UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

PROGRAMA DE MAESTRÍA Y DOCTORADO EN INGENIERÍA

FACULTAD DE INGENIERÍA

MODELOS CINEMÁTICO Y DINÁMICO DE UN ROBOT BÍPEDO DE DOCE GRADOS DE LIBERTAD INTERNOS

T E S I S

QUE PARA OPTAR POR EL GRADO DE:

MAESTRO EN INGENIERÍA

MECÁNICA - MECATRÓNICA

P R E S E N T A :

OCTAVIO NARVÁEZ AROCHE

TUTOR:
DR. EDMUNDO GABRIEL ROCHA CÓZATL

COTUTOR:
DR. FRANCISCO CUENCA JIMÉNEZ

2010

Jurado Asignado

Presidente: Dr. Carrera Bolaños Jorge

Secretario: Dr. Cuenca Jiménez Francisco

Vocal: Dr. Rocha Cózatl Edmundo Gabriel

1er.Suplente: Dr. Arteaga Pérez Marco

2do.Suplente: Dr. Dorador González Jesús Manuel

Ciudad Universitaria, México D.F.

Tutor de Tesis

Dr. Rocha Cózatl Edmundo Gabriel

_____

FIRMA

Índice general

Índice de figuras

Resumen

Este trabajo presenta el desarrollo de los modelos de cinemática y dinámica espacial para el robot bípedo de 12 GDL internos Scout.

La cinemática inversa se dedujo considerando a cada una de las piernas del robot como manipuladores; optándose por el uso de transformaciones homogéneas para la solución de la posición y del método vectorial para la velocidad y aceleración.

El modelo dinámico se basó en la formulación de Euler-Lagrange despreciando las fuerzas disipativas en las juntas rotacionales del bípedo, considerando colisiones inelásticas entre los pies y el piso y asumiendo que no existe deslizamiento entre los mismos.

En la simulación del ciclo de marcha del bípedo que ilustra la aplicación de su modelo cinemático, se usaron splines de segundo grado para la definición de las trayectorias deseadas de sus pies y torso.

En tanto que la generalidad en la formulación de ambos modelos permite describir el movimiento de bípedos con trece eslabones rígidos unidos por medio de juntas rotacionales actuadas y la base de sus pies constituida por polígonos; éstos pueden brindar información valiosa en la solución de los problemas de diseño de este tipo de robots, tales como las dimensiones con las que deben contar sus eslabones para tener un rango de movilidad determinado y los pares necesarios en los actuadores para realizar una buena gama de movimientos, con lo que podría escalarse hacia el diseño de una órtesis de miembro inferior que restituya la caminata a una persona.

Abstract

This master thesis presents the development of the kinematic and dynamic spatial models for the 12 internal DOF Scout biped.

The inverse kinematics was derived considering each of the legs of the robot as manipulators, using homogeneous transformations for the solution of the position and the vectorial method for velocity and acceleration.

The dynamic model was based on the Euler-Lagrange formulation neglecting dissipative forces in the rotational joints of the biped, assuming inelastic collisions between the feet and the floor and non-sliding conditions.

Splines of second degree for the definition of the desired trajectories of the feet and torso were used in the simulation of a gait cycle for the biped, illustrating the implementation of its kinematic model.

While the generality in the formulation of both models can describe the motion of a thirteen-rigid-links biped with rotational actuated joints and polygonal feet, these can provide valuable information in solving design problems of such robots; as the dimensions that would have to be considered to get a certain range of motion and the torque needed in the actuators for it to be versatile. These experiences could escalate into the design of a lower limb orthosis for restoring walking.

Nomenclatura

θ_{ni} Ángulo de rotación del eje del servomotor que transfiere su movimiento al eslabón n de la pierna i.

$\sin(\theta_{ni})$ Función seno.

x_{ni}, y_{ni}, z_{ni} Parámetros geométricos del bípedo correspondientes a las distancias con relevancia cinemática de los eslabones n de la pierna i.

m_{ni} Masa del eslabón n de la pierna i.

$\mathbf{I}_{ni}^{mi}$ Matriz de inercia del eslabón n de la pierna i en el marco de referencia mi.

$\mathbf{p}_{ni}$ Punto ni.

$(\mathbf{O}; \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ Marco de referencia inercial constituido por los vectores unitarios mutuamente ortogonales $\mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0$, con origen en el punto $\mathbf{O}$.

$(\mathbf{p}_{ni}; \mathbf{i}_{mi}, \mathbf{j}_{mi}, \mathbf{k}_{mi})$ Marco de referencia local mi constituido por los vectores unitarios mutuamente ortogonales $\mathbf{i}_{mi}, \mathbf{j}_{mi}, \mathbf{k}_{mi}$, con origen en el punto $\mathbf{p}_{ni}$.

$\mathbf{T}_{zi}$ Matriz de transformación homogénea básica.

$\mathbf{T}_{mi,(m+1)i}$ Matriz de transformaciones homogéneas del marco de referecia mi al $(m+1)i$.

$\mathbf{r}_{ni}^{mi}$ Vector de posición ni respecto al marco de referencia mi.

$\mathbf{r}_{Gni}^{mi}$ Vector de posición del centro de gravedad ni respecto al marco de referencia mi.

$\mathbf{R}_x$ Matriz de rotación básica.

$\mathbf{R}_{(m+1)i}^{mi}$ Matriz de rotación de la base $(m+1)i$ a la mi.

$\mathbf{v}_{ni}^{mi}$ Vector de velocidad ni respecto al marco de referencia mi.

$\boldsymbol{\omega}_{ni}^{mi}$ Vector de velocidad angular ni respecto al marco de referencia mi.

$\mathbf{a}_{ni}^{mi}$ Vector de aceleración ni respecto al marco de referencia mi.

$\boldsymbol{\alpha}_{ni}^{mi}$ Vector de aceleración angular ni respecto al marco de referencia mi.

$\mathbf{b}_{ni}^0$ Vector de posición del centro de gravedad del eslabón n de la pierna i respecto al marco inercial.

L Función Lagrangiana.

$\mathbf{q}$ Vector de coordenadas generalizadas.

δW Trabajo Virtual

$\delta \mathbf{q}$ Vector de cambios virtuales en las coordenadas generalizadas.

τ_{ni} Par del actuador que mueve el eslabón n de la pierna i.

$\mathbf{Q}$ Vector de fuerzas generalizadas.

$\boldsymbol{\Gamma}_k$ Vector de fuerzas de restricción en el intervalo de tiempo T_k.

$\mathbf{\Phi}_k^0$ Vector de ecuaciones de restricción para el intervalo T_k.

$\frac{\partial \mathbf{\Phi}_k^0}{\partial \mathbf{q}}$ Matriz Jacobiana de las ecuaciones de restricción en T_k.

$\mathbf{\lambda}_k^0$ Vector de multiplicadores de Lagrange en T_k.

Introducción

¿Cuántas personas se beneficiarían con la disposición comercial de órtesis activas de miembro inferior para la restitución de una caminata ortopédicamente adecuada?

De acuerdo con los datos publicados por el INEGI en el XII Censo General de Población y Vivienda, alrededor del 54.7 % de la población discapacitada del país poseía una discapacidad de tipo motriz en el año 2000; para el total de 2,241,193 personas censadas de este sector, esto implica que alrededor de 1.2 millones de mexicanos podrían mostrarse interesados en hacer uso de una órtesis activa (consultar Apéndice A). En los Estados Unidos de América se estima que 4.7 millones de personas con alteraciones de la locomoción derivadas de infartos al miocardio, 1 millón con trastornos post-polio, 400,000 con esclerosis múltiple, 200,000 con lesión medular y 100,000 con parálisis cerebral se beneficiarían del uso de estos dispositivos (Dollar and Herr, 2008).

Además de proporcionar indicadores del impacto social que posee la investigación en torno a tecnologías para la asistencia de la locomoción, la información anterior pone de manifiesto que las características particulares de cada una de las patologías vinculadas con trastornos ambulatorios, condicionará requerimientos distintos para el diseño de las órtesis que satisfagan las necesidades de sus usuarios; por lo que es difícil imaginar que un solo modelo de órtesis activa pueda ser suficiente para cubrir la demanda en el mercado de la ortopedia y rehabilitación.

Aunado a esta dificultad, todas las órtesis activas deberán disponer de los elementos necesarios para evitar que su usuario caiga al piso en cualquiera de los estados en que se encuentre durante su marcha; lo que detona las siguientes preguntas: ¿Cuáles son las dimensiones apropiadas para los eslabones de la órtesis?, ¿Dónde resulta deseable que se encuentren sus centros de gravedad?, ¿Entre qué velocidades y aceleraciones debe operar?, ¿Cuál es el patrón de caminata más apropiado para su funcionamiento?, ¿Cómo se pueden generar las trayectorias de sus articulaciones?, ¿Qué esquemas de control resultarán lo suficientemente robustos para garantizar que la caminata se desarrolle de manera estable?, ¿Con qué sensores y actuadores deberá contar?, ¿Cuál es la capacidad de cómputo necesaria?, ¿Qué fuente de energía será la más apropiada para esta aplicación?.

Entre las áreas de la ingeniería, la que ha mostrado mayores avances en la solución de problemas ligados a la movilidad estable de estructuras con dos extremidades inferiores es la robótica bípeda. Partiendo del paradigma señalado por algunos especialistas (Takanishi, 1995) de que el estudio de la locomoción de estructuras no antropométricas sentará las bases para el diseño sistemático de dispositivos enfocados en la caminata asistida del ser humano; se eligió como plataforma de investigación al robot bípedo Scout de la compañía Lynxmotion, debido a su disponibilidad comercial y número de grados de libertad (GDL) internos[1], pues la literatura reporta que doce es el mínimo necesario para lograr la síntesis de ciclos de marcha tridimensionales aproximados a la locomoción humana (Chevallereau et al., 2009).

En tanto que la revisión bibliográfica (ver sección 1.2) no aportó ningún modelo en $\mathbb{R}^3$ que pudiera usarse para describir el movimiento del bípedo citado, este trabajo tiene como objetivos:

> *Obtener un modelo cinemático espacial del bípedo de 12 GDL internos Scout que asista en la determinación de las características de posición, velocidad y aceleración angulares que deben poseer las rotaciones de sus eslabones para llevarlos a configuraciones espaciales con velocidades y aceleraciones específicas respecto a un marco inercial.*

---

[1] *Los grados de libertad internos de un robot bípedo son el número de juntas que posee en su estructura. El concepto se detalla en el capítulo 3.*

Desarrollar un modelo dinámico de acuerdo a la formulación Euler-Lagrange con el que se puedan determinar los pares necesarios en los actuadores del bípedo Scout para llevarlo a los estados previamente determinados con el modelo cinemático.

Simular un ciclo de marcha del bípedo en $\mathbb{R}^3$ que ilustre el uso del modelo cinemático deducido para programar las secuencias de movimiento del robot.

Se debe destacar que los modelos contarán con la generalidad suficiente para ser adaptados a la descripción del movimiento de bípedos con trece eslabones rígidos unidos por medio de juntas rotacionales actuadas y la base de sus pies constituida por polígonos, realizando los cambios pertinentes en las especificaciones expuestas en el apéndice B.

La disposición de los modelos cinemáticos y dinámicos del bípedo Scout busca dar pie al diseño de esquemas de control robustos que garanticen su equilibrio dinámico durante la caminata, lo que marcaría el inicio de este tipo de investigaciones en el Departamento de Mecatrónica de la UNAM, con miras a la implementación del conocimiento que se adquiera en el diseño de dispositivos para la restitución de la marcha en personas con discapacidades.

El capítulo 1 de esta tesis ofrece generalidades de la robótica bípeda con una breve revisión de su panorama histórico y el estado del arte de los prototipos más destacados, identificando al concepto de punto de momento cero (ZMP) como el marco teórico en el que se sustenta el desarrollo de los bípedos más versátiles de la actualidad. El capítulo 2 describe la arquitectura del bípedo Scout, esencial para la deducción de su cinemática inversa y dinámica con la formulación Euler-Lagrange; presentadas en los capítulos 3 y 4, respectivamente. El capítulo 5 pone de manifiesto la utilidad del modelo cinemático desarrollado con la simulación de un ciclo de marcha del robot, usando splines de segundo grado como herramientas de interpolación para la programación de las trayectorias seguidas por su torso y pies. Se da fin al texto exponiendo las conclusiones de la investigación y el trabajo a futuro.

Capítulo 1

Robótica bípeda

1.1. Panorama histórico

A pesar de la naturaleza de la locomoción del ser humano, su ambición por reproducir su caminata en una máquina bípeda es muy reciente; sobre todo si se le compara con el tiempo que ha invertido en la síntesis de soluciones alternativas para su transporte asistido en medios terrestres, marítimos e inclusive aéreos. Si bien se cita (Chevallereau et al., 2009) la existencia de algunos bocetos de Leonardo da Vinci para la fabricación de un mecanismo humanoide y la construcción de autómatas basados en máquinas de vapor -Steam Man por John Brainerd- y eléctricas -Electric Man por Frank Reade Junior- en el siglo XIX; no fue sino hasta la década de 1970 que se formalizó la inquietud de construir robots bípedos en instituciones académicas de Japón y Yugoslavia.

A grandes rasgos, un bípedo es estable si puede caminar sin caerse. Se considera que en los albores de la robótica bípeda japonesa los ciclos de marcha guardaban una estabilidad estática (Kajita and Espiau, 2008), pues usando pequeñas velocidades y aceleraciones y asumiendo que los puntos de contacto con el suelo se encontraban inmóviles en cada una de las etapas del ciclo de marcha, el profesor Ichiro Kato y su equipo de investigadores de la Universidad de Waseda presentaron en 1973 el primer robot antropomórfico caminante, el Wabot I (figura 1.1); cuya condición de equilibrio se reducía a garantizar que la proyección del centro de gravedad sobre su plano de desplazamiento siempre se mantuviera en el conjunto convexo formado por sus puntos de contacto con dicho plano.
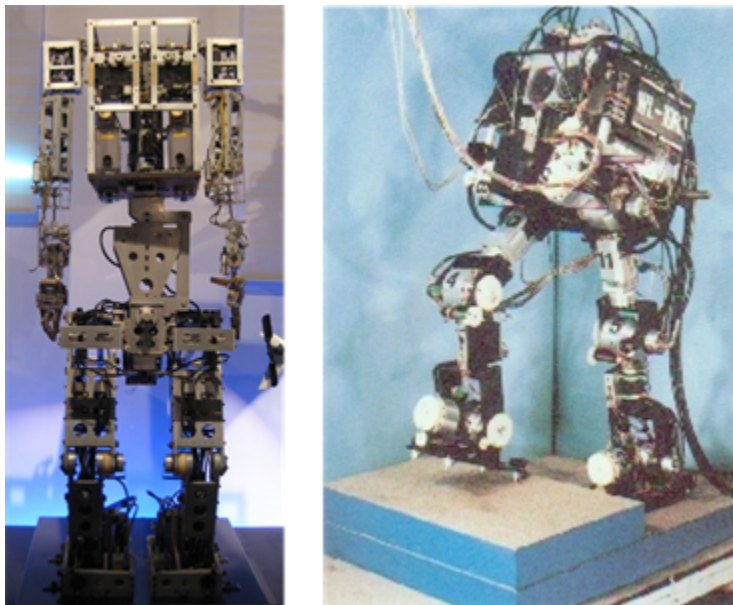
Figura 1.1: Bípedos diseñados por el Profesor Ichiro Kato. Wabot I, el primer robot antropomórfico caminante. WL-10RD, el primer bípedo que desarrolló una caminata dinámicamente estable.

De forma paralela al inicio de la prolífica producción de robots bípedos en la Universidad de Waseda, el equipo de Miomir Vukobratovic en el Instituto Mihailo Puppin de Belgrado diseñó los primeros exoeskeletos activos con propósitos de rehabilitación, aunque su contribución más importante a la robótica derivó de sus estudios relacionados con la estabilidad de la locomoción, los cuales dieron pie a la publicación del concepto de punto de momento cero (ZMP, por sus siglas en inglés *Zero-Moment Point*) en 1972. El ZMP sentó las bases teóricas de la estabilidad dinámica de la locomoción que ocupan los robots más versátiles de la actualidad y se relaciona intuitivamente a la idea de permitir que el bípedo reaccione a condiciones inesperadas en el seguimiento de sus trayectorias articulares para evitar su caída. Resulta paradójico que el primer robot basado en el cálculo del ZMP que logró una caminata dinámica exitosa fue el WL-10RD (figura 1.1), construido por Kato y Takanishi en 1985.

En la década de 1980 Marc Raibert fundó el MIT LegLab en el Instituto de Tecnología de Massachusetts, donde desde entonces han sido construidos una serie de robots para el estudio de ciclos de caminata y carrera dinámicamente estables, asi como la ejecución de saltos gimnásticos. Entre los bípedos más famosos de este laboratorio (figura 1.2) se encuentran Planar Biped (1985), 3D Biped (1989), Spring Turkey (1994), Geekbot (1994) y Spring Flamingo (1996).
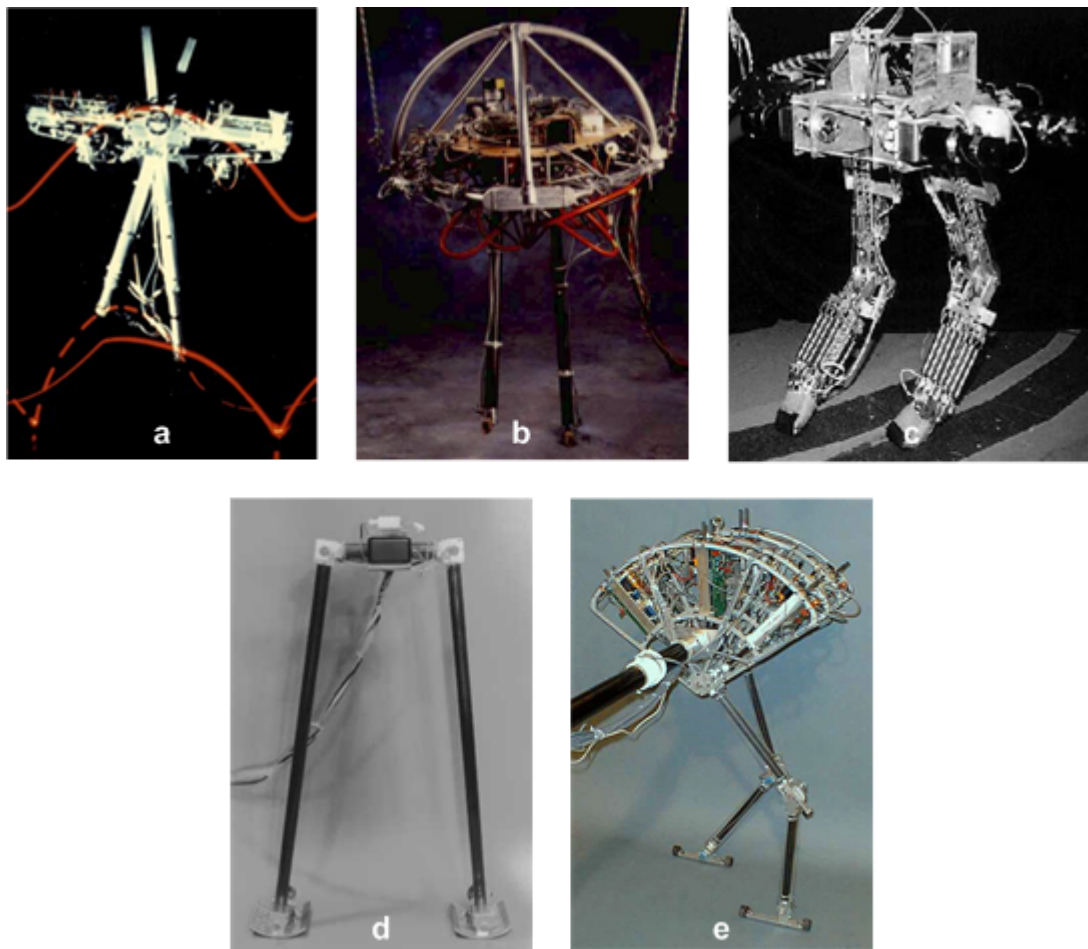
Figura 1.2: Bípedos construidos en el MIT LegLab. a) Planar Biped, b) 3D Biped, c) Spring Turkey, d) Geekbot, e) Spring Flamingo.

En 1990 Tad McGeer se convirtió en el pionero de la caminata dinámica pasiva demostrando matemáticamente que la marcha cíclica de un bípedo puede producirse de manera estable sin la incorporación de actuadores, siempre y cuando se modifique la geometría de sus pies para que las piernas se comporten como péndulos bajo la acción de la gravedad en un plano inclinado (Kuo, 2007).

El final del siglo XX fue testigo de la incorporación del sector industrial en la investigación de la robótica bípeda cuando Honda hizo público su humanoide P2 en diciembre de 1996.

1.2. Estado del Arte

De acuerdo a la complejidad de sus movimientos, los robots bípedos pueden describirse realizando combinaciones de los rasgos definidos en cada una de las siguientes categorías (Westervelt and de Wit, 2007):

Bidimensionales El movimiento de sus eslabones sólo tiene lugar en el plano sagital.

Tridimensionales El movimiento de sus eslabones tiene lugar dentro y fuera del plano sagital.

Minimalistas Poseen menos de seis grados de libertad (GDL) en cada una de sus piernas.

Antropomórficos Tienen seis o más GDL en cada una de sus piernas.

Pasivos Desarrollan su ciclo de marcha sin un control directo de la cinemática de sus eslabones, razón por la que constituyen sistemas subactuados.

Activos Caminan controlando la posición de los actuadores en cada uno de sus eslabones.

Adicionalmente, de acuerdo al problema de diseño mecánico que motiva su construcción, los bípedos se pueden clasificar como:

Eficientes Buscan que la intervención humana para asegurar la recarga de su fuente de energía se reduzca al mínimo.

Versátiles Persiguen ser suficientemente robustos para llevar a cabo su caminata de forma estable a pesar de su exposición a perturbaciones externas e incertidumbres, tales como la presencia de obstáculos en su superficie de desplazamiento.

En tanto que la caminata dinámica pasiva sólo describe pérdidas energéticas durante la colisión de la pierna oscilante con el piso, una gran cantidad de investigadores trabajan en extensiones de los resultados de McGeer para la síntesis de bípedos pasivos con la capacidad de caminar sobre superficies horizontales sin la imposición de trayectorias cinemáticas en las articulaciones de sus piernas. Entre estos bípedos destacan Cornell Efficient Biped y Denise.

Cornell Efficient Biped es el bípedo con el menor costo energético de transporte (Kuo, 2007) gracias a dos características primordiales; una curvatura proporcionada a sus pies, que reduce la velocidad de colisión con el piso, y resortes rotacionales en las articulaciones entre la pelvis y las piernas, que acentúan la oscilación de los miembros inferiores para proporcionar pasos rápidos y pequeños sin la necesidad de introducir energía adicional al sistema. Los únicos elementos que requieren del suministro de energía son dos motores que tensan resortes en los tobillos para aplicar la fuerza que separa los pies del piso y un sencillo controlador por eventos que coordina su acción durante el ciclo de marcha, gracias a la información proporcionada por sensores de contacto en la planta de los pies.

Denise es un bípedo bidimensional diseñado en la Universidad de Tecnología de Delft con cinco grados de libertad internos, uno en la cadera, dos en las rodillas y dos en los tobillos. Tiene un peso de 8 [kg] y se destaca por el uso de un tanque de CO_2 que permite el funcionamiento de un mecanismo con cadenas y bandas para mantener erecto su torso y sincronizar el movimiento de sus brazos y piernas para imitar la caminata humana. Dos pares antagónicos de músculos artificiales neumáticos -también conocidos como músculos de McKibben- proveen el torque necesario en las articulaciones de la cadera para producir la oscilación de las piernas y disminuir la probabilidad de que caiga. Sus rodillas tienen topes mecánicos para evitar la hiperextensión de las piernas (Wisse et al., 2007).
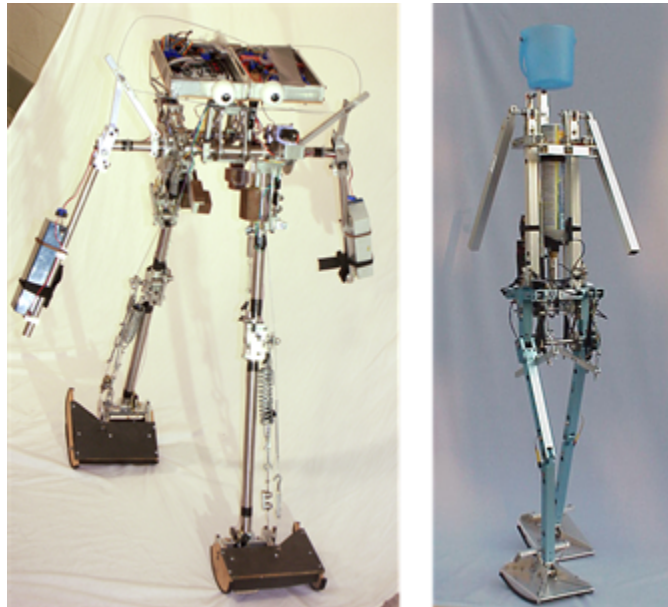
Figura 1.3: Dos bípedos minimalistas con una eficiencia sobresaliente: Cornell Efficient Biped y Denise.

El Cornell Efficient Biped y Denise (figura 1.3) son robots especializados en ser eficientes, pues tienen la capacidad de caminar económicamente en superficies libres de obstáculos y nada más, no pueden modificar la velocidad de sus pasos, dar vueltas, subir escaleras ni soportar cargas externas; de hecho la curvatura de sus pies ni siquiera les permite permanecer de pie. Actualmente los bípedos versátiles emplean una forma de control basada en el cálculo de su ZMP, de hecho el avance técnico ha sido tan vertiginoso que a los robots bípedos se les ha dotado de manipuladores superiores para la emulación de brazos humanos y un cuello con uno o varios grados de libertad para mover una cabeza; estas configuraciones estructurales reciben el nombre de humanoides. A continuación se hace una breve descripción de los bípedos versátiles más sobresalientes.

En la Universidad de Waseda, el conocimiento adquirido con la construcción de la familia de bípedos tridimensionales WL (siglas de Waseda Legged) dio paso a la construcción del humanoide WABIAN (WAseda BIpedal humANoid). Su versión más reciente, el WABIAN-2R (figura 1.4), tiene una altura de 1.5[m], un peso de 64.5[kg], 41 GDL internos, servomotores de corriente directa con engranes armónicos y encoders incrementales acoplados a sus ejes. Cada tobillo está equipado con sensores de fuerza y torque de seis ejes para la medición de las fuerzas de reacción del piso y el cálculo del ZMP. En su arquitectura resaltan dos articulaciones redundantes en la pelvis y una articulación pasiva en la planta de los pies; elementos que le permiten imitar con buena precisión la marcha normal de un ser humano e inclusive reproducir la de personas con patologías específicas. Su velocidad promedio es de 0.36[m/s] con un periodo de 0.96[s] por paso. El propósito principal del WABIAN-2R es constituir un simulador confiable de la caminata humana para la evaluación cuantitativa de equipo de asistencia para la movilidad (Atsuo-Takanishi-Lab, 2009a).
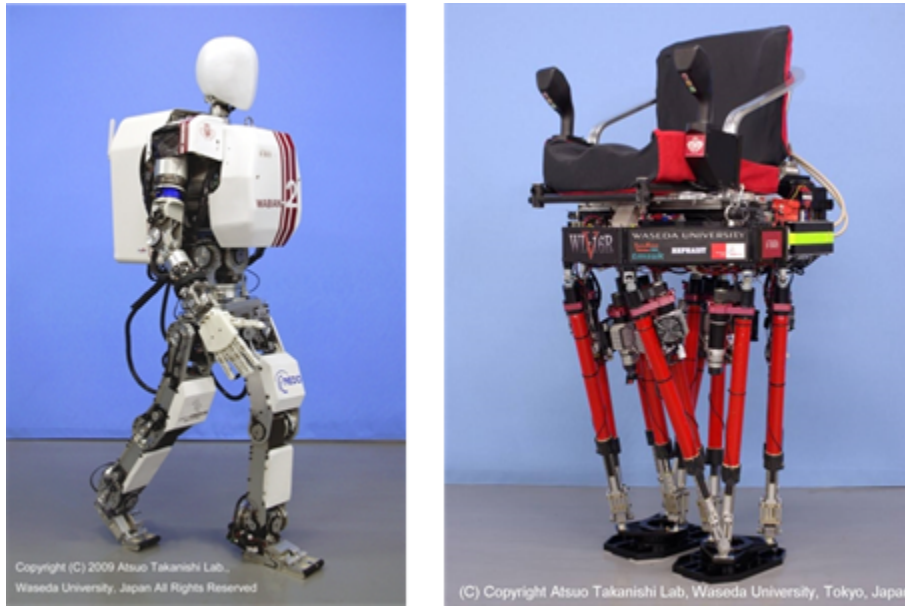
Figura 1.4: Bípedos de la Universidad de Waseda. WABIAN-2R y WL-16RV.

Otro de los robots destacados de la Universidad de Waseda es el bípedo WL-16RV (figura 1.4), una "silla caminante" capaz de cargar a un humano de hasta 94[kg]. Posee 12 GDL internos, 1.29[m] de altura y un peso de 55[kg]. Sus piernas tipo plataforma de Stewart son controladas por medio de actuadores lineales eléctricos. El propósito de las investigaciones a su alrededor es desarrollar un bípedo multipropósito cuya parte superior pueda modificarse de acuerdo a las necesidades locomotoras de su usuario (Atsuo-Takanishi-Lab, 2009b).

En el centro de investigación de robots humanoides del Instituto Avanzado de Ciencia y Tecnologia de Corea del Sur (por sus siglas en inglés KAIST) se desarrollan el humanoide Hubo KHR y el sillón bípedo Hubo FX.

El KHR-4 pesa 55 [kg], tiene 1.25[m] de altura, 41 GDL, puede caminar y correr con una rapidez de 1[m/s]. En su última actualización se sustituyó su cabeza por una reproducción del Dr. Albert Einstein, elaborada por Hanson-Robotics, que gracias a 35 articulaciones puede reproducir varias expresiones faciales con movimientos independientes de sus ojos y labios (Humanoid-Robot-Research-Center, 2009a). Desde entonces, el KHR-4 es identificado como Albert Hubo y se le considera el primer androide (figura 1.5).
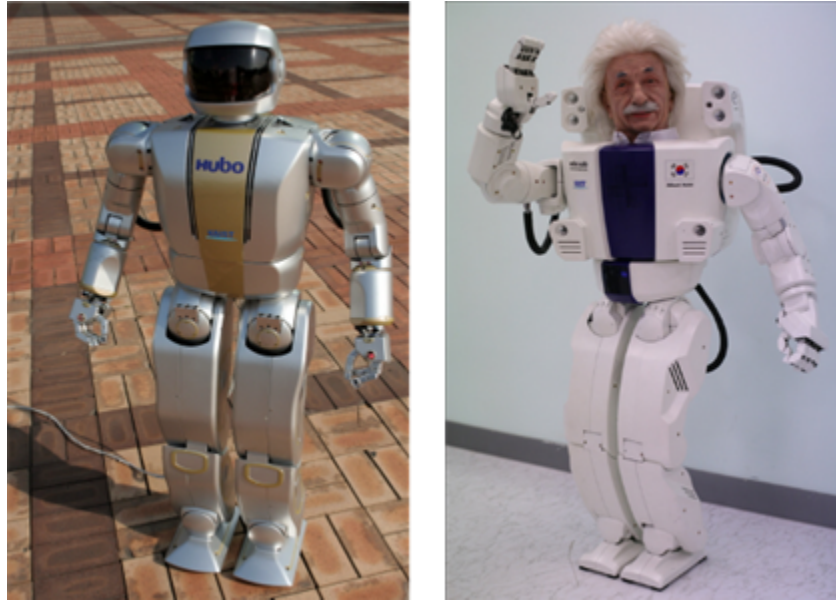
Figura 1.5: Bípedos del KAIST. KHR-4 y Albert Hubo, el primer androide.

El sillón bípedo FX-1 mide 1.75[m] de altura, pesa 150[kg], tiene 12 GDL y usa servomotores de corriente alterna para transportar con seguridad a su usuario de acuerdo a los comandos introducidos mediante un joystick (Humanoid-Robot-Research-Center, 2009b). Un bípedo tridimensional con la misma configuración fue presentado por Toyota en 2005 bajo el nombre de i-Foot (figura 1.6); se encuentra diseñado para mejorar la movilidad de personas con paraplejia.

Figura 1.6: Bípedos para el traslado de personas discapacitadas. FX-1 del Hubo Lab y i-Foot de Toyota.

Por iniciativa del Ministerio de Economía, Comercio e Industria de Japón (METI), el Instituto Nacional de Ciencia y Tecnología Industrial Avanzada (AIST) diseñó un humanoide, el HRP, y su plataforma de simulación, OpenHRP. La construcción del humanoide fue conferida a la compañía Kawada Industries, que elaboró

quince robots HRP-2. Catorce de estos fueron entregados en diferentes laboratorios de Japón y el restante fue transferido al Laboratorio de Análisis y Arquitectura de Sistemas (LAAS) en Toulouse (Chevallereau et al., 2009), como parte de un intento por hacer del HRP la plataforma académica estándar para el desarrollo de humanoides. En 2007 Kawada hizo público el HRP-3 Promet Mk-II (figura 1.7), tiene una altura de 1.6[m], un peso de 68[kg] y 42 GDL internos. Se le ha presentado caminando establemente sobre superficies con agua y arena, reconociendo objetos, bailando y usando taladros para la ejecución de tareas sencillas. Sus sistemas de enfriamiento y filtrado de polvo y humedad le permiten funcionar durante dos horas en condiciones de lluvia (Kawada-Industries-Inc., 2009).

Figura 1.7: Humanoide HRP-3 en el laboratorio de mecatrónica de Kawada, caminando sobre una superficie con poca fricción y manipulando un taladro para el ajuste de piezas.

De entre todos los humanoides de la actualidad, Honda ASIMO (Advanced Step in Innovative Mobility) es el que tiene la capacidad de realizar más funciones (Honda-Worldwide, 2010). Con una altura de 1.3[m], un peso total de 54[kg] y 34 GDL; puede caminar a 0.75[m/s], correr con una rapidez de 1.67[m/s], dar vueltas, subir y descender escaleras, manipular interruptores, abrir puertas, regular la fuerza de prensión de sus manos -provistas de cinco dedos móviles-, esquivar obstáculos en su trayectoria de movimiento, dirigir una orquesta sinfónica, cargar charolas de comida sin tirar su contenido y empujar un carro de compras (figura 1.8). Con las cámaras en su cabeza y su tecnología de inteligencia artificial puede reconocer su ambiente, objetos en movimiento, la postura y gestos de las personas a su alrededor. También puede identificar voces y los focos de producción de sonidos para ubicar a su interlocutor y seguir sus órdenes. En algunas de sus patentes se cita el uso de aleaciones de magnesio en sus eslabones, giroscopios, acelerómetros y una batería recargable de Li-ion que le permite trabajar tan solo por una hora.
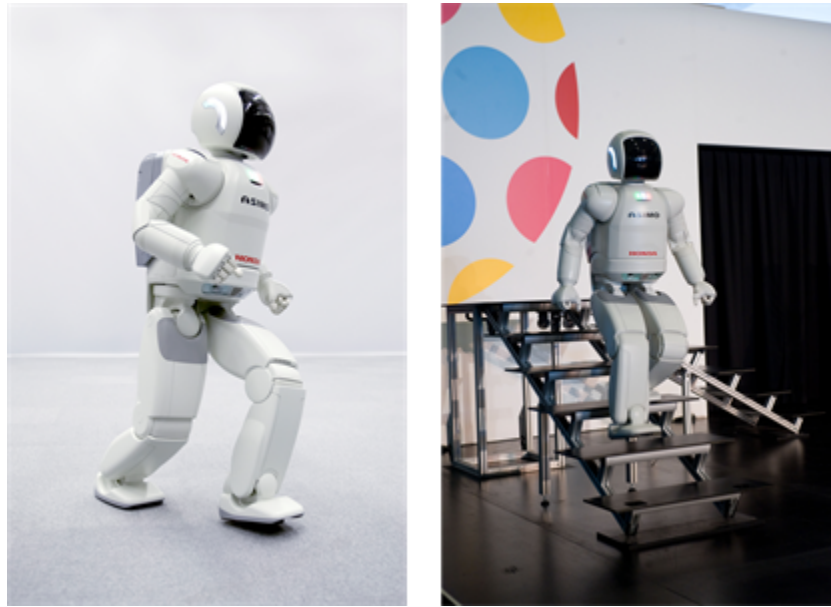
Figura 1.8: Honda ASIMO corriendo y descendiendo escaleras. Se le considera el humanoide más versátil.

Es importante resaltar que aunque ASIMO fue presentado por Honda Motor Co en el año 2000, el humanoide constituye el fruto de una investigación que comenzó en el año de 1986 con la construcción consecutiva de los bípedos E0, E1-E2-E3 (1987-1991), E4-E5-E6 (1991-1993) y los humanoides P1-P2-P3 (1993-1997).

Entre los robots humanoides de pequeña escala que pueden caminar figuran las creaciones de Tomotaka Takahashi, el QRIO de Sony y NAO de la empresa francesa Aldebaran Robotics.

El profesor Takahashi, de la Universidad de Tokyo, ha inventado humanoides originales con la capacidad de desempeñar tareas preprogramadas como caminar, trotar, correr, bailar, saltar y levantarse del piso, de acuerdo a comandos de voz. Entre sus creaciones más populares figuran Ropid (29 GDL), Chroino (24 GDL) y Female Type (23 GDL). Todos comparten la característica de caminar sin mantener constantemente dobladas sus rodillas -una crítica común contra algunos humanoides de escala humana- gracias a un sistema de control patentado por Robo Garage denominado SHIN-Walk (Tomotaka-Takahashi, 2010). Chroino fue elegido por Time Magazine como uno de los inventos más originales del 2004 (figura 1.9).

Figura 1.9: Humanoides de pequeña escala diseñados por el profesor Takahashi: Ropid, Chroino, y Female Type. Tienen una altura de 0.38[m], 0.35[m] y 0.35[m], respectivamente.

QRIO (Quest for cuRIOsity), presentado por Sony en 2003 como el primer humanoide capaz de correr con estabilidad dinámica (Sony-Global, 2010), tiene 28 GDL, 0.6[m] de altura y un peso de 6.5[kg]. Para lograr el desempeño del robot, la compañía japonesa desarrolló un actuador que patentó bajo el nombre de ISA (Intelligent Servo Actuator). Los planes para realizar su comercialización fueron cancelados junto con los del perro artificial AIBO en el 2006 (Block, 2010).

Figura 1.10: Dos humanoides que juegan fútbol. QRIO de Sony y NAO de Aldebaran Robotics. NAO es la plataforma oficial de RoboCup.

NAO (figura 1.10) es el robot oficial de la competencia internacional RoboCup en la liga de Plataforma Estándar desde que suplió al cuadrúpedo AIBO en el 2008. Con una altura de 0.58[m] y un peso de 4.3[kg], posee 25 GDL cuyo funcionamiento se encuentra mediado por la información captada con sus cámaras, micrófonos, acelerómetros, giroscopios, sensores de efecto Hall, ultrasónicos, infrarrojos y de fuerza. Se puede comunicar con una computadora personal y con otros robots NAO vía Wi-Fi. Los usuarios de NAO pueden programar sus movimientos y registrar los valores de sus sensores con una interfaz gráfica multiplataforma llamada Choregraphe, la cual incluye un simulador para la validación de las trayectorias de sus articulaciones compatible con otros entornos de programación comerciales como Microsoft Robotics Studio y Webots de Cyberbotics. La edición académica está acompañada de un SDK (Software Development Kit) que permite introducir código en C, C++, URBI, Python, Java y Matlab; flexibilidad que le ha valido su popularización en la pedagogía de la robótica (Aldebaran-Robotics, 2010).

Algunos de los bípedos que pueden encontrarse en el mercado son BRAT y Scout de la compañía estadounidense Lynxmotion y el humanoide Robonova de Hitec Robotics con 16 GDL, 0.4[m] de altura y un peso de 1.6[kg] (figura 1.11). La estructura de estos robots se encuentra restringida por la geometría de los servomotores comercializados por Hitec, empresa coreana de buena reputación en aplicaciones de radiocontrol. Se venden en paquetes en los que puede incluirse u omitir la electrónica para el control de los servomotores; aunque también es posible adquirir juegos de los eslabones para que el cliente construya su propio bípedo estilo Meccano. El Scout, seleccionado como bípedo de estudio en este trabajo, será descrito de forma detallada en el siguiente capítulo.

Figura 1.11: Bípedos al alcance del consumidor. BRAT (3 GDL por pierna), Scout (6 GDL por pierna) y Robonova.

