijar la cantidad mínima de acero de refuerzo para estructuras de concreto, para evitar el agrietamiento.

Apoyándose en los anteriores comentarios, es posible suponer que se desconoce en gran medida no solo la distribución y magnitud de las cargas dinámicas, sino en general todas las solicitaciones, y que estas se han tomado en cuenta de manera arbitraria y mediante consideraciones que están basadas en el buen funcionamiento de cimentaciones ya construidas, pues, dado el tamaño de las máquinas actuales, dichas especificaciones podrían resultar excesivamente conservadoras, lo que naturalmente implica problemas tanto constructivos como de costo de la estructura.

3. CRITERIOS ACTUALES DE ANALISIS DINAMICO

En este capítulo se discuten los criterios de análisis que con más frecuencia se emplean en Europa y en Estados Unidos de Norteamérica.

3.1 Criterios norteamericanos

Los criterios que en general se siguen en Estados Unidos para analizar las estructuras de apoyo de turbogeneradores, aparecen en publicaciones elaboradas por los propios fabricantes del equipo,[7,9] quienes aparentemente se apoyan en su experiencia previa, en datos proporcionados por sus departamentos de proyectos e ingeniería, o bien en las opiniones de sus consultores, ya que en las publicaciones no aclaran de dónde proviene la información que proporcionan, y solo mencionan que las estructuras diseñadas siguiendo tales normas han tenido éxito en muchas instalaciones. (Cabe mencionar que en la ref 7 se proporcionan diseños de estructuras de apoyo para turbogeneradores con una potencia que va desde 2 750 kw hasta 775 000 kw.)

En lo que respecta a los desplazamientos totales máximos permisibles de la estructura, en la ref 7 se establece, apoyándose en reglas empíricas, que se obtendrán estructuras suficientemente rígidas, para velocidades de operación de las máquinas de 1 800 rpm y mayores, si los desplazamientos verticales u horizontales totales de cualquier miembro estructural, producidos por la combinación máxima de cargas de diseño (que los fabricantes suministran dando su magnitud, dirección y punto de aplicación), se limitan a 20 mils (0.020 pulg), lo cual se logra con una frecuencia natural de los miembros que satisfaga la ecuación

$$\delta = \left(\frac{187.7}{N}\right)^2 \qquad (1)$$

donde δ es el desplazamiento, en pulg, y N la frecuencia natural del miembro, en rpm. Además, agregan que la frecuencia de cada miembro de la cimentación debe diferir por lo menos en 10 por ciento de la velocidad de rotación en operación normal de la máquina, para evitar las amplificaciones dinámicas grandes de los desplazamientos que ocurren en la zona de frecuencias cercanas a la de resonancia.

Este desplazamiento máximo admisible obliga a diseños que conducen a miembros estructurales muy robustos, dando por resultado estructuras tan rígidas que en la mayoría de los casos quedan clasificadas como de *alta sintonía,* concepto que se introducirá más adelante.

La ec 1 se obtiene a partir de

$$\omega = \sqrt{\frac{gk}{W}} \qquad (2)$$

tomando en cuenta que $W/k = \delta$; en esta ecuación, ω es la frecuencia angular de vibración, en rad/seg, de un sistema de un grado de libertad, g la aceleración de la gravedad, k la rigidez del sistema (fig 1), W el peso del marco más la parte que le corresponda del peso del turbogenerador, equipo auxiliar, etc., incrementado en un 25 o 50 por ciento (como se indicó en el cap 2) para tomar en cuenta en forma indirecta el efecto dinámico. Por lo tanto, en la ec 1, δ no representa el desplazamiento dinámico real sino uno estático. Por consiguiente, el modelo dado por esa ecuación deja mucho que desear en su aplicación a vibraciones verticales y horizontales, ya que no considera la amplificación dinámica real del desplazamiento de la estructura (fig 5).

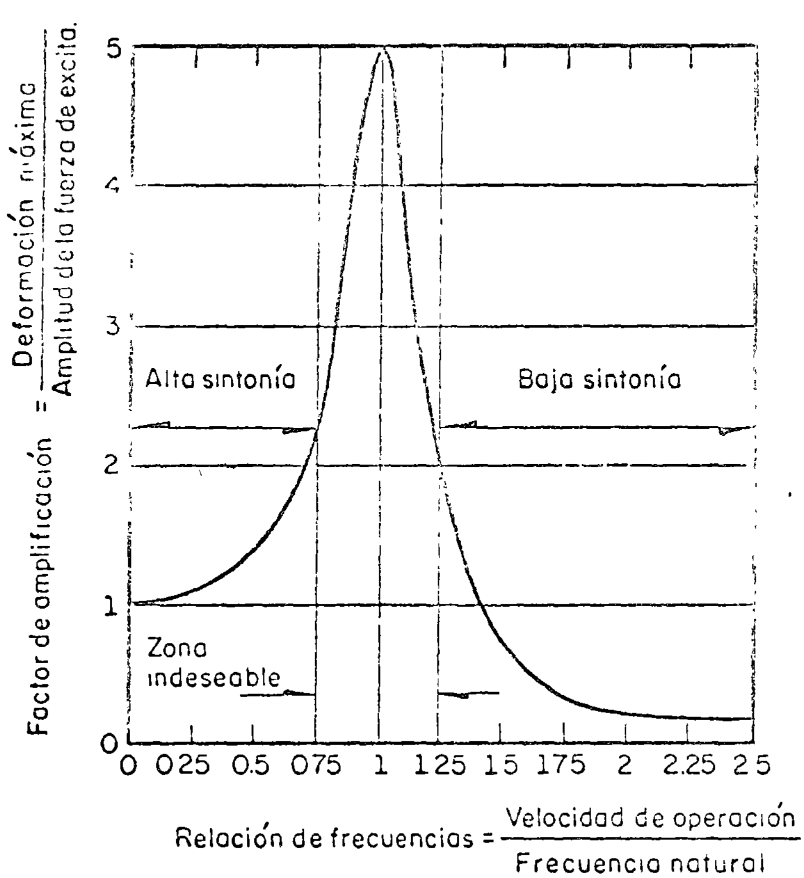


Fig 5. *Gráfica factor de amplificación-relación de frecuencias para un sistema de un grado de libertad*

Por otra parte, hay que tener presente que la ec 1 proviene de la ecuación para calcular la frecuencia circular de un sistema "masa-resorte" (fig 1) de un grado de libertad, que es una idealización bastante burda para representar el sistema continuo "estructura-cimentación-suelo", que tiene un número infinito de grados de libertad.

3.2 Criterios europeos

En Europa, aparentemente, se le ha prestado mayor atención al establecimiento de los criterios de análisis, principalmente en Alemania y Rusia, donde existen normas expedidas por oficinas gubernamentales, como las DIN[4] alemanas, que son las más empleadas.

Como se mencionó en el cap 1, de entre las publicaciones escritas sobre este problema destacan las refs 1 a 3, que de manera general presentan los criterios más usuales y las tendencias básicas seguidas actualmente en Europa.

Los criterios más comunes, que aparecen en la ref 2, se pueden clasificar de manera general en *método de resonancias* y *método de amplitudes*, los cuales se describirán brevemente a continuación, teniendo en cuenta que para analizar una estructura empleando cualquiera de estos métodos, es necesario conocer las cargas muertas y vivas que actúan sobre ella.

3.2.1 Método de resonancias (criterio alemán)

Al seguir este criterio se debe analizar la estructura de tal forma que el modelo que la represente tenga una frecuencia fundamental calculada, en rpm, que difiera en no menos del 10 por ciento de la velocidad de rotación en operación normal del turbogenerador. Si la frecuencia fundamental del modelo es mayor que la de rotación, se tiene una estructura de *alta sintonía*; en caso contrario será de *baja sintonía* (fig 5). En este último caso, la frecuencia de rotación estará entre la fundamental y una armónica superior. Como puede observarse, dependiendo de las dimensiones y disposición de los miembros de la estructura, se puede tener baja o alta sintonía en dirección vertical u horizontal. Para emplear el método de resonancias se idealiza la estructura mediante algún modelo matemático que, para el caso de vibraciones verticales, consiste en un sistema de dos grados de libertad que representa a uno de los marcos transversales de la propia estructura (fig 2); para que se justifique el empleo de dicho modelo se debe procurar que los marcos transversales de la estructura tengan frecuencias naturales semejantes entre sí, y suponer que las vigas longitudinales que unen los marcos son suficientemente flexibles en torsión, para poder analizarlos por separado.

El modelo de dos grados de libertad para vibraciones verticales tiene dos masas, una de ellas está dada por el peso correspondiente de la máquina sobre el marco y el de la viga transversal, y la otra por el peso de las vigas longitudinales y una fracción (33 por ciento) del peso propio de las columnas; los dos resortes del modelo corresponden, uno a la rigidez en flexión y cortante de la viga transversal, y el otro a la rigidez bajo fuerza axial de las columnas del marco.

Aunque este modelo de los marcos transversales tiene dos frecuencias naturales, en las refs 2 y 3 tan solo se calcula una, que es una aproximación, N_v, en rpm, de la frecuencia fundamental vertical del modelo, mediante la ecuación

$$N_v = \frac{300}{\sqrt{\delta_1 + \delta_2 + \delta_3}} \qquad (3)$$

donde δ_1 y δ_2 representan las deflexiones, en cm, del centro de la viga del marco debidas a flexión y cortante, respectivamente, y δ_3 corresponde a la deformación axial por compresión de las columnas del marco. Las fuerzas que producen las deflexiones δ_1, δ_2 y δ_3 corresponden al peso propio de la viga, de la máquina, del equipo auxiliar, de las vigas longitudinales y de las columnas.

La ec 3 proviene de aplicar la aproximación de Southwell-Dunkerley[10] al considerar que el modelo tiene tres tipos de deformación independientes: por flexión y cortante de la trabe y deformación axial de las columnas, y las distintas frecuencias, ω_i, estarían dadas por expresiones del tipo

$$\omega_i = \sqrt{\frac{k_i}{m_i}} = \sqrt{\frac{g}{\delta_i}} \qquad (4)$$

donde k_i son las distintas rigideces calculadas, m_i las masas que participan en la vibración, y δ_i los desplazamientos estáticos de las masas correspondientes.

Al aplicar la aproximación de Southwell-Dunkerley se tendrá

$$\frac{1}{\omega^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \frac{1}{\omega_3^2} = \frac{\delta_1 + \delta_2 + \delta_3}{g} \qquad (5)$$

luego

$$\omega = \frac{\sqrt{g}}{\sqrt{\delta_1 + \delta_2 + \delta_3}} \qquad (6)$$

Y, tomando en cuenta que $N = \frac{60}{2\pi}\omega$ (rpm), se llega a (ec 3)

$$N_v = \frac{300}{\sqrt{\delta_1 + \delta_2 + \delta_3}}$$

Se debe insistir en que aun cuando por su presentación la ec 3 aparente dar la frecuencia de un sistema de un grado de libertad, en realidad corresponde a una aproximación a la frecuencia fundamental de un sistema de dos grados de libertad. El hecho de solo considerar esta aproximación al primer modo es razonable para estructuras de alta sintonía, si se considera que la frecuencia correspondiente al segundo modo es muy superior a la del primero, y que por consiguiente su contribución a la respuesta de la estructura es pequeña. Cabe mencionar también que en este modelo no se ha considerado la interacción con el suelo debajo de la cimentación.

Mediante una modificación a la ec 3, que considera que la losa superior, el peso de la máquina y una parte de las columnas del marco forman una sola masa, y que el condensador, la parte inferior de las columnas y la losa de cimentación corresponden a otra masa, se puede tomar en cuenta la flexibilidad del suelo, considerando que este y las columnas se pueden idealizar como resortes en serie con las masas antes citadas.

Este modelo toma en cuenta la flexibilidad del suelo, pero no incluye la masa del mismo que participa en el problema dinámico; además, puesto que se sigue tratando de un modelo de dos grados de libertad, tal vez se siga sobresimplificando el problema.

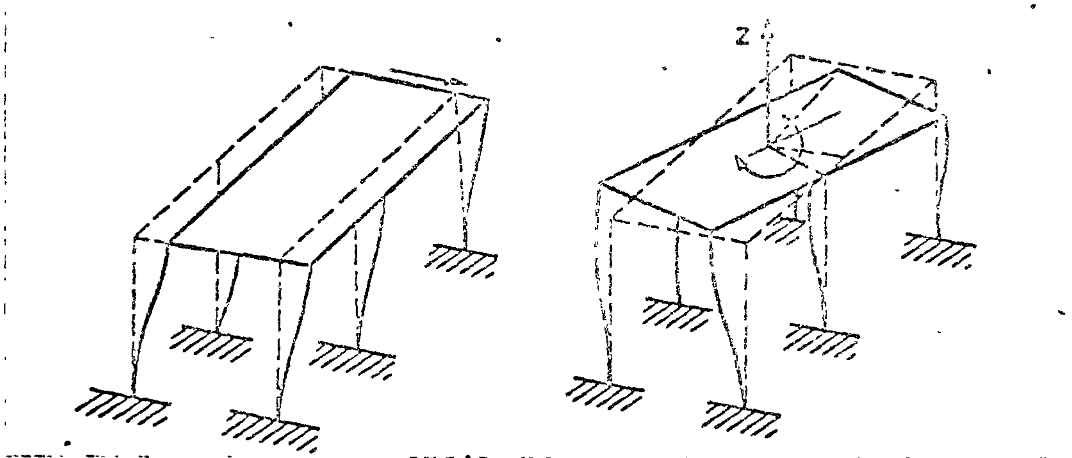
Para calcular las frecuencias naturales horizontales con el método de resonancias, se proponen dos maneras;[2] la primera permite calcular solo en forma aproximada las frecuencias naturales horizontales sin tomar en cuenta las propiedades del suelo, y acepta que la losa de apoyo de la máquina es infinitamente rígida y se encuentra sobre apoyos elásticos (representados por las columnas); además, el método requiere que las rigideces de los marcos transversales sean prácticamente iguales.

La frecuencia natural del sistema, N_h, en rpm, se calcula con la ecuación

$$N_h = \frac{300}{\sqrt{\Delta_h}} \qquad (7)$$

donde Δ_h, en cm, varía entre δ_h y $0.8\,\delta_h$, siendo δ_h el desplazamiento horizontal, considerando que el peso de la máquina, la losa superior y la tercera parte de las columnas actúan horizontalmente.

El segundo método propuesto, que puede tomar en cuenta las propiedades del suelo, supone que los marcos transversales están ligados entre sí por una losa superior infinitamente rígida, y que las columnas están empotradas en la losa de cimentación, y esta a su vez sobre suelo deformable, representado por resortes sin masa. El modelo matemático empleado puede ser de dos o cuatro grados de libertad acoplados, si no se toma en cuenta el suelo, uno corresponde al desplazamiento horizontal y el otro a un giro respecto a un eje vertical que pasa por el centro de masas (torsión) (fig 6). Cuando sí se toma en cuenta el suelo, se agregan dos grados de libertad y se tienen cuatro frecuencias naturales, dos correspondientes a desplazamiento horizontal y dos a giro en un plano horizontal (torsión), de las losas de cimentación y de apoyo de la máquina.



a) Traslación *b) Rotación*

Fig6. Modelo para calcular frecuencias naturales horizontales de la cimentación

La participación del suelo está dada en términos del coeficiente de compresión no uniforme, C_φ, el cual depende de las propiedades elásticas del suelo y del área de la base de la cimentación;[1] con este coeficiente se toma en cuenta la flexibilidad del suelo, pero no se considera la masa de este que participa en las vibraciones. Además, el coeficiente C_φ se determina de manera estática, y no depende de la frecuencia de la excitación.[11]

Finalmente, se calculan las amplitudes de los desplazamientos producidos por las fuerzas dinámicas,[2] para verificar que se encuentren por abajo de los valores permisibles. En general, las fuerzas dinámicas se transforman en cargas estáticas "equivalentes", mediante un coeficiente que supuestamente toma en cuenta los problemas de cargas repetidas y de amplificación dinámica. Para apoyar y reforzar lo anterior, a continuación se hacen algunos comentarios a lo escrito en el subcapítulo 3.6 de la ref 4 (ver Apéndice), respecto a la carga estática equivalente y al coeficiente que antes se mencionaron:

- Se acepta que para determinar lo que llaman "fuerza de excitación", K, se parte de apreciaciones muy toscas, ya que para determinar su valor hicieron encuestas entre los fabricantes de las máquinas y llegaron a la ec 1 de esa referencia, donde aparece un coeficiente, k, que toma en cuenta el desequilibrio de la máquina, cuyo valor han fijado en 10. Por otro lado, a lo que llaman "calidad de equilibrio", ew_n, le asignan 0.15 cm/seg. Además, aunque el valor de K depende del cuadrado de la frecuencia de rotación de la máquina, w_n^2, finalmente solo es función lineal de ella, aduciendo que se ha tomado así puesto que es más fácil equilibrar el rotor de la máquina para altas velocidades.

- Se menciona, además, que debido a la inseguridad con que se determina la fuerza de excitación, K, lo cual se hace de manera aproximada, para hacer cálculos es suficiente obtener una "fuerza estática equivalente", P; esta se obtiene mediante la ec 2 de esa referencia, en la que aparece un "coeficiente de fatiga", μ, al que asignan el valor 3, que relaciona la carga estática equivalente con la correspondiente dinámica. Cabe mencionar que el valor máximo de la fuerza estática equivalente aparece en la ec 3 de dicha referencia y tiene un valor de 15 veces el peso de la máquina para una velocidad de 3 000 rpm.

Por consiguiente, solo se tiene una idea aproximada del valor de la carga estática equivalente y del coeficiente mencionados.

Las amplitudes de las vibraciones, A, tanto verticales como horizontales, se calculan con la ecuación

$$A = \delta_c \nu \qquad (8)$$

donde δ_c es la deflexión vertical u horizontal del elemento considerado, debido al efecto estático de la fuerza de excitación, y ν el factor de amplificación dinámica, en el cual se toma en cuenta el amortiguamiento del sistema. Este valor de ν es el correspondiente a un sistema de un grado de libertad, cuyo valor depende del cociente de la frecuencia natural calculada del sistema y la velocidad de operación de la máquina. El valor de δ_c se obtiene multiplicando el desplazamiento debido al efecto estático de una fuerza unitaria por la fuerza de excitación, la cual es función de la masa de las partes móviles y de la velocidad de rotación de la máquina.

3.2.2 Método de amplitudes (criterio ruso)

En este criterio, la idea básica consiste en lograr que la respuesta del modelo matemático con que se idealiza la estructura no conduzca a desplazamientos dinámicos mayores que ciertos valores permisibles, los cuales aparecen en la ref 1, y a continuación se reproducen:[11]

Máquinas de	Amplitudes permisibles
3 000 rpm	
Vibraciones verticales	0.02 a 0.03 mm
Vibraciones horizontales	0.04 a 0.05 mm
1 500 rpm	
Vibraciones verticales	0.04 a 0.06 mm
Vibraciones horizontales	0.07 a 0.07 mm

Como se dijo con anterioridad, en el método de amplitudes, que en apariencia es el utilizado en la URSS, las amplitudes calculadas de las vibraciones se deben comparar con las permisibles para cada tipo de estructura. Si los valores obtenidos son menores que los establecidos, se acepta el diseño de la estructura; en caso contrario, esta se debe rediseñar.

Las hipótesis fundamentales del método de amplitudes también consisten en considerar que los marcos transversales tienen iguales deformaciones, y que la capacidad en torsión de las vigas longitudinales que unen los marcos es pequeña comparada con las deformaciones de las vigas transversales. Con estas suposiciones es admisible calcular, de manera individual, las frecuencias naturales de vibración vertical de cada marco.

Las frecuencias naturales verticales se obtienen empleando un modelo de dos grados de libertad, uno de los cuales es el desplazamiento del centro de la viga del marco, donde se concentran las masas formadas por el equipo y una fracción del peso propio de la viga, m_2, y el otro es el desplazamiento de la parte superior de las columnas, donde se concentran las masas correspondientes a fracciones de la viga longitudinal, la viga transversal incluyendo el equipo correspondiente y la columna, $m_1/2$ (fig 7). Las porciones de masas concentradas se toman de manera que tengan propiedades dinámicas "equivalentes" a las que tendrían si se les considerará distribuida; basan la equivalencia de propiedades dinámicas en que la carga concentrada que se considera debe tener igual energía cinética que si se le considera como uniformemente distribuida.

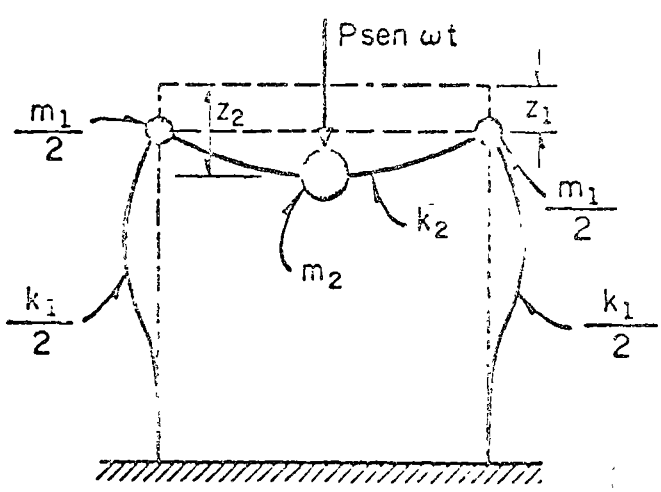


Fig 7. Modelo para calcular desplazamientos verticales de la cimentación

Las rigideces de los resortes del modelo corresponden a la fuerza necesaria, k_2, para producir un desplazamiento unitario en el centro de la viga, y la necesaria, $k_1/2$, para producir un acortamiento unitario en las columnas. Las frecuencias se determinan con la ecuación correspondiente a un sistema de dos grados de libertad, obteniéndose dos frecuencias naturales, en rad/seg, ω_1 y ω_2, tales que $\omega_1 < \omega_2$.

En la ref 1 se dice lo siguiente respecto a las frecuencias naturales, para justificar el hecho de que el modelo no considere la elasticidad del suelo de la cimentación:

"Si se considera cada marco como un sólido infinitamente rígido en su plano, apoyado sobre una base elástica, entonces su frecuencia natural de vibración vertical, ω_z, será, generalmente, mucho menor que la de operación del turbogenerador; además, llamando ω_1 a la menor de las frecuencias calculadas de vibración vertical del marco, y suponiendo que la base es absolutamente rígida, esta frecuencia depende solo de las propiedades elásticas y de inercia del marco. Para turbogeneradores diseñados con el criterio de alta sintonía, en general existe la relación

$$\omega_z < \omega < \omega_1 \tag{9}$$

donde ω es la velocidad de rotación de la máquina en rad/seg. Si se toma en cuenta la elasticidad del suelo de cimentación y la flexibilidad del marco, entonces las dos menores frecuencias de vibración vertical, ω_z^* y ω_1^*, del sistema marco-losa rígida-base elástica, tendrán la siguiente interrelación con las anteriores:

$$\omega_z^* < \omega_z < \omega < \omega_1 < \omega_1^* \tag{10}$$

"La desigualdad 10 muestra que al tomar en cuenta la elasticidad del suelo se aumenta la diferencia de las dos frecuencias de vibración respecto a la de rotación de la máquina, basada solo en las propiedades elásticas y de inercia del marco.

"Si se satisface la condición 10 en el diseño de una cimentación, el despreciar las propiedades elásticas del suelo en el cálculo de las vibraciones verticales en la cimentación contribuye a aumentar el factor de seguridad de la estabilidad dinámica de la misma. En tal caso, el problema de las vibraciones forzadas de una cimentación a base de marcos, se puede reducir al cálculo de vibraciones de marcos planos empotrados en una base inmóvil infinitamente rígida."

Se pueden hacer los siguientes comentarios de lo anterior:

- Hay que tener presente que el comentario acerca de que $\omega_z < \omega_i$ solo es cierto para suelos no muy rígidos, y que la desigualdad 10 solo es válida para estructuras de alta sintonía. Para las de baja sintonía, el efecto del suelo sí podría ser importante.

- Se debe tener presente que suponer que el suelo o los marcos sean infinitamente rígidos solo son casos extremos para acotar el problema, ya que si en realidad se supone que el suelo es infinitamente rígido, se está sobresimplificando el problema, pues no se considera la interacción suelo-estructura en términos de su rigidez real y de la masa de suelo que participa en la vibración, y tampoco es totalmente válido suponer que los marcos en la realidad sean infinitamente rígidos, pues a pesar de ser robustos, hay que tomar en cuenta que la estructura sí es flexible.

- Como se volverá a mencionar, el empleo de bloques de cimentación de masa mayor que la requerida acentúa los problemas de asentamiento y de costos.

Debido a que en el método de amplitudes la base de decisión acerca de un diseño es la amplitud de las vibraciones, para no tomar en cuenta el amortiguamiento del sistema, se debe verificar que las frecuencias fundamentales del modelo de la estructura difieran por lo menos en 30 por ciento de la frecuencia de rotación de la máquina en operación normal, de no ser así, sí se deberá incluir el efecto del amortiguamiento. En la ref 1, y sin dar explicación de cómo se obtuvo ese dato, se recomienda que se utilice un amortiguamiento del 5 al 10 por ciento del crítico.

La amplitud total de las vibraciones verticales se calcula como la suma de las dos amplitudes máximas, en valor absoluto, correspondientes a cada grado de libertad.

Para calcular la amplitud de las vibraciones transversales horizontales, se supone que la losa superior es absolutamente rígida en su plano y que las columnas están empotradas en una losa indeformable; esto es, no se considera la elasticidad de la losa superior ni la del bloque de cimentación y del suelo. Se trabaja con un modelo de dos grados de libertad, correspondientes al desplazamiento lateral de la losa superior, x, y el ángulo de rotación horizontal, ϕ, respecto a un eje vertical que pasa por el centro de masas (fig 8):

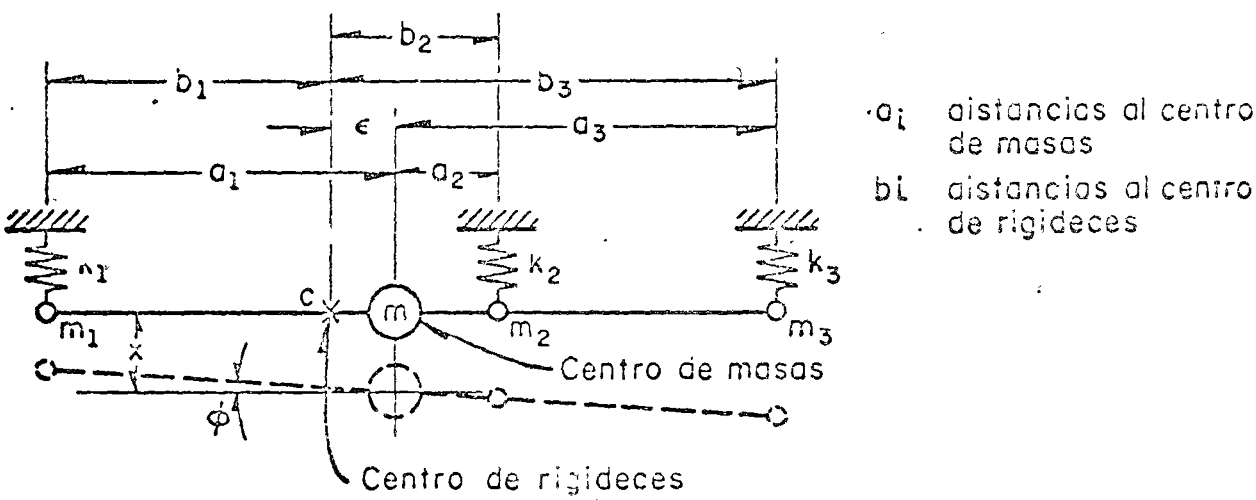


Fig 8. Modelo para calcular desplazamientos horizontales de la cimentación

Las masas que se toman en cuenta son: las cargas muertas que obran sobre las vigas transversales y longitudinales, su propio peso, y una porción de la masa de las columnas, m_i. La rigidez de cada uno de los marcos transversales, k_i, es la fuerza necesaria para producir un desplazamiento horizontal unitario del marco, aplicada en la parte superior de las columnas.

Como en el caso de vibraciones verticales, se debe cuidar que las frecuencias fundamentales difieran más de 30 por ciento de la de rotación de la máquina, pues de no ser así deberá tomarse en cuenta el amortiguamiento del sistema. Las amplitudes finales se calculan como la suma de las correspondientes al desplazamiento más las producidas por la rotación en torsión.

Cabe mencionar que para evaluar las amplitudes, se supone que las fuerzas de excitación en el generador y en la turbina actúan en la misma dirección en cada instante, y que la estructura se desplaza de manera rígida. En contra de esta suposición se ha podido verificar, mediante mediciones en estructuras,[5, 12] que los marcos no vibran en fase y, por lo tanto, el valor del momento de excitación se puede ver muy aumentado, con lo que también se incrementarían las amplitudes de vibración, que podrían excederse de las permisibles.

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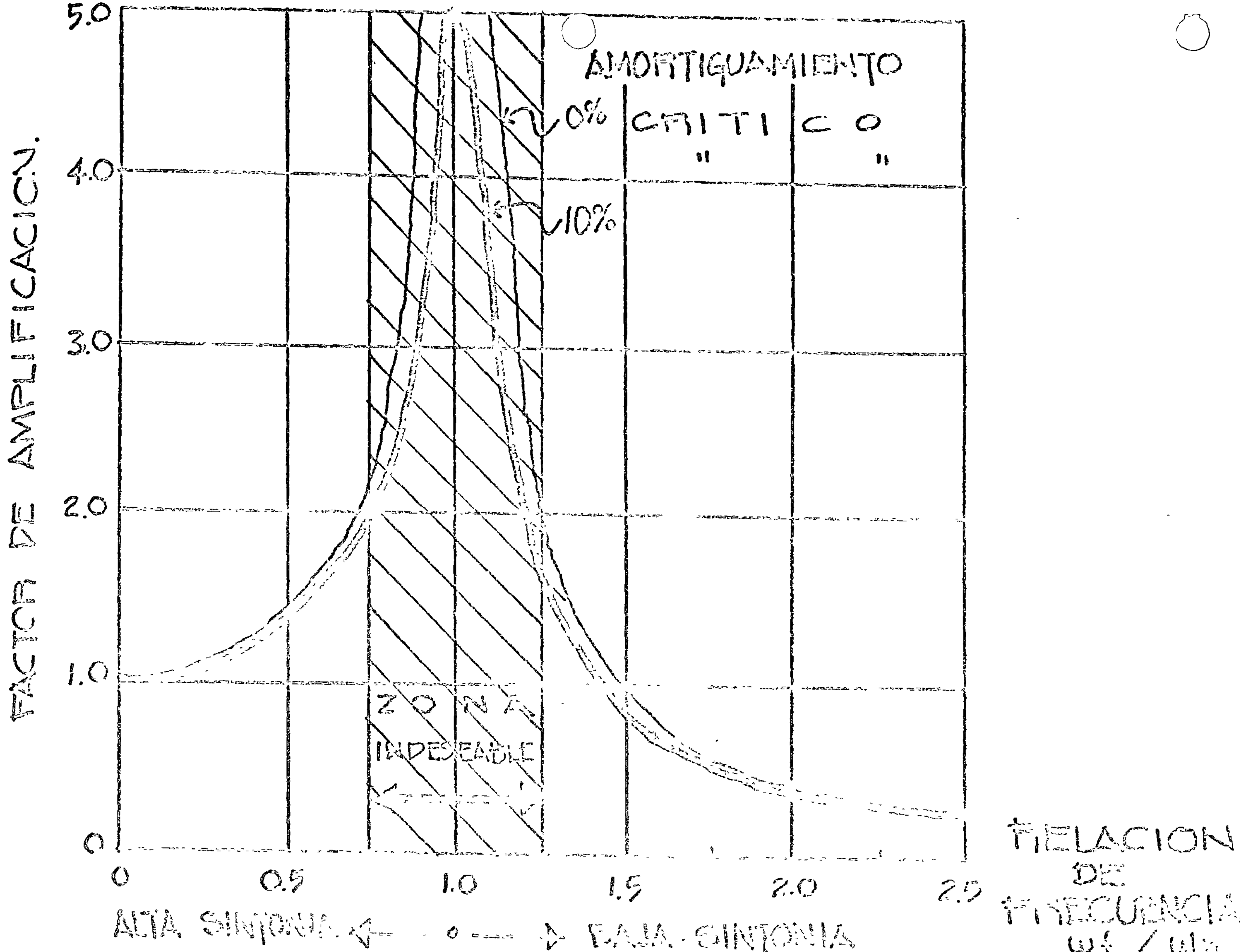
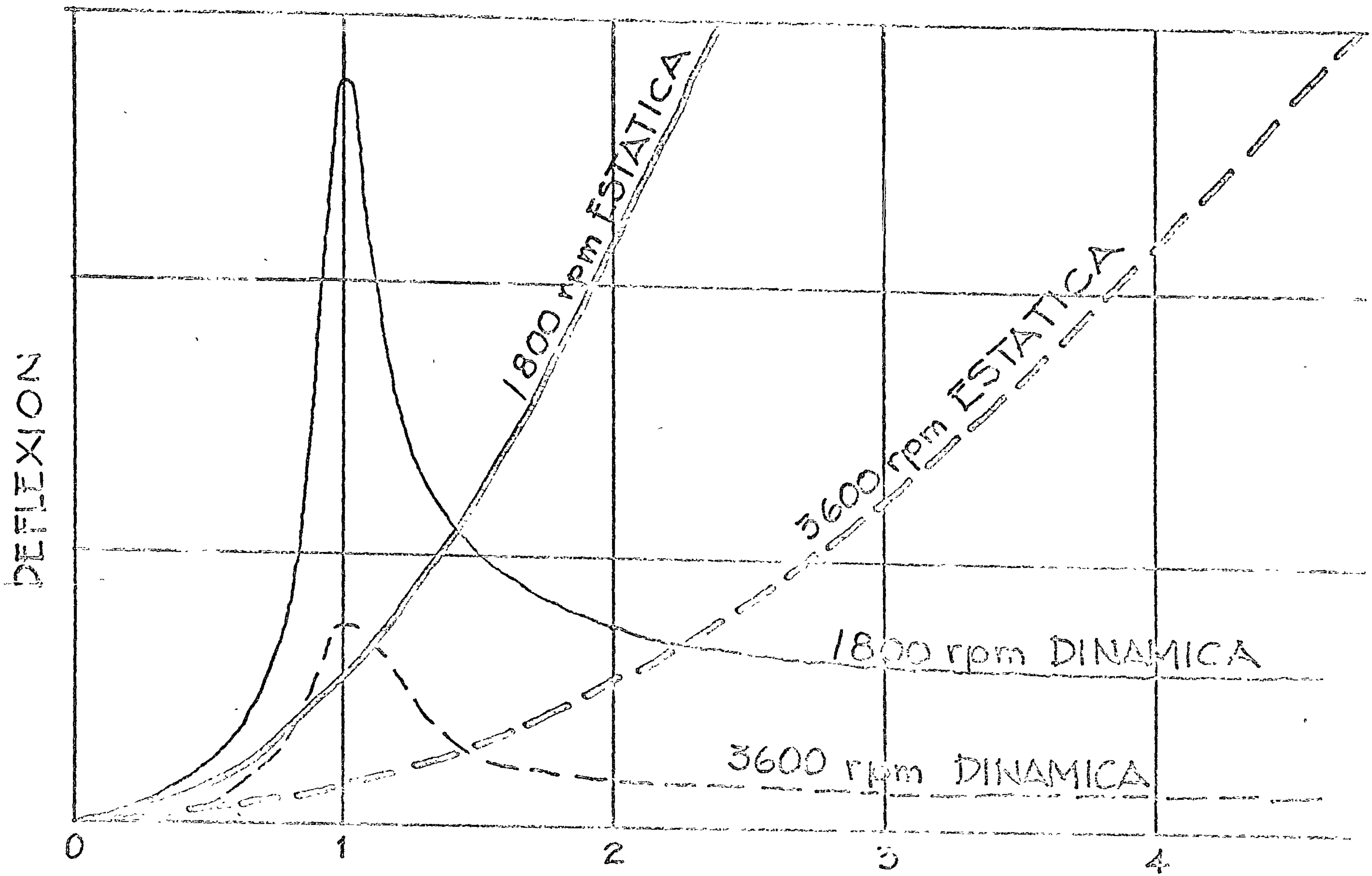


DEFLEXION EN Mils								
	3	6	11	17	25	44	68	1800 rpm
	1		3		6	11	17	3600 rpm

FACTOR DE AMPLIFICACION EN UN SISTEMA DE UN GRADO DE LIBERTAD.



AMORTIGUAMIENTO SUPUESTO:
10 % DEL CRITICO.

RELACION DE FRECUENCIAS
$\frac{\omega f}{\omega n}$

COMPARACION ENTRE DESPLAZAMIENTOS DINAMICOS Y ESTATICOS

Construcciones de Apoyo para Máquinas Rotatorias
(Preferentemente Cimentaciones de Mesa para Turbinas de Vapor)

Stützkonstruktionen für rotierende Maschinen (vorzugsweise Tisch-Fundamente fur Dampfturbinen)

Advertencia preliminar

Esta norma rige para cimentaciones de mesa, construcciones de apoyos aporticados y otras en acero, hormigón armado y sistemas de construccion analogos para admisión de turbo-generadores y turbo-compresores y otras máquinas de elevado número de revoluciones (unas 1000/min y más).
Para el proyecto de construcciones de apoyo de esta clase han de colaborar con la debida anticipación el ingeniero industrial y el ingeniero constructor. El ingeniero industrial indica la forma fundamental de la construcción y propone las dimensiones principales que se han previsto para las máquinas. El ingeniero constructor comprueba en el proyecto las condiciones técnicas de oscilación y las estáticas y propone las modificaciones necesarias debidas a la libertad de resonancia y la estabilidad. Se ha de llegar a un acuerdo entre ambos a más tardar durante el proyecto de las tuberías, de modo que aun se puedan tener en cuenta modificaciones.
Las construcciones de apoyo para maquinas de rotación rápida con dimensiones pequeñas se han de considerar también como construcciones de ingeniería importantes, a causa del valor y la importancia de las instalaciones de maquinaria que apoyan. Por consiguiente se han de observar las instrucciones especiales de las normas DIN competentes sobre elección del contratista, calidad, materiales, etc. Asimismo sólo puede encomendarse el proyecto a aquellos ingenieros que dispongan de los conocimientos especiales necesarios[1]).

A Instrucciones para el cálculo, construcción y ejecución

1 Instrucciones para el ingeniero industrial

1.1 El ingeniero industrial ha de conceder suficientes dimensiones para cada uno de los elementos de soporte, con el fin de hacer posible la absorción de las fuerzas estáticas y dinámicas y poder dar forma satisfactoria a la construcción también en sentido técnico de las oscilaciones. Se han de evitar adelgazamientos y entalladuras en los apoyos y vigas, sustituyéndolos mejor por orificios cerrados circundados por todos los lados.

1.2 Una forma constructiva corriente es la "cimentación de apoyos", en la que reposa un tablero sobre apoyos independientes. La estructura resistente del tablero consta aquí de vigas longitudinales y transversales. Los apoyos han de estar dispuestos en lo posible céntricamente debajo de las vigas, para que resulte un desarrollo de fuerzas claramente reconocible. Las placas de ménsula esbeltas, superficies de chapa grandes y análogas pueden oscilar espontáneamente y por consiguiente se han de evitar o dar forma, de modo que pueda ser eliminado un posible estado de resonancia.

1.3 El tablero se ha de separar del contorno por un resquicio de aire; la cubierta del resquicio ha de poder seguir los desplazamientos horizontales y verticales. También la placa de fundación ha de estar separada por una junta de las partes de la construcción que la rodean, sobre todo del pavimento de hormigón. Los pisos contiguos no han de estar apoyados en lo posible sobre la construcción de apoyo. Si esto es inevitable se recomienda un apoyo de forma adecuada.

1.4 La fábrica de maquinaria ha de reunir para el ingeniero constructor los siguientes datos:

1.41 Un gráfico de cargas detallado, donde se han de indicar separadamente el momento de cortocircuito sin aumento y la aspiración del vacío. El gráfico de cargas ha de estar averiguado cuidadosamente según la distribución real de las masas en la máquina y las disposiciones especiales que se hayan tomado para la transmision de la carga. Se han de indicar no sólo las cargas sinó también las superficies sobre las que se transmite la carga. Para generadores monofásicos se han de citar también los momentos oscilantes en el lado del generador y su número de oscilaciones. También se incluirán en el gráfico de cargas las posibles cargas adicionales en sentido horizontal u oblícuo que por ejemplo son transmitidas en los puntos fijos de la tubería de vapor a la construcción de apoyo. Además se indicará si el condensador está fijamente embridado con el empalme de escape de vapor de la turbina o por árbol flexible o prensaestopas.

1.42 Indicación de los pesos de rotores, para turbinas de engranajes también de las piezas rotatorias del engranaje.

1.43 Indicación de la potencia de la máquina.

1.44 Números de revoluciones de funcionamiento (eventualmente números de revoluciones de término rápido) de las turbomaquinas.

1.45 Los números de revoluciones críticos de los ejes en el acoplamiento y con los soportes, tal como existen en el funcionamiento.

1.5 Para poder tener en cuenta el efecto térmico en la construcción, se han de indicar datos sobre las temperaturas que se presentan en la proximidad de las partes de construccion (también temperaturas elevadas en el montaje o en el funcionamiento de prueba, p.e. en el secado del generador). Todas las partes calientes, sobre todo las tuberías, se han de aislar perfectamente ya antes del funcionamiento de prueba. Se tomaran medidas adecuadas contra la acumulación de calor en la construccion (p.e. orificios para escape del calor, cubierta de superficies con placas de aislamiento del calor).

2 Instrucciones para el ingeniero constructor

2.1 Se realizará un cálculo de oscilaciones con objeto de

a) evitar resonancia del sistema máquina y construccion de apoyo (eventualmente teniendo en cuenta la elasticidad del terreno de cimentación) con uno de los números de revoluciones de funcionamiento,

[1]) Véase "Rausch: Cimentaciones de máquinas y otros problemas dinámicos de la construcción", en la editorial VDI-Verlag Berlin, y la demas bibliografía indicada allí.

Continúa en páginas 2 a 4

b) averiguar para el cálculo estático de la construcción fuerzas supletorias estáticas en lugar de las fuerzas dinámicas. La fuerza supletoria estática se calcula por la magnitud de fuerza dinámica multiplicada por un coeficiente de fatiga dependiente del material y un coeficiente dinámico correspondiente a la sintonía.

2.11 Sintonía

2.111 Aclaración del concepto
Existe alta sintonía cuando el número de oscilaciones propias de la oscilación fundamental de la construcción se encuentra por encima del número de revoluciones de funcionamiento, y baja sintonía cuando el número de oscilaciones fundamentales se encuentra por debajo del número de revoluciones de funcionamiento, y baja sintonía cuando el número de oscilaciones fundamentales se encuentra por debajo del número de revoluciones de funcionamiento y por tanto está éste entre la oscilación fundamental y una oscilación armónica.

2.112 Los números de oscilaciones propias averiguados según párrafos 3.1 a 3.5 han de presentar una diferencia mínima de ± 20 % respecto al número de revoluciones de funcionamiento inmediato. Esta diferencia, en caso necesario, se ha de originar por modificaciones constructivas durante la preparación del proyecto. Sólo cuando para esto resulten dificultades especiales, se reducirá excepcionalmente la diferencia, en el supuesto de que se tomen disposiciones para un fuera de sintonía posterior según punto 2.44.

2.2 Se realizará el cálculo estático:

2.21 Para las cargas en reposo (peso propio de la construcción, cargas de la máquina, incluso peso del rotor, aspiración del vacío, etc.). Estas cargas se pondrán con el valor sencillo sin aumento. Las partes de construcción que no hayan de apoyar cargas de máquina, se calcularán para cargas de montaje.

2.22 Para el momento de cortocircuito en forma de un par de fuerzas vertical en ambos sentidos de rotación transmitido por la máquina con un aumento de 100 % a los valores sin aumento que ha indicado la fábrica de maquinaria.

2.23 Para la influencia dinámica de una fuerza centrífuga que pueda actuar en todas las direcciones perpendicularmente al eje del árbol. Para esto se introduce a continuación el concepto de una fuerza supletoria estática. La fuerza supletoria estática es proporcional en los mismos puntos que las cargas de la máquina, o sea se ha de poner de modo que en cada carga de máquina actúe vertical u horizontalmente una parte de la fuerza supletoria total proporcionalmente igual a esta carga. Para esto se han de tener en cuenta sólo las cargas de la máquina que ataquen a la propia obra aporticada. Estas partes de fuerza supletoria, según la forma de oscilación respectiva, pueden actuar en el mismo sentido o en sentido contrario. Para simplificación las fuerzas supletorias horizontales pueden ponerse por valor de los ejes de los travesaños.

2.24 Las partes de construcción no cargadas, teniendo en cuenta su oscilación conjunta, se han de calcular para una fuerza supletoria estática de 50 % de la carga propia vertical u horizontal, siempre que las cargas de montaje según párrafo 2.21 no den por resultado esfuerzos más desfavorables.

2.25 Para las construcciones de hormigón armado se han de considerar los efectos de temperatura y contracción de la obra aporticada como sigue:

2.251 Para la contracción del tablero respecto a la placa de fundación una caída de temperatura de 10 ºC; si la parte erigida sobre la placa de fundación se ejecuta pasados 2 meses después de la construcción, una de 15 ºC.

2.252 Para calentamiento uniforme entre el tablero y la placa de fundación, cuando se haya previsto una protección contra el calor según párrafo 1.5, pero no pudiéndose indicar datos más exactos, se supone un aumento de temperatura de 20 ºC, para lo que no obstante se puede deducir la medida de contracción.
La contracción y el calentamiento uniforme, por consiguiente, se han de considerar por el supuesto de un cambio de temperaturas desde − 10 hasta − 15º ó + 10 respectivamente + 5º.

2.253 En el lado de la turbina, además para protección contra el calor del hormigón armado según párrafo 1.5 y, si no se pueden dar más datos, se ha de calcular con una diferencia de temperatura dentro de las partes de construcción de 10º (dentro 20º más caliente que fuera). Para este cálculo se puede contar con la mitad del grado de elasticidad y con una sección según el estado II. Los momentos de flexión originados por esto en el bastidor cerrado horizontal del tablero exigen una armadura anular exterior.

2.26 Para el cálculo de las compresiones del terreno es suficiente considerar la mitad de las fuerzas supletorias aplicadas. Para cimentaciones profundas (p.e. cimentaciones de pilotes) pueden reducirse aun más las fuerzas supletorias, cuando se hayan comprobado coeficientes de oscilación propia situados profundamente respectivos de la cimentación.

2.27 El cálculo estático se realizará separadamente para cada caso de carga (cargas permanentes, cargas supletorias estáticas en dirección vertical y horizontal, doble momento de cortocircuito, temperatura y contracción). Para el cálculo sirve de norma la composición más desfavorable de los casos de carga, para lo que no obstante se ha de poner discrecionalmente, ya sea la fuerza supletoria vertical, la fuerza supletoria horizontal o el doble momento de cortocircuito.

2.28 La placa de fundación se calculará como viga empotrada en un extremo sobre el que actúan desde arriba las fuerzas de apoyo y desde abajo una contrapresión distribuida linealmente.

2.3 Otros puntos de vista para el cálculo estático:

2.31 Para el cálculo estático de construcciones de apoyo en hormigón armado rigen las disposiciones correspondientes especialmente la DIN 1045 (Disposiciones para la ejecución de obras de hormigón armado) y DIN 1048 (Disposiciones para ensayos del hormigón en la ejecución de obras de hormigón y hormigón armado), pero se empleará para la placa de fundación por lo menos hormigón B 160 y para la parte elevada por lo menos hormigón B 225; la tensión de tracción del acero para hormigón en todos los grupos de acero para hormigón no rebasará el límite admitido para acero para hormigón I y los aceros para hormigón especiales (véase DIN 1045, ed. 1943-xx, § 5, núm. 6 a) no deben ser empleados como armadura estática, no rigiendo, sin embargo, estas limitaciones para elementos pretensores de acero aplicarse hormigón pretensado. Para el cálculo de oscilaciones se pondrá en B 225 el grado de elasticidad E_b = 300 000 kg/cm². (Para mayor resistencia del hormigón se aumentarán en 10 % los grados de elasticidad indicados en la DIN 4227). Para averiguación de los momentos de inercia se tendrán en cuenta las armaduras de acero con su valor (múltiplo de n-1).

2.32 Para la ejecución en acero rige DIN 1050 (Bases de cálculo para acero en superestructuras) y DIN 4100 (Prescripciones para superestructuras de acero soldadas). Se cuidará de evitar efectos del impacto.

2.4 Además de los principios de construcción citados ya en párrafo A 1 se observarán los puntos de vista constructivos siguientes:

2.41 La máquina se acoplará a la construcción formando un todo.

2.42 Para evitar grietas en construcciones de hormigón armado, la armadura en cada elemento constructivo, excepto la placa de fundación, será como mínimo de 50 kg por cada m³ de hormigón sólido (sin tener en cuenta la clase de acero empleado) y estará dispuesta siempre con 3 ejes (cúbicos), aun cuando esto no sea necesario por cálculo. Se procurará un hormigón con una adición de agua reducida en lo posible, para evitar la tendencia a la contracción y para aumentar la resistencia a la tracción.

2.43 Para construcción de acero se preferirá la ejecución soldada.

2.44 Para baja sintonía se recomienda considerar la posibilidad de un fuera de sintonía posterior; en este caso son convenientes disposiciones para hormigón armado.

2.45 Para evitar un fuera de sintonía involuntario se recomienda mantener separada de la construcción las paredes de los canales de aire de los refrigeradores en circuito, plataformas intermedias y análogos.

2.46 Desde el punto de vista técnico de oscilaciones es ventajosa una placa de fundación pesada y gruesa, no debiendo ser su peso en general, incluido el hormigón árido posiblemente existente encima, inferior al peso de las máquinas, tablero y apoyos en conjunto; el peso del condensador no se toma en consideración para esto. El espesor de la placa de fundación en general no debe ser menor que 1/10 de la longitud.

2.47 La resultante del peso de la construcción y carga simple de la máquina (sin aspiración del vacío) debe pasar por el centro de gravedad del área de cimentación (cimentación profunda) para lograr una compresión del terreno uniforme.

2.48 Las condiciones del terreno de cimentación y de las aguas subterráneas bajo la placa de fundación se fijarán siempre según la DIN 1054, párrafo 3). Sirve de norma para apreciación del terreno de cimentación la DIN 1054[2]. Es conveniente una investigación del terreno correcta, porque los terrenos arenosos se vibran bajo el efecto de las vibraciones, los terrenos aglomerantes se comprimen al expulsar el contenido de agua, y de este modo pueden resultar asientos indeseados. En casos dudosos es conveniente hormigonar en la placa de fundación suficiente cantidad de tubos para poder realizar inyecciones para relleno de los espacios huecos resultantes o para elevar la placa de fundación.

Si el terreno de cimentación no es perfecto, se ha de prever una cimentación profunda. Asimismo cuando el agua subterránea se encuentra directamente bajo la placa de fundación, ya que en este caso el agua subterránea, a causa de su falta de capacidad de compresión, tiende especialmente a transmitir oscilaciones a los alrededores.

2.49 Si la placa de fundación de una construcción de alta sintonía se sumerge en el agua subterránea, es conveniente disponer una capa de amortiguación de oscilaciones debajo de la placa de fundación en una pila impermeabilizada contra el agua subterránea. En sentido análogo rige para la erección sobre roca.

B Indicaciones para el cálculo de oscilaciones

3 Para el cálculo hay que simplificar el sistema real. En general se considerarán separadamente las oscilaciones verticales y horizontales.