Aunque su propósito principal consiste en aumentar las capacidades físicas de su portador, algunos autores citan a los exoesqueletos (figura 1.12) como ejemplos de robótica bípeda (Chevallereau, et. al. 2009). Entre los proyectos en desarrollo más famosos figuran el Exoesqueleto de Extremidades Inferiores de Berkeley (BLEEX) y el de las compañías Sarcos y Raytheon, ambos financiados por la Agencia de Proyectos de Investigación de Defensa Avanzada de los Estados Unidos de América (DARPA). En contraste, HAL-5 (siglas de Hybrid

Assistive Limb) de Cyberdyne Inc, se lanzó en el mercado japonés desde noviembre de 2005 a un precio de 13,800 dólares como un dispositivo para asistir la caminata de adultos mayores y con problemas degenerativos en los músculos (Guizzo and Goldstein, 2005).
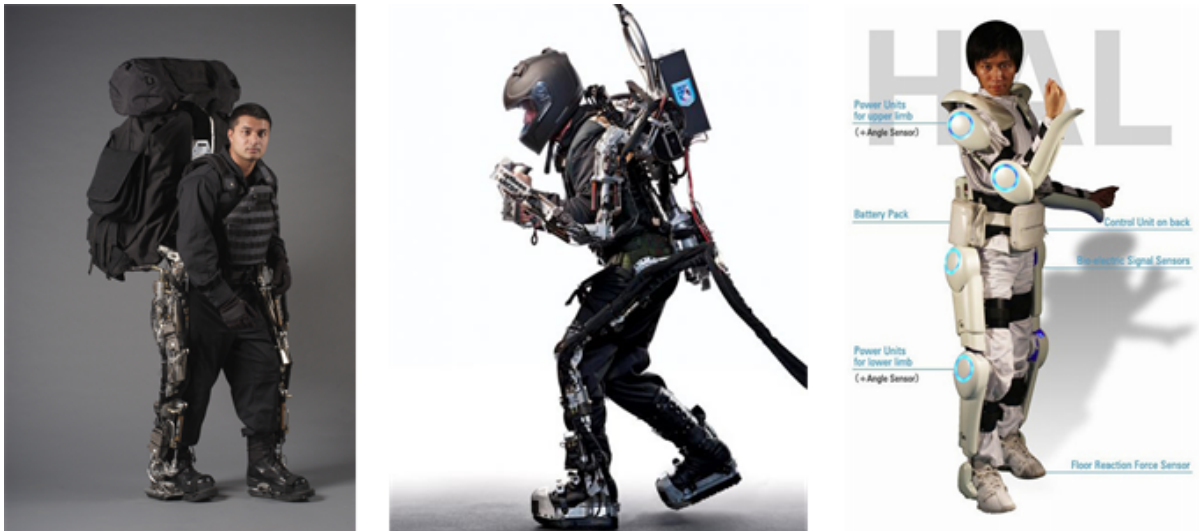
Figura 1.12: Exoesqueletos BLEEX, Sarcos-Raytheon y HAL.

La tendencia evolutiva trazada por los objetivos a largo plazo de los proyectos previamente mencionados apuntan a la integración de bípedos en:

- Agricultura.

- Ganadería.

- Construcción.

- Minería.

- Servicios de Limpieza.

- Exploración submarina y espacial.

- Protección ambiental.

- Trabajo doméstico.

- Asistencia hospitalaria.

- Rehabilitación.

- Aplicaciones militares.

La autonomía requerida para un robot bípedo que resulte útil en cualquiera de las aplicaciones anteriores demanda conjugar eficiencia energética y versatilidad; sin embargo, encontrar tanto un marco teórico como un diseño mecatrónico que permita obtener un balance entre las dos, sigue siendo uno de los principales retos de la ingeniería.

En las escasas publicaciones relacionadas a los bípedos anteriores se omite la presentación explícita de sus modelos dinámicos; optándose por la exposición de modelos de péndulo invertido parametrizados, en los casos en que se ilustran esquemas de control para garantizar la caminata estable de robots con el cálculo de la trayectoria deseada de su ZMP (Liu and Veloso, 2005),(Kajita et al., 2006),(Erbatur and Kurt, 2009). Si bien se encontraron los modelos cinemático y dinámico (basado en la formulación Newton-Euler) de un bípedo tridimensional con juntas esféricas en (Shih et al., 1993) y las ecuaciones dinámicas para el movimiento de un bípedo bidimensional en (Haavisto and Hyotyniemi, 2004), ninguno de estos proporcionaba la información adecuada para implementar esquemas de control basados en modelos Euler-Lagrange como los descritos en (Kajita and Espiau, 2008); de ahí que se considerara trascendente documentar la deducción de los modelos cinemático y dinámico del bípedo Scout, presentes en los capítulos siguientes.

Capítulo 2

Arquitectura del Bípedo Scout

El bípedo Scout está constituido por piernas de seis eslabones unidas entre sí mediante un eslabón central al que, equiparando la arquitectura del robot con la anatomía de un ser humano, se le denominará torso. Sus trece eslabones se conectan en serie a través de juntas rotacionales actuadas por servomotores.

En la figura 2.1 se presenta la perspectiva dimétrica posterior del ensamble CAD del bípedo en la configuración espacial que adopta con sus servomotores en posición neutral. Se caracterizan las piezas de sus eslabones por medio de diferentes colores y los servomotores en rojo. El torso se identifica con la letra B y los eslabones de las piernas con las etiquetas ni; donde $1 \leq n \leq 6$ se encuentra determinada por su distribución en relación al torso, correspondiendo $n = 1$ a los eslabones unidos a éste y $n = 6$ a los que cumplen el papel de pies. Se asigna $i = 1$ a los eslabones de la pierna izquierda e $i = 2$ a los de la derecha.

Para la elaboración de los análisis en este trabajo, resultó necesaria una simplificación del modelo CAD del bípedo que permitiera reconocer fácilmente sus juntas y los parámetros geométricos de los eslabones con relevancia cinemática. Esta simplificación puede apreciarse en la figura 2.2, donde las juntas se representan por medio de cilindros cuyos ejes coinciden con los de los servomotores y se unen entre sí a través de barras. La rotación de los ejes de los servomotores se describirá por medio de los ángulos θ_{ni}, siendo ni la etiqueta del eslabón sujeto a su eje y la de su barra correspondiente en el modelo simplificado. Recurriendo nuevamente a una analogía con la anatomía humana, es posible afirmar que los seis ángulos mostrados en rojo conceden al bípedo los GDL de la cadera, los dos en verde proporcionan el movimiento de las rodillas y los cuatro en azul se asocian a los tobillos. La geometría de las porciones de los eslabones $6i$ que realizan contacto con el plano sobre el que se realizará la caminata (el piso) no fue simplificada, debido a su importancia en la determinación del polígono de soporte del bípedo, de cuya relación con la posición del ZMP depende la estabilidad de la caminata (ver sección 6.2).

De acuerdo a la clasificación propuesta en (Westervelt and de Wit, 2007), Scout es un bípedo tridimensional antropomórfico; por lo que se requiere de sus vistas coronal y sagital para establecer los parámetros geométricos con relevancia cinemática, que en este caso corresponden a las distancias entre los puntos medios $\mathbf{p}_{ni}$ de los ejes de los cilindros señaladas en las figuras 2.3 y 2.4.
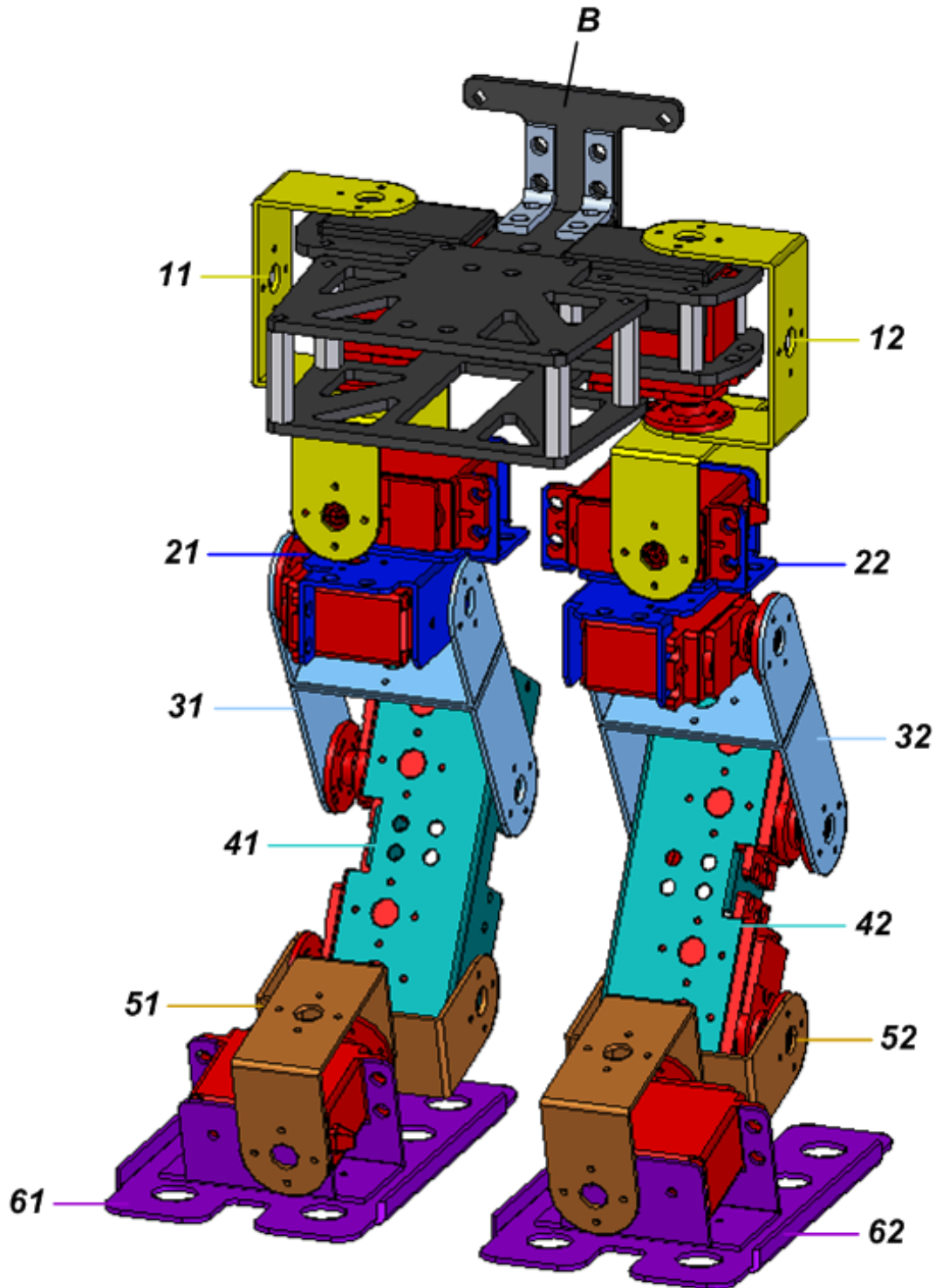
Figura 2.1: Perspectiva dimétrica posterior del ensamble CAD del bípedo Scout. El eslabón del torso se identifica con la letra B y los eslabones de las piernas con las etiquetas ni. Los servomotores acoplados a las juntas rotacionales se muestran en rojo.
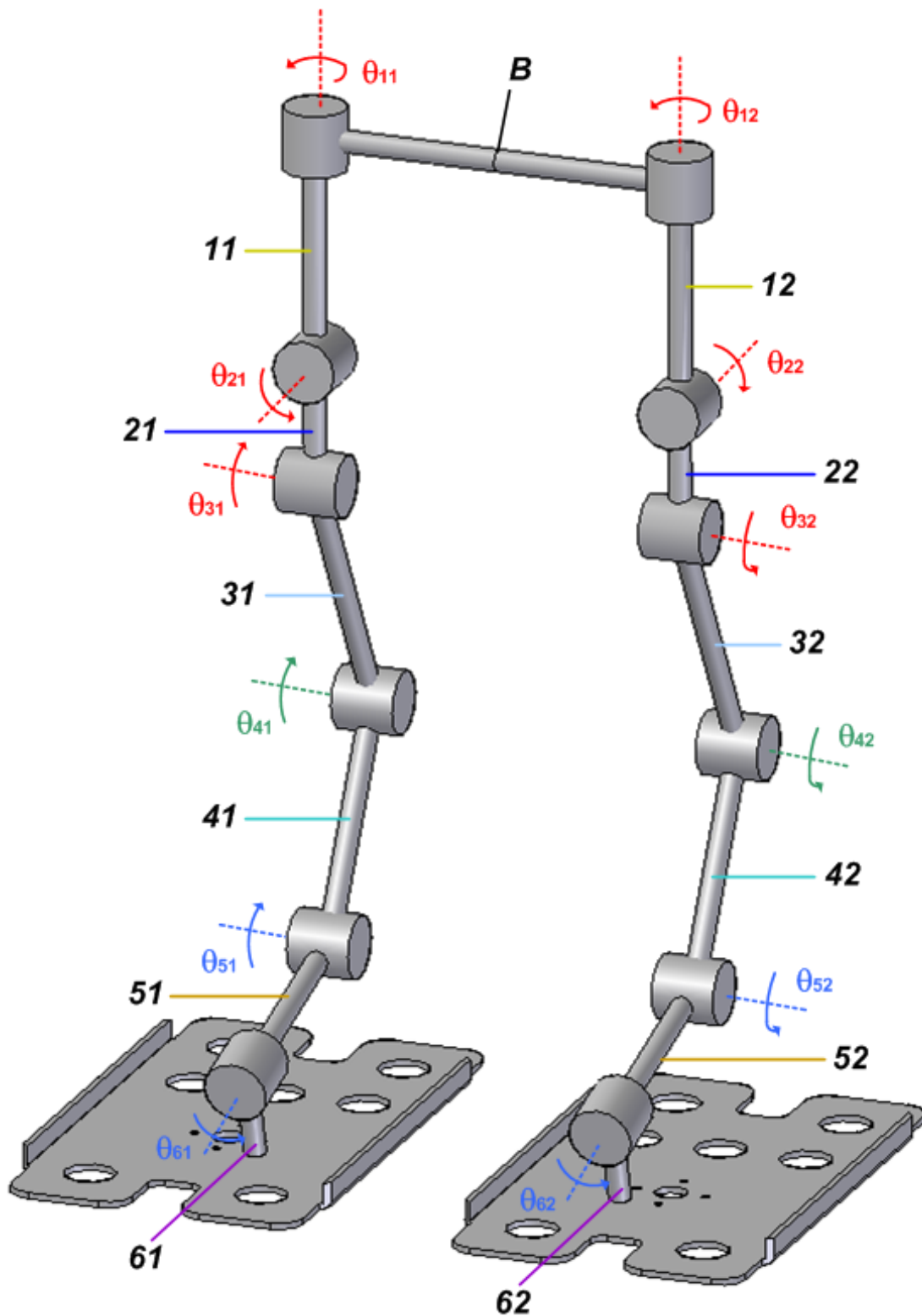
Figura 2.2: Modelo simplificado del bípedo Scout. Los ángulos θ_{ni} se asocian a las rotaciones de los servomotores en las juntas.

Figura 2.3: Vista coronal anterior del modelo simplificado del bípedo. Las cotas indican las distancias con relevancia cinemática.
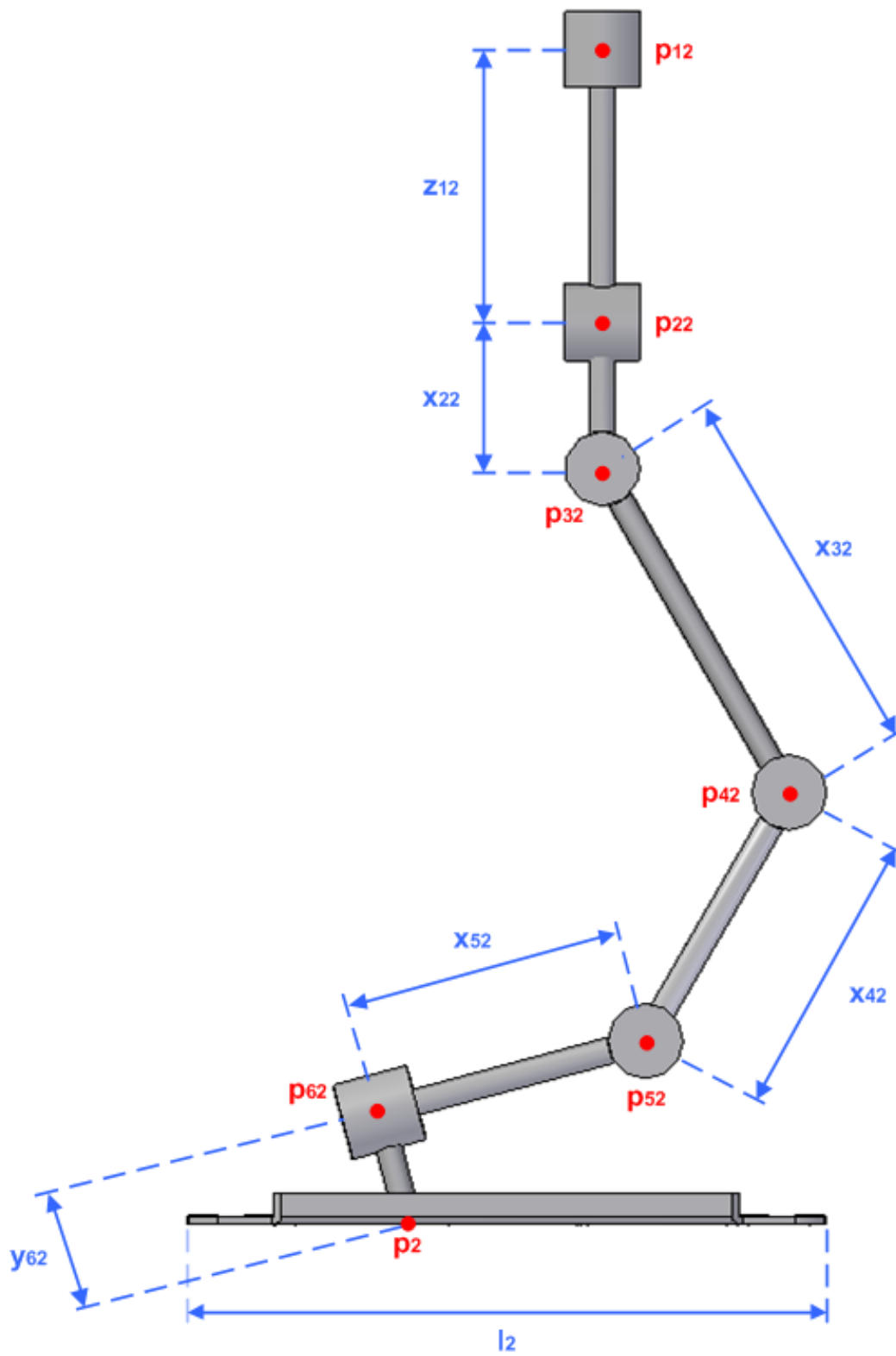
Figura 2.4: Vista sagital derecha del modelo simplificado del bípedo. Las cotas indican las distancias con relevancia cinemática.

Es importante destacar que los puntos $\mathbf{p}_i$ son la intersección entre las rectas perpendiculares a los ejes de las rotaciones θ_{6i} que contienen los puntos $\mathbf{p}_{6i}$ en planos paralelos al sagital y las superficies de los pies que hacen contacto con el piso; siendo y_{6i} las distancias que separan a los puntos $\mathbf{p}_{6i}$ y $\mathbf{p}_i$ a lo largo de las barras $6i$ en los pies izquierdo ($i = 1$) y derecho ($i = 2$). x_{0i} son las distancias entre el plano sagital y los planos paralelos a éste donde se encuentran los ejes de los servomotores fijos al torso, mientras que l_1 y l_2 caracterizan los rectángulos que delimitan las superficies de contacto entre cada uno de los pies y el piso.

En tanto que el bípedo cuenta con simetría sagital, se presentan las igualdades:

$$
\begin{aligned}
x_{01} &= x_{02} \\
z_{11} &= z_{12} \\
x_{21} &= x_{22} \\
x_{31} &= x_{32} \\
x_{41} &= x_{42} \\
x_{51} &= x_{52} \\
y_{61} &= y_{62}
\end{aligned}
$$

En lo sucesivo se aprovecharán estas igualdades para reducir la notación de los parámetros geométricos a $x_{0i}, z_{1i}, x_{2i}, x_{3i}, x_{4i}, x_{5i}, y_{6i}$.

En el apéndice B se pueden consultar las listas de elementos constituyentes del bípedo, planos de su ensamble y su modelo simplificado, los valores de los parámetros geométricos determinados mediante su modelado asistido por computadora, las masas de los eslabones, sus matrices de inercia y los vectores de posición de sus centros de gravedad; datos que cobrarán importancia en el análisis dinámico expuesto en el capítulo 4.

Capítulo 3

Ecuaciones Cinemáticas

El modelo cinemático empleado para el análisis de la caminata del bípedo se presenta en la figura 3.1.

El origen $\mathbf{O}$ del marco inercial fijo $(\mathbf{O}; \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ y los vectores unitarios $\mathbf{i}_0, \mathbf{j}_0$ se encuentran en el plano del piso. El vector $\mathbf{i}_0$ es paralelo a los bordes anteriores y posteriores de los pies, $\mathbf{j}_0$ se encuentra en la intersección con el plano sagital y $\mathbf{k}_0$ es el producto cruz $\mathbf{i}_0 \times \mathbf{j}_0$.

El punto $\mathbf{p}_B$ del marco local $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$ constituye el punto medio del segmento de recta que une los puntos $\mathbf{p}_{11}$ y $\mathbf{p}_{12}$. Los vectores $\mathbf{i}_B, \mathbf{j}_B$ se encuentran en el plano paralelo a las caras superiores de las piezas del eslabón B que contiene a $\mathbf{p}_B$; $\mathbf{i}_B$ es paralelo a los bordes anteriores y posteriores de dicho eslabón y $\mathbf{j}_B$ es paralelo a sus bordes laterales, mientras que $\mathbf{k}_B$ es el producto cruz $\mathbf{i}_B \times \mathbf{j}_B$.

De acuerdo a lo descrito en el capítulo anterior, los puntos $\mathbf{p}_i$ de los marcos locales $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ se encuentran en los planos de los pies que hacen contacto con el piso, junto con los vectores $\mathbf{i}_i, \mathbf{j}_i$. Los vectores $\mathbf{i}_i$ son paralelos a los bordes anteriores y posteriores de sus pies respectivos, $\mathbf{j}_i$ son paralelos a los bordes laterales y $\mathbf{k}_i$ son los productos cruz $\mathbf{i}_i \times \mathbf{j}_i$.

Con base en los marcos de referencia anteriores, el problema de cinemática inversa para el análisis de la posición del bípedo se define como:

> Dada la posición de los puntos $\mathbf{p}_B, \mathbf{p}_i$ y la orientación de los marcos locales $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$, $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ respecto al marco inercial $(\mathbf{O}; \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$, encontrar los valores de los ángulos θ_{1i}, θ_{2i}, θ_{3i}, θ_{4i}, θ_{5i}, θ_{6i} que permiten al bípedo asumir la configuración espacial determinada por dichas posiciones y orientaciones.

La posición de los puntos $\mathbf{p}_B, \mathbf{p}_i$ en el marco inercial se describe con los vectores:

$$
\begin{aligned}
\mathbf{r}_B^0 &= x_B \mathbf{i}_0 + y_B \mathbf{j}_0 + z_B \mathbf{k}_0 \\
\mathbf{r}_i^0 &= x_i \mathbf{i}_0 + y_i \mathbf{j}_0 + z_i \mathbf{k}_0
\end{aligned}
\tag{3.1}
$$

La orientación del marco local $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$ respecto al marco inercial queda caracterizada por medio de los ángulos de Euler:

$$
\theta_B, \phi_B, \psi_B
\tag{3.2}
$$

Mientras que a los marcos locales de los pies $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ corresponden los ángulos de Euler:

$$
\theta_i, \phi_i, \psi_i
\tag{3.3}
$$
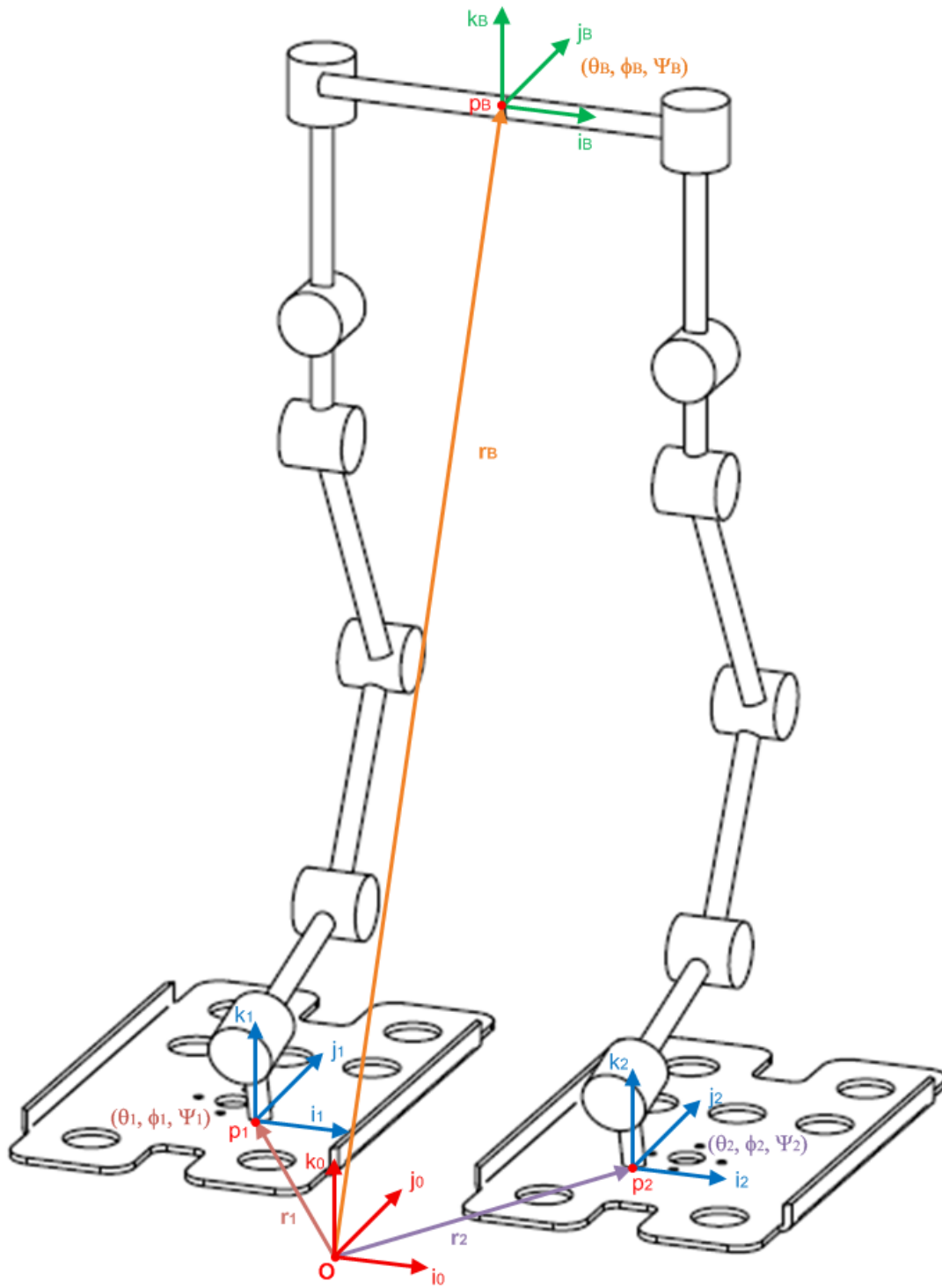
Figura 3.1: Modelo cinemático para el análisis de la caminata del bípedo.

En la figura 3.2 se ilustra el orden de las rotaciones representadas por los ángulos de Euler θ_i, ϕ_i, ψ_i para describir la orientación de los marcos locales $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ con respecto al marco inercial. El ángulo θ_i corresponde a una rotación sobre el eje $\mathbf{i}_0$, ϕ_i a una rotación sobre el eje $\mathbf{j}_{\theta i}$ y ψ_i a una sobre el eje $\mathbf{k}_{\phi i}$. De manera análoga, para la orientación del marco local $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$, θ_B corresponde a una rotación sobre el eje $\mathbf{i}_0$, ϕ_B a una rotación sobre el eje $\mathbf{j}_{\theta B}$ y ψ_B a una sobre el eje $\mathbf{k}_{\phi B}$.
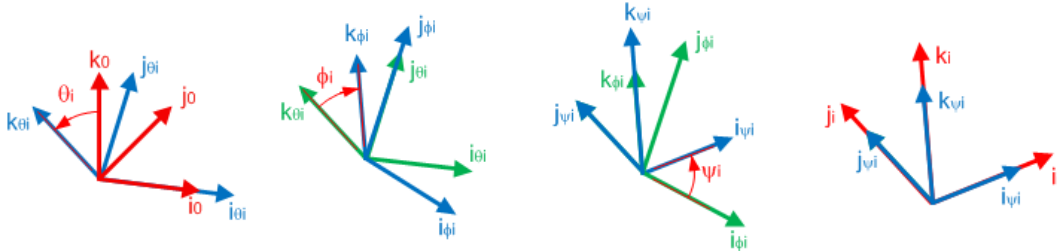
Figura 3.2: Orden de las rotaciones regidas por los ángulos de Euler.

Tomando en cuenta los cambios respecto al tiempo de las expresiones en (3.1) y los ángulos de Euler en (3.2) y (3.3), el problema cinemático inverso para la velocidad queda definido como:

Dadas las velocidades de traslación y rotación de los eslabones del torso y los pies con los vectores $\mathbf{v}_B^0 = \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0, \mathbf{v}_i^0 = \dot{x}_i \mathbf{i}_0 + \dot{y}_i \mathbf{j}_0 + \dot{z}_i \mathbf{k}_0$ y los valores de $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i$, encontrar los valores $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$ que determinan las velocidades angulares en las juntas rotacionales del bípedo.

Tomando en cuenta los cambios respecto al tiempo de $\mathbf{v}_B^0, \mathbf{v}_i^0$ y $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i$, el problema cinemático inverso para la aceleración queda definido como:

Dadas las aceleraciones de traslación y rotación de los eslabones del torso y los pies con los vectores $\mathbf{a}_B^0 = \ddot{x}_B \mathbf{i}_0 + \ddot{y}_B \mathbf{j}_0 + \ddot{z}_B \mathbf{k}_0, \mathbf{a}_i^0 = \ddot{x}_i \mathbf{i}_0 + \ddot{y}_i \mathbf{j}_0 + \ddot{z}_i \mathbf{k}_0$ y los valores de $\ddot{\theta}_B, \ddot{\phi}_B, \ddot{\psi}_B, \ddot{\theta}_i, \ddot{\phi}_i, \ddot{\psi}_i$, encontrar los valores de $\ddot{\theta}_{1i}, \ddot{\theta}_{2i}, \ddot{\theta}_{3i}, \ddot{\theta}_{4i}, \ddot{\theta}_{5i}, \ddot{\theta}_{6i}$ que determinan las aceleraciones angulares en las juntas rotacionales del bípedo.

En las siguientes secciones se describen los procedimientos realizados para deducir ecuaciones matriciales cuya solución numérica dará respuesta a los tres problemas de cinemática inversa definidos anteriormente.

Elaborando un conteo del número de valores que se requieren proporcionar para la solución del problema de la posición, se puede deducir que la configuración espacial del bípedo sólo puede definirse a través de dieciocho valores, tales como las nueve coordenadas cartesianas y nueve ángulos de Euler de (3.1),(3.2) y (3.3); por lo que se deduce que el robot posee 18 GDL.

Es común que en el material bibliográfico de robótica bípeda se caracterice a los robots por medio de su número de juntas, el cual se identifica como su número de grados de libertad internos (*internal degrees of freedom,* en inglés); es por ello que desde esta perspectiva se describe a Scout como un robot de doce GDL internos.

3.1. Ecuaciones de posición

De acuerdo a la notación en (Cuenca-Jiménez, 2010) para la solución del problema de cinemática inversa de la posición se usaron matrices de transformaciones homogéneas básicas de la forma:

$$
\mathbf{T}_{z1}(x) \;=\; \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_{z2}(y) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_{z3}(z) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_{z4}(\theta_x) \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_{z5}(\theta_y) \;=\; \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_{z6}(\theta_z) \;=\; \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Las transformaciones $\mathbf{T}_{z1}, \mathbf{T}_{z2}, \mathbf{T}_{z3}$ representan traslaciones con distancias x, y, z medidas sobre los ejes $\mathbf{i}, \mathbf{j}, \mathbf{k}$, respectivamente; mientras que $\mathbf{T}_{z4}, \mathbf{T}_{z5}, \mathbf{T}_{z6}$ constituyen rotaciones de ángulos $\theta_x, \theta_y, \theta_z$ sobre estos mismos ejes.

De acuerdo a las coordenadas cartesianas de los puntos $\mathbf{p}_B, \mathbf{p}_i$ en el marco inercial y los ángulos de Euler en (3.2) y (3.3), las transformaciones necesarias para trasladar y rotar el marco de referencia inercial a la posición y orientación de los marcos locales del torso $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B)$ y los pies $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ son:

$$
\begin{aligned}
\mathbf{T}_{0,B} &= \mathbf{T}_{z1}(x_B)\mathbf{T}_{z2}(y_B)\mathbf{T}_{z3}(z_B)\mathbf{T}_{z4}(\theta_B)\mathbf{T}_{z5}(\phi_B)\mathbf{T}_{z6}(\psi_B) \\
\mathbf{T}_{0,i} &= \mathbf{T}_{z1}(x_i)\mathbf{T}_{z2}(y_i)\mathbf{T}_{z3}(z_i)\mathbf{T}_{z4}(\theta_i)\mathbf{T}_{z5}(\phi_i)\mathbf{T}_{z6}(\psi_i)
\end{aligned}
$$

En las figuras 3.3, 3.4 y 3.5 se presentan los marcos locales establecidos con auxilio del modelo simplificado del bípedo para construir las transformaciones que trasladan y rotan el marco local del torso hasta los pies, pasando por cada una de las juntas rotacionales de las piernas.

Figura 3.3: Marcos locales empleados para la deducción de la ecuación matricial de posición del bípedo usando transformaciones homogéneas. El efecto de las matrices de transformación para trasladarse de un marco local a otro se ilustra con flechas moradas.
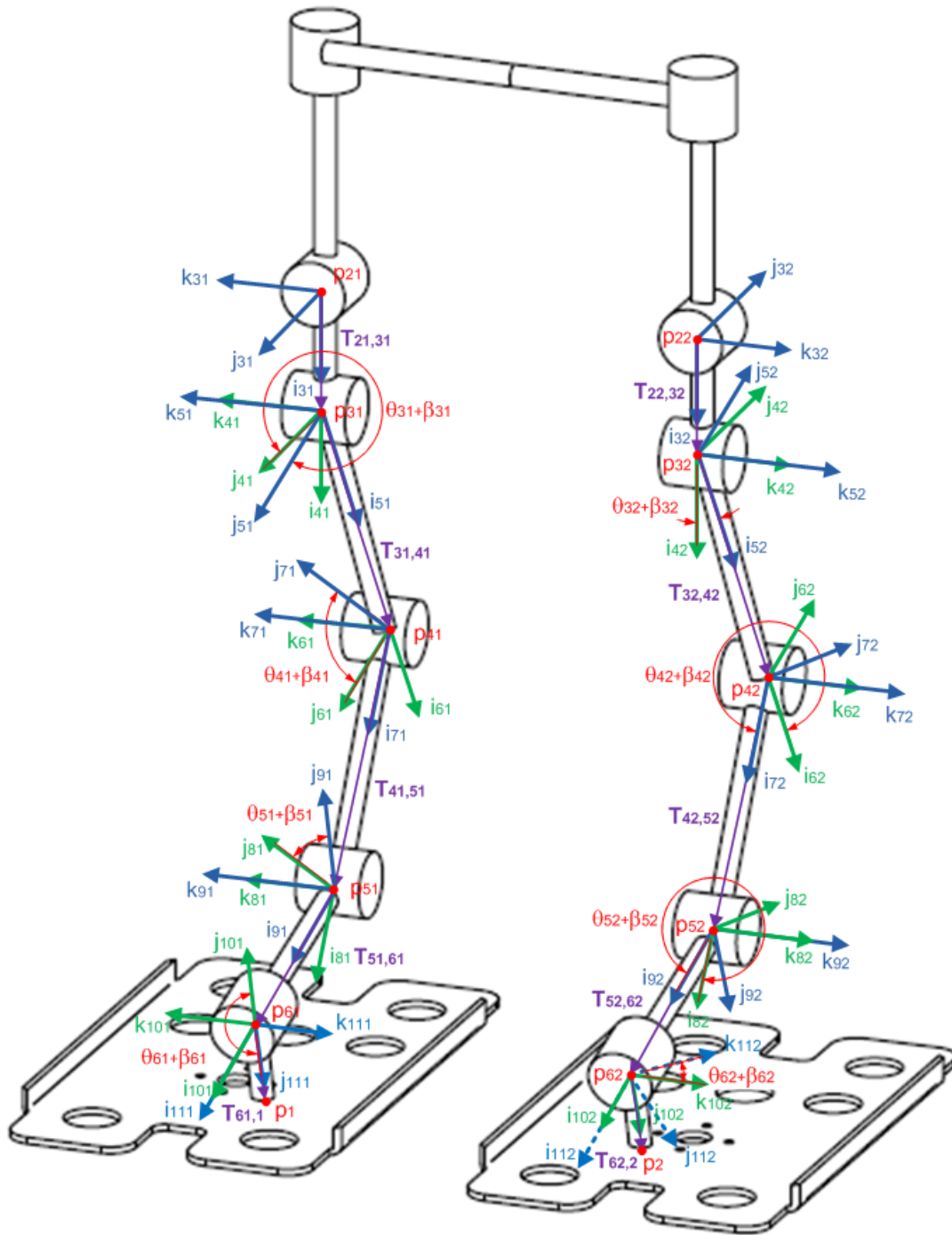
Figura 3.4: Marcos locales empleados para la deducción de la ecuación matricial de posición usando transformaciones homogéneas. Los marcos en verde se obtienen después de realizar una traslación, mientras que los azules se generan al rotar.
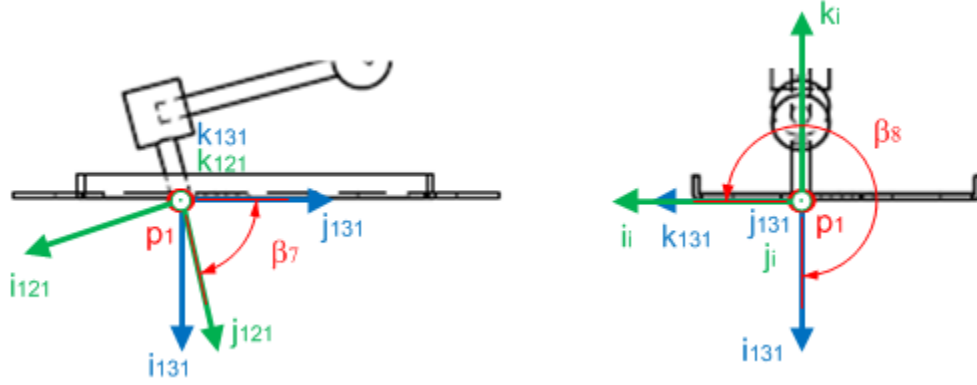
Figura 3.5: Marcos locales instalados en los pies, los ángulos β_7 y β_8 son iguales en ambos.

Las transformaciones $\mathbf{T}_{ni,(n+1)i}$ listadas a continuación, constituyen el producto de las transformaciones homogéneas básicas que permiten trasladar y rotar los marcos locales instalados en $\mathbf{p}_{ni}$ a los que se encuentran en $\mathbf{p}_{(n+1)i}$.

$$\begin{aligned}
\mathbf{T}_{B,11} &= \mathbf{T}_{z1}(-x_{01})\mathbf{T}_{z6}(\theta_{11} + \beta_{11}) \\
\mathbf{T}_{B,12} &= \mathbf{T}_{z1}(x_{02})\mathbf{T}_{z6}(\theta_{12} + \beta_{12}) \\
\mathbf{T}_{1i,2i} &= \mathbf{T}_{z3}(-z_{1i})\mathbf{T}_{z5}(\theta_{2i} + \beta_{2i}) \\
\mathbf{T}_{2i,3i} &= \mathbf{T}_{z1}(x_{2i})\mathbf{T}_{z6}(\theta_{3i} + \beta_{3i}) \\
\mathbf{T}_{3i,4i} &= \mathbf{T}_{z1}(x_{3i})\mathbf{T}_{z6}(\theta_{4i} + \beta_{4i}) \\
\mathbf{T}_{4i,5i} &= \mathbf{T}_{z1}(x_{4i})\mathbf{T}_{z6}(\theta_{5i} + \beta_{5i}) \\
\mathbf{T}_{5i,6i} &= \mathbf{T}_{z1}(x_{5i})\mathbf{T}_{z6}(\theta_{6i} + \beta_{6i}) \\
\mathbf{T}_{6i,i} &= \mathbf{T}_{z2}(y_{6i})\mathbf{T}_{z6}(\beta_7)\mathbf{T}_{z5}(\beta_8)
\end{aligned}$$

Las transformaciones $\mathbf{T}_{B,11}, \mathbf{T}_{B,12}$ corresponden a la alineación del marco del torso con los marcos locales $(\mathbf{p}_{1i}; \mathbf{i}_{1i}, \mathbf{j}_{1i}, \mathbf{k}_{1i})$, mientras que $\mathbf{T}_{6i,i}$ permiten la de los marcos locales $(\mathbf{p}_{6i}; \mathbf{i}_{11i}, \mathbf{j}_{11i}, \mathbf{k}_{11i})$ y $(\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$, de acuerdo a lo mostrado en la figura 3.5. Los ángulos β_{ni} son constantes para recuperar la posición que asume el bípedo con sus servomotores en posición neutral, cuando $\theta_{1i} = \theta_{2i} = \theta_{3i} = \theta_{4i} = \theta_{5i} = \theta_{6i} = 0$; sus valores pueden consultarse en el apéndice B.