3.1 Las oscilaciones verticales para construcción simétrica se pueden subdividir en simétricas y antimétricas (revoluciones alrededor de un eje longitudinal). También para simetría incompleta del grupo respecto al eje longitudinal se considerarán ambas separadas en primera aproximación. Los apoyos en general se elegirán de modo que todos los pares de apoyo tengan igual número de oscilaciones bajo las proporciones de carga que les corresponda. Para determinación de la elasticidad del tablero, además de la flexión, se ha de tener en cuenta también la deformación de cortadura y si la fuerza incide excéntricamente también la torsión. Las bancadas y cajas de las máquinas influyen en los números de oscilaciones propias, especialmente en las armónicas superiores. La aspiración de vacío del condensador no entra en el cálculo de oscilaciones como fuerza estática. Sin embargo, una parte del condensador se ha de considerar como masa oscilante, cuando el condensador está embridado fijamente con el empalme de escape de vapor. La magnitud de esta parte que se ha de elegir depende de las propiedades elásticas del condensador que no se puede considerar como completamente rígido. Hasta qué punto oscila conjuntamente la carga de agua de funcionamiento depende asimismo de la clase de construcción del condensador y de la frecuencia de la excitación; para frecuencias elevadas el agua ya no es comprometida en su totalidad. - Las masas de capas apoyadas con elasticidad suave o análogas no se consideran en el cálculo de oscilaciones.

3.2 Las oscilaciones horizontales pueden presentarse en dirección transversal y longitudinal. Las oscilaciones transversales son más importantes, aun cuando también debería tenerse en cuenta la posibilidad de una resonancia en dirección longitudinal.

El tablero tiene diversas posibilidades de oscilar horizontalmente: como forma rígida sobre los apoyos y de por sí; para las clases de construcción actualmente corrientes se pueden considerar separadamente ambas clases de oscilación. Para la influencia de la bancada y caja de la máquina rige en mayor grado lo dicho en párrafo 3.1. Dado el caso han de ser calculados aquí los números de oscilaciones propias de las armónicas superiores.

Las oscilaciones propias de flexión de los apoyos pueden ser comprobadas de por sí.

3.3 Influencia de la oscilación de la placa de fundación. Para la averiguación de los números de oscilaciones propias según párrafos 3.1 y 3.2 se supondrá primero como fija la placa de fundación. Sin embargo, los números de oscilación propia pueden ser influidos por el efecto de acoplamiento entre el tablero y la placa de fundación apoyada elásticamente sobre el terreno de cimentación, sobre una capa de amortiguación o sobre pilotes.

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[2]) Véase también: DIN 4021, Terreno de cimentación de agua subterránea. Principios para exploración; perforaciones, excavaciones, toma de muestras;
DIN 4022, Especificación de capas para investigaciones del terreno de cimentación.
Instrucciones para la disposición y para denominación de las clases de terreno.

3.4 Oscilaciones sobre el terreno de cimentación: para números de revoluciones de funcionamiento de unas 1000 hasta 1500/min se han de comprobar también las oscilaciones de la cimentación total sobre la base elástica (terreno de cimentación, capa de amortiguación, etc.). Aquí solo deben utilizarse las cifras de cimentación dinámicas[2]), porque las cifras de cimentación estáticas son demasiado bajas.
3.5 Otras influencias: en el cálculo de oscilaciones se ha de tener en cuenta también la posibilidad de que las condiciones reales difieran desfavorablemente de las hipótesis del cálculo, p.e. relativamente a la medida del grado de elasticidad, los momentos de inercia, longitudes de barras, puntos de sujeción, etc., y además porque las cargas están apoyadas excéntricas y las máquinas cedan elásticamente.
Las longitudes de barras de las construcciones de apoyo calculadas se han de reducir respectivamente a causa de las esquinas rígidas. El punto de empotramiento de los apoyos se encontrará generalmente debajo del borde superior de la placa de fundación, en cambio suce más arriba por un refuerzo de los pies de los apoyos.
3.6 Para poder determinar el esfuerzo de desequilibrio de la construcción hay que conocer primero los desequilibrios. Para el actual conocimiento hay que contar en este caso con apreciaciones muy toscas. Las encuestas en empresas dieron lugar a una fuerza de excitación K en estado de funcionamiento:

$$K = k\,\frac{L}{g}\,(e\,\omega_m)\,\omega_m \qquad (1)$$
$$= \text{apr. } 0{,}5\; L\,\frac{n_m}{3000}$$

donde son $\frac{L}{g}$ la masa del rotor, e el desequilibrio del rotor (excentricidad de la masa del rotor), ω_m la frecuencia del circuito. n_m el número de revoluciones por minuto de la máquina en funcionamiento. Para $(e\,\omega_m)$, la "calidad de equilibrio" se ha puesto 0,15 cm/seg. El coeficiente $k \approx 10$ tiene en cuenta el posible mal estado de equilibrio aun dudoso como límite superior en el funcionamiento.
La supresión de la cifra $\frac{n_m}{3000}$ hace posible la averiguación de K también para números de revoluciones de funcionamiento diferentes a 3000/min. Aunque la fuerza centrífuga para desequilibrio constante aumenta con el cuadrado del número de revoluciones, se ha tomado linealmente la dependencia de la fuerza de excitación K, porque se equilibra mejor con número de revoluciones superior ($e\omega$ = const.). Para deducir exactamente de esta fuerza de excitación el esfuerzo de la construcción, es necesario el cálculo de oscilaciones que no solo determinan los puntos de resonancia sino el curso de la oscilación forzada. A causa de la inseguridad de la fuerza de excitación sólo conocida muy aproximadamente según la clase y magnitud, es suficiente calcular aproximadamente con una fuerza supletoria estática como sigue:
Si la construcción es de alta sintonía[4]), o sea su coeficiente de oscilación propia se encuentra por encima del número de revoluciones de funcionamiento, se puede considerar como oscilador con un grado de libertad. Para este oscilador la fuerza supletoria estática depende de la sintonía $\zeta = n_e/n_m$ (n_e = coeficiente de oscilación propia, n_m = número de revoluciones de funcionamiento de la máquina) y es:

$$P = \mu \cdot \nu \cdot K = 3\,\frac{\zeta^2}{\zeta^2 - 1} \cdot K \qquad (2)$$
$$= 1{,}5\,\frac{\zeta^2}{\zeta^2 - 1} \cdot L\,\frac{n_m}{3000} \quad \text{(valor absoluto)},$$

pero no más que

$$\max P = 15\; L\,\frac{n_m}{3000}. \qquad (3)$$

La relación entre la fuerza supletoria P y la sintonía ζ está indicada por la línea de trazo fino en figura 1 para n_m = 3000. La figura 1 rige también para otros números de revoluciones de funcionamiento, cuando se pone la expresión $L \cdot \frac{n_m}{3000}$ en lugar del peso del rotor L. En la ecuación (2) significa $\mu = 3$ el coeficiente de fatiga (coeficiente entre la resistencia estática y la de oscilación). Con el conocimiento exacto de las propiedades del material puede variarse este valor respectivamente.
$\nu = \frac{\zeta^2}{\zeta^2 - 1}$ significa el coeficiente dinámico originado por balanceo (multiplicador).
Si la construcción es de baja sintonía, o sea se eligo su frecuencia básica inferior al número de revoluciones de funcionamiento, se encontrará el número de revoluciones de funcionamiento entre dos números de oscilaciones propias. La fuerza supletoria estática averiguada anteriormente rige aproximadamente también para esto, cuando en la averiguación de la sintonía ζ se aplica al número de oscilación propia inmediato al número de revoluciones de funcionamiento.
A causa de la inseguridad en la determinación de los coeficientes de oscilación propia (ante todo de los números de oscilación de orden superior) tanto para baja sintonía como para alta se ha de calcular con una sintonía ζ más desfavorable en 10 %. Las fuerzas supletorias estáticas aumentadas según esto, que sirven de norma, se indican por la línea de trazo grueso en figura 1.

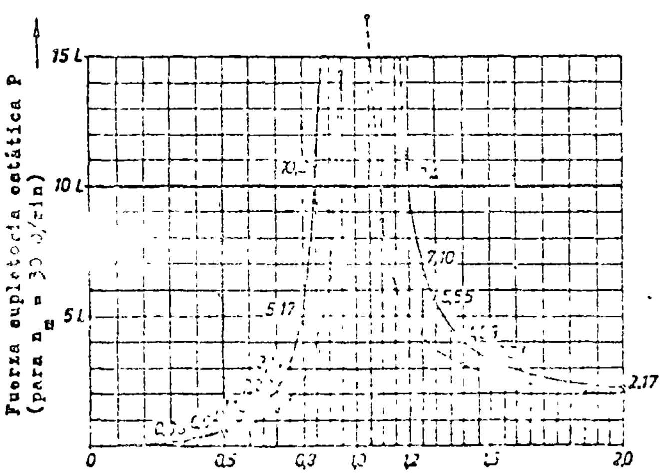


Figura 1

Sintonía $\zeta = n_e/n_m$ (trazo de línea fina)
o $(n_e + 10\,\%)/n_m$ (trazo de línea

3) Los valores actualmente conocidos para esto (véase nota al pie 1) son muy dispares. En casos importantes se recomienda una investigación del terreno dinámica.

4) Una sintonía alta se presenta prácticamente sólo en dirección vertical.

DISEÑO DE CIMENTACIONES SUJETAS A VIBRACION

Cimentaciones Reticulares

Ing. Alberto García Rubio

CIMENTACIONES RETICULARES PARA MAQUINARIA

Prediseño:

Una manera de proponer las escuadrías de los diferentes elementos de una cimentación reticular, consiste en la determinación aproximada de la frecuencia fundamental de vibración de la estructura en el sentido vertical. Esto puede lograrse, despreciando la contribución de los elementos de la plataforma superior y suponiendo que las rigideces de las columnas se suman directamente para obtener la rigidez total de la estructura en el sentido vertical;

Sabemos que $\omega_{nv} = \sqrt{\frac{\Sigma k_i}{\Sigma M_i}}$ ...... (1)

donde $k_i = \frac{E A_i}{L_i}$.- rigidez axial de las columnas.

$m_i = \frac{P}{g}$.- masas discretizadas del sistema.-

ω_{nv} .- Frecuencia natural fundamental de vibración aproximada.-

Deberá comprobarse que ω_{nv} difiera de la frecuencia de exitación (de la maquina) en ± 25%, valor que da un grado de confiabilidad adecuado para continuar con el diseño.-

Es conveniente, determinar en forma aproximada las frecuencias naturales de vibración horizontales de la estructura propuesta; para ello se parte del supuesto de que la plataforma superior es infinitamente rígida en su plano, de acuerdo con esta suposición se tendrán entonces tres grados de libertad horizontales, 2 desplazamientos (x, y) y un giro (θ_z). Se determina la matriz de rigidez horizontal de la estructura acoplándola como ya se vió, mediante un movimiento de cuerpo rígido; empleando la fórmula:

$$K = \begin{bmatrix} \Sigma \alpha_i^2 k_i & \Sigma \alpha_i \beta_i k_i & \Sigma (x_i \alpha_i \beta_i - y_i \alpha_i^2) k_i \\ \Sigma \alpha_i \beta_i k_i & \Sigma \beta_i^2 k_i & \Sigma (x_i \beta_i^2 - \alpha_i \beta_i y_i) k_i \\ \Sigma (x_i \alpha_i \beta_i - \alpha_i^2 y_i) k_i & \Sigma (x_i \beta_i^2 - \alpha_i \beta_i y_i) k_i & \Sigma (x_i^2 \beta_i^2 + \alpha_i^2 y_i^2 - 2 \alpha_i \beta_i x_i y_i) k_i \end{bmatrix} \quad \ldots (2)$$

Planteada la matriz (2), se define la matriz de masas del sistema

$$\begin{bmatrix} \Sigma m_i & & \\ & \Sigma m_i & \\ & & \Sigma m_i (x_i^2 + y_i^2) \end{bmatrix} = \begin{bmatrix} M & & \\ & M & \\ & & I_\theta \end{bmatrix} \quad \ldots (3)$$

donde M es la masa total del sistema -

I0 es el momento de inercia de la masa respecto al centro de masa -

Se ve entonces que las matrices ② y ③ deberán ser referidas al centro de masa del sistema.

Los valores de Ki de la matriz ② se pueden obtener fácilmente empleando el método de Kani, las ecuaciones de Wilbur ó cualquier otro método conocido para valuar desplazamientos horizontales bajo cargas unitarias.

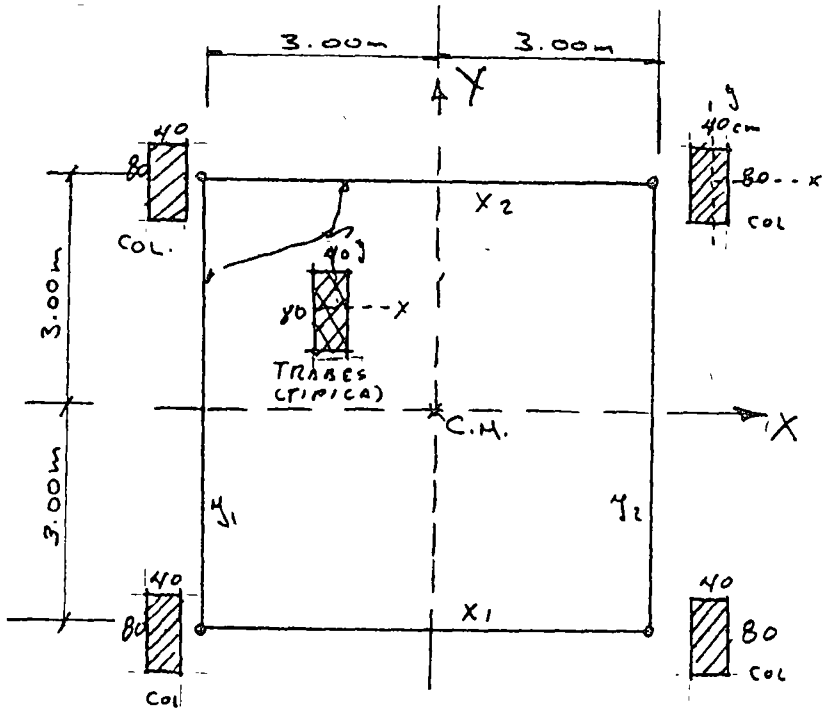
Las hipótesis que se hacen en esta etapa del diseño consisten en despreciar el amortiguamiento y suponer que el sistema estructural se comporta con un sistema de marcos ortogonales que se acoplan a través de un movimiento de cuerpo rígido. -

Ejemplo.-

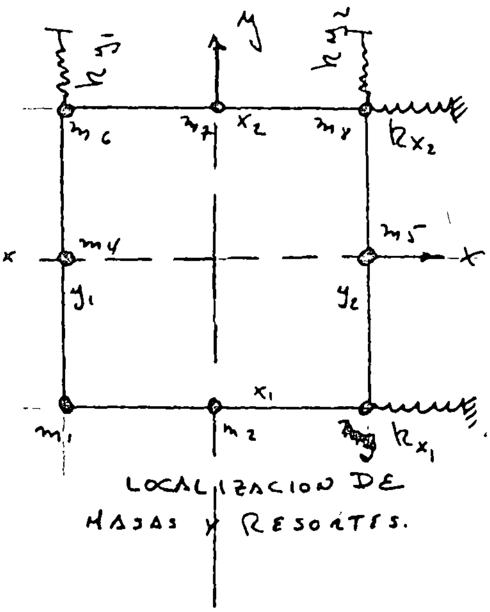
Dimensionar en forma preliminar la cimentación de una turbina centrífuga cuyas características son las siguientes:

ω operación	1800.00 rpm.
Peso total en operación de la Máquina	150.00 tons.
Potencia	10,000.00 H.P.

(Ver estructuración en hoja 4)



PLANTA.



LOCALIZACION DE MASAS Y RESORTES.

$m_1 = m_3 = m_6 = m_8$

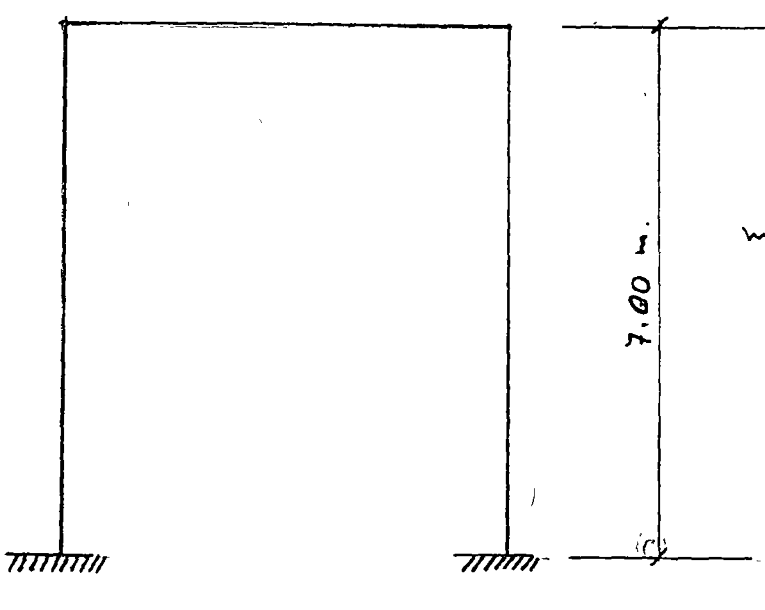
$= \frac{33.33 \times 10^3}{981} = 33.97$

$\doteq 34. \frac{Kg\ seg^2}{cm}$

$m_2 = m_4 = m_5 = m_7$

$= \frac{16.67 \times 10^3}{981} = 16.99$

$\doteq 17. \frac{Kg\ seg^2}{cm}$



ELEVACION

$M = (34+17)4$

$= 204 \frac{Kg\ seg^2}{cm}$

$P = 204 \times 981 = 200 \times 10^3\ Kg$

O.K.

Peso supuesto de la estructura = 50.00 tons.

(Plataforma Superior) + $\frac{1}{3}$ P. columnas

PROPIEDADES DE LOS MIEMBROS			
ELEMENTO	AREA cm^2	I_x cm^4	I_y cm^4
COLS	3,200.	1,706,666.	426,666.
TRABES.	3,200.	1,706,666.	426,666.

Cálculo de la frecuencia fundamental vertical aproximada: de (1)

$$k_{col} = \frac{EA}{L} \qquad E_{sup} = 200 \times 10^3 \ kg/cm^2$$

$$= \frac{200 \times 10^3 \times 3200}{700} = 914 \times 10^3 \ kg/cm$$

$$\Sigma m_i = M = 204. \ kg \ \frac{seg^2}{cm}$$

$$\omega_{nv} = \sqrt{\frac{4 \times 914 \times 10^3}{204}} = \sqrt{17920} = 133.87 \ seg^{-1}$$

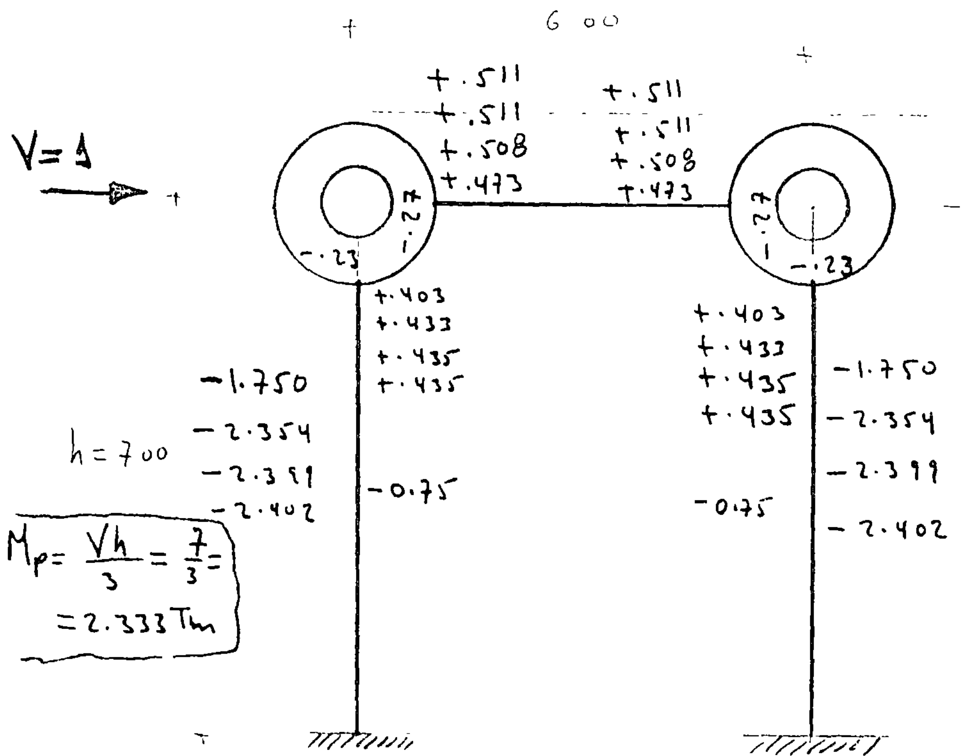
$$= 1278 \ rpm$$

Cálculo de las Frecuencias Naturales horizontales de vibración (3 grados de libertad)

1) Determinación de las rigideces horizontales de los marcos (constante de resorte)

Se puede emplear el Método de Kani.-

Marcos <u>Y_1 y Y_2</u>

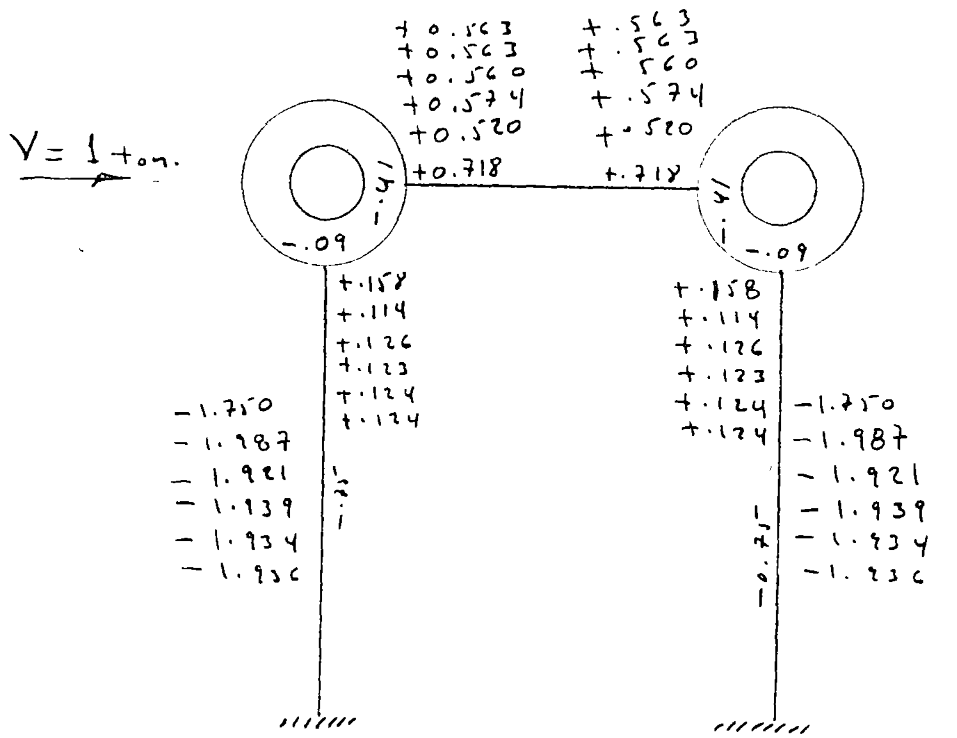


$$M_{FINAL} = M_{emp.} + 2M_{rot\ nudo} + M_{rot\ nudo\ op.} + M_{despl.}$$

$$M_{F\,TRABE} = 2\times0.511 + 0.511 = +\underline{\underline{1.533}}$$

$$M_{F\,col.} = 2\times0.435 - 2.402 = -\underline{\underline{1.532}} \therefore Ok.$$

MARCOS X_1 y X_2



$M_F = \frac{Vh}{3} = 2.333$

$M_{F\ TRABE} = 2 \times .563 + .563 = \underline{\underline{+1.689}}\ Tm$

$M_{F\ col} = 2 \times 0.124 - 1.936 = -\underline{\underline{1.688}}\ Tm$ ∴ Ok.

Obtención de las rigideces.

Sabemos que $\delta = \frac{m'' h}{6 E K}$

donde $\underline{m''}$ es el momento final en el entrepiso ó contribución por desplazamiento relativo de la col. del entrepiso

$K = \frac{I}{h}$

Para los marcos y_1 y y_2, se tiene:

$M'' = 2.402 \times 10^5$ kg cm

$E = 2 \times 10^5$ kg/cm²

$K = \frac{1,706\,666}{700} = 2438$ cm³

$h = 700$ cms

$\therefore \delta = \frac{2.402 \times 10^5 \times 700}{6 \times 2 \times 10^5 \times 2438} = 0.057$ cms

$\therefore k_{y_1} = \frac{V}{\delta} = \frac{1000}{0.057} = \underline{17\,400}$ kg/cm $= k_{y_2}$

Para los marcos x_1 y x_2

$M'' = 1.936 \times 10^5$ kg cm

$E = 2 \times 10^5$ kg/cm²

$K = \frac{426\,666}{700} = 610.$ cm³

$h = 700$ cm

$\delta = \frac{1.936 \times 10^5 \times 700}{6 \times 2 \times 10^5 \times 610} = 0.185$ cms

$\therefore k_{x_1} = \frac{V}{\delta} = \frac{1000}{0.185} = 5405$ kg/cm $= k_{x_2}$

Planteamiento de la matriz de masas.-

$M = 204$ kg s²/cm

$I_\theta = \Sigma m_i (x_i^2 + y_i^2)$

$\therefore I_{\theta\, m_1, m_3, m_6, m_8} = 4 \times 34 (300^2 + 300^2) = 24.48 \times 10^6$ kg seg² cm

$I\theta_{m_{2,4,5,7}} = 4\times17\times300^2 = 6.12\times10^6$ kg seg² cm

$\therefore I\theta_{total} = (24.48 + 6.12)\times10^6 = 30.60\times10^6$ kg seg² cm

∴ de ③

$$M = \begin{bmatrix} 204 & 0 & 0 \\ 0 & 204 & 0 \\ 0 & 0 & 30.60\times10^6 \end{bmatrix}$$

Sabemos de antes que:

$$[K] - \omega^2[M] = 0 \quad \ldots \quad (a)$$

de la ecuación ② obtener [K]; para ello obtendremos la K_i de c/marco.

Resorte k_{x_1} $x = 300.$ cm, $y = -300.$ cm, $\alpha = \cos 0° = 1.$, $\beta = \cos 90° = 0.$, $k_{x_1} = 5405$ kg/cm

$$K_{x_1} = \begin{bmatrix} 5405 & 0 & 1621.5\times10^3 \\ 0 & 0 & 0 \\ 1621.5\times10^3 & 0 & 486.45\times10^6 \end{bmatrix}$$

Resorte K_{x_2} $x = 300$ cm, $y = 300$ cm, $\alpha = 1.0$, $\beta = 0.$, $k_{x_2} = 5405$ kg/cm

$$K_{x_2} = \begin{bmatrix} 5405 & 0 & -1621.5\times10^3 \\ 0 & 0 & 0 \\ -1621.5\times10^3 & 0 & 486.45\times10^6 \end{bmatrix}$$

Resorte k_{y_1}

$x = -300$, $y = 300$, $\alpha = 0$, $\beta = 1$, $k_{y_1} = 17\,400$ kg/cm

$$K_{y_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 17400 & -5220\times10^3 \\ 0 & -5220.\times10^3 & 1566\times10^6 \end{bmatrix}$$

Resorte k_{y_2}

$x = 300$, $y = 300$, $\alpha = 0$, $\beta = 1$, $k_{y_2} = 17\,400$ kg/cm.

$$K_{y_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 17400 & 5220\times10^3 \\ 0 & 5220\times10^3 & 1566\times10^6 \end{bmatrix}$$

$$\therefore K_{TOTAL} = \begin{bmatrix} 10810.0 & 0 & 0 \\ 0 & 34800.0 & 0 \\ 0 & 0 & 4104.9\times10^6 \end{bmatrix}$$

Sustituyendo en ⓐ:
(haciendo $\omega^2 = \lambda$)

$$[[K] - \omega^2[M]]_1 = [K - \lambda M] = 0$$

$$\therefore \begin{bmatrix} 10810 - \lambda 204 & 0 & 0 \\ 0 & 34800 - \lambda 204 & 0 \\ 0 & 0 & 4104.9\times10^6 - \lambda 30.6\times10^6 \end{bmatrix} = 0$$

en este caso particular, el sistema se encuentra desacoplado como se puede ver en la matriz final.

desarrollando el determinante de esta matriz obtenemos la ecuación característica del sistema:

$$(10810 - 204\lambda)(34800 - 204\lambda)(4104.9 - 30.6\lambda)10^{6} = 0$$

$$-1.2748\times10^{6}\lambda^{3} + 4.5571\times10^{8}\lambda^{2} - 4.9702\times10^{10}\lambda + 1.5442\times10^{12} = 0$$

$$\lambda^{3} - 357.48\lambda^{2} + 38988.\lambda - 1,211327. = 0$$

Resolviendo la ecuación se tiene:

$$\lambda_1 = 53.03 \text{ seg}^{-2}$$
$$\lambda_2 = 134.08 \text{ seg}^{-2}$$
$$\lambda_3 = 188.38 \text{ seg}^{-2}$$

$$\therefore \quad \omega_1 = \sqrt{\lambda_1} = 7.282 \text{ seg}^{-1} = \underline{69.54 \text{ rpm}}$$
$$\omega_2 = \sqrt{\lambda_2} = 11.58 \text{ seg}^{-1} = \underline{110.57 \text{ rpm}}$$
$$\omega_3 = \sqrt{\lambda_3} = 13.725 \text{ seg}^{-1} = \underline{131.06 \text{ rpm}}$$

Dichos valores podían haberse obtenido directamente de la ecuación desacoplada.

Se ve que todas las frecuencias difieren de la frecuencia de excitación en + del 25% ∴ se acepta el prediseño. también se observa que la frecuencia mas crítica es la frecuencia vertical fundamental.

Determinación de la losa de Cimentación

Para determinar las dimensiones en planta de la losa de cimentación, deberá procurarse que el centro de carga estática, coincida con el centro de la losa a fin de no tener momentos permanentes de volteo debido a excentricidad en la carga.

El peralte de esta losa, puede elegirse de acuerdo a la tabla siguiente, en función de la potencia de la maquina; estas recomendaciones estan dadas por D.D. Barkan.

Maquinas con Potencia hasta 6,000 kw	0.8 @ 1.20 m
" " " de 6 @ 12,000 Kw	1.0 @ 1.60 "
" " " de 12 @ 25,000 Kw	1.60 @ 2.00 "
Maquinas con Potencias mayores	2.00 @ 4.00 "

Estas recomendaciones deben tomarse con las reservas del caso.

ANALISIS DE CIMENTACIONES NO MASIVAS PARA MAQUINARIA ROTATORIA.

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A. GARCIA R.

Subdirección de
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Se presenta un programa de computadora para el análisis matricial tridimensional a fin de determinar las frecuencias naturales de vibración y amplitudes máximas de los puntos de interés en las cimentaciones de maquinaria a partir de las fuerzas de exitación Este análisis en la mayoria de los casos satisface los requisitos impuestos por los fabricantes de la maquinaria

El análisis se divide en tres partes principales. la primera consiste en la determinación de las frecuencias caracteristicas de la parte superior de la estructura considerando tres grados de libertad y suponiendo un movimiento de cuerpo rígido de la misma en el plano horizontal. Los grados de libertad considerados son 2 desplazamientos y un giro en este mismo plano, la segunda consiste en determinar las frecuencias naturales de vibración de todas las columnas y puntos intermedios elegidos en sentido vertical, por último se presenta el análisis de respuesta máxima a una cierta exitación de acuerdo a la información proporcionada por el fabricante de la maquinaria Al final del articulo se presenta un ejemplo práctico

INTRODUCCION

Aunque el tratamiento de cimentaciones para maquinaria es un campo que aparentemente cae dentro de la Ingeniería Mecánica, podria pensarse que este tema no debía ser tratado dentro del campo de la Ingeniería Civil. sin embargo, lo importante aquí es el comportamiento de la estructura ante la presencia de solicitaciones impuestas por una maquinaria rotatoria Se trata entonces de analizar la estructura de apoyo y la respuesta de ésta a las solicitaciones antes mencionadas, pensamos que en este aspecto la Ingeniería Civil dispone de mayores herramientas para el ataque del problema Las cimentaciones reticuladas. no limitan al diseñador en la localización de las máquinas y sus equipos auxiliares como lo hacen las cimentaciones masivas Por ejemplo. los condensadores, líneas de tubería. sistemas de enfriamiento y alambrado eléctrico pueden ser arreglados más convenientemente si las máquinas se apoyan en marcos

El empleo de cimentaciones no masivas o reticuladas facilita considerablemente la inspección y acceso a todos los elementos de la máquina Las cimentaciones reticuladas se aplican generalmente en turbodinámos como son los turboexpansores, los turbocompresores y los turbogeneradores de diferentes potencias También pueden emplearse en máquinas eléctricas como motogeneradores, compensadores sincronos. dinámos de alta potencia y motores eléctricos. en los que no se presentan cargas repentinas

En este trabajo se presenta un programa que obtiene las frecuencias naturales de la estructura de apoyo. los modos de vibración de la misma y los desplazamientos que sufre la plataforma superior a partir de la geometria del sistema estructural propuesto y de las fuerzas de excitación impuestas por la máquina Los resultados antes mencionados se obtienen para el espacio de tres dimensiones

HIPOTESIS FUNDAMENTALES

1) El comportamiento de la estructura es elástico lineal sin discipación de energía, por lo tanto no se considera amortiguamiento.
2) El método de análisis empleado es el método de los desplazamientos o de las rigideces convencional para estructuras en el espacio mediante el uso de matrices.
3) La distribución de masas se hace en forma discreta, es decir suponemos una masa concentrada en los puntos donde suponemos un grado de libertad vertical, en los puntos que coinciden con una columna se toma la tercera parte de la masa de la misma
4) La estructura está empotrada en la losa inferior en contacto con el suelo; la experiencia ha demostrado que las amplitudes tanto horizontales como verticales de las losas inferiores de apoyo del sistema estructural andan del orden de 1 a 3 micras, siendo estas en general mucho menores que las amplitudes obtenidas en la losa superior de la estructura.

Los resultados instrumentales obtenidos en un gran número de cimentaciones reticuladas construidas en las plantas de PEMEX, llevan a la conclusión de que en la práctica, las losas inferiores en contacto con el suelo no están sujetas a vibración y consecuentemente no transmiten presión dinámica a la base Por lo tanto, la presión en el suelo bajo las cimentaciones de este tipo solo se determina considerando las cargas estáticas del sistema estructura-máquina. Por lo anteriormente expuesto podemos considerar que la estructura se encuentra empotrada en la losa inferior en contacto con el suelo.

5) Las fuerzas de excitación están concentradas en el centro de gravedad del rotor de la máquina considerada

DESCRIPCION DEL PROGRAMA

Como se advirtió en un principio. el procedimiento empleado en la solución del problema consiste primero en plantear la matriz de rigideces; dicha matriz toma en cuenta 6 grados de libertad por nudo (3 giros y 3 desplazamientos) Planteada la matriz de rigidez de cada miembro. se premultiplica por una matriz de transporte y se postmultiplica por la transpuesta de esta matriz, dicha matriz de transporte tiene por objeto referir la martiz de rigidez del miembro a un sistema general de ejes.

Una vez obtenidas las matrices de los miembros de la estructura respecto al sistema general se procede a ensamblar la matriz de rigideces de la estructura en orden creciente correspondiente al número de nudo y a los miembros que inciden en dicho nudo.

El programa está elaborado de modo que la numeración de los nudos de la estructura se haga principiando por los superiores y terminando con los apoyos que como ya se dijo se trata de empotramientos Los miembros pueden numerarse en cualquier orden

A partir de la matriz total de la estructura se pueden obtener matrices que corresponden con los grados de libertad de la misma según el modo de vibración correspondiente es decir, se tendrá una matriz de 3 × 3 para los modos de vibración horizontales (dos desplazamientos y un giro) y una matriz de n × n para los modos de vibración verticales, siendo n el número de puntos donde se considera una concentración de masa.

Las matrices antes mencionadas se obtienen haciendo una condensación de la matriz de rigidez total de la estructura. dicha condensación se efectúa del siguiente modo

La matriz de rigideces total de la estructura se ensambla de modo que el vector de desplazamiento se presenta ordenado conforme a los grados de libertad. es decir, el vector de desplazamiento para un nudo será de la forma

$$\begin{bmatrix} \delta_x \\ \delta_y \\ \delta_z \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}$$

La matriz de rigidez inicial deberá ordenarse mediante una matriz de ordenamiento que está formada por elementos cuyo valor es '0' y elementos

cuyo valor es '1' localizados en los lugares donde deseamos transportar los elementos de acuerdo con el nuevo orden requerido La matriz resultante es de la forma·

$$K \text{ ordenada} = [0]\ [K]\ [0]^T$$

donde [0] es una matriz formada por elementos '0' y elementos '1'.

Resulta conveniente ordenar la matriz de rigideces de modo que los grados de libertad horizontales $(\delta_x, \delta_y, \theta_z)$ se localicen en la parte superior izquierda de la misma y los valores correspondientes a los grados de libertad verticales (δ_z) en la parte inferior derecha de la matriz reordenada

Para lograr la condensación en función de los grados de libertad horizontales partimos de la siguiente matriz ordenada

$$K \text{ ordenada} = \left[\begin{array}{c|c} K_{hh} & K_{ha} \\ \hline K_{ah} & K_{aa} \end{array}\right]$$

donde K_{hh} es la matriz cuyos valores están en orden con los desplazamientos $(\delta_x, \delta_y, \theta_z)$ de cada nudo

La condensación se logra mediante el desarrollo siguiente

$$\text{Sea} \quad \begin{bmatrix} P \\ O \end{bmatrix} = \begin{bmatrix} K_{hh} & K_{ha} \\ K_{ah} & K_{aa} \end{bmatrix} \begin{bmatrix} \delta_h \\ \delta_a \end{bmatrix}$$

Donde P es el vector de fuerzas generalizado asociado a los grados de libertad para los cuales se desea hacer la condensación. δ_h es el vector de desplazamiento en el plano horizontal y δ_a es el vector de desplazamiento de los grados de libertad cuyos efectos queremos tomar en cuenta en la condensación

Desarrollando el sistema anterior se tiene·

$$P = K_{hh}\ \delta_h + K_{ha}\ \delta_a \qquad (1)$$

$$O = K_{ah}\ \delta_h + K_{aa}\ \delta_a \qquad (2)$$

$$\text{de (2)}\ \delta_a = -K_{aa}^{-1}\ K_{ah}\ \delta_h$$

Subst. en (1)

$$P = [\,K_{hh} - K_{ha}\ K_{aa}^{-1}\ K_{ah}\,]\ \delta_h$$

La matriz dentro del paréntesis constituye la matriz condensada donde sólo se tienen los grados de libertad horizontales Nótese que el orden de esta matriz es el mismo que el de K_{hh}

Una forma alternativa de lograr esta condensación es mediante inversiones sucesivas de acuerdo con el método de inversión por partición

Podemos escribir.

$$K \text{ ordenada} = \underbrace{\left[\begin{array}{c|c} K_{hh} & K_{ha} \\ \hline K_{ah} & K_{aa} \end{array}\right]}_{K} \underbrace{\left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right]}_{B} = \left[\begin{array}{c|c} I & O \\ \hline O & I \end{array}\right]$$

Desarrollando el producto se tiene·

$$K_{hh}\ B_1 + K_{ha}\ B_3 = I$$

$$K_{hh}\ B_2 + K_{ha}\ B_4 = O$$

$$K_{ah}\ B_1 + K_{aa}\ B_3 = O$$

$$K_{ah}\ B_2 + K_{aa}\ B_4 = I$$

Estas ecuaciones pueden resolverse a fin de expresar los valores de B en términos de A. se demuestra que

$$B_1 = [\,K_{hh} - K_{ha}\ K_{aa}^{-1}\ K_{ah}\,]^{-1}$$

Por lo tanto $B = K^{-1}$, e invirtiendo nuevamente la matriz B_1 puede obtenerse la condensación buscada

Obtenida la matriz de rigideces de orden $3n \times 3n$ (donde n es el número de nudos superiores) condensada de acuerdo con dos desplazamientos horizontales y un giro alrededor de un eje vertical. de acuerdo con la hipótesis de suponer que la losa superior es infinitamente rígida en su plano. podemos deducir que la parte superior de la estructura se mueve como cuerpo rígido y por lo tanto reducir el orden de la matriz de rigidez horizontal por medio de una matriz de transporte al centro de masa del sistema Es decir·

$$Ke = T\ K'_{hh}\ T^T$$

donde Ke es la matriz de rigidez final de la estructura en función de 3 grados de libertad por lo tanto el orden de esta matriz es de 3 $\times$ 3

T es una matriz que transporta las coordenadas de los nudos de los miembros al centro de masa del sistema cuyo orden es 3 $\times$ 3n siendo n el valor antes definido.

K'hh es la matriz de rigidez condensada de la estructura de orden 3n $\times$ 3n.

Se emplea una teoría similar para hacer la condensación en función de los grados de libertad verticales; partiendo de la misma matriz que se ordenó inicialmente sólo que la división en submatrices es la siguiente

$$\left[\begin{array}{c|c} K_{bb} & K_{bv} \\ \hline K_{vb} & K_{vv} \end{array}\right]$$

Donde Kvv es una submatriz diagonal de orden igual al número de grados de libertad verticales considerados; Kbb, Kbv y Kvb quedan obligados al realizar la división en submatrices.

La matriz condensada en función de los grados de libertad vertical, será una matriz simétrica de orden n $\times$ n.

Una vez obtenidas las matrices finales de la estructura (matriz de rigideces horizontales Ke y matriz de rigidez vertical K'vv), se plantean las matrices de peso del sistema, las cuales son de las formas siguientes.

Para el caso de los modos de vibración horizontal

$$M_H = \begin{bmatrix} M & & \\ & M & \\ & & IM \end{bmatrix}$$

Donde M es la masa total del sistema estructura máquina

IM es el Momento de Inercia de la masa respecto a un eje vertical que pasa por el centro de masa del sistema

Para el caso de los modos de vibración vertical se tiene

$$M_v = \begin{bmatrix} M_1 & & \\ & M_2 & \\ & & \bar{M}_n \end{bmatrix}$$

Donde M1, M2 Mn son las masas concentradas (sistema discreto) del sistema según el número de grados de libertad verticales considerados

Con las matrices de rigideces finales, una para la rigidez horizontal y otra para la rigidez vertical así como las matrices de peso antes mencionadas, se plantea un problema de valores característicos

Los valores característicos de este modelo serán las frecuencias naturales de la estructura al cuadrado y los vectores característicos representan las configuraciones de los modos de vibración

Dichos valores se encuentran empleando el método de Jacobi que consiste en diagonalizar la matriz mediante rotaciones sucesivas quedando en la diagonal precisamente los valores característicos. los vectores característicos se obtienen efectuando transformaciones unitarias sucesivas en la matriz original

OBTENCION DE LAS AMPLITUDES

De acuerdo con la hipótesis que supone que el sistema es no amortiguado puede suponerse que la ecuación siguiente es válida

$$M\ddot{\bar{X}} + K\bar{X} = F \operatorname{sen} \omega t$$

Supongamos una solución particular.

$$\bar{X} = \bar{A} \operatorname{sen} \omega t + \bar{B} \cos \omega t$$

$$\dot{\bar{X}} = \omega \bar{A} \cos \omega t - \omega \bar{B} \operatorname{sen} \omega t$$

$$\ddot{\bar{X}} = -\omega^2 (\bar{A} \operatorname{sen} \omega t + \bar{B} \cos \omega t)$$

Substituyendo en la ecuación inicial

$$-\omega^2 M(A \operatorname{sen} \omega t + B \cos \omega t) + K(A \operatorname{sen} \omega t +$$

$$+ B \cos \omega t) = F \operatorname{sen} \omega t$$

$$(-\omega^2 M + K)\bar{A} \operatorname{sen} \omega t + (\omega^2 M + K)\bar{B} \cos \omega t =$$

$$= F \operatorname{sen} \omega t$$

Por lo tanto

$$\bar{A} = (K - \omega^2 M)^{-1} \bar{F}, \quad \bar{B} = [0]$$

Si K es la matriz de rigideces horizontal, M la matriz de masas $[M_H]$ y $[F]$ el vector de exitación formado por una fuerza horizontal paralela a cualquiera de los ejes (X, Y) y un momento alrededor del eje vertical (Z) $\bar{A}$ nos dará los desplazamientos horizontales del centro de masa del sistema.

En el caso que $[K]$ sea la matriz de rigidez vertical, $[M]$ la matriz de Masas $[Mv]$ y $[F]$ el vector de fuerzas de exitación verticales discretizadas en los puntos de concentración de masa, A serán los desplazamientos verticales de estos puntos.

EJEMPLO DE APLICACION

El ejemplo de aplicación presentado corresponde a una cimentación de las descritas en el artículo para un turbocompresor centrífugo de 2588 HP con peso total de 27,262 kg con frecuencia de operación de 7330 r p m medida y de 6980 r p m nominal

Los resultados obtenidos pueden verse en los resultados del programa, los que muestran tanto las

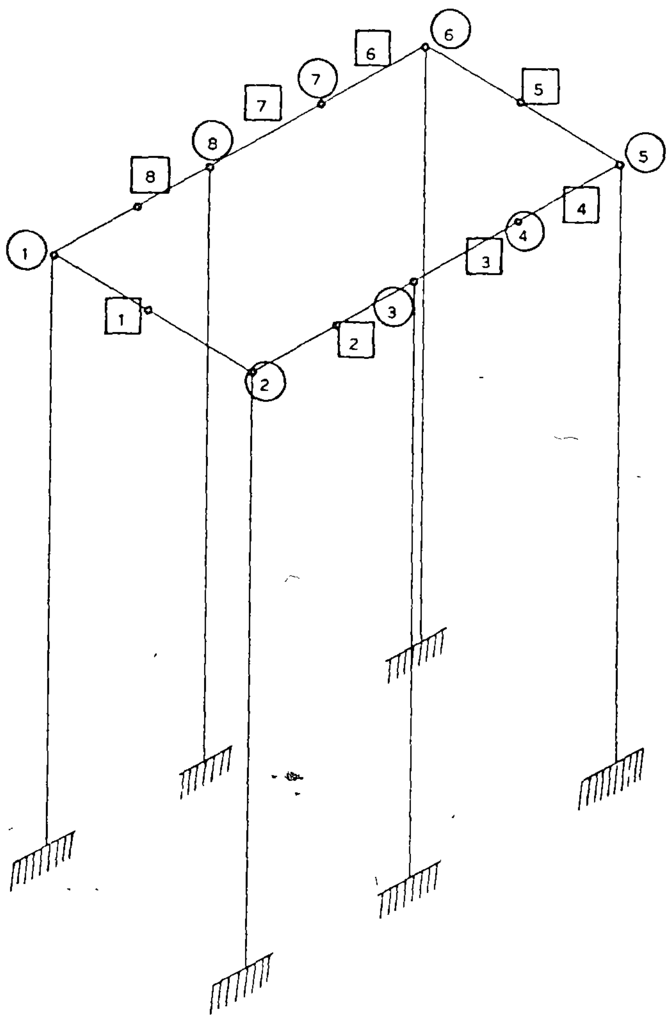

Fig. 1.- Idealización de la cimentación propuesta.

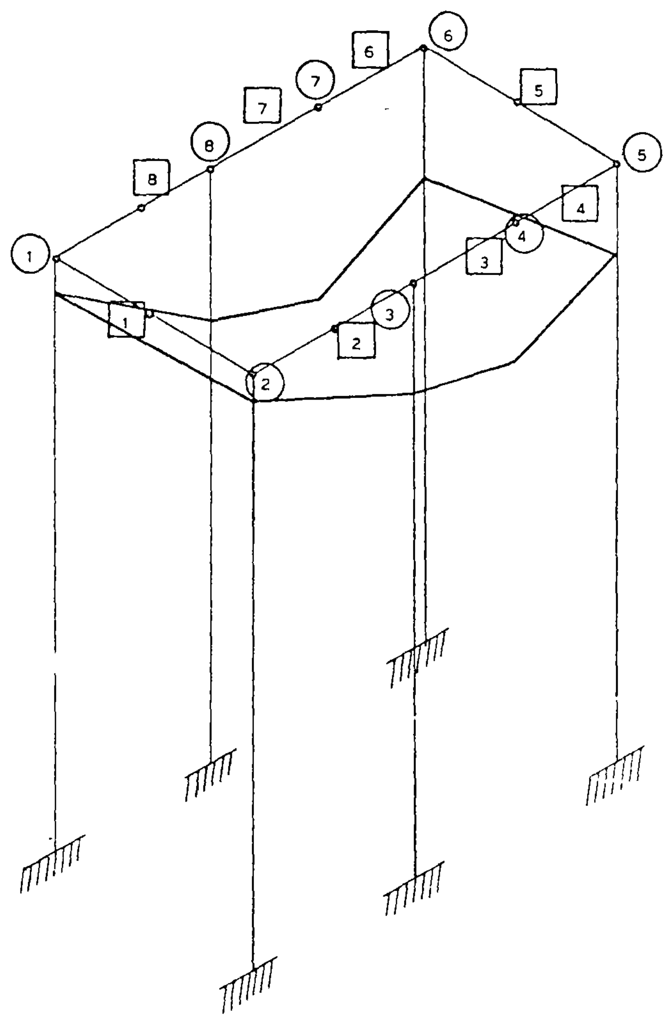

Fig. 2.- Configuración del primer modo.

ecuencias horizontales como verticales, por lo que pecta a estas últimas, el ASME (Amercian Society f Mechanical Engineers) propone analizarlas agru- ndo todas las columnas en un sólo grado de li- tad; lo que es equivalente a considerar una con- guración del primer modo con una misma defor- mación y sin considerar el efecto de rigidez relativo e las trabes; en contraste con este criterio se pre- ntan gráficamente los primeros tres modos de vi- ación verticales de la estructura propuesta para la cimentación en las figuras 2, 3 y 4.

El criterio adoptado por ASME para descrimi- nar la tolerancia que debe existir entre las frecuen- cias de operación y las naturales de la estructura es que deben diferir en un 20%, este criterio nos parece excesivamente simplista para el fenómeno que ocurre en la realidad, por lo que se recomienda efec- tuar una descomposición modal de la carga para po- der pesar los coeficientes de participación de cada modo, a fín de ponderar el peligro de resonancia real.

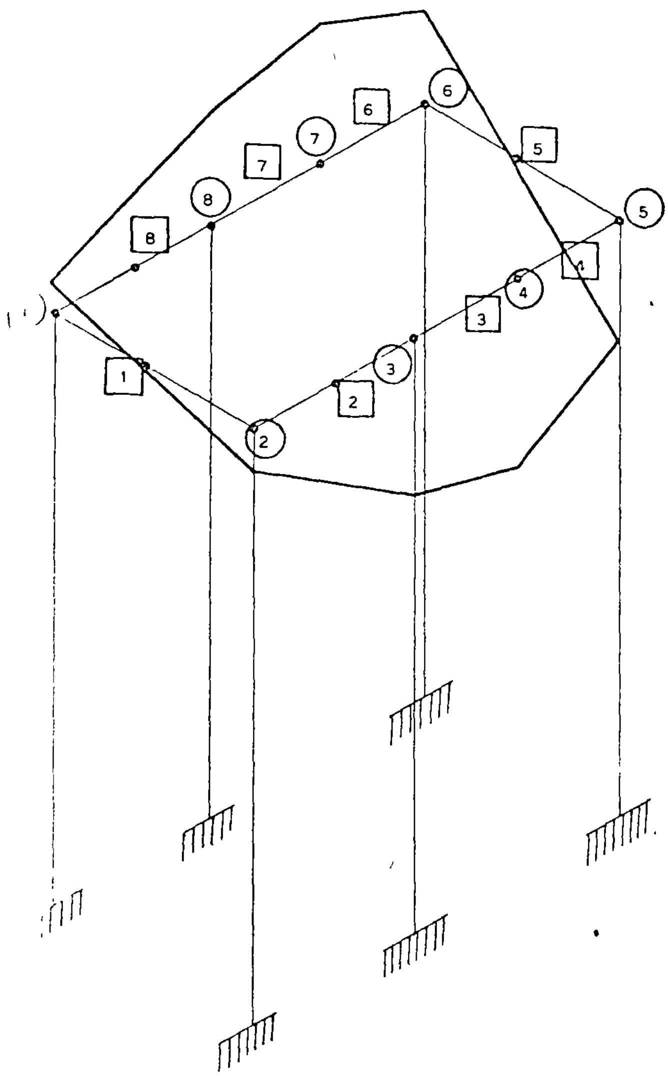

Fig. 3.- Configuración del segundo modo.