Partiendo del hecho que el producto de las transformaciones homogéneas para hacer coincidir el marco inercial con los marcos locales de los pies, pasando por el torso y las juntas rotacionales de cada pierna, debe ser igual al producto de las transformaciones en $\mathbf{T}_{0,i}$; se pueden definir las ecuaciones de lazo matricial:

$$\mathbf{T}_{0,B}\mathbf{T}_{B,1i}\mathbf{T}_{1i,2i}\mathbf{T}_{2i,3i}\mathbf{T}_{3i,4i}\mathbf{T}_{4i,5i}\mathbf{T}_{5i,6i}\mathbf{T}_{6i,i} = \mathbf{T}_{0,i} \tag{3.4}$$

A partir de (3.4) se tienen 12 ecuaciones escalares para la solución del problema cinemático inverso de la posición en cada una de las piernas del robot. Tomando 3 ecuaciones de la última columna y 3 de la imagen esférica, se constituyen sistemas de 6 ecuaciones con 6 incógnitas correspondientes a los ángulos $\theta_{11}, \theta_{21}, \theta_{31}, \theta_{41}, \theta_{51}, \theta_{61}$ en el caso de la pierna izquierda y $\theta_{12}, \theta_{22}, \theta_{32}, \theta_{42}, \theta_{52}, \theta_{62}$ en el de la derecha.

Para incluir las bases de los pies en las simulaciones de la caminata, es indispensable contar con los vectores que describen la posición de los vértices de los polígonos que los representan (figura 3.6).
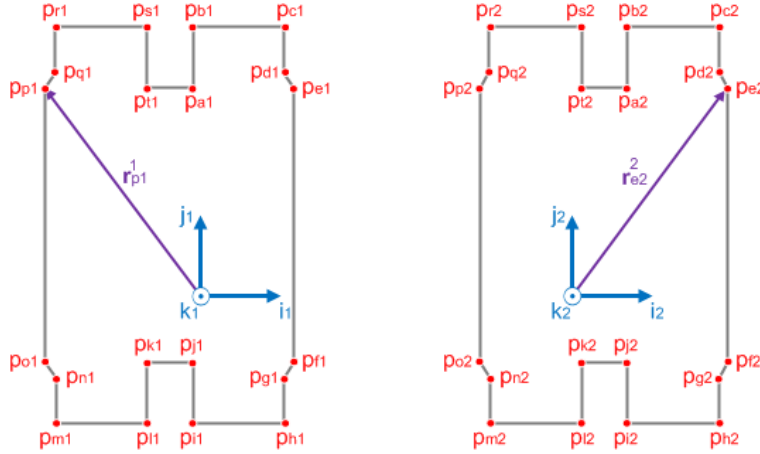
Figura 3.6: Vértices de los polígonos con los que se simulan las bases de los pies. Los vectores $\mathbf{r}_{p1}^1, \mathbf{r}_{e2}^2$ representan la posición de los puntos $\mathbf{p}_{p1}$ y $\mathbf{p}_{e2}$ en los marcos locales $(\mathbf{p}_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1)$ y $(\mathbf{p}_2; \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2)$.

Con los vectores $\mathbf{r}_{ai}^i, \mathbf{r}_{bi}^i, \ldots, \mathbf{r}_{ti}^i$ cuya definición se puede encontrar en el apéndice B, se calculan las coordenadas cartesianas de los vértices en el marco inercial $(ai_{x0}, ai_{y0}, ai_{z0}), (bi_{x0}, bi_{y0}, bi_{z0}), \ldots, (ti_{x0}, ti_{y0}, ti_{z0})$. Tomando como ejemplo los vértices $\mathbf{p}_{ai}$, se opera:

$$\mathbf{T}_{0,i}\mathbf{T}_{z1}(ai_x)\mathbf{T}_{z2}(ai_y) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} ai_{x0} \\ ai_{y0} \\ ai_{z0} \\ 1 \end{bmatrix}$$

Donde ai_x, ai_y, ai_z son las coordenadas de los vértices $\mathbf{p}_{ai}$ respecto a los ejes de los marcos locales $\mathbf{i}_i, \mathbf{j}_i$ y $\mathbf{k}_i$, respectivamente.

3.2. Ecuaciones de velocidad

Para la deducción de las ecuaciones de velocidad se optó por el uso del método vectorial (Cuenca-Jiménez, 2010) basado en las ecuaciones de posición:

$$\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0 = \mathbf{r}_i^0 \tag{3.5}$$

Donde los vectores de posición de la figura 3.7 en el marco inercial son:

$$\begin{aligned} \mathbf{r}_B^0 &= x_B \mathbf{i}_0 + y_B \mathbf{j}_0 + z_B \mathbf{k}_0 \\ \mathbf{r}_{01}^0 &= -x_{01} \mathbf{i}_B^0 \\ \mathbf{r}_{02}^0 &= x_{02} \mathbf{i}_B^0 \\ \mathbf{r}_{1i}^0 &= -z_{1i} \mathbf{k}_{1i}^0 \\ \mathbf{r}_{2i}^0 &= x_{2i} \mathbf{i}_{3i}^0 \\ \mathbf{r}_{3i}^0 &= x_{3i} \mathbf{i}_{5i}^0 \\ \mathbf{r}_{4i}^0 &= x_{4i} \mathbf{i}_{7i}^0 \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
\mathbf{r}_{5i}^0 &= x_{5i}\mathbf{i}_{9i}^0 \\
\mathbf{r}_{6i}^0 &= y_{6i}\mathbf{j}_{11i}^0 \\
\mathbf{r}_i^0 &= x_i\mathbf{i}_0 + y_i\mathbf{j}_0 + z_i\mathbf{k}_0
\end{aligned}
$$
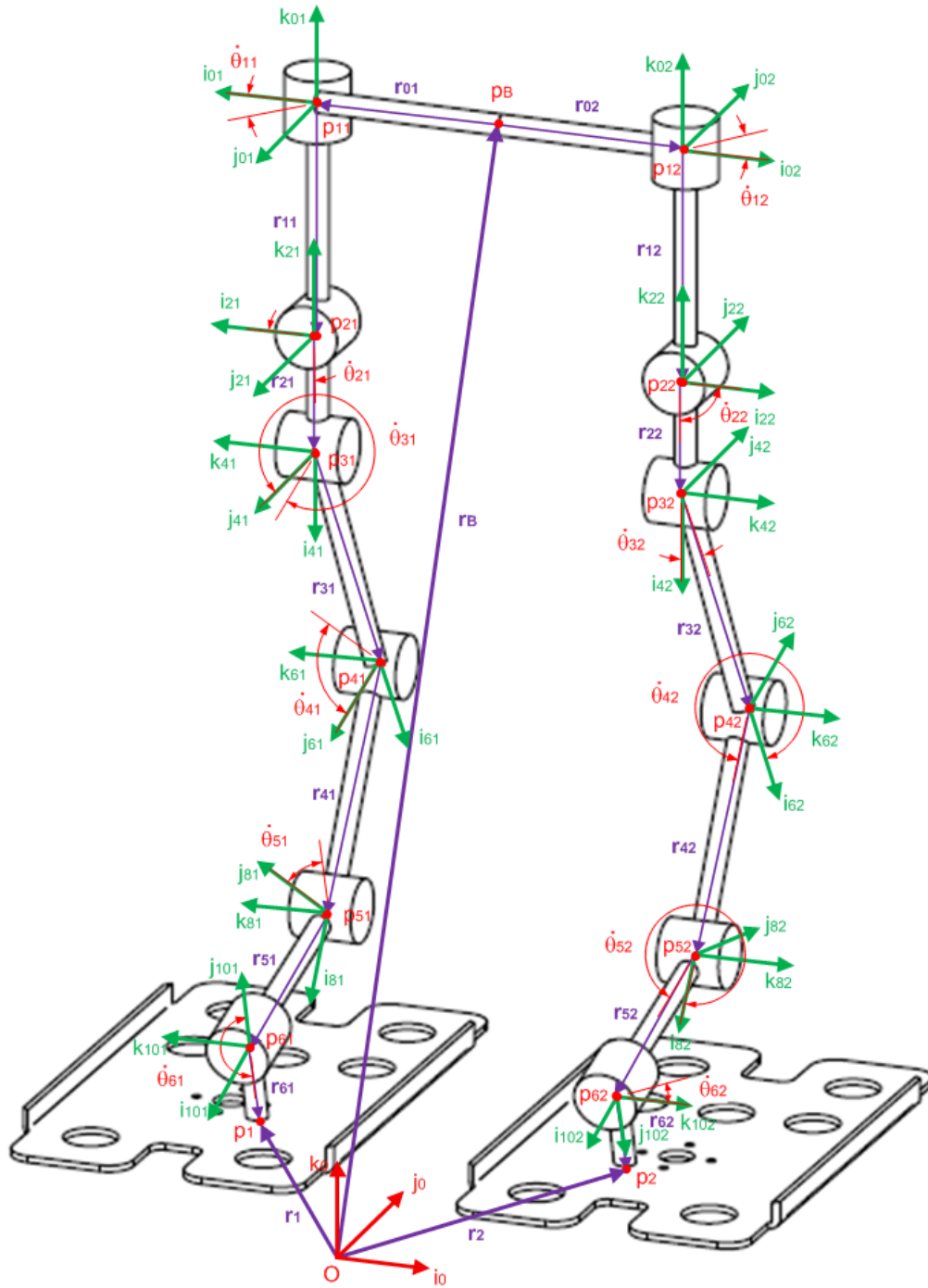
Figura 3.7: Vectores para la deducción de las ecuaciones de velocidad y aceleración (en morado).

Usando matrices para la descripción de rotaciones básicas de ángulos $\theta_x, \theta_y, \theta_z$ sobre los ejes $\mathbf{i}, \mathbf{j}, \mathbf{k}$ de la forma:

$$\mathbf{R}_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}$$

$$\mathbf{R}_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$$\mathbf{R}_z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Los vectores unitarios $\mathbf{i}_B, \mathbf{k}_{1i}, \mathbf{i}_{3i}, \mathbf{i}_{5i}, \mathbf{i}_{7i}, \mathbf{i}_{9i}, \mathbf{j}_{11i}$ se describen en el marco inercial como:

$$
\begin{aligned}
\mathbf{i}_B^0 &= \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{i}_B \\
&= \mathbf{R}_B^0\mathbf{i}_B \\
\mathbf{k}_{1i}^0 &= \mathbf{R}_B^0\mathbf{R}_z(\theta_{1i} + \beta_{1i})\mathbf{k}_{1i} \\
&= \mathbf{R}_{1i}^0\mathbf{k}_{1i} \\
\mathbf{i}_{3i}^0 &= \mathbf{R}_{1i}^0\mathbf{R}_y(\theta_{2i} + \beta_{2i})\mathbf{i}_{3i} \\
&= \mathbf{R}_{3i}^0\mathbf{i}_{3i} \\
\mathbf{i}_{5i}^0 &= \mathbf{R}_{3i}^0\mathbf{R}_z(\theta_{3i} + \beta_{3i})\mathbf{i}_{5i} \\
&= \mathbf{R}_{5i}^0\mathbf{i}_{5i} \\
\mathbf{i}_{7i}^0 &= \mathbf{R}_{5i}^0\mathbf{R}_z(\theta_{4i} + \beta_{4i})\mathbf{i}_{7i} \\
&= \mathbf{R}_{7i}^0\mathbf{i}_{7i} \\
\mathbf{i}_{9i}^0 &= \mathbf{R}_{7i}^0\mathbf{R}_z(\theta_{5i} + \beta_{5i})\mathbf{i}_{9i} \\
&= \mathbf{R}_{9i}^0\mathbf{i}_{9i} \\
\mathbf{j}_{11i}^0 &= \mathbf{R}_{9i}^0\mathbf{R}_x(\theta_{6i} + \beta_{6i})\mathbf{j}_{11i} \\
&= \mathbf{R}_{11i}^0\mathbf{j}_{11i}
\end{aligned}
\tag{3.7}
$$

Derivando con respecto al tiempo las ecuaciones de posición vectorial en (3.5) se escribe:

$$\dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{2i}^0 + \dot{\mathbf{r}}_{3i}^0 + \dot{\mathbf{r}}_{4i}^0 + \dot{\mathbf{r}}_{5i}^0 + \dot{\mathbf{r}}_{6i}^0 = \dot{\mathbf{r}}_i^0$$

Asignando $\dot{\mathbf{r}} = \mathbf{v}$, se tienen las ecuaciones vectoriales de velocidad lineal para cada pierna:

$$\mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \mathbf{v}_{5i}^0 + \mathbf{v}_{6i}^0 = \mathbf{v}_i^0 \tag{3.8}$$

Donde los vectores de velocidad en el marco inercial están dados por:

$$
\begin{aligned}
\mathbf{v}_B^0 &= \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 \\
\mathbf{v}_{0i}^0 &= \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v}_{1i}^0 &= \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 \\
\mathbf{v}_{2i}^0 &= \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 \\
\mathbf{v}_{3i}^0 &= \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 \\
\mathbf{v}_{4i}^0 &= \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 \\
\mathbf{v}_{5i}^0 &= \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0 \\
\mathbf{v}_{6i}^0 &= \boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0 \\
\mathbf{v}_{i}^0 &= \dot{x}_i \mathbf{i}_0 + \dot{y}_i \mathbf{j}_0 + \dot{z}_i \mathbf{k}_0
\end{aligned}
\tag{3.9}
$$

El vector de velocidad angular del torso $\boldsymbol{\omega}_B^0$ es:

$$
\begin{aligned}
\boldsymbol{\omega}_B^0 &= \boldsymbol{\omega}_{\theta B}^0 + \boldsymbol{\omega}_{\phi B}^0 + \boldsymbol{\omega}_{\psi B}^0 \\
\boldsymbol{\omega}_{\theta B}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 \\
\boldsymbol{\omega}_{\phi B}^0 &= \dot{\phi}_B \mathbf{j}_{\phi B}^0 \\
\boldsymbol{\omega}_{\psi B}^0 &= \dot{\psi}_B \mathbf{k}_{\psi B}^0
\end{aligned}
$$

donde

$$
\begin{aligned}
\mathbf{i}_{\theta B}^0 &= \mathbf{i}_0 \\
\mathbf{j}_{\phi B}^0 &= \mathbf{R}_x(\theta_B)\mathbf{j}_{\phi B} \\
\mathbf{k}_{\psi B}^0 &= \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{k}_{\psi B}
\end{aligned}
$$

Los vectores de velocidad angular en las juntas rotacionales $\boldsymbol{\omega}_{ni}^0$ son:

$$
\begin{aligned}
\boldsymbol{\omega}_{1i}^0 &= \boldsymbol{\omega}_B^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 \\
\boldsymbol{\omega}_{2i}^0 &= \boldsymbol{\omega}_{1i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0 \\
\boldsymbol{\omega}_{3i}^0 &= \boldsymbol{\omega}_{2i}^0 + \dot{\theta}_{3i} \mathbf{k}_{4i}^0 \\
\boldsymbol{\omega}_{4i}^0 &= \boldsymbol{\omega}_{3i}^0 + \dot{\theta}_{4i} \mathbf{k}_{6i}^0 \\
\boldsymbol{\omega}_{5i}^0 &= \boldsymbol{\omega}_{4i}^0 + \dot{\theta}_{5i} \mathbf{k}_{8i}^0 \\
\boldsymbol{\omega}_{6i}^0 &= \boldsymbol{\omega}_{5i}^0 + \dot{\theta}_{6i} \mathbf{i}_{10i}^0
\end{aligned}
\tag{3.10}
$$

donde

$$
\begin{aligned}
\mathbf{k}_{0i}^0 &= \mathbf{R}_B^0 \mathbf{k}_{0i} \\
\mathbf{j}_{2i}^0 &= \mathbf{R}_{1i}^0 \mathbf{j}_{2i} \\
\mathbf{k}_{4i}^0 &= \mathbf{R}_{3i}^0 \mathbf{k}_{4i} \\
\mathbf{k}_{6i}^0 &= \mathbf{R}_{5i}^0 \mathbf{k}_{6i} \\
\mathbf{k}_{8i}^0 &= \mathbf{R}_{7i}^0 \mathbf{k}_{8i} \\
\mathbf{i}_{10i}^0 &= \mathbf{R}_{9i}^0 \mathbf{i}_{10i}
\end{aligned}
$$

Mientras que los vectores de velocidad angular asociados con los pies $\boldsymbol{\omega}_i^0$ son:

$$\begin{aligned}
\boldsymbol{\omega}_i^0 &= \boldsymbol{\omega}_{\theta i}^0 + \boldsymbol{\omega}_{\phi i}^0 + \boldsymbol{\omega}_{\psi i}^0 \\
\boldsymbol{\omega}_{\theta i}^0 &= \dot{\theta}_i \mathbf{i}_{\theta i}^0 \\
\boldsymbol{\omega}_{\phi i}^0 &= \dot{\phi}_i \mathbf{j}_{\phi i}^0 \\
\boldsymbol{\omega}_{\psi i}^0 &= \dot{\psi}_i \mathbf{k}_{\psi i}^0
\end{aligned}$$

donde

$$\begin{aligned}
\mathbf{i}_{\theta i}^0 &= \mathbf{i}_0 \\
\mathbf{j}_{\phi i}^0 &= \mathbf{R}_x(\theta_i)\mathbf{j}_{\phi i} \\
\mathbf{k}_{\psi i}^0 &= \mathbf{R}_x(\theta_i)\mathbf{R}_y(\phi_i)\mathbf{k}_{\psi i}
\end{aligned}$$

Se recuerda que $\theta_B, \phi_B, \theta_i, \phi_i$ son ángulos de Euler para la descripción de la orientación de los marcos locales en el torso y los pies.

Sustituyendo los vectores (3.9) en las ecuaciones (3.8), se tiene:

$$\dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0 + \boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0 = \dot{x}_i\mathbf{i}_0 + \dot{y}_i\mathbf{j}_0 + \dot{z}_i\mathbf{k}_0$$

En tanto que:

$$\begin{aligned}
\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0) \times \mathbf{r}_{0i}^0 \\
\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0) \times \mathbf{r}_{1i}^0 \\
\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0) \times \mathbf{r}_{2i}^0 \\
\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0) \times \mathbf{r}_{3i}^0 \\
\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0) \times \mathbf{r}_{4i}^0 \\
\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0) \times \mathbf{r}_{5i}^0 \\
\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0 &= (\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0 + \dot{\theta}_{6i}\mathbf{i}_{10i}^0) \times \mathbf{r}_{6i}^0
\end{aligned}$$

Las ecuaciones vectoriales de velocidad lineal se pueden reescribir como:

$$\begin{aligned}
\dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\left[\mathbf{i}_{\theta B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\phi}_B\left[\mathbf{j}_{\phi B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\psi}_B\left[\mathbf{k}_{\psi B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{1i}\left[\mathbf{k}_{0i}^0 \times (\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{2i}\left[\mathbf{j}_{2i}^0 \times (\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{3i}\left[\mathbf{k}_{4i}^0 \times (\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{4i}\left[\mathbf{k}_{6i}^0 \times (\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{5i}\left[\mathbf{k}_{8i}^0 \times (\mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0)\right] &+ \\
\dot{\theta}_{6i}\left[\mathbf{i}_{10i}^0 \times \mathbf{r}_{6i}^0\right] &= \dot{x}_i\mathbf{i}_0 + \dot{y}_i\mathbf{j}_0 + \dot{z}_i\mathbf{k}_0 \quad (3.11)
\end{aligned}$$

Definiendo los vectores:

$$
\begin{aligned}
\mathbf{u}_{\theta_B}^0 &= \mathbf{i}_{\theta B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\phi_B}^0 &= \mathbf{j}_{\phi B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\psi_B}^0 &= \mathbf{k}_{\psi B}^0 \times (\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{1i}}^0 &= \mathbf{k}_{0i}^0 \times (\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{2i}}^0 &= \mathbf{j}_{2i}^0 \times (\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{3i}}^0 &= \mathbf{k}_{4i}^0 \times (\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{4i}}^0 &= \mathbf{k}_{6i}^0 \times (\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{5i}}^0 &= \mathbf{k}_{8i}^0 \times (\mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0) \\
\mathbf{u}_{\theta_{6i}}^0 &= \mathbf{i}_{10i}^0 \times \mathbf{r}_{6i}^0
\end{aligned}
\tag{3.12}
$$

Es posible simplificar la notación de las ecuaciones en (3.11) como:

$$
\dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{u}_{\theta_B}^0 + \dot{\phi}_B\mathbf{u}_{\phi_B}^0 + \dot{\psi}_B\mathbf{u}_{\psi_B}^0 + \dot{\theta}_{1i}\mathbf{u}_{\theta_{1i}}^0 + \dot{\theta}_{2i}\mathbf{u}_{\theta_{2i}}^0 + \dot{\theta}_{3i}\mathbf{u}_{\theta_{3i}}^0 + \dot{\theta}_{4i}\mathbf{u}_{\theta_{4i}}^0 + \dot{\theta}_{5i}\mathbf{u}_{\theta_{5i}}^0 + \dot{\theta}_{6i}\mathbf{u}_{\theta_{6i}}^0 = \dot{x}_i\mathbf{i}_0 + \dot{y}_i\mathbf{j}_0 + \dot{z}_i\mathbf{k}_0
$$

Colocando todos los términos que incluyen los datos $\dot{x}_B, \dot{y}_B, \dot{z}_B, \dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{x}_i, \dot{y}_i, \dot{z}_i$ en el lado derecho, se tienen las ecuaciones vectoriales de velocidad lineal:

$$
\dot{\theta}_{1i}\mathbf{u}_{\theta_{1i}}^0 + \dot{\theta}_{2i}\mathbf{u}_{\theta_{2i}}^0 + \dot{\theta}_{3i}\mathbf{u}_{\theta_{3i}}^0 + \dot{\theta}_{4i}\mathbf{u}_{\theta_{4i}}^0 + \dot{\theta}_{5i}\mathbf{u}_{\theta_{5i}}^0 + \dot{\theta}_{6i}\mathbf{u}_{\theta_{6i}}^0 = \dot{x}_i\mathbf{i}_0 + \dot{y}_i\mathbf{j}_0 + \dot{z}_i\mathbf{k}_0 - \dot{x}_B\mathbf{i}_0 - \dot{y}_B\mathbf{j}_0 - \dot{z}_B\mathbf{k}_0 - \dot{\theta}_B\mathbf{u}_{\theta_B}^0 - \dot{\phi}_B\mathbf{u}_{\phi_B}^0 - \dot{\psi}_B\mathbf{u}_{\psi_B}^0
\tag{3.13}
$$

Las ecuaciones de velocidad angular para cada una de las piernas son:

$$
\boldsymbol{\omega}_{6i}^0 = \boldsymbol{\omega}_i^0 \tag{3.14}
$$

$$
\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0 + \dot{\theta}_{6i}\mathbf{i}_{10i}^0 = \dot{\theta}_i\mathbf{i}_{\theta i}^0 + \dot{\phi}_i\mathbf{j}_{\phi i}^0 + \dot{\psi}_i\mathbf{k}_{\psi i}^0
$$

Pasando todos los términos que incluyen los datos $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i$ del lado derecho, las ecuaciones vectoriales de velocidad angular se escriben:

$$
\dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0 + \dot{\theta}_{6i}\mathbf{i}_{10i}^0 = \dot{\theta}_i\mathbf{i}_{\theta i}^0 + \dot{\phi}_i\mathbf{j}_{\phi i}^0 + \dot{\psi}_i\mathbf{k}_{\psi i}^0 - \dot{\theta}_B\mathbf{i}_{\theta B}^0 - \dot{\phi}_B\mathbf{j}_{\phi B}^0 - \dot{\psi}_B\mathbf{k}_{\psi B}^0 \tag{3.15}
$$

Considerando que la solución numérica de las ecuaciones (3.4) ya ha proporcionado los valores de posición angular $\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}$; con (3.13) y (3.15) se pueden constituir sistemas de 6 ecuaciones escalares con 6 incógnitas correspondientes a $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$ que en forma matricial son:

$$
\begin{bmatrix} \mathbf{u}_{\theta_{1i}}^0 & \mathbf{u}_{\theta_{2i}}^0 & \mathbf{u}_{\theta_{3i}}^0 & \mathbf{u}_{\theta_{4i}}^0 & \mathbf{u}_{\theta_{5i}}^0 & \mathbf{u}_{\theta_{6i}}^0 \\ \mathbf{k}_{0i}^0 & \mathbf{j}_{2i}^0 & \mathbf{k}_{4i}^0 & \mathbf{k}_{6i}^0 & \mathbf{k}_{8i}^0 & \mathbf{i}_{10i}^0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_{1i} \\ \dot{\theta}_{2i} \\ \dot{\theta}_{3i} \\ \dot{\theta}_{4i} \\ \dot{\theta}_{5i} \\ \dot{\theta}_{6i} \end{bmatrix} = \mathbf{V}_i \tag{3.16}
$$

donde

$$\mathbf{V}_i = \left[\begin{array}{cccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta i}^0 & \mathbf{j}_{\phi i}^0 & \mathbf{k}_{\psi i}^0 \end{array} \right] \left[\begin{array}{c} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \\ \dot{\theta}_i \\ \dot{\phi}_i \\ \dot{\psi}_i \end{array} \right] - \left[\begin{array}{cccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{u}_{\theta_B}^0 & \mathbf{u}_{\phi_B}^0 & \mathbf{u}_{\psi_B}^0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 \end{array} \right] \left[\begin{array}{c} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \\ \dot{\theta}_B \\ \dot{\phi}_B \\ \dot{\psi}_B \end{array} \right]$$

Todos los vectores se escriben en columna para la constitución de matrices cuadradas de 6×6, por lo que el vector $\mathbf{0}$ tiene la forma:

$$\mathbf{0} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]$$

La solución numérica de las ecuaciones matriciales en (3.16) proporcionará los valores de velocidad angular $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$.

3.3. Ecuaciones de aceleración

Continuando con el método en (Cuenca-Jiménez, 2010) y tomando en consideración el cambio respecto al tiempo de (3.8), se tiene:

$$\dot{\mathbf{v}}_B^0 + \dot{\mathbf{v}}_{0i}^0 + \dot{\mathbf{v}}_{1i}^0 + \dot{\mathbf{v}}_{2i}^0 + \dot{\mathbf{v}}_{3i}^0 + \dot{\mathbf{v}}_{4i}^0 + \dot{\mathbf{v}}_{5i}^0 + \dot{\mathbf{v}}_{6i}^0 = \dot{\mathbf{v}}_i^0$$

Asignando $\dot{\mathbf{v}} = \mathbf{a}$, las ecuaciones vectoriales de aceleración lineal son:

$$\mathbf{a}_B^0 + \mathbf{a}_{0i}^0 + \mathbf{a}_{1i}^0 + \mathbf{a}_{2i}^0 + \mathbf{a}_{3i}^0 + \mathbf{a}_{4i}^0 + \mathbf{a}_{5i}^0 + \mathbf{a}_{6i}^0 = \mathbf{a}_i^0 \tag{3.17}$$

Donde los vectores de aceleración en el marco inercial están dados por:

$$\begin{aligned} \mathbf{a}_B^0 &= \ddot{x}_B \mathbf{i}_0 + \ddot{y}_B \mathbf{j}_0 + \ddot{z}_B \mathbf{k}_0 \\ \mathbf{a}_{0i}^0 &= \boldsymbol{\alpha}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_B^0 \times \left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 \right) \\ \mathbf{a}_{1i}^0 &= \boldsymbol{\alpha}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{1i}^0 \times \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 \right) \\ \mathbf{a}_{2i}^0 &= \boldsymbol{\alpha}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{2i}^0 \times \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 \right) \\ \mathbf{a}_{3i}^0 &= \boldsymbol{\alpha}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{3i}^0 \times \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 \right) \\ \mathbf{a}_{4i}^0 &= \boldsymbol{\alpha}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\omega}_{4i}^0 \times \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 \right) \\ \mathbf{a}_{5i}^0 &= \boldsymbol{\alpha}_{5i}^0 \times \mathbf{r}_{5i}^0 + \boldsymbol{\omega}_{5i}^0 \times \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0 \right) \\ \mathbf{a}_{6i}^0 &= \boldsymbol{\alpha}_{6i}^0 \times \mathbf{r}_{6i}^0 + \boldsymbol{\omega}_{6i}^0 \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0 \right) \\ \mathbf{a}_i^0 &= \ddot{x}_i \mathbf{i}_0 + \ddot{y}_i \mathbf{j}_0 + \ddot{z}_i \mathbf{k}_0 \end{aligned}$$

Sustituyendo estos vectores en (3.17), las ecuaciones de aceleración lineal para cada pierna quedan:

$$\ddot{x}_B \mathbf{i}_0 + \ddot{y}_B \mathbf{j}_0 + \ddot{z}_B \mathbf{k}_0 \quad +$$

$$\begin{aligned}
\boldsymbol{\alpha}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_B^0 \times \left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) & + \\
\boldsymbol{\alpha}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{1i}^0 \times \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) & + \\
\boldsymbol{\alpha}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{2i}^0 \times \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) & + \\
\boldsymbol{\alpha}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{3i}^0 \times \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) & + \\
\boldsymbol{\alpha}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\omega}_{4i}^0 \times \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) & + \\
\boldsymbol{\alpha}_{5i}^0 \times \mathbf{r}_{5i}^0 + \boldsymbol{\omega}_{5i}^0 \times \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) & + \\
\boldsymbol{\alpha}_{6i}^0 \times \mathbf{r}_{6i}^0 + \boldsymbol{\omega}_{6i}^0 \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right) & = \quad \ddot{x}_i \mathbf{i}_0 + \ddot{y}_i \mathbf{j}_0 + \ddot{z}_i \mathbf{k}_0
\end{aligned}$$

Agrupando los términos del lado izquierdo de estas ecuaciones que incluyen productos cruz en los vectores $\mathbf{x}_i^0, \mathbf{y}_i^0$, de manera que:

$$\mathbf{x}_i^0 = \boldsymbol{\alpha}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\alpha}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\alpha}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\alpha}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\alpha}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\alpha}_{5i}^0 \times \mathbf{r}_{5i}^0 + \boldsymbol{\alpha}_{6i}^0 \times \mathbf{r}_{6i}^0$$

$$\begin{aligned}
\mathbf{y}_i^0 = {} & \boldsymbol{\omega}_B^0 \times \left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \boldsymbol{\omega}_{1i}^0 \times \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \boldsymbol{\omega}_{2i}^0 \times \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \boldsymbol{\omega}_{3i}^0 \times \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \\
& \boldsymbol{\omega}_{4i}^0 \times \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \boldsymbol{\omega}_{5i}^0 \times \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \boldsymbol{\omega}_{6i}^0 \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)
\end{aligned}$$

Las ecuaciones de aceleración lineal se pueden reescribir como:

$$\ddot{x}_B \mathbf{i}_0 + \ddot{y}_B \mathbf{j}_0 + \ddot{z}_B \mathbf{k}_0 + \mathbf{x}_i^0 + \mathbf{y}_i^0 = \ddot{x}_i \mathbf{i}_0 + \ddot{y}_i \mathbf{j}_0 + \ddot{z}_i \mathbf{k}_0 \tag{3.18}$$

El vector de aceleración angular del torso $\boldsymbol{\alpha}_B^0$ es:

$$\begin{aligned}
\boldsymbol{\alpha}_B^0 & = \dot{\boldsymbol{\omega}}_B^0 \\
& = \boldsymbol{\alpha}_{\theta_B}^0 + \boldsymbol{\alpha}_{\phi_B}^0 + \boldsymbol{\alpha}_{\psi_B}^0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\alpha}_{\theta_B}^0 & = \dot{\boldsymbol{\omega}}_{\theta_B}^0 \\
& = \ddot{\theta}_B \mathbf{i}_0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\alpha}_{\phi_B}^0 & = \dot{\boldsymbol{\omega}}_{\phi_B}^0 \\
& = \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\alpha}_{\psi_B}^0 & = \dot{\boldsymbol{\omega}}_{\psi_B}^0 \\
& = \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 + \boldsymbol{\omega}_{\phi_B}^0\right) \times \boldsymbol{\omega}_{\psi_B}^0
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\alpha}_B^0 & = \ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right) + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 + \boldsymbol{\omega}_{\phi_B}^0\right) \times \boldsymbol{\omega}_{\psi_B}^0 \\
& = \ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right) + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) + \left(\boldsymbol{\omega}_{\phi_B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) \\
& = \ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right) + \left(\boldsymbol{\omega}_{\phi_B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) \\
& = \ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right) + \left(\dot{\phi}_B \mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) + \left(\boldsymbol{\omega}_{\theta_B}^0 \times \dot{\psi}_B \mathbf{k}_{\psi B}^0\right) \\
& = \ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0\right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0\right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0\right)
\end{aligned}$$

Los vectores de aceleración angular en las juntas rotacionales $\boldsymbol{\alpha}_{ni}^0$ son:

$$
\begin{aligned}
\boldsymbol{\alpha}_{1i}^0 &= \dot{\boldsymbol{\omega}}_{1i}^0 \\
&= \boldsymbol{\alpha}_B^0 + \ddot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{1i}\left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{2i}^0 &= \dot{\boldsymbol{\omega}}_{2i}^0 \\
&= \boldsymbol{\alpha}_{1i}^0 + \ddot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{2i}\left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{3i}^0 &= \dot{\boldsymbol{\omega}}_{3i}^0 \\
&= \boldsymbol{\alpha}_{2i}^0 + \ddot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{3i}\left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{4i}^0 &= \dot{\boldsymbol{\omega}}_{4i}^0 \\
&= \boldsymbol{\alpha}_{3i}^0 + \ddot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{4i}\left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{5i}^0 &= \dot{\boldsymbol{\omega}}_{5i}^0 \\
&= \boldsymbol{\alpha}_{4i}^0 + \ddot{\theta}_{5i}\mathbf{k}_{8i}^0 + \dot{\theta}_{5i}\left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{k}_{8i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{6i}^0 &= \dot{\boldsymbol{\omega}}_{6i}^0 \\
&= \boldsymbol{\alpha}_{5i}^0 + \ddot{\theta}_{6i}\mathbf{i}_{10i}^0 + \dot{\theta}_{6i}\left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{i}_{10i}^0\right)
\end{aligned}
$$

Mientras que los vectores de aceleración angular en los pies $\boldsymbol{\alpha}_i^0$ son:

$$
\boldsymbol{\alpha}_i^0 = \boldsymbol{\alpha}_{\theta i}^0 + \boldsymbol{\alpha}_{\phi i}^0 + \boldsymbol{\alpha}_{\psi i}^0
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{\theta i}^0 &= \dot{\boldsymbol{\omega}}_{\theta i}^0 \\
&= \ddot{\theta}_i \mathbf{i}_0
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{\phi i}^0 &= \dot{\boldsymbol{\omega}}_{\phi i}^0 \\
&= \ddot{\phi}_i \mathbf{j}_{\phi_i}^0 + \left(\boldsymbol{\omega}_{\theta_i}^0 \times \boldsymbol{\omega}_{\phi_i}^0\right) \\
&= \ddot{\phi}_i \mathbf{j}_{\phi_i}^0 + \left(\dot{\theta}_i \mathbf{i}_{\theta_i}^0 \times \boldsymbol{\omega}_{\phi_i}^0\right) \\
&= \ddot{\phi}_i \mathbf{j}_{\phi_i}^0 + \dot{\theta}_i \left(\mathbf{i}_{\theta_i}^0 \times \boldsymbol{\omega}_{\phi_i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\alpha}_{\psi i}^0 &= \dot{\boldsymbol{\omega}}_{\psi i}^0 \\
&= \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \left(\boldsymbol{\omega}_{\theta_i}^0 + \boldsymbol{\omega}_{\phi_i}^0\right) \times \boldsymbol{\omega}_{\psi_i}^0 \\
&= \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \left(\boldsymbol{\omega}_{\theta_i}^0 \times \boldsymbol{\omega}_{\psi_i}^0\right) + \left(\boldsymbol{\omega}_{\phi_i}^0 \times \boldsymbol{\omega}_{\psi_i}^0\right) \\
&= \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \left(\boldsymbol{\omega}_{\theta_i}^0 \times \dot{\psi}_i \mathbf{k}_{\psi i}^0\right) + \left(\dot{\phi}_i \mathbf{j}_{\phi i}^0 \times \boldsymbol{\omega}_{\psi_i}^0\right) \\
&= \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \dot{\psi}_i \left(\boldsymbol{\omega}_{\theta_i}^0 \times \mathbf{k}_{\psi_i}^0\right) + \dot{\phi}_i \left(\mathbf{j}_{\phi i}^0 \times \boldsymbol{\omega}_{\psi_i}^0\right)
\end{aligned}
$$

$$\boldsymbol{\alpha}_i^0 = \ddot{\theta}_i \mathbf{i}_0 + \ddot{\phi}_i \mathbf{j}_{\phi_i}^0 + \dot{\theta}_i \left(\mathbf{i}_{\theta i}^0 \times \boldsymbol{\omega}_{\phi_i}^0 \right) + \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \dot{\psi}_i \left(\boldsymbol{\omega}_{\theta_i}^0 \times \mathbf{k}_{\psi i}^0 \right) + \dot{\phi}_i \left(\mathbf{j}_{\phi i}^0 \times \boldsymbol{\omega}_{\psi_i}^0 \right)$$

$$= \ddot{\theta}_i \mathbf{i}_0 + \ddot{\phi}_i \mathbf{j}_{\phi_i}^0 + \ddot{\psi}_i \mathbf{k}_{\psi_i}^0 + \dot{\theta}_i \left(\mathbf{i}_{\theta i}^0 \times \boldsymbol{\omega}_{\phi_i}^0 \right) + \dot{\phi}_i \left(\mathbf{j}_{\phi i}^0 \times \boldsymbol{\omega}_{\psi_i}^0 \right) + \dot{\psi}_i \left(\boldsymbol{\omega}_{\theta_i}^0 \times \mathbf{k}_{\psi i}^0 \right)$$

Desarrollando los productos cruz de los vectores de aceleración angular y posición, se tiene:

$$\boldsymbol{\alpha}_B^0 \times \mathbf{r}_{0i}^0 = \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) \right) \times \mathbf{r}_{0i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 \right) \times \mathbf{r}_{0i}^0 + \left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) \right) \times \mathbf{r}_{0i}^0$$

$$\boldsymbol{\alpha}_{1i}^0 \times \mathbf{r}_{1i}^0 = \left(\boldsymbol{\alpha}_B^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) \right) \times \mathbf{r}_{1i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 \right) \times \mathbf{r}_{1i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) + \dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) \right) \times \mathbf{r}_{1i}^0$$

$$\boldsymbol{\alpha}_{2i}^0 \times \mathbf{r}_{2i}^0 = \left(\boldsymbol{\alpha}_{1i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 + \dot{\theta}_{2i} \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0 \right) \right) \times \mathbf{r}_{2i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 \right) \times \mathbf{r}_{2i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) + \dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0 \right) \right) \times \mathbf{r}_{2i}^0$$

$$\boldsymbol{\alpha}_{3i}^0 \times \mathbf{r}_{3i}^0 = \left(\boldsymbol{\alpha}_{2i}^0 + \ddot{\theta}_{3i} \mathbf{k}_{4i}^0 + \dot{\theta}_{3i} \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0 \right) \right) \times \mathbf{r}_{3i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 + \ddot{\theta}_{3i} \mathbf{k}_{4i}^0 \right) \times \mathbf{r}_{3i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) \right) \times \mathbf{r}_{3i}^0 +$$

$$\left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0 \right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0 \right) \right) \times \mathbf{r}_{3i}^0$$

$$\boldsymbol{\alpha}_{4i}^0 \times \mathbf{r}_{4i}^0 = \left(\boldsymbol{\alpha}_{3i}^0 + \ddot{\theta}_{4i} \mathbf{k}_{6i}^0 + \dot{\theta}_{4i} \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0 \right) \right) \times \mathbf{r}_{4i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 + \ddot{\theta}_{3i} \mathbf{k}_{4i}^0 + \ddot{\theta}_{4i} \mathbf{k}_{6i}^0 \right) \times \mathbf{r}_{4i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) \right) \times \mathbf{r}_{4i}^0 +$$

$$\left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0 \right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0 \right) + \dot{\theta}_{4i} \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0 \right) \right) \times \mathbf{r}_{4i}^0$$

$$\boldsymbol{\alpha}_{5i}^0 \times \mathbf{r}_{5i}^0 = \left(\boldsymbol{\alpha}_{4i}^0 + \ddot{\theta}_{5i} \mathbf{k}_{8i}^0 + \dot{\theta}_{5i} \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{k}_{8i}^0 \right) \right) \times \mathbf{r}_{5i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 + \ddot{\theta}_{3i} \mathbf{k}_{4i}^0 + \ddot{\theta}_{4i} \mathbf{k}_{6i}^0 + \ddot{\theta}_{5i} \mathbf{k}_{8i}^0 \right) \times \mathbf{r}_{5i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi_B}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi_B}^0 \right) + \dot{\psi}_B \left(\boldsymbol{\omega}_{\theta_B}^0 \times \mathbf{k}_{\psi B}^0 \right) \right) \times \mathbf{r}_{5i}^0 +$$

$$\left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0 \right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0 \right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0 \right) + \dot{\theta}_{4i} \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0 \right) + \dot{\theta}_{5i} \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{k}_{8i}^0 \right) \right) \times \mathbf{r}_{5i}^0$$

$$\boldsymbol{\alpha}_{6i}^0 \times \mathbf{r}_{6i}^0 = \left(\boldsymbol{\alpha}_{5i}^0 + \ddot{\theta}_{6i} \mathbf{i}_{10i}^0 + \dot{\theta}_{6i} \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{i}_{10i}^0 \right) \right) \times \mathbf{r}_{6i}^0$$

$$= \left(\ddot{\theta}_B \mathbf{i}_0 + \ddot{\phi}_B \mathbf{j}_{\phi_B}^0 + \ddot{\psi}_B \mathbf{k}_{\psi_B}^0 + \ddot{\theta}_{1i} \mathbf{k}_{0i}^0 + \ddot{\theta}_{2i} \mathbf{j}_{2i}^0 + \ddot{\theta}_{3i} \mathbf{k}_{4i}^0 + \ddot{\theta}_{4i} \mathbf{k}_{6i}^0 + \ddot{\theta}_{5i} \mathbf{k}_{8i}^0 + \ddot{\theta}_{6i} \mathbf{i}_{10i}^0 \right) \times \mathbf{r}_{6i}^0 +$$

$$\left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta_B} \times \boldsymbol{\omega}^0_{\phi_B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi_B} \times \boldsymbol{\omega}^0_{\psi_B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta_B} \times \mathbf{k}^0_{\psi_B}\right)\right) \times \mathbf{r}^0_{6i} +$$

$$\left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}\right)\right) \times \mathbf{r}^0_{6i} +$$

$$\left(\dot{\theta}_{4i} \left(\boldsymbol{\omega}^0_{3i} \times \mathbf{k}^0_{6i}\right) + \dot{\theta}_{5i} \left(\boldsymbol{\omega}^0_{4i} \times \mathbf{k}^0_{8i}\right) + \dot{\theta}_{6i} \left(\boldsymbol{\omega}^0_{5i} \times \mathbf{i}^0_{10i}\right)\right) \times \mathbf{r}^0_{6i}$$