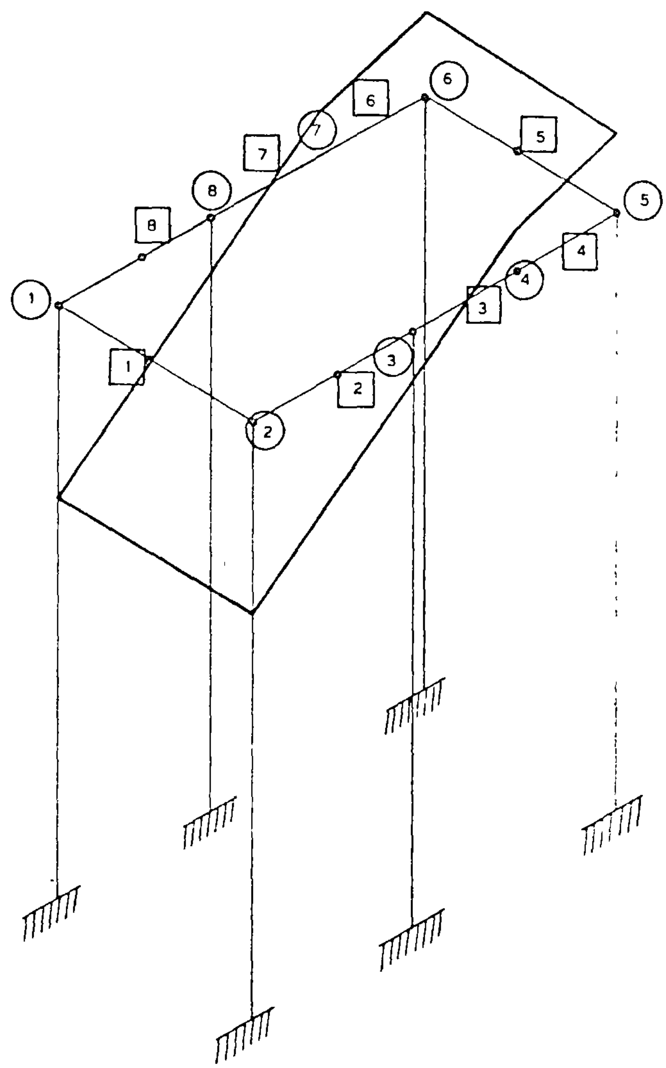

Fig. 4.- Configuración del tercer modo.

PROBLEMA EJEMPLO

CIMENTACION COMPRESORA GP-501, CTO.1072 2A ALTERNATIVA

NUDO	X	Y	Z
1	0.0	0.0	0.7430E 03
2	0.2700E 03	0.0	0.7430E 03
3	0.2700E 03	0.2200E 03	0.7430E 03
4	0.2700E 03	0.3600E 03	0.7430E 03
5	0.2700E 03	0.[illegible]000E 03	0.7430E 03
6	0.0	0.5000E 03	0.7430E 03
7	0.0	0.3600E 03	0.7430E 03
8	0.0	0.2200E 03	0.7430E 03
9	0.0	0.0	0.0
10	0.2700E 03	0.0	0.0
11	0.2700E 03	0.2200E 03	0.0
12	0.2700E 03	0.[illegible]000E 03	0.0
13	0.0	0.5000E 03	0.0
14	0.0	0.2200E 03	0.0

MIEMBRO	N1	N2	ANG	AREA X	AREA Y	AREA Z	IX	IY	IZ
1	1	2	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
2	2	[illegible]	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
3	3	4	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
4	4	5	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
5	5	6	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
6	7	[illegible]	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
7	[illegible]	7	0.0	[illegible]600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
8	1	[illegible]	0.0	3600.000	3600.000	3600.000	666000.000	3839808.000	540000.000
9	9	1	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000
10	10	2	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000
11	11	[illegible]	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000
12	12	[illegible]	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000
13	13	6	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000
14	14	8	0.0	4800.000	4800.000	4800.000	3058560.000	1440000.000	2560000.000

MATRIZ DE RIGIDEZ FINAL DE LA ESTRUCTURA

0.3671E 02	-0.733[illegible]-05	-0.8959E 04
0.137[illegible]-06	0.7[illegible]02	0.1065E 05
-0.8959E 04	0.1065E 05	0.9064E 07

MASAS CONSIDERADAS

MASA	X	Y
0.4000E-02	0.0	0.0
0.9000E-03	0.0	0.1100E 03
0.3900E-02	0.0	0.2200E 03
0.1100E-02	0.0	0.3600E 03
0.4300E-02	0.0	0.[illegible]000E 03
0.1600E-02	0.1350E 03	0.5000E 03
0.4300E-02	0.2700E 03	0.5000E 03
0.1100E-02	0.2700E 03	0.3600E 03
0.3900E-02	0.2700E 03	0.2200E 03
0.9000E-03	0.2700E 03	0.1100E 03
0.4000E-02	0.2700E 03	0.0
0.1400E-02	-0.1350E 03	0.0
0.5600E-02	0.4360E 02	0.5480E 02
0.5600E-02	0.2264E 03	0.5480E 02
0.6500E-02	0.4360E 02	0.2650E 03
0.6500E-02	0.2264E 03	0.2650E 03
0.2000E-02	0.4360E 02	0.4485E 03
0.1700E-02	0.2264E 03	0.4485E 03

COORDENADAS DEL CENTRO DE MASA

XRAYA = 134.[illegible] YRAYA = 228.0642

MATRIX DE R. HORIZONTAL REFERIDA AL CENTRO DE M.

0.3671E 02	-0.7[illegible]E-05	-0.1733E 05
0.1378E-06	0.7862E 02	0.2125E 05
-0.1733E 05	0.2125E 05	0.1935E 08

MATRIZ SUMII DE MASAS Y DE INERCIA

0.5930E-01	0.0	0.0
0.0	0.5930E-01	0.0
0.0	0.0	0.2592E 04

EL NUMERO DE ROTACIONES ES = 8
(PARA EL CALCULO DE FREC. HORIZONTALES)

FRECUENCIAS NATURALES DE VIBRACION HORIZONTALES
Y VECTORES CARACTERISTICOS ASOCIADOS

FRECUENCIA NATURAL (1) = 362.455 R.P.M.

VECTOR CARACTERISCO
0.72700924E 00 -0.9[illegible]303E 00 -0.1875141[illegible]E-01

FRECUENCIA NATURAL (2) = [illegible]10.67[illegible] R.P.M.

VECTOR CARACTERISCO
0.1825[illegible]59[illegible]E 01 0.[illegible]63[illegible]49150 01 -0.27[illegible]46726 -02

FRECUENCIA NATURAL (3) = 135.050 R.P.M.

VECTOR CARACTERISCO
0.3603595612E 01 -0.104[illegible]660E 01 0.51690094E-02

MATRIZ DE MASAS
0.9[illegible]23E-02 [illegible]2[illegible]E-02 0.3972E-02 0.4[illegible]E-02 0.[illegible]E-02 0.726[illegible]E-02
0.4939E-02 0.[illegible]20E-02

EL NUMERO DE ROTACIONES ES = [illegible]
(PARA EL CALCULO DE FREC. VERTICALES)

FRECUENCIAS NATURALES DE VIBRACION VERTICALES

FRECUENCIA NATURAL (1) = 9019.955 R.P.M.

FRECUENCIA NATURAL (2) = [illegible]900.[illegible] R.P.M.

FRECUENCIA NATURAL (3) = [illegible]64[illegible].18[illegible] R.P.M.

FRECUENCIA NATURAL (4) = 45[illegible]3.[illegible]2[illegible] R.P.M.

FRECUENCIA NATURAL (5) = 3904.[illegible] R.P.M.

FRECUENCIA NATURAL (6) = 376[illegible].368 R.P.M.

FRECUENCIA NATURAL (7) = [illegible]2[illegible]1.722 R.P.M.

FRECUENCIA NATURAL (8) = 3233.655 R.P.M.

VECTORES CARACTERISTICOS ASOCIADOS A LAS FRECUENCIAS VERTICALES DE VIBRACION

```
VECTOR CARACTERISCO
0.73240405E 00 -0.90528276E 00  0.42065552E 01 -0.88323651E 01  0.31471329E 01

VECTOR CARACTERISCO
-0.24572918E 01  0.63814697E 01 -0.33027792E 01

VECTOR CARACTERISCO
0.94656780E 00  0.64289965E 00 -0.33362589E 01  0.69485502E 01 -0.25791695E 01

VECTOR CARACTERISCO
-0.30366907E 01  0.38091125E 01 -0.42604055E 01

VECTOR CARACTERISCO
-0.25927414E 01  0.33242607E 01 -0.45037327E 01 -0.10059519E 01  0.67225227E 01

VECTOR CARACTERISCO
-0.5377[illegible]234E 01  0.32591413E 00  0.36302376E 01

VECTOR CARACTERISCO
0.34516640E 01  0.27582502E 01 -0.36158657E 01 -0.92181540E 00  0.53676596E 01

VECTOR CARACTERISCO
0.66073262E 01 -0.11666726E 01 -0.44867554E 01

VECTOR CARACTERISCO
-0.67835930E 01  0.69[illegible]226E 01  0.13740435E 01 -0.16736374E 01 -0.30751026E 01

VECTOR CARACTERISCO
0.30146494E 01  0.1[illegible]934095E 01 -0.13982210E 01

VECTOR CARACTERISCO
0.69656067E 01  0.68223448E 01  0.16043062E 01 -0.11872129E 01 -0.29338617E 01

VECTOR CARACTERISCO
-0.29974556E 01 -0.11692276E 01  0.16917849E 01

VECTOR CARACTERISCO
-0.10794230E 01  0.14044019E 01  0.54707928E 01  0.68500843E 01  0.44490242E 01

VECTOR CARACTERISCO
-0.31687786E 01 -0.49201566E 01 -0.39472198E 01

VECTOR CARACTERISCO
0.10763845E 01  0.70642592E 00  0.38552532E 01  0.49851570E 01  0.33967257E 01

VECTOR CARACTERISCO
0.46499405E 01  0.63819075E 01  0.53469372E 01
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AMPLITUDES HORIZONTALES DEL C. DE MASAS

AMPLITUD X -0.1270E-03 CMS.

AMPLITUD Y 0.9[illegible]E-07 CMS.

ROTACION R 0.[illegible]181E-07 RAD.

AMPLITUDES HORIZONTALES DE LOS NUDOS SUPERIORES
DE LA ESTRUCTURA RELATIVAS AL CENTRO DE MASAS

NUDO	DESPL X (CMS)	DESPL Y (CMS)
1	-0.1173E-03	-0.5530E-05
2	-0.117[illegible]E-03	0.5756E-05
3	-0.1267[illegible]-03	0.5758E-05
4	-0.1[illegible]26E-03	0.5758E-05
5	-0.1384E-03	0.57[illegible]E-05
6	-0.1[illegible]4E-03	-0.5530E-05
7	-0.1326E-03	-0.5530E-05
8	-0.1267E-03	-0.5530E-05

AMPLITUDES VERTICALES MAXIMAS DE LOS PUNTOS DE CONCENTRACION DE MASAS (CMS.)

NUDO	AMPLITUD
1	0.1129E-[illegible]
2	0.[illegible]72[illegible]E-[illegible]
[illegible]	0.[illegible]22[illegible]E-0[illegible]
4	0.2194E-03
[illegible]	-0.[illegible]60E-06
6	-0.[illegible]9[illegible]2E-0[illegible]
7	-0.[illegible]28E-0[illegible]
8	-0.97[illegible]-0[illegible]

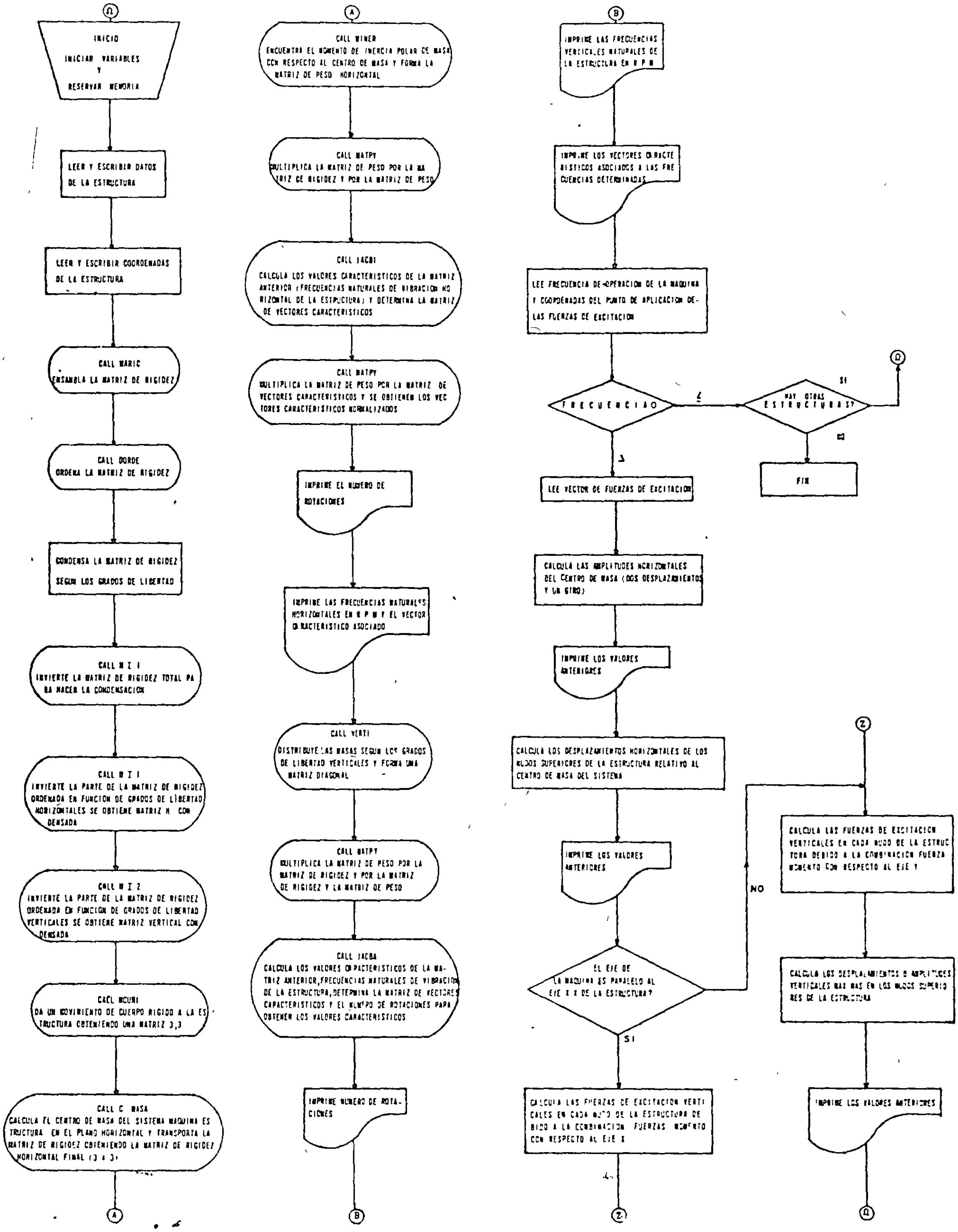

Diagrama de Flujo Condensado del Programa

IV CONGRESO NACIONAL DE INGENIERIA SISMICA
OAXACA, OAX.

COMPORTAMIENTO DE CIMENTACIONES RETICULARES PARA MAQUINARIA CON TRABES INTERMEDIAS

Alberto García Rubio

COMPORTAMIENTO DE CIMENTACIONES RETICULARES PARA MAQUINARIA CON TRABES INTERMEDIAS

(*) Ing. Alberto García Rubio

Aunque resulta aparente que el comportamiento de cimentaciones para maquinaria es un campo que debe ser tratado dentro de la Ingeniería Mecánica, podria pensarse que este tema no debía verse dentro de la Ingeniería Civil; sin embargo resulta importante analizar el comportamiento de las cimentaciones ante la presencia de las solicitaciones impuestas por una maquinaria rotatoria. Las cimentaciones reticulares o reticuladas en el caso de maquinaria, se emplean principalmente para dar apoyo a turbodinámos como pueden ser los turboexpansores, los turbocompresores y los turbogeneradores.

También se emplean con éxito en maquinas eléctricas como motogeneradores, compensadores sincronos, dinámos de alta potencia y motores eléctricos, en los que no se presentan cargas repentinas.

Se emplea un programa de computadora, del cual se obtienen las frecuencias naturales de vibración del sistema estructural en cuestión, los modos de vibración de éste y los desplazamientos (amplitudes) que sufre la plataforma superior a partir de la geometría del sistema propuesto y de las fuerzas de excitación

(*) Ingeniero de la Subdirección de Ingeniería de Proyecto del Instituto Mexicano del Petróleo

impuestas por la máquina. Los resultados mencionados se obtienen para un espacio de tres dimensiones.

Se adjunta el diagrama de flujo condensado de este programa.

Las hipótesis fundamentales son las siguientes:

1) El comportamiento de la estructura es elástico lineal sin disipación de energía, por lo tanto no se considera amortiguamiento.

2) El método de análisis empleado es el método de los desplazamientos o de las rigideces, convencional para estructuras en el espacio mediante el uso de matrices.

3) La distribución de masas se hace en forma discreta, es decir se supone una masa concentrada en los puntos donde se supone un grado de libertad vertical; en los puntos que coinciden con una columna, se toma la tercera parte de la masa de la misma.

4) La estructura está empotrada en la losa inferior en contacto con el suelo; la experiencia ha demostrado que las amplitudes tanto verticales como horizontales de las losas inferiores de apoyo del sistema estructural, son el orden de 0.5 a 2.5 micras, siendo estas en general mucho menores que las obtenidas en la losa superior de la estructura; por lo anterior puede emplearse este programa tanto en cimentaciones por su -

perficie como en cimentaciones donde se empleen pilotes.

Empleando entonces el programa de computadora mencionado se - analizó una estructura de soporte (ver figura 1) donde se dieron varias alternativas para la posición de elementos interme dios horizontales (trabes intermedias); tambien se varió el - número de estos elementos y las inercias de los mismos.

Los resultados obtenidos se muestran en las tablas 1 y 2, 3 y 4; los valores de las tablas 1 y 2 se graficarón a fín de poder ver el comportamiento de la cimentación de manera mas objetiva (Figs. 2 y 3). En los 6 primeros casos se propone la- misma inercia para la trabe intermedia variando la altura de- ésta con respecto a los apoyos de las columnas, el caso 7 se- refiere a una posición de trabe a una altura de 6.00 mts. - empleando la misma escuadría que para los casos anteriores só lo que eliminando las trabes interiores y el caso 8 muestra - el comportamiento de la estructura de apoyo para una altura - de trabes intermedias de 500 cms, sólo que con una escuadría- igual a la de las columnas de la estructura en cuestión.

De la observación de la gráfica en el caso de los modos de vibración verticales, se desprende que la variación de frecuencias para un mismo modo no es significativo y es aproximada - mente del 2%; lo cual dentro de los rangos de operación de ma

quinarias rotatorias no implica una aportación importante a los modos de vibración del sistema; la figura 1 muestra las escuadrías empleadas en la estructura que se tomó como ejemplo. Los puntos marcados con λ en las gráficas muestran el caso de las frecuencias naturales de la estructura con un sistema de trabes a una altura intermedia de 5.0 mts. (Ver Fig. 2) y con una escuadría igual a la de las columnas de la cimentación; se observa que la variación en los tres primeros modos de vibración es muy pequeña; aunque esta aumenta en los modos superiores.

En la Fig. 3, se muestra la gráfica que da las frecuencias de los modos horizontales de vibración que corresponden a tres grados de libertad en el plano horizontal (2 desplazamientos y un giro) para diferentes alturas de la trabe intermedia, se observa que las diferencias máximas para un mismo modo de vibración son del orden del 15%; en el caso marcado con + en donde se emplearon trabes de la misma escuadría que las columnas; las diferencias resultan mayores, sin embargo el empleo de escuadrías de este tipo es poco común en la práctica.

Las figuras 4 y 5, muestran las configuraciones de los 2 primeros modos verticales de vibración para el caso de estructura sin trabe intermedia y con trabe uniendo los puntos medios

de las columnas, se observa que prácticamente no existe variación.

De lo anteriormente expuesto, podemos concluir; que el empleo de trabes intermedias no reporta cambios importantes en el -- comportamiento dinámico de las estructuras de apoyo para la - quinaria rotatoria, por lo que su uso no implica una ayuda -- real al mejoramiento de dicho comportamiento. Solamente se - recomienda el empleo de trabes intermedias cuando se requiera por condiciónes estáticas de la estructura.

En el caso de emplear trabes intermedias de gran rigidéz, se- puede lograr incrementar los valores de las frecuencias naturales horizontales en porcentajes mayores; sin embargo los valores de estas frecuencias conviene conservarlos bajos, ya que - las frecuencias de operación de las máquinas, son casi siempre mayores a las frecuencias naturales horizontales del sistema.

Se anexa copia de los resultados del programa de computadora - para el caso de una estructura con trabe intermedia, a una altura de 6.00 mts. sobre el desplante de las columnas.

TABLA 1

FRECUENCIAS NATURALES PARA LOS MODOS VERTICALES DE VIBRACION A DIFERENTES ALTURAS DE LA TRABE INTERMEDIA

FRC. MODO / mts. ALTURA	FN1	FN2	FN3	FN4	FN5	FN6	FN7	FN8	FN9	FN10
0	1962.9	2053.1	2130.1	2215.2	2235.0	2336.1	2429.6	2561.9	2778.2	3048.6
2.50	1966.0	2058.1	2137.6	2220.0	2245.2	2347.9	2440.7	2573.9	2791.7	3079.2
5.00	1972.14	2064.98	2152.65	2229.94	2263.27	2369.41	2462.05	2596.94	2821.3	3132.10
6.00	1975.1	2066.8	2160.1	2235.6	2270.5	2378.0	2471.7	2606.4	2837.3	3151.7
8.00	1979.5	2067.9	2170.74	2247.55	2280.2	2391.7	2490.6	2621.9	2862.8	3167.6
9.60	1969.56	2062.31	2145.99	2226.6	2262.4	2378.7	2475.9	2612.68	2811.4	3111.09
* 6.00	1974.8	2061.9	2159.6	2234.4	2261.7	2368.7	2462.6	2585.3	2837.2	3133.0
** 5.00	1983.27	2081.46	2175.67	2253.56	2310.4	2438.45	2523.11	2687.28	2884.99	3280.85

* Trabes intermedias sin elementos interiores

** Trabes intermedias con escuadrias iguales a las columnas

TABLA 2

FRECUENCIAS NATURALES PARA LOS MODOS HORIZONTALES DE VIBRACION CON VARIACION DE ALTURAS EN LA POSICION DE -- LA TRABE INTERMEDIA

ALTURA mts. \ FREC. rpm	FN1	FN2	FN3
0	74.03	165.06	731.72
2.50	81.17	199.40	803.8
5.00	87.74	229.2	875.4
6.00	87.6	225.6	876.9
8.00	82.5	196.26	829.8
9.60	76.77	170.19	770.9
*6.00	84.64	225.0	805.29
**5.00	110.92	292.61	1114.56

* Trabes intermedias sin elementos interiores

** Trabes intermedias con escuadrias iguales a las columnas

TABLA 3

AMPLITUDES VERTICALES MAXIMAS DE LOS NUDOS SUPERIORES DE LA ESTRUCTURA (μX 10^{-2}) DEBIDAS A CARGAS DINAMICAS

NUDO \ ALTURA	0	250	500	600	800	960	* 600	** 500
1	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40	-2.40
2	-2.46	-2.46	-2.46	-2.46	-2.46	-2.46	-2.46	-2.46
3	-1.10	-1.10	-1.10	-1.10	-1.10	-1.10	-1.10	-1.10
4	-0.76	-0.76	-0.76	-0.76	-0.76	-0.76	-0.76	-0.76
5	0.30	0.30	0.30	0.30	0.30	0.30	0.30	0.30
6	-0.46	-0.46	-0.46	-0.46	-0.46	-0.46	-0.46	-0.46
7	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
8	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
9	1.94	1.94	1.94	1.94	1.94	1.94	1.94	1.94
10	2.64	2.64	2.64	2.64	2.64	2.64	2.64	2.64

* Trabes intermedias sin elementos interiores

** Trabes intermedias con escuadrias iguales a las columnas

TABLA 4

AMPLITUDES HORIZONTALES DE LOS NUDOS SUPERIORES DE LA ESTRUCTURA RELATIVOS AL CENTRO DE MASAS DEL SISTEMA. (μX 10^{-2})
DEBIDAS A CARGAS DINAMICAS

ALTURA cms. / NUDO	0		250		500		600		800		960		* 600		** 500	
	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y	X	Y
1	-6.31	-.399	-6.31	-.40	-6.31	-.40	-6.31	-.40	-6.31	-.40	6.31	-.40	-6.31	-.40	-6.33	-.404
2	-5.49	-.399	-5.49	-.40	-5.49	-.40	-5.49	-.40	-5.49	-.40	-5.49	-.40	-5.49	-.40	-5.50	-.404
3	-4.34	-.399	-4.34	-.40	-4.34	-.40	-4.34	-.40	-4.34	-.40	-4.34	-.40	-4.34	-.40	-4.34	-.404
4	-3.30	-.399	-3.30	-.40	-3.30	-.40	-3.30	-.40	-3.30	-.40	-3.30	-.40	-3.30	-.40	-3.30	-.404
5	-2.02	-.399	-2.02	-.40	-2.02	-.40	-2.02	-.40	-2.02	-.40	-2.02	-.40	-2.02	-.40	-2.01	-.404
6	-6.31	0.400	-6.31	0.40	-6.31	0.40	-6.31	0.40	-6.31	0.40	-6.31	0.40	-6.31	0.40	-6.33	0.401
7	-5.49	0.400	-5.49	0.40	-5.49	0.40	-5.49	0.40	-5.49	0.40	-5.49	0.40	-5.49	0.40	-5.50	0.401
8	-4.34	0.400	-4.34	0.40	-4.34	0.40	-4.34	0.40	-4.34	0.40	-4.34	0.40	-4.34	0.40	-4.34	0.401
9	-3.30	0.400	-3.30	0.40	-3.30	0.40	-3.30	0.40	-3.30	0.40	-3.30	0.40	-3.30	0.40	-3.30	0.401
10	-2.02	0.400	-2.02	0.40	-2.02	0.40	-2.02	0.40	-2.02	0.40	-2.02	0.40	-2.02	0.40	-2.01	0.401

* Trabes intermedias sin elementos interiores

** Trabes intermedias con escuadrias iguales a las columnas

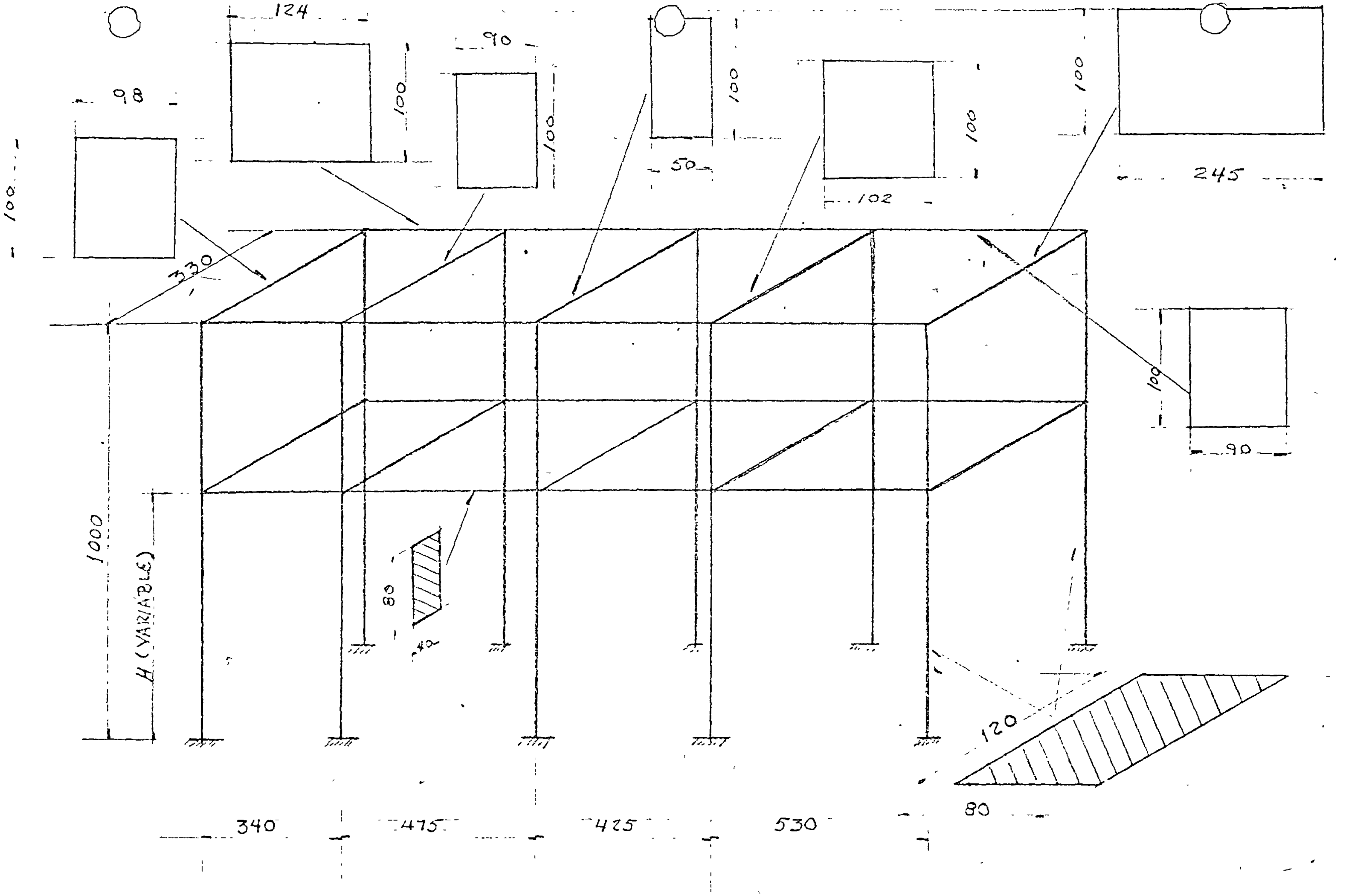

FIG. 1

Estructura empleada como ejemplo (Dimensiones y Secciones)

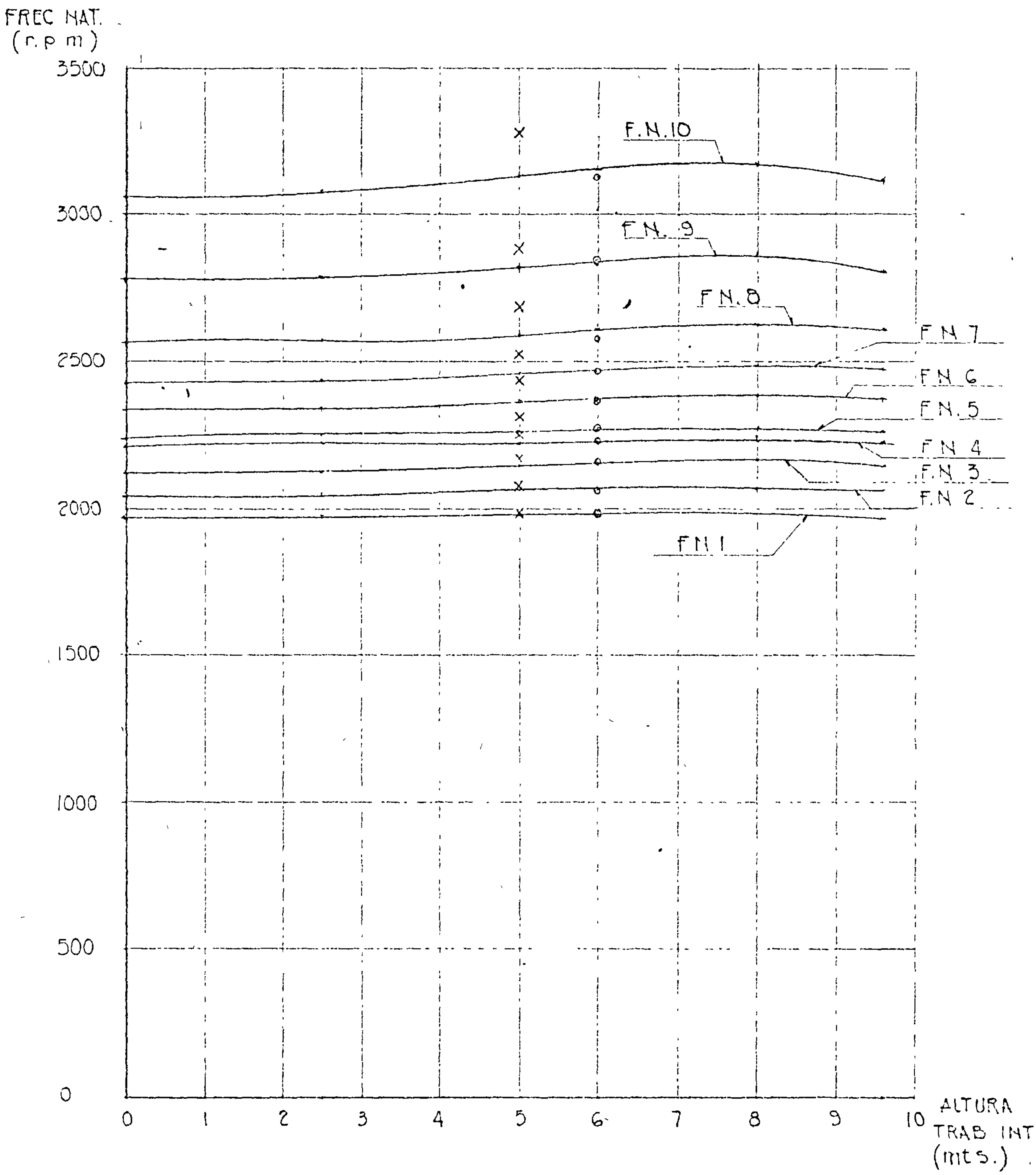

FIG. 2

Frecuencias naturales verticales para diferentes casos de altura de trabe intermedia

°Trabes intermedias sin elementos interiores

x Trabes intermedias con escuadrias iguales a las columnas

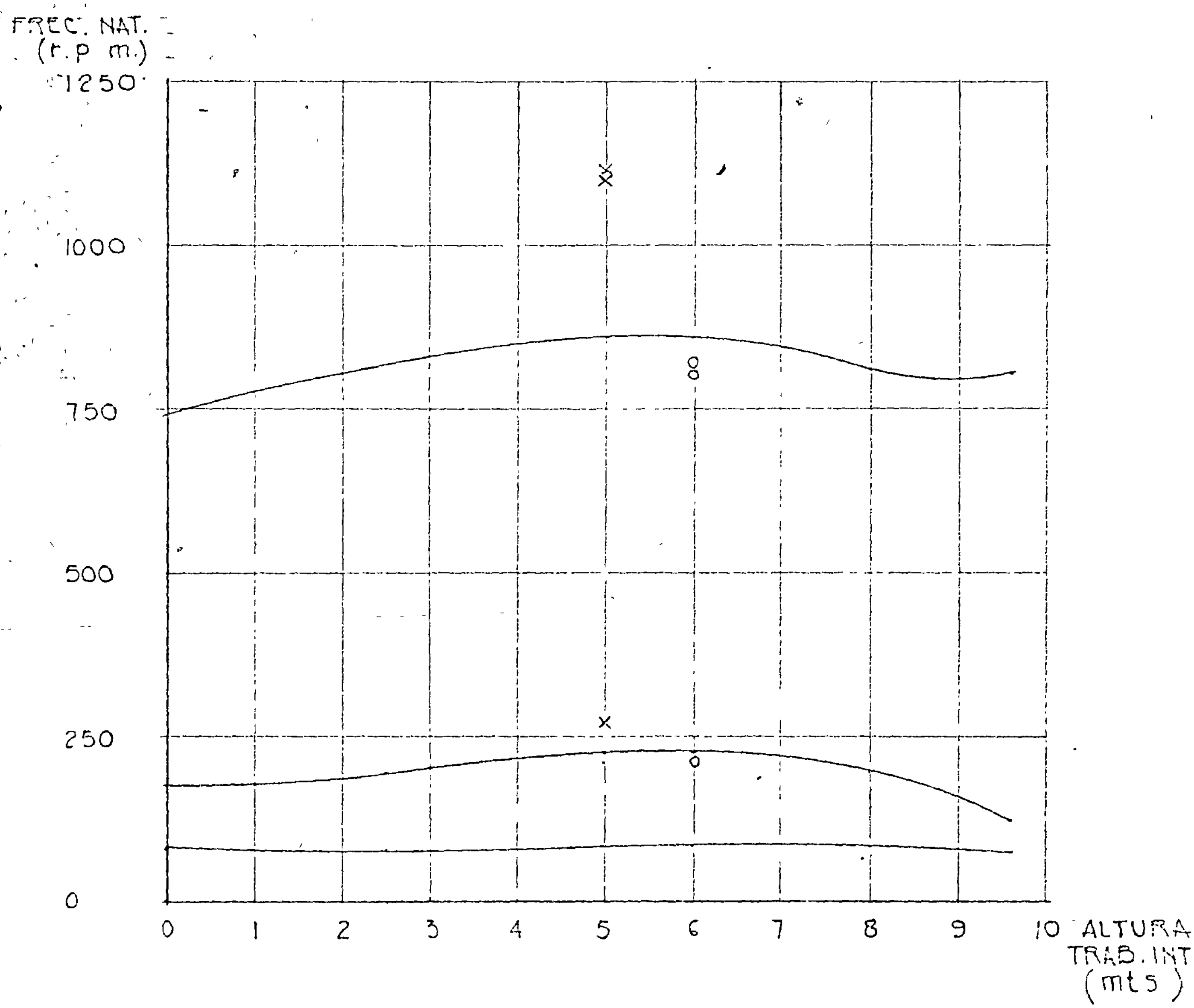

FIG. 3

Frecuencias naturales horizontales para diferentes casos de altura de trabe intermedia

- ° Trabes intermedias sin elementos interiores
- x Trabes intermedias con escuadrias iguales a las columnas

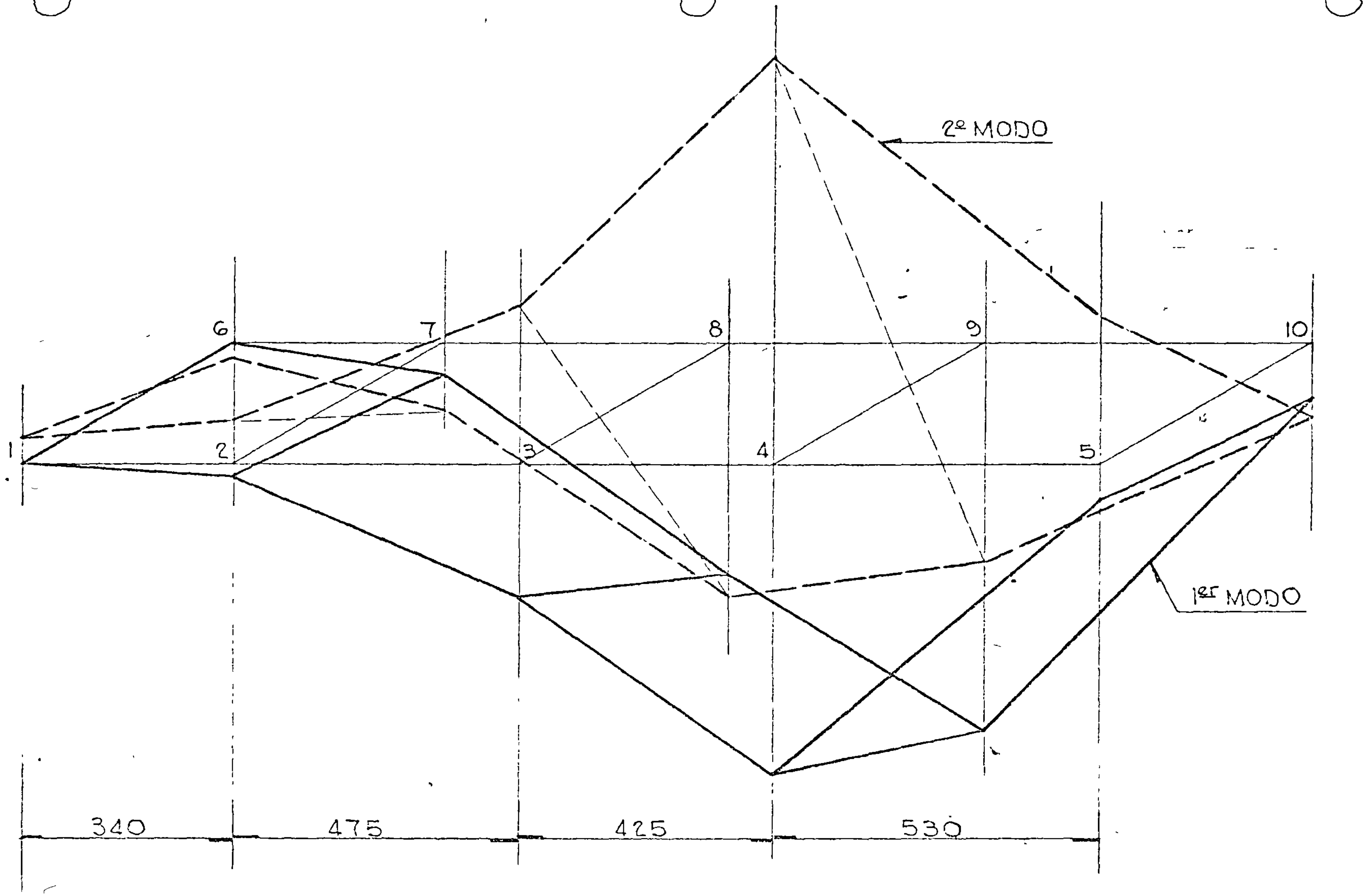

FIGURA - 4

CONFIGURACION MODAL "ESTRUCTURA SIN TRABE INTERMEDIA"

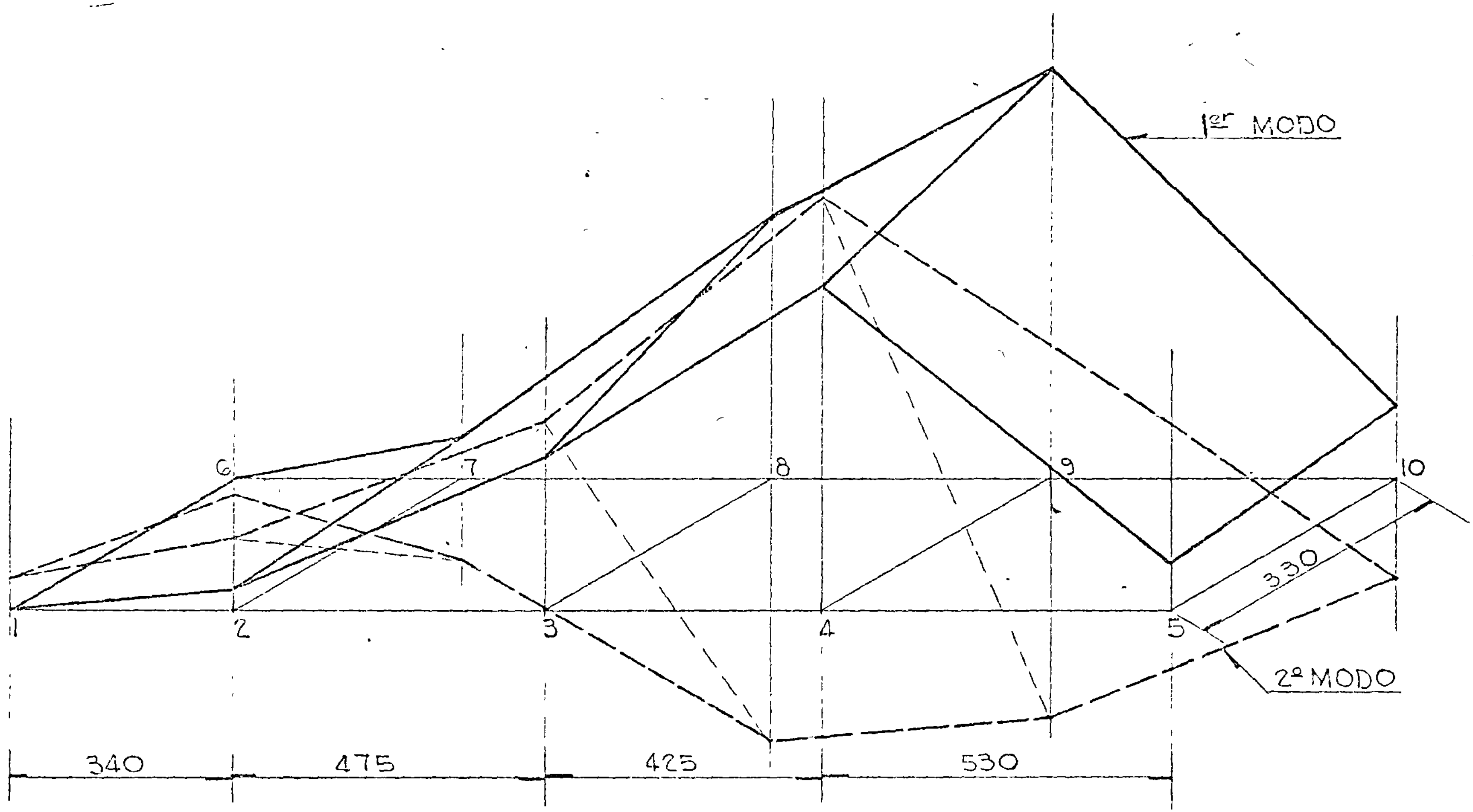

FIGURA - 5

CONFIGURACION MODAL "ESTRUCTURA CON TRABE INTERMEDIA H=5.00 m."

C1 [illegible] TACION DE COMPRESORA DE PRUEBA ALT 1 [illegible]

NUDO	X	Y	Z
1	.3300+03	.0000	.1000+04
2	.3300+03	.3400+03	.1000+04
3	.3300+03	.8150+03	.1000+04
4	.3300+03	.1240+04	.1000+04
5	.3300+03	.1770+04	.1000+04
6	.0000	.0000	.1000+04
7	.0000	.3400+03	.1000+04
8	.0000	.8150+03	.1000+04
9	.0000	.1240+04	.1000+04
10	.0000	.1770+04	.1000+04
11	.3300+03	.0000	.6000+03
12	.3300+03	.3400+03	.6000+03
13	.3300+03	.8150+03	.6000+03
14	.3300+03	.1240+04	.6000+03
15	.3300+03	.1770+04	.6000+03
16	.0000	.0000	.6000+03
17	.0000	.3400+03	.6000+03
18	.0000	.8150+03	.6000+03
19	.0000	.1240+04	.6000+03
20	.0000	.1770+04	.6000+03
21	.3300+03	.0000	.0000
22	.3300+03	.3400+03	.0000
23	.3300+03	.8150+03	.0000
24	.3300+03	.1240+04	.0000
25	.3300+03	.1770+04	.0000
26	.0000	.0000	.0000
27	.0000	.3400+03	.0000
28	.0000	.8150+03	.0000
29	.0000	.1240+04	.0000
30	.0000	.1770+04	.0000

MIEMBRO	N1	N2	ANG	AREA X	AREA Y	AREA Z	IX	IY	IZ
1	1	2	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
2	2	3	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
3	3	4	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
4	4	5	.0000	9000.000	9000.000	9000.000	11153700.000	7500000.000	6075000.000
5	6	1	.0000	9800.000	9800.000	9800.000	13459046.000	8166666.000	7843267.000
6	7	2	.0000	9000.000	9000.000	9000.000	11153700.000	7500000.000	6075000.000
7	8	3	.0000	5000.000	5000.000	5000.000	2862500.000	4166666.000	1041666.000
8	9	4	.0000	10600.000	10600.000	10600.000	15688000.000	8833333.000	9925133.000
9	10	5	.0000	24500.000	24500.000	24500.000	60515000.000	2041666.000	99999999.000
10	6	7	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
11	7	8	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
12	8	9	.0000	11500.000	11500.000	11500.000	18170000.000	9583333.000	12673958.000
13	9	10	.0000	9000.000	9000.000	9000.000	11153700.000	7500000.000	6075000.000
14	11	1	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
15	12	2	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
16	13	3	.0000	9500.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
17	14	4	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
18	15	5	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
19	16	6	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
20	17	7	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
21	18	8	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000

22	19	9	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
23	20	10	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
24	16	11	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
25	19	14	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
26	20	15	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
27	21	11	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
28	22	12	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
29	23	13	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
30	24	14	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
31	25	15	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
32	26	16	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
33	27	17	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
34	28	18	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
35	29	19	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
36	30	20	.0000	9600.000	9600.000	9600.000	12042240.000	11520000.000	5120000.000
37	11	12	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
38	12	13	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
39	13	14	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
40	16	17	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
41	17	18	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
42	18	19	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
43	14	15	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
44	19	20	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
45	17	12	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000
46	13	13	.0000	3200.000	3200.000	3200.000	1172500.000	1706666.000	426666.000

MATRIZ DE RIGIDEZ FINAL DE LA ESTRUCTURA

.2737+03	-.1802-04	-.2190+06
-.5275-06	.2097+03	.3460+05
-.2190+06	.3460+05	.3054+09

MASA	X	Y
.1549-01	.3300+03	.0000
.2222-01	.3300+03	.3400+03
.2212-01	.3300+03	.8150+03
.2569-01	.3300+03	.1240+04
.2181-01	.3300+03	.1770+04
.1549-01	.0000	.0000
.2222-01	.0000	.3400+03
.2212-01	.0000	.8150+03
.2569-01	.0000	.1240+04
.2181-01	.0000	.1770+04
.4600-02	.3300+03	.1050+03
.4600-02	.0000	.1050+03
.6130-02	.1650+03	.3400+03
.1024-01	.3300+03	.4050+03
.1024-01	.3300+03	.4050+03
.1024-01	.0000	.7180+03
.1024-01	.0000	.7180+03
.1495-01	.3300+03	.9200+03
.1496-01	.0000	.9200+03

.[illegible]0-02	.8000+02	.1240+04
.[illegible]0-02	.2500+03	.1240+04
.5610-02	.1650+03	.1240+04
.1869-01	.3300+03	.1337+04
.1869-01	.0000	.1337+04
.8130-02	.8500+02	.1770+04
.1019-01	.2450+03	.1770+04

XRAYA = 165.4352 YRAYA = -943.6839

MATRIX DE R. HORIZONTAL REFERIDA AL CENTRO DE M.

.2737+03	-.1802-04	-.4773+06
-.5275-06	.2097+03	.6930+05
-.4773+06	.6930+05	.9797+09

MATRIZ SUMI1 DE MASAS Y DE INERCIA

.3787+00	.0000	.0000
.0000	.3787+00	.0000
.0000	.0000	.1256+06

EL NUMERO DE ROTACIONES ES = 7

FRECUENCIAS NATURALES DE VIBRACION HORIZONTALES

FRECUENCIA NATURAL (1) = 876.981 R.P.M.

VECTOR CARACTERISCO
.44329901+00 -.62984975-01 -.27123858-02

FRECUENCIA NATURAL (2) = 225.654 R.P.M.

VECTOR CARACTERISCO
.36814750+00 .15953443+01 .40148660-04

FRECU[illegible]IA NATURAL (3) = 87.599 R.P.M.

VECTOR CARACTERISCO
.15326917+01 -.30252669+00 .77642895-03

MATRIZ DE MASAS

.2469-01	.4175-01	.3985-01	.4784-01	.3671-01	.2442-01
.3402-01	.4534-01	.4500-01	.3608-01		

EL NUMERO DE ROTACIONES ES = 146

FRECUENCIAS NATURALES DE VIBRACION VERTICALES

FRECUENCIA NATURAL (1) = 3151.725 R.P.M.

FRECUENCIA NATURAL (2) = 2837.305 R.P.M.

FRECUENCIA NATURAL (3) = 2606.387 R.P.M.

FRECUENCIA NATURAL (4) = 2471.677 R.P.M.

FRECUENCIA NATURAL (5) = 2378.004 R.P.M.

FRECUENCIA NATURAL (6) = 2270.472 R.P.M.

FRECUENCIA NATURAL (7) = 2235.568 R.P.M.

FRECUENCIA NATURAL (8) = 2160.137 R.P.M.

FRECUENCIA NATURAL (9) = 2066.868 R.P.M.

FRECUENCIA NATURAL (10) = 1975.101 R.P.M.

VECTOR CARACTERISCO
.35690378+01 -.12317386+01 .35368896+00 -.10645634+00 .49542385-01

VECTOR CARACTERISCO
-.42395387+01 .21159833+01 -.37533862+00 .12947874+00 -.65371005-01

VECTOR CARACTERISCO
-.47149394+01 .10060568+01 -.48892854+00 .13244633+00 -.10814767+00

VECTOR CARACTERISCO
-.35753864+01 .15093402+01 -.40617958+00 .10430693+00 -.94348673-01

NUDO	DESPL X (CMS)	DESPL Y (CMS)
1	-.6317-05	-.4015-06
2	-.5491-05	-.4015-06
3	-.4337-05	-.4015-06
4	-.3304-05	-.4015-06
5	-.2017-05	-.4015-06
6	-.6317-05	.4002-06
7	-.5491-05	.4002-06
8	-.4337-05	.4002-06
9	-.3304-05	.4002-06
10	-.2017-05	.4002-06

AMPLITUDES VERTICALES MAXIMAS DE LOS PUNTOS DE CONCENTRACION DE MASAS (CMS.)

NUDO	AMPLITUD
1	-.2398-05
2	-.2459-05
3	-.1500-05
4	-.759e-06
5	.3005-06
6	-.4520-06
7	.5102-07
8	.1021-05
9	.1936-05
10	.2635-05


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          AMPLITUDES HORIZONTALES DEL C. DE MASAS

     AMPLITUD X    -.4024-05 CMS.

     AMPLITUD Y    -.1698-08 CMS.

     ROTACION R    -.2429-08 RAD.


  AMPLITUDES HORIZONTALES DE LOS NUDOS SUPERIORES
E LA ESTRUCTURA RELATIVOS AL CENTRO-DE MASAS
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VI

FRAME FOUNDATIONS FOR MACHINERY

VI-1. Instructions for the Design and Construction of Frame Foundations

a. Field of Application of Frame Foundations. Frame foundations do not limit a designer in the location of the engine and its auxiliary equipment as do massive foundations. For example, condensers, pipelines, air vents, and electric wiring for turbodynamos and electromotors can be arranged much more conveniently if the machines are mounted on frame foundations.

The use of frame foundations facilitates considerably the inspection of and access to all parts of the machine. Therefore frame foundations are often employed for turbodynamos (turboblowers, turbocompressors, and turbogenerators) of varying power. In the course of recent years a tendency has appeared, in the practice of foundation design for these engines, to limit the use of frame foundations to low-power turbodynamos only (up to 10 to 12,000 kw), and to use massive foundations for turbodynamos of higher power. However, this tendency is not at all justified, since observations of frame foundations under high-power dynamos (up to 100,000 kw) show that these foundations are in many cases more economical than massive foundations and, as has been indicated, they are advantageous in many respects in regard to the mounting and maintenance of the engine: In addition, investigations established that very often cracks are formed in massive foundations under turbodynamos due to the stresses induced by settlement or by temperature changes, while no cracks due to these causes are observed in frame foundations.

Frame foundations can also be successfully used for various electrical machines, such as motor generators, synchronous compensators, high-power dynamos, and electromotors, in which no sudden changes in load occur.

Lately there have been cases in industrial design practice where frame foundations were used for reciprocating engines, in particular for compressors. This use of frame foundations is most rational in cases where for some reason a foundation should have a considerable height; this may happen, for example, if it cuts through a basement.

We shall not consider frame foundations under reciprocating engines, because these foundations are of such high rigidity that they should be computed as rigid bodies resting on elastic bases.

b. Design Assignment. In addition to data on soil conditions, the following information is required for the design of a foundation:

1. Foundation diagrams showing dimensions, distribution and sizes of pipelines, tunnels, channels, grooves, and openings in the foundation, and distribution and sizes of foundation bolts and pads under bolts

2. A Design Assignment for the installation of the condensation floor within the limits of the edge of the lower slab of the foundation

3. A Design Assignment for the installation of a platform around the turbosystem at floor level of the machine room

4. Data concerning the layout of auxiliary equipment, in particular chambers of the air-cooling apparatus and the generator outlets

5. A diagram of static loads acting on the foundation, imposed by both stationary and rotating parts (the magnitudes of loads and the points of their application should be indicated)

6. Power of the engine in kilowatts and speed

7. The distribution of hot pipelines and the temperatures at the outer insulation surfaces

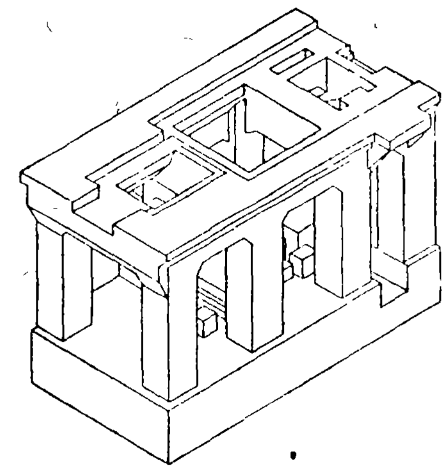


Fig VI-1 Frame foundation

c. Instructions for the Design. A frame foundation (Fig VI-1) is usually designed to be built of three or more transverse frames embedded in a sufficiently thick foundation slab. At the top these frames are tied together by longitudinal girders and an upper (erection) platform having openings necessary for stationary machine parts. Often a layout of the frame foundation is more complicated. Transverse walls are inserted between columns of the transverse frames, or two-story frames are used. Sometimes the rigidity of transverse frames is increased by structural measures to such a degree that the foundation cannot be considered an elastic frame system, but should be treated as an absolutely rigid body.

Figure VI-2 shows an isometric projection of a frame foundation with transverse walls, designed for a 100,000-kw turbogenerator. The computations for such a foundation, particularly dynamic computations, are very complicated. Therefore the foundation should be designed so that the diagram of stresses transferred from the machine to the base is as simple as possible and secures the most efficient distribution of internal stresses in the foundation, as well as the simplest forms of foundation

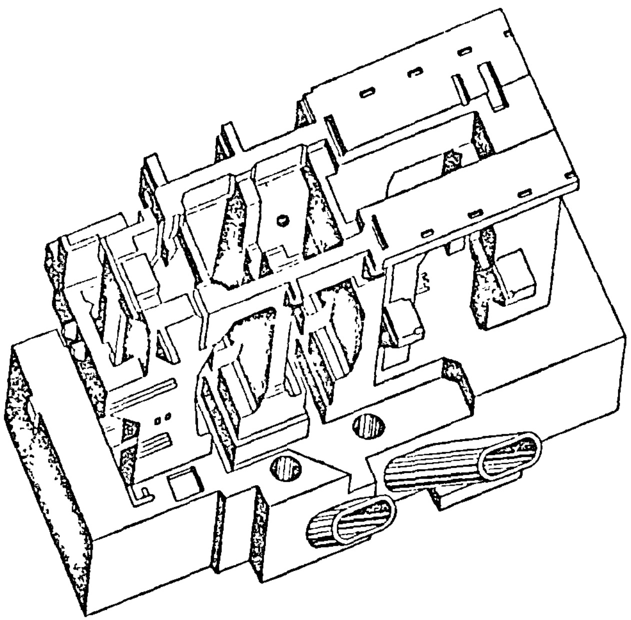


FIG. VI-2. Isometric view of a frame foundation for a 100,000-kw turbogenerator.

vibrations. In this respect, the foundation design should satisfy the following conditions:

The geometric layout of the foundation, the shapes of girder cross sections, and their reinforcement should be basically symmetric with respect to a vertical plane passing through the rotation axis of the engine. The frame beams should be placed directly under bearings, so that centrifugal forces which develop during engine operation are transmitted directly to the transverse frames. Axes of columns and transverse frame beams should lie in the same vertical plane perpendicular to the rotation axis of the motor. To prevent the appearance of torsional stress in transverse girders, eccentric loading of the latter should be avoided as much as possible. The direction of the load should, if possible, pass through the center of gravity of the beam cross section. The beams and

girders should be designed of rectangular or T-shaped cross sections. In accordance with the official *Technical Rules and Construction Code*, the minimum cross-sectional dimensions of unloaded elements should be 15 cm for slabs and 25 cm for girders.

The upper erection platform of the foundation should be as rigid as possible in its plane. One method of achieving this is to extend the longitudinal and edge transverse beams towards the outer faces of the foundation. If one attempts to increase the rigidity of the upper platform by extending the dimensions of the horizontal elements of the foundation in the direction of surfaces which limit the space assigned for the installation of machine parts, this change in dimensions should be coordinated with the machine manufacturer.

In order to increase the general rigidity of the frame foundation, haunches should be provided at the intersections of beams and columns.

Turbodynamos and electrical machinery are relatively safe in regard to the transmission of vibrations to buildings. No cases are on record of vibrations of entire buildings induced by these machines. However, occasionally it happens that turbodynamos cause objectionable local vibrations in columns, isolated wall sections, and especially floors and other building elements. An extensive instrumental investigation of foundations under turbogenerators was conducted by the author.[4] In the course of this investigation, considerable vertical floor vibrations were found in places where the foundation was rigidly connected with the floor of the machine room. These vibrations, especially when caused by high-frequency machines with speeds of, for example, 3,000 rpm, produce a very adverse effect on people standing on the vibrating sections, as they cause an unpleasant feeling in the soles of the feet. The vibrations also result in the displacement of pieces of equipment not tied to the floor. These phenomena are observed during floor vibrations with an amplitude of 0.02 mm. For this amplitude and a frequency of 3,000 oscillations per minute the vibration acceleration is about 0.2g.

In order to decrease the transfer of vibrations from the upper erection platform of the foundation under the turbogenerator to the building, and particularly to the floor of the machine room, it is recommended that a gap be provided around the entire contour of the upper foundation platform. The floor beams should be placed on separate columns supported by footings independent of the machine foundation.

Foundations under low-frequency electrical machines cannot produce the floor vibrations described above, since the frequency of natural vibrations of the foundations is considerably higher than the operational frequencies of the machines. Therefore in the design of these foundations there is no necessity to provide a gap between the foundation and the floor of the machine room. Bearing floor elements may be supported

directly by the frame beams and the columns of the foundation under low-frequency electrical machines In some cases such a support may be effective in decreasing the amplitudes of machine foundation vibrations.

Maintenance records of foundations under high-frequency turbo-systems indicate cases of relatively large vibrations of cantilevered parts of the erection platform of the foundation.

Figure VI-3 gives graphs of the distribution of amplitudes of vertical vibrations of cantilevered elements along one of the foundations investigated. These graphs show that in some places the amplitudes of vibrations reached 0.06 mm, which corresponds to an acceleration of vibrations equaling some 0.5g. Vibrations with such high acceleration resulted in the formation of cracks in the erection platform. Figure VI-3 indicates

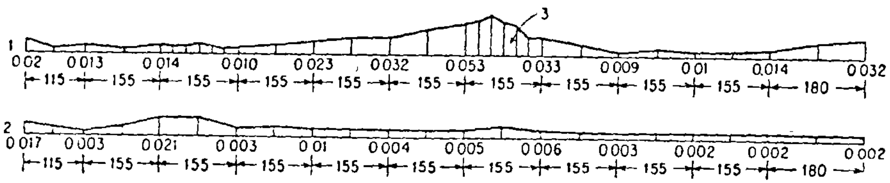


FIG. VI-3. Recorded vibrations (in millimeters) of a floor platform cantilevered around a machine foundation.

(1) a crack in the cantilevered slab and in the edge girder; (2) the zone of assumed deformation in the platform; (3) a crack in the edge girder.

Such vibrations occur only when the frequency of natural vibrations of the erection platform, acting as a cantilever of variable cross section, is close to the frequency of machine rotation Therefore the cantilever elements of the foundation erection platform should be designed to be much more rigid than is required by static computations; their frequencies of natural vibrations should be much higher than the frequency of machine rotation

The cantilevered elements of the erection platform usually are T beams of variable cross sections; therefore the computation of the frequencies of natural vibrations of these elements involves some difficulties and is extremely laborious. The design of the erection platform should ensure sufficient rigidity of such cantilever elements. This may be achieved by the installation of a rigid circumferential edge beam resting directly on the cantilevers; another method consists in the installation of special rigid stiffeners. The cross-sectional height of the cantilever at the embedment point should be no less than 60 to 75 per cent of its span.

Turbodynamo and electrical-machine bearings should be thoroughly adjusted, and the shafts should be in strictly horizontal position. Therefore designs of foundations under these machines should ensure proper

centering of the masses. For many machine foundations an eccentricity in mass distribution is permissible up to 5 per cent of the side of the foundation area in contact with soil, in the direction in which displacement of the center of gravity occurs. The eccentricity in turbodynamos and electrical machines should, if possible, come close to zero; in any case, its value should not exceed 1 to 2 per cent.

The author studied 36 foundations under turbogenerators and found that only in 2 foundations the amplitudes of vertical vibrations of the lower slabs were 0 002 to 0 003 mm; in 3 foundations the amplitudes of vertical vibrations of the slabs were on the order of 0.001 mm; the amplitudes of vibrations of the lower slabs of the remaining foundations were smaller than 1 micron (0.001 mm). The vibration amplitudes of the foundation slabs were much smaller than the amplitudes of vertical vibrations of the upper parts of the foundations.

The results of instrumental measurements of foundation vibrations lead to the conclusion that in practice the lower slabs of foundations under turbogenerators are not subjected to vibrations and consequently do not transmit any dynamic pressure to the base. Therefore the pressure on the soil under turbogenerator foundations is determined only by static loads, i e., by the weight of the foundation and equipment thereon. Hence it is clear that it is not necessary to follow the traditions of recent practice in assigning design pressures under turbogenerators not to exceed 0.5 to 0 6 of the permissible pressure on soils determined with respect to static loading only.

The introduction of a coefficient equaling 0.5 to 0.6 and the reduction of permissible pressure on the soil led to the necessity for employing piles, and consequently to considerable rise in construction cost It should be noted that the above-mentioned extensive investigation of machine foundations established that the use of pile foundations did not safeguard against considerable settlements and tilting of foundations under turbogenerators.

The lower foundation slabs under turbogenerators practically do not vibrate at all; therefore the coefficient of reduction of permissible pressure on soil may be taken to equal 0.8 to 1 0.

The depth of foundation under turbodynamos and electrical machines has no effect on the transmission of vibrations to adjacent structures. Therefore, when necessary because of design considerations or other reasons, the depth of foundation under the machine may be made even smaller than the depth of footings under walls or columns. If a foundation under a turbodynamo is to be erected close to footings under walls, columns, and other machines, then special care should be taken to protect it from nonuniform stresses imposed by adjacent footings. Hence foundations under turbodynamos and electrical machines should be

placed at such distances from adjacent foundations that the pearshaped lines of stresses (i.e., the "pressure bulb") in the soil imposed by the latter do not distort significantly the symmetry of the lines of stresses under the machine foundations in question. For the same reason, in some cases it may be useful to increase somewhat the depth of machine foundations with respect to the depth of adjacent footings under walls or columns.

The lower foundation slab should be sufficiently rigid to secure proper embedment of the foundation columns and prevent their nonuniform settlement. In addition, the presence of a lower foundation slab having considerable thickness decreases the height of the common center of gravity of the machine and foundation. Therefore the thickness of the lower foundation slab is usually taken larger than required by static computations. Tentative values of the height of the foundation slab, depending on the power of the machine, are taken as follows:

For machines with power up to 6,000 kw:	0.8 to 1.2 m
For machines with power of 6 to 12,000 kw:	1 to 1.6 m
For machines with power of 12 to 25,000 kw:	1.6 to 2 m
For machines with greater power:	2 to 4 m

Modern turbodynamos use steam of high temperature; consequently proper thermic insulation of steam pipes and air lines conducting hot air should be provided. The pipes should be insulated at least until they leave the foundation. The temperature at the outside surface of the insulation should not exceed 40 to 50°C; otherwise considerable local temperature stresses may develop in the foundation. Therefore the installation of steam and air pipes directly inside the foundation is objectionable.

Frame columns and beams are either reinforced according to design computations or the reinforcement is fitted to field conditions.

In foundation slabs having a thickness of 1.0 m, the vertical reinforcing rods should reach the area in contact with soil. In higher slabs, it is permissible to cut 50 per cent of the reinforcing rods at the half height of the foundation slab. Relevant chapters of the official *Technical Rules and Construction Code for Design of Reinforced-Concrete Structures* should be used in the design of foundation units, and, in addition, the following directions should be taken into account: All units of the foundation should be provided with double reinforcement. A symmetric reinforcement should in all cases be employed in the columns. Reinforcing rods should also be installed along the other two sides of cross sections of beams and columns, even if they are not required by design computations. The amount of reinforcement in separate foundation units should

be no less than 30 kg/m³ of concrete. The distance between stirrups in beams should not exceed 25 cm, and in columns 35 cm. To resist stresses induced by settlement, reinforcing rods of 8 to 10 mm diameter are to be installed along three mutually perpendicular directions in massive units of the foundation and are to be spaced 50 to 60 cm apart. The upper and lower reinforcements of the lower foundation slab should be tied together by stirrups (dowels) and spaced 50 to 50 cm apart in a checkerboard pattern. Hooks are to be provided at the ends of steel rods, subjected both to tensile and to compressive stresses. When the reinforcement for the columns is designed, it should be kept in mind that the total steel area of vertical reinforcing rods in a column should be smaller than the total cross-sectional area of the anchor foundation bolts. Additional reinforcing rods should be placed in sections where the foundation is weakened by openings, ducts, etc.

Concrete type 110† is employed for the upper parts of frame foundations, and concrete type 90† is used for the lower foundation slabs.

d. Instructions for Construction Operations. The construction of foundations under turbodynamos and electrical machines should proceed in accordance with all requirements of the applicable official *Technical Rules and Construction Code.*

It should be noted that large cracks observed in foundations under operating turbodynamos and electrical machines are in most cases caused by careless construction work. The construction of foundations under these machines should be carried out with particular care, since the performance of these machines affects the normal work of many plants. Special care should be taken in meeting the following requirements in regard to construction procedures:

Concrete employed for the erection of the foundation should be of plastic consistency, without excessive water; a slump test should show that the cone slump is around 10 to 12 cm (4 to 5 in.). The same concrete mix should be used throughout the construction of the whole upper part of the foundation. The forms for the upper part of the foundation should be fitted with grooves and planed on their inner surface.

Concrete should be poured continuously in horizontal layers. In an emergency, an interruption may be permitted at the level of the upper edge of the lower slab, or at the level of one-third of the column height, where the bending moment has a minimum value. If an interruption in the work occurs, the following measures should be taken to secure the monolithic character of the foundation:

1. Along the cross section of the foundation, where the pouring of concrete was interrupted, 16-mm reinforcing rods should be added to those installed according to the design. Short dowels should be

† See footnote in Art. IV-1-c, p. 132.

embedded to a depth of not less than 0.5 m on both sides of the joint, and their spacing should not exceed 0 2 m

2. The surface of the joint should be rough. Prior to placing a new layer of concrete, the previously laid surface should be thoroughly cleaned, washed by water, and covered with a rich cement mixture.

As a rule, the placing of the foundation concrete should be mechanized, and a uniform distribution of the concrete aggregates should be assured. A segregation of concrete aggregates into layers usually occurs in places where it is delivered from considerable height. If anchor bolts are embedded into the foundation to considerable depth, it is recommended that pipes of corresponding cross sections be inserted for these bolts; these pipes remain in the concrete permanently. During the concreting of the foundation, the quality control of concrete and of its aggregates is essential, and sample cubes of concrete are to be taken for investigation of its strength properties in accordance with special instructions All essential points in the process of foundation construction should be recorded in special documents. In any case, the following documents should be compiled (1) a record of the nature of the soil in the excavation made for the foundation; (2) a record of changes in the type of concrete used for the foundation; it should be noted at what elevation such changes took place, (3) a record concerning the interruption in concreting, if such an interruption occurred; the place where this interruption took place should be noted with a description of measures taken to secure a proper joint; (4) a record of the condition of the concrete after the forms were removed; the length of time the concrete remained in the forms should be noted

In the process of machine assembly, prior to pouring cement under the machine bedplate, the adjoining foundation surface should be cleaned thoroughly. This surface (erection platform) should be rough to secure the best possible binding of the additionally poured cement to the foundation

The location of all openings, recesses, etc., should be carefully checked against design drawings

VI-2. Computations of Forced Vibrations of Frame Foundations

a Exciting Loads Imposed by Turbodynamos and Electrical Machines. The exciting loads imposed by turbodynamos and electrical machines, unlike those of reciprocating engines and impact mechanisms, cannot be established by computations.

The main moving units of these machines are rotors which execute simple rotating movements. Theoretically the center of gravity of the rotor coincides with the axis of rotation, and consequently the theoretically established values of unbalanced inertia forces equal zero.

However, actual conditions are different In any engine containing rotating parts, even if this engine is well balanced, there remains a certain unbalanced state caused by the fact that the center of gravity of the rotating parts does not exactly coincide with the axis of rotation This residual unbalanced state cannot be completely eliminated, and in the course of the machine operation there appear unbalanced inertial forces which induce foundation vibrations

The magnitude of these exciting loads is proportional to the eccentricity of the rotating parts, the magnitudes of their masses, and the square of the frequency of machine rotation. Rotors of high-power turbodynamos and electrical machines weigh tens of tons, and their speeds can be very large—up to 10,000 rpm Therefore even for minute eccentricities of rotating masses the magnitudes of the exciting loads may be very large Consequently, their influence should be taken into account in the design of foundations For a long time the magnitudes of exciting loads imposed by turbodynamos were unknown, therefore in computations of foundations for turbodynamos some "temporary" loads were taken into account The static action of these loads was assumed to be equivalent to the dynamic action of actual exciting loads caused by the unbalanced state of the engine

Many suggestions were offered concerning the selection of these equivalent loads However, all these suggestions were equally ungrounded, and design computations of foundations were reduced to static stress analyses of the action of arbitrarily selected loads

However, in the course of recent years, voluminous material has been collected in the U S S.R. concerning the balancing of turbodynamos and electromotors, as well as measurements of vibrations of these machines This material makes it possible to establish design values of exciting forces caused by these machines with a degree of accuracy sufficient for practical purposes. Thus it is no longer necessary to introduce into computations the previously mentioned static equivalents of loads For the same reason, the method of foundation design changes instead of computations taking into account static-equivalent loads, computations are performed of forced vibrations of foundations produced by exciting forces and moments Consequently the foundation may be so designed that forced vibration amplitudes do not exceed permissible values.

This method of design of foundations under turbodynamos and electrical machines does not differ in principle from methods of design accepted, for instance, for foundations under reciprocating engines