De acuerdo a estas expresiones, los vectores $\mathbf{x}^0_i$ se puede obtener como la suma $\mathbf{x}^0_i = \mathbf{x}^0_{1i} + \mathbf{x}^0_{2i}$, donde:

$$
\begin{aligned}
\mathbf{x}^0_{1i} = {} & \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B}\right) \times \mathbf{r}^0_{0i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i}\right) \times \mathbf{r}^0_{1i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i} + \ddot{\theta}_{2i} \mathbf{j}^0_{2i}\right) \times \mathbf{r}^0_{2i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i} + \ddot{\theta}_{2i} \mathbf{j}^0_{2i} + \ddot{\theta}_{3i} \mathbf{k}^0_{4i}\right) \times \mathbf{r}^0_{3i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i} + \ddot{\theta}_{2i} \mathbf{j}^0_{2i} + \ddot{\theta}_{3i} \mathbf{k}^0_{4i} + \ddot{\theta}_{4i} \mathbf{k}^0_{6i}\right) \times \mathbf{r}^0_{4i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i} + \ddot{\theta}_{2i} \mathbf{j}^0_{2i} + \ddot{\theta}_{3i} \mathbf{k}^0_{4i} + \ddot{\theta}_{4i} \mathbf{k}^0_{6i} + \ddot{\theta}_{5i} \mathbf{k}^0_{8i}\right) \times \mathbf{r}^0_{5i} + \\
& \left(\ddot{\theta}_B \mathbf{i}^0_{\theta B} + \ddot{\phi}_B \mathbf{j}^0_{\phi B} + \ddot{\psi}_B \mathbf{k}^0_{\psi B} + \ddot{\theta}_{1i} \mathbf{k}^0_{0i} + \ddot{\theta}_{2i} \mathbf{j}^0_{2i} + \ddot{\theta}_{3i} \mathbf{k}^0_{4i} + \ddot{\theta}_{4i} \mathbf{k}^0_{6i} + \ddot{\theta}_{5i} \mathbf{k}^0_{8i} + \ddot{\theta}_{6i} \mathbf{i}^0_{10i}\right) \times \mathbf{r}^0_{6i}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}^0_{2i} = {} & \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right)\right) \times \mathbf{r}^0_{0i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right) + \dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right)\right) \times \mathbf{r}^0_{1i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right) + \dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right)\right) \times \mathbf{r}^0_{2i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right)\right) \times \mathbf{r}^0_{3i} + \\
& \left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}\right)\right) \times \mathbf{r}^0_{3i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right)\right) \times \mathbf{r}^0_{4i} + \\
& \left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}\right) + \dot{\theta}_{4i} \left(\boldsymbol{\omega}^0_{3i} \times \mathbf{k}^0_{6i}\right)\right) \times \mathbf{r}^0_{4i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right)\right) \times \mathbf{r}^0_{5i} + \\
& \left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}\right) + \dot{\theta}_{4i} \left(\boldsymbol{\omega}^0_{3i} \times \mathbf{k}^0_{6i}\right) + \dot{\theta}_{5i} \left(\boldsymbol{\omega}^0_{4i} \times \mathbf{k}^0_{8i}\right)\right) \times \mathbf{r}^0_{5i} + \\
& \left(\dot{\theta}_B \left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) + \dot{\phi}_B \left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) + \dot{\psi}_B \left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right)\right) \times \mathbf{r}^0_{6i} + \\
& \left(\dot{\theta}_{1i} \left(\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}\right) + \dot{\theta}_{2i} \left(\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}\right) + \dot{\theta}_{3i} \left(\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}\right) + \dot{\theta}_{4i} \left(\boldsymbol{\omega}^0_{3i} \times \mathbf{k}^0_{6i}\right) + \dot{\theta}_{5i} \left(\boldsymbol{\omega}^0_{4i} \times \mathbf{k}^0_{8i}\right) + \dot{\theta}_{6i} \left(\boldsymbol{\omega}^0_{5i} \times \mathbf{i}^0_{10i}\right)\right) \times \mathbf{r}^0_{6i}
\end{aligned}
$$

Con los vectores definidos en (3.12), los vectores $\mathbf{x}^0_{1i}$ se escriben:

$$\mathbf{x}^0_{1i} = \ddot{\theta}_B \mathbf{u}^0_{\theta_B} + \ddot{\phi}_B \mathbf{u}^0_{\phi_B} + \ddot{\psi}_B \mathbf{u}^0_{\psi_B} + \ddot{\theta}_{1i} \mathbf{u}^0_{\theta_{1i}} + \ddot{\theta}_{2i} \mathbf{u}^0_{\theta_{2i}} + \ddot{\theta}_{3i} \mathbf{u}^0_{\theta_{3i}} + \ddot{\theta}_{4i} \mathbf{u}^0_{\theta_{4i}} + \ddot{\theta}_{5i} \mathbf{u}^0_{\theta_{5i}} + \ddot{\theta}_{6i} \mathbf{u}^0_{\theta_{6i}}$$

Agrupando los términos comunes a $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$, los vectores $\mathbf{x}^0_{2i}$ son:

$$
\begin{aligned}
\mathbf{x}^0_{2i} = {} & \dot{\theta}_B \left(\left(\mathbf{i}^0_{\theta B} \times \boldsymbol{\omega}^0_{\phi B}\right) \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i}\right)\right) + \\
& \dot{\phi}_B \left(\left(\mathbf{j}^0_{\phi B} \times \boldsymbol{\omega}^0_{\psi B}\right) \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i}\right)\right) + \\
& \dot{\psi}_B \left(\left(\boldsymbol{\omega}^0_{\theta B} \times \mathbf{k}^0_{\psi B}\right) \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i}\right)\right) +
\end{aligned}
$$

$$\dot{\theta}_{1i}\left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0\right) \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) +$$

$$\dot{\theta}_{2i}\left(\left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0\right) \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) +$$

$$\dot{\theta}_{3i}\left(\left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0\right) \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) +$$

$$\dot{\theta}_{4i}\left(\left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0\right) \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) +$$

$$\dot{\theta}_{5i}\left(\left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{k}_{8i}^0\right) \times \left(\mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) +$$

$$\dot{\theta}_{6i}\left(\left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{i}_{10i}^0\right) \times \mathbf{r}_{6i}^0\right) \tag{3.19}$$

Desarrollando los productos cruz en los vectores $\mathbf{y}_i^0$:

$$\boldsymbol{\omega}_B^0 \times \left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0\right) \times \left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right)$$

$$\boldsymbol{\omega}_{1i}^0 \times \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0\right) \times \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right)$$

$$\boldsymbol{\omega}_{2i}^0 \times \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0\right) \times \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right)$$

$$\boldsymbol{\omega}_{3i}^0 \times \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0 + \dot{\theta}_{3i} \mathbf{k}_{4i}^0\right) \times \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right)$$

$$\boldsymbol{\omega}_{4i}^0 \times \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0 + \dot{\theta}_{3i} \mathbf{k}_{4i}^0 + \dot{\theta}_{4i} \mathbf{k}_{6i}^0\right) \times \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right)$$

$$\boldsymbol{\omega}_{5i}^0 \times \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0 + \dot{\theta}_{3i} \mathbf{k}_{4i}^0 + \dot{\theta}_{4i} \mathbf{k}_{6i}^0 + \dot{\theta}_{5i} \mathbf{k}_{8i}^0\right) \times \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right)$$

$$\boldsymbol{\omega}_{6i}^0 \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right) = \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0 + \dot{\theta}_{2i} \mathbf{j}_{2i}^0 + \dot{\theta}_{3i} \mathbf{k}_{4i}^0 + \dot{\theta}_{4i} \mathbf{k}_{6i}^0 + \dot{\theta}_{5i} \mathbf{k}_{8i}^0 + \dot{\theta}_{6i} \mathbf{i}_{10i}^0\right) \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)$$

Agrupando los términos comunes a $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$ en la suma de los productos cruz anteriores, el vector $\mathbf{y}_i^0$ se escribe:

$$
\begin{aligned}
\mathbf{y}_i^0 = \ & \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \left(\left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{3i} \left(\mathbf{k}_{4i}^0 \times \left(\left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{4i} \left(\mathbf{k}_{6i}^0 \times \left(\left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{5i} \left(\mathbf{k}_{8i}^0 \times \left(\left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right) + \\
& \dot{\theta}_{6i} \left(\mathbf{i}_{10i}^0 \times \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)
\end{aligned} \tag{3.20}
$$

Definiendo los vectores $\mathbf{s}$ como la suma de los vectores que multiplican a $\dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B, \dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$ en (3.19) y (3.20) se tiene:

$$
\begin{aligned}
\mathbf{s}_{\theta_B}^0 = \ & \left(\left(\mathbf{i}_{\theta B}^0 \times \boldsymbol{\omega}_{\phi B}^0\right) \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) + \\
& \left(\mathbf{i}_{\theta B}^0 \times \left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{s}_{\phi_B}^0 = \ & \left(\left(\mathbf{j}_{\phi B}^0 \times \boldsymbol{\omega}_{\psi B}^0\right) \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{6i}^0\right)\right) + \\
& \left(\mathbf{j}_{\phi B}^0 \times \left(\left(\boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0\right) + \left(\boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0\right) + \left(\boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0\right) + \left(\boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0\right) + \left(\boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0\right) + \left(\boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0\right) + \left(\boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{6i}^0\right)\right)\right)
\end{aligned}
$$

$$\mathbf{s}^0_{\psi_B} = \left((\boldsymbol{\omega}^0_{\theta_B} \times \mathbf{k}^0_{\psi B}) \times (\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) +$$
$$\left(\mathbf{k}^0_{\psi B} \times \left((\boldsymbol{\omega}^0_B \times \mathbf{r}^0_{0i}) + (\boldsymbol{\omega}^0_{1i} \times \mathbf{r}^0_{1i}) + (\boldsymbol{\omega}^0_{2i} \times \mathbf{r}^0_{2i}) + (\boldsymbol{\omega}^0_{3i} \times \mathbf{r}^0_{3i}) + (\boldsymbol{\omega}^0_{4i} \times \mathbf{r}^0_{4i}) + (\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{1i}} = \left((\boldsymbol{\omega}^0_B \times \mathbf{k}^0_{0i}) \times (\mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) +$$
$$\left(\mathbf{k}^0_{0i} \times \left((\boldsymbol{\omega}^0_{1i} \times \mathbf{r}^0_{1i}) + (\boldsymbol{\omega}^0_{2i} \times \mathbf{r}^0_{2i}) + (\boldsymbol{\omega}^0_{3i} \times \mathbf{r}^0_{3i}) + (\boldsymbol{\omega}^0_{4i} \times \mathbf{r}^0_{4i}) + (\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{2i}} = \left((\boldsymbol{\omega}^0_{1i} \times \mathbf{j}^0_{2i}) \times (\mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) +$$
$$\left(\mathbf{j}^0_{2i} \times \left((\boldsymbol{\omega}^0_{2i} \times \mathbf{r}^0_{2i}) + (\boldsymbol{\omega}^0_{3i} \times \mathbf{r}^0_{3i}) + (\boldsymbol{\omega}^0_{4i} \times \mathbf{r}^0_{4i}) + (\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{3i}} = \left((\boldsymbol{\omega}^0_{2i} \times \mathbf{k}^0_{4i}) \times (\mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) + \left(\mathbf{k}^0_{4i} \times \left((\boldsymbol{\omega}^0_{3i} \times \mathbf{r}^0_{3i}) + (\boldsymbol{\omega}^0_{4i} \times \mathbf{r}^0_{4i}) + (\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{4i}} = \left((\boldsymbol{\omega}^0_{3i} \times \mathbf{k}^0_{6i}) \times (\mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) + \left(\mathbf{k}^0_{6i} \times \left((\boldsymbol{\omega}^0_{4i} \times \mathbf{r}^0_{4i}) + (\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{5i}} = \left((\boldsymbol{\omega}^0_{4i} \times \mathbf{k}^0_{8i}) \times (\mathbf{r}^0_{5i} + \mathbf{r}^0_{6i})\right) + \left(\mathbf{k}^0_{8i} \times \left((\boldsymbol{\omega}^0_{5i} \times \mathbf{r}^0_{5i}) + (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)\right)$$

$$\mathbf{s}^0_{\theta_{6i}} = \left((\boldsymbol{\omega}^0_{5i} \times \mathbf{i}^0_{10i}) \times \mathbf{r}^0_{6i}\right) + \left(\mathbf{i}^0_{10i} \times (\boldsymbol{\omega}^0_{6i} \times \mathbf{r}^0_{6i})\right)$$

Con base en los vectores $\mathbf{s}$, las sumas $\mathbf{x}^0_{2i} + \mathbf{y}^0_i$ son:

$$\mathbf{x}^0_{2i} + \mathbf{y}^0_i = \dot{\theta}_B \mathbf{s}^0_{\theta_B} + \dot{\phi}_B \mathbf{s}^0_{\phi_B} + \dot{\psi}_B \mathbf{s}^0_{\psi_B} + \dot{\theta}_{1i} \mathbf{s}^0_{\theta_{1i}} + \dot{\theta}_{2i} \mathbf{s}^0_{\theta_{2i}} + \dot{\theta}_{3i} \mathbf{s}^0_{\theta_{3i}} + \dot{\theta}_{4i} \mathbf{s}^0_{\theta_{4i}} + \dot{\theta}_{5i} \mathbf{s}^0_{\theta_{5i}} + \dot{\theta}_{6i} \mathbf{s}^0_{\theta_{6i}}$$

Por lo que $\mathbf{x}^0_i + \mathbf{y}^0_i$ queda:

$$\mathbf{x}^0_i + \mathbf{y}^0_i = \mathbf{x}^0_{1i} + \mathbf{x}^0_{2i} + \mathbf{y}^0_i$$
$$= \ddot{\theta}_B \mathbf{u}^0_{\theta_B} + \ddot{\phi}_B \mathbf{u}^0_{\phi_B} + \ddot{\psi}_B \mathbf{u}^0_{\psi_B} + \ddot{\theta}_{1i} \mathbf{u}^0_{\theta_{1i}} + \ddot{\theta}_{2i} \mathbf{u}^0_{\theta_{2i}} + \ddot{\theta}_{3i} \mathbf{u}^0_{\theta_{3i}} + \ddot{\theta}_{4i} \mathbf{u}^0_{\theta_{4i}} + \ddot{\theta}_{5i} \mathbf{u}^0_{\theta_{5i}} + \ddot{\theta}_{6i} \mathbf{u}^0_{\theta_{6i}} +$$
$$\dot{\theta}_B \mathbf{s}^0_{\theta_B} + \dot{\phi}_B \mathbf{s}^0_{\phi_B} + \dot{\psi}_B \mathbf{s}^0_{\psi_B} + \dot{\theta}_{1i} \mathbf{s}^0_{\theta_{1i}} + \dot{\theta}_{2i} \mathbf{s}^0_{\theta_{2i}} + \dot{\theta}_{3i} \mathbf{s}^0_{\theta_{3i}} + \dot{\theta}_{4i} \mathbf{s}^0_{\theta_{4i}} + \dot{\theta}_{5i} \mathbf{s}^0_{\theta_{5i}} + \dot{\theta}_{6i} \mathbf{s}^0_{\theta_{6i}} \quad (3.21)$$

Sustituyendo (3.21) en (3.18):

$$\ddot{x}_B \mathbf{i}_0 + \ddot{y}_B \mathbf{j}_0 + \ddot{z}_B \mathbf{k}_0 \quad +$$
$$\ddot{\theta}_B \mathbf{u}^0_{\theta_B} + \ddot{\phi}_B \mathbf{u}^0_{\phi_B} + \ddot{\psi}_B \mathbf{u}^0_{\psi_B} + \ddot{\theta}_{1i} \mathbf{u}^0_{\theta_{1i}} + \ddot{\theta}_{2i} \mathbf{u}^0_{\theta_{2i}} + \ddot{\theta}_{3i} \mathbf{u}^0_{\theta_{3i}} + \ddot{\theta}_{4i} \mathbf{u}^0_{\theta_{4i}} + \ddot{\theta}_{5i} \mathbf{u}^0_{\theta_{5i}} + \ddot{\theta}_{6i} \mathbf{u}^0_{\theta_{6i}} \quad +$$
$$\dot{\theta}_B \mathbf{s}^0_{\theta_B} + \dot{\phi}_B \mathbf{s}^0_{\phi_B} + \dot{\psi}_B \mathbf{s}^0_{\psi_B} + \dot{\theta}_{1i} \mathbf{s}^0_{\theta_{1i}} + \dot{\theta}_{2i} \mathbf{s}^0_{\theta_{2i}} + \dot{\theta}_{3i} \mathbf{s}^0_{\theta_{3i}} + \dot{\theta}_{4i} \mathbf{s}^0_{\theta_{4i}} + \dot{\theta}_{5i} \mathbf{s}^0_{\theta_{5i}} + \dot{\theta}_{6i} \mathbf{s}^0_{\theta_{6i}} = \ddot{x}_i \mathbf{i}_0 + \ddot{y}_i \mathbf{j}_0 + \ddot{z}_i \mathbf{k}_0$$

Llevando los términos con datos $\ddot{x}_i, \ddot{y}_i, \ddot{z}_i, \ddot{x}_B, \ddot{y}_B, \ddot{z}_B, \ddot{\theta}_B, \ddot{\phi}_B, \ddot{\psi}_B, \dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B$ del lado derecho de la igualdad anterior, las ecuaciones de aceleración lineal para cada una de las piernas se escriben:

$$\ddot{\theta}_{1i} \mathbf{u}^0_{\theta_{1i}} + \ddot{\theta}_{2i} \mathbf{u}^0_{\theta_{2i}} + \ddot{\theta}_{3i} \mathbf{u}^0_{\theta_{3i}} \quad +$$
$$\ddot{\theta}_{4i} \mathbf{u}^0_{\theta_{4i}} + \ddot{\theta}_{5i} \mathbf{u}^0_{\theta_{5i}} + \ddot{\theta}_{6i} \mathbf{u}^0_{\theta_{6i}} \quad +$$
$$\dot{\theta}_{1i} \mathbf{s}^0_{\theta_{1i}} + \dot{\theta}_{2i} \mathbf{s}^0_{\theta_{2i}} + \dot{\theta}_{3i} \mathbf{s}^0_{\theta_{3i}} \quad +$$

$$\dot{\theta}_{4i}\mathbf{s}^0_{\theta_{4i}} + \dot{\theta}_{5i}\mathbf{s}^0_{\theta_{5i}} + \dot{\theta}_{6i}\mathbf{s}^0_{\theta_{6i}} \;=\; \ddot{x}_i\mathbf{i}_0 + \ddot{y}_i\mathbf{j}_0 + \ddot{z}_i\mathbf{k}_0 - \ddot{x}_B\mathbf{i}_0 - \ddot{y}_B\mathbf{j}_0 - \ddot{z}_B\mathbf{k}_0$$
$$-\;\; \ddot{\theta}_B\mathbf{u}^0_{\theta_B} - \ddot{\phi}_B\mathbf{u}^0_{\phi_B} - \ddot{\psi}_B\mathbf{u}^0_{\psi_B} - \dot{\theta}_B\mathbf{s}^0_{\theta_B} - \dot{\phi}_B\mathbf{s}^0_{\phi_B} - \dot{\psi}_B\mathbf{s}^0_{\psi_B} \qquad (3.22)$$

Las ecuaciones de aceleración angular se obtienen derivando con respecto al tiempo (3.14) :

$$\dot{\boldsymbol{\omega}}^0_{6i} \;=\; \dot{\boldsymbol{\omega}}^0_i$$

$$\boldsymbol{\alpha}^0_{6i} \;=\; \boldsymbol{\alpha}^0_i$$

$$
\begin{aligned}
\ddot{\theta}_B\mathbf{i}_0 + \ddot{\phi}_B\mathbf{j}^0_{\phi_B} + \ddot{\psi}_B\mathbf{k}^0_{\psi_B} + \dot{\theta}_B\left(\mathbf{i}^0_{\theta B}\times\boldsymbol{\omega}^0_{\phi_B}\right) &\;+ \\
\dot{\phi}_B\left(\mathbf{j}^0_{\phi B}\times\boldsymbol{\omega}^0_{\psi_B}\right) + \dot{\psi}_B\left(\boldsymbol{\omega}^0_{\theta_B}\times\mathbf{k}^0_{\psi B}\right) &\;+ \\
\ddot{\theta}_{1i}\mathbf{k}^0_{0i} + \dot{\theta}_{1i}\left(\boldsymbol{\omega}^0_B\times\mathbf{k}^0_{0i}\right) &\;+ \\
\ddot{\theta}_{2i}\mathbf{j}^0_{2i} + \dot{\theta}_{2i}\left(\boldsymbol{\omega}^0_{1i}\times\mathbf{j}^0_{2i}\right) &\;+ \\
\ddot{\theta}_{3i}\mathbf{k}^0_{4i} + \dot{\theta}_{3i}\left(\boldsymbol{\omega}^0_{2i}\times\mathbf{k}^0_{4i}\right) &\;+ \\
\ddot{\theta}_{4i}\mathbf{k}^0_{6i} + \dot{\theta}_{4i}\left(\boldsymbol{\omega}^0_{3i}\times\mathbf{k}^0_{6i}\right) &\;+ \\
\ddot{\theta}_{5i}\mathbf{k}^0_{8i} + \dot{\theta}_{5i}\left(\boldsymbol{\omega}^0_{4i}\times\mathbf{k}^0_{8i}\right) &\;+ \\
\ddot{\theta}_{6i}\mathbf{i}^0_{10i} + \dot{\theta}_{6i}\left(\boldsymbol{\omega}^0_{5i}\times\mathbf{i}^0_{10i}\right) &\;=\; \ddot{\theta}_i\mathbf{i}_0 + \ddot{\phi}_i\mathbf{j}^0_{\phi_i} + \ddot{\psi}_i\mathbf{k}^0_{\psi_i} + \dot{\theta}_i\left(\mathbf{i}^0_{\theta i}\times\boldsymbol{\omega}^0_{\phi_i}\right) + \dot{\phi}_i\left(\mathbf{j}^0_{\phi i}\times\boldsymbol{\omega}^0_{\psi_i}\right) + \dot{\psi}_i\left(\boldsymbol{\omega}^0_{\theta_i}\times\mathbf{k}^0_{\psi i}\right)
\end{aligned}
$$

Definiendo los vectores $\mathbf{t}$:

$$
\begin{aligned}
\mathbf{t}^0_{\theta_B} &\;=\; \mathbf{i}^0_{\theta B}\times\boldsymbol{\omega}^0_{\phi_B} \\
\mathbf{t}^0_{\phi_B} &\;=\; \mathbf{j}^0_{\phi B}\times\boldsymbol{\omega}^0_{\psi_B} \\
\mathbf{t}^0_{\psi_B} &\;=\; \boldsymbol{\omega}^0_{\theta_B}\times\mathbf{k}^0_{\psi B} \\
\mathbf{t}^0_{\theta_{1i}} &\;=\; \boldsymbol{\omega}^0_B\times\mathbf{k}^0_{0i} \\
\mathbf{t}^0_{\theta_{2i}} &\;=\; \boldsymbol{\omega}^0_{1i}\times\mathbf{j}^0_{2i} \\
\mathbf{t}^0_{\theta_{3i}} &\;=\; \boldsymbol{\omega}^0_{2i}\times\mathbf{k}^0_{4i} \\
\mathbf{t}^0_{\theta_{4i}} &\;=\; \boldsymbol{\omega}^0_{3i}\times\mathbf{k}^0_{6i} \\
\mathbf{t}^0_{\theta_{5i}} &\;=\; \boldsymbol{\omega}^0_{4i}\times\mathbf{k}^0_{8i} \\
\mathbf{t}^0_{\theta_{6i}} &\;=\; \boldsymbol{\omega}^0_{5i}\times\mathbf{i}^0_{10i} \\
\mathbf{t}^0_{\theta_i} &\;=\; \mathbf{i}^0_{\theta i}\times\boldsymbol{\omega}^0_{\phi_i} \\
\mathbf{t}^0_{\phi_i} &\;=\; \mathbf{j}^0_{\phi i}\times\boldsymbol{\omega}^0_{\psi_i} \\
\mathbf{t}^0_{\psi_i} &\;=\; \boldsymbol{\omega}^0_{\theta_i}\times\mathbf{k}^0_{\psi i}
\end{aligned}
$$

y escribiendo los términos con $\ddot{\theta}_i, \ddot{\phi}_i, \ddot{\psi}_i, \ddot{\theta}_B, \ddot{\phi}_B, \ddot{\psi}_B, \dot{\theta}_i, \dot{\phi}_i, \dot{\psi}_i, \dot{\theta}_B, \dot{\phi}_B, \dot{\psi}_B$ del lado derecho, las ecuaciones de aceleración angular son:

$$
\begin{aligned}
\ddot{\theta}_{1i}\mathbf{k}^0_{0i} + \ddot{\theta}_{2i}\mathbf{j}^0_{2i} + \ddot{\theta}_{3i}\mathbf{k}^0_{4i} &\;+ \\
\ddot{\theta}_{4i}\mathbf{k}^0_{6i} + \ddot{\theta}_{5i}\mathbf{k}^0_{8i} + \ddot{\theta}_{6i}\mathbf{i}^0_{10i} &\;+ \\
\dot{\theta}_{1i}\mathbf{t}^0_{\theta_{1i}} + \dot{\theta}_{2i}\mathbf{t}^0_{\theta_{2i}} + \dot{\theta}_{3i}\mathbf{t}^0_{\theta_{3i}} &\;+
\end{aligned}
$$

$$\dot{\theta}_{4i}\mathbf{t}^0_{\theta_{4i}} + \dot{\theta}_{5i}\mathbf{t}^0_{\theta_{5i}} + \dot{\theta}_{6i}\mathbf{t}^0_{\theta_{6i}} = \ddot{\theta}_i\mathbf{i}_0 + \ddot{\phi}_i\mathbf{j}^0_{\phi_i} + \ddot{\psi}_i\mathbf{k}^0_{\psi_i} - \ddot{\theta}_B\mathbf{i}_0 - \ddot{\phi}_B\mathbf{j}^0_{\phi_B} - \ddot{\psi}_B\mathbf{k}^0_{\psi_B} +$$
$$\dot{\theta}_i\mathbf{t}^0_{\theta_i} + \dot{\phi}_i\mathbf{t}^0_{\phi_i} + \dot{\psi}_i\mathbf{t}^0_{\psi_i} - \dot{\theta}_B\mathbf{t}^0_{\theta_B} - \dot{\phi}_B\mathbf{t}^0_{\phi_B} - \dot{\psi}_B\mathbf{t}^0_{\psi_B} \tag{3.23}$$

Tomando en cuenta que la solución numérica de las ecuaciones (3.4) y (3.16) ya ha proporcionado los valores de posición $\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}$ y velocidad angulares $\dot{\theta}_{1i}, \dot{\theta}_{2i}, \dot{\theta}_{3i}, \dot{\theta}_{4i}, \dot{\theta}_{5i}, \dot{\theta}_{6i}$; con las ecuaciones vectoriales (3.22) y (3.23) se pueden constituir sistemas de 6 ecuaciones escalares con 6 incógnitas correspondientes a $\ddot{\theta}_{1i}, \ddot{\theta}_{2i}, \ddot{\theta}_{3i}, \ddot{\theta}_{4i}, \ddot{\theta}_{5i}, \ddot{\theta}_{6i}$ que en forma matricial son:

$$\begin{bmatrix} \mathbf{u}^0_{\theta_{1i}} & \mathbf{u}^0_{\theta_{2i}} & \mathbf{u}^0_{\theta_{3i}} & \mathbf{u}^0_{\theta_{4i}} & \mathbf{u}^0_{\theta_{5i}} & \mathbf{u}^0_{\theta_{6i}} \\ \mathbf{k}^0_{0i} & \mathbf{j}^0_{2i} & \mathbf{k}^0_{4i} & \mathbf{k}^0_{6i} & \mathbf{k}^0_{8i} & \mathbf{i}^0_{10i} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_{1i} \\ \ddot{\theta}_{2i} \\ \ddot{\theta}_{3i} \\ \ddot{\theta}_{4i} \\ \ddot{\theta}_{5i} \\ \ddot{\theta}_{6i} \end{bmatrix} + \begin{bmatrix} \mathbf{s}^0_{\theta_{1i}} & \mathbf{s}^0_{\theta_{2i}} & \mathbf{s}^0_{\theta_{3i}} & \mathbf{s}^0_{\theta_{4i}} & \mathbf{s}^0_{\theta_{5i}} & \mathbf{s}^0_{\theta_{6i}} \\ \mathbf{t}^0_{\theta_{1i}} & \mathbf{t}^0_{\theta_{2i}} & \mathbf{t}^0_{\theta_{3i}} & \mathbf{t}^0_{\theta_{4i}} & \mathbf{t}^0_{\theta_{5i}} & \mathbf{t}^0_{\theta_{6i}} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{1i} \\ \dot{\theta}_{2i} \\ \dot{\theta}_{3i} \\ \dot{\theta}_{4i} \\ \dot{\theta}_{5i} \\ \dot{\theta}_{6i} \end{bmatrix} = \mathbf{A}_i$$
$$\tag{3.24}$$

donde

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta_i} & \mathbf{j}^0_{\phi_i} & \mathbf{k}^0_{\psi_i} \end{bmatrix} \begin{bmatrix} \ddot{x}_i \\ \ddot{y}_i \\ \ddot{z}_i \\ \ddot{\theta}_i \\ \ddot{\phi}_i \\ \ddot{\psi}_i \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{t}^0_{\theta_i} & \mathbf{t}^0_{\phi_i} & \mathbf{t}^0_{\psi_i} \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \\ \dot{\theta}_i \\ \dot{\phi}_i \\ \dot{\psi}_i \end{bmatrix} -$$

$$- \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{u}^0_{\theta_B} & \mathbf{u}^0_{\phi_B} & \mathbf{u}^0_{\psi_B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta_i} & \mathbf{j}^0_{\phi_B} & \mathbf{k}^0_{\psi_B} \end{bmatrix} \begin{bmatrix} \ddot{x}_B \\ \ddot{y}_B \\ \ddot{z}_B \\ \ddot{\theta}_B \\ \ddot{\phi}_B \\ \ddot{\psi}_B \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{s}^0_{\theta_B} & \mathbf{s}^0_{\phi_B} & \mathbf{s}^0_{\psi_B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{t}^0_{\theta_B} & \mathbf{t}^0_{\phi_B} & \mathbf{t}^0_{\psi_B} \end{bmatrix} \begin{bmatrix} \dot{x}_B \\ \dot{y}_B \\ \dot{z}_B \\ \dot{\theta}_B \\ \dot{\phi}_B \\ \dot{\psi}_B \end{bmatrix}$$

La solución numérica de las ecuaciones matriciales en (3.24) proporcionará los valores de aceleración angular $\ddot{\theta}_{1i}, \ddot{\theta}_{2i}, \ddot{\theta}_{3i}, \ddot{\theta}_{4i}, \ddot{\theta}_{5i}, \ddot{\theta}_{6i}$.

Capítulo 4

Ecuaciones dinámicas con la formulación Euler-Lagrange

Elaborar el análisis dinámico de un sistema mecánico implica encontrar las relaciones matemáticas entre sus coordenadas $\mathbf{q}$ y fuerzas $\mathbf{Q}$ generalizadas. De acuerdo a (Spong et al., 2006), en la deducción de dichas relaciones se pueden distinguir dos objetivos principales:

1. Obtener ecuaciones de forma cerrada que describan la evolución en el tiempo de las coordenadas generalizadas $\mathbf{q}(t)$.

2. Conocer cuáles son las fuerzas generalizadas $\mathbf{Q}(t)$ que deben ser aplicadas para producir una evolución particular en el tiempo de las coordenadas generalizadas $\mathbf{q}(t)$, sin preocuparse de la relación funcional entre ambas.

Aunque se sabe que las formulaciones de Euler-Lagrange y Newton-Euler para deducir las ecuaciones dinámicas de un sistema conducen a los mismos resultados; se considera que el método basado en energía de Euler-Lagrange es superior para alcanzar el objetivo 1, mientras que el método recursivo de Newton-Euler resulta más apropiado para el 2.

En tanto (Chevallereau et al., 2009) reporta que la caminata estable de un bípedo depende de un delicado equilibrio entre los efectos gravitatorios y los esfuerzos derivados del contacto entre el piso y los pies, se ha considerado que la disposición de las relaciones funcionales entre $\mathbf{q}(t)$ y $\mathbf{Q}(t)$ sentará las bases para coordinar el movimiento de las juntas rotacionales del robot y proporcionar los efectos propulsivos adecuados para evitar su caída durante la ejecución del ciclo de marcha; por lo que se ha optado por la deducción de las ecuaciones dinámicas de Euler-Lagrange. Vale la pena mencionar que la mayor parte de los esquemas de control óptimo, predictivo y por retroalimentación de los estados descritos en (Kajita and Espiau, 2008) para garantizar la caminata dinámicamente estable de robots con piernas, se encuentran basados en modelos de Euler-Lagrange. En particular, se refiere que esta formulación presenta ventajas para extender los estudios a fenómenos mecánicos más avanzados asociados a la deformación elástica de los eslabones y la colisión viscoelástica de los pies contra el piso.

De acuerdo a la formulación Euler-Lagrange presentada en (Westervelt et al., 2007), el problema de dinámica inversa para el bípedo es:

Establecer ecuaciones diferenciales de la forma:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j + \Gamma_j^k \tag{4.1}$$

cuya solución proporcione los valores de torque necesarios en los actuadores para llevar al bípedo a los estados definidos por sus coordenadas generalizadas q_j.

De acuerdo a los 18 GDL del robot, las coordenadas generalizadas q_j, $1 \leq j \leq 18$, son:

$$
\begin{aligned}
q_1 &= x_B & q_2 &= y_B & q_3 &= z_B \\
q_4 &= \theta_B & q_5 &= \phi_B & q_6 &= \psi_B \\
q_7 &= \theta_{11} & q_8 &= \theta_{21} & q_9 &= \theta_{31} \\
q_{10} &= \theta_{41} & q_{11} &= \theta_{51} & q_{12} &= \theta_{61} \\
q_{13} &= \theta_{12} & q_{14} &= \theta_{22} & q_{15} &= \theta_{32} \\
q_{16} &= \theta_{42} & q_{17} &= \theta_{52} & q_{18} &= \theta_{62}
\end{aligned}
\tag{4.2}
$$

Derivando cada una de estas coordenadas generalizadas con respecto al tiempo, se obtienen $\dot{q}_j$:

$$
\begin{aligned}
\dot{q}_1 &= \dot{x}_B & \dot{q}_2 &= \dot{y}_B & \dot{q}_3 &= \dot{z}_B \\
\dot{q}_4 &= \dot{\theta}_B & \dot{q}_5 &= \dot{\phi}_B & \dot{q}_6 &= \dot{\psi}_B \\
\dot{q}_7 &= \dot{\theta}_{11} & \dot{q}_8 &= \dot{\theta}_{21} & \dot{q}_9 &= \dot{\theta}_{31} \\
\dot{q}_{10} &= \dot{\theta}_{41} & \dot{q}_{11} &= \dot{\theta}_{51} & \dot{q}_{12} &= \dot{\theta}_{61} \\
\dot{q}_{13} &= \dot{\theta}_{12} & \dot{q}_{14} &= \dot{\theta}_{22} & \dot{q}_{15} &= \dot{\theta}_{32} \\
\dot{q}_{16} &= \dot{\theta}_{42} & \dot{q}_{17} &= \dot{\theta}_{52} & \dot{q}_{18} &= \dot{\theta}_{62}
\end{aligned}
$$

La función Lagrangiana L del sistema está dada por la expresión:

$$L = L_B + \sum_{i=1}^{2}\left(\sum_{n=1}^{6} L_{ni}\right) \tag{4.3}$$

Donde L_B representa la energía relacionada con el movimiento del torso y L_{ni} la correspondiente a cada uno de los seis eslabones de las piernas izquierda ($i = 1$) y derecha ($i = 2$). A su vez, L_B, L_{ni} constituyen la diferencia entre su energía cinética T_B, T_{ni} y potencial V_B, V_{ni}:

$$
\begin{aligned}
L_B &= T_B - V_B \\
L_{ni} &= T_{ni} - V_{ni}
\end{aligned}
$$

Q_j son los elementos del vector de fuerzas generalizadas $\mathbf{Q}$ restringido a los efectos de los pares de los actuadores τ_{ni}, mientras que Γ_j^k son los elementos del vector de fuerzas de restricción $\mathbf{\Gamma}_k$ ligados a las fuerzas y momentos de reacción derivados del contacto de los pies del bípedo con el piso durante su caminata.

A continuación se presentan los procedimientos realizados para obtener cada uno de los términos en las ecuaciones (4.1) y la constitución explícita del modelo dinámico para el bípedo Scout.