Let us assume that the exciting loads developed by the machine under consideration can be reduced to one unbalanced centrifugal force F, whose plane of action coincides with the plane of symmetry of the machine rotor This unbalanced state is generally called the static unbalanced

state. The unbalanced state of the rotor may be caused by the fact that in addition to the exciting force there exists also an exciting moment (the dynamic unbalanced state) Rotors of electromotor generators, as well as rotors of steam turbines, are short; if the machine under consideration has one rotor whose plane of symmetry coincides with a vertical plane passing through the center of gravity of the foundation, then the exciting moment will be small and may be neglected.

The determination of unbalanced loads is much more difficult for machines with several rotors. If a machine has two rotors (which occurs in the majority of cases), then the force F acting in the vertical transverse plane of the whole installation (the foundation and machine) can be considered in computations as a design exciting load. Then the exciting moment equals

$$M = Fl_e \tag{VI-2-1}$$

where l_e is the distance along the axis of the main shaft between the resultant of exciting forces and the center of mass of the whole installation.

The exciting force of the rotor, being the unbalanced centrifugal inertial force, will rotate with the same frequency as the machine. Therefore the vertical and horizontal components of the exciting force will equal

$$\begin{aligned} F_z &= r_0 m_0 \omega^2 \sin \omega t \\ F_x &= r_0 m_0 \omega^2 \cos \omega t \end{aligned} \tag{VI-2-2}$$

where r_0 = eccentricity of machine rotor
m_0 = mass of rotor
ω = rotation frequency
F_z, F_x = vertical, horizontal components of exciting force (which act in a plane perpendicular to machine shaft)

The exciting moment can be resolved in the same manner into its vertical and horizontal components. Under the action of the vertical component of this moment, the foundation will undergo forced vibrations in the plane parallel to the main shaft of the machine. Measurements show that foundation vibrations often occur in this plane. However, the amplitudes of these vibrations usually are small in comparison with the amplitudes of vertical and horizontal vibrations in a direction perpendicular to the shaft of the machine. Therefore dynamic computation of the foundations under turbodynamos and electrical machines may be limited to computation of the amplitudes of vibrations induced by the exciting force and the horizontal component of the exciting moment, which equals

$$M_x = F_x l_e \tag{VI-2-3}$$

The frequency of rotation and the mass of rotating parts of a turbodynamo or an electrical machine are known; consequently, the eccentricity (unbalance) should be known in order to determine the exciting loads acting on the foundation. The eccentricity can be tentatively determined only from the results of balancing of machines and from measurements of vibrations before and after balancing

Let us assume that the rotor of the machine under consideration has a static unbalanced state, defined by the force F_0, which causes an amplitude of forced vibrations equal to A_0. Let us further assume that in the process of balancing an additional mass was attached to the rotor at some distance from the axis of rotation This mass produced a centrifugal force F. Then we assume that as a result of balancing, the amplitude of forced vibrations decreased to the value A.

There exists a simple proportional relationship between the magnitude of the exciting force and the amplitude of forced vibrations it produces; therefore

$$F_0 - F = \frac{A}{A_0} F_0$$

Hence,

$$F_0 = \frac{A_0}{A_0 - A} F$$

With this relationship it is possible to determine the value of the initial unbalanced exciting force from the results of balancing and measurements of vibrations before and after balancing. From the value of the initial unbalanced exciting force, the mass of the rotor, and the frequency of its rotation, it is easy to determine the eccentricity:

$$r_0 = \frac{F_0}{m_0 \omega^2}$$

Table VI-1 presents data on the balancing of several turbogenerators of various types and powers. From these data we computed values of F_0 and r_0.

All the data presented in Table VI-1 refer to balancing carried out under operating conditions, at times when, in the opinion of workers, machines vibrated with increased amplitudes. Therefore the computed values of exciting forces and eccentricities lie within the range of maximum permissible values.

The amplitudes of vibrations were measured, not on the foundations, but on the bearings, at the same place and in the same directions, both before and after balancing Vibrations were measured by means of a Geiger vibrograph, which in some cases yields considerably exaggerated readings. In spite of this, the presented values of F_0 and r_0 permit some

conclusions which are of interest for dynamic computations of foundations under machines of the type being considered.

Table VI-1 shows that the eccentricity r_0 depends on both the power of the machine and its speed. For machines characterized by 1,500 rpm

TABLE VI-1. RESULTS OF BALANCING MACHINE ROTORS

No.	Power, kw	Weight of rotor, tons	Amplitude of vibrations, mm		F, tons	F_0, tons	r_0, mm
			Prior to balancing A_0	After balancing A			
	Machines with 1,500 rpm						
1	5,000	12 0	0 160	0 035	1 86	2 38	0 086
2	3,000	5 0	0 075	0 012	0 43	0 97	0 085
3	3,000	5 0	0 125	0 023	0 92	1 12	0 097
4	3,000	5 0	0.125	0 025	0 76	0 95	0 082
5	3,000	5 0	0 092	0 027	0 30	0 43	0 038†
6	3,000	5 0	0 600†	0 058	1 23	1 36	0 118
7	3,000	6 0	0 062	0 026	0 52	0 80	0 070
8	50,000	70 0	0 260	0 017	11 41	12 2	0 076
9	50,000	70 0	0 350†	0 030	23 5	25 8	0 160
10	50,000	70 0	0 43†	0 035	25 4	27 3	0 170
11	50,000	37 5	0 150	0 058	3 97	6 5	0 075
	Machines with 3,000 rpm						
12	17,500	18 3	0 087	0 066	1 01	4 3	0 025
13	16,000	18 0	0 157	0 060	1 77	2 87	0 017
14	16,000	7 0	0 133	0 056	1 64	2 84	0 015
15	16,000	18 0	0 170	0 042	2 36	3 10	0 019
16	16,000	18 0	0 140	0 025	8 25	10 0	0 060†
17	25,000	20 0	0 170	0 058	5 10	7.75	0 042
18	25,000	17 5	0 120	0 040	3 02	4 53	0 029
19	6,000	7 2	0.180	0 030	2.18	2 62	0 040
20	6,000	7 2	0 125	0 021	1 00	1 20	0 018

† This figure is not reliable

and 3,000 kw power, the value of eccentricity lies in the relatively narrow range from 0.070 to 0.118 mm. In only one case the eccentricity was much smaller (0.038 mm). For machines with the same speed but 50,000 kw power, the value of eccentricity varies within a somewhat wider range: from 0.075 to 0.170 mm. However, the order of eccentricity values for these machines remains similar to that for low-power engines. Consequently, the increase in the magnitude of the exciting

force observed with an increase in power is mainly explained by the increase in the mass of rotating parts

The data of Table VI-1 which refer to machines characterized by 3,000 rpm show that the eccentricity for these machines lies in the range 0 017 to 0 015 mm For these machines, as for machines with speeds of 1,500 rpm, the value of the exciting force grows with an increase in power, and consequently with the weight of the rotor.

Thus there is a difference between the orders of eccentricity for machines with speeds of 1,500 rpm and 3,000 rpm The maximum value of eccentricity for the low-speed machines may be taken as 0 20 mm, but for machines running at 3,000 rpm the maximum eccentricity does not exceed 0 05 mm. Consequently, it can be held that the eccentricities of the rotating masses of turbodynamos are approximately inversely proportional to the squares of their speeds As the number of revolutions increases, the weight of rotating machine parts (provided the power is the same) decreases; therefore high-speed turbodynamos are better balanced.

Generalizing the above relationship between the eccentricity and the number of revolutions, and selecting 0.20×10^{-3} m as the design value of eccentricity for machines having a speed of 1,500 rpm, we obtain the following expression for a machine running at N rpm.

$$r_N = \frac{500}{N^2} \quad \text{meters} \tag{VI-2-4}$$

This relationship may serve as a basis for the selection of tentative design values of eccentricity for rotating machine masses characterized by different speeds

b. Modulus of Elasticity of Reinforced Concrete. In the design of reinforced-concrete structures subjected only to the action of static loads, the computations mainly determine maximum stresses and deformations appearing under the action of primary loading. In this connection, it is interesting to analyze the behavior of reinforced concrete subjected to primary loading, or loading of the same sign

The official *Technical Rules and Construction Code* gives values of the modulus of elasticity of concrete established as a result of tests performed on concrete samples under increasing loading. These tests established a mean value of the modulus of elasticity of concrete $E_c = 210{,}000$ kg/cm² to be used in the design of reinforced-concrete structures.

When concrete is subjected to primary loading, the relationship between load and deformation is nonlinear; therefore the modulus of elasticity depends on the magnitude of the load and on its sign.

An experimental study of the behavior of reinforced concrete under imposed loads shows that even small stresses result in a simultaneous

appearance of elastic and residual deformations. The relative value of residual deformations grows with increase in load; it grows faster than the load. Therefore a "loading-strain" curve tends to turn in the direction of deformations, and the stress-strain relationship in concrete, above a certain value of stresses, has a nonlinear character; therefore the modulus of elasticity depends on the magnitude of stress.

Consequently, the modulus of elasticity established as a result of investigations in which the irreversible part of deformation was not separated from the total deformation is not the actual modulus of elasticity of the material, just as the coefficient of subgrade reaction of soil is not the coefficient of elastic uniform compression of soil. The modulus of elasticity of concrete, which is usually employed in design computations for stresses smaller than the proportionality limit, represents its modulus of linear deformability. The modulus of elasticity may be established after determining the relationship between stresses and the elastic part of deformation. Corresponding static investigations should be carried out by means of repeated loading and unloading of samples.

The amplitude of vibrations and natural frequencies of reinforced-concrete structures depend on the elastic properties of the material, but not on its characteristics corresponding to residual deformations; therefore the modulus of elasticity of concrete may be determined in the simplest way from natural or forced vibrations.

As a result of measurements of vibrations of one frame foundation under a pump and two frame foundations under turbogenerators, the following values of the modulus of elasticity of reinforced concrete were found:

For the foundation under the pump:

$$3 \times 10^6 \text{ tons/m}^2$$

For the first foundation under the turbogenerator:

$$4.2 \times 10^6 \text{ tons/m}^2$$

For the second foundation under the turbogenerator:

$$5.78 \times 10^6 \text{ tons/m}^2$$

These values of E_c are much higher than those usually used for static computations.

N. P. Pavliuk and O. A. Savinov investigated a two-column frame made of concrete type 160† and found the modulus of elasticity to be some

† See footnote in Art. IV-1-c, p. 132.

4×10^6 tons/m². As a result of the investigation of a four-column frame of concrete of about the same type, it was established that the value of E_c lay within the range 4.62 to 3.5 $\times 10^6$ tons/m².

From results of static investigations of reinforced-concrete beams made of the same concrete, the modulus of elasticity was established to be within the range 3.77 to 3.17 $\times 10^6$ tons/m².

The modulus of elasticity of concrete may be established by the electro-acoustic method, in which a sound generator excites in the sample

TABLE VI-2. RESULTS OF ACOUSTIC DETERMINATION OF YOUNG'S MODULUS OF CONCRETE

Composition of concrete	Age of concrete, days	Young's modulus, tons/m²
1:2.55:2.55	7 28	3.6×10^6 3.81×10^6
1:3.0:3.0	7 28	3.02×10^6 3.81×10^6
1:1.93:3.23	7 28	3.53×10^6 4.11×10^6
1:2.6:4.05	7 28	4.32×10^6 3.96×10^6
1:3.76:3.0	7 28	3.10×10^6 3.67×10^6
1:4.65:6.18	7 28	2.95×10^6 3.31×10^6

under investigation longitudinal or transverse vibrations of varying frequency. In this way the frequency of natural vibrations of the sample is determined, from which the modulus of elasticity can be easily established. In the course of recent years, this method has been widely used for the determination of elastic constants of very different materials.

Table VI-2 presents results of one such determination of the moduli of elasticity of various types of concrete. It is seen from the table that the modulus of elasticity does not change much with changes in the composition of concrete. At the same time, the test results show that the modulus of elasticity increases with an increase in the age of concrete. Absolute values of the modulus of elasticity at an age of 28 days were in no case smaller than 3.0×10^6 tons/m²; the average value established from six determinations was 3.78×10^6 tons/m².

Thus the experimental data show that the actual modulus of elasticity of concrete is much larger than 2×10^6 tons/m²; i.e., it is larger than the value established by the official *Technical Rules and Construction Code.*

The foregoing discussion makes it possible to consider that the actual value of Young's modulus for concrete (at an age not less than 28 to 30 days) is not less than 3×10^6 tons/m². This value of the modulus should be taken for design computations.

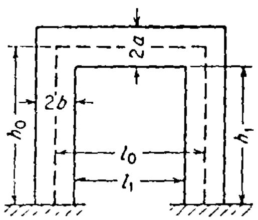


Fig. VI-4. Design values for computations of frame vibration frequencies

c. Design Dimensions of the Upper Part of the Foundation. The cross-sectional dimensions of foundation units are usually much larger than the spans between them; hence the influence of the rigidity of corner sections of the frame should be taken into account. If one is to consider the corner sections as being absolutely flexible, then in the determination of deflections and bending moments of separate frame elements, the span l_0 (Fig. VI-4) and the height h_0 should be introduced into the computations.

However, if one considers the frame corner sections as being absolutely rigid, it becomes necessary to use in computations the inside free span l_1 and the inside free height h_1 of the frame. For usual foundation sizes, the value of l_0 often exceeds the value of l_1 by 25 per cent. Formulas for deflection computations contain the value of the span in the third or fourth power. Therefore design values of the span and height considerably affect the results of computations, in particular the value of the natural vibrations of the frame. For example, if one is to calculate frequencies of natural vibrations of a frame having $l_0 = 5.50$ m and $l_1 = 4.00$ m, then for different design values of the span (from $l = l_0$ to $l = l_1$), the frequency of vibrations computed for $l_0 = 5.50$ m will be (for the case in which the frame is loaded only by its own weight) approximately two times smaller than the frequency for the case where $l = 4.00$ m.

As a matter of fact, the frame corner sections are neither absolutely flexible nor absolutely rigid; therefore design values of the span and height of the frame should be smaller than l_0 and h_0, and larger than l_1 and h_1. They should be determined from the following expressions:

$$l = l_0 - 2\alpha b \qquad h = h_0 - 2\alpha a \qquad \text{(VI-2-4a)}$$

where $2a$ = height of frame beam (Fig. VI-4)
$2b$ = width of column

The value of coefficient α is taken from the graph (Fig. VI-5). When this graph is used, intermediate values of h_0 and l_0 should be determined by interpolation.

If haunches are provided in frame or beam corner sections, the values of a and b are taken as shown in Fig. VI-6.

d. Rigidity of the Upper Platform of the Foundation. The upper foundation platform on which the machine is placed is formed by longitudinal beams which tie the transverse frames together and a reinforced-

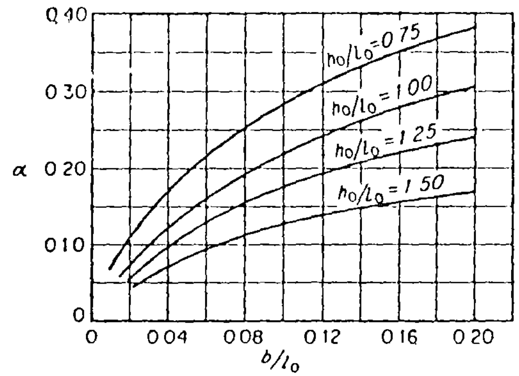


Fig. VI-5. Graph for determination of coefficient α in Eqs. (VI-2-4a)

concrete slab provided with openings required for pipes, machine parts, condensers, and so on. The rigidity of this platform depends essentially on the rigidity and relative position of the bedframe of the machine.

The upper foundation platform together with the bedframe of the machine represents a structure which is extremely rigid in the horizontal direction; therefore, for an approximate determination of the amplitude of horizontal vibrations of the foundation, the upper platform may be considered to be absolutely rigid. This assumption simplifies computations of horizontal vibrations of the frame foundation but does not involve large errors in the results of computations.

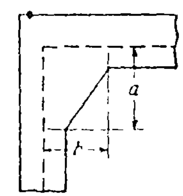


Fig. VI-6. Values of a and b to be used in Eqs. (VI-2-4a) if a frame has haunches

e. Computation of Forced Vertical Vibrations. The foregoing assumptions reduce a dynamic computation of the frame foundation to computations of the amplitudes of forced vibrations of a three-dimensional frame system consisting of thin beams and columns embedded in an absolutely rigid slab. The latter rests on an elastic base, i.e., on soil.

Although it is possible to obtain rigorous solutions to the problem of forced vibrations of such a system,[30] the solutions obtained are so cumbersome and lead to such complicated calculations that they are of little use for practical purposes. Therefore several assumptions are necessary

which will simplify the solution of the problem without influencing its accuracy and will make the solution practical.

When vertical vibrations occur, the most intensive deformations are observed in the beams of the transverse frames. The columns of the frames are deformed less, and the corner haunch sections of the plane transverse frames do not deform at all. If a separate frame vibrates at a frequency not too large in comparison with its smallest natural frequency, then the form of vertical vibrations of this frame approximately corresponds to that shown in Fig. VI-7.

Let us assume that the transverse frames of the foundation are subjected to vertical vibrations in one phase. Let us also assume that differences in vertical deformations of the columns of separate frames are small. Then the elastic resistance of longitudinal beams, developing as a result of their bending, will be small in comparison with the elastic resistance against longitudinal compression of the columns. Longitudinal beams are usually fixed only at the corner sections of the transverse frames. Therefore the vertical vibrations of a transverse frame are also affected by the resistance to torsion of the longitudinal beams. This resistance is also small, as is the resistance resulting from the bending of the longitudinal beams. Therefore the influence of longitudinal beams may be disregarded.

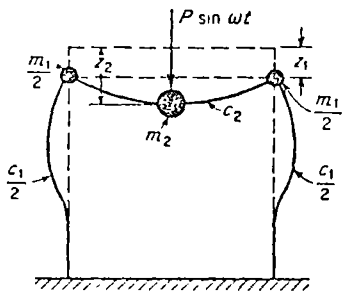


FIG. VI-7. Vibrations of a separate frame.

By neglecting the influence of longitudinal beams on the vertical vibrations of the transverse frames, it is possible to consider the vibrations independently of each other. The natural frequencies of vertical vibrations of separate frames, calculated on the basis of such an assumption, will be somewhat smaller than the actual values. In order to compensate for the influence of the longitudinal beams, we shall disregard the actions of other factors which affect natural frequencies in an opposite manner. These factors include the shearing forces and the inertia of rotation of cross sections of the units of the frame.

A. I. Lur'ye[29] showed that the frequencies of natural vibrations of the frames may be strongly influenced by the elasticity of the base under the foundation. Computations show that the frequencies of natural vertical vibrations, computed with consideration of the elasticity of the base under the foundation, may differ by 10 to 20 per cent from the values computed without taking this factor into account.

The frequency of natural vertical vibrations f_{nz} of the frame considered to be a rigid solid resting on an elastic base is usually much smaller than

the operational frequency of the turbodynamo. We denote by f_{n1} the smallest frequency of natural vertical vibrations of the frame, assuming that the base is absolutely rigid; this frequency depends only on the elastic and inertial properties of the frame. For turbodynamos there usually exists the following interrelationship:

$$f_{nz} < \omega < f_{n1} \qquad \text{(VI-2-5)}$$

where ω is the frequency of rotation of the machine.

For low-speed electrical machines (for example, motor generators) the frequency of rotation may also be smaller than f_{nz}; therefore the following interrelationship may exist:

$$\omega < f_{nz} < f_{n1} \qquad \text{(VI-2-6)}$$

If one is to consider the elasticity of the base under the foundation, then the two smallest frequencies of vertical vibrations of the system "frame–rigid-slab–elastic-base" will have the following interrelationship with the frequencies f_{nz}, ω, and f_{n1}:

$$f_{nz}^{*} < f_{nz} < \omega < f_{n1} < f_{n1}^{*}$$

The latter inequalities show that consideration of the elasticity of soil leads to an increase in the fundamental frequency of natural vibrations based only on the elastic and inertial properties of the frame.

If requirement (VI-2-5) or (VI-2-6) is satisfied in the design of a foundation, then neglecting the elastic properties of soil in the computation of vertical vibrations of the foundation contributes to an increase in the safety factor of the dynamic stability of the foundation. Consequently, the computation of forced vibrations of a frame foundation may be reduced to the computation of vibrations of plane frames resting on an absolutely rigid base. Then the columns of each frame may be considered as being rigidly embedded in an immovable foundation slab.

Let us consider the loads acting on a transverse foundation frame. Each frame usually supports one of the machine bearings. The width of bearings supporting the rotating machine parts usually is small in comparison with the length of the beams. Therefore the load transmitted by the bearings may be considered concentrated and located in the middle of the frame beam. The static load is the part of the rotor weight resting on this bearing. In addition, the same bearing transmits to the frame an exciting vertical force $P \sin \omega t$. The frame beam is also subjected to the action of a uniform load imposed by its own weight. Let us replace it by an equivalent concentrated load located at the center of the frame beam span. In order that this change should not influence the results of dynamic computations, the magnitude of this equivalent load should be

selected so that the kinetic energy of the system will not change. Without furnishing proof here, let us note that the magnitude of an equivalent mass, selected on the basis of the above condition, corresponds to 45 per cent of the dead weight of the frame beam.

The frame columns are subjected to the following loads:

1 The loads imposed by adjacent longitudinal frame beams, including their own weight These loads, falling on each frame, are computed according to the laws of statics Loads imposed by longitudinal beams may be considered concentrated at the tops of the columns, since the longitudinal beams are supported by the corner sections of the transverse frames.

2 The loads imposed by the weight of the transverse beam, also concentrated at the tops of the columns On the basis of the same considerations which governed the selection of the equivalent concentrated mass at the center of the beam span, the dead weight loads imposed by the adjacent transverse beam on each column are taken to equal 25 5 per cent of the weight of this beam.

3 The weight of the column, replaced by an equivalent weight load concentrated at the top of the column. It follows from the theory of longitudinal vibrations of prismatic bars that the value of this load should be equal to 33 per cent of the column weight

As a result of reducing the dead weight loads, we come to the consideration of vibrations of a plane frame whose elements are weightless and whose masses are concentrated in two places (Fig VI-7): one mass m_2 at the center of the frame beam span, and two other masses, each equaling $m_1/2$, at the tops of the columns. Vibrations of all frame units are determined by the vertical displacements z_1 and z_2 of these masses. In this manner, the problem of vertical forced vibrations of transverse frames of the foundation is reduced to the problem of vibrations of a system with two degrees of freedom. It is assumed that the exciting vertical force acts on mass m_2.

Let us denote by c_2 the coefficient of rigidity of the frame beam; this coefficient represents a vertical force which should be applied to the center of the frame beam span in order to cause a deflection of unit length, i.e.,

$$c_2 = \frac{1}{l_2} \qquad \text{(VI-2-7)}$$

The value of l_2 is determined by the formula

$$l_2 = \frac{l^3(1 + 2k)}{96EI_l(2 + k)} + \frac{3l}{8GA_l} \qquad \text{(VI-2-8)}$$

$$k = \frac{hI_l}{lI_h} \qquad \text{(VI-2-9)}$$

where E, G = Young's modulus, modulus of elasticity in shear of material of frame beam

A_l, I_l = cross-sectional area, moment of inertia of frame beam

A_h, I_h = cross-sectional area, moment of inertia of column

Let us denote by $c_1/2$ the coefficient of rigidity of a column, this coefficient represents the vertical force which must be applied to the column in order to cause a unit change in its length; it is evident that

$$\frac{c_1}{2} = \frac{EA_h}{h} \qquad \text{(VI-2-10)}$$

The differential equations of forced vibrations of the system shown in Fig. VI-7 will be exactly the same as Eqs (IV-6-1), and the solutions for the amplitudes are determined by Eqs (IV-6-4) and (IV-6-5)

The differential equations (IV-6-1) do not take into account the influence of damping reactions; so the computations of amplitudes of forced vibrations, presented in Chap IV, will produce adequate results only in cases in which the fundamental frequencies of the system (Fig VI-7) differ by at least ±30 per cent from the frequency of machine rotation If this condition is not satisfied, and forced vibrations of the foundation occur in the resonance zone, then the use of the foregoing equations leads to large errors in the determination of the amplitudes of forced vibrations

The natural frequencies of vertical vibrations of the foundation will be determined as roots of Eq. (IV-6-8) The solution of this equation will provide the two natural frequencies f_{n1} and f_{n2} of the frame under consideration. Let us assume that the frequency of excitation is close to one of these frequencies.

The following relationship usually exists between the limiting frequencies f_l and f_{nz}:

$$f_l < f_{nz}$$

Therefore the frequencies of natural vertical vibrations lie in the range

$$f_{n2} < f_l < f_{nz} < f_{n1}$$

If the frequency of excitation lies close to the lower natural frequency, then the form of the frame vibrations does not differ much from the form of vibrations of frequency f_l, and it can be considered with sufficient accuracy that the system under consideration has one degree of freedom; therefore the amplitude of forced vibrations of the system is determined by the expression

$$A_2^* = \frac{P}{m_2\sqrt{(f_{n2}^2 - \omega^2)^2 + 4c^2\omega^2}} \qquad \text{(VI-2-11)}$$

where c is the damping constant. If the difference $f_{n2}^2 - \omega^2$ is small, then, approximately,

$$A_2^* = \frac{P}{2c\omega m_2} \qquad \text{(VI-2-12)}$$

An approximate value of the damping constant is 5 to 10 per cent of f_{n2}.

If the frequency of excitation lies close to the higher natural frequency f_{n1}, and the form of frame vibrations does not differ much from the form of vibrations of frequency f_{n2}, then A_2 is determined from formula (VI-2-11) or (VI-2-12), in which f_{n2} and m_2 are replaced by f_{n1} and $(m_1 + m_2)$.

f. Computation of Horizontal Transverse Vibrations. In the computation of forced horizontal vibrations of the frame foundation we neglect the elasticity of the upper slab and the soil; i.e., we assume that the slab is absolutely rigid and the frame columns are embedded in an unyielding foundation

Let us consider, for example, a foundation having six columns and three transverse frames. We replace all vibrating masses of the foundation and machine by three equivalent masses m_1, m_2, and m_3, each located at the center of one transverse frame beam span. Each of these masses is computed by adding the following:

1. The mass of the concentrated and distributed deadweight load on the frame beam, including its own weight
2. The mass formed by 30 per cent of the weight of the two columns of the transverse frame
3. The mass of the deadweight transferred by longitudinal frame beams adjacent to the transverse frame under consideration, their own weight included

The amplitudes of forced transverse vibrations depend on the sizes of these masses and on values characterizing the rigidity of transverse frames

Let us replace each transverse frame by an equivalent spring (Fig. VI-8), and the upper slab by a prismatic bar which is assumed to be absolutely rigid. The motion of the system (Fig. VI-8) is determined by x, the lateral displacement of the center of mass of the prismatic bar, and by φ, the angle of its rotation with respect to the center of mass. Consequently the system has two degrees of freedom; i.e., it has two natural frequencies of vibration.

The differential equations describing the transverse horizontal vibra-

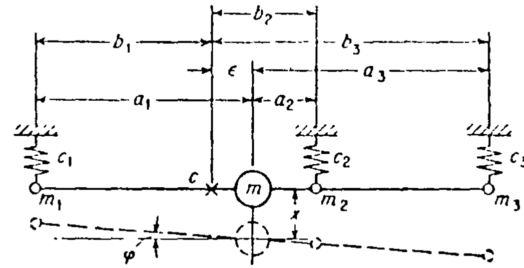


FIG. VI-8. Derivation of Eq (VI-2-13).

tions of the foundation in the case under consideration read as follows:

$$\begin{aligned} m\ddot{x} + \sum_{i=1}^{n} R_i &= P \sin \omega t \\ M_m \ddot{\varphi} + \sum_{i=1}^{n} M_i &= M \sin \omega t \end{aligned} \qquad \text{(VI-2-13)}$$

where $m = \sum_{i=1}^{n} m_i$ = sum of all equivalent masses

$M_m = \sum_{i=1}^{n} m_i a_i^2$ = total moment of inertia of all equivalent masses with respect to common center of mass

a_i = distance between common center of mass and mass m; we consider these distances to be positive in one direction and negative in opposite direction

R_i = elastic force acting on mass m_i during its forward displacement up to value x_i

M_i = moment of inertia of force R_i with respect to an axis passing through center of mass

P, M = exciting force, moment

The summation should be performed for all transverse frames, so that n denotes the number of transverse foundation frames (usually n equals 3 or 4).

When the mass m_i is displaced by the value x_i, then the elastic force of the equivalent springs acting thereon equals

$$R_i = c_i x_i$$

where c_i is the rigidity of the ith spring.

The moment of the elastic force is

$$M_i = c_i a_i x_i$$

Since

$$x_i = x + a_i \varphi$$

we may write:

$$R_i = c_i(x + a_i\varphi)$$

and

$$R_i = (x + \epsilon\varphi)C \qquad \text{(VI-2-14)}$$

where ϵ = distance between center of mass and center of rigidity
$C = \Sigma c_i$ = total rigidity of equivalent springs

To determine the rigidity c_i of the ith transverse frame, let us apply a horizontal force to the center of the frame beam span. This force equals unity and is directed along the axis of the frame beam. It is known that the lateral displacement δ caused by this force equals

$$\delta = \frac{h^3(2 + 3k)}{12EI_h(1 + 6k)} \qquad \text{(VI-2-15)}$$

Here, as before,

$$k = \frac{hI_l}{lI_h}$$

where h = height of column
l = length of frame beam
I_h, I_l = moments of inertia of cross sections of column, frame beam

Having determined the deflection δ for the ith transverse frame, we find the rigidity c_i characterizing the equivalent spring corresponding to this frame:

$$c_i = \frac{1}{\delta} \qquad \text{(VI-2-16)}$$

In the same way we find

$$\Sigma M_i = \Sigma c_i a_i x_i = C(x + \epsilon\varphi)\epsilon + \gamma\varphi \qquad \text{(VI-2-17)}$$

where

$$\gamma = \Sigma c_i b_i^2$$

is the moment of a couple causing the rotation of the prismatic bar through a unit angle.

Substituting the computed values of ΣR_i and ΣM_i into Eqs. (VI-2-13), we obtain two differential equations of forced vibration of the foundation:

$$\begin{aligned} mx + Cx + c\epsilon\varphi &= P \sin \omega t \\ M_m\ddot{\varphi} + C\epsilon x + (C\epsilon^2 + \gamma)\varphi &= M \sin \omega t \end{aligned} \qquad \text{(VI-2-18)}$$

or

$$\begin{aligned} x + f_{nx}^2 x + f_{nx}^2\epsilon\varphi &= p \sin \omega t \\ \ddot{\varphi} + \frac{\epsilon}{r^2} f_{nx}^2 x + \left(\frac{\epsilon^2}{r^2} f_{nx}^2 + f_{n\varphi}^2\right)\varphi &= R \sin \omega t \end{aligned} \qquad \text{(VI-2-19)}$$

The following designations were used in the foregoing formulas:

$$f_{nx}^2 = \frac{C}{m} \qquad \text{(VI-2-20)}$$

where f_{nx} is the limiting frequency of natural lateral vibrations of the foundation when the center of rigidity of the foundation coincides with the center of mass, i.e., when $\epsilon = 0$;

$$f_{n\varphi}^2 = \frac{\gamma}{M_m} \qquad \text{(VI-2-21)}$$

where $f_{n\varphi}$ is the limiting frequency of natural rocking vibrations of the foundation when $\epsilon = 0$;

$$r^2 = \frac{M_m}{m}$$

where r is the radius of gyration; finally

$$p = \frac{P}{m} \qquad R = \frac{M}{M_m}$$

We seek solutions of the system of Eqs. (VI-2-19) corresponding only to the forced vibrations of foundations in the form

$$x = A_x \sin \omega t \qquad \varphi = A_\varphi \sin \omega t$$

Inserting these values of x and φ into Eq. (VI-2-19), we obtain the following two equations containing the amplitudes of forced vibrations A_x and A_φ of the foundation as unknown values:

$$\begin{aligned} (f_{nx}^2 - \omega^2)A_x + f_{nx}^2\epsilon A_\varphi &= p \\ \frac{\epsilon^2}{r^2} f_{nx}^2 A_x + \left(\frac{\epsilon^2}{r^2} f_{nx}^2 + f_{n\varphi}^2 - \omega^2\right) A_\varphi &= R \end{aligned}$$

Solving these equations for A_x and A_φ, we obtain

$$A_x = \frac{[(\epsilon^2/r^2)f_{nx}^2 + f_{n\varphi}^2 - \omega^2]p - f_{nx}^2 R}{\Delta(\omega^2)} \qquad \text{(VI-2-22)}$$

$$A_\varphi = \frac{(\epsilon^2/r^2)f_{nx}^2 p - (f_{nx}^2 - \omega^2)R}{\Delta(\omega^2)} \qquad \text{(VI-2-23)}$$

where

$$\Delta(\omega^2) = \omega^4 - (\alpha f_{nx}^2 + f_{n\varphi}^2)\omega^2 + f_{nx}^2 f_{n\varphi}^2 \qquad \text{(VI-2-24)}$$

$$\alpha = 1 + \frac{\epsilon^2}{r^2}$$

The natural frequencies f_{n1} and f_{n2} of foundation vibrations are determined as roots of the equation

$$\Delta(\omega^2) = 0 \qquad \text{(VI-2-25)}$$

It is clear that the above formulas for the determination of amplitudes of forced vibration of the foundation may be applied only when the frequency of excitement ω differs from the fundamental frequencies of

the foundation. Otherwise, the amplitudes of vibration should be computed analogously to the computation of the amplitudes of vertical vibrations when one of the natural frequencies is close to the engine frequency.

For example, if ω lies close to the lower frequency f_{n2} determined as the root of Eq. (VI-2-25), and if f_{n2} does not differ much from f_{nx}, then the foundation will be subjected predominantly to vibrations accompanied by lateral displacements as a rigid body. The amplitude of vibrations may be approximately established from Eq. (VI-2-11) where m_2 is assumed to equal m. If resonance occurs because the frequency of machine rotation lies close to the second frequency f_{n1} and the latter does not differ much from f_φ, then the foundation will undergo chiefly rocking vibrations.

In this case, an approximate value of the amplitude of vibrations may be found from the same Eq. (VI-2-11), in which P, m_2, and f_{n2} should be replaced respectively by M, M_m, and f_{n1}.

g. Design Values of the Permissible Amplitude of Vibrations. Design values of the permissible amplitude of vibrations of foundations for turbodynamos and electrical machines should be established on the basis of data derived from the study of operating machines. It is hardly possible to establish these limits on the basis of any theoretical premises.

As a matter of fact, if a permissible amplitude of vibrations is established on the basis of permissible stresses for the foundation materials, it is found that the computed amplitudes are tens of times larger than those permissible for normal machine operation. Therefore a selection of the design value of vibration amplitude should be based on the amplitudes accepted by machine operators as permissible for a given machine type. Table VI-3 presents data on the permissible values of amplitudes of vibrations of turbogenerator bearings.

The absolute values of permissible amplitudes of vibrations may be much larger for machines running at 1,500 rpm than for 3,000-rpm machines. If one admits that the permissible amplitude of vibrations of bearings may be taken as the arithmetic mean of the values given in Table VI-3 for machines with certain speeds, then the following values may be accepted:

For 3,000-rpm machines:
Vertical vibrations: 0.02 to 0.03 mm
Horizontal vibrations: 0.04 to 0.05 mm
For 1,500-rpm machines:
Vertical vibrations: 0.04 to 0.06 mm
Horizontal vibrations: 0.07 to 0.09 mm

Vibration investigations of 36 foundations under turbodynamos established that actual vibration amplitudes do not exceed the above per-

missible values. Only in 1 foundation was an amplitude found as high as 0.016 mm. In 4 foundations, the amplitude of vibrations of the upper part of the foundation lay within the range 0.010 to 0.016 mm. In all other foundations the amplitudes of vibrations were within the range

TABLE VI-3. PERMISSIBLE VIBRATION AMPLITUDE VALUES OF TURBOGENERATOR BEARINGS

Type of vibrations	Location of measurements	Evaluation of the engine	Amplitudes† of vibrations, mm, corresponding to speeds, rpm, of:				
			3,000	2,500	2,000	1,500	1,000 and less
Vertical	Extreme bearings	Is fit for operation	0.02	0.03	0.04	0.06	0.08
		No adjustment is needed	0.03	0.05	0.06	0.09	0.11
		An adjustment is desirable	0.04	0.08	0.09	0.13	0.15
	Central bearings	Is fit for operation	0.01	0.02	0.03	0.04	0.05
		No adjustment is needed	0.02	0.03	0.05	0.06	0.08
		An adjustment is desirable	0.03	0.04	0.08	0.09	0.13
Horizontal and transverse	Extreme bearings	Is fit for operation	0.05	0.07	0.08	0.09	0.12
		No adjustment is needed	0.08	0.10	0.11	0.12	0.15
		An adjustment is desirable	0.13	0.14	0.15	0.17	0.20
	Central bearings	Is fit for operation	0.03	0.04	0.05	0.07	0.09
		No adjustment is needed	0.05	0.06	0.08	0.10	0.12
		An adjustment is desirable	0.08	0.09	0.13	0.14	0.17

† The largest permissible values of amplitudes are presented

0.004 to 0.010 mm; i.e., they were considerably smaller than permissible values.

Vibrations with an amplitude of 0.016 mm, had no influence on the normal operation of the turbogenerator.

Permissible values of foundation vibration amplitudes under electric machines having speeds close to 1,500 or 3,000 rpm may be the same as

those for turbogenerators. It is very difficult to select even tentative design values of permissible amplitudes of vibrations for electric machines with speeds lower than 1,500 rpm, since scarcely any data are available on the results of vibration investigations of these machines. For low-frequency electrical machines (less than 500 rpm) such as motor generators and Leonard generators, a design value of permissible vibration amplitudes may be selected on the basis of values established for reciprocating engines (around 0.20 mm).

VI-3. Examples of Dynamic Computations of Foundations of the Frame Type

Example 1. Illustrative design of a frame foundation under a 1,500-kw turbogenerator

1. DATA. The speed of the turbogenerator is $N = 3{,}000$ rpm; i.e., the frequency of excitation is $\omega = 300\ \text{sec}^{-1}$; $\omega^2 = 9 \times 10^4\ \text{sec}^{-2}$.

The foundation to be designed will have six columns and three transverse frames. Table VI-4 gives the initial data required for the dynamic computations of the foundation which are taken from the Design Assignment. Figure VI-9 shows the geometry of the foundation with indication of the loads imposed by the stationary and rotating parts of the machine.

TABLE VI-4. DESIGN DATA FOR COMPUTATIONS OF EXAMPLE VI-3-1 AND FIG. VI-9

Dimensions and design parameters	Frame I	Frame II	Frame III
Height of transverse frames h_0, m	4 30	4.30	4.30
Span of transverse frames l_0, m	3·20	3.20	3 20
Height of cross section of transverse frame $2a$, m	0 95	1 00	1.00
Height of cross section of column $2b$, m	0 80	0 83	0 80
Area of cross section of column A_h, m²	0.76	0.78	0.76
Moment of inertia of cross-sectional area of column I_h, m⁴	0 039	0.041	0 039
Cross-sectional area of frame beam A_l, m²	0 58	0 83	0.64
Moment of inertia of cross-sectional area of beam I_l, m⁴	0 0425	0 069	0 053
Weight of frame beam W_l, tons	4 26	6 10	4.70
Weight of column W_h, tons	6 53	6 67	6 53

The loads imposed by the rotating parts of the turbogenerator act only on the beams of the transverse frames. The loads imposed by the stationary parts (the stator of the generator and the cover of the turbine) are transmitted to the longitudinal beams.

The design value of the modulus of elasticity for the material of the upper part of the foundation (concrete type 110†) we assume to equal $E = 3 \times 10^6$ tons/m².

2. COMPUTATIONS. The computation of forced vertical vibrations of the foundation is begun by some preliminary computations. Table VI-5 gives their results. Table VI-6 presents results of computations of equivalent masses m_1 and m_2 for each of the transverse frames.

† See footnote, Art. IV-1-c, p. 132.

Results of computations of equivalent rigidities are given in Table VI-7, and those of limiting frequencies of each frame in Table VI-8.

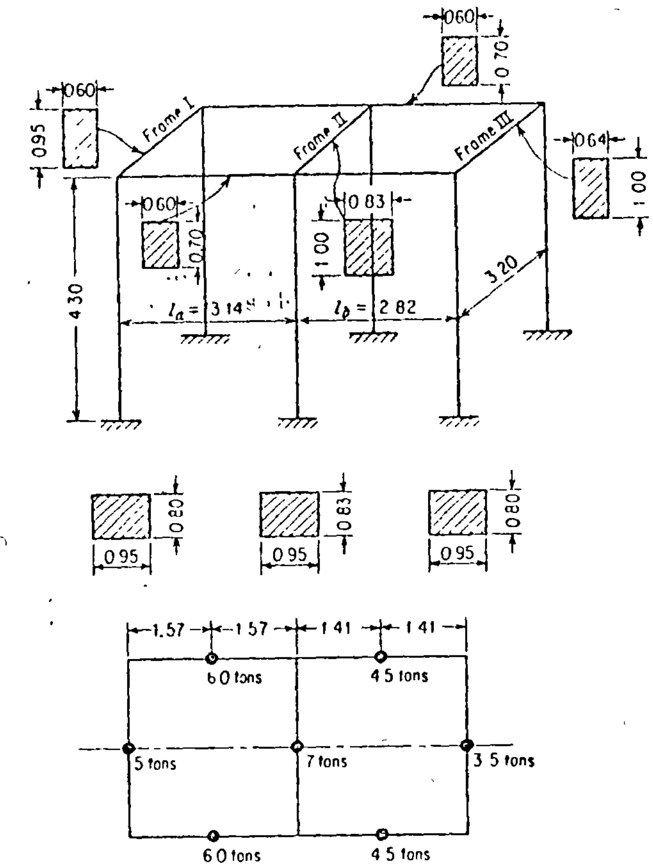


FIG. VI-9. Foundation of Example VI-3-1.

Let us now compute the amplitudes of forced vertical vibrations of the foundation. According to the foregoing data, the design value of the eccentricity of rotating masses of the turbogenerator is taken to equal

$$r_0 = 0.05 \times 10^{-3}\ \text{m}$$

The weights of rotating parts falling on each frame equal

$$W_1 = 1.5\ \text{tons} \qquad W_2 = 2.0\ \text{tons} \qquad W_3 = 1.0\ \text{ton}$$

The magnitude of exciting vertical load acting on each transverse frame of the foundation is:

Frame I: $P_1 = 0.05 \times 10^{-3} \times 1.5/9.81 \times 9 \times 10^4 = 0.69$ ton
Frame II: $P_2 = 0.05 \times 10^{-3} \times 2.0/9.81 \times 9 \times 10^4 = 0.92$ ton
Frame III: $P_3 = 0.05 \times 10^{-3} \times 1.0/9.81 \times 9 \times 10^4 = 0.46$ ton

TABLE VI-5 RESULTS OF PRELIMINARY COMPUTATIONS FOR EXAMPLE VI-3-1, FIG VI-9

Design parameters	Frame I	Frame II	Frame III
$\frac{h_0}{l_0}$	$\frac{4\ 30}{3\ 20} = 1.35$	$\frac{4\ 30}{3\ 20} = 1.35$	$\frac{4\ 30}{3\ 20} = 1.35$
$\frac{b}{l_0}$	$\frac{0.40}{3\ 20} = 0.125$	$\frac{0\ 415}{3\ 20} = 0.130$	$\frac{0\ 40}{3\ 20} = 0.125$
α (from Fig VI-5)	0.17	0 17	0 17
Reduced height h, m [from Eq (VI-2-4a)]	$4\ 30 - 0.17 \times 0.48 = 4.22$	$4\ 30 - 0.17 \times 0\ 50 = 4.22$	$4\ 30 - 0.17 \times 0\ 50 = 4.22$
Reduced length , m [from Eq (VI-2-4a)]	$3.20 - 0.17 \times 0.80 = 3.06$	$3\ 20 - 0.17 \times 0.83 = 3\ 06$	$3.20 - 0.17 \times 0\ 80 = 3.06$
$l_r = \frac{h}{l}$	$\frac{4\ 22}{3\ 06} = 1.39$	$\frac{4.22}{3.06} = 1\ 38$	$\frac{4.22}{3.06} = 1\ 39$
l_r^2	1.94	1.91	1.94
$\frac{I_l}{I_h}$	$\frac{0.0425}{0\ 0390} = 1.09$	$\frac{0.069}{0.041} = 1.62$	$\frac{0.053}{0.039} = 1.36$
$k = \frac{hI_l}{lI_h}$	$1.09 \times 1.39 = 1.51$	$1.62 \times 1.38 = 2.23$	$1.36 \times 1.39 = 1.89$
k^2	2.27	5.00	3.57

We then determine the amplitude of forced vibrations of each frame.

FRAME I. We find the value of coefficient $\Delta(\omega^2)$ from Eq. (IV-6-8):

$$\Delta(\omega^2) = 81 \times 10^8 - (1 + 0.55)(30.2 + 75\ 0)9 \times 10^8 + (1 + 0.55)30.2 \times 75.0 \times 10^8 = 21.2 \times 10^{10}$$

The amplitude of longitudinal vibrations of the column we find from Eq. (IV-6-4):

$$A_1 = \frac{30.2 \times 10^4 \times 0.69}{1.28 \times 21.2 \times 10^{10}} = 0.71 \times 10^{-6}\ \text{m}$$

The amplitude of vertical vibrations of the center of the frame beam span we find from Eq. (IV-6-5):

$$A_2 = \frac{(1 + 0.55) \times 75.0 \times 10^4 + 0.55 \times 30.2 \times 10^4 - 9 \times 10^4}{0.71 \times 21.2 \times 10^{10}} 0.69 = 4.7 \times 10^{-6}\ \text{m}$$

TABLE VI-6 COMPUTATIONS OF EQUIVALENT MASSES FOR EXAMPLE VI-3-1, FIG VI-9

Loads and equivalent masses	Frame I	Frame II	Frame III
Concentrated load imposed on the frame beam, tons	5 0	7 0	3 5
Equivalent load from frame beam weight, $0.45 \times W_l$, tons.	$0\ 45 \times 4\ 26 = 1.91$	$0\ 45 \times 6\ 10 = 2\ 71$	$0\ 15 \times 4\ 70 = 2\ 12$
Equivalent mass m_2 reduced to center of frame beam span, tons $\times$ sec²/m	$\frac{5.0 + 1.91}{9\ 81} = 0\ 71$	$\frac{7.0 + 2\ 74}{9\ 81} = 1\ 00$	$\frac{3\ 5 + 2\ 12}{9\ 81} = 0\ 58$
Equivalent load of columns' weight $0\ 33 \times W_h$, tons	$0\ 33 \times 6\ 53 = 2\ 20$	$0\ 33 \times 6\ 67 = 2\ 25$	$0\ 33 \times 6\ 53 = 2\ 20$
Load imposed by longitudinal beams, tons	$6.0 + 2.4 \times 0.42 \times 3.14 = 9.16$	$6.0 + 4.5 + 2\ 4 \times 0.42 \times (3\ 14 + 2\ 82) = 16.50$	$4\ 5 + 2\ 4 \times 0\ 42 \times 2\ 82 = 7\ 36$
Equivalent load imposed on columns by transverse beams of frames, tons	1 09	1 55	1 20
Equivalent mass m_1 reduced to top of column, tons $\times$ sec²/m	$\frac{2.20 + 9.16 + 1.09}{9\ 81} = 1\ 28$	$\frac{2\ 25 + 16\ 5 + 1\ 55}{9\ 81} = 2.08$	$\frac{2\ 20 + 7\ 36 + 1\ 20}{9\ 81} = 1\ 11$

Thus the amplitude of total vertical vibrations of the frame under consideration equals

$$A_t = A_1 + A_2 = (0\ 71 + 4\ 7)10^{-6}\ \text{m} = 0.005\ \text{mm}$$

FRAME II

$$\Delta(\omega^2) = 81 \times 10^8 - (1 + 0.48)(30\ 8 + 49.8)9 \times 10^8 + (1 + 0\ 48)30\ 8 \times 49\ 8 \times 10^8 = 12\ 8 \times 10^{10}$$

$$A_1 = \frac{30.8 \times 10^4 \times 0.92}{2\ 08 \times 12.8 \times 10^{10}} = 1.0 \times 10^{-6}\ \text{m}$$

$$A_2 = \frac{(1 + 0.48)49.8 \times 10^4 + 0.48 \times 30.8 \times 10^4 - 9 \times 10^4}{\text{[illegible]}\ 1\ 0 = 12.8 \times 10^{10}} 0.92 = 5.7 \times 10^{-6}\ \text{m}$$

$$A_t = A_1 + A_2 = (1.0 + 5.7)10^{-6}\ \text{m} = 0.007\ \text{mm}$$

TABLE VI-7. RESULTS OF COMPUTATIONS OF EQUIVALENT RIGIDITIES FOR EXAMPLE VI-3-1, FIG. VI-9

Parameters and equivalent rigidities	Frame I	Frame II	Frame III
$\frac{l^3(1+2k)}{96EI_l(2+k)}$	$\frac{3.06^3(1+2\times 1.51)}{96\times 3\times 10^6\times 0.0425(2+1.51)} = 2.70\times 10^{-6}$	$\frac{3.06^3(1+2\times 2.23)}{96\times 3\times 10^6\times 0.069(2+2.23)} = 1.88\times 10^{-6}$	$\frac{3.06^3(1+2\times 1.89)}{96\times 3\times 10^6\times 0.053(2+1.89)} = 2.35\times 10^{-6}$
$\frac{3l}{8GA_l}$	$\frac{3\times 3.06}{8\times 10^6\times 0.58} = 1.98\times 10^{-6}$	$\frac{3\times 3.06}{8\times 10^6\times 0.83} = 1.38\times 10^{-6}$	$\frac{3\times 3.06}{8\times 10^6\times 0.64} = 1.80\times 10^{-6}$
Deflection of frame beam at center of span under action of unit force, m. . .	$(2.70+1.98)10^{-6} = 4.68\times 10^{-6}$	$(1.88+1.38)10^{-6} = 3.26\times 10^{-6}$	$(2.35+1.80)10^{-6} = 4.15\times 10^{-6}$
Rigidity c_2 of frame beam, tons/m. .	21.4×10^4	30.8×10^4	24.0×10^4
Rigidity c_1 of column, tons/m $\left(\frac{2EA_h}{h}\right)$	$\frac{2\times 3\times 10^6\times 0.76}{3.06} = 149\times 10^4$	$\frac{2\times 3\times 10^6\times 0.78}{3.06} = 153\times 10^4$	$\frac{2\times 3\times 10^6\times 0.76}{3.06} = 149\times 10^4$

TABLE VI-8. RESULTS OF COMPUTATIONS OF LIMITING FREQUENCIES FOR EXAMPLE VI-3-1 AND FIG. VI-9

Ratios between masses and the limiting frequencies	Frame I	Frame II	Frame III
$\alpha = \frac{m_2}{m_1}$	$\frac{0.71}{1.28} = 0.55$	$\frac{1.00}{2.08} = 0.48$	$\frac{0.58}{1.11} = 0.52$
Square of frequency of natural vibrations of frame beam, considering columns to be absolutely rigid f_l^2, sec^{-2}	$\frac{21.4\times 10^4}{0.71} = 30.2\times 10^4$	$\frac{30.8\times 10^4}{1.00} = 30.8\times 10^4$	$\frac{24.0\times 10^4}{0.58} = 41.5\times 10^4$
Square of frequency of natural vibrations of columns, considering frame beams to be absolutely rigid $f_{n_2}^2$, sec^{-2}	$\frac{149\times 10^4}{1.28+0.71} = 75.0\times 10^4$	$\frac{153\times 10^4}{2.08+1.00} = 49.8\times 10^4$	$\frac{149\times 10^4}{1.11+0.58} = 88.3\times 10^4$

FRAME III

$$\Delta(\omega^2) = 81 \times 10^8 - (1 + 0.48)(41.5 + 88.3)9 \times 10^{10} + (1 + 0.48)41.5 \times 88.3 \times 10^8 = 37.8 \times 10^{10}$$

$$A_1 = \frac{41.5 \times 10^4 \times 0.16}{1.1 \times 37.8 \times 10^{10}} = 0.16 \times 10^{-6}\ \text{m}$$

$$A_2 = \frac{(1 + 0.48)88.3 \times 10^4 \times 0.48 \times 41.5 \times 10^4 - 9 \times 10^4}{0.58 \times 37.8 \times 10^{10}} 0.46$$

$$= 1.6 \times 10^{-6}\ \text{m}$$

$$A_z = A_1 + A_2 = (0.46 + 1.6)10^{-6}\ \text{m} \cong 0.002\ \text{mm}$$

It is clear that the computed amplitudes of vertical vibrations of the foundation are much smaller than the permissible values.

We compute the forced horizontal (transverse) vibrations of the foundation. First we determine the equivalent mass of each frame. As has been indicated, the magnitude of each of these masses is determined by the following:

1 Concentrated and distributed loading imposed on the frame beam, including its own weight

2 30 per cent of the column weights

3. Loads imposed by longitudinal frame beams adjacent to the transverse frame under consideration, the deadweight of longitudinal beams also included

Thus we have, for frame I:

$$m_1 = \frac{5.0 + 4.26 + 0.30 \times 6.53 \times 2 + 9.16}{9.81} = 2.13\ \text{tons} \times \text{sec}^2/\text{m}$$

for frame II:

$$m_2 = \frac{7.00 + 6.10 + 0.30 \times 6.67 \times 2 + 16.50}{9.81} = 3.25\ \text{tons} \times \text{sec}^2/\text{m}$$

and for frame III:

$$m_3 = \frac{3.5 + 4.70 + 0.30 \times 6.53 \times 2 + 3.5}{9.81} = 1.40\ \text{tons} \times \text{sec}^2/\text{m}$$

The total equivalent mass equals

$$m = m_1 + m_2 + m_3 = 2.13 + 3.25 + 1.40 = 6.78\ \text{tons} \times \text{sec}^2/\text{m}$$

We determine the distance from the axis of frame I, along the foundation, to the total equivalent mass:

$$a_1 = \frac{3.14 \times 3.25 + (3.14 + 2.82)1.40}{6.78} = 2.95\ \text{m}$$

The distances from the common center of mass to the axes of frames II and III are:

$$a_2 = 2.95 - 3.14 = -0.19\ \text{m}$$
$$a_3 = 2.95 - 5.96 = -3.01\ \text{m}$$

The moment of inertia of all the equivalent masses with respect to the common center of mass equals

$$M_m = m_1 a_1^2 + m_2 a_2^2 + m_3 a_3^2$$
$$= 2.13 \times 2.95^2 + 3.25 \times 0.19^2 + 1.40 \times 3.01^2$$
$$= 31.2\ \text{tons} \times \text{m} \times \text{sec}^2$$

From Eqs (VI-2-15) and (VI-2-16) we find the equivalent rigidities for frame I, the displacement δ_1 caused by a unit horizontal force directed along the axis of the frame beam is

$$\delta_1 = \frac{4.25^3(2 + 3 \times 1.51)}{12 \times 3 \times 10^6 \times 0.039(1 + 6 \times 1.51)} = 3.44 \times 10^{-3}\ \text{m}$$

The equivalent rigidity equals

$$c_1 = \frac{1}{\delta} = \frac{1}{3.44 \times 10^{-3}} = 0.29 \times 10^3\ \text{tons/m}$$

For frame II

$$\delta_2 = \frac{4.22^3(2 + 3 \times 2.23)}{12 \times 3 \times 10^6 \times 0.041(1 + 6 \times 2.23)} = 3.12 \times 10^{-3}\ \text{m}$$

$$c_2 = \frac{1}{3.12 \times 10^{-3}} = 0.32 \times 10^3\ \text{tons/m}$$

and for frame III.

$$\delta_3 = \frac{4.22^3(2 + 3 \times 1.89)}{12 \times 3 \times 10^6 \times 0.039(1 + 6 \times 1.89)} = 3.85 \times 10^{-3}\ \text{m}$$

$$c_3 = \frac{1}{3.85 \times 10^{-3}} = 0.26 \times 10^3\ \text{tons/m}$$

The total rigidity of all three frames equals

$$C = (0.29 + 0.32 + 0.26)10^3 = 0.87 \times 10^3\ \text{tons/m}$$

We determine the distance to the center of rigidity, along the foundation, from the axis of frame I:

$$b_1 = \frac{3.14 \times 0.32 \times 10^3 + (3.14 + 2.82)0.26 \times 10^3}{0.87 \times 10^3} = 2.95\ \text{m}$$

The distance between the center of mass and the center of rigidity of the frame is

$$\epsilon = 2.95 - 2.95 = 0$$

Consequently, the center of inertia and the center of rigidity coincide For this particular case, Eqs. (VI-2-12) and (VI-2-13) for determining the amplitudes of vibration are simplified.

The amplitude of lateral horizontal (transverse) vibrations of the foundation equals

$$A_x = \frac{P}{m(f_{nx}^2 - \omega^2)} \qquad \text{(VI-2-26)}$$

The amplitude of rocking vibrations around the common center of gravity in the plane of the upper platform is

$$A_\varphi = \frac{M}{M_m(f_{n\varphi} - \omega^2)} \qquad \text{(VI-2-27)}$$

We determine the limiting natural frequencies of the foundation. From Eq (VI-2-20) we have:

$$f_{nx}^2 = \frac{0.87 \times 10^3}{6.78} = 0.128 \times 10^3\ \text{sec}^{-2}$$

The total exciting force equals

$$P = 0.69 + 0.92 + 0.46 = 2.07\ \text{tons}$$

The amplitude of transverse vibrations of the foundation is

$$A_z = \frac{2.07}{6.78(0.128 \times 10^3 - 9 \times 10^3)} = 0.034 \text{ mm}$$

To determine the rigidity of the foundation against torsion, we have

$$b_1 = 2.95 \text{ m} \qquad b_2 = a_2 = -0.19 \text{ m} \qquad b_3 = a_3 = -3.01 \text{ m}$$

It follows that

$$\begin{aligned} \gamma &= c_1 b_1^2 + c_2 b_2^2 + c_3 b_3^2 \\ &= (0.29 \times 2.95^2 + 0.32 \times 19^2 + 0.26 \times 3.01^2)10^3 \\ &= 4.89 \times 10^3 \text{ tons} \times \text{m} \end{aligned}$$

From Eq. (VI-2-21) we compute the frequency of natural rocking vibrations of the foundation.

$$f_{n\varphi}^2 = \frac{\gamma}{M_m} = \frac{4.89 \times 10^3}{31.2} = 0.157 \times 10^3 \text{ sec}^{-2}$$

Let us assume that exciting forces in the generator and the turbine act in the same direction at each instant of time. Then the magnitude of the exciting moment will equal

$$\begin{aligned} M &= P_1 a_1 + P_2 a_2 + P_3 a_3 \\ &= 0.69 \times 2.95 - 0.92 \times 0.19 - 0.46 \times 3.01 \\ &= 0.48 \text{ tons} \times \text{m} \end{aligned}$$

The amplitude of rocking vibrations determined from Eq. (VI-2-27) is

$$A_\varphi = \frac{0.48}{31.2(0.157 - 9.0)10^3} = 0.175 \times 10^{-5} \text{ radians}$$

The largest horizontal displacement as a result of rocking vibrations is

$$a_3 A_\varphi = 3.01 \times 0.175 \times 10^{-5} = 0.005 \times 10^{-3} \text{ m} = 0.005 \text{ mm}$$

Thus the total maximum amplitude of horizontal displacement of the foundation in the direction perpendicular to the axis of the main shaft of the turbine equals

$$a_x = A_x + a_3 A_\varphi = 0.034 + 0.005 = 0.039 \text{ mm}$$

The latter value lies within the range of permissible design values.

Example 2. Illustrative design of a foundation for a 500-kw generator

1. DATA. The motor generator runs at 750 rpm; consequently the frequency of forced vibrations will be

$$\omega = 75 \text{ sec}^{-1} \qquad \omega^2 = 5.6 \times 10^3 \text{ sec}^{-2}$$

The foundation is designed to consist of six columns and three transverse frames. Figure VI-10 gives a diagram of the foundation and the loads acting thereon. The weight of the rotating part of the motor generator is 5.9 tons; its total weight is 13.2 tons.

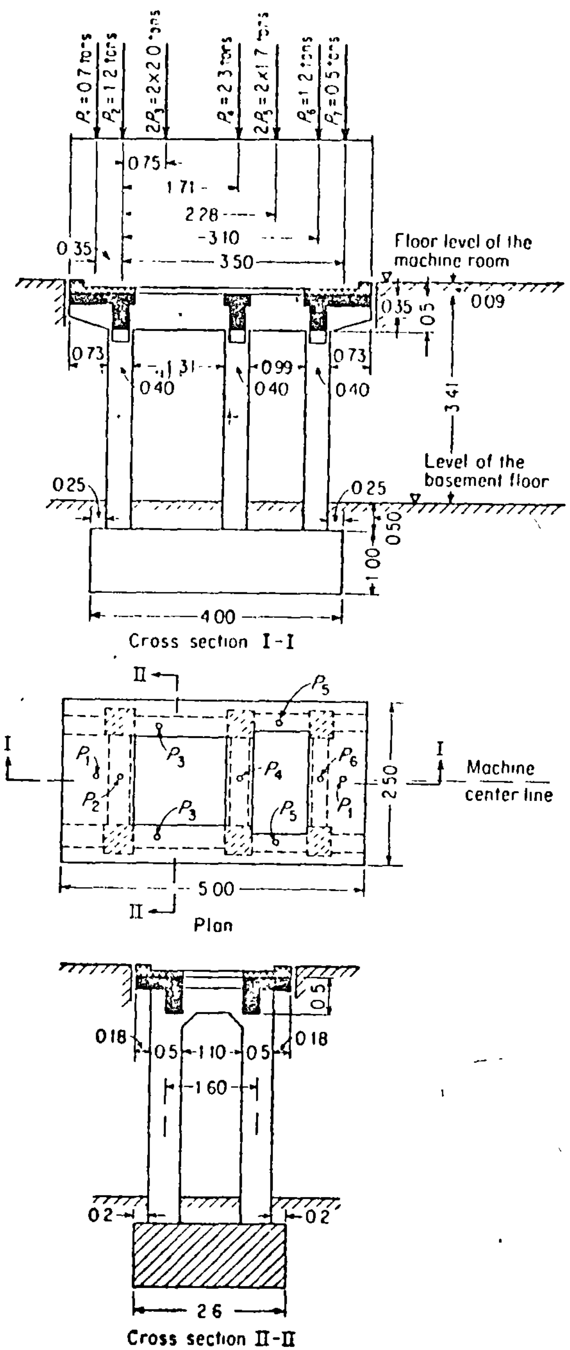


FIG. VI-10. Foundation of Example VI-3-2.

Table VI-9 gives design dimensions of transverse frames of the foundation determined in accordance with Art. VI-2. The same symbols are used here as in the preceding example.

TABLE VI-9. DESIGN DIMENSIONS FOR EXAMPLE VI-3-2 AND FIG. VI-10

Frame	h, m	l, m	I_h, m⁴	I_l, m⁴	k
I	3.6	1.5	4.2×10^{-3}	9.4×10^{-3}	5.4
II	3.6	1.5	4.2×10^{-3}	6.1×10^{-3}	3.5
III	3.6	1.5	4.2×10^{-3}	9.4×10^{-3}	5.4

2. COMPUTATIONS. Foundations under motor generators vibrate mostly in a transverse direction. Therefore the dynamic computations of the foundation may be limited to determining the amplitudes of forced horizontal vibrations.

TABLE VI-10. MAIN LOADS ACTING ON FOUNDATION OF EXAMPLE VI-3-2 AND FIG. VI-10

Loads, tons	Frame I	Frame II	Frame III
Concentrated load in center of frame beam span	1.9	2.3	1.7
Weight of frame beam	1.2	0.7	1.2
Loads imposed by longitudinal beams adjacent to transverse frame	2.6	4.5	3.6
Weight of columns	1.7	1.7	1.7

Table VI-10 gives the magnitudes of the mass loads (in tons) acting on the foundation.

The equivalent masses are: for frame I:

$$m_1 = \frac{1.9 + 1.2 + 0.30 \times 1.7 \times 2 + 2.6}{9.81} = 0.66 \text{ ton} \times \text{sec}^2/\text{m}$$

for frame II:

$$m_2 = \frac{2.3 + 0.70 + 0.30 \times 1.7 \times 2 + 4.5}{9.81} = 0.88 \text{ ton} \times \text{sec}^2/\text{m}$$

and for frame III:

$$m_3 = \frac{1.7 + 1.2 + 0.30 \times 1.7 \times 2 + 3.6}{9.81} = 0.76 \text{ ton} \times \text{sec}^2/\text{m}$$

The total equivalent mass is

$$m = 0.66 + 0.88 + 0.76 = 2.30 \text{ tons} \times \text{sec}^2/\text{m}$$

We now determine the distance to the total equivalent mass, along the foundation, from the axis of frame I:

$$a_1 = \frac{1.71 \times 0.88 + 3.10 \times 0.76}{2.30} = 1.67 \text{ m}$$

The distances from the common center of mass to the axes of frames II and III are

$$a_2 = 1.67 - 1.71 = -0.04 \text{ m}$$
$$a_3 = 1.67 - 3.50 = -1.83 \text{ m}$$

The moment of inertia of all equivalent masses with respect to the common center of mass equals

$$M_m = m_1 a_1^2 + m_2 a_2^2 + m_3 a_3^2$$
$$= 0.66 \times 1.67^2 + 0.88 \times 0.04^2 + 0.76 \times 1.83^2$$
$$= 4.3 \text{ tons} \times \text{m} \times \text{sec}^2$$

We compute the equivalent rigidities of each frame, for frame I:

$$\delta_1 = \frac{3.6^3(2 + 3 \times 5.4)}{12 \times 3 \times 10^6 \times 4.2 \times 10^{-3}(1 + 6 \times 5.4)} = 1.68 \times 10^{-4} \text{ m}$$

$$c_1 = \frac{1}{1.68 \times 10^{-4}} = 0.59 \times 104 \text{ tons/m}$$

for frame II:

$$\delta_1 = \frac{3.6^3(2 + 3 \times 3.5)}{12 \times 3 \times 10^6 \times 4.2 \times 10^{-3}(1 + 6 \times 3.5)} = 1.74 \times 10^{-4} \text{ m}$$

$$c_2 = \frac{1}{1.74 \times 10^{-4}} = 0.57 \times 10^{-4} \text{ tons/m}$$

and for frame III

$$\delta_1 = \frac{3.6^3(2 + 3 \times 5.4)}{12 \times 3 \times 10^6 \times 4.2 + 10^{-3}(1 + 6 \times 5.4)} = 1.68 \times 10^{-4} \text{ m}$$

$$c_3 = \frac{1}{1.68 \times 10^{-4}} = 0.59 \times 10^4 \text{ tons/m}$$

The total rigidity of all frames is

$$C = (0.59 + 0.57 + 0.59)10^4 = 1.75 \times 10^4 \text{ tons/m}$$

We determine the distance to the center of rigidity, along the foundation, from the axis of frame I:

$$b_1 = \frac{(1.71 \times 0.57 + 3.10 \times 0.59)10^4}{1.75 \times 10^4} = 1.61 \text{ m}$$

The distance between the centers of mass and rigidity of the foundation is

$$\epsilon = 1.67 - 1.61 = 0.05 \text{ m}$$

To determine the rigidity of the foundation against torsion, we have:

$$b_1 = 1.61 \text{ m}$$
$$b_2 = a_2 + \epsilon = -0.04 + 0.06 = 0.02 \text{ m}$$
$$b_3 = a_3 + \epsilon = -1.83 + 0.06 = -1.77 \text{ m}$$

The rigidity of the foundation against torsion is then

$$\gamma = (0.59 \times 1.61^2 + 0.57 \times 0.02^2 + 0.59 \times 1.77^2)10^4$$
$$= 3.65 \times 10^4 \text{ tons} \times \text{m}$$

From Eqs. (VI-2-20) and (VI-2-21) we compute the limiting frequencies of natural vibrations of the foundation:

$$f_{nx}^2 = \frac{1.75 \times 10^4}{2.30} = 7.6 \times 10^3 \text{ sec}^{-2}$$

$$f_{n\varphi}^2 = \frac{3.65 \times 10^4}{4.30} = 8.5 \times 10^3 \text{ sec}^{-2}$$

The square of the radius of gyration equals

$$r^2 = \frac{M_m}{m} = 4.30/2.30 = 1.86$$

We find the value

$$\alpha = 1 + \frac{2}{r^2} = 1 + 0.06^2/1.86^2 \approx 1$$

In the case under consideration, α does not differ much from unity; therefore the natural frequencies f_{n1} and f_{n2} of the foundation, determined as roots of Eq. (VI-2-35), differ very little from the limiting frequencies f_{nx} and $f_{n\varphi}$.

According to Eq (VI-2-4) we determine the design value of the radius of unbalance (eccentricity) of the rotating masses of the motor generator:

$$r_0 = 500/750^2 = 0.9 \times 10^{-3}\ \text{m}$$

The exciting force imposed by all rotating masses of the motor generator equals

$$P = 0.9 \times 10^{-3} \times 5.9/9.81 \times 9 \times 10^3 = 4.8\ \text{tons}$$

Assuming, as in the preceding example, that the unbalanced state refers only to the static loads, we obtain for the exciting moment

$$M = 1.9 \times 1.67 - 2.3 \times 0.04 - 1.7 \times 1.83 = 0.06\ \text{ton} \times \text{m}$$

The influence of this moment on the amplitudes of forced vibrations of the foundation may be neglected.

The amplitude of horizontal displacement is obtained from Eq. (VI-2-26).

$$A_x = \frac{4.8}{2.3(7.6 - 6.1)10^3} = 1.4 \times 10^{-3}\ \text{m} = 1.4\ \text{mm}$$

The foregoing computations show that the value of one of the natural vibration frequencies is close to the operational frequency of the engine; it follows that the amplitudes of vibrations considerably exceed permissible values. The foundation should be brought out of the zone of resonance; this may be done by increasing the cross sections of the columns to 0.6 by 0.7 m instead of the dimensions 0.4 by 0.5 m accepted in the design.

VII

MASSIVE FOUNDATIONS

VII-1. Massive Foundations for Motor Generators

General Remarks in Regard to Design and Design Computations Motor generators usually operate at much lower speeds than turbogenerators. Therefore according to Eq. (VI-2-4) the eccentricity of rotating masses in motor generators and analogous electric machines is considerably larger than in turbogenerators. This is indirectly confirmed by the results of measurements of foundation vibrations.

If the eccentricities in motor generators were indeed of the same order as in turbogenerators, the exciting forces imposed by motor generators would be so small that they could not induce appreciable vibrations, even under conditions of resonance, when the frequency of vibrations approaches one of the natural frequencies of the foundation. Then no dynamic computations of foundation vibrations under motor generators would be needed. Observations show, however, that the foundations under low-speed motor generators (up to 300 to 400 rpm) often undergo vibrations with amplitudes of the order of 0.1 to 0.3 mm. Foundations weighing several hundred tons may undergo forced vibrations with such amplitudes only when exciting loads are large. If one is to take the eccentricity of rotating masses in the motor generator as having the same value as in turbogenerators (0.2 mm), then for a 75-ton weight of rotating masses of a motor generator with a speed of 300 rpm, the value of exciting forces generated will equal only 1.5 tons. Such an exciting load, even under conditions of resonance, cannot induce vibrations with an amplitude of the order of 0.1 to 0.3 mm in a foundation weighing several hundred tons.

Thus the results of instrumental measurements of vibrations of low-frequency motor generators provide a basis for the assumption that in these machines the eccentricity of rotating masses is much larger than in turbogenerators.

Stress analysis of a massive foundation is not required because of the small magnitude of stresses imposed by static and dynamic external loads. In addition to the computation of amplitudes of transverse vibrations, it is necessary to center the foundation and check the magnitude of the pressure imposed on the soil by its weight. The permissible pressure on the soil may be taken to equal the permissible pressure under conditions of static loading only.

The foundation is built of concrete; the upper part is made of concrete type 100,† and the lower slab may be made of concrete type 60† or cyclopean concrete. In portions weakened by openings and grooves, the foundation is reinforced to fit the field conditions; approximately 20 to 30 kg of steel are used for 1 m³.

Example. Dynamic computation

1. DATA A dynamic computation of the massive foundation under a 3,000-kw motor generator running at 300 rpm is to be performed. Figure VII-1 shows the main dimensions of the foundation selected on the basis of construction assignments from the engine manufacturer and the client who ordered the design The static loads and points of application are also shown in the figure.

Geologic conditions are as follows: loessial clay with some sand extends to a depth of 28 m, its moisture content is about 9 to 10 per cent; it is underlaid by dense brown clays. The ground-water level is at a depth of 14 m. The following coefficients of elasticity of the soil have been established from investigations of its elastic properties: Coefficient of elastic uniform compression:

$$c_u = 5 \times 10^3 \text{ tons/m}^3$$

Coefficient of elastic nonuniform compression:

$$c_\varphi = 10 \times 10^3 \text{ tons/m}^3$$

Coefficient of elastic uniform shear:

$$c_\tau = 3.5 \times 10^3 \text{ tons/m}^3$$

The foundation is to be erected in a machine room with several operating motor generators. The width of the building does not permit increasing the width of the foundation beyond that shown in Fig. VII-1. The distance between footings under columns and motor generators is 25 cm.

The following values necessary for dynamic computations were established from calculations:

Weight of the foundation (taking into account all cavities) and engine:

$$W = 1{,}136 \text{ tons}$$

Mass of the foundation and engine:

$$m = 115.7 \text{ tons} \times \text{sec}^2/\text{m}$$

† See footnote, Art. IV-1-c, p. 132.

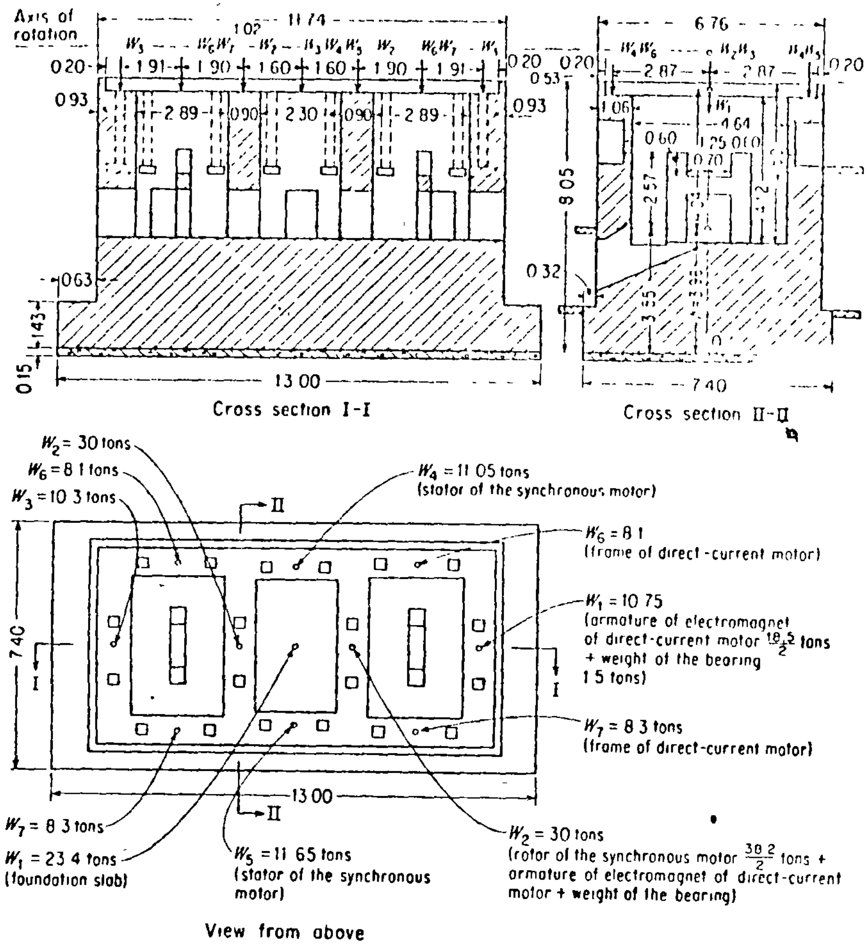


FIG VII-1. Massive foundation for 3,000-kw 300-rpm motor generator of example of Art. VII-1.

Foundation area in contact with soil:

$$A = 96.0 \text{ m}^2$$

Distance between the level of the foundation contact area with soil and the common center of mass:

$$h = 4.6 \text{ m}$$

Moment of inertia of the foundation contact area with respect to the longitudinal axis passing through the centroid of the contact area:

$$I = 440 \text{ m}^4$$

Moment of inertia of the mass of the foundation and engine with respect to the same axis:

$$I_m = 3{,}974.6 \text{ tons} \times \text{m} \times \text{sec}^2$$

Moment of inertia of the mass of the foundation and engine with respect to the axis which passes through the center of gravity and is perpendicular to the plane of vibrations:

$$I_0 = 1{,}510.6 \text{ tons} \times \text{m} \times \text{sec}^2$$

Ratio between the moments of inertia of masses:

$$\gamma = 0.38$$

2. Computations. Using these data, we begin by establishing the frequency of natural vibrations of the foundation. The frequency of natural vertical vibrations (from Eq. III-1-5) is

$$f_{nz}^2 = \frac{5 \times 10^3 \times 96.0}{115.7} = 4.4 \times 10^3 \text{ sec}^{-2}$$
$$f_{nz} = 64.3 \text{ sec}^{-1}$$

The number of natural vertical vibrations of the foundation is

$$N_z = 9.55 f_{nz} = 9.55 \times 64.3 = 614 \text{ min}^{-1}$$

The difference between the numbers of natural vibrations and forced vibrations equals per cent

$$\eta_z = \frac{614 - 300}{300} 100 = 105 \text{ per cent}$$

Hence, the design of the foundation is satisfactory in regard to vertical vibrations.

In order to determine the frequencies of natural vibrations of the foundation f_{n1} and f_{n2} in a transverse plane, the limiting frequencies $f_{n\varphi}$ and f_{nx} of the foundation should first be established. From Eqs. (III-2-6) and (III-3-2), we have

$$f_{n\varphi}^2 = \frac{10 \times 10^3 \times 440}{3974.6} = 1.11 \times 10^3 \text{ sec}^{-2}$$
$$f_{nx}^2 = \frac{3.5 \times 10^3 \times 96}{115.7} = 2.91 \times 10^3 \text{ sec}^{-2}$$

Then we obtain

$$\frac{f_{n\varphi}^2 + f_{nx}^2}{2\gamma} = \frac{1.11 \times 10^3 + 2.91 \times 10^3}{2 \times 0.38} = 5.3 \times 10^3$$
$$\frac{f_{n\varphi}^2 f_{nx}^2}{\gamma} = \frac{1.11 \times 10^3 \times 2.91 \times 10^3}{0.38} = 8.5 \times 10^6$$

According to Eq. (III-4-8),

$$f_{n1,2}^2 = 5.3 \times 10^3 \pm \sqrt{28.0 \times 10^6 - 8.5 \times 10^6}$$
$$= (5.3 \pm 4.43)10^3 \text{ sec}^{-2}$$

Hence

$$f_{n2}^2 = 0.87 \times 10^3 \text{ sec}^{-2}$$
$$f_{n2} = 29.5 \text{ sec}^{-1}$$

Thus the minimum number of natural vibrations of the foundation is

$$N_2 = 9.55 \times 29.5 = 282 \text{ min}^{-1}$$

This differs from the operational speed of the engine by only

$$\eta_2 = \frac{282 - 300}{300} 100 = -3 \text{ per cent}$$

Therefore it can be assumed that if the rotating masses of the motor generator are only slightly out of balance, considerable transverse vibrations of the foundation may develop. It follows that the design of the foundation is not satisfactory in regard to these vibrations

3. Modification. The dimensions of the machine and the building do not permit any considerable changes in foundation width Changes in the depth of the foundation or an increase in the foundation length has very little influence on the frequencies of natural vibrations of the foundation; therefore the only way to increase their value is to increase artificially the rigidity of the base under the foundation. In the case under consideration, the best way to achieve an increase in the rigidity of the base would be the provision of short conical precast reinforced-concrete piles However, the use of these piles would require a comparatively long time for their casting and curing. In addition, the driving of piles inside a building with machines in operation would cause considerable inconvenience

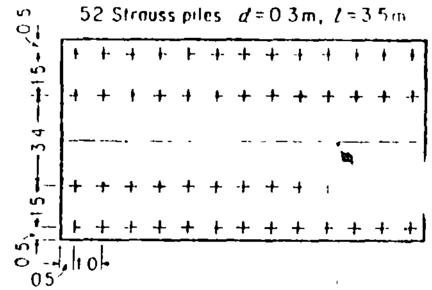


Fig VII-2 Distribution of situ-cast concrete bore piles under the foundation of Fig VII-1

Therefore the decision was taken to increase the rigidity of the base by the installation of 52 situ-cast bore piles system Strauss, each 3.5 m long Figure VII-2 shows the distribution of these piles in plan

The coefficients of elasticity of such a pile base are about three times larger than those of the natural base under the foundation One can take for the pile base

$$c_\varphi = 30 \times 10^3 \text{ tons/m}^3$$
$$c_x = 10.5 \times 10^3 \text{ tons/m}^3$$

Let us compute the forced vibrations of the foundation in a transverse plane when piles are used:

$$f_\varphi^2 = 3 \times 1.11 \times 10^3 = 3.3 \times 10^3 \text{ sec}^{-2}$$
$$f_x^2 = 3 \times 2.91 \times 10^3 = 8.72 \times 10^3 \text{ sec}^{-2}$$

From Eq. (III-4-8),

$$f_{1,2}^2 = \left(\frac{3.3 + 8.72}{2 \times 0.38} \pm \sqrt{\frac{3.3 + 8.72^2}{2 \times 0.38} - \frac{3.3 \times 8.72}{0.38}}\right) 10^3$$
$$= (15.7 \pm 12.3)10^3$$

Hence we have

$$f_1^2 = 29.0 \times 10^3 \text{ sec}^{-2}$$
$$f_2^2 = 3.4 \times 10^3 \text{ sec}^{-2}$$
$$f_2 = 58 \text{ sec}^{-1}$$

We find the multiplier,

$$\Delta(\omega^2) = mI_0(f_1^2 - \omega^2)(f_2^2 - \omega^2)$$
$$= 115.7 \times 1510.6(29 - 0.9)(3.4 - 0.9)10^6 = 12.3 \times 10^{12}$$

According to Eqs. (III-4-1) and (III-4-12), the amplitudes of forced vibrations of the foundation will equal

$$A_x = \frac{30 \times 10^3 \times 440 - 1136 \times 4.6 + 10.5 \times 10^3 \times 96 \times (4.6)^2 - 1510.6 \times 0.9 \times 10^3}{12.3 \times 10^{12}} P$$
$$+ \frac{10.5 \times 10^3 \times 96 \times 4.6}{12.3 \times 10^{12}} M_s = (2.7P + 0.37M_s)10^{-6} \text{ m}$$
$$A_\varphi = \frac{10.5 \times 10^3 \times 96 \times 4.6}{12.3 \times 10^{12}} P + \frac{10.5 \times 10^3 \times 96 - 115.7 \times 0.9 \times 10^3}{12.3 \times 10^{12}} M_s$$
$$= (1.71P + 0.00M_s)10^{-6}$$

Thus the total amplitude of horizontal vibrations of the upper part of the foundation equals

$$A = A_x + h_1 A_\varphi$$

where $h_1 = 3.45$ m is the distance from the common center of the engine and foundation to the top of the foundation. Inserting the numerical values of A_x and A_φ, we obtain

$$A = (2.7P + 6.27M_s)10^{-6} \text{ m}$$

The exciting moment equals

$$M_s = PH$$

where $H = 5.2$ m is the distance from the axis of the machine shaft to the common center of gravity, thus

$$A = 27.7 \times 10^6 P$$

According to the data from the plant, the design value of the rotor weight is around 60 tons, an approximate value of the eccentricity r_0 of rotating masses may be taken as ten times that for turbogenerators with speeds of 1,500 rpm; i.e., $r_0 = 2$ mm. Then the design value of the exciting force equals

$$P = 2 \times 10^{-3} \times 60/9.81 \times 0.9 \times 10^3 = 10.8 \text{ tons}$$

Inserting this value of P into the expression for the amplitude, we obtain

$$A = 27.7 \times 10^{-6} \times 10^6 = 0.3 \times 10^{-3} \text{ m} = 0.3 \text{ mm}$$

For low-frequency motor generators, the permissible design value of amplitude of vibrations may be taken to equal 0.30 mm. It follows that the foundation under consideration satisfies the conditions of dynamic stability.

VII-2. Massive Foundations under Turbodynamos

Basically, massive foundations under turbodynamos are blocks with cavities and grooves for individual parts of the machine or for mounting auxiliary equipment. Such a foundation consists of an upper part designed as a very rigid box or as two walls with grooves and openings, and a lower foundation slab transmitting the load to the soil. Special design features of massive foundations for turbodynamos are seen in Fig. VII-3a and b, which shows a general view and a longitudinal section of a foundation for a 1,200-kw turbodynamo.

The upper part of this foundation consists of a complicated combination of individual structural units: girders, columns, walls, slabs, and others. Dynamic and static computations of this part of a foundation therefore involve a high degree of approximation. The dimensions of the upper part of the foundation and its individual units are usually determined by the construction assignment prepared by the machine manufacturer. Thus a designer's task is limited to determining the dimensions of the lower foundation slab and designing the reinforcement

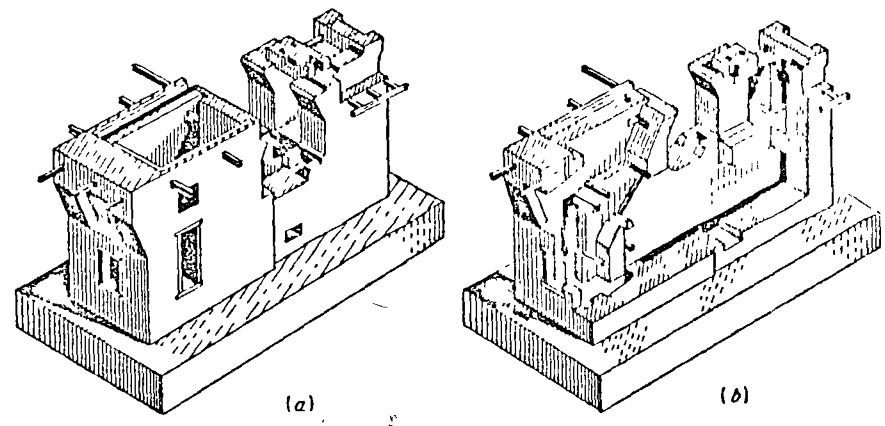


FIG. VII-3 Foundation for a 1,200-kw turbodynamo. (a) general view, (b) longitudinal section.

for the foundation. Essential points of instruction for the construction of frame foundations given in Art. VI-1 should be followed in the design of massive foundations for turbodynamos. Concrete type 150† is used for the upper part of a massive foundation, and concrete type 100† for the lower foundation slab.

All structural units of the upper part of the foundation are designed so that their numbers of natural vibrations should not be smaller than 3,000 min-1.

Unbalanced inertial forces of turbodynamos may induce vibrations of the foundation as a rigid body on an elastic base, as well as vibrations of the separate structural elements constituting the foundation.

Experience in operating high-frequency turbodynamos has not revealed any cases of significant vibrations of massive foundations acting as rigid bodies on elastic bases for the following reasons:

† See footnote, Art. IV-1-c, p. 132.

The natural frequencies of foundation vibrations are usually smaller than the operational frequencies of high-frequency turbodynamos, and it is hardly possible that these two frequencies will coincide. High-frequency turbodynamos are well balanced in regard to both static and dynamic loads. Their actual eccentricities (Art. VI-2) do not exceed 2 mm. Therefore the exciting loads inducing foundation vibrations are relatively small.

Even under the most unfavorable conditions, the exciting loads cannot induce vibrations with impermissible amplitudes, because the foundation mass is large in comparison with the mass of rotating machine parts. In the case of high-frequency vibrations, there is considerable influence of damping forces. In order to approximately evaluate this influence, let us compute the amplitude of foundation vibrations under the most unfavorable conditions—at resonance (i.e., when $\omega = f_{nz}$). The amplitude of vertical vibrations of the foundation as a rigid body on an elastic base can then be established from Eq. (III-1-21):

$$A_z = \frac{P_z}{2mc\omega}$$

where m = mass of foundation and machine

c = damping constant whose value may be taken as proportional to frequency of vibrations, i.e., $c = \eta\omega$

The maximum value of the vertical component of the exciting force equals

$$P_z = r_0 m_0 \omega^2$$

where r_0 = eccentricity

m_0 = mass of rotating parts of machine

Inserting expressions for P_z and c in the formulas for A_z, we obtain the following expression for the amplitude of vertical vibrations of the foundation at resonance:

$$A_z = \frac{1}{2}\frac{r_0\mu}{\eta}$$

where $\mu = m_0/m$ is the ratio between the rotating machine masses and the total mass of the foundation and machine.

For turbodynamos μ equals approximately 0.05; the value of the coefficient of proportionality η may be taken as 0.5. For these values of μ and η, we obtain

$$A_z = 5 \times 10^{-2} r_0$$

Even for r_0 = 0.2 mm, the maximum amplitude of vertical vibrations of the foundation to be expected under conditions of resonance will be on the order of only 0.010 mm. Vibrations of such an amplitude are not dangerous. Actual amplitudes of foundation vibrations are much smaller, because it is hardly likely that the frequencies of natural vertical vibrations and the frequencies of the machine will coincide. Therefore in design computations of massive foundations under turbodynamos with speeds greater than 1,000 rpm dynamic computation of the foundation as a rigid body on an elastic base is not required.

In order to prevent the vibration of individual units constituting the foundation, it is advisable to check them as to danger of resonance, then design them so that their natural frequencies will be larger than the operational frequencies of the machine.

Investigations of resonance should be performed on transverse girders of the foundation which support the machine bearings, because it is these girders which carry the dynamic loads imposed by the machine. In the computation of frequencies of natural vibrations of these elements, formulas for single-degree-of-freedom systems may be used in ... with a sufficient degree of approximation. When computing deflections one should consider only the dead loads carried directly by the element studied.

High frequencies are characterized by large damping forces developing as a result of vibrations of individual units. Therefore, for foundations under turbodynamos with speeds greater than 3,000 rpm, there is no necessity to check individual units as to danger of resonance. It suffices to perform a static computation of the foundation elements directly supporting the loads. The same computations should be made also for machines having speeds below 3,000 rpm.

Dynamic stresses in bases under foundations are very small because the amplitudes of vibrations of massive foundations under turbodynamos are very small. Therefore the permissible bearing value of soils under foundations for high-frequency engines may be taken to equal about 80 to 100 per cent of the permissible bearing value for static load only.

VII-3. Foundations for Rolling Mills

In the process of hot-rolling operations, in addition to constant (with respect to time) loads acting on the foundation, there appear also variable loads. These loads may induce foundation oscillations and dynamic stresses in both the soil and the foundation.

The larger the rolling mill, the larger the alternate loads imposed on the foundation and soil. Of the heavy rolling mills, reversible double-level mills are the ones most commonly used in engineering metallurgy. Therefore the computations outlined below refer to this type of mill. However, the data presented here may be easily applied to other types of rolling mills, such as three-level types and nonreversible types.

Modern rolling mills consist of the following main units:

1. Driving roll motor, whose foundation in some cases is rigidly tied to the foundation of the stands; occasionally no tie exists between these foundations
2. Ilgner power system, which is always mounted on a separate foundation
3. Operating and drive-gear stands, usually having a common foundation

a. Dynamic Loads Imposed on the Foundation by the Driving Roll Motor. Reversible direct-current motors are commonly used for the operation of rolling mills. The operational speed of these motors is rather low—around 58 rpm. The maximum (switching-off) moment at the shaft of the motor may reach several hundred tons × meters. The power is supplied from the Ilgner power system; the motor is mounted on a massive foundation. It will be assumed in further discussions that the motor is rigidly tied to the foundation, which is considered to be an absolutely rigid body resting on a fully elastic soil whose essential constants are known.

If a torsional moment M is applied to the rotor shaft, then the stator, and consequently the foundation, will be under the action of a moment whose magnitude equals $|M|$, but whose direction is opposite to that of M. This moment is the only alternating load acting on the foundation.

Changes in the torsional moment M applied to the rotor shaft, and consequently changes in the alternating moment acting on the foundation and soil, are a complicated function of many independent variables whose influence is difficult to evaluate. Therefore calculations are usually based on several assumptions. First of all some assumptions should be made concerning the distribution of the so-called reduced pressure of metal on the rolls. The magnitude of this pressure essentially affects the magnitude of M.

In computations of power consumed by the rolling mill, it is customarily assumed that the reduced pressure along the arc of contact between the ingot and the rolls remains constant. Usually it is assumed that the angular speed of the rotor is constant. Under these conditions, the magnitude of the moment of rotation may be expressed at any instant as a linear function of time.

Figure VII-4 shows a graph of changes in M for one of the first passes of the ingot through the rolls, plotted on the basis of the preceding assumptions concerning the reduced pressure of metal on the rolls and the angular speed of the mill. The horizontal axis of the graph shows periods of time t corresponding to successive stages in the passage of the ingot.

These stages are as follows (1) no-load speeding up of the rolls; (2) the ingot is gripped and forced through; (3) rolling with acceleration; (4) rolling at constant speed, (5) slowing down of the rolls; (6) exit of the ingot; (7) stoppage of the mill Figure VII-5 gives a graph of changes in M during the whole process of rolling of an ingot.

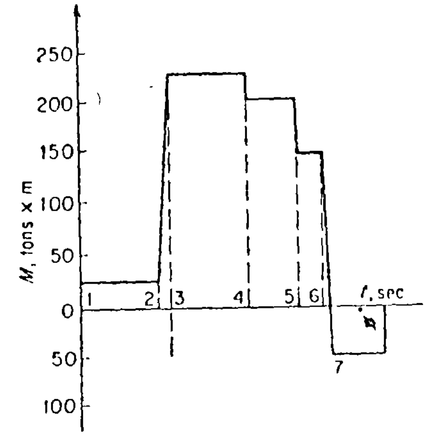


FIG VII-4 Graph of changes in the torsional moment of the shaft during one passage of an ingot on a rolling mill.

It is seen from graphs VII-4 and VII-5 that the external torsional moment, and consequently the exciting moment acting on the foundation of the motor, do not change much in the course of a pass of the ingot, except for the periods of its entry and exit. Therefore, instead of the diagram of changes in M shown in Fig VII-4, one can assume that changes will occur according to the diagram in Fig. VII-6a.

When the ingot emerges from the rolls, the absolute value of change occurring in M in practice may be considered to equal the change occur-

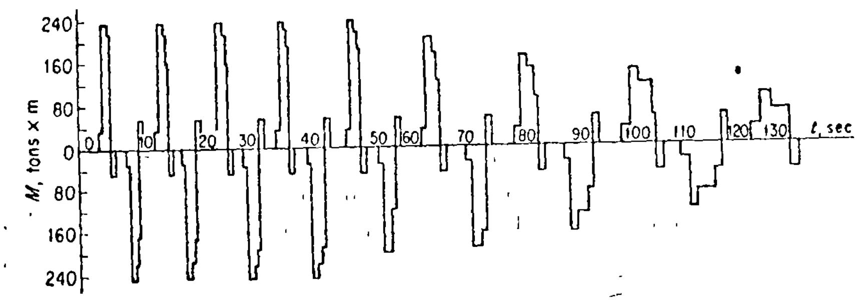


FIG. VII-5. Graph of changes in the torsional moment of the shaft during entire rolling process of an ingot.

ring when the ingot is gripped by the rolls. The exit of an ingot from the rolls is accompanied by foundation vibration. Due to a decrease in loading, the magnitudes of stresses induced by the vibration will not exceed those observed during the steady process of rolling. This makes it possible to base calculations not on Fig. VII-6a, but on Fig VII-6b.

In conformity with this diagram, let us set up the following conditions for the exciting moment M:

1. When $t = 0$, $M = 0$;

2. At the time the ingot is gripped by the rolls ($t \leq \tau$),

$$\frac{d}{dt} M = \text{constant} > 0$$

3. For the steady process of rolling ($t > \tau$),

$$M = M_{\max} = \text{constant}$$

Under the action of the torsional moment, the foundation will rotate around an axis passing through the center of gravity of the foundation

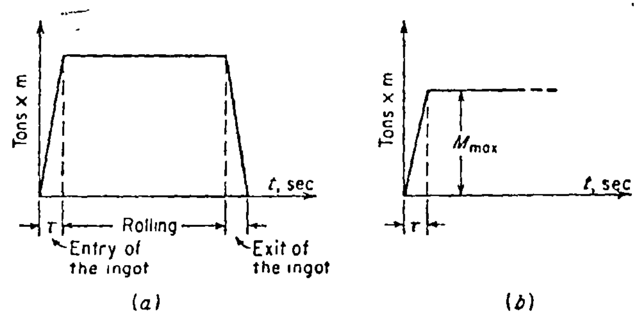


FIG. VII-6. Simplified design diagram of changes in the torsional moment of a rolling mill shaft.

area in contact with soil, perpendicular to the plane in which the moment acts. Therefore the stresses in the soil along the contact area will vary, and the maximum stress $p_{\max}$ at the foundation edge will equal

$$p_{\max} = \frac{W}{A} + c_\varphi a \varphi_{\max} \qquad \text{(VII-3-1)}$$

where W = weight of foundation and equipment thereon
A = foundation area in contact with soil
c_φ = coefficient of elastic nonuniform compression of soil
$2a$ = foundation width in plane of action of moment

Let us compute $\varphi_{\max}$ for the interval of time corresponding to the gripping of the ingot by the rolls. The equations of motion of the foundation are as follows:

$$\begin{aligned} \ddot{x} + \alpha_{11}x + \alpha_{12}\varphi &= 0 \\ \ddot{\varphi} + \alpha_{21}x + \alpha_{22}\varphi &= M_s t \end{aligned} \qquad \text{(VII-3-2)}$$

where

$$M_s = \frac{M_{\max}}{M_\tau}$$

and M_τ is the moment of inertia of the installation mass (machine and foundation).

The coefficients of this system of equations depend on the elastic properties of soil and the dimensions and mass of the foundation and motor [Eqs. (III-4-5)].

Assuming that at time zero the displacement x_0 and the angle of rotation φ of the center of gravity of the foundation equal zero, we obtain the general solution for φ:

$$\varphi = \frac{\alpha_{11} - f_{n1}^2}{f_{n1}^2(f_{n2}^2 - f_{n1}^2)} M \sin f_{n1}t - \frac{\alpha_{11} - f_{n2}^2}{f_{n2}^2(f_{n2}^2 - f_{n1}^2)} M \sin f_{n2}t + \varphi_s \qquad \text{(VII-3-3)}$$

where f_{n1} and f_{n2} are the natural frequencies of the foundation established from Eq. (III-4-8). The expression

$$\varphi_s = \frac{\alpha_{11}}{f_{n1}^2 f_{n2}^2} M_s t \qquad \text{(VII-3-4)}$$

gives the value of φ for the condition that the torsional moment has only a static effect. The other terms in Eq. (VII-3-3) evaluate the dynamic action of the external torsional moment applied to the foundation. Assuming

$$\varphi = \mu \varphi_s$$

we obtain for the dynamic coefficient

$$\mu = 1 + \frac{(\alpha_{11} - f_{n1}^2) f_{n2}^2}{\alpha_{11}(f_{n2}^2 - f_{n2}^2) f_{n1}t} \sin f_{n1}t - \frac{(\alpha_{11} - f_{n1}^2) f_{n1}^2}{\alpha_{11}(f_{n2}^2 - f_{n1}^2) f_{n2}t} \sin f_{n2}t \qquad \text{(VII-3-5)}$$

If the gripping period is small in comparison with the periods T_1 and T_2 of natural vibrations of the foundation, then, assuming that

$$\sin f_{n1}t = f_{n1}t$$
$$\sin f_{n2}t = f_{n2}t$$

we obtain

$$\mu = 2$$

In this case the maximum rotation of the foundation under the action of the alternating torsional moment will not exceed the twofold value of displacement caused by the static action of the same moment.

As the ingot gripping time increases, the value $\mu - 1$ decreases, approaching zero for high values of t. Consequently, if the gripping time is large in comparison with the periods T_1 and T_2, then the action of the alternating torsional moment upon the foundation does not differ much from the static pressures.

The value of μ may be computed with a comparatively high degree of accuracy as soon as one knows the periods of natural vibrations of the foundation in the plane of action of the alternating torsional moment;

then one will also know the time required to grip the ingots. In practice this time varies within a comparatively wide range. Only approximate values of the periods T_1 and T_2 may be established by computations. The calculation of T_1 and T_2 involves laborious arithmetic operations; therefore practical design computations of the foundation under the driving roll motor should be based on the most unfavorable conditions by setting $\mu = 2$.

Evidently, for $t > \tau$, i.e., for the steady process of rolling, φ will not exceed the maximum value characterizing the times of gripping of the ingot by the rolls and its emergence from the rolls. Therefore the stresses in the soil along the foundation contact area during the process of rolling will not exceed the stresses developing at the time of entry of the ingot. The values of these stresses should be used in calculations of the base under the foundation for the rolling mill.

b. Dynamic Action on the Foundation by the Ilgner-Power System. The purpose of the Ilgner power system is to feed power to the motor driving the rolls. The power system consists of one or several direct-current generators and a flywheel, mounted on the same shaft. The generators are set in motion by an electric motor.

The power W_1 of an asynchronous motor, taken from the line, remains almost constant during power-system operation. It equals the average quadratic power required for rolling during one cycle (15 to 19 passes). The power supplied by the direct-current generators is almost constant. The power W_2 taken from these two generators by a motor driving the rolls undergoes extremely sharp changes. The range of these changes is from zero, which corresponds to a pause in the rolling, to the maximum power required by the motor. The maximum power may be considerably larger than the power which at a given instant is supplied to the generators by the asynchronous motor. The flywheel and other rotating masses increase the amount of energy which is yielded by the generators, since their kinetic energy changes as a result of changes in the speed of the shaft. Thus a deficiency in energy required for rolling is made up. In addition, during the operation interval in which the generators do not supply energy to the drive motor, the flywheel and other rotating masses accumulate the energy which is taken by the asynchronous motor from the line.