4.1. Función Lagrangiana

Según (Spong and Vidyasagar, 1989) la energía cinética del torso T_B y cada uno de los eslabones T_{ni} puede calcularse como:

$$
\begin{aligned}
T_B &= \frac{1}{2}m_B \left(\dot{\mathbf{b}}_B^0\right)^T \dot{\mathbf{b}}_B^0 + \frac{1}{2}\left(\boldsymbol{\omega}_B^0\right)^T \left(\mathbf{I}_B^0 \boldsymbol{\omega}_B^0\right) \\
&= \frac{1}{2}\left(m_B \left(\dot{\mathbf{b}}_B^0\right)^T \dot{\mathbf{b}}_B^0 + \left(\boldsymbol{\omega}_B^0\right)^T \left(\mathbf{I}_B^0 \boldsymbol{\omega}_B^0\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
T_{ni} &= \frac{1}{2}m_{ni} \left(\dot{\mathbf{b}}_{ni}^0\right)^T \dot{\mathbf{b}}_{ni}^0 + \frac{1}{2}\left(\boldsymbol{\omega}_{ni}^0\right)^T \left(\mathbf{I}_{ni}^0 \boldsymbol{\omega}_{ni}^0\right) \\
&= \frac{1}{2}\left(m_{ni} \left(\dot{\mathbf{b}}_{ni}^0\right)^T \dot{\mathbf{b}}_{ni}^0 + \left(\boldsymbol{\omega}_{ni}^0\right)^T \left(\mathbf{I}_{ni}^0 \boldsymbol{\omega}_{ni}^0\right)\right)
\end{aligned}
$$

Mientras que la energía potencial de cada uno de ellos es:

$$
\begin{aligned}
V_B &= -m_B \mathbf{g}^T \mathbf{b}_B^0 \\
V_{ni} &= -m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0
\end{aligned}
$$

Por lo que la función Lagrangiana en (4.3) queda:

$$
\begin{aligned}
L &= (T_B - V_B) + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}(T_{ni} - V_{ni})\right) \\
&= \left(\frac{1}{2}\left(m_B \left(\dot{\mathbf{b}}_B^0\right)^T \dot{\mathbf{b}}_B^0 + \left(\boldsymbol{\omega}_B^0\right)^T \left(\mathbf{I}_B^0 \boldsymbol{\omega}_B^0\right)\right) + m_B \mathbf{g}^T \mathbf{b}_B^0\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\frac{1}{2}\left(m_{ni} \left(\dot{\mathbf{b}}_{ni}^0\right)^T \dot{\mathbf{b}}_{ni}^0 + \left(\boldsymbol{\omega}_{ni}^0\right)^T \left(\mathbf{I}_{ni}^0 \boldsymbol{\omega}_{ni}^0\right)\right) + m_{ni}\mathbf{g}^T \mathbf{b}_{ni}^0\right)\right)
\end{aligned} \tag{4.4}
$$

Donde $\mathbf{g}$ es el vector que describe la aceleración gravitatoria en el marco inercial

$$
\begin{aligned}
\mathbf{g} &= -g\mathbf{k}_0 = -9.81\left[\frac{m}{s^2}\right]\mathbf{k}_0 \\
\mathbf{g}^T &= \begin{bmatrix} 0 & 0 & -9.81\left[\frac{m}{s^2}\right] \end{bmatrix}
\end{aligned}
$$

m_B, m_{ni} son las masas de los eslabones (consultar apéndice B.5) y las matrices $\mathbf{I}_B^0$, $\mathbf{I}_{ni}^0$ representan sus momentos de inercia referidos al marco inercial de acuerdo a las transformaciones:

$$
\begin{aligned}
\mathbf{I}_B^0 &= \mathbf{R}_B^0 \mathbf{I}_B^B \left(\mathbf{R}_B^0\right)^T \\
\mathbf{I}_{1i}^0 &= \mathbf{R}_{1i}^0 \mathbf{I}_{1i}^{1i} \left(\mathbf{R}_{1i}^0\right)^T \\
\mathbf{I}_{2i}^0 &= \mathbf{R}_{3i}^0 \mathbf{I}_{2i}^{3i} \left(\mathbf{R}_{3i}^0\right)^T \\
\mathbf{I}_{3i}^0 &= \mathbf{R}_{5i}^0 \mathbf{I}_{3i}^{5i} \left(\mathbf{R}_{5i}^0\right)^T \\
\mathbf{I}_{4i}^0 &= \mathbf{R}_{7i}^0 \mathbf{I}_{4i}^{7i} \left(\mathbf{R}_{7i}^0\right)^T \\
\mathbf{I}_{5i}^0 &= \mathbf{R}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\mathbf{R}_{9i}^0\right)^T \\
\mathbf{I}_{6i}^0 &= \mathbf{R}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\mathbf{R}_{11i}^0\right)^T
\end{aligned}
$$

$\mathbf{I}_B^B, \mathbf{I}_{1i}^{1i}, \mathbf{I}_{2i}^{3i}, \mathbf{I}_{3i}^{5i}, \mathbf{I}_{4i}^{7i}, \mathbf{I}_{5i}^{9i}, \mathbf{I}_{6i}^{11i}$ son matrices simétricas de la forma:

$$\mathbf{I}_B^B = \begin{bmatrix} I_{xx_B} & I_{xy_B} & I_{xz_B} \\ I_{xy_B} & I_{yy_B} & I_{yz_B} \\ I_{xz_B} & I_{yz_B} & I_{zz_B} \end{bmatrix}$$

$$\mathbf{I}_{ni}^{mi} = \begin{bmatrix} I_{xx_{ni}} & I_{xy_{ni}} & I_{xz_{ni}} \\ I_{xy_{ni}} & I_{yy_{ni}} & I_{yz_{ni}} \\ I_{xz_{ni}} & I_{yz_{ni}} & I_{zz_{ni}} \end{bmatrix}$$

calculadas vía software a partir de los modelos CAD para cada eslabón respecto a los ejes de los marcos locales $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B), (\mathbf{p}_{ni}; \mathbf{i}_{mi}, \mathbf{j}_{mi}, \mathbf{k}_{mi})$; por lo que sus elementos son constantes (ver apéndice B.6). En las figuras 4.1-4.3 se presentan los marcos locales $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B), (\mathbf{p}_{n2}; \mathbf{i}_{m2}, \mathbf{j}_{m2}, \mathbf{k}_{m2})$, que también fueron aprovechados para describir los centros de gravedad del torso y cada uno de los eslabones de la pierna derecha con los vectores $\mathbf{r}_{GB}, \mathbf{r}_{Gn2}$. Se debe tener en cuenta que las matrices de inercia de los eslabones de la pierna izquierda y de los vectores $\mathbf{r}_{Gn1}$ se realizó de manera análoga respetando la orientación de los marcos locales $(\mathbf{p}_{n1}; \mathbf{i}_{m1}, \mathbf{j}_{m1}, \mathbf{k}_{m1})$, los cuales pueden observarse en las figuras 3.3 y 3.4.
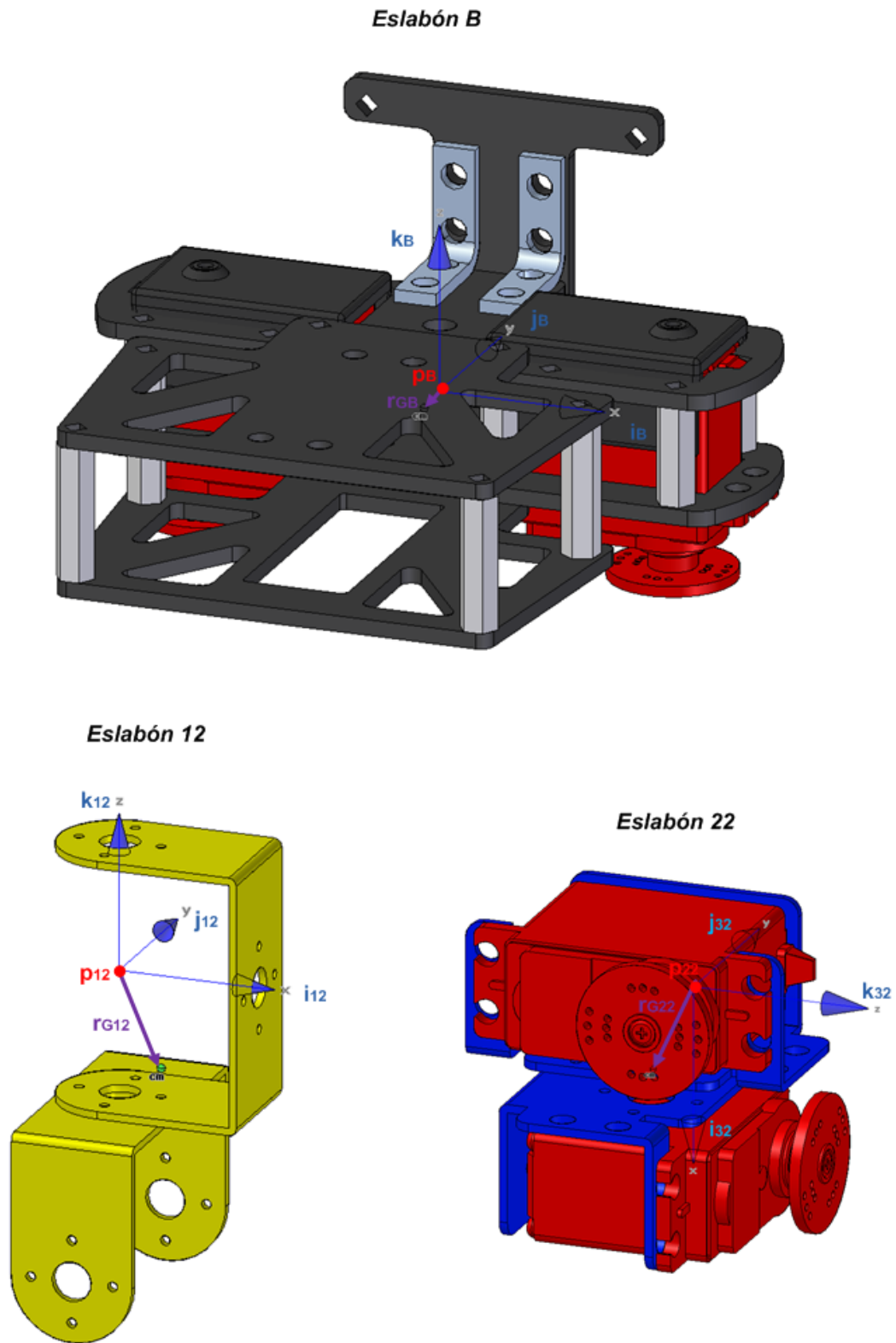
Figura 4.1: Marcos locales $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B), (\mathbf{p}_{12}; \mathbf{i}_{12}, \mathbf{j}_{12}, \mathbf{k}_{12}), (\mathbf{p}_{22}; \mathbf{i}_{32}, \mathbf{j}_{32}, \mathbf{k}_{32})$ en los eslabones B,12 y 22. Los centros de gravedad son identificados por el software con esferas etiquetadas como cm.

Figura 4.2: Marcos locales $(\mathbf{p}_{32}; \mathbf{i}_{52}, \mathbf{j}_{52}, \mathbf{k}_{52}), (\mathbf{p}_{42}; \mathbf{i}_{72}, \mathbf{j}_{72}, \mathbf{k}_{72})$ en los eslabones 32 y42. El cálculo de los momentos de inercia $I_{xx_{ni}}, I_{yy_{ni}}, I_{zz_{ni}}, I_{xy_{ni}}, I_{xz_{ni}}, I_{yz_{ni}}$ se elabora respecto a los ejes x, y, z definidos en el software de modelado CAD.
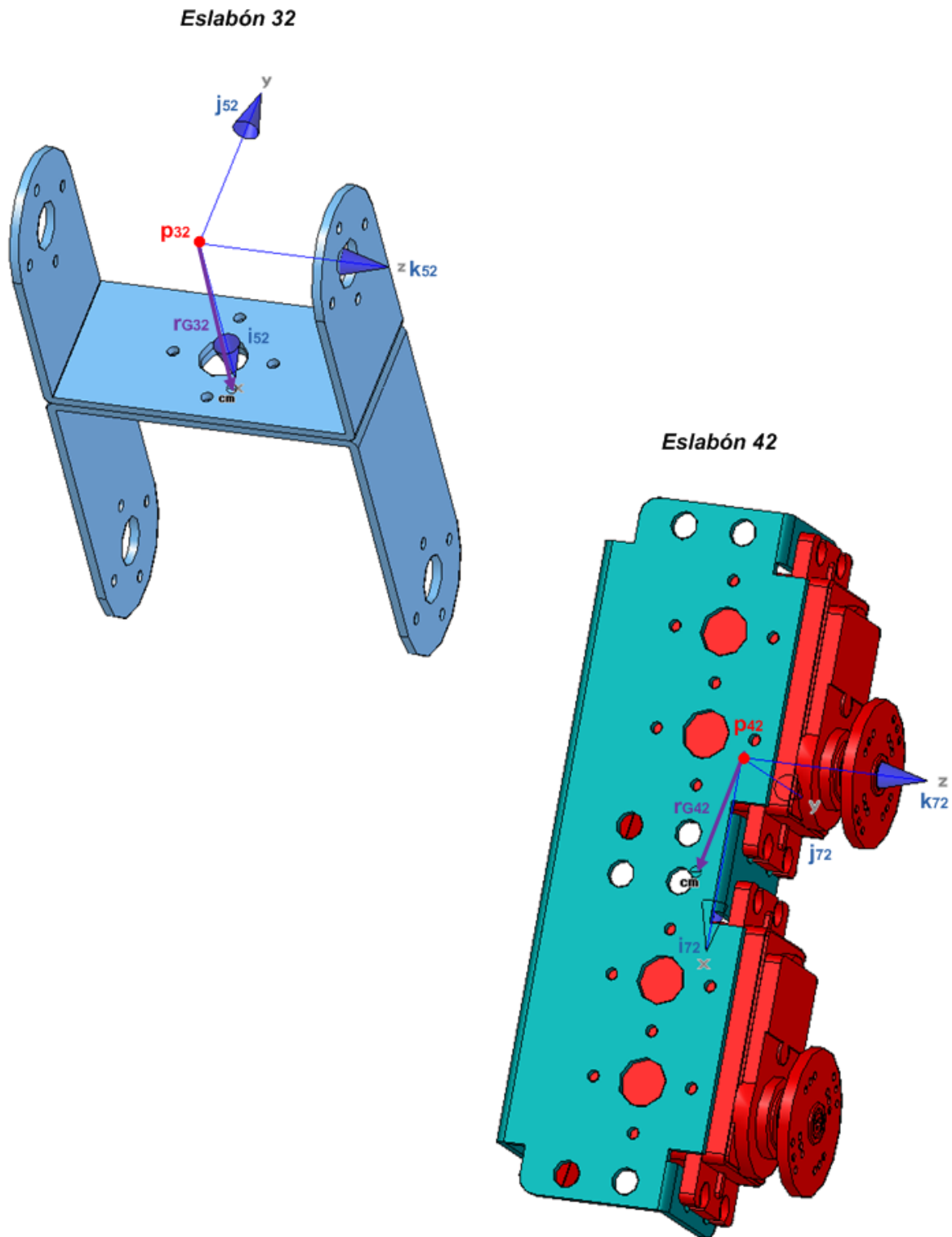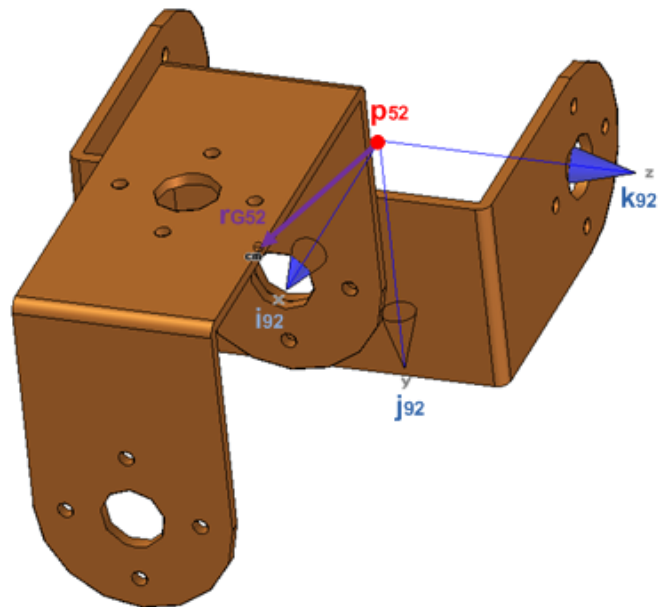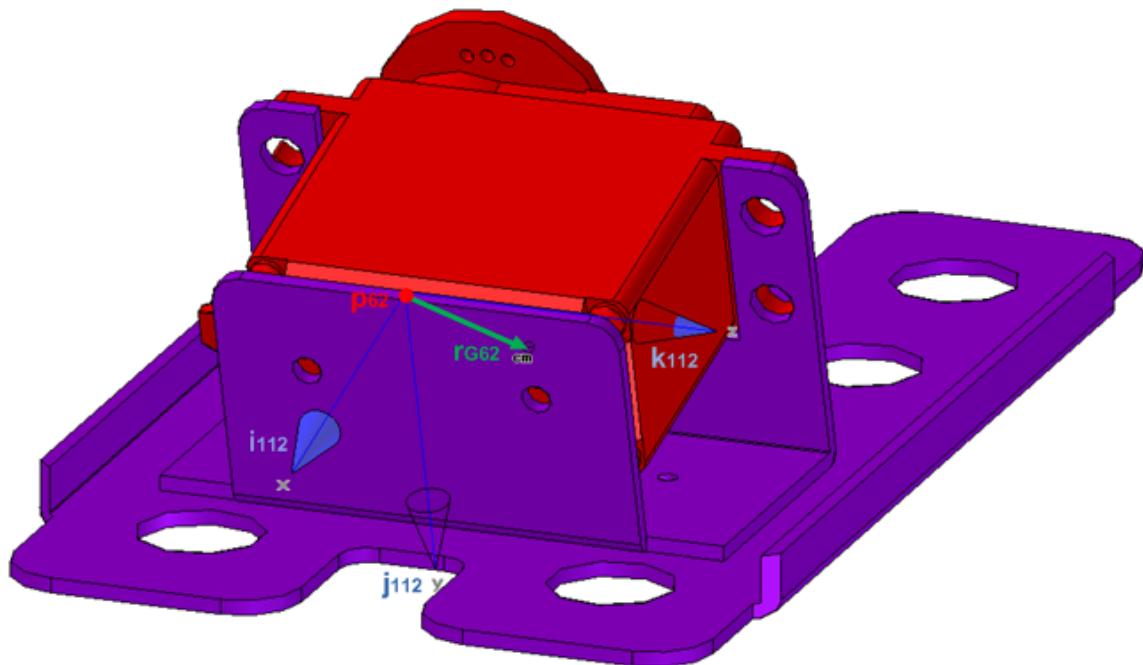
Figura 4.3: Marcos locales $(\mathbf{p}_{52}; \mathbf{i}_{92}, \mathbf{j}_{92}, \mathbf{k}_{92}), (\mathbf{p}_{62}; \mathbf{i}_{112}, \mathbf{j}_{112}, \mathbf{k}_{112})$ en los eslabones 52 y 62. La distribución de la masa de los servomotores se consideró homogénea en tanto no se cuenta con un modelo detallado de su interior.

De acuerdo a los vectores $\mathbf{r}_{GB}, \mathbf{r}_{Gni}$ y los definidos en (3.6), la posición en el marco inercial de los centros de gravedad del torso y los eslabones ni está dada por los vectores $\mathbf{b}_B^0, \mathbf{b}_{ni}^0$:

$$
\begin{aligned}
\mathbf{b}_B^0 &= \mathbf{r}_B^0 + \mathbf{r}_{GB}^0 \\
\mathbf{b}_{1i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \\
\mathbf{b}_{2i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \\
\mathbf{b}_{3i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \\
\mathbf{b}_{4i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \\
\mathbf{b}_{5i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \\
\mathbf{b}_{6i}^0 &= \mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0
\end{aligned}
$$

donde

$$
\begin{aligned}
\mathbf{r}_{GB}^0 &= \mathbf{R}_B^0 \mathbf{r}_{GB}^B \\
\mathbf{r}_{G1i}^0 &= \mathbf{R}_{1i}^0 \mathbf{r}_{G1i}^{1i} \\
\mathbf{r}_{G2i}^0 &= \mathbf{R}_{3i}^0 \mathbf{r}_{G2i}^{3i} \\
\mathbf{r}_{G3i}^0 &= \mathbf{R}_{5i}^0 \mathbf{r}_{G3i}^{5i} \\
\mathbf{r}_{G4i}^0 &= \mathbf{R}_{7i}^0 \mathbf{r}_{G4i}^{7i} \\
\mathbf{r}_{G5i}^0 &= \mathbf{R}_{9i}^0 \mathbf{r}_{G5i}^{9i} \\
\mathbf{r}_{G6i}^0 &= \mathbf{R}_{11i}^0 \mathbf{r}_{G6i}^{11i}
\end{aligned}
$$

La expresión de los vectores $\mathbf{r}_{GB}^B, \mathbf{r}_{G1i}^{1i}, \mathbf{r}_{G2i}^{3i}, \mathbf{r}_{G3i}^{5i}, \mathbf{r}_{G4i}^{7i}, \mathbf{r}_{G5i}^{9i}, \mathbf{r}_{G6i}^{11i}$ en sus marcos locales respectivos se encuentra en el apéndice B.7. Los vectores $\mathbf{b}_B^0, \mathbf{b}_{ni}^0$ se ilustran en la figura 4.4.
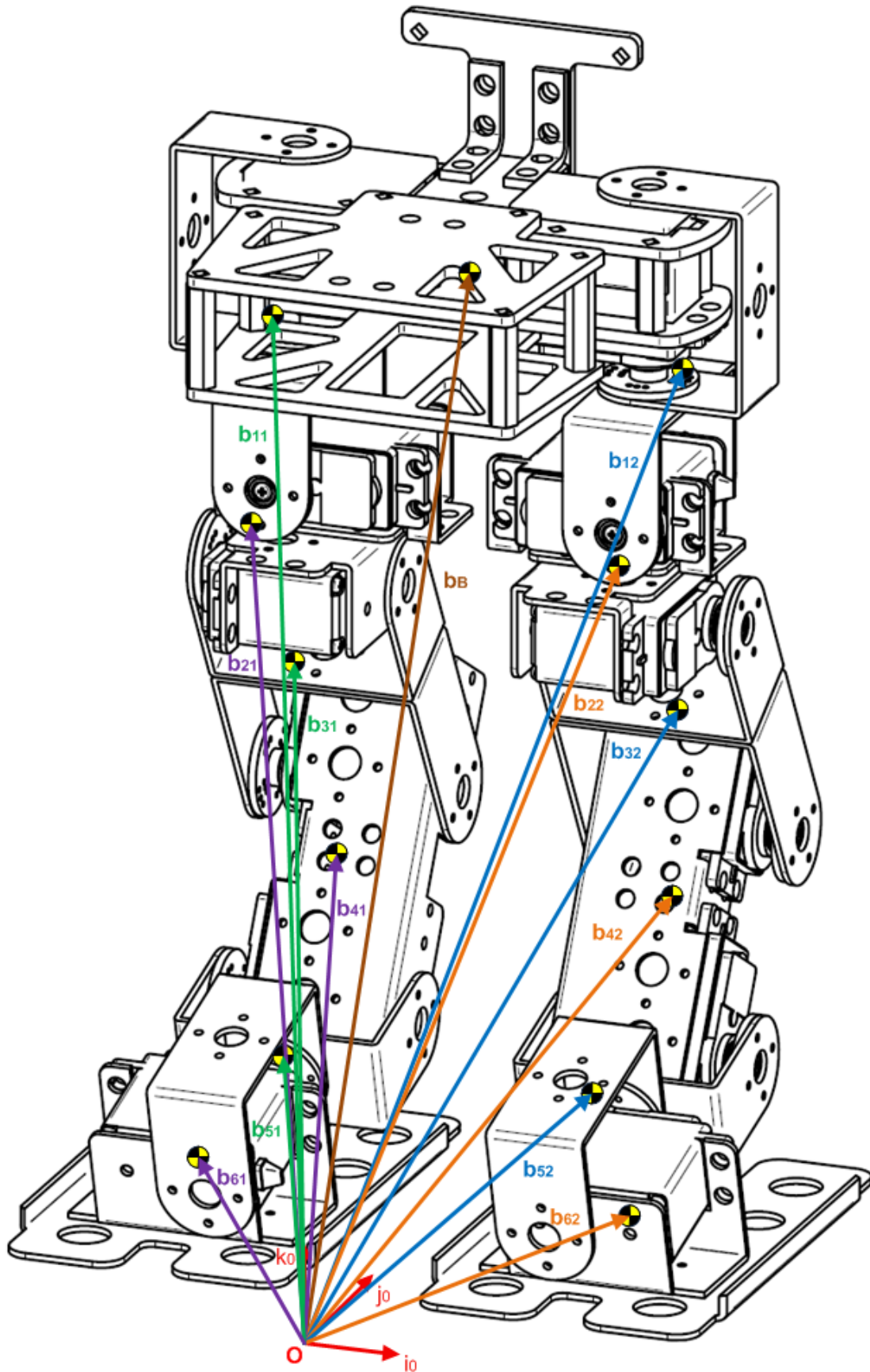
Figura 4.4: Vectores asociados a la posición de los centros de gravedad de los eslabones del bípedo en el marco inercial.

Derivando los vectores $\mathbf{b}_B^0, \mathbf{b}_{ni}^0$ con respecto al tiempo se tiene:

$$
\begin{aligned}
\dot{\mathbf{b}}_B^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{GB}^0 \\
&= \mathbf{v}_B^0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{GB}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + (\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0) \times \mathbf{r}_{GB}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \mathbf{r}_{GB}^0 \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \mathbf{r}_{GB}^0 \right) + \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \mathbf{r}_{GB}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{1i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{G1i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{G1i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{G1i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + (\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0) \times \mathbf{r}_{0i}^0 + (\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i} \mathbf{k}_{0i}^0) \times \mathbf{r}_{G1i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \right) + \\
&\quad \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \right) + \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \mathbf{r}_{G1i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{2i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{G2i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{G2i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{G2i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \right) + \\
&\quad \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \right) + \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \right) + \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \mathbf{r}_{G2i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{3i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{G3i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{G3i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right) + \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right) + \\
&\quad \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right) + \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right) + \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right) + \\
&\quad \dot{\theta}_{3i} \left(\mathbf{k}_{4i}^0 \times \mathbf{r}_{G3i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{4i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{2i}^0 + \dot{\mathbf{r}}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{G4i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{G4i}^0 \\
&= \dot{x}_B \mathbf{i}_0 + \dot{y}_B \mathbf{j}_0 + \dot{z}_B \mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \\
&\quad \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \\
&\quad \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \dot{\theta}_{3i} \left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) + \\
&\quad \dot{\theta}_{4i} \left(\mathbf{k}_{6i}^0 \times \mathbf{r}_{G4i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{5i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{2i}^0 + \dot{\mathbf{r}}_{3i}^0 + \dot{\mathbf{r}}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{G5i}^0
\end{aligned}
$$

$$
\begin{aligned}
&= \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{G5i}^0 \\
&= \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \\
&\quad \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \\
&\quad \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \\
&\quad \dot{\theta}_{3i} \left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \dot{\theta}_{4i} \left(\mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) + \dot{\theta}_{5i} \left(\mathbf{k}_{8i}^0 \times \mathbf{r}_{G5i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{6i}^0 &= \dot{\mathbf{r}}_B^0 + \dot{\mathbf{r}}_{0i}^0 + \dot{\mathbf{r}}_{1i}^0 + \dot{\mathbf{r}}_{2i}^0 + \dot{\mathbf{r}}_{3i}^0 + \dot{\mathbf{r}}_{4i}^0 + \dot{\mathbf{r}}_{5i}^0 + \dot{\mathbf{r}}_{G6i}^0 \\
&= \mathbf{v}_B^0 + \mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \mathbf{v}_{5i}^0 + \boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{G6i}^0 \\
&= \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \boldsymbol{\omega}_B^0 \times \mathbf{r}_{0i}^0 + \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{1i}^0 + \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{2i}^0 + \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{3i}^0 + \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{4i}^0 + \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{5i}^0 + \boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{G6i}^0 \\
&= \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \\
&\quad \dot{\phi}_B \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \dot{\psi}_B \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \\
&\quad \dot{\theta}_{1i} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \dot{\theta}_{2i} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \\
&\quad \dot{\theta}_{3i} \left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \dot{\theta}_{4i} \left(\mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \\
&\quad \dot{\theta}_{5i} \left(\mathbf{k}_{8i}^0 \times \left(\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right) + \dot{\theta}_{6i} \left(\mathbf{i}_{10i}^0 \times \mathbf{r}_{G6i}^0 \right)
\end{aligned}
$$

Definiendo:

$$
\begin{aligned}
\mathbf{e}_{\theta_B}^0 &= \mathbf{i}_{\theta B}^0 \times \mathbf{r}_{GB}^0 \\
\mathbf{e}_{\phi_B}^0 &= \mathbf{j}_{\phi B}^0 \times \mathbf{r}_{GB}^0 \\
\mathbf{e}_{\psi_B}^0 &= \mathbf{k}_{\psi B}^0 \times \mathbf{r}_{GB}^0 \\
\mathbf{e}_{1i}^0 &= \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \\
\mathbf{e}_{2i}^0 &= \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \\
\mathbf{e}_{3i}^0 &= \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0 \right) \\
\mathbf{e}_{4i}^0 &= \mathbf{k}_{0i}^0 \times \mathbf{r}_{G1i}^0 \\
\mathbf{e}_{5i}^0 &= \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \\
\mathbf{e}_{6i}^0 &= \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \\
\mathbf{e}_{7i}^0 &= \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \\
\mathbf{e}_{8i}^0 &= \mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \\
\mathbf{e}_{9i}^0 &= \mathbf{j}_{2i}^0 \times \mathbf{r}_{G2i}^0 \\
\mathbf{e}_{10i}^0 &= \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \\
\mathbf{e}_{11i}^0 &= \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \\
\mathbf{e}_{12i}^0 &= \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \\
\mathbf{e}_{13i}^0 &= \mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \\
\mathbf{e}_{14i}^0 &= \mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \\
\mathbf{e}_{15i}^0 &= \mathbf{k}_{4i}^0 \times \mathbf{r}_{G3i}^0 \\
\mathbf{e}_{16i}^0 &= \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \\
\mathbf{e}_{17i}^0 &= \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right)
\end{aligned}
$$

$$\mathbf{e}_{18i}^0 = \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)$$

$$\mathbf{e}_{19i}^0 = \mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)$$

$$\mathbf{e}_{20i}^0 = \mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)$$

$$\mathbf{e}_{21i}^0 = \mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)$$

$$\mathbf{e}_{22i}^0 = \mathbf{k}_{6i}^0 \times \mathbf{r}_{G4i}^0$$

$$\mathbf{e}_{23i}^0 = \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{24i}^0 = \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{25i}^0 = \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{26i}^0 = \mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{27i}^0 = \mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{28i}^0 = \mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{29i}^0 = \mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)$$

$$\mathbf{e}_{30i}^0 = \mathbf{k}_{8i}^0 \times \mathbf{r}_{G5i}^0$$

$$\mathbf{e}_{31i}^0 = \mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{32i}^0 = \mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{33i}^0 = \mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{34i}^0 = \mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{35i}^0 = \mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{36i}^0 = \mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{37i}^0 = \mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{38i}^0 = \mathbf{k}_{8i}^0 \times \left(\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)$$

$$\mathbf{e}_{39i}^0 = \mathbf{i}_{10i}^0 \times \mathbf{r}_{G6i}^0$$

Las velocidades de los centros de gravedad pueden escribirse:

$$\dot{\mathbf{b}}_B^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{\theta_B}^0 + \dot{\phi}_B\mathbf{e}_{\phi_B}^0 + \dot{\psi}_B\mathbf{e}_{\psi_B}^0$$

$$\dot{\mathbf{b}}_{1i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{1i}^0 + \dot{\phi}_B\mathbf{e}_{2i}^0 + \dot{\psi}_B\mathbf{e}_{3i}^0 + \dot{\theta}_{1i}\mathbf{e}_{4i}^0$$

$$\dot{\mathbf{b}}_{2i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{5i}^0 + \dot{\phi}_B\mathbf{e}_{6i}^0 + \dot{\psi}_B\mathbf{e}_{7i}^0 + \dot{\theta}_{1i}\mathbf{e}_{8i}^0 + \dot{\theta}_{2i}\mathbf{e}_{9i}^0$$

$$\dot{\mathbf{b}}_{3i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{10i}^0 + \dot{\phi}_B\mathbf{e}_{11i}^0 + \dot{\psi}_B\mathbf{e}_{12i}^0 + \dot{\theta}_{1i}\mathbf{e}_{13i}^0 + \dot{\theta}_{2i}\mathbf{e}_{14i}^0 + \dot{\theta}_{3i}\mathbf{e}_{15i}^0$$

$$\dot{\mathbf{b}}_{4i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{16i}^0 + \dot{\phi}_B\mathbf{e}_{17i}^0 + \dot{\psi}_B\mathbf{e}_{18i}^0 + \dot{\theta}_{1i}\mathbf{e}_{19i}^0 + \dot{\theta}_{2i}\mathbf{e}_{20i}^0 + \dot{\theta}_{3i}\mathbf{e}_{21i}^0 + \dot{\theta}_{4i}\mathbf{e}_{22i}^0$$

$$\dot{\mathbf{b}}_{5i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{23i}^0 + \dot{\phi}_B\mathbf{e}_{24i}^0 + \dot{\psi}_B\mathbf{e}_{25i}^0 + \dot{\theta}_{1i}\mathbf{e}_{26i}^0 + \dot{\theta}_{2i}\mathbf{e}_{27i}^0 + \dot{\theta}_{3i}\mathbf{e}_{28i}^0 + \dot{\theta}_{4i}\mathbf{e}_{29i}^0 + \dot{\theta}_{5i}\mathbf{e}_{30i}^0$$

$$\dot{\mathbf{b}}_{6i}^0 = \dot{x}_B\mathbf{i}_0 + \dot{y}_B\mathbf{j}_0 + \dot{z}_B\mathbf{k}_0 + \dot{\theta}_B\mathbf{e}_{31i}^0 + \dot{\phi}_B\mathbf{e}_{32i}^0 + \dot{\psi}_B\mathbf{e}_{33i}^0 + \dot{\theta}_{1i}\mathbf{e}_{34i}^0 + \dot{\theta}_{2i}\mathbf{e}_{35i}^0 + \dot{\theta}_{3i}\mathbf{e}_{36i}^0 + \dot{\theta}_{4i}\mathbf{e}_{37i}^0 + \dot{\theta}_{5i}\mathbf{e}_{38i}^0 + \dot{\theta}_{6i}\mathbf{e}_{39i}^0$$

Reuniendo las coordenadas generalizadas en (4.2) y sus derivadas respecto al tiempo en los vectores $\mathbf{q}, \dot{\mathbf{q}}$

y $\ddot{\mathbf{q}}$:

$$
\mathbf{q} =
\begin{bmatrix}
x_B \\
y_B \\
z_B \\
\theta_B \\
\phi_B \\
\psi_B \\
\theta_{11} \\
\theta_{21} \\
\theta_{31} \\
\theta_{41} \\
\theta_{51} \\
\theta_{61} \\
\theta_{12} \\
\theta_{22} \\
\theta_{32} \\
\theta_{42} \\
\theta_{52} \\
\theta_{62}
\end{bmatrix}
, \quad
\dot{\mathbf{q}} =
\begin{bmatrix}
\dot{x}_B \\
\dot{y}_B \\
\dot{z}_B \\
\dot{\theta}_B \\
\dot{\phi}_B \\
\dot{\psi}_B \\
\dot{\theta}_{11} \\
\dot{\theta}_{21} \\
\dot{\theta}_{31} \\
\dot{\theta}_{41} \\
\dot{\theta}_{51} \\
\dot{\theta}_{61} \\
\dot{\theta}_{12} \\
\dot{\theta}_{22} \\
\dot{\theta}_{32} \\
\dot{\theta}_{42} \\
\dot{\theta}_{52} \\
\dot{\theta}_{62}
\end{bmatrix}
, \quad
\ddot{\mathbf{q}} =
\begin{bmatrix}
\ddot{x}_B \\
\ddot{y}_B \\
\ddot{z}_B \\
\ddot{\theta}_B \\
\ddot{\phi}_B \\
\ddot{\psi}_B \\
\ddot{\theta}_{11} \\
\ddot{\theta}_{21} \\
\ddot{\theta}_{31} \\
\ddot{\theta}_{41} \\
\ddot{\theta}_{51} \\
\ddot{\theta}_{61} \\
\ddot{\theta}_{12} \\
\ddot{\theta}_{22} \\
\ddot{\theta}_{32} \\
\ddot{\theta}_{42} \\
\ddot{\theta}_{52} \\
\ddot{\theta}_{62}
\end{bmatrix}
$$

los vectores de velocidad de los centros de gravedad pueden escribirse en forma matricial como:

$$
\begin{aligned}
\dot{\mathbf{b}}_B^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{\theta_B}^0 & \mathbf{e}_{\phi_B}^0 & \mathbf{e}_{\psi_B}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_B \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{11}^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{11}^0 & \mathbf{e}_{21}^0 & \mathbf{e}_{31}^0 & \mathbf{e}_{41}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_{11} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{21}^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{51}^0 & \mathbf{e}_{61}^0 & \mathbf{e}_{71}^0 & \mathbf{e}_{81}^0 & \mathbf{e}_{91}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_{21} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{31}^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{101}^0 & \mathbf{e}_{111}^0 & \mathbf{e}_{121}^0 & \mathbf{e}_{131}^0 & \mathbf{e}_{141}^0 & \mathbf{e}_{151}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_{31} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{41}^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{161}^0 & \mathbf{e}_{171}^0 & \mathbf{e}_{181}^0 & \mathbf{e}_{191}^0 & \mathbf{e}_{201}^0 & \mathbf{e}_{211}^0 & \mathbf{e}_{221}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_{41} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{b}}_{51}^0 &= \begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{231}^0 & \mathbf{e}_{241}^0 & \mathbf{e}_{251}^0 & \mathbf{e}_{261}^0 & \mathbf{e}_{271}^0 & \mathbf{e}_{281}^0 & \mathbf{e}_{291}^0 & \mathbf{e}_{301}^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{J}_{51} \dot{\mathbf{q}}
\end{aligned}
$$

$$\dot{\mathbf{b}}^0_{61} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{311} & \mathbf{e}^0_{321} & \mathbf{e}^0_{331} & \mathbf{e}^0_{341} & \mathbf{e}^0_{351} & \mathbf{e}^0_{361} & \mathbf{e}^0_{371} & \mathbf{e}^0_{381} & \mathbf{e}^0_{391} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{61}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{12} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{12} & \mathbf{e}^0_{22} & \mathbf{e}^0_{32} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{42} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{12}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{22} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{52} & \mathbf{e}^0_{62} & \mathbf{e}^0_{72} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{82} & \mathbf{e}^0_{92} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$\mathbf{J}_{22}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{32} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{102} & \mathbf{e}^0_{112} & \mathbf{e}^0_{122} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{132} & \mathbf{e}^0_{142} & \mathbf{e}^0_{152} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{32}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{42} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{162} & \mathbf{e}^0_{172} & \mathbf{e}^0_{182} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{192} & \mathbf{e}^0_{202} & \mathbf{e}^0_{212} & \mathbf{e}^0_{222} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{42}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{52} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{232} & \mathbf{e}^0_{242} & \mathbf{e}^0_{252} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{262} & \mathbf{e}^0_{272} & \mathbf{e}^0_{282} & \mathbf{e}^0_{292} & \mathbf{e}^0_{302} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{52}\dot{\mathbf{q}}$$

$$\dot{\mathbf{b}}^0_{62} = \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{312} & \mathbf{e}^0_{322} & \mathbf{e}^0_{332} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{342} & \mathbf{e}^0_{352} & \mathbf{e}^0_{362} & \mathbf{e}^0_{372} & \mathbf{e}^0_{382} & \mathbf{e}^0_{392} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{J}_{62}\dot{\mathbf{q}}$$

Todos los vectores se escriben en columnas para la constitución de matrices de 3×18.