Let us investigate the dynamic loads acting on the foundation during the operation of the power system. If

W_1 = power taken by motor from line
ω = angular speed of aggregate shaft (a varying value)
M_1 = torsional moment of motor shaft

then

$$W_1 = M_1\omega$$

In electrical motors, the stators (and consequently the foundation) are under the action of a reactive moment whose absolute value equals that of the torsional moment M_1, but whose direction is opposite.

In addition to this moment M_1, a moment M_2 is also acting on the foundation, induced by the generators. This moment has the same sign as the moment of the generator shaft. If W_2 is the power yielded by the generators, then

$$W_2 = M_2\omega$$

The resulting external moment M_t acting on the foundation evidently will equal the difference between the moments; i.e.,

$$M_t = M_2 - M_1$$

Neglecting power losses in the engine, we obtain

$$W_2 = W_1 - \frac{d\omega}{dt}\,\omega \sum I_t$$

where $\Sigma I_t = I$ is the sum of the moments of inertia of all the rotating masses of the power system, i.e., of the flywheel, motor generators, and armatures of electromagnets. Since

$$W_2 - W_1 = \omega(M_2 - M_1) = M_t\omega$$

it follows that

$$M_t = \frac{d\omega}{dt} I \qquad \text{(VII-3-6)}$$

when the energy yielded by the power system equals the energy taken from the line, i.e., when

$$\frac{d\omega}{dt} = 0$$

the external moment acting on the foundation also equals zero. At the same time, the foundation will be subjected to internal moments tending to produce torsion in it.

Let us assume that the Ilgner power system consists of two alternating-current generators, an asynchronous motor, and a flywheel. The total flywheel moment GD^2 of all the rotating masses of the aggregate is about 870 tons $\times$ m²; hence

$$I = \frac{GD^2}{4g} = \frac{870}{4 \times 9.81} = 22.3 \text{ tons} \times \text{m} \times \text{sec}^2$$

If N is the number of rpm of the power system, then

$$\omega = \frac{2\pi}{60} N \qquad \frac{d\omega}{dt} = \frac{2\pi}{60}\frac{dN}{dt}$$

Therefore,

$$M_t = \frac{2\pi}{60} I \frac{dN}{dt} = 2.24 \frac{dN}{dt}$$

The rate of change of N, i e., dN/dt, varies within the range 2.8 to 10.4. The magnitude of the external moment acting on the power-system foundation during the whole cycle of rolling of one ingot changes from 2.6 tons × m (periods of running idle) up to 24 tons × m (periods of rolling).

The design value of the exciting moment should be taken to equal $2M_t$ for the most unfavorable case. The angle of foundation rotation, induced by this moment, is determined from Eq. (VII-3-3).

In addition to the exciting moments caused by changes in the kinetic energy of the power system, the foundation may be subjected to periodic exciting loads caused by the unbalanced state of the engine with respect to magnetic forces and static equilibrium. The computation of forced vibrations of the foundation caused by these loads is performed in the same way as for foundations under motor generators.

c. Dynamic Loads on the Common Foundation of Working and Gear Stands. In the process of the rolling-mill operation, the frame of the driving-gear stand, and consequently its foundation, are subjected to the action of a varying exciting moment equal in magnitude and sign to the moment of the shaft of the driving roll motor.

The forces appearing as a result of the acceleration of the ingot may be neglected because of their minute magnitudes; hence it may be considered that stresses occur only in the working stand during the rolling operations. These stresses have a tendency to rupture the stand. The sum of all the external alternating loads equals zero.

The drive-gear and working stands may be mounted on a separate foundation, not tied to that under the driving roll motor. In this case, the dynamic influences of external loads on the foundations are evaluated separately but similarly.

If the drive-gear stand, working stand, and driving roll motor are mounted on a common foundation, then the drive-gear stand is subjected to the action of a torsional moment whose sign is opposite that of the moment acting on the stator of the driving roll motor. Therefore the sum of all the external dynamic loads transmitted to the foundation and soil equals zero. The foundation will be under the action of internal torsional moments whose magnitude equals the moment of the shaft of the motor, as well as under the action of the equipment weight. These

loads should be considered in the stress analysis of the foundation and its separate elements. The dynamic nature of the internal moments is taken into account by introducing in the calculations the twofold magnitude of the maximum torsional moment of the shaft of the driving roll motor.

In the case under consideration, stresses in the soil are determined for a design load consisting of the combined weight of the foundation and the equipment mounted thereon.

d. Remarks concerning Design. The foundations for the principal rolling equipment (stands, reducer, gear) are always built as massive

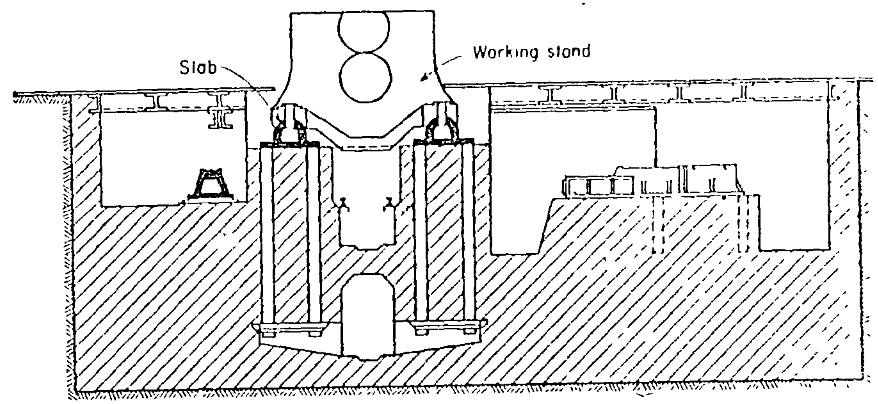


FIG. VII-7 Foundation for a stand of a sheet-rolling mill

units which either are monolithic or are provided with deformation joints. As illustration, diagrams of massive foundations are shown as follows: Fig. VII-7: a foundation for the stand of a sheet-rolling mill; Fig. VII-8: a foundation for a light-section steel mill, Fig. VII-9: a foundation for a drive-gear stand.

The main part of the foundation under the drive-gear and working stands is always designed as one block. This part of the foundation usually has two tunnels, located along the axis of the stand at different heights The upper tunnel serves for the removal of mill scale and for the runoff of cooling water under the working stand, as well as for the inspection of equipment and the runoff of lubricant under the drive-gear stand The lower tunnel serves for the inspection of anchor bolts; it is provided with several recesses to facilitate access to anchor plates In the central part of the foundation are located spindle benches which are provided with wells for counterweights and an appliance for changing the first roller of the stand.

The foundation under the driving roll motor (Fig. VII-10) is built as a separate massive block or as a block forming one monolith with the

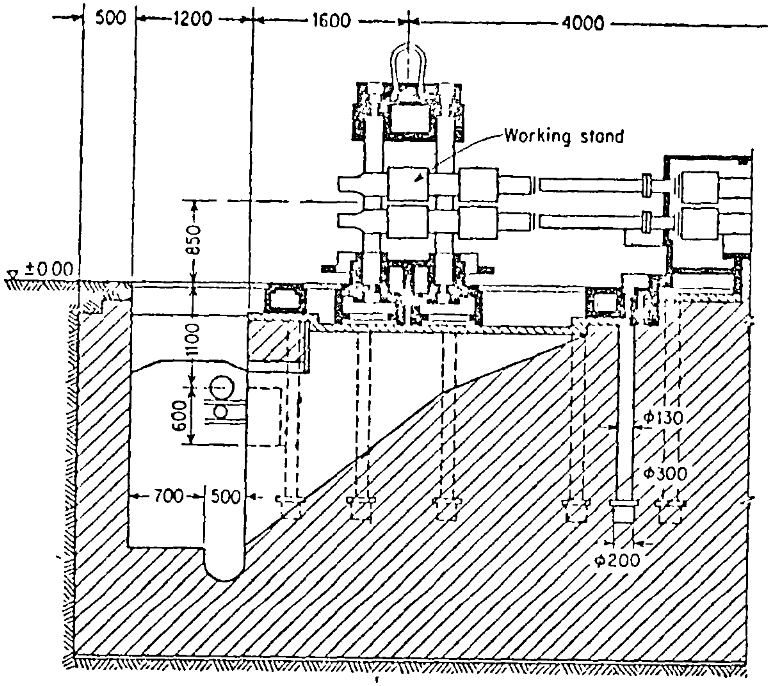


Fig. VII-8. Foundation for a light-section steel mill.

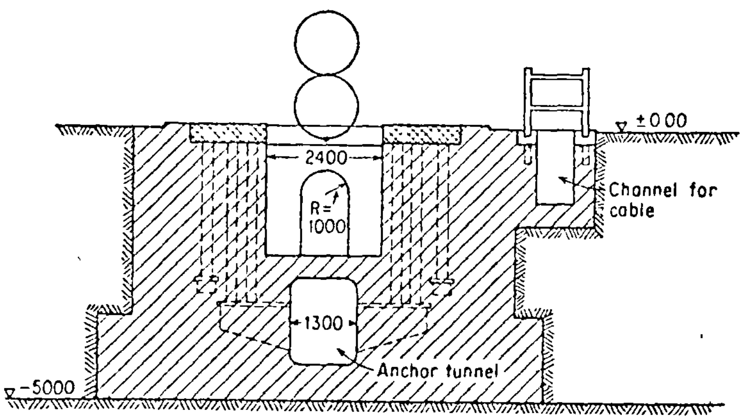


Fig. VII-9. Foundation for a drive-gear stand.

foundation under the drive-gear stand. The foundation has a deep groove for the inspection and mounting of the equipment and a channel for the air-cooling of the motor.

On both sides of the working stand, along the rolling-mill axis, are located the foundations for manipulators and roller conveyors. Usually

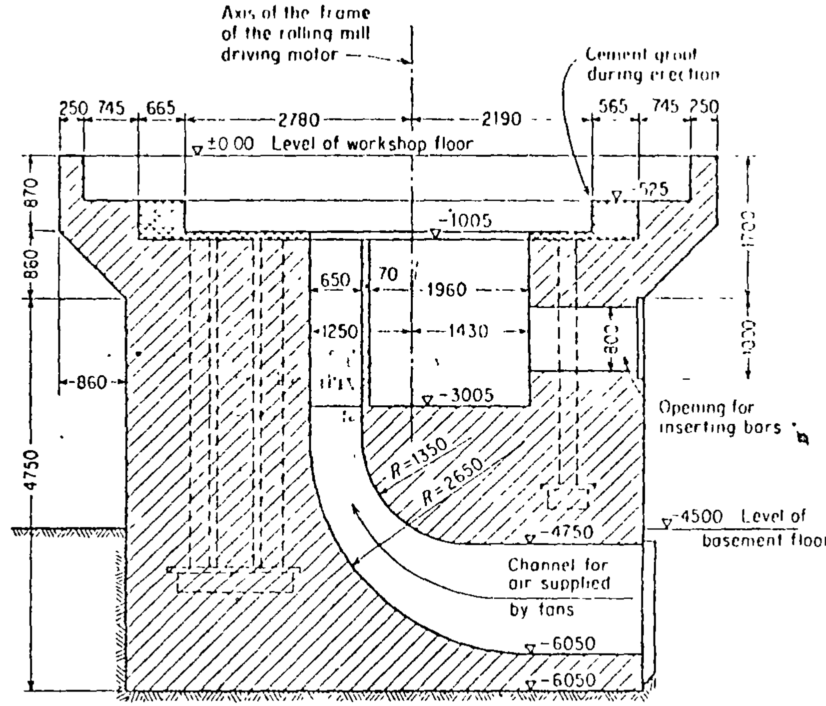


Fig. VII-10. Foundation for the motor driving the rolls.

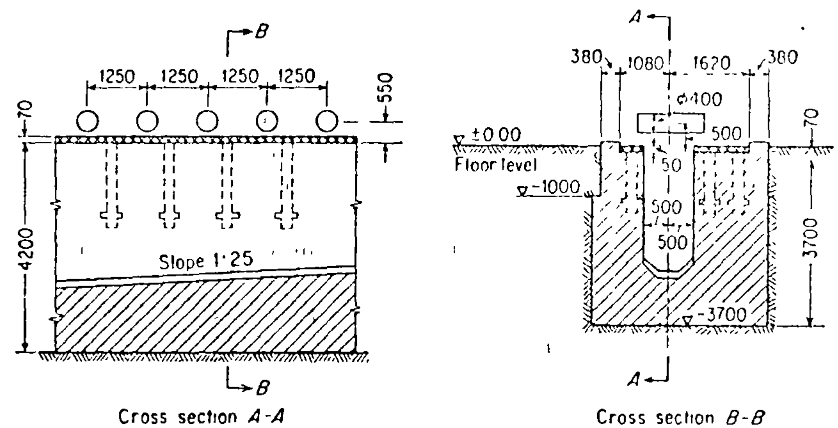
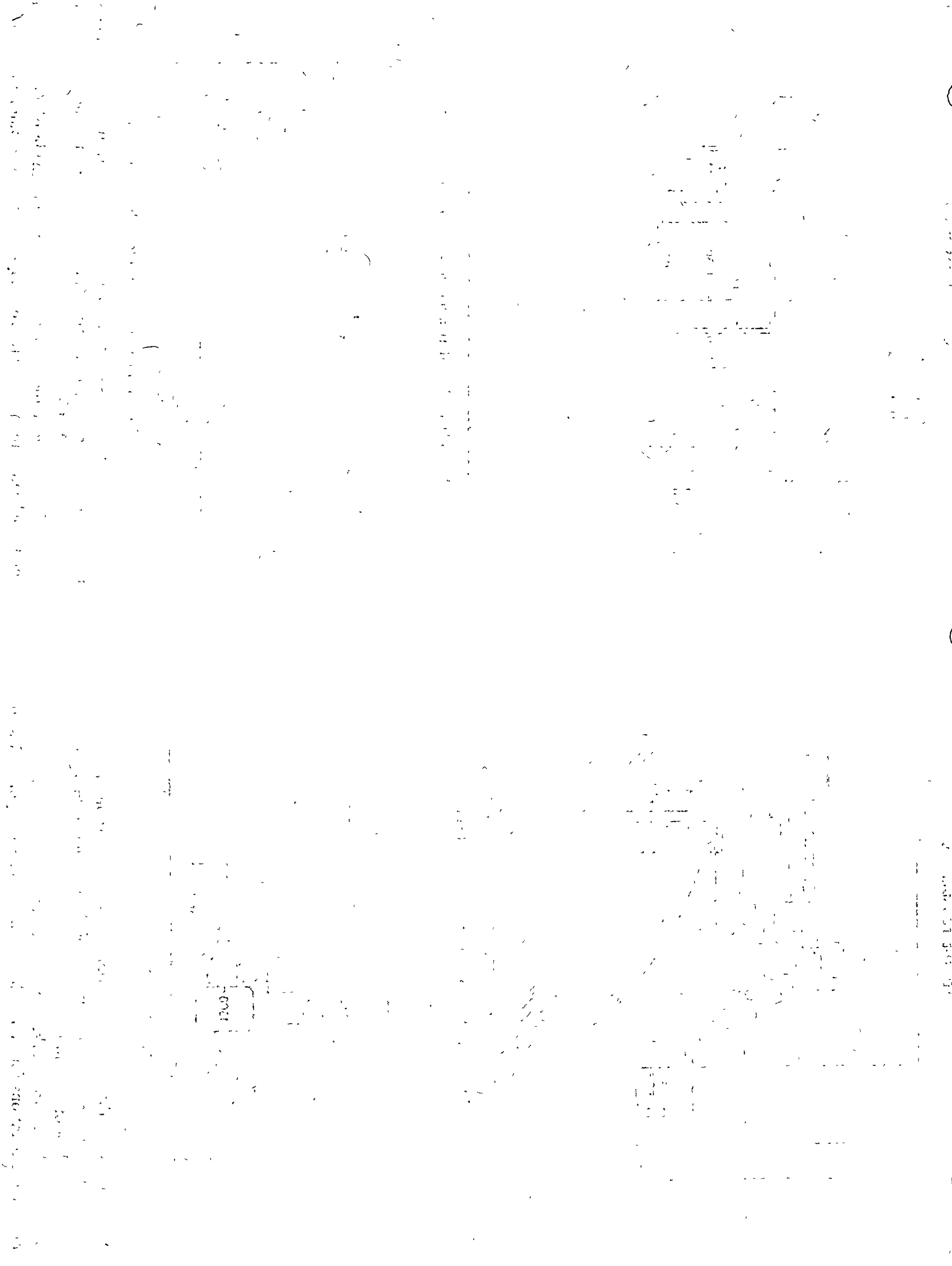


Fig. VII-11. Block foundation for roller conveyors

these foundations are also built as massive blocks with required channels and grooves (Fig. VII-11). Sometimes they are designed as frame foundations (Fig. VII-12).

Foundations under rolling-mills equipment are made of concrete and reinforced concrete. Concrete is employed for massive foundations which



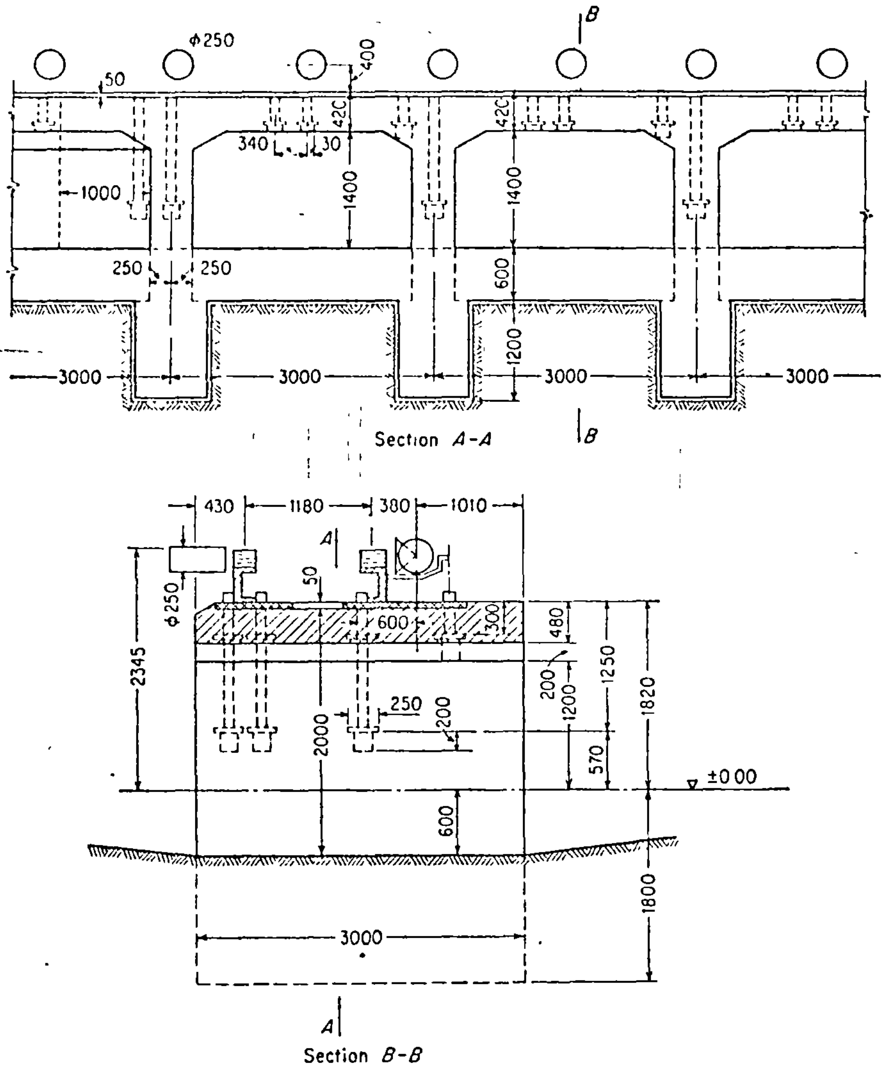


FIG VII-12. Frame foundation for roller conveyors.

are not weakened by large openings and channels and which are erected on sufficiently rigid and homogeneous bases. Otherwise, reinforced concrete is used. As a rule, concrete type 100† is employed.

The foundation area in contact with soil should be, as far as possible, all on the same level. Large differences in depths of separate sections of

† See footnote, Art. IV-1-c, p. 132.

the foundation should not be permitted. If locating all foundation contact areas on the same level leads to a considerable overexpenditure of material, then deformation joints may be provided between sections lying at different depths.

The location of expansion, shrinkage, and settlement joints in foundations under rolling-mill equipment is determined by the distribution of the equipment, the depth of separate foundation sections, the soil bearing value, and the temperature regime of rolling. Distances between deformation joints are selected according to the official *Technical Rules and Construction Code*. Joints should be located so that they divide the foundation into separate sections which support units of equipment not connected with each other. For example, in order to avoid uneven settlement, the foundations under working and drive-gear stands should not be separated.

Continuous footings longer than 20 to 30 m and foundation sections under stands larger than 15 by 15 m or 20 by 20 m should be provided with deformation joints. If a large section of the foundation cannot be divided by deformation joints, then, in order to prevent the appearance of shrinkage cracks, such a foundation may be divided by temporary joints with reinforcement extending beyond the joints Later these joints are filled with concrete of the same type. The projecting reinforcement is overlapped and welded.

c. Design Loads. For the analysis of stresses within the foundation and for the determination of pressure on the base, the following loads should be considered:

Weight of the rolling-mill equipment
Weight of the driving roll motor
Maximum disconnection moment at the motor shaft
Horizontal force transmitted to the footings under manipulators and tilting devices
Erection loads
Foundation weight

Static computations of the foundation may be limited to:

1. Stress analysis of separate units of the foundation, such as units weakened by openings, cantilevers, and others
2. Computation of local stresses under supporting slabs
3. Analysis of stresses within the foundation
4. Computation of pressures transmitted to the soil

The foundation is considered to be a girder of varying rigidity resting on an elastic base.

For calculations listed in points 1 and 2, a value of the dynamic coefficient equaling 2 is introduced in the calculations of the weight of the rolling mill and of the driving roll motor. For calculations listed in points 3 and 4 the actual weight of the same machines is taken, without introducing a dynamic coefficient.

If a foundation is treated in design computations as a beam resting on an elastic base, then, in order to simplify operations, it is permissible to consider separate comparatively rigid units of the foundation as being absolutely rigid. An uneven settlement at the contact of the foundation under the roller conveyers with the foundation under the rolling mill leads to the appearance of stresses along this contact. To determine these stresses, it is permitted to consider the foundation under the rolling mill to be an absolutely rigid unit.

The permissible pressure on the soil under the foundations of rolling mills and driving roll motors for dynamic loads may be taken to equal the corresponding permissible pressure for static loads only.

In concrete or lightly reinforced foundations, the soil pressure imposed by separate machinery units and established for conditionally separated foundation sections without considering the influence of other foundation units should not exceed the permissible bearing value of soil.

Foundations subjected to horizontal impacts, such as those under manipulators and tilting devices, should be designed for the double value of the maximum horizontal force.

f. Data on Performance of Existing Foundations under Rolling Mills. The author and B. M. Terenin investigated several foundations under rolling mills at one of the Soviet plants. These foundations were built of concrete, and each consisted of a single massive block supporting the driving roll motor as well as the drive-gear and working stands.

The foundations investigated were not reinforced at places weakened by recesses, openings, and channels. Results of laboratory tests showed that concrete had been used which, at the age of 28 days, had a temporary compressive strength of 90 kg/cm^2, with slight deviations in some parts of the foundation. Concrete type 60† was used for the foundations under lifting platforms of rolling mill "750," and concrete type 130† for the foundation under the first working stand of the same mill.

The foundations were placed on loessial clays with some sand. Owing to the wetting of the soil, for different reasons the foundations underwent uneven settlements resulting in the appearance of cracks. In the block of the central part of the foundation under rolling mill "750" several

† See footnote, Art. IV-1-c, p. 132.

cracks were observed in the tunnel under the driving gear and operation stands, in the tunnels under the lifting platforms, in wells at the contact between the foundations of roller conveyors and the foundations of rolling mills, and in the foundation unit under the driving roll motor. The appearance of these cracks was due to two causes:

1. A horizontal foliation of the foundation under the drive-gear and operating stands developed at the level of the anchor plates. The most distinct crack was observed in the tunnel under the drive-gear stand. Under the operating stands were found slightly developed small horizontal cracks coinciding with working joints. These cracks indicate that a long interruption had occurred in the concreting of the foundation and that no measures were taken to secure the monolithic character of the foundation.

2. There was a differential settlement of the foundation under the rolling mill and the foundation under adjacent auxiliary equipment. This settlement was caused by the wetting of soil and resulted in cracks in the tunnels of the rear and front lifting platforms, in the wall of the middle platform of the staircase, in the arch near the lifting platform of the second operating stand, and under the decelerator of the driving roll motor.

A vertical crack was observed approximately in the middle of the tunnel of rolling mill "450." This crack ran along the walls in places where they were weakened by niches, and along the arch.

A vertical crack was found in the tunnel of rolling mill "360" near the inlet opening; two vertical cracks were found in niches, one of them running along the arch.

In the tunnel of rolling mill "280" a vertical crack was found under the operating stand through which water was flowing abundantly. Channels of rolling mills "360" and "280," especially in their lower sections, were filled with water.

An instrumental investigation of vibrations of foundations under the rolling mills was performed at several points along the foundation axis and along its height: on the slabs of the operating and drive-gear stands, at the level of niches where anchor slabs of the foundation were located, and at points on the floor of the tunnels.

Results of the measurements are shown in Table VII-1. It is seen from this table that the largest amplitudes of vibrations were found directly on the slab under the drive-gear stand of rolling mill "750." The measurements performed here showed that the foundation underwent extremely irregular high-frequency vibrations with amplitudes of the order of 0.006 to 0.010 mm, caused by impacts of the gear.

These measured values of vibration amplitudes under rolling mills show that the additional pressure on the soil and the stresses within the

foundation caused by dynamic loads are small in comparison with stresses imposed by the weight of equipment and foundation. Therefore a value of 3 for the dynamic coefficient, often taken in design computations of such foundations, is exaggerated.

TABLE VII-1. RESULTS OF VIBRATION MEASUREMENTS ON ROLLING-MILL FOUNDATIONS

Rolling mill	Vibrations measured at:	Nature of vibrations and amplitude
"750"	Station (1 at slab of gear stand)	Extremely irregular high-frequency vibrations with amplitudes 0.005–0.010 mm. At time of entry and exit of ingot, vibrometer records impacts inducing vertical and horizontal vibrations with amplitude 0.030–0.050 mm.
"750"	Station 2 (at edge of foundation near slab of gear stand)	High-frequency vibrations with amplitude less than 0.003 mm. At time of entry and exit of ingot, vertical impacts are recorded with amplitudes of some 0.006–0.010 mm.
"750"	Station 3 (housing under gear stand)	The same as for Station 2
"750"	Station 4 (on floor of tunnel under gear stand)	The same as for Station 2
"750"	Station 5 (at surface of foundation near rolling-mill driving motor)	High-frequency vibrations with amplitude 0.003 mm. At time of entry and exit of ingot, impacts are recorded inducing vibrations with amplitude of 0.010 mm
"450"	Station 6 (at surface of foundation near rolling-mill driving motor)	Quickly damped vibrations were recorded, with amplitudes on the order of 0.0015 mm. Impacts at time of entry and exit of ingot are only slightly noticeable.
"360"	Station 7 (at surface of foundation near rolling-mill driving motor)	Vibrations of same nature as those at Station 6
"280"	Station 8 (at surface of foundation near rolling-mill driving motor)	The same as for Station 6
Slabbing	Near working stand	The same as for Station 2

Measurements of vibrations of the foundation under rolling mill "750," performed on the upper and lower parts of the foundation divided by a horizontal crack, established that these two sections underwent vibrations of the same character with the same amplitude. This indicates that the complete foundation vibrated as one block. It followed that foliation of the foundation is not dangerous for rolling-mill operations.

VII-4. Foundations for Crushing Equipment

a. Design Computations of Foundations under Jaw Crushers. There are many different arrangements of jaw-crusher operating mechanisms. However, one common feature of these crushers is that, analogously to reciprocating engines, they create unbalanced inertial forces varying with time. These inertial forces form exciting loads which induce forced vibrations of the foundation.

The most common arrangement of the operating units of the jaw crusher is one in which the motion of the mechanism is due to the action of so-called lower couples of rotation. Some typical arrangements of jaw crushers of this group are shown in Fig. VII-13. Approximate formulas for the determination of unbalanced inertia forces are also given. Accurate methods of computation of the exciting loads imposed by jaw crushers may be found in specialized publications.[2]

It follows from the equations in Fig. VII-13 that exciting loads imposed by jaw crushers are of the same nature as exciting loads imposed by reciprocating engines. Therefore all directives outlined in Chap. IV concerning the design of foundations for reciprocating engines may be applied to the dynamic computation and design of foundations for jaw crushers.

b. Computations of a Foundation under a Gyratory Crusher. In gyratory crushers the ore is pulverized between the crushing head of the main shaft, undergoing a rocking motion along a circle, and the armored jacket of the upper stationary part.

Under the action of frictional forces, the crushing cone moves around the axis of the crusher and develops an angular velocity whose value is close to that of the movement but has opposite sign. As a result of this, the frame of the machine, and consequently the foundation, is subjected to the action of gyroscopic and inertial loads which may be approximately expressed by one resultant exciting force:

$$R = (m_1 r_1 - m_2 r_2)\omega^2 \qquad \text{(VII-4-1)}$$

where m_1 = total mass of main shaft and crushing cone attached to it
m_2 = mass of camshaft and units rigidly connected with it (gears, counterweights, and others)
r_1 = distance between crusher axis and center of gravity of main shaft
r_2 = distance between another axis and center of gravity of eccentric shaft
ω = frequency of rotation of crusher

This force, rotating at a constant angular speed, acts in a horizontal plane

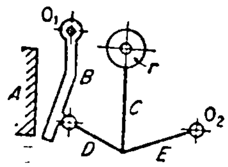
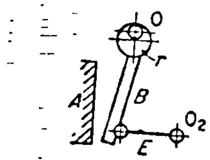
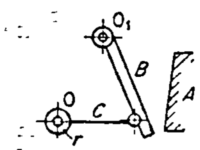
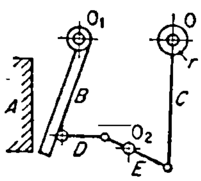
No.	Diagram of the crusher	Approximate values of inertia forces	Designations
1		$P_z = (M_0 + M_c)r\omega^2 \sin \omega t$ $P_x = (M_0 + 0.8M_c)r\omega^2 \cos \omega t$ } I $P_z = [(M_0 + M_c)r - M_d r_1]\omega^2 \sin \omega t$ $P_x = 0.25M_B r\omega^2 \sin \omega t$ } II	M_B = mass of moving (crushing) jaw M_c = mass of connecting rod
2		$P_z = (M_0 + M_B)r\omega^2 \sin \omega t$ $P_x = (M_0 + 0.5M_B)r\omega^2 \sin \omega t$ } I $P_z = [(M_0 + M_B)r - M_d r_1]\omega^2 \sin \omega t$ $P_x = [(M_0 + 0.5M_B)r - M_d r_1]\omega^2 \cos \omega t$ } II	M_0 = mass of eccentric (or 50% of crankshaft mass) M_d = total mass of counterweights
3		$P_z = (M_0 + 0.7M_c)r\omega^2 \sin \omega t$ $P_x = (M_0 + M_c + 0.5M_B)r\omega^2 \cos \omega t$ } I $P_z = 0$ $P_x = [(M_0 + M_c + 0.5M_B)r - M_d r_1]\omega^2 \cos \omega t$ } II	r = eccentricity r_1 = distance from axis of rotation to center of gravity of counterweights center
4		$P_z = (M_0 + M_c)r\omega^2 \sin \omega t$ $P_x = (M_0 + 0.8M_c)r\omega^2 \cos \omega t$ } $P_z = [(M_0 + M_c)r - M_d r_1]\omega^2 \sin \omega t$ $P_x = 0.25M_B r\omega^2 \sin \omega t$ } II	ω = angular speed

FIG. VII-13. Data on jaw crushers.

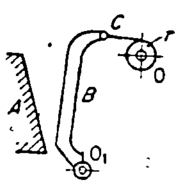
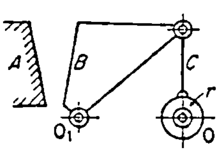
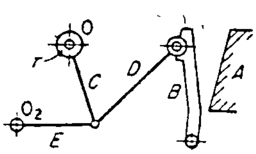
No.	Diagram of the crusher	Approximate values of inertia forces	Designations
5		$P_z = (M_0 + 0.7M_c)r\omega^2 \sin \omega t$ $P_x = (0.5M_B + M_c + M_0)r\omega^2 \cos \omega t$ } I $P_z = 0$ $P_x = [(0.5M_B + M_c + M_0)r - M_d r_1]\omega^2 \cos \omega t$ } II	P_z = vertical component of resultant inertia force P_x = horizontal component of resultant inertia force
6		$P_z = (M_0 + M_c + 0.5M_B)r\omega^2 \sin \omega t$ $P_x = (M_0 + 0.7M_c + 0.5M_B)r\omega^2 \cos \omega t$ } I $P_z = [(M_0 + M_c + 0.5M_B)r - M_d r_1]\omega^2 \sin \omega t$ $P_x = [(M_0 + 0.7M_c + 0.5M_B)r - M_d r_1]\omega^2 \cos \omega t$ } II	*Notes:* 1. Forces P_z and P_x are applied to axis of main shaft. 2. Equations I refer to crushers without counterweights.
7		$P_z = (M_0 + M_c)r\omega^2 \sin \omega t$ $P_x = (M_0 + 0.8M_c)r\omega^2 \cos \omega t$ } I $P_z = [(M_0 + M_c)r - M_d r_1]\omega^2 \sin \omega t$ $P_x = 0.25M_B r\omega^2 \sin \omega t$ } II	Equations II refer to crushers with counterweights

FIG. VII-13 (*Continued*)

passing through the center of the main shaft (in crushers with a sharp cone) or through the point of rest (in crushers with a flat cone).

Resolving R into components along the horizontal axes x and y, the principal inertial axes of the installation, we obtain

$$P_x = R \sin \omega t$$
$$P_y = R \cos \omega t$$

The dynamic computation of a foundation under a gyratory crusher is reduced to the determination of amplitudes of forced vibrations imposed on principal vertical planes of the foundation by exciting forces P_x and P_y and moments $P_x h_1$ and $P_y h_1$.

Thus, dynamic computation of a foundation under a gyratory crusher in principle does not differ at all from the dynamic computation of a foundation under a jaw crusher.

VIII

PROPAGATION OF ELASTIC WAVES IN SOIL

INTRODUCTION

As stated in Chap. I, there are several reasons why the application of Hooke's law to soils is limited. For example, it has been indicated that the elastic constants of soil depend on normal stresses and that elastic deformations may affect the initial internal stresses which always exist in soil. It should also be noted that the solution of problems related to the propagation of waves may be greatly influenced by dissipative properties of soil which govern the absorption of wave energy.

When solving problems related to the propagation of waves in soils, one has to start with models of the phenomenon, which are very far from reality. For example, the investigation of waves emanating from machine foundations leads to a composite dynamic theory-of-elasticity problem which starts with displacements in a certain section of the soil surface—while the rest of the soil is free of stresses. In the simplest case the soil is considered to be a semi-infinite elastic solid. The solution of such a composite problem involves considerable mathematical difficulties. Therefore, a source of waves is represented as an alternating force, either concentrated or distributed over the given soil surface area. This model of the source of waves is far from reality, and the results of such a solution may differ (sometimes considerably) from the results of experimental investigations of wave propagation from an actual source of waves such as a vibrating foundation.

However, in spite of the indicated limitations, the development of the theory of propagation of waves in soils on the basis of the theory of elasticity, even for highly abstract conditions, gives us a chance to investigate several very important specific features of wave propagation

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Foundations for High-Speed Machinery

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This paper covers the dynamic aspects of foundation design for high-speed rotating machinery such as turbines and compressors. Part 1 covers: General considerations, resonant frequencies and dynamic response, and practical considerations. Part 2 covers: Simplified calculation procedure, vibration test results.

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Foundations for High-Speed Machinery

J. S. SOHRE

PART 1

INTRODUCTION

Special requirements must be considered in the design of foundations for high-speed machinery such as compressors, turbines, and generators. This paper deals primarily with problems of vibration and alignment. The purpose of this presentation is strictly to provide practical information which can immediately be applied to design a workable foundation; refinements are not included because of space limitations. It is, therefore, absolutely necessary to observe the initial assumptions and safety margins as given in the paper.

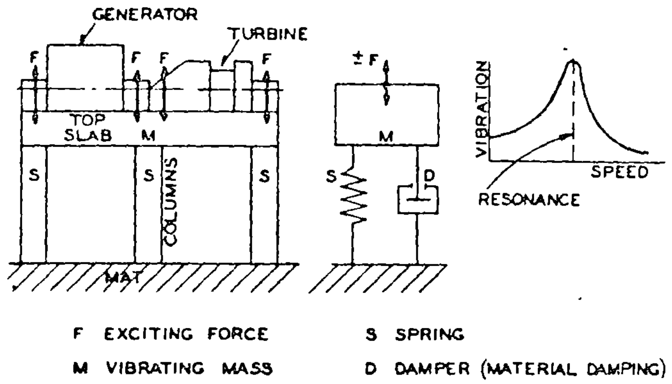


Fig. 1

1 General Considerations

Foundations must be designed for:

. Minimum Vibration at Operating Speed. This means: Vibration for a given rotor unbalance should be a minimum. This depends primarily on the following factors:

1 Ratio of resonant frequency to operating speed, Fig.1.

2 Ratio of vibrating mass to rotating mass, Figs.2 and 3.

3 Stiffness of supporting structure.

4 Dynamic properties of soil and structural materials.

Minimum dynamic response is obtained by tuning the foundation to a frequency safely above or below the operating speed range. See "Simplified Calculation Procedure," Part 2 of this paper.

Minimum Vibration Transmission to and From the Unit. This will prevent:

1 Oil-whirl resonance in unit bearings, excited by vibrations transmitted into the foundation from pumps, fans, mills, or other equipment running elsewhere in the plant at approximately 1/2, 1/4, 1/6, 1/8 .... and so on, of unit speed (the shaft whirls at frequency between 0.4 and 0.5 of unit speed).

2 Excitation of rotor criticals, foundation resonance, and so forth, by units running at these respective speeds elsewhere in the plant.

3 Vibrations in buildings, which may be transmitted over long distances, and which can become severe if members of the building are resonant.

Vibration transmission is minimized by isolating vibrating components from building members and floors by air gaps or vibration joints, spring supports for piping, and so on; refer to section on "Practical Considerations." Usually involved are operating and basement floors, piping, building columns, stairways. Must also isolate from bed rock or ground water.

The phenomena of resonance and vibration transmission have nothing to do with the degree of rotor balance ("roughness") of the machine. The inherent vibration can be either amplified or reduced by the structure, depending on its dynamic characteristics. In other words, a unit which runs perfectly smooth on one foundation may run very rough on another, although the rotor unbalance is still the same. This means a unit on a resonant foundation will need an extremely well-balanced rotor to operate satisfactorily. Whenever this degree of balance is disturbed, even momentarily, rough operation will result and damage to the machine (ranging from a slight packing rub to a complete wreck) may occur. This danger should never be underestimated, because upsets such as rapid load changes, slugs of fluid, thermal shocks, will be experienced during the life of almost any machine, and how much of this the machine can take will depend to a high degree on the dynamic characteristics of the foundation. Where a machine must be started and stopped frequently, and where rapid changes in operating conditions (peaking turbines) occur, this factor will determine the availability of a unit, and the degree of

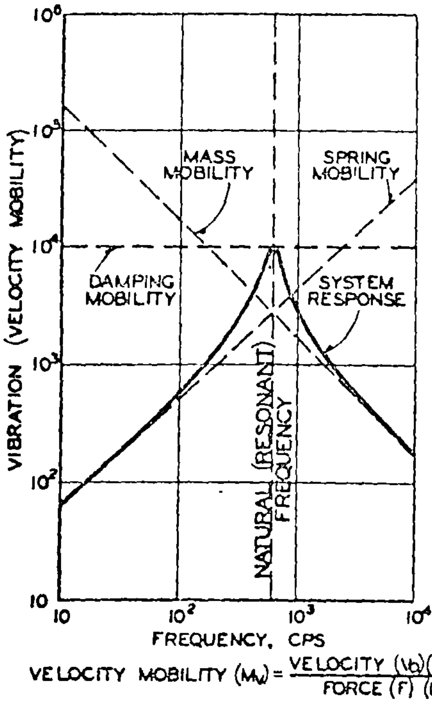


Fig. 2 Response of single-degree-of-freedom systems (Church, reference 1)

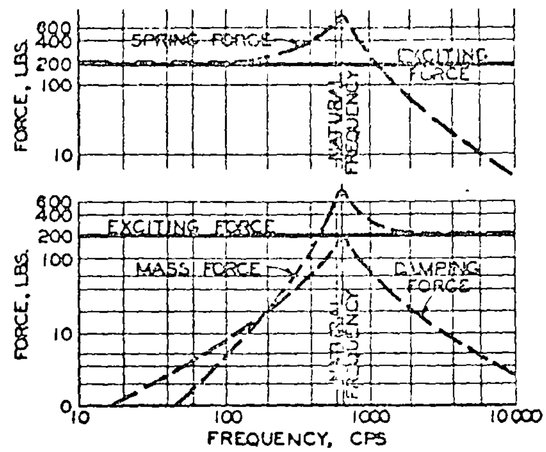


Fig. 3 Component forces of vibrating system (Church, reference 1)

confidence the operators can have in it.

How vitally important this problem can become under emergency conditions (blade failure, slug of fluid, packing rub, thermal shaft bow, bearing failure, and so on) can be seen from reference (1),[1] where several 50-MW units experienced packing rubs by just coasting down through the resonant ranges of structural-steel foundations with insufficient system damping.

Sufficient Rigidity. This must be provided to maintain shaft alignment within 0.002 to 0.005 in. (depending on speed) during all operating conditions and over long periods of time. Considering concrete shrink (about 0.006 in/ft during first 6 years), creep (about three to four times static deflection, during first 2 years), soil settling, and temperature changes (expansion coefficient for concrete about same as for steel, 0.65×10^{-3} in/in deg F), this is not an easy task. However, if the structure is designed properly from the dynamic angle, and the recommendations in this paper are followed, there will be little extra consideration required.

Mechanical Strength. This will be covered only so far as dynamic loading is concerned, see "Practical Considerations." Again, if properly designed for dynamic properties, little needs to be added. Resonant foundations, however, can easily be damaged by vibratory stresses.

RESONANT FREQUENCIES AND DYNAMIC RESPONSE, GENERAL CONCEPTS

A system is "resonant" when the frequency of excitation (RPM of a machine, and so on) coincides with the natural frequency of this system. How the system (single degree of freedom) responds to other exciting frequencies is shown in Fig.2; how the components participate is shown in Fig.3 (2). Note that the dominating factor is the spring when exciting below natural frequency, the mass above natural frequency. Damping alone controls vibration at resonance, and its influence is insignificant at any other point.

The major modes of vibration are shown in Fig. 4 for a turbine foundation.

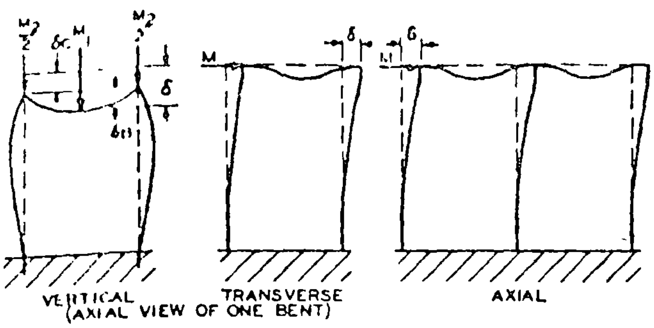
The vertical vibration is usually of the greatest importance and controls the design. Therefore, a foundation which is "tuned low," has its vertical resonant frequency below operating speed. If "tuned high," the vertical resonant frequency is above operating speed.

How a foundation can be regarded as a single-degree-of-freedom system is shown in Fig.1 for vertical vibration. The resonant frequency for such a system is

$$N = 187.7 \left(\frac{1000}{\delta}\right)^{1/2} \qquad (1)$$

where δ is the static deflection (in mils) under the total weight, as shown in Fig.4. This is not an actual deflection in the cases of horizontal vibration, but just a parameter to express the combination of mass (weight) and spring (stiffness) of the structure.

[1] Underlined numbers in parentheses designate References at the end of Part 2 of the paper.



$\delta - \delta_B + \delta_C$

δ_B - BEAM DEFLECTION (SHEAR AND BENDING)

δ_C - COLUMN DEFLECTION

Fig. 4

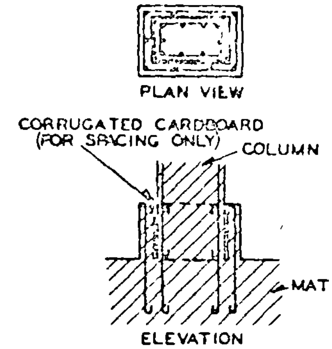


Fig. 5 Arrangement of connecting steel to permit changing of foundation frequency in case of resonance (Rausch, reference 3)

Resonant frequencies of the structure should be clear of the following exciting frequencies, in order of importance:

(a) Operating speed range, clear by ± 20 percent.

(b) Oil-whirl frequency range: Avoid 40 to 55 percent of operating speed range.

(c) Two times operating speed range. Clear by ± 15 percent margin. Harmonic excitation and field reactions of electrical machines occur at this frequency (7200 cpm for 3600-rpm machines).

(d) Frequencies of background vibration, such as caused by other machinery and transmitted through piping and soil. Try to avoid by ± 10 to ± 20 percent margin, depending on severity and provisions for vibration isolation of the new structure.

(e) Rotor critical speeds. Try to avoid by 10 percent where possible. It is often unavoidable to have foundation resonance and rotor criticals in the same area. This will not result in an extremely severe peak but rather cause two peaks instead (1).

(f) On bed rock or with foundation mat reaching into or near ground-water level: Tune low, or set mat on elastic material (cork, rubber, and the like, which must be contained in a separate trough to eliminate pumping action of mat in water). See (3) for examples. This is required to prevent vibration transmission over long distances.

Evidently, there is only a rather limited choice concerning location of foundation resonant frequencies. This narrows down even further when design limitations, such as height, cost, and space requirements are taken into account. The design is practically dictated by these circumstances, often to the extent that there is only one reasonable solution. Practical rules and graphs to find this solution are given in Part 2 of the paper in outline form.

PRACTICAL CONSIDERATIONS

Limitations of Calculation Procedure

From the foregoing it will be realized that a rigorous analysis in the mathematical sense is quite impossible for practical reasons and that we are only trying to predict the general area in which resonant frequencies can be expected. This is, however, extremely valuable for all practical purposes, provided an eye is kept on the assumptions on which the calculation is based, and provided sufficient margin is left for inherent inaccuracies. When these points are observed, it will be noted that the calculation, given in Part 2 of this paper, can predict resonant frequencies with good accuracy, considering all circumstances. The secret is to avoid anything that looks cramped and unreasonable, and to keep the lines of the design as clean and simple as possible.

Tuning After Erection

It is often advisable to provide means for tuning the foundation at a later date, in case of errors or faulty construction. Boots may be provided around the column feet as shown in Fig. 5. In case of resonance, the concrete of the boot can be broken down, and vertical steel connections are then available for building up the column with a larger cross section. More devices for tuning are shown in (3). Some builders let the extra steel stick out of the mat until the foundation has been vibration-tested and the unit is in operation, and then the steel is cut off and the basement grouted. Another possibility is to provide room and reinforcement for additional concrete in the top slab (must check stresses and soil loading).

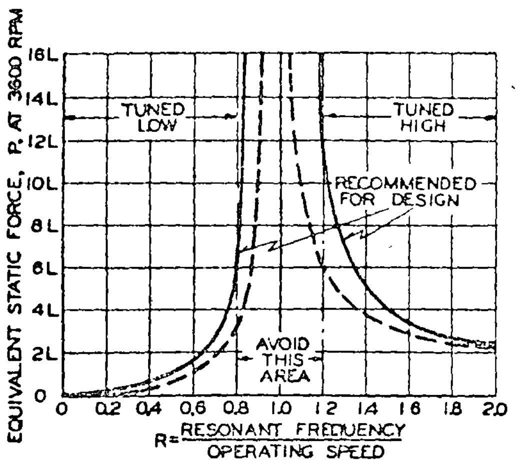


Fig. 6 Dynamic forces

Vibration Joints or Air Gaps

These should be used between foundation and building structure, to prevent transmission of vibration to and from the machine. A 1/2 to 3/4-in. mastic joint is usually provided to separate the mat from the basement floor. Air gaps (1/2 in.) are provided at the operating and intermediate floor levels. Rubber or cork joints (must be quite soft) should be used at stairs, rails, and so on, and also under grating, instrument panels, and steel plates on the foundation, to prevent rattling and drumming. Piping should be spring-supported. If on bed rock, gap should still be provided around the mat, and so forth, to prevent strains in mat.

Overhanging Platforms (Catwalks)

These should be avoided because they can easily become resonant. Where unavoidable, strong ribbing should be provided and platform should be of substantial thickness relative to its overhanging length.

Stresses and Strength

The magnitude of dynamic forces in the foundation depends mainly upon two factors: Closeness to resonance, and balance of rotor, in conjunction with rotor weight L. Damping does not normally enter into the picture since it is only effective at resonance, Figs. 2 and 3. The curves in Fig 6 show both the actual dynamic load and the recommended de-ign load for the structure. This load P is expressed as a multiple of the rotor weight and it takes the reduced strength of the material in fatigue loading into account (by a factor of 3).

This load can now simply be applied as a static load, in addition to the true static loads. Hence the term "Equivalent Static Force " Distribution of this load can be assumed to be the same as for shaft-bearing reactions. This force acts in both the horizontal and vertical directions

The chart is calculated for 3600 rpm, but may also be used for other speeds by applying a factor to rotor weight L:

$$L' = L \times \frac{n}{3600}; \text{ where } n = \text{operating speed}$$

These data have been adopted from German Industrial Standards, DIN 4025 "Supporting Structures for Rotating Machinery (especially Table-Type Steam-Turbine Foundations)." The curve is based on maximum allowable rotor vibration as recommended by turbine manufacturers.

Thermal Stress and Distortion, Shrinkage Stresses. Assume a temperature difference ranging from -30 to +10 deg F between top slab and mat, the slab varying between these limits with respect to the mat, creating cycling, biaxial bending stresses in the columns, for which reinforcement must be provided. In addition, assume inside of top slab bay to be 35 deg F hotter than outside, due to radiation, and provide reinforcement for the resulting (cycling) bending stresses.

To reduce thermal distortion, special care must be taken to protect the structure from radiation and uneven heating and cooling, especially at columns and top slab. Steam lines passing in the vicinity should have full insulation (even small lines) and heat shields (stainless steel on asbestos) should be provided on the concrete where lines pass close to the structure Long runs parallel and close to the top slab or columns must be avoided.

It is extremely important to minimize thermal distortion since much operating trouble has been caused by this source. For example, heating or cooling a 20-ft concrete column only 5 deg F (open door, and so on) will distort the top slab by 7.8 mils, which could cause vibration of the unit. Coefficient of expansion for concrete is about the same as for steel (0.65 mils per in. per 100 deg F).

Stresses due to expansion of turbine, generator, or compressor cause cycling bending in slab and top beams, in addition to tension-compression in top beams, especially the middle beam, and bending in the front beam. Here, as always with turbine foundations, it must be kept in mind that the limiting criterion of design is deflection of

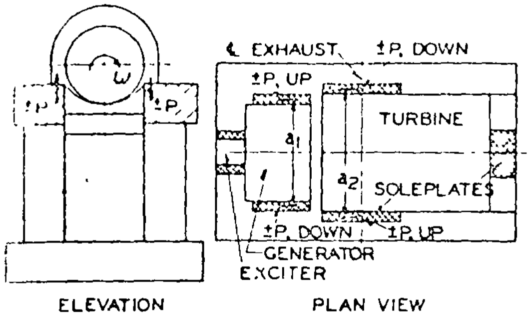


Fig. 7

only a few thousandths of an inch, not ultimate strength.

Short-circuit torque causes sudden cycling load in the top slab. Use 30 x nameplate torque, applied at turbine and generator soleplates. There will be no "overturning" moment in the foundation because turbine casing torque will oppose generator stator torque, but the top slab must be dimensioned and reinforced to withstand the resulting stresses without significant deflections or cracking.

There are two frequencies involved here: One force varies with system frequency, the other with twice system frequency. Superposition of the two forces leads to the value of 30 x torque (3) including a factor for impact, but no significant safety factor.

Nameplate torque: $M_t = \dfrac{7040 \times (KW)}{n}$ ft-lb

$$M_t = \frac{5250 \times (HP)}{n}$$

Short circuit: $P = \dfrac{\pm 30\, M_t}{a}$, lb

Force at Soleplates:

n = speed, rpm

kw = nameplate rating of generator

a = distance between supports, ft, Fig.7

It will be realized that stresses and deflections may be amplified by resonant conditions in the top slab. Since some of these frequencies would be very difficult to predict, a generous factor of safety is indicated. Conditions similar to short circuit may be experienced when the generator is synchronized more or less out of phase with the rest of the system. Experience indicates that this must be expected to happen a certain number of times during the life of the unit.

Vacuum pull and piping forces must be considered in the stress analysis, but not in the resonant-frequency calculation. Vacuum pull exists only where an expansion joint is used, for example at the turbine exhaust. The top slab (or baseplate, if used) must not only be able to withstand the resulting stresses, but the resulting deformations must be kept within 1 to 5 mils (thousandths of an inch), depending on unit RPM, to prevent vibration spells during start-up, peaking, and emergency conditions (noncondensing). These forces do not represent vibrating masses, and therefore they will not affect the resonant frequencies of the foundation.

Concrete strength specified should not be less than 3000 psi at 28 days for which the dynamic modulus of elasticity has been found to be around 4.3×10^6 psi (this figure has not much to do with the static E-modulus). Where higher concrete strength is used, it can be assumed that the dy-

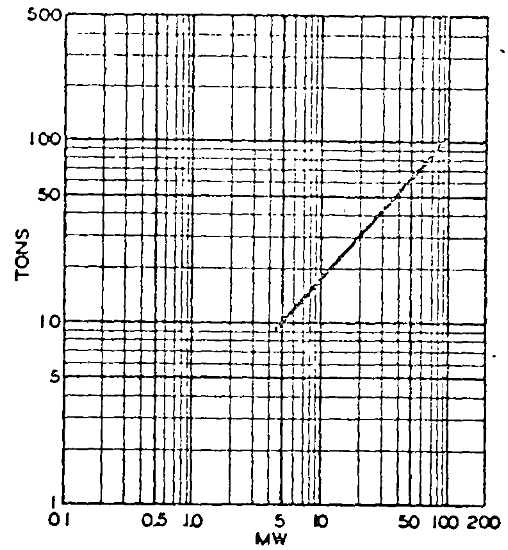


Fig. 8(a) Concrete required

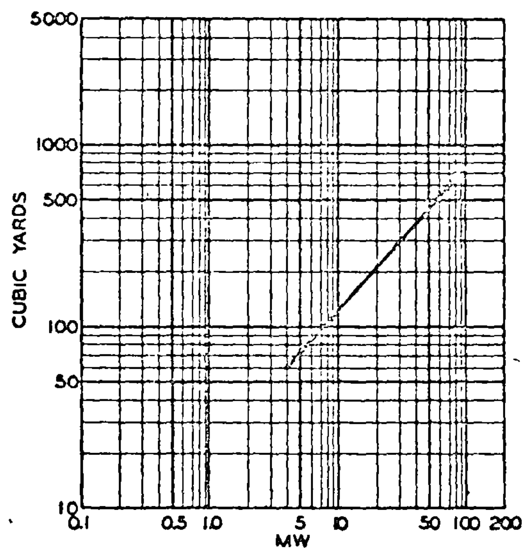


Fig. 8(b) Reinforcing steel required

namic E-modulus increases, but not proportional to the strength (6). Therefore, higher strengths should be used with caution when tuning low, while for rigid design (tuning high), a higher strength may give some additional margin in resonant frequency. The 4.3×10^6 figure includes reinforcement.

When designing reinforcement it must be kept in mind that the structure will be exposed to thermal, dynamic, and settling stresses and that, therefore, the standards used in building design are insufficient for high-speed machinery foundations.

Stress Levels. Reinforcing

Stress levels in columns are usually quite conservative, seldom exceeding 120 psi (static loading only). Reinforcement is much heavier than for other structures, especially in the columns. The mat should be heavily reinforced all around.

Approximate concrete and reinforcing steel requirements are shown in Fig.8, these should be used for estimating purposes only. Considerably more concrete may be required where attempts are made to shift resonant frequencies to higher levels where tuning low would normally be indicated. Also, turbine-compressor units often require more material, up to twice the amounts shown. Curves are adopted from reference (3), and have been compared and slightly modified with domestic data.

Construction should be in accordance with applicable Codes, keeping in mind that requirements for this type of foundation are entirely different from what is customary for building construction. The contractor should have experience in this field, or should at least be aware that special knowledge and procedures are required. Most of the unacceptable foundations are a result of the inexperience of the contractor with the special requirements involved. The contract specifications should be written accordingly.

Joints. Most difficulties occur at the construction joints, which are often not properly prepared and bonded. Without good bond, cracks may open at these locations due to vibration. These cracks may lead to complete separation at the joint due to relative vibration, which is pounding the concrete (laitance is often present in such joints) to a fine powder. The load is then transmitted through the reinforcing steel which, of course, has different elastic properties than a sound concrete section. The resonant frequencies of such a structure are indeterminate. It can be expected that the new foundation will have frequencies close to the calculated ones. As the separation progresses, new frequencies may show up and shift, as will the original ones, until finally vibration becomes excessive. The process is aggravated by thermal and settling stresses. It is then usually extremely difficult to correct the situation, especially when the joint at the bottom of a column is involved.

Although this is often not recognized, good bond can be obtained by proper procedure. Tests of joints (5) have shown tensile, shear, bending and compressive strengths almost equal to or exceeding the strength of the basic concrete. Proper preparation is, of course, essential. For above tests, the concrete was roughened while still fresh, but after partial setting, and the top layer (laitance) removed with a strong water jet. The surface should be kept wet as long as possible. Before pouring is continued, a single or double layer of cement-water paste (consistency of paint) is brushed in. Then, about 15 to 20 in. of concrete is placed on the joint and carefully worked and vibrated into the surface before pouring is continued in the usual manner. It appears that cleanliness and strict adherence to all phases of the procedure are of critical importance. A more expensive, but probably safer method is described in (6). Good results were obtained with the following procedure:

Prepare joint by chipping old concrete-1 to 2 in. deep.

Keep wet for several days before continuing.

Remove all water and foreign material.

Brush on cement paste (2:1).

Place several inches of mortar on joint (1 to 6 in.) depending on height of next lift. Consistency of paint, or at least 6 in. slump. Work around.

Continue pour

See also reference (7).

Grouting of top slab (floor) calls for basically the same procedure, except that chipping is usually limited to 1 in. The effect of poor bond can be studied on many installations where the floor came loose, or buckled under vibratory and thermal stresses. Raking of the fresh concrete is not sufficient to guarantee bonding. Above all, laitance and watery concrete must be removed by chipping.

Location of Construction Joints. These are usually at top and bottom of columns, to facilitate placing of concrete and to prevent shrinkage cracking. Large foundations have been poured in one continuous pour from the mat up, a practice which calls for very rigid planning and supervision to be successful, in addition to a generous amount of equipment (trucks, cranes, vibrators) and labor. Maximum time limits for the completion of the job must be established, to prevent partial setting of concrete while it is still being

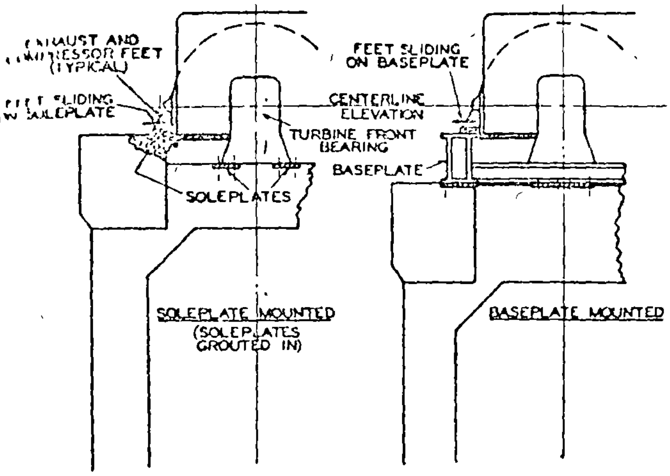


Fig. 9

worked. Such continuous pours should not be interrupted for lunch, concrete delivery, and so on, and should be finished in about 8 hr or less. Special attention must be paid to prevent separation, excessive concrete drop, over or under-vibration, seams or nonuniform sections due to partial setting while concrete is placed at other locations. Extra fillets and reinforcement are required to prevent shrinkage cracking where the columns join beams or the top slab. Simultaneous pouring at several locations will usually be required.

Baseplate-mounted Units. Baseplate-mounting of turbine-compressor units is only advantageous for units of smaller size. Beyond certain physical dimensions, soleplate mounting is more reliable, more economical, and easier to design and install. Speed of machine, weight of rotor, size of piping, operating temperatures, critical speeds, weight of foundation--all enter into the picture.

To demonstrate the limitations of baseplate construction, consider a 24,000 hp turbine-compressor; speed range 3000 to 4200 rpm, stiff shaft design. Such a machine would be about 32 ft long, weigh about 200,000 lb. Assume foundation height of about 29 ft (top of mat to operating floor). Fig.9 shows the two possible designs. As can be seen, for soleplate mounting, lines are much cleaner, appearance and accessibility better, installation easier, and cost lower. Even more important is the shorter and better defined path of force between casing supports and concrete (force-mass connection, with fewer joints. The effect of baseplate flexibility would be very difficult to include in the frequency calculation (in fact, an additional resonant frequency will be introduced because the baseplate represents a spring between the unit and top slab masses.) Thermal gradients between baseplate and concrete due to radiation and conduction from turbine and compressor casings, piping, and so on, can introduce very serious warpage and cracking problems. Furthermore, because of the additional top slab height, it will be found difficult to provide enough beam between turbine and compressor and to provide enough clearance for piping and condenser. The effective column height will be reduced, making it difficult to get the vertical foundation resonant frequency below the operating range (while tuning high is virtually impossible because of the height of the structure).

Effect of baseplate flexibility on critical speeds can also be very serious (1,2,8,9), especially for large, high-speed, stiff-shaft machines, as the one quoted above.

Foundation design is basically the same for both types of mounting. The concrete mass ratios should still be observed, and strength and rigidity should still come from the foundation, not from the baseplate, because any stress in the baseplate will cause deflection and consequently misalignment and possibly operating difficulties, unless the baseplate is specifically designed for column mounting. See references (10,11,12,13,14) for applicable standards.

Levelling and grouting procedures are similar for both types of mounting.

Practically all direct-drive turbine-generator units are mounted on soleplates.

As can be seen from the foregoing, baseplates, while advantageous during transit and erection for smaller units, have very definite limitations. Also, costs of baseplates are often as high, or higher, than the cost of a good concrete foundation.

Clearance Requirements

Provide ample clearances for installation of all equipment, including piping; access for operation and space required for dismantling during maintenance operations. Minimum clearance between machinery (flanges) and foundation is 1.5 in.

NOMENCLATURE

- A = cross-sectional area of structural member, sq ft; also soil contact area
- b = smaller side of column or beam, ft
- d = mat or beam thickness, ft
- E = dynamic modulus of elasticity, psi; 4.3×10^6 for reinforced concrete
- f = static deflection of a component under its own weight, mils (1 mil = 1/1000 in.)
- G = vibrating lead on one bent, including 1/2 column weight, lb
- G' = vibrating lead on one bent, including 1/2 column weight, horizontal
- H' = horizontal spring constant of one bent, lb/mil
- h = active column height, ft
- I = cross-sectional moment of inertia, ft^4
- k = bent factor, equation (5)
- L = rotor weight, lb
- ℓ = active bent width, ft, also length of beams and slabs
- ℓ_o = open width between columns, ft
- m = beam width, ft
- N = natural frequency of structure, cycles per minute (cpm)
- n = beam height, ft
- P = force or external load, lb
- p = soil pressure, psf
- Q = weight of structural components, lb
- q = uniformly distributed load, lb/ft
- s = center of gravity height, ft
- W = total unit weight, lb; also weight of vibrating and stationary masses
- w = width of mat, ft
- α = bent correction factor, Fig.15
- δ = static deflection of structure or component under vibrating mass loading, to be used for determination of resonant frequencies, mils (1 mil = 1/1000 in.)
- δ' = horizontal static deflection of one bent, mils
- δ_H = horizontal static deflection of structure, mils
- δ^* = indicator of horizontal rigidity, mils

INTRODUCTION

The calculation procedure outlined in this paper is necessarily only in skeleton form, covering only the absolutely vital steps. For refinements, applicable literature such as reference (3) or (15) must be consulted. The method presented here is based primarily on these references.

The method given is for concrete structures, but the same basic considerations hold for structural-steel design, except that the top slab is much lighter, unless filled with concrete.

SIMPLIFIED CALCULATION PROCEDURE, STEP BY STEP, FOR CONCRETE FOUNDATION

Preliminary Assumptions for First Layout

1 Select vertical resonant frequency as follows:

From Fig.10 for given column height (subtract about 3 to 5 ft from total floor height above mat) find column stress level required at 1.2 times maximum operating speed, RPM. If stress level is above 50 psi, try to tune low. If stress level is below 50 psi, try to tune high.

The foregoing is obviously only a rule of thumb, and especially around 50 psi further consideration may be necessary to come to a decision.

Remember: To raise resonant frequency: Heavier columns

To lower resonant frequency: Thinner columns, more weight in top slab.

Select resonant frequency: Tuning high: 1.2 x maximum speed or higher

Tuning low: about 0.6 x operating speed

2 Assume approximate distribution of machine weight per bent (a "bent" consists of two columns and a transverse beam).

3 Assume weight ratio of top slab/unit:

Tuning high: 0.75 to 1.25

Tuning low: 1.5 to 2.25

Do not include: Oil tank, condenser, columns, piping.

Include: Upper beams and slabs. See Fig. 11

4 Assume factor for bent flexibility:

	Based on frequency	Based on static deflection
Tuning high	0.80 to 0.85	0.64 to 0.73
Tuning low	0.90 to 0.95	0.80 to 0.90

NOTE: The higher the columns, the higher the bent flexibility factor.

5 Calculate approximate column dimensions as follows: Column loading consists of all weight supported above 1/2 column height, = vibrating weight (see Fig.11).

Condenser. If hung or spring supported, include condenser weight in vibrating weight, but

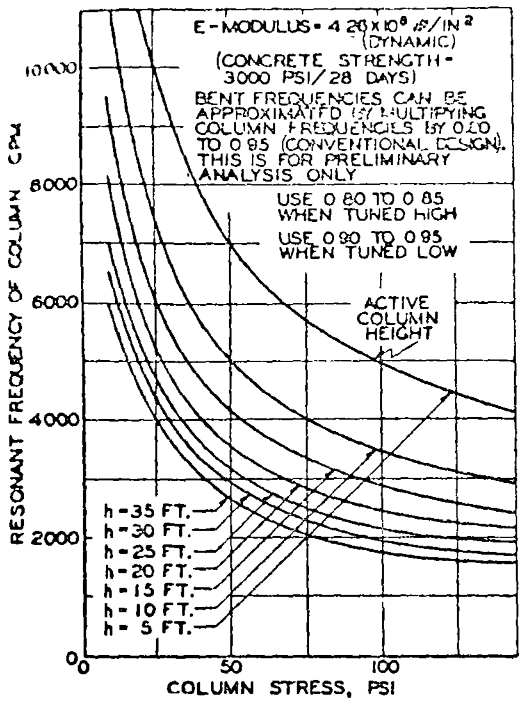


Fig. 10 Vertical-resonant frequencies of reinforced-concrete columns under compressive load

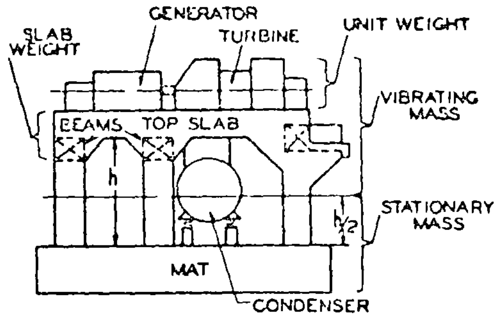


Fig. 11

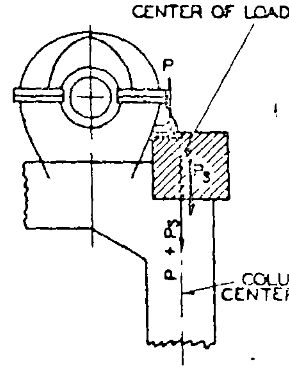


Fig. 12

not in slab/unit weight ratio. If expansion joint is used at exhaust, include condenser weight in stationary mass. Disregard vacuum pull, since it does not represent a mass. Divide desired resonant frequency by bent flexibility factor. Using Fig.11, find corresponding column stress level, using h for column height.

Using weight distribution as assumed above, find required cross-sectional areas of columns.

6 Complete preliminary layout on basis of foregoing figures. Provide generous fillets at long spans and at other points of stress concentration; also provide beams and ribs where necessary. Avoid overhangs and large, unsupported areas (floors). If unavoidable: Provide very generous ribbing, fillets and bracing, to avoid drumming.

Make stationary weight (mat) equal to or larger than vibrating weight. Make top slab as rigid as possible, especially in vertical direction. Arrange column centerlines under center of load, to reduce torsion, Fig.12.

7 Provide heavily reinforced beam between turbine and generator (or compressor), with generous fillets. This beam will be loaded in tension-compression when turbine and generator feet slide in and out with changes of operating temperature. Assume friction coefficient of 50 percent for beam and slab-reinforcement calculation. The same condition exists at the turbine front support, where the force is axial. See Fig.13 for location of friction forces. "P" designates the reaction at the respective support. Note that friction forces on opposite sides will oppose each other, but will cause bending in top slab.

NOTE: P_1, P_2, P_3 are the actual loads on the respective supports, including vacuum and piping loads. Only loads affecting foundation design are shown in Fig.13.

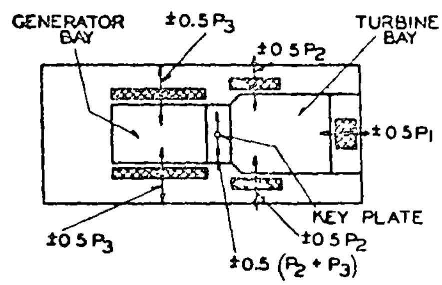


Fig. 13

Keep foundation lines as clean and simple as possible. This will reduce cost, simplify final calculation, and make results more reliable.

Simplified Calculation Procedure, Step by Step

1 Find actual load distribution on columns, using preliminary layout.

NOTE: The top slab should actually be regarded as a vibrating beam on flexible supports, but where design is conventional and the top slab sufficiently rigid as compared to columns, assuming statically determinate load distribution (zero bending moment on column centerline) usually gives satisfactory results.

All bents must have the same resonant frequency (static deflection), within about 3 percent. Otherwise, the calculation procedure presented here cannot be used.

2 Find active column height (Fig.14).

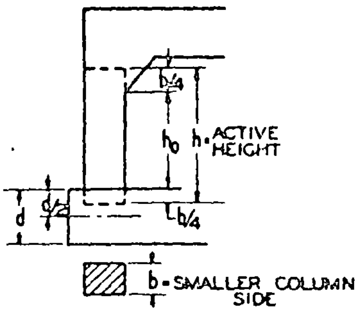


Fig. 14 (Rausch, reference 3)

$$h = h_o + \frac{b}{2} \tag{2}$$

NOTE: The column foot should not be assumed below d/2. If b/4 > d/2, use

$$h = h_o + \left\{\frac{b}{4} + \frac{d}{2}\right. \tag{3}$$

3 Find corrected bent widths for bending deflection (Fig.15).

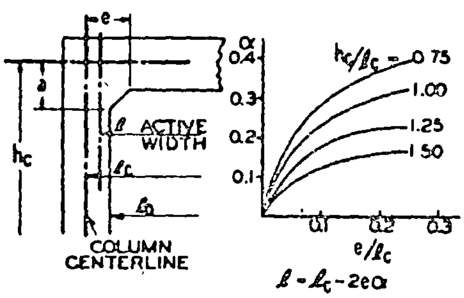


Fig. 15 (Rausch, reference 3)

Vertical Vibration of Structure

4 Distribution of beam load for bending and shear deflections (Fig.16).

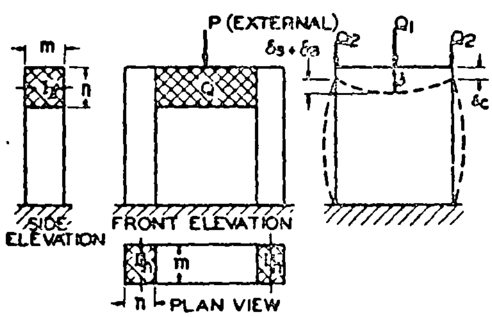


Fig. 16

$$Q_1 = P + \frac{Q}{2}$$

$$Q_2 = \frac{Q}{4} \tag{4}$$

5 Deflections, vertical:

Use bent factor $$k = \frac{h}{\ell} \frac{I_\ell}{I_h} \tag{5}$$

where

h as in Figure 14, feet

ℓ as in Figure 15, feet

I_ℓ = moment of inertia of beam cross-section, ft.^{4},

See Figure 16 ; $= \frac{m\,n^3}{12}$

I_h = moment of inertia of column cross-section, ft.^{4},

See Figure 16 ; $= \frac{m\,n^3}{12}$

Note: Where more accurate prediction of bending deflections is required, reinforcement must be taken into account, see (3).

Bending deflection of beam, mils:

$$\delta_b = \frac{Q_1 \ell^3 (8k+4)}{384\,E\,I_\ell (k+2)} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}}$$

$$= 0.217 \frac{Q_1 \ell^3 (8k+4)}{E \cdot I_\ell (k+2)} \tag{6}$$

Shear deflection of beam, mils:

$$\delta_s = \frac{3}{5}\frac{Q_1 \ell_o}{E\,A_\ell} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} = 50.0 \times \frac{Q_1 \ell_o}{E\,A_\ell} \tag{7}$$

Column compression, mils:

$$\delta_c = \frac{G}{2} \times \frac{h}{E\,A_h} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} = 41.7 \frac{G\,h}{E\,A_h} \tag{8}$$

where

P = external load, lb } See Fig.16, equation (4)
Q = beam weight, lb }

$\frac{G}{2}$ = load on one column, as calculated in step 1, plus 1/2 column weight. = "Vibrating" weight on column, see Fig.11.

E = dynamic modulus of elasticity, psi; use 4.3×10^6 psi for 3000 psi, reinforced concrete

ℓ_o = open width of bent, ft, see Fig.15.

A_ℓ = cross-sectional area of beam, sq ft

A_h = cross-sectional area of columns, sq ft

One half of the beam weight is applied as concentrated load at mid-span, the other half at the two columns.

Bending and shear deflections of top slab: To simplify frequency calculation, design slab for deflections smaller or equal to beam deflections. For rough comparison can use:

Slab bending, mils:

$$\delta_b' = \frac{Q_1 \ell^3}{192\ E\ I_s} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft}}$$

$$= 0.434\ \frac{Q_1 \ell^3}{192\ E\ I_s} \qquad (9)$$

For Q_1 use equation (4) in conjunction with Figs.16 and 17.

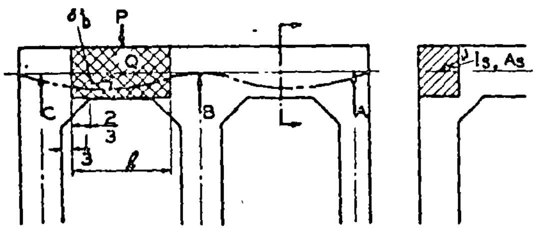


Fig. 17

I_s = moment of inertia about axis shown in Fig.17.

Slab, shear deflection:

$$\delta_s' = \frac{3}{5}\frac{Q_1 \ell}{E\ A_s} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} = 50.0\ \frac{Q_1 \ell}{E\ A_s}$$

$$\delta_b + \delta_s > \delta_b' + \delta_s' \qquad (10)$$

7 Resonant frequencies of bents, vertical:

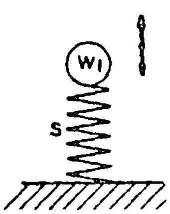


Fig. 18

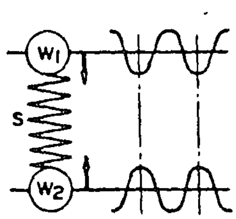


Fig. 19

Assuming mat rigid (Fig.18):

$$N_v = 187.7 \sqrt{\frac{1000 \text{ (mils/in.)}}{\delta_b + \delta_s + \delta_c \text{ (mils)}}} \qquad (11)$$

Assuming mat suspended (coupled vibration) (Fig.19) (3)

$$N_v' = N_v \sqrt{\frac{W_1 + W_2}{W_2}} \qquad (12)$$

The actual resonant frequency will be found between N_v and N_v', depending on soil characteristics. For $W_1 = W_2$:

$$N_v' = N_v \sqrt{2}$$

NOTE: For this simplified analysis it was assumed that all bents have equal resonant frequencies; it is therefore absolutely necessary to design the bents accordingly.

The entire resonant frequency range as defined by N_v and N_v must be either 20 percent above or 20 percent below the speed range of the unit. For other frequencies to be avoided see Part 1 of this paper.

Horizontal Vibration of Structure

8 Deflections, horizontal:

To get simple equations, assume slab to be stiff in horizontal direction, design accordingly.

All bents should have approximately equal horizontal stiffness, as expressed by δ (Fig.20).

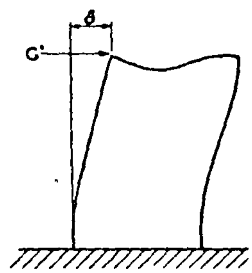


Fig. 20

Spring constant of each bent:

$$H' = \frac{6EI_h}{h^3} \times \frac{1 + 6\ k}{1 + 1.5\ k} \times \frac{12 \text{ in./ft.}}{1000 \text{ mils/in.}} \qquad (13)$$

$$= 0.072 \times \frac{EI_h\ (1 + 6\ k)}{h^3\ (1 + 1.5\ k)}$$

where

H' = horizontal spring constant of one bent, lb/mil

E = dynamic modulus of elasticity, psi (4.3×10^6 for 3000 psi concrete)

I_h = see Fig.16

$h = h_c - a\ \alpha$ see Fig.15

k = see equation (5).

Deflection of each bent; mils:

$$\delta' = \frac{G'}{H'} \qquad (14)$$

Deflection of whole framework; mils:

$$\delta_H = \frac{\sum G'}{\sum H'} \qquad (15)$$

where

G' = total weight carried on a bent (lb), including 1/2 column weight, see Fig 11.

$\delta' ; \delta_H$ = static deflections, horizontal, if vibration loads were applied in horizontal direction. This deflection represents dynamic stiffness only, there will be no actual horizontal deflection. As a measure of adequate stiffness check:

$$\delta^* = 0.5\,\delta_H \times \frac{\text{Weight of unit alone}}{\sum G'} \qquad (16)$$

δ^* should be less than 15 mils (using dynamic E-modulus). Provide reinforcement for loading accordingly.

9 Horizontal frequency of foundation (cpm):

Mat rigid:

$$N_H = 187.7\sqrt{\frac{1000}{\delta_H \times (0.8 \text{ to } 1.0)}} \qquad (17)$$

Mat suspended:

$$N_H' = N_H \sqrt{\frac{W_1 + W_2}{W_2}} \qquad (18)$$

For explanations see equations (11) and (12)

NOTE: Horizontal frequencies are usually safely below operating speed, but other dangerous frequencies must be avoided.

Component Frequencies

Basic frequency equation for single-degree-of-freedom systems:

$$N = 187.7 \sqrt{\frac{1000}{\delta}} \quad \text{(cpm)}$$

Formulas below give δ for uniformly distributed load, any material, and also for reinforced concrete with $E = 4.3 \times 10^6$ psi

- f = static deflection, mils
- δ = static deflection, mils, corrected for dynamic load distribution
- d = beam thickness, ft.
- ℓ = beam length
- q = weight per unit length, lbs/ft

Component Frequencies:

1 Both ends fixed (Fig.21):

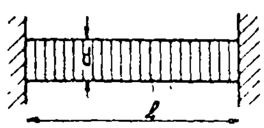


Fig.21

$$N_1 = 709{,}000\,\frac{d}{\ell^2} \quad \text{(For concrete only)} \qquad (19)$$

$$f = \frac{q\ell^4}{384EI} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} \qquad \delta = 0.77f$$

2 Both ends simply supported (Fig.22):

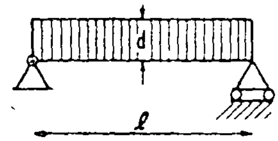


Fig.22

$$N_1 = 313{,}200\,\frac{d}{\ell^2} \quad \text{(Concrete only)} \qquad (20)$$

$$f = \frac{5\,q\ell^4}{384EI} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} \qquad \delta = 0.79f$$

$$N_2 \approx 4N_1 \qquad N_3 \approx 9N_1 \qquad N_4 \approx 16N_1$$

Most beams and columns cannot be regarded as either fixed or simply supported. For these cases, experience has indicated a value of 450,000 to 500,000:

$$N_1 \approx 470{,}000 \times \frac{d}{\ell^2} \quad \text{(Concrete only)}$$

$N_2 \approx 4 \times N_1$ Should provide for at least $\pm$ 30% when using this formula

3 Cantilever Beam (Fig.23):

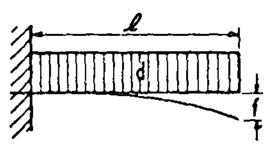


Fig. 23

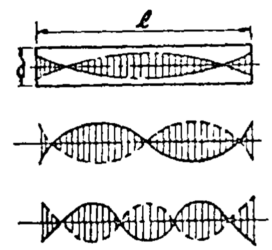


Fig. 24

$$N_1 = 111{,}500\,\frac{d}{\ell^2} \quad \text{(Concrete only)} \qquad (21)$$

$$f = \frac{q\ell^4}{8\,EI} \times \frac{1000 \text{ mils/in.}}{12 \text{ in./ft.}} \qquad \delta = 0.65f$$

4 Free Beam (Fig.24):

Concrete Only. (22)

1st mode: $N_1 = 0.709 \times 10^6 \times \frac{d}{\ell^2}$

2nd mode: $N_2 = 1.995 \times 10^6 \times \frac{d}{\ell^2}$

3rd mode: $N_3 = 3.83 \times 10^6 \times \frac{d}{\ell^2}$

General: $$N = \beta^2 \times 187.7 \sqrt{\frac{EI}{q\ell^4}} \times 12 \text{ in./ft}$$

where

β^2 = 22.38 first mode
61.65 second mode
120.9 third mode

5 Beam evenly supported by springs (Fig.25):

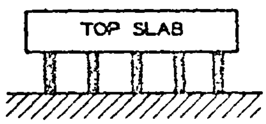


Fig.25

$$N = \sqrt{N_v{}^2 + N^2{}_1} \quad (23)$$

where

N = combined frequency
N_v = vertical frequency of springs alone, assuming slab infinitely stiff
N_1 = bending frequency of slab alone (equation 22) first mode only.

This equation can be used to find bending frequencies of top slab.

Approximate Resonant Frequencies of Foundation on Soil

These frequencies must be checked where the superstructure is tuned high, and also for solid (block-type) foundations.

Fig.26* further simplifies the calculation as given by Rausch (3). However, strict adherence to the following assumptions is mandatory when using this chart:

Center of gravity of load must be in vertical line with the geometric center of the load-carrying area, to get even soil pressure distribution.

Ground water table not to extend higher than 1/2 mat width below mat.

Foundation must be separated from surrounding structures and floors by a gap filled with soft, plyable material (mastic).

Soil must be reasonably uniform and undisturbed

Furthermore, it is highly recommended that provisions for more uniform soil loading be made (precompression of soil by temporary overloading of foundation, using ballast; relief of center of loaded area, imbedded pipes for grout injection after settling). Details about such provisions can be found in (3), for example.

Soil Damping. Increases with contact area and inversely to soil loading. For areas exceeding approximately 250 sq ft and light soil loading, damping is usually sufficient to prevent significant vibration amplification. However, resonance should still be avoided because of the high levels of energy transmitted into the soil and the consequent settling and vibration transmission. If resonant conditions cannot be avoided, dynamic forces must be applied according to Fig.6.

Resonant Frequency Calculation, Simplified. The method given here gives only a rough estimate. For more precise and detailed calculation, the literature must be consulted (3,15,16).

Modes of vibration:

Vertical.

Torsional, about vertical axis through center of gravity.

Horizontal. There are 4 horizontal modes, two sideways and two lengthwise. They are actually rolling motions, one about an axis below the center of gravity, the other about an axis above the center of gravity, in each direction.

Calculation Procedure, step by step:

1 Calculate contact area A of foundation, make sure vertical center of gravity axis of all weights intersects center of contact area. Find $\sqrt{A}$ (ft).

2 Calculate soil loading:

$$p = W/A \text{ (psf)}$$

where W = total weight supported on A, lb.

3 Calculate vertical distance s (ft) of center of gravity with respect to A, see Figs.26 and 27.

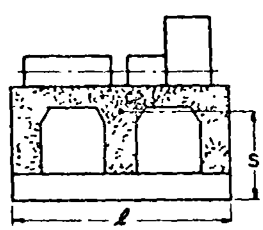


Fig.27

4 Calculate ratios ℓ/s and w/s, where
ℓ = length of base area (A), ft

* Because of exigencies of makeup of paper Fig.26 appears as full page plate on the following page. Fig.27 is then with its text reference.

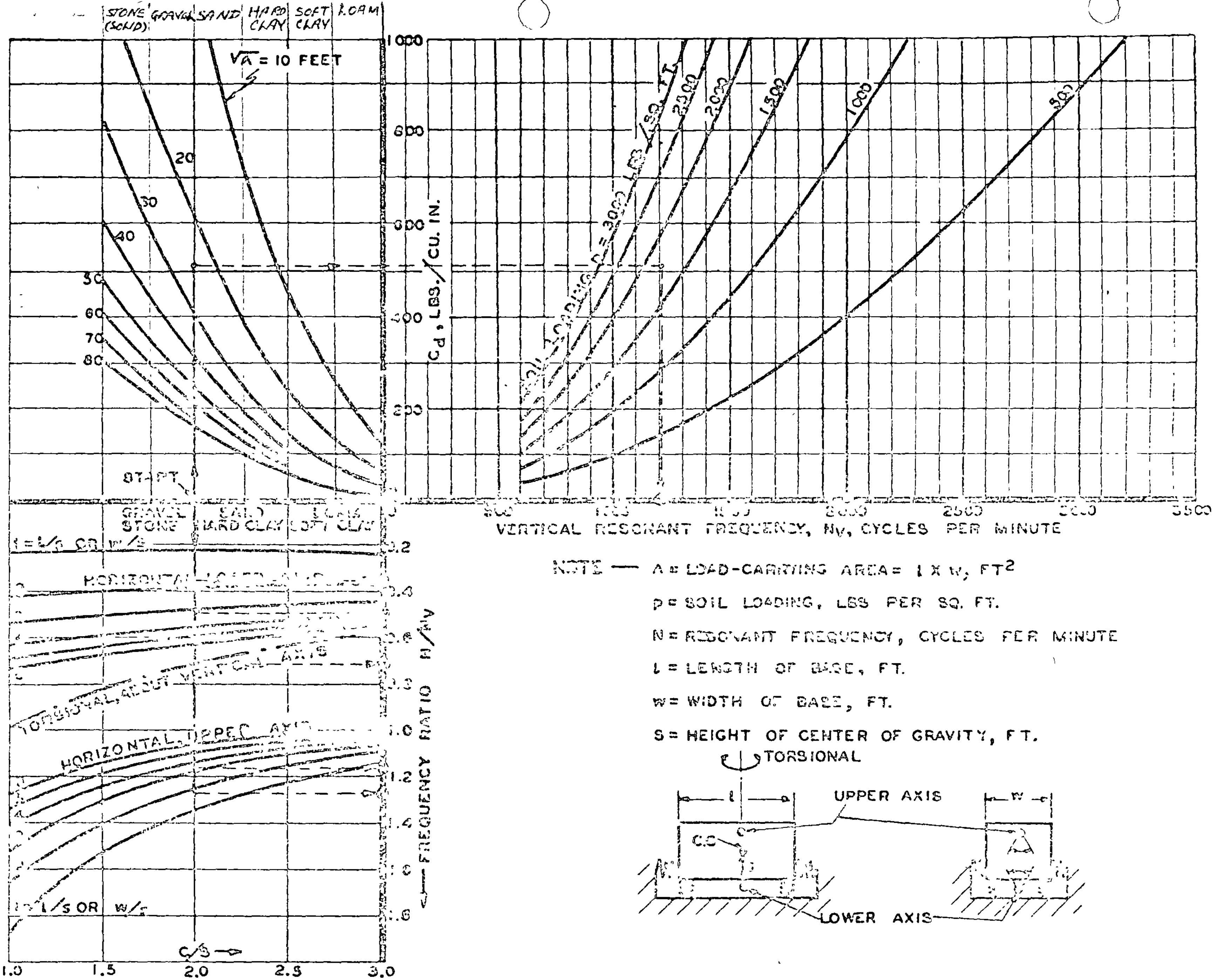


Fig. 26 Resonant frequencies of foundation on soil

w = width of base area (A), ft

s = height of center of gravity over A, ft

5 Use Fig.26 to determine resonant frequencies as shown, starting on the left side horizontal axis. To find vertical frequency, go up to $\sqrt{A}$ line, then horizontal to p-line (disregard c_d scale, which is for reference only), then go down and read vertical resonant frequency N_V

6 To find horizontal and torsional frequencies, go back to starting point, then go down to the ℓ/s (or w/s) lines for the various modes (horizontal, about lower axis, torsional, horizontal about upper axis). Read corresponding frequency ratio N/N_V for each mode

7 Multiply N/N_V ratio by vertical frequency N_V (found in Step 5), to get horizontal and torsional frequencies in cycles/minute.

Note: When selecting the starting point on the soil scale, it must be kept in mind that the plastic and adhesive characteristics of the soil are important for the horizontal and torsional vibrations, as expressed by the compression/shear ratio (C/S). Nonbinding soils (sand, gravel) have a C/S ratio around 1.5, while binding soils (clay, loam) have ratios around 3. This should be considered when selecting the starting point in Fig. 26. The coordination of soil type and C/S in Fig. 26 is an approximation. If better data are available, the torsional and horizontal frequencies should be found for the correct C/S value, for which a scale is shown below these curves.

All frequencies may vary as much as ± 20 percent because of variations of soil, settling displacement, precompression, and other unknowns.

Example

1 Foundation length: ℓ = 33 ft

Foundation width: w = 18.5 ft

Loaded area: A = 33 x 18.5 = 610 sq ft

$\sqrt{A}$ = 24.7 ft

2 Total weight, foundation and machinery:

W = 1,080,000 lb

$$p = \frac{W}{A} = \frac{1,080,000}{610} = 1,770 \text{ psf}$$

3 Center of gravity locations: s = 9.8 ft above loaded area

4 ℓ/s = 33/9.8 = 3.37, w/s = 18.5/9.8 = 1.89

5 Type of soil: Primarily sand and gravel, with some clay.

From Fig.26: N_V = 1206 cpm

Horizontal, sideways (w/s = 1.89):

About upper axis: N/N_V = 1 26; N = 1205 x 1.26 = 1520 cpm ±20%

About lower axis: N/N_V = 0.36, N = 1205 x 0.36 = 434 cpm ±20%

Horizontal, lengthwise(w/s = 3.37):

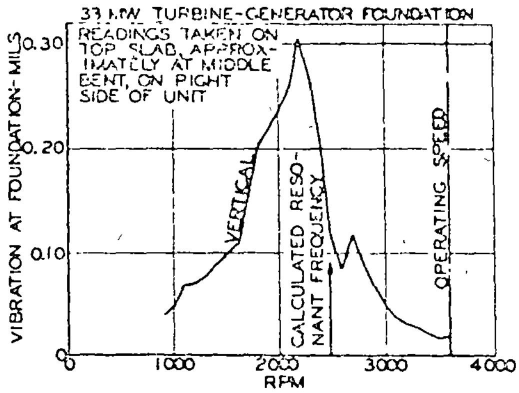


Fig. 28 Foundation resonance spectrum

About upper axis: N/N_V = 1.16, N = 1205 x 1.16 = 1400 cpm ±20%

About lower axis: N/N_V = 0.51, N = 1205 x 0.51 = 615 cpm ±20%

Torsional (about vertical axis):

N/N_V = 0.72; N_T = 1205 x 0.72 = 868 cpm ±20%

This foundation would not be suitable for a 1200-rpm machine.

Summary of Stress Calculation

1 Check horizontal stability S*, using formula (16).

2 Calculate vertical and horizontal stresses including dynamic forces, Fig.6, vacuum pull and piping forces, short circuit torque.

3 Add shrinkage and thermal stresses.

4 Add unit expansion stresses.

5 Consider seismic conditions (earthquake).

6 Consider settling stresses.

7 Determine soil loading. If foundation is tuned low and not resonant on soil (soil resonance usually in the 900 to 1500-cpm range,) only 1/2 of the dynamic forces, Fig.6, need to be applied to the soil loading. If tuned high, check soil frequencies and apply dynamic forces using closest resonant frequency.

VIBRATION TEST RESULTS

Figs.28 to 30 show some typical test results obtained by varying the speed of the unit and plotting vibration versus speed, either by hand or automatically, using an X-Y plotter in conjunction with a tachometer (for the X-axis) and a vibration analyzer (for the Y-axis). These plots show where points of resonance are located and, naturally, this includes the critical speeds of the machines In questionable cases the latter may be eliminated by using a variable-frequency shaker and testing the foundation with the rotor removed.

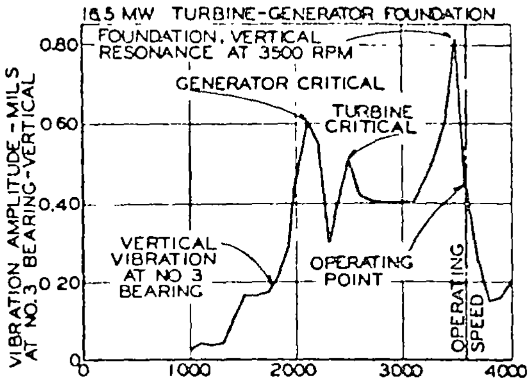


Fig. 29 Resonance spectrum

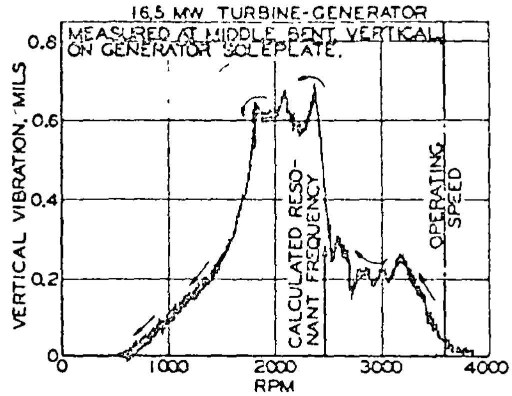


Fig. 30 Foundation resonance spectrum

The first plot, Fig.28, shows the vertical response of a properly designed foundation for a 33-MW unit. The vertical peak shows up very strong at the predicted frequency for an infinitely stiff mat (this mat was 6 ft thick), and this seems to hold for most tests run so far. The small peak at 2700 cpm could be either one of the critical speeds of the machine or the effect of finite mat mass, both can be expected in this area. The criticals are obscured by the much stronger foundation resonances, which often fall into the same speed range. It can be seen that the operating speed is at a favorable point on the spectrum, as it should be. Horizontal and axial vibrations were negligible, and for this reason are omitted here.

The foundation in the next plot, Fig.29, was not designed according to the principles outlined here, but rather by the old rule of thumb, trying to make the structure rigid, without checking natural frequencies. The resonant frequencies show up very close to operating speed, at 3500 cpm. Here the rotor criticals can be clearly identified. It is obvious that this machine will always run with 3 to 5 times the vibration it would normally have on a properly tuned foundation, and that this will be a tricky machine to handle if it should ever pick up some minor unbalance due to temporary thermal distortion (during startup, peaking, rapid load change, tripout, and so on) contamination or erosion, or during emergencies (load dump, quick start, blade failure, packing rub, slug of water). Even so, we are very fortunate indeed that the peak drops off so sharply. Otherwise the vibration could easily be 8 to 12 times normal. It is interesting to note here that, although this design was impossible to calculate accurately (walls, unequal column deflections, and the like), the bent frequencies calculated, roughly, as follows:

Front:	3644 to 4795
Middle:	2428 to 3195
Rear:	2767 to 3640

The sharp drop between peaks is often characteristic between two adjacent resonant frequencies. It should be remembered here that actual behavior is not easily predicted, because of the coupling effect of the top slab, which is probably responsible for the raised middle bent frequency.

For comparison, Fig.30 shows a resonance plot of an identical machine, but this time supported on a foundation designed in accordance with the rules given in this paper. It can be seen that, again, the main vertical peak comes in strongly at the predicted frequencies, together with the rotor criticals. There is again a somewhat puzzling secondary peak at about 3000 cpm. This peak, this time, cannot represent a rotor critical because these are lower (and well known from calculation and shop test). Although these foundations (there are two identical installations) are very satisfactory, the vibration level could still be cut in half by shifting this secondary peak downwards by about 300 cpm.

It may seem exaggerated to try to improve on a machine with only 0.05-mil vibration, but again it must be remembered that cutting vibration in half means one half, be it of 5 mils or 0.05 mils, and the reliability of the machine will improve proportionately when, under emergency conditions, vibration may reach higher levels. Therefore, this secondary peak is now under investigation. Until we can find other means to shift it downscale, it is recommended to select the basic frequency around 0.58 to 0.61 times operating speed, even if this sometimes means more mass in the top slab We would probably be operating exactly at the bottom of the vibration curve (now at 3850 cpm) if this had been done on this foundation.

Comparison of Fig 29 and Fig 30 would not be

complete without a comparison of construction costs for these foundations. Here, we were told that costs for a tuned foundation are somewhat lower than for the conventional design, mainly because of the more elaborate form work required for the latter. Concrete and steel requirements are about equal.

ACKNOWLEDGMENT

I wish to thank Mr. J.H. Mulholland of the Elliott Company for his thorough review of this paper.

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1. INTRODUCCION

Se presentan en este informe los resultados del estudio del subsuelo realizado por Geotec, S.A. tendiente a determinar el tipo de cimentación más apropiado para una compresora que Multi-Tek, S.A. proyecta instalar en su planta ubicada en el Fracc. Industrial Naucalpan, Edo. de México.

El estudio se basó en las características y propiedades del subsuelo en el sitio, determinadas en base a un sondeo mixto y a los ensayes de laboratorio efectuados, así como en las características de funcionamiento y descargas de la máquina por cimentar, proporcionadas por el fabricante.

En los incisos 2 a 5 del informe se describen las características del sitio y de la máquina, los trabajos de campo, los ensayes de laboratorio y la estratigrafía y propiedades del subsuelo. En el Inciso 6 se analiza el tipo de cimentación más conveniente; finalmente, en los incisos 7 y 8 se presentan las conclusiones derivadas del estudio y las recomendaciones para el diseño y construcción de la cimentación, estos últimos de carácter general.

2. DESCRIPCION DEL SITIO Y DE LA MAQUINA

La máquina se proyecta instalar bajo una mezzanine localizada en el interior de una nave tipo industrial. La nave, estructuralmente consiste en una armadura metálica apoyada en columnas de concreto reforzado y presenta las dimensiones mostradas en la Fig. 1. Debe señalarse que se desconoce el tipo y características de su cimentación, datos que no pudieron ser proporcionados por Multi-Tek, S.A. ni por la compañía a cuyo cargo estuvo su construcción.

En cuanto a las características de la máquina por cimentar, éstas fueron proporcionadas por el fabricante y son las que se señalan a continuación: se trata de un compresor tipo XLE 7" carrera, no lubricado, tamaño 17 y 10 x 7, de dos pistones, 200 H.P., que pesa aproximadamente 4.4 ton y ocupa en planta una superficie de unas 3.3 x 1.6 m. Su frecuencia de operación es de 750 RPM para las fuerzas primarias, las cuales, al igual que los momentos

primarios y secundarios, son nulos. Para las fuerzas secundarias la frecuencia es del orden del doble de la mencionada anteriormente y de una magnitud igual a 1.1 ton. En la Fig. 2 se muestran vistas frontal, lateral y planta de la máquina.

3. TRABAJOS DE CAMPO

Los trabajos de campo consistieron en la exploración y muestreo del subsuelo en un sondeo, denominado SM-1, cuya localización en planta se indica en la Fig. 1; en su ejecución se alternó el muestreo inalterado y la recuperación de muestras alteradas representativas, llevándose hasta 8.53 m de profundidad.

Para la obtención de muestras "inalteradas" se empleó barril Denison accionado a rotación, equipado en su interior con un tubo de pared delgada tipo Shelby de 10 cm (4") de diámetro.

El procedimiento de penetración estándar se empleó para la obtención de muestras alteradas representativas y determinación de la resistencia a la penetración, midiéndose ésta por el número de golpes de un martinete de 63.5 kg (140 lb) de peso y 76 cm (30") de altura de caída libre, necesarios para hincar los 30 cm (1') intermedios del penetrómetro estándar de 5 cm (2") de diámetro exterior; 3.5 cm (1 3/8") de diámetro interior y 60 cm (2') de longitud. La variación con la profundidad de la resistencia a la penetración se reporta en la gráfica "número de golpes" de la Fig. 3.

Durante la exploración no se detectaron cavidades ni irregularidades que manifestasen su existencia; lo anterior corroboró las observaciones e informaciones recabadas durante un reconocimiento detallado del lugar y de las cañadas y cortes cercanos. En lo que respecta al nivel freático, éste no se detectó en ninguno de los sondeos.

4. ENSAYES DE LABORATORIO

Los ensayes de laboratorio efectuados a las muestras recuperadas, además de su clasificación manual y visual conforme al Sistema Unificado de Clasificación de Suelos (SUCS)*, consistieron en la determinación de las siguientes propiedades índice y mecánicas.

a) contenido natural de agua *;

b) límites de consistencia líquido y plástico *;

c) porcentaje de partículas finas (material que pasa la malla No. 200)*;

ch) resistencia al corte en compresión no confinada de probetas en estado natural **;

d) resistencia al corte en compresión triaxial no consolidada-no drenada de probetas en estado natural **;

---

* en muestras alteradas representativas

** sólo en muestras inalteradas

e) peso específico relativo o densidad de sólidos, relación de vacíos, peso volumétrico y grado de saturación, calculados para las probetas ensayadas en las pruebas indicadas de (ch) a (e)**.

La variación con la profundidad de las propiedades mencionadas en los puntos (a) a (ch) y (e) anteriores, se reporta en la Fig. 3.

Los diagramas de Mohr-Coulomb resultantes de las pruebas del tipo señalado en el punto (d), se dibujaron en las Figs. 4 a 6.

5. ESTRATIGRAFIA Y PROPIEDADES

Con los datos obtenidos en campo y laboratorio se construyó el perfil estratigráfico de suelos a lo largo del sondeo, el cual se presenta en la Fig. 3. En base a esta figura se concluye que el subsuelo en el sitio es típico de la llamada zona de lomas, una de las tres en que tradicionalmente se ha dividido el subsuelo del área urbana de la Ciudad de México (Ref. 1), estando constituido como a continuación se describe:

Superficialmente, bajo una losa de concreto reforzado, existen rellenos artificiales de unos 30 cm de espesor, constituidos por suelos areno-limosos color gris y pedacería de tabique. Bajo ellos, hasta la profundidad de 1.5 m, se encontraron arena pumítica color café claro, con grava fina y un contenido natural de agua comprendido entre 51 y 60%. Presentaron una resistencia a la penetración estándar alta, aunque debido a la presencia de la grava fina, la correlación de esta propiedad con la compacidad relativa de estos suelos resulta muy incierta.

Finalmente, subyaciendo los estratos anteriores y hasta la máxima profundidad explorada, existen depósitos limo-arenosos y areno-limosos cementados con algunas gravas finas, presentando apariencia de tobas. Su color es café claro y sus propiedades determinadas fueron: contenido natural de agua variable de 19 a 38 %; relación de vacíos oscilando de 0.7 a 0.9; peso volumétrico medio de 1.8 ton/m^3; resistencia al corte en compresión no confinada de probetas en estado natural mínima de 10 kg/cm^2 y máxima de 25 kg/cm^2; en pruebas de compresión triaxial no consolidada-no drenada sus parámetros de resistencia al corte variaron dentro de los intervalos $4 \leq c \leq 7$ kg/cm^2 y $49 \leq \emptyset \leq 57$, mientras que el módulo de deformación estuvo comprendido entre 1172 y 1794 kg/cm^2.

6. ANALISIS DE LA CIMENTACION

En base a la estratigrafía y propiedades del subsuelo en el sitio (veáse Inciso 5), la distribución y magnitud de las fuerzas que transmitirá la máquina al terreno (veáse Inciso 2), así como su costo y ubicación dentro de la planta, se propone cimentar ésta mediante un macizo de concreto desplantado a 1.5 m bajo la superficie actual del terreno, en los materiales limo-arenosos resistentes detectados en el sondeo a esa profundidad.

Para un cimiento de 5 m de largo, 2 m de ancho y 1.5 m de espesor, dimensiones mayores que las mínimas especificadas por el fabricante, se tendrá un peso total del bloque del orden de las 36 ton, que agregado al peso de la máquina, origina una descarga estática al subsuelo de unas 4.0 ton/m^2. Dada la alta capacidad de carga y baja compresibilidad de los suelos que constituirán el estrato de apoyo, esa descarga es aceptable considerando inclusive la contribución de las fuerzas dinámicas inducidas en la cimentación por la operación de la compresora.

De un análisis dinámico efectuado se obtuvo que los desplazamiento teóricos máximos que pueden generarse durante el funcionamiento de la máquina son del orden de 0.05 mm, valor que se encuentra dentro de los límites tolerables para este tipo de cimentación (Ref. 2). Sin embargo, debe mencionarse que dichos movimientos pueden ser perceptibles para las personas que laboran en su cercanía.

Cabe aquí señalar que, además del análisis realizado para el macizo propuesto, se estudiaron y revisaron bloques de concreto de otras dimensiones. El cimiento presentado por el fabricante se chó debido a que la profundidad de desplante necesaria era de 1.5 m como mínimo; bloques de mayores dimensiones además de encarecer el costo, se ven limitados por el espacio disponible en la nave, presentan pocas ventajas en cuanto a la reducción de vibraciones generadas por la máquina y ofrecen un factor de seguridad menor contra el fenómeno de la resonancia.

7. CONCLUSIONES

Del estudio realizado se derivan las siguientes conclusiones:

a) El subsuelo en el sitio está constituido superficialmente, bajo la losa de concreto de la nave industrial, por rellenos artificiales de unos 30 cm de espesor, subyacidos por arena pumítica, la cual se detectó hasta 1.5 m de profundidad. Bajo los suelos anteriores y hasta la máxima profundidad explorada, existen depósitos limo-arenosos y areno-limosos cementados, de alta resistencia al corte y baja compresibilidad.

b) El tipo de cimentación que se considera más apropiada consiste en un macizo de concreto desplantado a 1.50 m de profundidad, de las dimensiones que se recomiendan en el Inciso 8. Según análisis teóricos esta cimentación, además de su economía, ofrece la ventaja de tener un factor de seguridad mayor contra el fenómeno de la resonancia, respecto a las formadas por bloques similares de otras dimensiones (veáse Inciso 6).

8. RECOMENDACIONES PARA EL DISEÑO Y CONSTRUCCION

a) La cimentación consistirá en un macizo de concreto de 5.0 x 2.0 m en planta, desplantado a 1.50 m de profundidad sobre los suelos limo-arenosos

detectados a esa profundidad.

b) La parte superior del macizo quedará a nivel de la losa de piso de la nave, a fin de evitar que aumente el efecto de cabeceo en la cimentación.

c) El centro de gravedad común de la máquina y la cimentación deberá encontrarse en la vertical del centroide del área de contacto entre el cimiento y el suelo.

d) Entre las paredes del bloque de cimentación y el terreno deberá colocarse una capa de material aislante (por ejemplo corcho), de por lo menos 3 cm de espesor. Este material deberá quedar protegido contra posibles infiltraciones de agua que modifiquen su comportamiento. En la superficie de contacto entre la base del cimiento y el terreno no deberá colocarse dicho material.

f) La excavación del foso que alojará al bloque podrá hacerse según cortes verticales sin ademar, siempre que se mantenga abierta el menor tiempo posible.

Atentamente

M en I. Jaime Martínez M.

Ing. Guillermo Springall C.
Director General

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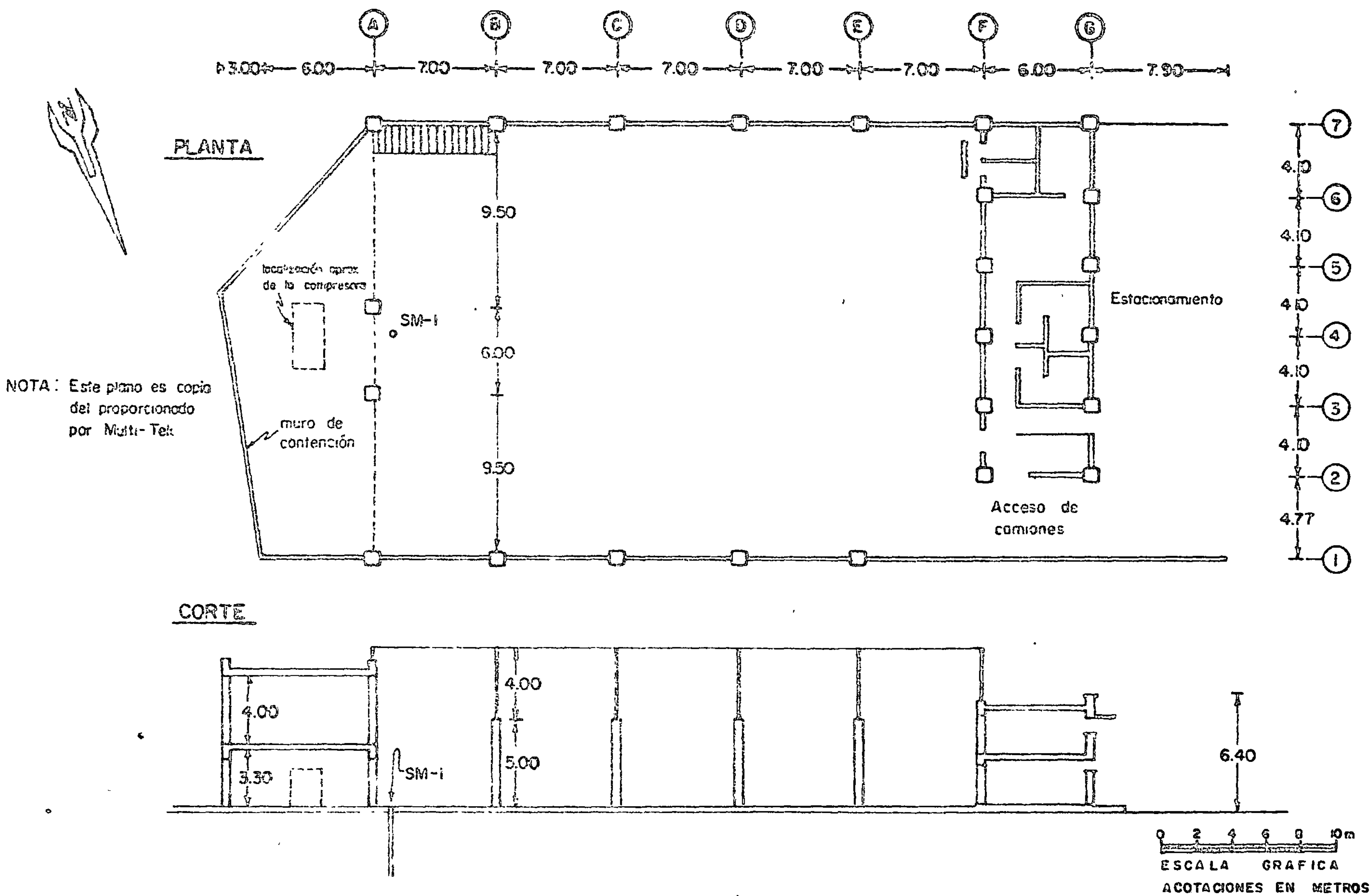
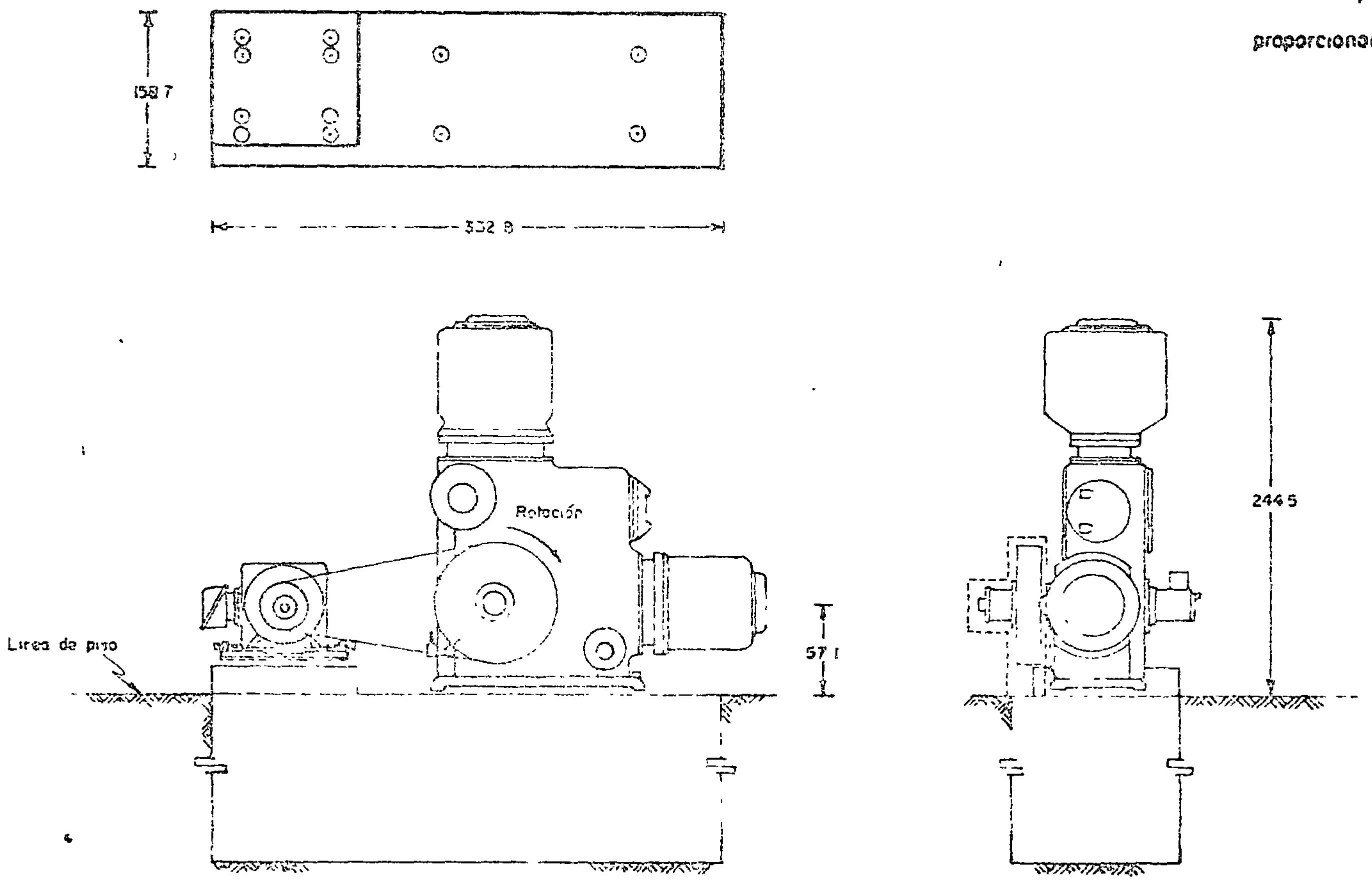


GEOTEC

Fig. 1 Planta y corte de las instalaciones y localización del sondeo.

NOTA Este dibujo es copia del proporcionado por Multi-Tek, S.A



Dibujo fuera de escala
Acotaciones en cm

Fig 2 Planta, vista lateral y frontal de la compresora .

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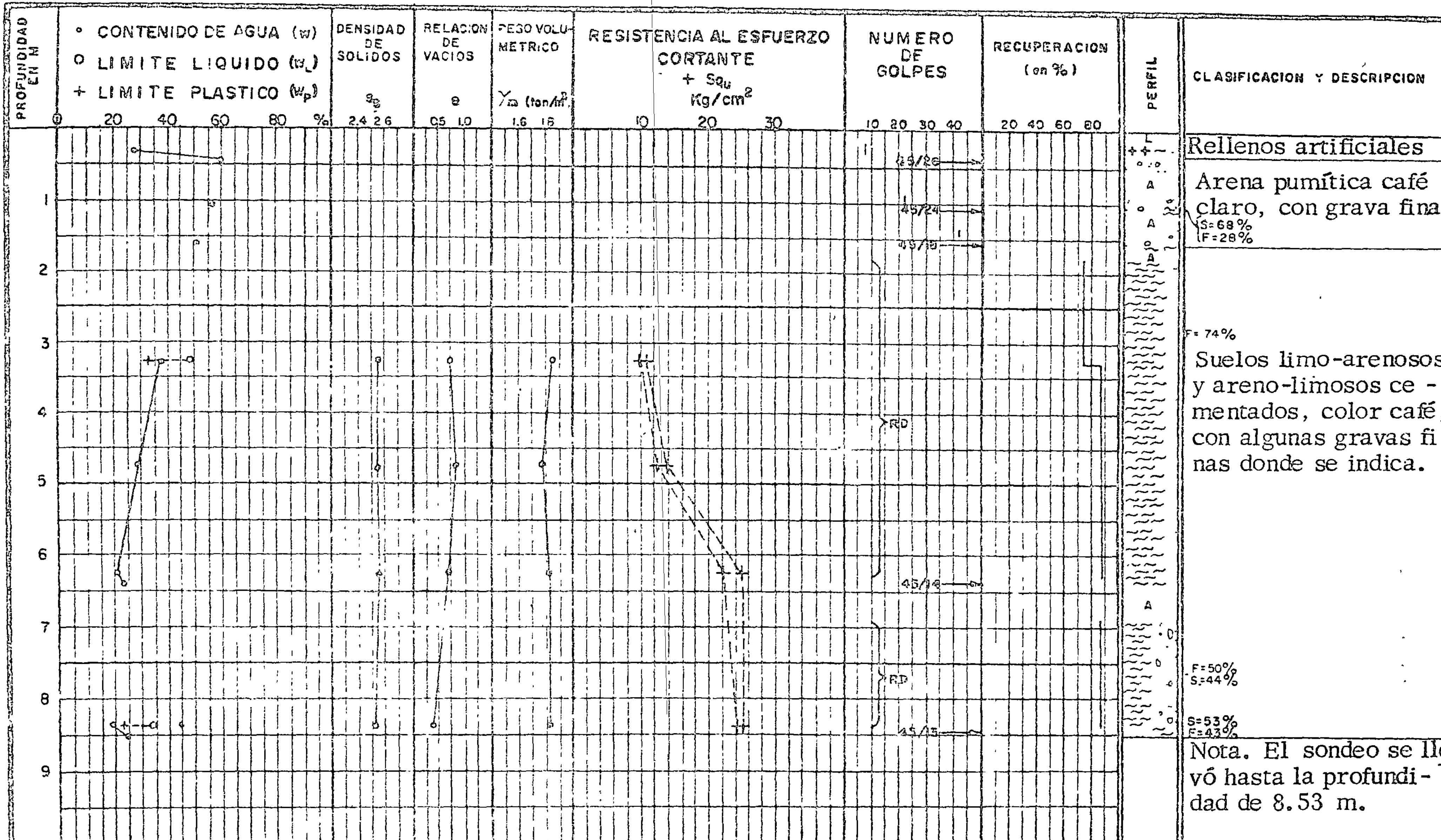

ARCILLA
LIMO
ARENA
GRAVA

A Avance
RD Rotación con barril denison
45/26 Golpes/cm
L Losa de concreto
Relleno

FIG. 3 ESTRATIGRAFIA Y PROPIEDADES DEL SUBSUELO EN EL SONDEO SM-I

MULTI-TEK. S.A.

GEOTEC, S.A.

PRUEBAS DE COMPRESION TRIAXIAL
NO CONSOLIDADA - NO DRENADA (Q)

SONDEO : SM-1

MUESTRA : 4

PROFUNDIDAD : 2.40 m

PROBETA	σ_{III} kg/cm²	σ_I kg/cm²	ε_r %	M kg/cm²	s_s	e_i	w_i %	G_i %
1	0.25	19.9	1.23	1268	2.56	0.83	27.6	85.1
2	0.50	35.7	1.12	1912	2.56	0.95	34.2	92.2
3	1.00	35.3	1.90	1632	2.56	0.99	35.0	90.6

DIAGRAMA DE MOHR

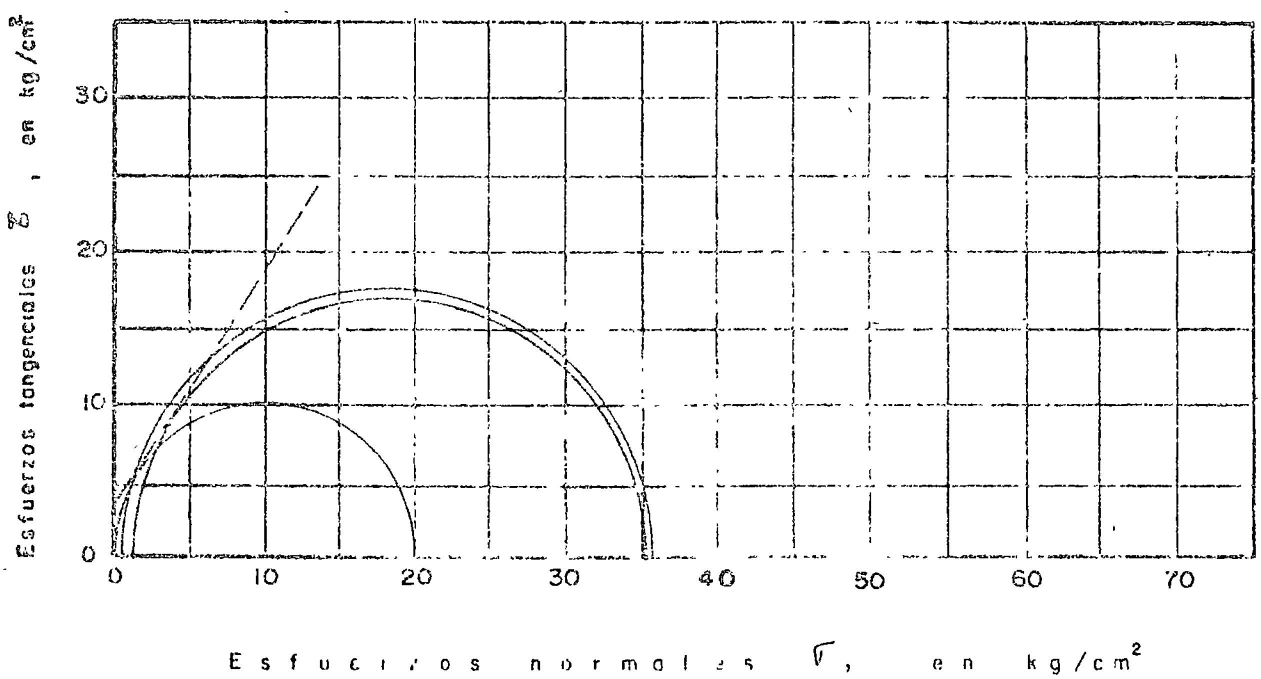

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FIG. 4

PRUEBAS DE COMPRESION TRIAXIAL
NO CONSOLIDADA — NO DRENADA (Q)

SONDEO : SM-1
MUESTRA : 5
PROFUNDIDAD : 4.10 m

PROBETA	σ_{III} kg/cm²	σ_{I} kg/cm²	ε_r %	M kg/cm²	s_s	e_i	w_i %	G_i %
1	0.50	27.6	1.14	1406	2.46	0.89	30.4	84.1
2	1.00	33.8	1.95	1172	2.46	0.82	30.3	90.9

DIAGRAMA DE MOHR

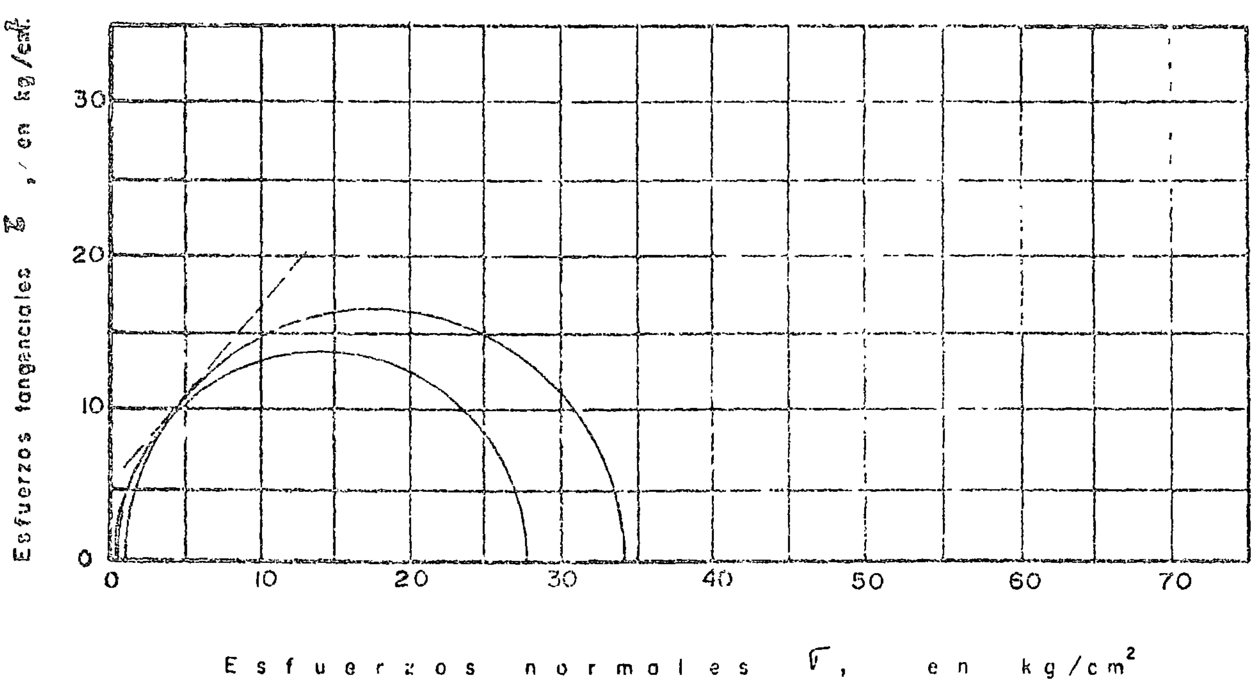

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FIG 5

PRUEBAS DE COMPRESION TRIAXIAL
NO CONSOLIDADA – NO DRENADA (Q)

SONDEO : SM-1
MUESTRA : 8
PROFUNDIDAD : 8.20 m

PROBETA	σ_{III} kg/cm²	σ_{I} kg/cm²	ε_r %	M kg/cm²	s_s	e_i	w_i %	G_i %
1	0.25	45.9	2.05	1793	2.58	0.58	18.3	81.5
2	0.50	51.3	1.93	1349	2.58	0.70	19.8	73.1
3	1.50	59.2	1.87	1794	2.58,	0.64	19.4	78.2

DIAGRAMA DE MOHR

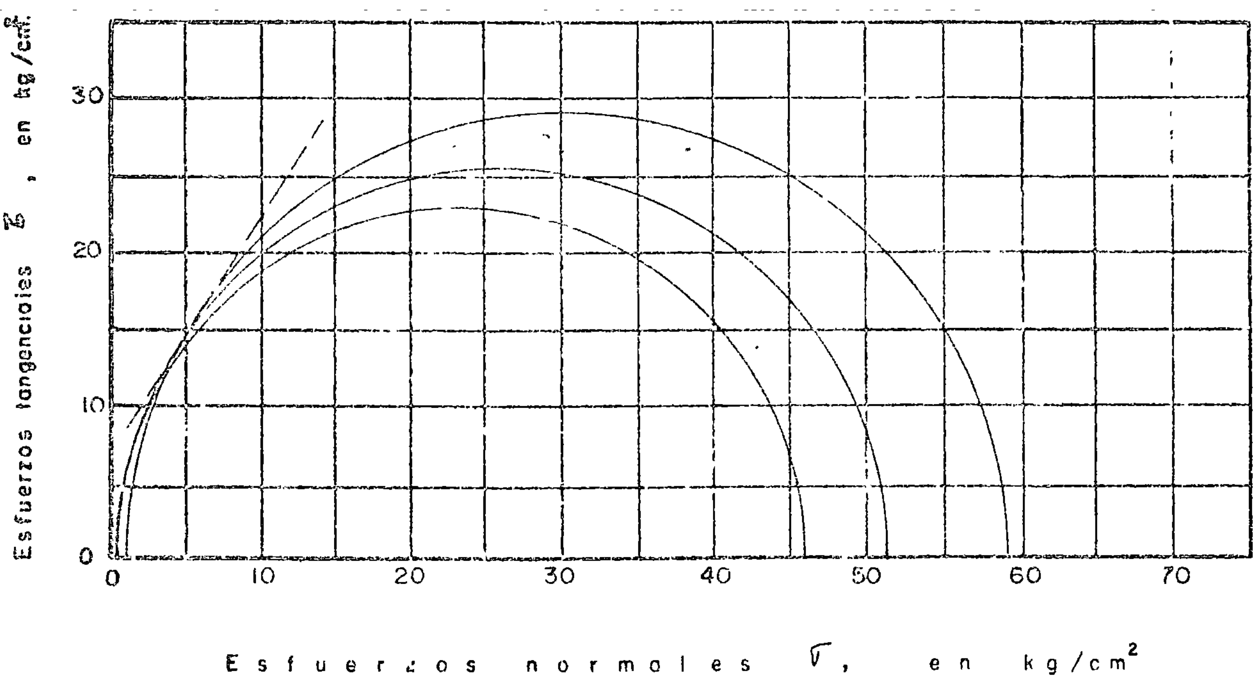

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FIG. 6

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24. ING. HECTOR F. TENA MARTINEZ Misión de San Agustín No. 4 Col. Misiones Naucalpan, Edo. de México Tel: 5-62-58-59	COMISION FEDERAL DE ELECTRICIDAD Ródano No. 14-506 Col. Cuauhtémoc México, D. F. Tel: 5-53-71-33 Ext. 2649
25. ING. HECTOR TOLEDO GARCI Pto. México No. 40-402 Col. Roma Sur México 7, D. F. Tel: 5-84-74-10	INVESTIGACION Y PLANEACION DE INVERSIONES Plaza Miravalle No. 3 Col. Roma Sur México 7, D. F. Tel: 5-33-46-37
26. ING. ALFONSO TOVAR S. Calz. Nonoalco No. 205 Unidad Nonoalco México 3, D. F. Tel: 5-83-29-37	ESCUELA SUPERIOR DE INGENIERIA Y ARQUITECTURA - I.P.N. Zacatenco Col. Lindavista México, D. F.