También combiene disponer de la descripción de los vectores de velocidad angular en términos matriciales de acuerdo a:

$$\boldsymbol{\omega}^0_B = \dot{\theta}_B\mathbf{i}^0_{\theta B} + \dot{\phi}_B\mathbf{j}^0_{\phi B} + \dot{\psi}_B\mathbf{k}^0_{\psi B}$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}^0_{\theta B} & \mathbf{j}^0_{\phi B} & \mathbf{k}^0_{\psi B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{W}_B\dot{\mathbf{q}}$$

$$\boldsymbol{\omega}^0_{11} = \dot{\theta}_B\mathbf{i}^0_{\theta B} + \dot{\phi}_B\mathbf{j}^0_{\phi B} + \dot{\psi}_B\mathbf{k}^0_{\psi B} + \dot{\theta}_{11}\mathbf{k}^0_{01}$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}^0_{\theta B} & \mathbf{j}^0_{\phi B} & \mathbf{k}^0_{\psi B} & \mathbf{k}^0_{01} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{W}_{11}\dot{\mathbf{q}}$$

$$\boldsymbol{\omega}^0_{21} = \dot{\theta}_B\mathbf{i}^0_{\theta B} + \dot{\phi}_B\mathbf{j}^0_{\phi B} + \dot{\psi}_B\mathbf{k}^0_{\psi B} + \dot{\theta}_{11}\mathbf{k}^0_{01} + \dot{\theta}_{21}\mathbf{j}^0_{21}$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}^0_{\theta B} & \mathbf{j}^0_{\phi B} & \mathbf{k}^0_{\psi B} & \mathbf{k}^0_{01} & \mathbf{j}^0_{21} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]\dot{\mathbf{q}}$$
$$= \mathbf{W}_{21}\dot{\mathbf{q}}$$

$$
\begin{aligned}
\boldsymbol{\omega}_{31}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{11} \mathbf{k}_{01}^0 + \dot{\theta}_{21} \mathbf{j}_{21}^0 + \dot{\theta}_{31} \mathbf{k}_{41}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{31} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{41}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{11} \mathbf{k}_{01}^0 + \dot{\theta}_{21} \mathbf{j}_{21}^0 + \dot{\theta}_{31} \mathbf{k}_{41}^0 + \dot{\theta}_{41} \mathbf{k}_{61}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{41} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{51}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{11} \mathbf{k}_{01}^0 + \dot{\theta}_{21} \mathbf{j}_{21}^0 + \dot{\theta}_{31} \mathbf{k}_{41}^0 + \dot{\theta}_{41} \mathbf{k}_{61}^0 + \dot{\theta}_{51} \mathbf{k}_{81}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{k}_{81}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{51} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{61}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{11} \mathbf{k}_{01}^0 + \dot{\theta}_{21} \mathbf{j}_{21}^0 + \dot{\theta}_{31} \mathbf{k}_{41}^0 + \dot{\theta}_{41} \mathbf{k}_{61}^0 + \dot{\theta}_{51} \mathbf{k}_{81}^0 + \dot{\theta}_{61} \mathbf{i}_{101}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{k}_{81}^0 & \mathbf{i}_{101}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{61} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{12}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{12} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{22}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{22} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{32}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 + \dot{\theta}_{32} \mathbf{k}_{42}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{32} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{42}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 + \dot{\theta}_{32} \mathbf{k}_{42}^0 + \dot{\theta}_{42} \mathbf{k}_{62}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{0} & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{42} \dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_{52}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 + \dot{\theta}_{32} \mathbf{k}_{42}^0 + \dot{\theta}_{42} \mathbf{k}_{62}^0 + \dot{\theta}_{52} \mathbf{k}_{82}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{k}_{82}^0 & \mathbf{0} \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{52} \dot{\mathbf{q}}
\end{aligned}
$$

$$\begin{aligned}
\boldsymbol{\omega}_{62}^0 &= \dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 + \dot{\theta}_{32} \mathbf{k}_{42}^0 + \dot{\theta}_{42} \mathbf{k}_{62}^0 + \dot{\theta}_{52} \mathbf{k}_{82}^0 + \dot{\theta}_{62} \mathbf{i}_{102}^0 \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{k}_{82}^0 & \mathbf{i}_{102}^0 \end{bmatrix} \dot{\mathbf{q}} \\
&= \mathbf{W}_{62} \dot{\mathbf{q}}
\end{aligned}$$

Aprovechando la notación de las velocidades de los centros de gravedad de los eslabones con las matrices $\mathbf{J}_B$ y $\mathbf{J}_{ni}$ y de los vectores de velocidad angular con $\mathbf{W}_B$ y $\mathbf{W}_{ni}$, la función Lagrangiana (4.4) se puede reescribir como:

$$\begin{aligned}
L &= \left(\frac{1}{2} \left(m_B \left(\mathbf{J}_B \dot{\mathbf{q}} \right)^T \mathbf{J}_B \dot{\mathbf{q}} + \left(\mathbf{W}_B \dot{\mathbf{q}} \right)^T \left(\mathbf{I}_B^0 \mathbf{W}_B \dot{\mathbf{q}} \right) \right) + m_B \mathbf{g}^T \mathbf{b}_B^0 \right) + \\
&\quad \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \left(\frac{1}{2} \left(m_{ni} \left(\mathbf{J}_{ni} \dot{\mathbf{q}} \right)^T \mathbf{J}_{ni} \dot{\mathbf{q}} + \left(\mathbf{W}_{ni} \dot{\mathbf{q}} \right)^T \left(\mathbf{I}_{ni}^0 \mathbf{W}_{ni} \dot{\mathbf{q}} \right) \right) + m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0 \right) \right) \tag{4.5}
\end{aligned}$$

4.2. Desarrollo del término $\frac{\partial L}{\partial \dot{q}_j}$

Desarrollando la derivada parcial con respecto a $\dot{q}_j$ de la función Lagrangiana en (4.5) se tiene:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(m_B \left(\mathbf{J}_B \dot{\mathbf{q}} \right)^T \mathbf{J}_B \dot{\mathbf{q}} + \left(\mathbf{W}_B \dot{\mathbf{q}} \right)^T \left(\mathbf{I}_B^0 \mathbf{W}_B \dot{\mathbf{q}} \right) \right) + m_B \mathbf{g}^T \mathbf{b}_B^0 \right) + \\
&\quad \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(m_{ni} \left(\mathbf{J}_{ni} \dot{\mathbf{q}} \right)^T \mathbf{J}_{ni} \dot{\mathbf{q}} + \left(\mathbf{W}_{ni} \dot{\mathbf{q}} \right)^T \left(\mathbf{I}_{ni}^0 \mathbf{W}_{ni} \dot{\mathbf{q}} \right) \right) + m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0 \right) \right) \\
&= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(m_B \dot{\mathbf{q}}^T \left(\mathbf{J}_B^T \mathbf{J}_B \right) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \left(\mathbf{W}_B^T \mathbf{I}_B^0 \mathbf{W}_B \right) \dot{\mathbf{q}} \right) + m_B \mathbf{g}^T \mathbf{b}_B^0 \right) + \\
&\quad \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(m_{ni} \dot{\mathbf{q}}^T \left(\mathbf{J}_{ni}^T \mathbf{J}_{ni} \right) \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \left(\mathbf{W}_{ni}^T \mathbf{I}_{ni}^0 \mathbf{W}_{ni} \right) \dot{\mathbf{q}} \right) + m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0 \right) \right) \\
&= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(\dot{\mathbf{q}}^T \left(m_B \mathbf{J}_B^T \mathbf{J}_B + \mathbf{W}_B^T \mathbf{I}_B^0 \mathbf{W}_B \right) \dot{\mathbf{q}} \right) + m_B \mathbf{g}^T \mathbf{b}_0^0 \right) + \\
&\quad \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \left(\dot{\mathbf{q}}^T \left(m_{ni} \mathbf{J}_{ni}^T \mathbf{J}_{ni} + \mathbf{W}_{ni}^T \mathbf{I}_{ni}^0 \mathbf{W}_{ni} \right) \dot{\mathbf{q}} \right) + m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0 \right) \right)
\end{aligned}$$

Definiendo las matrices:

$$\begin{aligned}
\mathbf{N}_B &= m_B \mathbf{J}_B^T \mathbf{J}_B + \mathbf{W}_B^T \mathbf{I}_B^0 \mathbf{W}_B \\
\mathbf{N}_{ni} &= m_{ni} \mathbf{J}_{ni}^T \mathbf{J}_{ni} + \mathbf{W}_{ni}^T \mathbf{I}_{ni}^0 \mathbf{W}_{ni}
\end{aligned}$$

la derivada parcial de la función Lagrangiana L con respecto a $\dot{q}_j$ se escribe como:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_B \dot{\mathbf{q}} + m_B \mathbf{g}^T \mathbf{b}_0^0 \right) + \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{\partial}{\partial \dot{q}_j} \left(\frac{1}{2} \dot{\mathbf{q}}^T \mathbf{N}_{ni} \dot{\mathbf{q}} + m_{ni} \mathbf{g}^T \mathbf{b}_{ni}^0 \right) \right) \\
&= \frac{1}{2} \frac{\partial}{\partial \dot{q}_j} \left(\dot{\mathbf{q}}^T \mathbf{N}_B \dot{\mathbf{q}} \right) + \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{1}{2} \frac{\partial}{\partial \dot{q}_j} \left(\dot{\mathbf{q}}^T \mathbf{N}_{ni} \dot{\mathbf{q}} \right) \right) \\
&= \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_B \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_B \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) + \sum_{i=1}^{2} \left(\sum_{n=1}^{6} \frac{1}{2} \left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j} \mathbf{N}_{ni} \dot{\mathbf{q}} + \dot{\mathbf{q}}^T \mathbf{N}_{ni} \frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} \right) \right) \tag{4.6}
\end{aligned}$$

Desarrollando $\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}$ para $j = 1, 2, ..., 18$ se tiene:

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_1} = \frac{\partial}{\partial \dot{x}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_2} = \frac{\partial}{\partial \dot{y}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_3} = \frac{\partial}{\partial \dot{z}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_4} = \frac{\partial}{\partial \dot{\theta}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_5} = \frac{\partial}{\partial \dot{\phi}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_6} = \frac{\partial}{\partial \dot{\psi}_B} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_7} = \frac{\partial}{\partial \dot{\theta}_{11}} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_8} = \frac{\partial}{\partial \dot{\theta}_{21}} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_9} = \frac{\partial}{\partial \dot{\theta}_{31}} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{10}} = \frac{\partial}{\partial \dot{\theta}_{41}} \begin{bmatrix} \dot{x}_B & \dot{y}_B & \dot{z}_B & \dot{\theta}_B & \dot{\phi}_B & \dot{\psi}_B & \dot{\theta}_{11} & \dot{\theta}_{21} & \dot{\theta}_{31} & \dot{\theta}_{41} & \dot{\theta}_{51} & \dot{\theta}_{61} & \dot{\theta}_{12} & \dot{\theta}_{22} & \dot{\theta}_{32} & \dot{\theta}_{42} & \dot{\theta}_{52} & \dot{\theta}_{62} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{11}} \;=\; \frac{\partial}{\partial \dot{\theta}_{51}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{12}} \;=\; \frac{\partial}{\partial \dot{\theta}_{61}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{13}} \;=\; \frac{\partial}{\partial \dot{\theta}_{12}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{14}} \;=\; \frac{\partial}{\partial \dot{\theta}_{22}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{15}} \;=\; \frac{\partial}{\partial \dot{\theta}_{32}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{16}} \;=\; \frac{\partial}{\partial \dot{\theta}_{42}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{17}} \;=\; \frac{\partial}{\partial \dot{\theta}_{52}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\right]$$

$$\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_{18}} \;=\; \frac{\partial}{\partial \dot{\theta}_{62}} \left[\; \dot{x}_B \;\; \dot{y}_B \;\; \dot{z}_B \;\; \dot{\theta}_B \;\; \dot{\phi}_B \;\; \dot{\psi}_B \;\; \dot{\theta}_{11} \;\; \dot{\theta}_{21} \;\; \dot{\theta}_{31} \;\; \dot{\theta}_{41} \;\; \dot{\theta}_{51} \;\; \dot{\theta}_{61} \;\; \dot{\theta}_{12} \;\; \dot{\theta}_{22} \;\; \dot{\theta}_{32} \;\; \dot{\theta}_{42} \;\; \dot{\theta}_{52} \;\; \dot{\theta}_{62} \;\right]$$

$$=\; \left[\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\right]$$

4.3. Desarrollo del término $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$

Derivando con respecto al tiempo la expresión en (4.6) se tiene:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \;=\; \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{N}_B\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right) + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\mathbf{N}_{ni}\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right)\right)$$

$$=\; \frac{1}{2}\left(\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\right)\mathbf{N}_B\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_B\dot{\mathbf{q}} + \mathbf{N}_B\ddot{\mathbf{q}}\right) + \left(\ddot{\mathbf{q}}^T\mathbf{N}_B + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_B\right)\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} + \dot{\mathbf{q}}^T\mathbf{N}_B\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right)\right) +$$

$$+\; \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\right)\mathbf{N}_{ni}\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_{ni}\dot{\mathbf{q}} + \mathbf{N}_{ni}\ddot{\mathbf{q}}\right) + \left(\ddot{\mathbf{q}}^T\mathbf{N}_{ni} + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_{ni}\right)\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} + \dot{\mathbf{q}}^T\mathbf{N}_{ni}\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right)\right)\right)$$

En tanto $\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right) = \mathbf{0}$ y $\frac{d}{dt}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\right) = \mathbf{0}^T$:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) &= \frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_B\dot{\mathbf{q}} + \mathbf{N}_B\ddot{\mathbf{q}}\right) + \left(\ddot{\mathbf{q}}^T\mathbf{N}_B + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_B\right)\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_{ni}\dot{\mathbf{q}} + \mathbf{N}_{ni}\ddot{\mathbf{q}}\right) + \left(\ddot{\mathbf{q}}^T\mathbf{N}_{ni} + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_{ni}\right)\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_B\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B\ddot{\mathbf{q}} + \ddot{\mathbf{q}}^T\mathbf{N}_B\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_B\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_{ni}\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni}\ddot{\mathbf{q}} + \ddot{\mathbf{q}}^T\mathbf{N}_{ni}\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j} + \dot{\mathbf{q}}^T\dot{\mathbf{N}}_{ni}\frac{\partial \dot{\mathbf{q}}}{\partial \dot{q}_j}\right)\right)
\end{aligned}
$$

Recurriendo a la propiedad de matrices transpuestas $(\mathbf{ABC})^T = \mathbf{C}^T\mathbf{B}^T\mathbf{A}^T$:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) &= \frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_B\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B^T\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_B^T\dot{\mathbf{q}}\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_{ni}\dot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni}\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni}^T\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_{ni}^T\dot{\mathbf{q}}\right)\right) \\
&= \frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\mathbf{N}_B + \mathbf{N}_B^T\right)\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_B + \dot{\mathbf{N}}_B^T\right)\dot{\mathbf{q}}\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\mathbf{N}_{ni} + \mathbf{N}_{ni}^T\right)\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(\dot{\mathbf{N}}_{ni} + \dot{\mathbf{N}}_{ni}^T\right)\dot{\mathbf{q}}\right)\right)
\end{aligned}
\tag{4.7}
$$

Aprovechando la misma propiedad en la expresión de la matriz $\mathbf{N}_B$:

$$
\begin{aligned}
\mathbf{N}_B^T &= \left(m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\mathbf{I}_B^0\mathbf{W}_B\right)^T \\
&= m_B\left(\mathbf{J}_B^T\mathbf{J}_B\right)^T + \left(\mathbf{W}_B^T\mathbf{I}_B^0\mathbf{W}_B\right)^T \\
&= m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\left(\mathbf{I}_B^0\right)^T\mathbf{W}_B
\end{aligned}
$$

donde

$$
\begin{aligned}
\left(\mathbf{I}_B^0\right)^T &= \left(\mathbf{R}_B^0\mathbf{I}_B^B\left(\mathbf{R}_B^0\right)^T\right)^T \\
&= \mathbf{R}_B^0\left(\mathbf{I}_B^B\right)^T\left(\mathbf{R}_B^0\right)^T
\end{aligned}
$$

En tanto que las matrices de inercia en su base local son simétricas:

$$
\begin{aligned}
\left(\mathbf{I}_B^0\right)^T &= \mathbf{R}_B^0\mathbf{I}_B^B\left(\mathbf{R}_B^0\right)^T \\
&= \mathbf{I}_B^0
\end{aligned}
$$

se deduce:

$$
\begin{aligned}
\mathbf{N}_B^T &= m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\left(\mathbf{I}_B^0\right)^T\mathbf{W}_B \\
&= m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\mathbf{I}_B^0\mathbf{W}_B \\
&= \mathbf{N}_B
\end{aligned}
$$

De forma análoga se puede demostrar que $\mathbf{N}_{ni} = \mathbf{N}_{ni}^T$.

Una vez demostrado que las matrices $\mathbf{N}_B$ y $\mathbf{N}_{ni}$ son simétricas, la derivada con respecto al tiempo en (4.7) se reduce a:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) &= \frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}(2\mathbf{N}_B)\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(2\dot{\mathbf{N}}_B\right)\dot{\mathbf{q}}\right) + \\
&\quad \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{1}{2}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}(2\mathbf{N}_{ni})\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\left(2\dot{\mathbf{N}}_{ni}\right)\dot{\mathbf{q}}\right)\right) \\
&= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_B\dot{\mathbf{q}} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni}\ddot{\mathbf{q}} + \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_{ni}\dot{\mathbf{q}}\right)\right)
\end{aligned}
$$

Para simplificar la notación de esta derivada se recurre a los vectores renglón $\mathbf{D}_{Bj}$, $\mathbf{V}_{Bj}$, $\mathbf{D}_{nij}$ y $\mathbf{V}_{nij}$:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \mathbf{D}_{Bj}\ddot{\mathbf{q}} + \mathbf{V}_{Bj}\dot{\mathbf{q}} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{D}_{nij}\ddot{\mathbf{q}} + \mathbf{V}_{nij}\dot{\mathbf{q}}\right)\right) \tag{4.8}
$$

donde

$$
\begin{aligned}
\mathbf{D}_{Bj} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_B \\
\mathbf{V}_{Bj} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_B \\
\mathbf{D}_{nij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\mathbf{N}_{ni} \\
\mathbf{V}_{nij} &= \frac{\partial \dot{\mathbf{q}}^T}{\partial \dot{q}_j}\dot{\mathbf{N}}_{ni}
\end{aligned}
$$

A continuación se exponen los cálculos para la determinación de las matrices $\dot{\mathbf{N}}$ de cada uno de los eslabones del bípedo.

4.3.1. Eslabón B

$$
\begin{aligned}
\dot{\mathbf{N}}_B &= \frac{d}{dt}\left(m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\mathbf{I}_B^0\mathbf{W}_B\right) \\
&= m_B\left(\dot{\mathbf{J}}_B^T\mathbf{J}_B + \mathbf{J}_B^T\dot{\mathbf{J}}_B\right) + \dot{\mathbf{W}}_B^T\mathbf{I}_B^0\mathbf{W}_B + \mathbf{W}_B^T\left(\dot{\mathbf{I}}_B^0\mathbf{W}_B + \mathbf{I}_B^0\dot{\mathbf{W}}_B\right) \\
&= m_B\left(\dot{\mathbf{J}}_B^T\mathbf{J}_B + \mathbf{J}_B^T\dot{\mathbf{J}}_B\right) + \dot{\mathbf{W}}_B^T\mathbf{I}_B^0\mathbf{W}_B + \mathbf{W}_B^T\dot{\mathbf{I}}_B^0\mathbf{W}_B + \mathbf{W}_B^T\mathbf{I}_B^0\dot{\mathbf{W}}_B
\end{aligned}
$$

donde

$$
\begin{aligned}
\dot{\mathbf{I}}_B^0 &= \frac{d}{dt}\left(\mathbf{R}_B^0\mathbf{I}_B^B\left(\mathbf{R}_B^0\right)^T\right) \\
&= \dot{\mathbf{R}}_B^0\mathbf{I}_B^B\left(\mathbf{R}_B^0\right)^T + \mathbf{R}_B^0\mathbf{I}_B^B\left(\dot{\mathbf{R}}_B^0\right)^T
\end{aligned}
$$

$$
\dot{\mathbf{R}}_B^0 = \dot{\mathbf{R}}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B) + \mathbf{R}_x(\theta_B)\dot{\mathbf{R}}_y(\phi_B)\mathbf{R}_z(\psi_B) + \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\dot{\mathbf{R}}_z(\psi_B)
$$

Definiendo las matrices

$$
\boldsymbol{\Omega}_x(\dot{\theta}_x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_x \\ 0 & \dot{\theta}_x & 0 \end{bmatrix}
$$

$$
\boldsymbol{\Omega}_y(\dot{\theta}_y) = \begin{bmatrix} 0 & 0 & \dot{\theta}_y \\ 0 & 0 & 0 \\ -\dot{\theta}_y & 0 & 0 \end{bmatrix} \tag{4.9}
$$

$$
\boldsymbol{\Omega}_z(\dot{\theta}_z) = \begin{bmatrix} 0 & -\dot{\theta}_z & 0 \\ \dot{\theta}_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$\dot{\mathbf{R}}_x(\theta_B)$, $\dot{\mathbf{R}}_y(\phi_B)$ y $\dot{\mathbf{R}}_z(\psi_B)$ se calculan como:

$$
\begin{aligned}
\dot{\mathbf{R}}_x(\theta_B) &= \boldsymbol{\Omega}_x(\dot{\theta}_B)\mathbf{R}_x(\theta_B) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}_B \\ 0 & \dot{\theta}_B & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_B & -\sin\theta_B \\ 0 & \sin\theta_B & \cos\theta_B \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta}_B\sin\theta_B & -\dot{\theta}_B\cos\theta_B \\ 0 & \dot{\theta}_B\cos\theta_B & -\dot{\theta}_B\sin\theta_B \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{R}}_y(\phi_B) &= \boldsymbol{\Omega}_y(\dot{\phi}_B)\mathbf{R}_y(\phi_B) \\
&= \begin{bmatrix} 0 & 0 & \dot{\phi}_B \\ 0 & 0 & 0 \\ -\dot{\phi}_B & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi_B & 0 & \sin\phi_B \\ 0 & 1 & 0 \\ -\sin\phi_B & 0 & \cos\phi_B \end{bmatrix} \\
&= \begin{bmatrix} -\dot{\phi}_B\sin\phi_B & 0 & \dot{\phi}_B\cos\phi_B \\ 0 & 0 & 0 \\ -\dot{\phi}_B\cos\phi_B & 0 & -\dot{\phi}_B\sin\phi_B \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{R}}_z(\psi_B) &= \boldsymbol{\Omega}_z(\dot{\psi}_B)\mathbf{R}_z(\psi_B) \\
&= \begin{bmatrix} 0 & -\dot{\psi}_B & 0 \\ \dot{\psi}_B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\psi_B & -\sin\psi_B & 0 \\ \sin\psi_B & \cos\psi_B & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\dot{\psi}_B\sin\psi_B & -\dot{\psi}_B\cos\psi_B & 0 \\ \dot{\psi}_B\cos\psi_B & -\dot{\psi}_B\sin\psi_B & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Para las matrices $\dot{\mathbf{J}}_B, \dot{\mathbf{W}}_B$ se tiene:

$$
\dot{\mathbf{J}}_B = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{\theta_B} & \mathbf{e}^0_{\phi_B} & \mathbf{e}^0_{\psi_B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{\theta_B}^0 & \dot{\mathbf{e}}_{\phi_B}^0 & \dot{\mathbf{e}}_{\psi_B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{W}}_B = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{i}}_{\theta B}^0 & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

donde

$$\begin{aligned}
\dot{\mathbf{e}}_{\theta_B}^0 &= \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \dot{\mathbf{i}}_{\theta B}^0 \times \mathbf{r}_{GB}^0 + \mathbf{i}_{\theta B}^0 \times \dot{\mathbf{r}}_{GB}^0 \\
&= (\mathbf{0}) \times \mathbf{r}_{GB}^0 + \mathbf{i}_{\theta B}^0 \times \dot{\mathbf{r}}_{GB}^0 \\
&= \mathbf{i}_{\theta B}^0 \times \dot{\mathbf{r}}_{GB}^0
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{e}}_{\phi_B}^0 &= \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \dot{\mathbf{j}}_{\phi B}^0 \times \mathbf{r}_{GB}^0 + \mathbf{j}_{\phi B}^0 \times \dot{\mathbf{r}}_{GB}^0
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{e}}_{\psi_B}^0 &= \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \dot{\mathbf{k}}_{\psi B}^0 \times \mathbf{r}_{GB}^0 + \mathbf{k}_{\psi B}^0 \times \dot{\mathbf{r}}_{GB}^0
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{i}}_{\theta B}^0 &= \mathbf{0} \\
\dot{\mathbf{j}}_{\phi B}^0 &= \boldsymbol{\omega}_{\theta B}^0 \times \mathbf{j}_{\phi B}^0 \\
\dot{\mathbf{k}}_{\psi B}^0 &= \left(\boldsymbol{\omega}_{\theta B}^0 + \boldsymbol{\omega}_{\phi B}^0\right) \times \mathbf{k}_{\psi B}^0 \\
\dot{\mathbf{r}}_{GB}^0 &= \boldsymbol{\omega}_B^0 \times \mathbf{r}_{GB}^0
\end{aligned}$$

4.3.2. Eslabones $1i$

$$\dot{\mathbf{N}}_{1i} = m_{1i}\left(\dot{\mathbf{J}}_{1i}^T \mathbf{J}_{1i} + \mathbf{J}_{1i}^T \dot{\mathbf{J}}_{1i}\right) + \dot{\mathbf{W}}_{1i}^T \mathbf{I}_{1i}^0 \mathbf{W}_{1i} + \mathbf{W}_{1i}^T \dot{\mathbf{I}}_{1i}^0 \mathbf{W}_{1i} + \mathbf{W}_{1i}^T \mathbf{I}_{1i}^0 \dot{\mathbf{W}}_{1i}$$

donde

$$\begin{aligned}
\dot{\mathbf{I}}_{1i}^0 &= \frac{d}{dt}\left(\mathbf{R}_{1i}^0 \mathbf{I}_{1i}^{1i}\left(\mathbf{R}_{1i}^0\right)^T\right) \\
&= \dot{\mathbf{R}}_{1i}^0 \mathbf{I}_{1i}^{1i}\left(\mathbf{R}_{1i}^0\right)^T + \mathbf{R}_{1i}^0 \mathbf{I}_{1i}^{1i}\left(\dot{\mathbf{R}}_{1i}^0\right)^T
\end{aligned}$$

$$\dot{\mathbf{R}}_{1i}^0 = \dot{\mathbf{R}}_B^0 \mathbf{R}_z(\theta_{1i} + \beta_{0i})\mathbf{R}_y(\beta_{1i}) + \mathbf{R}_B^0 \dot{\mathbf{R}}_z(\theta_{1i} + \beta_{0i})\mathbf{R}_y(\beta_{1i})$$

Con las matrices en (4.9):

$$\dot{\mathbf{R}}_z(\theta_{1i} + \beta_{0i}) = \boldsymbol{\Omega}_z(\dot{\theta}_{1i})\mathbf{R}_z(\theta_{1i} + \beta_{0i})$$

En particular, para cada eslabón

$$\dot{\mathbf{J}}_{11} = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{11}^0 & \mathbf{e}_{21}^0 & \mathbf{e}_{31}^0 & \mathbf{e}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{11}^0 & \dot{\mathbf{e}}_{21}^0 & \dot{\mathbf{e}}_{31}^0 & \dot{\mathbf{e}}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{W}}_{11} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \dot{\mathbf{k}}_{01}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{J}}_{12} = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{12}^0 & \mathbf{e}_{22}^0 & \mathbf{e}_{32}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{12}^0 & \dot{\mathbf{e}}_{22}^0 & \dot{\mathbf{e}}_{32}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{W}}_{12} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{k}}_{02}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

donde

$$\dot{\mathbf{e}}_{1i}^0 = \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$
$$= \dot{\mathbf{i}}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\mathbf{v}_{0i}^0 + \dot{\mathbf{r}}_{G1i}^0\right)$$

$$\dot{\mathbf{e}}_{2i}^0 = \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$
$$= \dot{\mathbf{j}}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\mathbf{v}_{0i}^0 + \dot{\mathbf{r}}_{G1i}^0\right)$$

$$\dot{\mathbf{e}}_{3i}^0 = \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\mathbf{v}_{0i}^0 + \dot{\mathbf{r}}_{G1i}^0\right)$$

$$\dot{\mathbf{e}}_{4i}^0 = \frac{d}{dt}\left(\mathbf{k}_{0i}^0 \times \mathbf{r}_{G1i}^0\right)$$
$$= \dot{\mathbf{k}}_{0i}^0 \times \mathbf{r}_{G1i}^0 + \mathbf{k}_{0i}^0 \times \dot{\mathbf{r}}_{G1i}^0$$

$$\dot{\mathbf{k}}_{0i}^0 = \boldsymbol{\omega}_B^0 \times \mathbf{k}_{0i}^0$$
$$\dot{\mathbf{r}}_{G1i}^0 = \boldsymbol{\omega}_{1i}^0 \times \mathbf{r}_{G1i}^0$$

4.3.3. Eslabones $2i$

$$\dot{\mathbf{N}}_{2i} = m_{2i}\left(\dot{\mathbf{J}}_{2i}^T \mathbf{J}_{2i} + \mathbf{J}_{2i}^T \dot{\mathbf{J}}_{2i}\right) + \dot{\mathbf{W}}_{2i}^T \mathbf{I}_{2i}^0 \mathbf{W}_{2i} + \mathbf{W}_{2i}^T \dot{\mathbf{I}}_{2i}^0 \mathbf{W}_{2i} + \mathbf{W}_{2i}^T \mathbf{I}_{2i}^0 \dot{\mathbf{W}}_{2i}$$

donde

$$
\begin{aligned}
\dot{\mathbf{I}}_{2i}^0 &= \frac{d}{dt}\left(\mathbf{R}_{3i}^0 \mathbf{I}_{2i}^{3i}\left(\mathbf{R}_{3i}^0\right)^T\right) \\
&= \dot{\mathbf{R}}_{3i}^0 \mathbf{I}_{2i}^{3i}\left(\mathbf{R}_{3i}^0\right)^T + \mathbf{R}_{3i}^0 \mathbf{I}_{2i}^{3i}\left(\dot{\mathbf{R}}_{3i}^0\right)^T
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{R}}_{3i}^0 &= \frac{d}{dt}\left(\mathbf{R}_{1i}^0 \mathbf{R}_y(\theta_{2i})\right) \\
&= \dot{\mathbf{R}}_{1i}^0 \mathbf{R}_y(\theta_{2i}) + \mathbf{R}_{1i}^0 \dot{\mathbf{R}}_y(\theta_{2i})
\end{aligned}
$$

$$
\dot{\mathbf{R}}_y(\theta_{2i}) = \boldsymbol{\Omega}_y(\dot{\theta}_{2i})\mathbf{R}_y(\theta_{2i})
$$

En particular, para cada eslabón:

$$
\begin{aligned}
\dot{\mathbf{J}}_{21} &= \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{51}^0 & \mathbf{e}_{61}^0 & \mathbf{e}_{71}^0 & \mathbf{e}_{81}^0 & \mathbf{e}_{91}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{51}^0 & \dot{\mathbf{e}}_{61}^0 & \dot{\mathbf{e}}_{71}^0 & \dot{\mathbf{e}}_{81}^0 & \dot{\mathbf{e}}_{91}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{W}}_{21} &= \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \dot{\mathbf{k}}_{01}^0 & \dot{\mathbf{j}}_{21}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{J}}_{22} &= \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{52}^0 & \mathbf{e}_{62}^0 & \mathbf{e}_{72}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{82}^0 & \mathbf{e}_{92}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{52}^0 & \dot{\mathbf{e}}_{62}^0 & \dot{\mathbf{e}}_{72}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{82}^0 & \dot{\mathbf{e}}_{92}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{W}}_{22} &= \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{k}}_{02}^0 & \dot{\mathbf{j}}_{22}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{array}\right]
\end{aligned}
$$

donde

$$
\begin{aligned}
\dot{\mathbf{e}}_{5i}^0 &= \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right) \\
&= \dot{\mathbf{i}}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \dot{\mathbf{r}}_{G2i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{e}}_{6i}^0 &= \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right) \\
&= \dot{\mathbf{j}}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \dot{\mathbf{r}}_{G2i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{e}}_{7i}^0 &= \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right) \\
&= \dot{\mathbf{k}}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \dot{\mathbf{r}}_{G2i}^0\right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{e}}_{8i}^0 &= \frac{d}{dt}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right) \\
&= \dot{\mathbf{k}}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\mathbf{v}_{1i}^0 + \dot{\mathbf{r}}_{G2i}^0\right)
\end{aligned}
$$

$$\dot{\mathbf{e}}_{9i}^0 = \frac{d}{dt}\left(\mathbf{j}_{2i}^0 \times \mathbf{r}_{G2i}^0\right)$$
$$= \dot{\mathbf{j}}_{2i}^0 \times \mathbf{r}_{G2i}^0 + \mathbf{j}_{2i}^0 \times \dot{\mathbf{r}}_{G2i}^0$$

$$\dot{\mathbf{j}}_{2i}^0 = \boldsymbol{\omega}_{1i}^0 \times \mathbf{j}_{2i}^0$$
$$\dot{\mathbf{r}}_{G2i}^0 = \boldsymbol{\omega}_{2i}^0 \times \mathbf{r}_{G2i}^0$$

4.3.4. Eslabones $3i$

$$\dot{\mathbf{N}}_{3i} = m_{3i}\left(\dot{\mathbf{J}}_{3i}^T \mathbf{J}_{3i} + \mathbf{J}_{3i}^T \dot{\mathbf{J}}_{3i}\right) + \dot{\mathbf{W}}_{3i}^T \mathbf{I}_{3i}^0 \mathbf{W}_{3i} + \mathbf{W}_{3i}^T \dot{\mathbf{I}}_{3i}^0 \mathbf{W}_{3i} + \mathbf{W}_{3i}^T \mathbf{I}_{3i}^0 \dot{\mathbf{W}}_{3i}$$

donde

$$\dot{\mathbf{I}}_{3i}^0 = \frac{d}{dt}\left(\mathbf{R}_{5i}^0 \mathbf{I}_{3i}^{5i}\left(\mathbf{R}_{5i}^0\right)^T\right)$$
$$= \dot{\mathbf{R}}_{5i}^0 \mathbf{I}_{3i}^{5i}\left(\mathbf{R}_{5i}^0\right)^T + \mathbf{R}_{5i}^0 \mathbf{I}_{3i}^{5i}\left(\dot{\mathbf{R}}_{5i}^0\right)^T$$

$$\dot{\mathbf{R}}_{5i}^0 = \frac{d}{dt}\left(\mathbf{R}_{3i}^0 \mathbf{R}_z(\theta_{3i}+\beta_{2i})\right)$$
$$= \dot{\mathbf{R}}_{3i}^0 \mathbf{R}_z(\theta_{3i}+\beta_{2i}) + \mathbf{R}_{3i}^0 \dot{\mathbf{R}}_z(\theta_{3i}+\beta_{2i})$$

$$\dot{\mathbf{R}}_z(\theta_{3i}+\beta_{2i}) = \boldsymbol{\Omega}_z(\dot{\theta}_{3i})\mathbf{R}_z(\theta_{3i}+\beta_{2i})$$

En particular, para cada eslabón:

$$\dot{\mathbf{J}}_{31} = \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{101}^0 & \mathbf{e}_{111}^0 & \mathbf{e}_{121}^0 & \mathbf{e}_{131}^0 & \mathbf{e}_{141}^0 & \mathbf{e}_{151}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{101}^0 & \dot{\mathbf{e}}_{111}^0 & \dot{\mathbf{e}}_{121}^0 & \dot{\mathbf{e}}_{131}^0 & \dot{\mathbf{e}}_{141}^0 & \dot{\mathbf{e}}_{151}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\dot{\mathbf{W}}_{31} = \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \dot{\mathbf{k}}_{01}^0 & \dot{\mathbf{j}}_{21}^0 & \dot{\mathbf{k}}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\dot{\mathbf{J}}_{32} = \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{102}^0 & \mathbf{e}_{112}^0 & \mathbf{e}_{122}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{132}^0 & \mathbf{e}_{142}^0 & \mathbf{e}_{152}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{102}^0 & \dot{\mathbf{e}}_{112}^0 & \dot{\mathbf{e}}_{122}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{132}^0 & \dot{\mathbf{e}}_{142}^0 & \dot{\mathbf{e}}_{152}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\dot{\mathbf{W}}_{32} = \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$
$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{k}}_{02}^0 & \dot{\mathbf{j}}_{22}^0 & \dot{\mathbf{k}}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

donde

$$\dot{\mathbf{e}}_{10i}^0 = \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$

$$= \dot{\mathbf{i}}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0\right)$$

$$\dot{\mathbf{e}}_{11i}^0 = \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$
$$= \dot{\mathbf{j}}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0\right)$$

$$\dot{\mathbf{e}}_{12i}^0 = \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0\right)$$

$$\dot{\mathbf{e}}_{13i}^0 = \frac{d}{dt}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0\right)$$

$$\dot{\mathbf{e}}_{14i}^0 = \frac{d}{dt}\left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$
$$= \dot{\mathbf{j}}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{j}_{2i}^0 \times \left(\mathbf{v}_{2i}^0 + \dot{\mathbf{r}}_{G3i}^0\right)$$

$$\dot{\mathbf{e}}_{15i}^0 = \frac{d}{dt}\left(\mathbf{k}_{4i}^0 \times \mathbf{r}_{G3i}^0\right)$$
$$= \dot{\mathbf{k}}_{4i}^0 \times \mathbf{r}_{G3i}^0 + \mathbf{k}_{4i}^0 \times \dot{\mathbf{r}}_{G3i}^0$$

$$\dot{\mathbf{k}}_{4i}^0 = \boldsymbol{\omega}_{2i}^0 \times \mathbf{k}_{4i}^0$$
$$\dot{\mathbf{r}}_{G3i}^0 = \boldsymbol{\omega}_{3i}^0 \times \mathbf{r}_{G3i}^0$$

4.3.5. Eslabones $4i$

$$\dot{\mathbf{N}}_{4i} = m_{4i}\left(\dot{\mathbf{J}}_{4i}^T \mathbf{J}_{4i} + \mathbf{J}_{4i}^T \dot{\mathbf{J}}_{4i}\right) + \dot{\mathbf{W}}_{4i}^T \mathbf{I}_{4i}^0 \mathbf{W}_{4i} + \mathbf{W}_{4i}^T \dot{\mathbf{I}}_{4i}^0 \mathbf{W}_{4i} + \mathbf{W}_{4i}^T \mathbf{I}_{4i}^0 \dot{\mathbf{W}}_{4i}$$

donde

$$\dot{\mathbf{I}}_{4i}^0 = \frac{d}{dt}\left(\mathbf{R}_{7i}^0 \mathbf{I}_{4i}^{7i}\left(\mathbf{R}_{7i}^0\right)^T\right)$$
$$= \dot{\mathbf{R}}_{7i}^0 \mathbf{I}_{4i}^{7i}\left(\mathbf{R}_{7i}^0\right)^T + \mathbf{R}_{7i}^0 \mathbf{I}_{4i}^{7i}\left(\dot{\mathbf{R}}_{7i}^0\right)^T$$

$$\dot{\mathbf{R}}_{7i}^0 = \frac{d}{dt}\left(\mathbf{R}_{5i}^0 \mathbf{R}_z(\theta_{4i} + \beta_{3i})\right)$$
$$= \dot{\mathbf{R}}_{5i}^0 \mathbf{R}_z(\theta_{4i} + \beta_{3i}) + \mathbf{R}_{5i}^0 \dot{\mathbf{R}}_z(\theta_{4i} + \beta_{3i})$$

$$\dot{\mathbf{R}}_z(\theta_{4i} + \beta_{3i}) = \boldsymbol{\Omega}_z(\dot{\theta}_{4i})\mathbf{R}_z(\theta_{4i} + \beta_{3i})$$

En particular, para dada eslabón:

$$\dot{\mathbf{J}}_{41} = \frac{d}{dt}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{161}^0 & \mathbf{e}_{171}^0 & \mathbf{e}_{181}^0 & \mathbf{e}_{191}^0 & \mathbf{e}_{201}^0 & \mathbf{e}_{211}^0 & \mathbf{e}_{221}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{161}^0 & \dot{\mathbf{e}}_{171}^0 & \dot{\mathbf{e}}_{181}^0 & \dot{\mathbf{e}}_{191}^0 & \dot{\mathbf{e}}_{201}^0 & \dot{\mathbf{e}}_{211}^0 & \dot{\mathbf{e}}_{221}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{W}}_{41} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \dot{\mathbf{k}}_{01}^0 & \dot{\mathbf{j}}_{21}^0 & \dot{\mathbf{k}}_{41}^0 & \dot{\mathbf{k}}_{61}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{J}}_{42} = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{162}^0 & \mathbf{e}_{172}^0 & \mathbf{e}_{182}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{192}^0 & \mathbf{e}_{202}^0 & \mathbf{e}_{212}^0 & \mathbf{e}_{222}^0 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{162}^0 & \dot{\mathbf{e}}_{172}^0 & \dot{\mathbf{e}}_{182}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}_{192}^0 & \dot{\mathbf{e}}_{202}^0 & \dot{\mathbf{e}}_{212}^0 & \dot{\mathbf{e}}_{222}^0 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\dot{\mathbf{W}}_{42} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}_{\phi B}^0 & \dot{\mathbf{k}}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{k}}_{02}^0 & \dot{\mathbf{j}}_{22}^0 & \dot{\mathbf{k}}_{42}^0 & \dot{\mathbf{k}}_{62}^0 & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

donde

$$\dot{\mathbf{e}}_{16i}^0 = \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{i}}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{17i}^0 = \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{j}}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{18i}^0 = \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{19i}^0 = \frac{d}{dt}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{20i}^0 = \frac{d}{dt}\left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{j}}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{j}_{2i}^0 \times \left(\mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{21i}^0 = \frac{d}{dt}\left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)\right)$$
$$= \dot{\mathbf{k}}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right) + \mathbf{k}_{4i}^0 \times \left(\mathbf{v}_{3i}^0 + \dot{\mathbf{r}}_{G4i}^0\right)$$

$$\dot{\mathbf{e}}_{22i}^0 = \frac{d}{dt}\left(\mathbf{k}_{6i}^0 \times \mathbf{r}_{G4i}^0\right)$$
$$= \dot{\mathbf{k}}_{6i}^0 \times \mathbf{r}_{G4i}^0 + \mathbf{k}_{6i}^0 \times \dot{\mathbf{r}}_{G4i}^0$$

$$\dot{\mathbf{k}}_{6i}^0 = \boldsymbol{\omega}_{3i}^0 \times \mathbf{k}_{6i}^0$$

$$\dot{\mathbf{r}}_{G4i}^0 = \boldsymbol{\omega}_{4i}^0 \times \mathbf{r}_{G4i}^0$$

4.3.6.   Eslabones $5i$

$$\dot{\mathbf{N}}_{5i} = m_{5i} \left(\dot{\mathbf{J}}_{5i}^T \mathbf{J}_{5i} + \mathbf{J}_{5i}^T \dot{\mathbf{J}}_{5i} \right) + \dot{\mathbf{W}}_{5i}^T \mathbf{I}_{5i}^0 \mathbf{W}_{5i} + \mathbf{W}_{5i}^T \dot{\mathbf{I}}_{5i}^0 \mathbf{W}_{5i} + \mathbf{W}_{5i}^T \mathbf{I}_{5i}^0 \dot{\mathbf{W}}_{5i}$$

donde

$$
\begin{aligned}
\dot{\mathbf{I}}_{5i}^0 &= \frac{d}{dt}\left(\mathbf{R}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\mathbf{R}_{9i}^0\right)^T \right) \\
&= \dot{\mathbf{R}}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\mathbf{R}_{9i}^0\right)^T + \mathbf{R}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\dot{\mathbf{R}}_{9i}^0\right)^T
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{R}}_{9i}^0 &= \frac{d}{dt}\left(\mathbf{R}_{7i}^0 \mathbf{R}_z(\theta_{5i} + \beta_{4i}) \right) \\
&= \dot{\mathbf{R}}_{7i}^0 \mathbf{R}_z(\theta_{5i} + \beta_{4i}) + \mathbf{R}_{7i}^0 \dot{\mathbf{R}}_z(\theta_{5i} + \beta_{4i})
\end{aligned}
$$

$$\dot{\mathbf{R}}_z(\theta_{5i} + \beta_{4i}) = \boldsymbol{\Omega}_z(\dot{\theta}_{5i}) \mathbf{R}_z(\theta_{5i} + \beta_{4i})$$

En particular, para cada eslabón:

$$
\begin{aligned}
\dot{\mathbf{J}}_{51} &= \frac{d}{dt}\left[\; \mathbf{i}_0 \quad \mathbf{j}_0 \quad \mathbf{k}_0 \quad \mathbf{e}_{231}^0 \quad \mathbf{e}_{241}^0 \quad \mathbf{e}_{251}^0 \quad \mathbf{e}_{261}^0 \quad \mathbf{e}_{271}^0 \quad \mathbf{e}_{281}^0 \quad \mathbf{e}_{291}^0 \quad \mathbf{e}_{301}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}\;\right] \\
&= \left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{e}}_{231}^0 \quad \dot{\mathbf{e}}_{241}^0 \quad \dot{\mathbf{e}}_{251}^0 \quad \dot{\mathbf{e}}_{261}^0 \quad \dot{\mathbf{e}}_{271}^0 \quad \dot{\mathbf{e}}_{281}^0 \quad \dot{\mathbf{e}}_{291}^0 \quad \dot{\mathbf{e}}_{301}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}\;\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{W}}_{51} &= \frac{d}{dt}\left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{i}_{\theta B}^0 \quad \mathbf{j}_{\phi B}^0 \quad \mathbf{k}_{\psi B}^0 \quad \mathbf{k}_{01}^0 \quad \mathbf{j}_{21}^0 \quad \mathbf{k}_{41}^0 \quad \mathbf{k}_{61}^0 \quad \mathbf{k}_{81}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}\;\right] \\
&= \left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{j}}_{\phi B}^0 \quad \dot{\mathbf{k}}_{\psi B}^0 \quad \dot{\mathbf{k}}_{01}^0 \quad \dot{\mathbf{j}}_{21}^0 \quad \dot{\mathbf{k}}_{41}^0 \quad \dot{\mathbf{k}}_{61}^0 \quad \dot{\mathbf{k}}_{81}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}\;\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{J}}_{52} &= \frac{d}{dt}\left[\; \mathbf{i}_0 \quad \mathbf{j}_0 \quad \mathbf{k}_0 \quad \mathbf{e}_{232}^0 \quad \mathbf{e}_{242}^0 \quad \mathbf{e}_{252}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{e}_{262}^0 \quad \mathbf{e}_{272}^0 \quad \mathbf{e}_{282}^0 \quad \mathbf{e}_{292}^0 \quad \mathbf{e}_{302}^0 \quad \mathbf{0}\;\right] \\
&= \left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{e}}_{232}^0 \quad \dot{\mathbf{e}}_{242}^0 \quad \dot{\mathbf{e}}_{252}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{e}}_{262}^0 \quad \dot{\mathbf{e}}_{272}^0 \quad \dot{\mathbf{e}}_{282}^0 \quad \dot{\mathbf{e}}_{292}^0 \quad \dot{\mathbf{e}}_{302}^0 \quad \mathbf{0}\;\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{W}}_{52} &= \frac{d}{dt}\left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{i}_{\theta B}^0 \quad \mathbf{j}_{\phi B}^0 \quad \mathbf{k}_{\psi B}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{k}_{02}^0 \quad \mathbf{j}_{22}^0 \quad \mathbf{k}_{42}^0 \quad \mathbf{k}_{62}^0 \quad \mathbf{k}_{82}^0 \quad \mathbf{0}\;\right] \\
&= \left[\; \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{j}}_{\phi B}^0 \quad \dot{\mathbf{k}}_{\psi B}^0 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \dot{\mathbf{k}}_{02}^0 \quad \dot{\mathbf{j}}_{22}^0 \quad \dot{\mathbf{k}}_{42}^0 \quad \dot{\mathbf{k}}_{62}^0 \quad \dot{\mathbf{k}}_{82}^0 \quad \mathbf{0}\;\right]
\end{aligned}
$$

donde

$$
\begin{aligned}
\dot{\mathbf{e}}_{23i}^0 &= \frac{d}{dt}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) \\
&= \dot{\mathbf{i}}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{i}_{\theta B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0 \right)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{e}}_{24i}^0 &= \frac{d}{dt}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right) \\
&= \dot{\mathbf{j}}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{j}_{\phi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0 \right)
\end{aligned}
$$

$$\dot{\mathbf{e}}_{25i}^0 = \frac{d}{dt}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)\right)$$

$$= \dot{\mathbf{k}}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\mathbf{v}_{0i}^0 + \mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0\right)$$

$$\dot{\mathbf{e}}_{26i}^0 = \frac{d}{dt}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)\right)$$

$$= \dot{\mathbf{k}}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\mathbf{v}_{1i}^0 + \mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0\right)$$

$$\dot{\mathbf{e}}_{27i}^0 = \frac{d}{dt}\left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)\right)$$

$$= \dot{\mathbf{j}}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right) + \mathbf{j}_{2i}^0 \times \left(\mathbf{v}_{2i}^0 + \mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0\right)$$

$$\dot{\mathbf{e}}_{28i}^0 = \frac{d}{dt}\left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)\right)$$

$$= \dot{\mathbf{k}}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right) + \mathbf{k}_{4i}^0 \times \left(\mathbf{v}_{3i}^0 + \mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0\right)$$

$$\dot{\mathbf{e}}_{29i}^0 = \frac{d}{dt}\left(\mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right)\right)$$

$$= \dot{\mathbf{k}}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0\right) + \mathbf{k}_{6i}^0 \times \left(\mathbf{v}_{4i}^0 + \dot{\mathbf{r}}_{G5i}^0\right)$$

$$\dot{\mathbf{e}}_{30i}^0 = \frac{d}{dt}\left(\mathbf{k}_{8i}^0 \times \mathbf{r}_{G5i}^0\right)$$

$$= \dot{\mathbf{k}}_{8i}^0 \times \mathbf{r}_{G5i}^0 + \mathbf{k}_{8i}^0 \times \dot{\mathbf{r}}_{G5i}^0$$

$$\dot{\mathbf{k}}_{8i}^0 = \boldsymbol{\omega}_{4i}^0 \times \mathbf{k}_{8i}^0$$

$$\dot{\mathbf{r}}_{G5i}^0 = \boldsymbol{\omega}_{5i}^0 \times \mathbf{r}_{G5i}^0$$

4.3.7. Eslabones $6i$

$$\dot{\mathbf{N}}_{6i} = m_{6i}\left(\dot{\mathbf{J}}_{6i}^T \mathbf{J}_{6i} + \mathbf{J}_{6i}^T \dot{\mathbf{J}}_{6i}\right) + \dot{\mathbf{W}}_{6i}^T \mathbf{I}_{6i}^0 \mathbf{W}_{6i} + \mathbf{W}_{6i}^T \dot{\mathbf{I}}_{6i}^0 \mathbf{W}_{6i} + \mathbf{W}_{6i}^T \mathbf{I}_{6i}^0 \dot{\mathbf{W}}_{6i}$$

donde

$$\dot{\mathbf{I}}_{6i}^0 = \frac{d}{dt}\left(\mathbf{R}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\mathbf{R}_{11i}^0\right)^T\right)$$

$$= \dot{\mathbf{R}}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\mathbf{R}_{11i}^0\right)^T + \mathbf{R}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\dot{\mathbf{R}}_{11i}^0\right)^T$$

$$\dot{\mathbf{R}}_{11i}^0 = \frac{d}{dt}\left(\mathbf{R}_{9i}^0 \mathbf{R}_x(\theta_{6i})\right)$$

$$= \dot{\mathbf{R}}_{9i}^0 \mathbf{R}_x(\theta_{6i}) + \mathbf{R}_{9i}^0 \dot{\mathbf{R}}_x(\theta_{6i})$$

$$\dot{\mathbf{R}}_x(\theta_{6i}) = \boldsymbol{\Omega}_x(\dot{\theta}_{6i}) \mathbf{R}_x(\theta_{6i})$$

En particular, para cada eslabón:

$$
\dot{\mathbf{J}}_{61} = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{311} & \mathbf{e}^0_{321} & \mathbf{e}^0_{331} & \mathbf{e}^0_{341} & \mathbf{e}^0_{351} & \mathbf{e}^0_{361} & \mathbf{e}^0_{371} & \mathbf{e}^0_{381} & \mathbf{e}^0_{391} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}^0_{311} & \dot{\mathbf{e}}^0_{321} & \dot{\mathbf{e}}^0_{331} & \dot{\mathbf{e}}^0_{341} & \dot{\mathbf{e}}^0_{351} & \dot{\mathbf{e}}^0_{361} & \dot{\mathbf{e}}^0_{371} & \dot{\mathbf{e}}^0_{381} & \dot{\mathbf{e}}^0_{391} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
\dot{\mathbf{W}}_{61} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}^0_{\theta B} & \mathbf{j}^0_{\phi B} & \mathbf{k}^0_{\psi B} & \mathbf{k}^0_{01} & \mathbf{j}^0_{21} & \mathbf{k}^0_{41} & \mathbf{k}^0_{61} & \mathbf{k}^0_{81} & \mathbf{i}^0_{101} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}^0_{\phi B} & \dot{\mathbf{k}}^0_{\psi B} & \dot{\mathbf{k}}^0_{01} & \dot{\mathbf{j}}^0_{21} & \dot{\mathbf{k}}^0_{41} & \dot{\mathbf{k}}^0_{61} & \dot{\mathbf{k}}^0_{81} & \dot{\mathbf{i}}^0_{101} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}
$$

$$
\dot{\mathbf{J}}_{62} = \frac{d}{dt}\begin{bmatrix} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}^0_{312} & \mathbf{e}^0_{322} & \mathbf{e}^0_{332} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^0_{342} & \mathbf{e}^0_{352} & \mathbf{e}^0_{362} & \mathbf{e}^0_{372} & \mathbf{e}^0_{382} & \mathbf{e}^0_{392} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}^0_{312} & \dot{\mathbf{e}}^0_{322} & \dot{\mathbf{e}}^0_{332} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{e}}^0_{342} & \dot{\mathbf{e}}^0_{352} & \dot{\mathbf{e}}^0_{362} & \dot{\mathbf{e}}^0_{372} & \dot{\mathbf{e}}^0_{382} & \dot{\mathbf{e}}^0_{392} \end{bmatrix}
$$

$$
\dot{\mathbf{W}}_{62} = \frac{d}{dt}\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}^0_{\theta B} & \mathbf{j}^0_{\phi B} & \mathbf{k}^0_{\psi B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}^0_{02} & \mathbf{j}^0_{22} & \mathbf{k}^0_{42} & \mathbf{k}^0_{62} & \mathbf{k}^0_{82} & \mathbf{i}^0_{102} \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{j}}^0_{\phi B} & \dot{\mathbf{k}}^0_{\psi B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dot{\mathbf{k}}^0_{02} & \dot{\mathbf{j}}^0_{22} & \dot{\mathbf{k}}^0_{42} & \dot{\mathbf{k}}^0_{62} & \dot{\mathbf{k}}^0_{82} & \dot{\mathbf{i}}^0_{102} \end{bmatrix}
$$

donde

$$
\dot{\mathbf{e}}^0_{31i} = \frac{d}{dt}\left(\mathbf{i}^0_{\theta B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{i}}^0_{\theta B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{i}^0_{\theta B} \times \left(\mathbf{v}^0_{0i} + \mathbf{v}^0_{1i} + \mathbf{v}^0_{2i} + \mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{32i} = \frac{d}{dt}\left(\mathbf{j}^0_{\phi B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{j}}^0_{\phi B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{j}^0_{\phi B} \times \left(\mathbf{v}^0_{0i} + \mathbf{v}^0_{1i} + \mathbf{v}^0_{2i} + \mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{33i} = \frac{d}{dt}\left(\mathbf{k}^0_{\psi B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{k}}^0_{\psi B} \times \left(\mathbf{r}^0_{0i} + \mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{k}^0_{\psi B} \times \left(\mathbf{v}^0_{0i} + \mathbf{v}^0_{1i} + \mathbf{v}^0_{2i} + \mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{34i} = \frac{d}{dt}\left(\mathbf{k}^0_{0i} \times \left(\mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{k}}^0_{0i} \times \left(\mathbf{r}^0_{1i} + \mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{k}^0_{0i} \times \left(\mathbf{v}^0_{1i} + \mathbf{v}^0_{2i} + \mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{35i} = \frac{d}{dt}\left(\mathbf{j}^0_{2i} \times \left(\mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{j}}^0_{2i} \times \left(\mathbf{r}^0_{2i} + \mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{j}^0_{2i} \times \left(\mathbf{v}^0_{2i} + \mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{36i} = \frac{d}{dt}\left(\mathbf{k}^0_{4i} \times \left(\mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{k}}^0_{4i} \times \left(\mathbf{r}^0_{3i} + \mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{k}^0_{4i} \times \left(\mathbf{v}^0_{3i} + \mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$
\dot{\mathbf{e}}^0_{37i} = \frac{d}{dt}\left(\mathbf{k}^0_{6i} \times \left(\mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right)\right)
$$

$$
= \dot{\mathbf{k}}^0_{6i} \times \left(\mathbf{r}^0_{4i} + \mathbf{r}^0_{5i} + \mathbf{r}^0_{G6i}\right) + \mathbf{k}^0_{6i} \times \left(\mathbf{v}^0_{4i} + \mathbf{v}^0_{5i} + \dot{\mathbf{r}}^0_{G6i}\right)
$$

$$\dot{\mathbf{e}}_{38i}^0 \;=\; \frac{d}{dt}\left(\mathbf{k}_{8i}^0 \times (\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0)\right)$$
$$\;=\; \dot{\mathbf{k}}_{8i}^0 \times (\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0) + \mathbf{k}_{8i}^0 \times (\mathbf{v}_{5i}^0 + \dot{\mathbf{r}}_{G6i}^0)$$

$$\dot{\mathbf{e}}_{39i}^0 \;=\; \frac{d}{dt}\left(\mathbf{i}_{10i}^0 \times \mathbf{r}_{G6i}^0\right)$$
$$\;=\; \dot{\mathbf{i}}_{10i}^0 \times \mathbf{r}_{G6i}^0 + \mathbf{i}_{10i}^0 \times \dot{\mathbf{r}}_{G6i}^0$$

$$\dot{\mathbf{i}}_{10i}^0 \;=\; \boldsymbol{\omega}_{5i}^0 \times \mathbf{i}_{10i}^0$$
$$\dot{\mathbf{r}}_{G6i}^0 \;=\; \boldsymbol{\omega}_{6i}^0 \times \mathbf{r}_{G6i}^0$$

4.4. Desarrollo del término $\frac{\partial L}{\partial q_j}$

Derivando la función Lagrangiana en (4.5) respecto a las coordenadas generalizadas q_j se tiene:

$$\frac{\partial L}{\partial q_j} \;=\; \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{N}_B\dot{\mathbf{q}} + m_B\mathbf{g}^T\mathbf{b}_B^0\right) + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{N}_{ni}\dot{\mathbf{q}} + m_{ni}\mathbf{g}^T\mathbf{b}_{ni}^0\right)\right)$$
$$\;=\; \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial\mathbf{N}_B}{\partial q_j}\dot{\mathbf{q}} + m_B\mathbf{g}^T\frac{\partial\mathbf{b}_B^0}{\partial q_j} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial\mathbf{N}_{ni}}{\partial q_j}\dot{\mathbf{q}} + m_{ni}\mathbf{g}^T\frac{\partial\mathbf{b}_{ni}^0}{\partial q_j}\right)\right)$$
$$\;=\; \mathbf{V'}_{\mathrm{Bj}}\dot{\mathbf{q}} + \mathbf{C'}_{\mathrm{Bj}} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{V'}_{nij}\dot{\mathbf{q}} + \mathbf{C'}_{nij}\right)\right) \tag{4.10}$$

donde

$$\mathbf{V'}_{\mathrm{Bj}} \;=\; \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial\mathbf{N}_B}{\partial q_j}$$
$$\mathbf{C'}_{\mathrm{Bj}} \;=\; m_B\mathbf{g}^T\frac{\partial\mathbf{b}_B^0}{\partial q_j}$$
$$\mathbf{V'}_{\mathrm{Bj}} \;=\; \frac{1}{2}\dot{\mathbf{q}}^T\frac{\partial\mathbf{N}_{ni}}{\partial q_j}$$
$$\mathbf{C'}_{\mathrm{nij}} \;=\; m_{ni}\mathbf{g}^T\frac{\partial\mathbf{b}_{ni}^0}{\partial q_j}$$

En las siguientes secciones se desarrollan $\frac{\partial\mathbf{N}_B}{\partial q_j}$, $\frac{\partial\mathbf{b}_B^0}{\partial q_j}$, $\frac{\partial\mathbf{N}_{ni}}{\partial q_j}$ y $\frac{\partial\mathbf{b}_{ni}^0}{\partial q_j}$ para cada uno de los eslabones.

4.4.1. Eslabón B

$$\frac{\partial\mathbf{N}_B}{\partial q_j} \;=\; \frac{\partial}{\partial q_j}\left(m_B\mathbf{J}_B^T\mathbf{J}_B + \mathbf{W}_B^T\mathbf{I}_B^0\mathbf{W}_B\right)$$
$$\;=\; m_B\left(\frac{\partial\mathbf{J}_B^T}{\partial q_j}\mathbf{J}_B + \mathbf{J}_B^T\frac{\partial\mathbf{J}_B}{\partial q_j}\right) + \frac{\partial\mathbf{W}_B^T}{\partial q_j}\mathbf{I}_B^0\mathbf{W}_B + \mathbf{W}_B^T\frac{\partial\mathbf{I}_B^0}{\partial q_j}\mathbf{W}_B + \mathbf{W}_B^T\mathbf{I}_B^0\frac{\partial\mathbf{W}_B}{\partial q_j}$$

donde

$$
\begin{aligned}
\frac{\partial \mathbf{b}_B^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{r}_B^0 + \mathbf{r}_{GB}^0\right) \\
&= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{GB}^0}{\partial q_j} \\
\frac{\partial \mathbf{I}_B^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{R}_B^0 \mathbf{I}_B^B \left(\mathbf{R}_B^0\right)^T\right) \\
&= \frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{I}_B^B\left(\mathbf{R}_B^0\right)^T + \mathbf{R}_B^0 \mathbf{I}_B^B\left(\frac{\partial \mathbf{R}_B^0}{\partial q_j}\right)^T \\
\frac{\partial \mathbf{r}_B^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(x_B \mathbf{i}_0 + y_B \mathbf{j}_0 + z_B \mathbf{k}_0\right) \\
\frac{\partial \mathbf{r}_{GB}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{R}_B^0 \mathbf{r}_{GB}^B\right) \\
&= \frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{r}_{GB}^B
\end{aligned}
$$

Para las matrices $\mathbf{J}_B, \mathbf{W}_B$, se tiene:

$$
\begin{aligned}
\frac{\partial \mathbf{J}_B}{\partial q_j} &= \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{\theta_B}^0 & \mathbf{e}_{\phi_B}^0 & \mathbf{e}_{\psi_B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{\theta_B}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{\phi_B}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{\psi_B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
\frac{\partial \mathbf{W}_B}{\partial q_j} &= \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]
\end{aligned}
$$

donde

$$
\begin{aligned}
\frac{\partial \mathbf{e}_{\theta_B}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{i}_{\theta B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \mathbf{r}_{GB}^0 + \mathbf{i}_{\theta B}^0 \times \frac{\partial \mathbf{r}_{GB}^0}{\partial q_j} \\
\frac{\partial \mathbf{e}_{\phi_B}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{j}_{\phi B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \mathbf{r}_{GB}^0 + \mathbf{j}_{\phi B}^0 \times \frac{\partial \mathbf{r}_{GB}^0}{\partial q_j} \\
\frac{\partial \mathbf{e}_{\psi_B}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{k}_{\psi B}^0 \times \mathbf{r}_{GB}^0\right) \\
&= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \mathbf{r}_{GB}^0 + \mathbf{k}_{\psi B}^0 \times \frac{\partial \mathbf{r}_{GB}^0}{\partial q_j} \\
\frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} &= \mathbf{0}
\end{aligned}
$$

$$\frac{\partial \mathbf{j}^0_{\phi B}}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_x(\theta_B)\mathbf{j}_{\phi_B}\right)$$

$$= \frac{\partial \mathbf{R}_x(\theta_B)}{\partial q_j}\mathbf{j}_{\phi_B}$$

$$\frac{\partial \mathbf{k}^0_{\psi B}}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{k}_{\psi_B}\right)$$

$$= \frac{\partial \mathbf{R}_x(\theta_B)}{\partial q_j}\mathbf{R}_y(\phi_B)\mathbf{k}_{\psi_B} + \mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial q_j}\mathbf{k}_{\psi_B}$$

Definiendo las matrices:

$$\frac{\partial \mathbf{R}_x(\alpha)}{\partial \alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\alpha & -\cos\alpha \\ 0 & \cos\alpha & -\sin\alpha \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_y(\beta)}{\partial \beta} = \begin{bmatrix} -\sin\beta & 0 & \cos\beta \\ 0 & 0 & 0 \\ -\cos\beta & 0 & -\sin\beta \end{bmatrix} \qquad (4.11)$$

$$\frac{\partial \mathbf{R}_z(\gamma)}{\partial \gamma} = \begin{bmatrix} -\sin\gamma & -\cos\gamma & 0 \\ \cos\gamma & -\sin\gamma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Se calcula:

$$\frac{\partial \mathbf{R}_x(\theta_B)}{\partial q_j} = \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\frac{\partial \theta_B}{\partial q_j}$$

$$\frac{\partial \mathbf{R}_y(\phi_B)}{\partial q_j} = \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B}\frac{\partial \phi_B}{\partial q_j}$$

con lo que se escribe:

$$\frac{\partial \mathbf{R}^0_B}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\right)$$

$$= \frac{\partial \mathbf{R}_x(\theta_B)}{\partial q_j}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B) + \mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial q_j}\mathbf{R}_z(\psi_B) +$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial q_j}$$

$$= \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\frac{\partial \theta_B}{\partial q_j}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B) + \mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B}\frac{\partial \phi_B}{\partial q_j}\mathbf{R}_z(\psi_B) +$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B}\frac{\partial \psi_B}{\partial q_j}$$

4.4.2. Eslabones $1i$

$$
\begin{aligned}
\frac{\partial \mathbf{N}_{1i}}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(m_{1i}\mathbf{J}_{1i}^T\mathbf{J}_{1i} + \mathbf{W}_{1i}^T\mathbf{I}_{1i}^0\mathbf{W}_{1i}\right) \\
&= m_{1i}\left(\frac{\partial \mathbf{J}_{1i}^T}{\partial q_j}\mathbf{J}_{1i} + \mathbf{J}_{1i}^T\frac{\partial \mathbf{J}_{1i}}{\partial q_j}\right) + \frac{\partial \mathbf{W}_{1i}^T}{\partial q_j}\mathbf{I}_{1i}^0\mathbf{W}_{1i} + \mathbf{W}_{1i}^T\frac{\partial \mathbf{I}_{1i}^0}{\partial q_j}\mathbf{W}_{1i} + \mathbf{W}_{1i}^T\mathbf{I}_{1i}^0\frac{\partial \mathbf{W}_{1i}}{\partial q_j}
\end{aligned}
$$

donde

$$
\begin{aligned}
\frac{\partial \mathbf{b}_{1i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) \\
&= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j} \\
\frac{\partial \mathbf{I}_{1i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{R}_{1i}^0\mathbf{I}_{1i}^{1i}\left(\mathbf{R}_{1i}^0\right)^T\right) \\
&= \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j}\mathbf{I}_{1i}^{1i}\left(\mathbf{R}_{1i}^0\right)^T + \mathbf{R}_{1i}^0\mathbf{I}_{1i}^{1i}\left(\frac{\partial \mathbf{R}_{1i}^0}{\partial q_j}\right)^T \\
\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} &= \frac{\partial x_{0i}\mathbf{R}_B^0\mathbf{i}_B}{\partial q_j} \\
&= x_{0i}\frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{i}_B \\
\frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j} &= \frac{\partial \mathbf{R}_{1i}^0\mathbf{r}_{G1i}^{1i}}{\partial q_j} \\
&= \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j}\mathbf{r}_{G1i}^{1i}
\end{aligned}
$$

Con las matrices definidas en (4.11):

$$
\begin{aligned}
\frac{\partial \mathbf{R}_{1i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\mathbf{R}_B^0\mathbf{R}_z(\theta_{1i}+\beta_{1i})\right) \\
&= \frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{R}_z(\theta_{1i}+\beta_{1i}) + \mathbf{R}_B^0\frac{\partial \mathbf{R}_z(\theta_{1i}+\beta_{1i})}{\partial q_j} \\
&= \frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{R}_z(\theta_{1i}+\beta_{1i}) + \mathbf{R}_B^0\frac{\partial \mathbf{R}_z(\theta_{1i}+\beta_{1i})}{\partial(\theta_{1i}+\beta_{1i})}\frac{\partial(\theta_{1i}+\beta_{1i})}{\partial q_j}
\end{aligned}
$$

En particular, para cada eslabón:

$$
\begin{aligned}
\frac{\partial \mathbf{J}_{11}}{\partial q_j} &= \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{11}^0 & \mathbf{e}_{21}^0 & \mathbf{e}_{31}^0 & \mathbf{e}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{11}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{21}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{31}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{41}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
\frac{\partial \mathbf{W}_{11}}{\partial q_j} &= \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]
\end{aligned}
$$

$$\frac{\partial \mathbf{J}_{12}}{\partial q_j} = \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{12}^0 & \mathbf{e}_{22}^0 & \mathbf{e}_{32}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{12}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{22}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{32}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{42}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\frac{\partial \mathbf{W}_{12}}{\partial q_j} = \frac{\partial}{\partial q_j}\left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

Donde,

$$\frac{\partial \mathbf{e}_{1i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$

$$= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$

$$= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{G1i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{0i}^0 \times \mathbf{r}_{G1i}^0\right)$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \mathbf{r}_{G1i}^0 + \mathbf{k}_{0i}^0 \times \frac{\partial \mathbf{r}_{G1i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_B^0 \mathbf{k}_{0i}\right)$$

$$= \frac{\partial \mathbf{R}_B^0}{\partial q_j}\mathbf{k}_{0i}$$

4.4.3. Eslabones $2i$

$$\frac{\partial \mathbf{N}_{2i}}{\partial q_j} = \frac{\partial}{\partial q_j}\left(m_{2i}\mathbf{J}_{2i}^T\mathbf{J}_{2i} + \mathbf{W}_{2i}^T\mathbf{I}_{2i}^0\mathbf{W}_{2i}\right)$$

$$= m_{2i}\left(\frac{\partial \mathbf{J}_{2i}^T}{\partial q_j}\mathbf{J}_{2i} + \mathbf{J}_{2i}^T\frac{\partial \mathbf{J}_{2i}}{\partial q_j}\right) + \frac{\partial \mathbf{W}_{2i}^T}{\partial q_j}\mathbf{I}_{2i}^0\mathbf{W}_{2i} + \mathbf{W}_{2i}^T\frac{\partial \mathbf{I}_{2i}^0}{\partial q_j}\mathbf{W}_{2i} + \mathbf{W}_{2i}^T\mathbf{I}_{2i}^0\frac{\partial \mathbf{W}_{2i}}{\partial q_j}$$

donde

$$\frac{\partial \mathbf{b}_{2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)$$

$$= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{I}_{2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{3i}^0 \mathbf{I}_{2i}^{3i} \left(\mathbf{R}_{3i}^0 \right)^T \right)$$

$$= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{I}_{2i}^{3i} \left(\mathbf{R}_{3i}^0 \right)^T + \mathbf{R}_{3i}^0 \mathbf{I}_{2i}^{3i} \left(\frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \right)^T$$

$$\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(x_{1i} \mathbf{R}_{1i}^0 \mathbf{i}_{1i} \right)$$

$$= x_{1i} \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j} \mathbf{i}_{1i}$$

$$\frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{3i}^0 \mathbf{r}_{G2i}^{3i} \right)$$

$$= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{r}_{G2i}^{3i}$$

$$\frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{1i}^0 \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \right)$$

$$= \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j} \mathbf{R}_y(\theta_{2i} + \beta_{2i}) + \mathbf{R}_{1i}^0 \frac{\partial \mathbf{R}_y(\theta_{2i} + \beta_{2i})}{\partial q_j}$$

$$= \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j} \mathbf{R}_y(\theta_{2i} + \beta_{2i}) + \mathbf{R}_{1i}^0 \frac{\partial \mathbf{R}_y(\theta_{2i} + \beta_{2i})}{\partial \theta_{2i}} \frac{\partial(\theta_{2i} + \beta_{2i})}{\partial q_j}$$

En particular, para cada eslabón:

$$\frac{\partial \mathbf{J}_{21}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{51}^0 & \mathbf{e}_{61}^0 & \mathbf{e}_{71}^0 & \mathbf{e}_{81}^0 & \mathbf{e}_{91}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{51}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{61}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{71}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{81}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{91}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{21}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{21}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{J}_{22}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{52}^0 & \mathbf{e}_{62}^0 & \mathbf{e}_{72}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{82}^0 & \mathbf{e}_{92}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{52}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{62}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{72}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{82}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{92}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{22}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{22}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

donde

$$\frac{\partial \mathbf{e}_{5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{6i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right)$$

$$= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{7i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{8i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{G2i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{9i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{2i}^0 \times \mathbf{r}_{G2i}^0\right)$$

$$= \frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} \times \mathbf{r}_{G2i}^0 + \mathbf{j}_{2i}^0 \times \frac{\partial \mathbf{r}_{G2i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{1i}^0 \mathbf{j}_{2i}\right)$$

$$= \frac{\partial \mathbf{R}_{1i}^0}{\partial q_j} \mathbf{j}_{2i}$$

4.4.4. Eslabones $3i$

$$\frac{\partial \mathbf{N}_{3i}}{\partial q_j} = \frac{\partial}{\partial q_j} \left(m_{3i} \mathbf{J}_{3i}^T \mathbf{J}_{3i} + \mathbf{W}_{3i}^T \mathbf{I}_{3i}^0 \mathbf{W}_{3i}\right)$$

$$= m_{3i} \left(\frac{\partial \mathbf{J}_{3i}^T}{\partial q_j} \mathbf{J}_{3i} + \mathbf{J}_{3i}^T \frac{\partial \mathbf{J}_{3i}}{\partial q_j}\right) + \frac{\partial \mathbf{W}_{3i}^T}{\partial q_j} \mathbf{I}_{3i}^0 \mathbf{W}_{3i} + \mathbf{W}_{3i}^T \frac{\partial \mathbf{I}_{3i}^0}{\partial q_j} \mathbf{W}_{3i} + \mathbf{W}_{3i}^T \mathbf{I}_{3i}^0 \frac{\partial \mathbf{W}_{3i}}{\partial q_j}$$

$$\frac{\partial \mathbf{b}_{3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)$$

$$= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{I}_{3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{5i}^0 \mathbf{I}_{3i}^{5i} \left(\mathbf{R}_{5i}^0\right)^T\right)$$

$$= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{I}_{3i}^{5i} \left(\mathbf{R}_{5i}^0\right)^T + \mathbf{R}_{5i}^0 \mathbf{I}_{3i}^{5i} \left(\frac{\partial \mathbf{R}_{5i}^0}{\partial q_j}\right)^T$$

$$\frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(x_{2i} \mathbf{R}_{3i}^0 \mathbf{i}_{3i}\right)$$

$$= x_{2i} \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{i}_{3i}$$

$$\frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{5i}^0 \mathbf{r}_{G3i}^{5i} \right)$$

$$= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{r}_{G3i}^{5i}$$

$$\frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{3i}^0 \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \right)$$

$$= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{R}_z(\theta_{3i} + \beta_{3i}) + \mathbf{R}_{3i}^0 \frac{\partial \mathbf{R}_z(\theta_{3i} + \beta_{3i})}{\partial q_j}$$

$$= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j} \mathbf{R}_z(\theta_{3i} + \beta_{3i}) + \mathbf{R}_{3i}^0 \frac{\partial \mathbf{R}_z(\theta_{3i} + \beta_{3i})}{\partial(\theta_{3i} + \beta_{3i})} \frac{\partial(\theta_{3i} + \beta_{3i})}{\partial q_j}$$

En particular, para cada eslabón

$$\frac{\partial \mathbf{J}_{31}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{101}^0 & \mathbf{e}_{111}^0 & \mathbf{e}_{121}^0 & \mathbf{e}_{131}^0 & \mathbf{e}_{141}^0 & \mathbf{e}_{151}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{101}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{111}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{121}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{131}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{141}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{151}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{31}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{21}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{41}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{J}_{32}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{102}^0 & \mathbf{e}_{112}^0 & \mathbf{e}_{122}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{132}^0 & \mathbf{e}_{142}^0 & \mathbf{e}_{152}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{102}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{112}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{122}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{132}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{142}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{152}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{32}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{22}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{42}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

donde

$$\frac{\partial \mathbf{e}_{10i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) + \mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{11i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) + \mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{12i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{13i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{14i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right)\right)$$

$$= \frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{G3i}^0\right) + \mathbf{j}_{2i}^0 \times \left(\frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{15i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{4i}^0 \times \mathbf{r}_{G3i}^0\right)$$

$$= \frac{\partial \mathbf{k}_{4i}^0}{\partial q_j} \times \mathbf{r}_{G3i}^0 + \mathbf{k}_{4i}^0 \times \frac{\partial \mathbf{r}_{G3i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{k}_{4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_{3i}^0 \mathbf{k}_{4i}\right)$$

$$= \frac{\partial \mathbf{R}_{3i}^0}{\partial q_j}\mathbf{k}_{4i}$$

4.4.5. Eslabones $4i$

$$\frac{\partial \mathbf{N}_{4i}}{\partial q_j} = \frac{\partial}{\partial q_j}\left(m_{4i}\mathbf{J}_{4i}^T\mathbf{J}_{4i} + \mathbf{W}_{4i}^T\mathbf{I}_{4i}^0\mathbf{W}_{4i}\right)$$

$$= m_{4i}\left(\frac{\partial \mathbf{J}_{4i}^T}{\partial q_j}\mathbf{J}_{4i} + \mathbf{J}_{4i}^T\frac{\partial \mathbf{J}_{4i}}{\partial q_j}\right) + \frac{\partial \mathbf{W}_{4i}^T}{\partial q_j}\mathbf{I}_{4i}^0\mathbf{W}_{4i} + \mathbf{W}_{4i}^T\frac{\partial \mathbf{I}_{4i}^0}{\partial q_j}\mathbf{W}_{4i} + \mathbf{W}_{4i}^T\mathbf{I}_{4i}^0\frac{\partial \mathbf{W}_{4i}}{\partial q_j}$$

donde

$$\frac{\partial \mathbf{b}_{4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0\right)$$

$$= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{I}_{4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_{7i}^0\mathbf{I}_{4i}^{7i}\left(\mathbf{R}_{7i}^0\right)^T\right)$$

$$= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j}\mathbf{I}_{4i}^{7i}\left(\mathbf{R}_{7i}^0\right)^T + \mathbf{R}_{7i}^0\mathbf{I}_{4i}^{7i}\left(\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j}\right)^T$$

$$\frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(x_{3i}\mathbf{R}_{5i}^0\mathbf{i}_{5i}\right)$$

$$= x_{3i}\frac{\partial \mathbf{R}_{5i}^0}{\partial q_j}\mathbf{i}_{5i}$$

$$\frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_{7i}^0\mathbf{r}_{G4i}^{7i}\right)$$

$$
\begin{aligned}
&= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{r}_{G4i}^{7i}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{R}_{5i}^0 \mathbf{R}_z(\theta_{4i} + \beta_{4i}) \right) \\
&= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \mathbf{R}_{5i}^0 \frac{\partial \mathbf{R}_z(\theta_{4i} + \beta_{4i})}{\partial q_j} \\
&= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \mathbf{R}_{5i}^0 \frac{\partial \mathbf{R}_z(\theta_{4i} + \beta_{4i})}{\partial(\theta_{4i} + \beta_{4i})} \frac{\partial(\theta_{4i} + \beta_{4i})}{\partial q_j}
\end{aligned}
$$

En particular, para cada eslabón:

$$
\begin{aligned}
\frac{\partial \mathbf{J}_{41}}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{161}^0 & \mathbf{e}_{171}^0 & \mathbf{e}_{181}^0 & \mathbf{e}_{191}^0 & \mathbf{e}_{201}^0 & \mathbf{e}_{211}^0 & \mathbf{e}_{221}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{161}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{171}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{181}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{191}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{201}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{211}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{221}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{W}_{41}}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{21}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{41}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{61}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{J}_{42}}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{162}^0 & \mathbf{e}_{172}^0 & \mathbf{e}_{182}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{192}^0 & \mathbf{e}_{202}^0 & \mathbf{e}_{212}^0 & \mathbf{e}_{222}^0 & \mathbf{0} & \mathbf{0} \end{array} \right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{162}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{172}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{182}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{192}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{202}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{212}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{222}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} \end{array} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{W}_{42}}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{0} & \mathbf{0} \end{array} \right] \\
&= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{i}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{22}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{42}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{62}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} \end{array} \right]
\end{aligned}
$$

donde

$$
\begin{aligned}
\frac{\partial \mathbf{e}_{16i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) \\
&= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right) \\[2mm]
\frac{\partial \mathbf{e}_{17i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) \\
&= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right) \\[2mm]
\frac{\partial \mathbf{e}_{18i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right) \\
&= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right) \\[2mm]
\frac{\partial \mathbf{e}_{19i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right)
\end{aligned}
$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{k}_{0i}^0 \times \left(\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{20i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{j}_{2i}^0 \times \left(\frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{21i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{4i}^0}{\partial q_j} \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{G4i}^0 \right) + \mathbf{k}_{4i}^0 \times \left(\frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{22i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{6i}^0 \times \mathbf{r}_{G4i}^0 \right)$$

$$= \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \times \mathbf{r}_{G4i}^0 + \mathbf{k}_{6i}^0 \times \frac{\partial \mathbf{r}_{G4i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{5i}^0 \mathbf{k}_{6i} \right)$$

$$= \frac{\partial \mathbf{R}_{5i}^0}{\partial q_j} \mathbf{k}_{6i}$$

4.4.6. Eslabones $5i$

$$\frac{\partial \mathbf{N}_{5i}}{\partial q_j} = \frac{\partial}{\partial q_j} \left(m_{5i} \mathbf{J}_{5i}^T \mathbf{J}_{5i} + \mathbf{W}_{5i}^T \mathbf{I}_{5i}^0 \mathbf{W}_{5i} \right)$$

$$= m_{5i} \left(\frac{\partial \mathbf{J}_{5i}^T}{\partial q_j} \mathbf{J}_{5i} + \mathbf{J}_{5i}^T \frac{\partial \mathbf{J}_{5i}}{\partial q_j} \right) + \frac{\partial \mathbf{W}_{5i}^T}{\partial q_j} \mathbf{I}_{5i}^0 \mathbf{W}_{5i} + \mathbf{W}_{5i}^T \frac{\partial \mathbf{I}_{5i}^0}{\partial q_j} \mathbf{W}_{5i} + \mathbf{W}_{5i}^T \mathbf{I}_{5i}^0 \frac{\partial \mathbf{W}_{5i}}{\partial q_j}$$

donde

$$\frac{\partial \mathbf{b}_{5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right)$$

$$= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{I}_{5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\mathbf{R}_{9i}^0 \right)^T \right)$$

$$= \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{I}_{5i}^{9i} \left(\mathbf{R}_{9i}^0 \right)^T + \mathbf{R}_{9i}^0 \mathbf{I}_{5i}^{9i} \left(\frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \right)^T$$

$$\frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(x_{4i} \mathbf{R}_{7i}^0 \mathbf{i}_{7i} \right)$$

$$= x_{4i} \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{i}_{7i}$$

$$\frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{9i}^0 \mathbf{r}_{G5i}^{9i} \right)$$

$$= \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{r}_{G5i}^{9i}$$

$$\frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{7i}^0 \mathbf{R}_z(\theta_{5i} + \beta_{5i}) \right)$$

$$= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{R}_z(\theta_{5i} + \beta_{5i}) + \mathbf{R}_{7i}^0 \frac{\partial \mathbf{R}_z(\theta_{5i} + \beta_{5i})}{\partial q_j}$$

$$= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{R}_z(\theta_{5i} + \beta_{5i}) + \mathbf{R}_{7i}^0 \frac{\partial \mathbf{R}_z(\theta_{5i} + \beta_{5i})}{\partial (\theta_{5i} + \beta_{5i})} \frac{\partial (\theta_{5i} + \beta_{5i})}{\partial q_j}$$

En particular, para cada eslabón:

$$\frac{\partial \mathbf{J}_{51}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{231}^0 & \mathbf{e}_{241}^0 & \mathbf{e}_{251}^0 & \mathbf{e}_{261}^0 & \mathbf{e}_{271}^0 & \mathbf{e}_{281}^0 & \mathbf{e}_{291}^0 & \mathbf{e}_{301}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{231}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{241}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{251}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{261}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{271}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{281}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{291}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{301}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{51}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{k}_{81}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{21}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{41}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{61}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{81}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{J}_{52}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{232}^0 & \mathbf{e}_{242}^0 & \mathbf{e}_{252}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{262}^0 & \mathbf{e}_{272}^0 & \mathbf{e}_{282}^0 & \mathbf{e}_{292}^0 & \mathbf{e}_{302}^0 & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{232}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{242}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{252}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{262}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{272}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{282}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{292}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{302}^0}{\partial q_j} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{52}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{k}_{82}^0 & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{22}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{42}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{62}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{82}^0}{\partial q_j} & \mathbf{0} \end{array} \right]$$

donde

$$\frac{\partial \mathbf{e}_{23i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{24i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{25i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{26i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{k}_{0i}^0 \times \left(\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{27i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{j}_{2i}^0 \times \left(\frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{28i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{4i}^0}{\partial q_j} \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{k}_{4i}^0 \times \left(\frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{29i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{G5i}^0 \right) + \mathbf{k}_{6i}^0 \times \left(\frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{30i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{k}_{8i}^0 \times \mathbf{r}_{G5i}^0 \right)$$

$$= \frac{\partial \mathbf{k}_{8i}^0}{\partial q_j} \times \mathbf{r}_{G5i}^0 + \mathbf{k}_{8i}^0 \times \frac{\partial \mathbf{r}_{G5i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{k}_{8i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{7i}^0 \mathbf{k}_{8i} \right)$$

$$= \frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} \mathbf{k}_{8i}$$

4.4.7. Eslabones $6i$

$$\frac{\partial \mathbf{N}_{6i}}{\partial q_j} = \frac{\partial}{\partial q_j} \left(m_{6i} \mathbf{J}_{6i}^T \mathbf{J}_{6i} + \mathbf{W}_{6i}^T \mathbf{I}_{6i}^0 \mathbf{W}_{6i} \right)$$

$$= m_{6i} \left(\frac{\partial \mathbf{J}_{6i}^T}{\partial q_j} \mathbf{J}_{6i} + \mathbf{J}_{6i}^T \frac{\partial \mathbf{J}_{6i}}{\partial q_j} \right) + \frac{\partial \mathbf{W}_{6i}^T}{\partial q_j} \mathbf{I}_{6i}^0 \mathbf{W}_{6i} + \mathbf{W}_{6i}^T \frac{\partial \mathbf{I}_{6i}^0}{\partial q_j} \mathbf{W}_{6i} + \mathbf{W}_{6i}^T \mathbf{I}_{6i}^0 \frac{\partial \mathbf{W}_{6i}}{\partial q_j}$$

donde

$$\frac{\partial \mathbf{b}_{6i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{r}_B^0 + \mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right)$$

$$= \frac{\partial \mathbf{r}_B^0}{\partial q_j} + \frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{I}_{6i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\mathbf{R}_{11i}^0 \right)^T \right)$$

$$= \frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} \mathbf{I}_{6i}^{11i} \left(\mathbf{R}_{11i}^0 \right)^T + \mathbf{R}_{11i}^0 \mathbf{I}_{6i}^{11i} \left(\frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} \right)^T$$

$$\frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(x_{5i} \mathbf{R}_{9i}^0 \mathbf{i}_{9i} \right)$$

$$= x_{5i} \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{i}_{9i}$$

$$\frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{11i}^0 \mathbf{r}_{G6i}^{11i} \right)$$

$$= \frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} \mathbf{r}_{G6i}^{11i}$$

$$\frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{R}_{9i}^0 \mathbf{R}_x(\theta_{6i} + \beta_{6i}) \right)$$

$$= \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{R}_x(\theta_{6i} + \beta_{6i}) + \mathbf{R}_{9i}^0 \frac{\partial \mathbf{R}_x(\theta_{6i} + \beta_{6i})}{\partial q_j}$$

$$= \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{R}_x(\theta_{6i} + \beta_{6i}) + \mathbf{R}_{5i}^0 \frac{\partial \mathbf{R}_x(\theta_{6i} + \beta_{6i})}{\partial(\theta_{6i} + \beta_{6i})} \frac{\partial(\theta_{6i} + \beta_{6i})}{\partial q_j}$$

En particular, para cada eslabón

$$\frac{\partial \mathbf{J}_{61}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{311}^0 & \mathbf{e}_{321}^0 & \mathbf{e}_{331}^0 & \mathbf{e}_{341}^0 & \mathbf{e}_{351}^0 & \mathbf{e}_{361}^0 & \mathbf{e}_{371}^0 & \mathbf{e}_{381}^0 & \mathbf{e}_{391}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{311}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{321}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{331}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{341}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{351}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{361}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{371}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{381}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{391}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{61}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{k}_{01}^0 & \mathbf{j}_{21}^0 & \mathbf{k}_{41}^0 & \mathbf{k}_{61}^0 & \mathbf{k}_{81}^0 & \mathbf{i}_{101}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{01}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{21}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{41}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{61}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{81}^0}{\partial q_j} & \frac{\partial \mathbf{i}_{101}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array} \right]$$

$$\frac{\partial \mathbf{J}_{62}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{i}_0 & \mathbf{j}_0 & \mathbf{k}_0 & \mathbf{e}_{312}^0 & \mathbf{e}_{322}^0 & \mathbf{e}_{332}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}_{342}^0 & \mathbf{e}_{352}^0 & \mathbf{e}_{362}^0 & \mathbf{e}_{372}^0 & \mathbf{e}_{382}^0 & \mathbf{e}_{392}^0 \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{312}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{322}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{332}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{e}_{342}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{352}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{362}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{372}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{382}^0}{\partial q_j} & \frac{\partial \mathbf{e}_{392}^0}{\partial q_j} \end{array} \right]$$

$$\frac{\partial \mathbf{W}_{62}}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{i}_{\theta B}^0 & \mathbf{j}_{\phi B}^0 & \mathbf{k}_{\psi B}^0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{k}_{02}^0 & \mathbf{j}_{22}^0 & \mathbf{k}_{42}^0 & \mathbf{k}_{62}^0 & \mathbf{k}_{82}^0 & \mathbf{i}_{102}^0 \end{array} \right]$$

$$= \left[\begin{array}{cccccccccccccccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{k}_{02}^0}{\partial q_j} & \frac{\partial \mathbf{j}_{22}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{42}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{62}^0}{\partial q_j} & \frac{\partial \mathbf{k}_{82}^0}{\partial q_j} & \frac{\partial \mathbf{i}_{102}^0}{\partial q_j} \end{array} \right]$$

donde

$$\frac{\partial \mathbf{e}_{31i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{i}_{\theta B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{i}_{\theta B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) +$$

$$\mathbf{i}_{\theta B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{32i}^0}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\mathbf{j}_{\phi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) \right)$$

$$= \frac{\partial \mathbf{j}_{\phi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0 \right) +$$

$$\mathbf{j}_{\phi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j} \right)$$

$$\frac{\partial \mathbf{e}_{33i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{\psi B}^0 \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{\psi B}^0}{\partial q_j} \times \left(\mathbf{r}_{0i}^0 + \mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) +$$

$$\mathbf{k}_{\psi B}^0 \times \left(\frac{\partial \mathbf{r}_{0i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{34i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{0i}^0 \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{0i}^0}{\partial q_j} \times \left(\mathbf{r}_{1i}^0 + \mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) + \mathbf{k}_{0i}^0 \times \left(\frac{\partial \mathbf{r}_{1i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{35i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{j}_{2i}^0 \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{j}_{2i}^0}{\partial q_j} \times \left(\mathbf{r}_{2i}^0 + \mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) + \mathbf{j}_{2i}^0 \times \left(\frac{\partial \mathbf{r}_{2i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{36i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{4i}^0 \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{4i}^0}{\partial q_j} \times \left(\mathbf{r}_{3i}^0 + \mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) + \mathbf{k}_{4i}^0 \times \left(\frac{\partial \mathbf{r}_{3i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{37i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{6i}^0 \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{6i}^0}{\partial q_j} \times \left(\mathbf{r}_{4i}^0 + \mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) + \mathbf{k}_{6i}^0 \times \left(\frac{\partial \mathbf{r}_{4i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{38i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{k}_{8i}^0 \times \left(\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right)\right)$$

$$= \frac{\partial \mathbf{k}_{8i}^0}{\partial q_j} \times \left(\mathbf{r}_{5i}^0 + \mathbf{r}_{G6i}^0\right) + \mathbf{k}_{8i}^0 \times \left(\frac{\partial \mathbf{r}_{5i}^0}{\partial q_j} + \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}\right)$$

$$\frac{\partial \mathbf{e}_{39i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{i}_{10i}^0 \times \mathbf{r}_{G6i}^0\right)$$

$$= \frac{\partial \mathbf{i}_{10i}^0}{\partial q_j} \times \mathbf{r}_{G6i}^0 + \mathbf{i}_{10i}^0 \times \frac{\partial \mathbf{r}_{G6i}^0}{\partial q_j}$$

$$\frac{\partial \mathbf{i}_{10i}^0}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\mathbf{R}_{9i}^0 \mathbf{i}_{10i}\right)$$

$$= \frac{\partial \mathbf{R}_{9i}^0}{\partial q_j} \mathbf{i}_{10i}$$

4.5.   Fuerzas generalizadas

De acuerdo a lo expuesto en (Greenwood, 1965), el trabajo virtual δW restringido al par ejercido por cada uno de los actuadores en las juntas rotacionales del bípedo está dado por[1]:

$$\begin{aligned}
\delta W &= \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\boldsymbol{\tau}_{ni}^{0T}\delta\mathbf{Q}_{ni}^{0}\right) \\
&= \mathbf{Q}^{T}\delta\mathbf{q}
\end{aligned}$$

(4.12)

De acuerdo a la figura 4.5, los vectores para describir los pares de los actuadores en las juntas rotacionales del bípedo respecto al marco inercial $\boldsymbol{\tau}_{ni}^{0}$ son:

$$\begin{aligned}
\boldsymbol{\tau}_{1i}^{0} &= \tau_{1i}\mathbf{k}_{0i}^{0} \\
\boldsymbol{\tau}_{2i}^{0} &= \tau_{2i}\mathbf{j}_{2i}^{0} \\
\boldsymbol{\tau}_{3i}^{0} &= \tau_{3i}\mathbf{k}_{4i}^{0} \\
\boldsymbol{\tau}_{4i}^{0} &= \tau_{4i}\mathbf{k}_{6i}^{0} \\
\boldsymbol{\tau}_{5i}^{0} &= \tau_{5i}\mathbf{k}_{8i}^{0} \\
\boldsymbol{\tau}_{6i}^{0} &= \tau_{6i}\mathbf{i}_{10i}^{0}
\end{aligned}$$

---

[1]La expresión presentada originalmente por el autor es $\delta\mathbf{W} = \mathbf{F}^{T}\delta\mathbf{r}+\mathbf{M}^{T}\delta\mathbf{Q} = \mathbf{Q}\delta\mathbf{q}$, donde $\mathbf{F}$ es el vector asociado a la fuerza aplicada en el punto del cuerpo identificado con el vector de posición $\mathbf{r}$ y $\mathbf{M}$ es el momento correspondiente a las coordenadas generalizadas ligadas al cambio de orientación del cuerpo contempladas en $\delta\mathbf{Q}$. En tanto que las fuerzas de reacción del piso se considerarán en términos de los multiplicadores de Lagrange, el cálculo del trabajo virtual para identificar el vector de fuerzas generalizadas se simplifica a la ecuación (4.12).
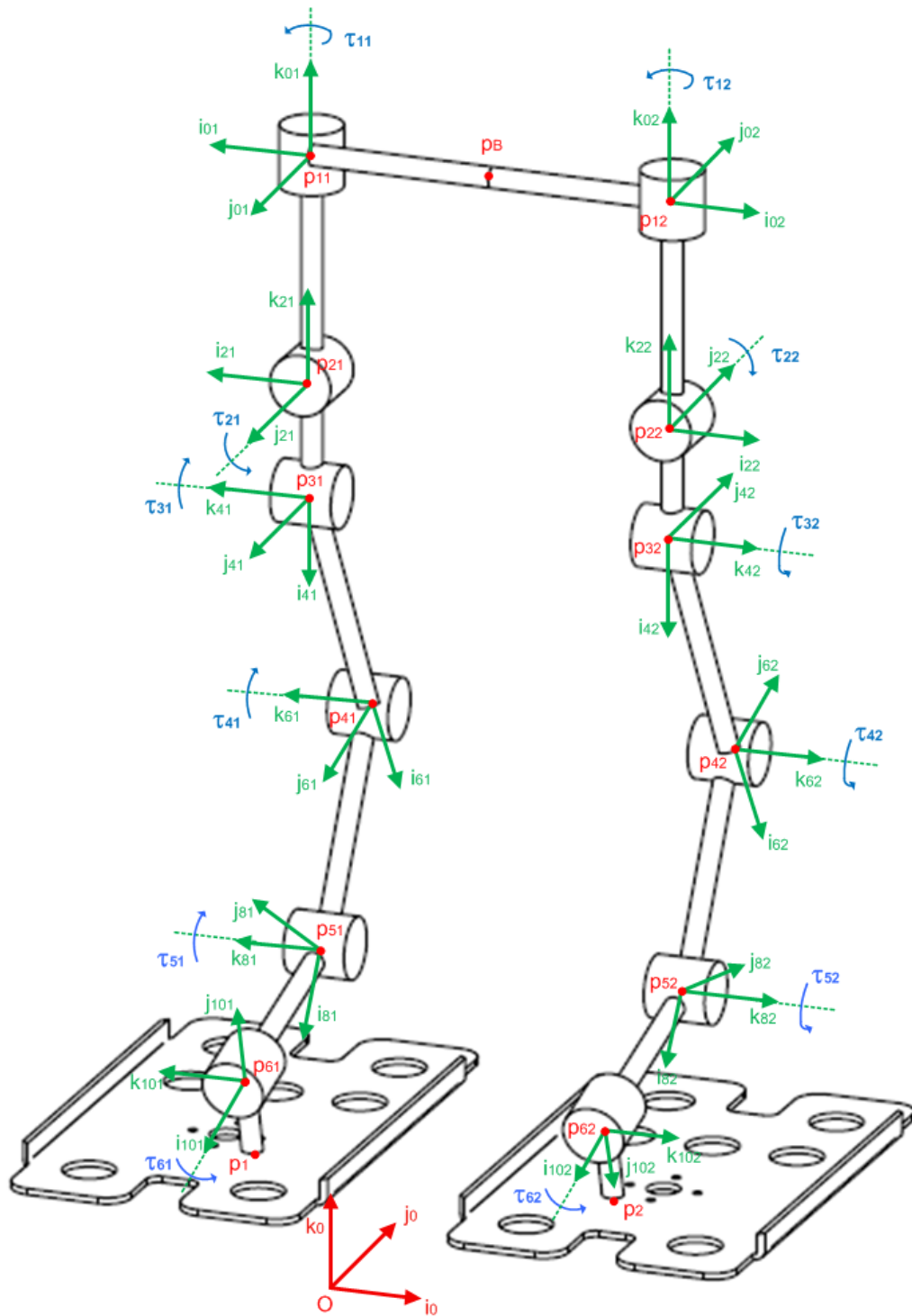
Figura 4.5: Pares τ_{ni} ejercidos por cada uno de los actuadores en las juntas rotacionales del bípedo.

$\mathbf{Q}$ es el vector de fuerzas generalizadas y $\delta\mathbf{q}$ el vector de cambios virtuales en las coordenadas generalizadas:

$$\delta\mathbf{q} = \begin{bmatrix} \delta x_B \\ \delta y_B \\ \delta z_B \\ \delta\theta_B \\ \delta\phi_B \\ \delta\psi_B \\ \delta\theta_{11} \\ \delta\theta_{21} \\ \delta\theta_{31} \\ \delta\theta_{41} \\ \delta\theta_{51} \\ \delta\theta_{61} \\ \delta\theta_{12} \\ \delta\theta_{22} \\ \delta\theta_{32} \\ \delta\theta_{42} \\ \delta\theta_{52} \\ \delta\theta_{62} \end{bmatrix}$$

Los cambios virtuales de las coordenadas generalizadas ligadas a la orientación de los eslabones del bípedo se relacionan a los vectores de velocidad angular de acuerdo a:

$$\delta\mathbf{Q}_{ni}^0 = \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{ni}^0}{\partial\dot{q}_j}\delta q_j$$

Con esta expresión y los vectores de velocidad angular en (3.10) se plantea:

$$\begin{aligned}
\delta\mathbf{Q}_{1i}^0 &= \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{1i}^0}{\partial\dot{q}_j}\delta q_j \\
&= \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0\right)}{\partial\dot{q}_j}\delta q_j \\
\delta\mathbf{Q}_{2i}^0 &= \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{2i}^0}{\partial\dot{q}_j}\delta q_j \\
&= \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0\right)}{\partial\dot{q}_j}\delta q_j \\
\delta\mathbf{Q}_{3i}^0 &= \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{3i}^0}{\partial\dot{q}_j}\delta q_j \\
&= \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0\right)}{\partial\dot{q}_j}\delta q_j
\end{aligned}$$

$$\delta\mathbf{Q}_{4i}^0 \;=\; \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{4i}^0}{\partial\dot{q}_j}\delta q_j$$

$$=\; \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$\delta\mathbf{Q}_{5i}^0 \;=\; \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{5i}^0}{\partial\dot{q}_j}\delta q_j$$

$$=\; \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$\delta\mathbf{Q}_{6i}^0 \;=\; \sum_{j=1}^{18} \frac{\partial\boldsymbol{\omega}_{6i}^0}{\partial\dot{q}_j}\delta q_j$$

$$=\; \sum_{j=1}^{18} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{1i}\mathbf{k}_{0i}^0 + \dot{\theta}_{2i}\mathbf{j}_{2i}^0 + \dot{\theta}_{3i}\mathbf{k}_{4i}^0 + \dot{\theta}_{4i}\mathbf{k}_{6i}^0 + \dot{\theta}_{5i}\mathbf{k}_{8i}^0 + \dot{\theta}_{6i}\mathbf{i}_{10i}^0\right)}{\partial\dot{q}_j}\delta q_j$$

Omitiendo la derivación respecto a los términos $\dot{q}_j$ que no aparecen en cada uno de los vectores de velocidad angular $\boldsymbol{\omega}_{ni}^0$, se calcula:

$$\delta\mathbf{Q}_{11}^0 \;=\; \sum_{j=4}^{7} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$=\; \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11}$$

$$\delta\mathbf{Q}_{21}^0 \;=\; \sum_{j=4}^{8} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0 + \dot{\theta}_{21}\mathbf{j}_{21}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$=\; \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11} + \mathbf{j}_{21}^0\delta\theta_{21}$$

$$\delta\mathbf{Q}_{31}^0 \;=\; \sum_{j=4}^{9} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0 + \dot{\theta}_{21}\mathbf{j}_{21}^0 + \dot{\theta}_{31}\mathbf{k}_{41}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$=\; \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11} + \mathbf{j}_{21}^0\delta\theta_{21} + \mathbf{k}_{41}^0\delta\theta_{31}$$

$$\delta\mathbf{Q}_{41}^0 \;=\; \sum_{j=4}^{10} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0 + \dot{\theta}_{21}\mathbf{j}_{21}^0 + \dot{\theta}_{31}\mathbf{k}_{41}^0 + \dot{\theta}_{41}\mathbf{k}_{61}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$=\; \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11} + \mathbf{j}_{21}^0\delta\theta_{21} + \mathbf{k}_{41}^0\delta\theta_{31} + \mathbf{k}_{61}^0\delta\theta_{41}$$

$$\delta\mathbf{Q}_{51}^0 \;=\; \sum_{j=4}^{11} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0 + \dot{\theta}_{21}\mathbf{j}_{21}^0 + \dot{\theta}_{31}\mathbf{k}_{41}^0 + \dot{\theta}_{41}\mathbf{k}_{61}^0 + \dot{\theta}_{51}\mathbf{k}_{81}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$=\; \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11} + \mathbf{j}_{21}^0\delta\theta_{21} + \mathbf{k}_{41}^0\delta\theta_{31} + \mathbf{k}_{61}^0\delta\theta_{41} + \mathbf{k}_{81}^0\delta\theta_{51}$$

$$\delta\mathbf{Q}_{61}^0 = \sum_{j=4}^{12} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{11}\mathbf{k}_{01}^0 + \dot{\theta}_{21}\mathbf{j}_{21}^0 + \dot{\theta}_{31}\mathbf{k}_{41}^0 + \dot{\theta}_{41}\mathbf{k}_{61}^0 + \dot{\theta}_{51}\mathbf{k}_{81}^0 + \dot{\theta}_{61}\mathbf{i}_{101}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{01}^0\delta\theta_{11} + \mathbf{j}_{21}^0\delta\theta_{21} + \mathbf{k}_{41}^0\delta\theta_{31} + \mathbf{k}_{61}^0\delta\theta_{41} + \mathbf{k}_{81}^0\delta\theta_{51} + \mathbf{i}_{101}^0\delta\theta_{61}$$

$$\delta\mathbf{Q}_{12}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0\right)}{\partial\dot{q}_{13}}\delta q_{13}$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{02}^0\delta\theta_{12}$$

$$\delta\mathbf{Q}_{22}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\sum_{j=13}^{14} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{02}^0\delta\theta_{12} + \mathbf{j}_{22}^0\delta\theta_{22}$$

$$\delta\mathbf{Q}_{32}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\sum_{j=13}^{15} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{02}^0\delta\theta_{12} + \mathbf{j}_{22}^0\delta\theta_{22} + \mathbf{k}_{42}^0\delta\theta_{32}$$

$$\delta\mathbf{Q}_{42}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0 + \dot{\theta}_{42}\mathbf{k}_{62}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\sum_{j=13}^{16} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0 + \dot{\theta}_{42}\mathbf{k}_{62}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{02}^0\delta\theta_{12} + \mathbf{j}_{22}^0\delta\theta_{22} + \mathbf{k}_{42}^0\delta\theta_{32} + \mathbf{k}_{62}^0\delta\theta_{42}$$

$$\delta\mathbf{Q}_{52}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0 + \dot{\theta}_{42}\mathbf{k}_{62}^0 + \dot{\theta}_{52}\mathbf{k}_{82}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\sum_{j=13}^{17} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0 + \dot{\theta}_{42}\mathbf{k}_{62}^0 + \dot{\theta}_{52}\mathbf{k}_{82}^0\right)}{\partial\dot{q}_j}\delta q_j$$

$$= \mathbf{i}_{\theta B}^0\delta\theta_B + \mathbf{j}_{\phi B}^0\delta\phi_B + \mathbf{k}_{\psi B}^0\delta\psi_B + \mathbf{k}_{02}^0\delta\theta_{12} + \mathbf{j}_{22}^0\delta\theta_{22} + \mathbf{k}_{42}^0\delta\theta_{32} + \mathbf{k}_{62}^0\delta\theta_{42} + \mathbf{k}_{82}^0\delta\theta_{52}$$

$$\delta\mathbf{Q}_{62}^0 = \sum_{j=4}^{6} \frac{\partial\left(\dot{\theta}_B\mathbf{i}_{\theta B}^0 + \dot{\phi}_B\mathbf{j}_{\phi B}^0 + \dot{\psi}_B\mathbf{k}_{\psi B}^0 + \dot{\theta}_{12}\mathbf{k}_{02}^0 + \dot{\theta}_{22}\mathbf{j}_{22}^0 + \dot{\theta}_{32}\mathbf{k}_{42}^0 + \dot{\theta}_{42}\mathbf{k}_{62}^0 + \dot{\theta}_{52}\mathbf{k}_{82}^0 + \dot{\theta}_{62}\mathbf{i}_{102}^0\right)}{\partial\dot{q}_j}\delta q_j +$$

$$\sum_{j=13}^{18} \frac{\partial \left(\dot{\theta}_B \mathbf{i}_{\theta B}^0 + \dot{\phi}_B \mathbf{j}_{\phi B}^0 + \dot{\psi}_B \mathbf{k}_{\psi B}^0 + \dot{\theta}_{12} \mathbf{k}_{02}^0 + \dot{\theta}_{22} \mathbf{j}_{22}^0 + \dot{\theta}_{32} \mathbf{k}_{42}^0 + \dot{\theta}_{42} \mathbf{k}_{62}^0 + \dot{\theta}_{52} \mathbf{k}_{82}^0 + \dot{\theta}_{62} \mathbf{i}_{102}^0 \right)}{\partial \dot{q}_j} \delta q_j$$

$$\delta \mathbf{Q}_{62}^0 \;=\; \mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22} + \mathbf{k}_{42}^0 \delta\theta_{32} + \mathbf{k}_{62}^0 \delta\theta_{42} + \mathbf{k}_{82}^0 \delta\theta_{52} + \mathbf{i}_{102}^0 \delta\theta_{62}$$

Con el desarrollo de los términos $\delta \mathbf{Q}_{ni}^0$ y los vectores $\boldsymbol{\tau}_{ni}^0$, se determina el trabajo virtual en (4.12):

$$
\begin{aligned}
\delta W \;=\; & \left(\tau_{11}\mathbf{k}_{01}^0\right)^T \delta\mathbf{Q}_{11}^0 + \left(\tau_{21}\mathbf{j}_{21}^0\right)^T \delta\mathbf{Q}_{21}^0 + \left(\tau_{31}\mathbf{k}_{41}^0\right)^T \delta\mathbf{Q}_{31}^0 + \left(\tau_{41}\mathbf{k}_{61}^0\right)^T \delta\mathbf{Q}_{41}^0 + \left(\tau_{51}\mathbf{k}_{81}^0\right)^T \delta\mathbf{Q}_{51}^0 + \\
& \left(\tau_{61}\mathbf{i}_{101}^0\right)^T \delta\mathbf{Q}_{61}^0 + \left(\tau_{12}\mathbf{k}_{02}^0\right)^T \delta\mathbf{Q}_{12}^0 + \left(\tau_{22}\mathbf{j}_{22}^0\right)^T \delta\mathbf{Q}_{22}^0 + \left(\tau_{32}\mathbf{k}_{42}^0\right)^T \delta\mathbf{Q}_{32}^0 + \left(\tau_{42}\mathbf{k}_{62}^0\right)^T \delta\mathbf{Q}_{42}^0 + \\
& \left(\tau_{52}\mathbf{k}_{82}^0\right)^T \delta\mathbf{Q}_{52}^0 + \left(\tau_{62}\mathbf{i}_{102}^0\right)^T \delta\mathbf{Q}_{62}^0
\end{aligned}
$$

$$
\begin{aligned}
\;=\; & \left(\tau_{11}\mathbf{k}_{01}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11}\right) + \\
& \left(\tau_{21}\mathbf{j}_{21}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11} + \mathbf{j}_{21}^0 \delta\theta_{21}\right) + \\
& \left(\tau_{31}\mathbf{k}_{41}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11} + \mathbf{j}_{21}^0 \delta\theta_{21} + \mathbf{k}_{41}^0 \delta\theta_{31}\right) + \\
& \left(\tau_{41}\mathbf{k}_{61}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11} + \mathbf{j}_{21}^0 \delta\theta_{21} + \mathbf{k}_{41}^0 \delta\theta_{31} + \mathbf{k}_{61}^0 \delta\theta_{41}\right) + \\
& \left(\tau_{51}\mathbf{k}_{81}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11} + \mathbf{j}_{21}^0 \delta\theta_{21} + \mathbf{k}_{41}^0 \delta\theta_{31} + \mathbf{k}_{61}^0 \delta\theta_{41} + \mathbf{k}_{81}^0 \delta\theta_{51}\right) + \\
& \left(\tau_{61}\mathbf{i}_{101}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{01}^0 \delta\theta_{11} + \mathbf{j}_{21}^0 \delta\theta_{21} + \mathbf{k}_{41}^0 \delta\theta_{31} + \mathbf{k}_{61}^0 \delta\theta_{41} + \mathbf{k}_{81}^0 \delta\theta_{51} + \mathbf{i}_{101}^0 \delta\theta_{61}\right) + \\
& \left(\tau_{12}\mathbf{k}_{02}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12}\right) + \\
& \left(\tau_{22}\mathbf{j}_{22}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22}\right) + \\
& \left(\tau_{32}\mathbf{k}_{42}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22} + \mathbf{k}_{42}^0 \delta\theta_{32}\right) + \\
& \left(\tau_{42}\mathbf{k}_{62}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22} + \mathbf{k}_{42}^0 \delta\theta_{32} + \mathbf{k}_{62}^0 \delta\theta_{42}\right) + \\
& \left(\tau_{52}\mathbf{k}_{82}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22} + \mathbf{k}_{42}^0 \delta\theta_{32} + \mathbf{k}_{62}^0 \delta\theta_{42} + \mathbf{k}_{82}^0 \delta\theta_{52}\right) + \\
& \left(\tau_{62}\mathbf{i}_{102}^{0T}\right) \left(\mathbf{i}_{\theta B}^0 \delta\theta_B + \mathbf{j}_{\phi B}^0 \delta\phi_B + \mathbf{k}_{\psi B}^0 \delta\psi_B + \mathbf{k}_{02}^0 \delta\theta_{12} + \mathbf{j}_{22}^0 \delta\theta_{22} + \mathbf{k}_{42}^0 \delta\theta_{32} + \mathbf{k}_{62}^0 \delta\theta_{42} + \mathbf{k}_{82}^0 \delta\theta_{52} + \mathbf{i}_{102}^0 \delta\theta_{62}\right)
\end{aligned}
$$

Agrupando los términos comunes a los cambios virtuales δq_j se tiene:

$$
\begin{aligned}
\delta W \;=\; & \left(\tau_{11}\mathbf{k}_{01}^{0T} + \tau_{21}\mathbf{j}_{21}^{0T} + \tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{i}_{\theta B}^0 \delta\theta_B + \\
& \left(\tau_{12}\mathbf{k}_{02}^{0T} + \tau_{22}\mathbf{j}_{22}^{0T} + \tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right) \mathbf{i}_{\theta B}^0 \delta\theta_B + \\
& \left(\tau_{11}\mathbf{k}_{01}^{0T} + \tau_{21}\mathbf{j}_{21}^{0T} + \tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{j}_{\phi B}^0 \delta\phi_B + \\
& \left(\tau_{12}\mathbf{k}_{02}^{0T} + \tau_{22}\mathbf{j}_{22}^{0T} + \tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right) \mathbf{j}_{\phi B}^0 \delta\phi_B + \\
& \left(\tau_{11}\mathbf{k}_{01}^{0T} + \tau_{21}\mathbf{j}_{21}^{0T} + \tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{k}_{\psi B}^0 \delta\psi_B + \\
& \left(\tau_{12}\mathbf{k}_{02}^{0T} + \tau_{22}\mathbf{j}_{22}^{0T} + \tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right) \mathbf{k}_{\psi B}^0 \delta\psi_B + \\
& \left(\tau_{11}\mathbf{k}_{01}^{0T} + \tau_{21}\mathbf{j}_{21}^{0T} + \tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{k}_{01}^0 \delta\theta_{11} + \\
& \left(\tau_{21}\mathbf{j}_{21}^{0T} + \tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{j}_{21}^0 \delta\theta_{21} + \\
& \left(\tau_{31}\mathbf{k}_{41}^{0T} + \tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{k}_{41}^0 \delta\theta_{31} + \\
& \left(\tau_{41}\mathbf{k}_{61}^{0T} + \tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{k}_{61}^0 \delta\theta_{41} + \\
& \left(\tau_{51}\mathbf{k}_{81}^{0T} + \tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{k}_{81}^0 \delta\theta_{51} + \\
& \left(\tau_{61}\mathbf{i}_{101}^{0T}\right) \mathbf{i}_{101}^0 \delta\theta_{61} + \\
& \left(\tau_{12}\mathbf{k}_{02}^{0T} + \tau_{22}\mathbf{j}_{22}^{0T} + \tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right) \mathbf{k}_{02}^0 \delta\theta_{12} +
\end{aligned}
$$

$$\left(\tau_{22}\mathbf{j}_{22}^{0T} + \tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right)\mathbf{j}_{22}^{0}\delta\theta_{22} +$$

$$\left(\tau_{32}\mathbf{k}_{42}^{0T} + \tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right)\mathbf{k}_{42}^{0}\delta\theta_{32} +$$

$$\left(\tau_{42}\mathbf{k}_{62}^{0T} + \tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right)\mathbf{k}_{62}^{0}\delta\theta_{42} +$$

$$\left(\tau_{52}\mathbf{k}_{82}^{0T} + \tau_{62}\mathbf{i}_{102}^{0T}\right)\mathbf{k}_{82}^{0}\delta\theta_{52} +$$

$$\left(\tau_{62}\mathbf{i}_{102}^{0T}\right)\mathbf{i}_{102}^{0}\delta\theta_{62}$$

$$\delta W \quad = \quad \left(\tau_{11}\mathbf{k}_{01}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{21}\mathbf{j}_{21}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{31}\mathbf{k}_{41}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{i}_{\theta B}^{0}\right)\delta\theta_B +$$

$$\left(\tau_{12}\mathbf{k}_{02}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{22}\mathbf{j}_{22}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{32}\mathbf{k}_{42}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{i}_{\theta B}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{i}_{\theta B}^{0}\right)\delta\theta_B +$$

$$\left(\tau_{11}\mathbf{k}_{01}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{21}\mathbf{j}_{21}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{31}\mathbf{k}_{41}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{j}_{\phi B}^{0}\right)\delta\phi_B +$$

$$\left(\tau_{12}\mathbf{k}_{02}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{22}\mathbf{j}_{22}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{32}\mathbf{k}_{42}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{j}_{\phi B}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{j}_{\phi B}^{0}\right)\delta\phi_B +$$

$$\left(\tau_{11}\mathbf{k}_{01}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{21}\mathbf{j}_{21}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{31}\mathbf{k}_{41}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{k}_{\psi B}^{0}\right)\delta\psi_B +$$

$$\left(\tau_{12}\mathbf{k}_{02}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{22}\mathbf{j}_{22}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{32}\mathbf{k}_{42}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{k}_{\psi B}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{k}_{\psi B}^{0}\right)\delta\psi_B +$$

$$\left(\tau_{11}\mathbf{k}_{01}^{0T}\mathbf{k}_{01}^{0} + \tau_{21}\mathbf{j}_{21}^{0T}\mathbf{k}_{01}^{0} + \tau_{31}\mathbf{k}_{41}^{0T}\mathbf{k}_{01}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{k}_{01}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{k}_{01}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{k}_{01}^{0}\right)\delta\theta_{11} +$$

$$\left(\tau_{21}\mathbf{j}_{21}^{0T}\mathbf{j}_{21}^{0} + \tau_{31}\mathbf{k}_{41}^{0T}\mathbf{j}_{21}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{j}_{21}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{j}_{21}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{j}_{21}^{0}\right)\delta\theta_{21} +$$

$$\left(\tau_{31}\mathbf{k}_{41}^{0T}\mathbf{k}_{41}^{0} + \tau_{41}\mathbf{k}_{61}^{0T}\mathbf{k}_{41}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{k}_{41}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{k}_{41}^{0}\right)\delta\theta_{31} +$$

$$\left(\tau_{41}\mathbf{k}_{61}^{0T}\mathbf{k}_{61}^{0} + \tau_{51}\mathbf{k}_{81}^{0T}\mathbf{k}_{61}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{k}_{61}^{0}\right)\delta\theta_{41} +$$

$$\left(\tau_{51}\mathbf{k}_{81}^{0T}\mathbf{k}_{81}^{0} + \tau_{61}\mathbf{i}_{101}^{0T}\mathbf{k}_{81}^{0}\right)\delta\theta_{51} +$$

$$\left(\tau_{61}\mathbf{i}_{101}^{0T}\mathbf{i}_{101}^{0}\right)\delta\theta_{61} +$$

$$\left(\tau_{12}\mathbf{k}_{02}^{0T}\mathbf{k}_{02}^{0} + \tau_{22}\mathbf{j}_{22}^{0T}\mathbf{k}_{02}^{0} + \tau_{32}\mathbf{k}_{42}^{0T}\mathbf{k}_{02}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{k}_{02}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{k}_{02}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{k}_{02}^{0}\right)\delta\theta_{12} +$$

$$\left(\tau_{22}\mathbf{j}_{22}^{0T}\mathbf{j}_{22}^{0} + \tau_{32}\mathbf{k}_{42}^{0T}\mathbf{j}_{22}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{j}_{22}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{j}_{22}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{j}_{22}^{0}\right)\delta\theta_{22} +$$

$$\left(\tau_{32}\mathbf{k}_{42}^{0T}\mathbf{k}_{42}^{0} + \tau_{42}\mathbf{k}_{62}^{0T}\mathbf{k}_{42}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{k}_{42}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{k}_{42}^{0}\right)\delta\theta_{32} +$$

$$\left(\tau_{42}\mathbf{k}_{62}^{0T}\mathbf{k}_{62}^{0} + \tau_{52}\mathbf{k}_{82}^{0T}\mathbf{k}_{62}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{k}_{62}^{0}\right)\delta\theta_{42} +$$

$$\left(\tau_{52}\mathbf{k}_{82}^{0T}\mathbf{k}_{82}^{0} + \tau_{62}\mathbf{i}_{102}^{0T}\mathbf{k}_{82}^{0}\right)\delta\theta_{52} +$$

$$\left(\tau_{62}\mathbf{i}_{102}^{0T}\mathbf{i}_{102}^{0}\right)\delta\theta_{62}$$

Definiendo el vector $\boldsymbol{\tau}$ y la matriz $\mathbf{A}_\tau$ como:

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \\ \tau_{41} \\ \tau_{51} \\ \tau_{61} \\ \tau_{12} \\ \tau_{22} \\ \tau_{32} \\ \tau_{42} \\ \tau_{52} \\ \tau_{62} \end{bmatrix}$$

$$
\mathbf{A}_\tau = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} & A_{49} & A_{410} & A_{411} & A_{412} \\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55} & A_{56} & A_{57} & A_{58} & A_{59} & A_{510} & A_{511} & A_{512} \\
A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} & A_{68} & A_{69} & A_{610} & A_{611} & A_{612} \\
A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_{93} & A_{94} & A_{95} & A_{96} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_{104} & A_{105} & A_{106} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{115} & A_{116} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_{126} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{137} & A_{138} & A_{139} & A_{1310} & A_{1311} & A_{1312} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{148} & A_{149} & A_{1410} & A_{1411} & A_{1412} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{159} & A_{1510} & A_{1511} & A_{1512} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1610} & A_{1611} & A_{1612} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1711} & A_{1712} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{1812}
\end{bmatrix}
$$

donde

$$
\begin{array}{llllllll}
A_{41} & = & \mathbf{k}_{01}^{0T}\mathbf{i}_{\theta B}^{0} & A_{42} & = & \mathbf{j}_{21}^{0T}\mathbf{i}_{\theta B}^{0} & A_{43} & = & \mathbf{k}_{41}^{0T}\mathbf{i}_{\theta B}^{0} & A_{44} & = & \mathbf{k}_{61}^{0T}\mathbf{i}_{\theta B}^{0} \\
A_{45} & = & \mathbf{k}_{81}^{0T}\mathbf{i}_{\theta B}^{0} & A_{46} & = & \mathbf{i}_{101}^{0T}\mathbf{i}_{\theta B}^{0} & A_{47} & = & \mathbf{k}_{02}^{0T}\mathbf{i}_{\theta B}^{0} & A_{48} & = & \mathbf{j}_{22}^{0T}\mathbf{i}_{\theta B}^{0} \\
A_{49} & = & \mathbf{k}_{42}^{0T}\mathbf{i}_{\theta B}^{0} & A_{410} & = & \mathbf{k}_{62}^{0T}\mathbf{i}_{\theta B}^{0} & A_{411} & = & \mathbf{k}_{82}^{0T}\mathbf{i}_{\theta B}^{0} & A_{412} & = & \mathbf{i}_{102}^{0T}\mathbf{i}_{\theta B}^{0} \\
A_{51} & = & \mathbf{k}_{01}^{0T}\mathbf{j}_{\phi B}^{0} & A_{52} & = & \mathbf{j}_{21}^{0T}\mathbf{j}_{\phi B}^{0} & A_{53} & = & \mathbf{k}_{41}^{0T}\mathbf{j}_{\phi B}^{0} & A_{54} & = & \mathbf{k}_{61}^{0T}\mathbf{j}_{\phi B}^{0} \\
A_{55} & = & \mathbf{k}_{81}^{0T}\mathbf{j}_{\phi B}^{0} & A_{56} & = & \mathbf{i}_{101}^{0T}\mathbf{j}_{\phi B}^{0} & A_{57} & = & \mathbf{k}_{02}^{0T}\mathbf{j}_{\phi B}^{0} & A_{58} & = & \mathbf{j}_{22}^{0T}\mathbf{j}_{\phi B}^{0} \\
A_{59} & = & \mathbf{k}_{42}^{0T}\mathbf{j}_{\phi B}^{0} & A_{510} & = & \mathbf{k}_{62}^{0T}\mathbf{j}_{\phi B}^{0} & A_{511} & = & \mathbf{k}_{82}^{0T}\mathbf{j}_{\phi B}^{0} & A_{512} & = & \mathbf{i}_{102}^{0T}\mathbf{j}_{\phi B}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{61} & = & \mathbf{k}_{01}^{0T}\mathbf{k}_{\psi B}^{0} & A_{62} & = & \mathbf{j}_{21}^{0T}\mathbf{k}_{\psi B}^{0} & A_{63} & = & \mathbf{k}_{41}^{0T}\mathbf{k}_{\psi B}^{0} & A_{64} & = & \mathbf{k}_{61}^{0T}\mathbf{k}_{\psi B}^{0} \\
A_{65} & = & \mathbf{k}_{81}^{0T}\mathbf{k}_{\psi B}^{0} & A_{66} & = & \mathbf{i}_{101}^{0T}\mathbf{k}_{\psi B}^{0} & A_{67} & = & \mathbf{k}_{02}^{0T}\mathbf{k}_{\psi B}^{0} & A_{68} & = & \mathbf{j}_{22}^{0T}\mathbf{k}_{\psi B}^{0} \\
A_{69} & = & \mathbf{k}_{42}^{0T}\mathbf{k}_{\psi B}^{0} & A_{610} & = & \mathbf{k}_{62}^{0T}\mathbf{k}_{\psi B}^{0} & A_{611} & = & \mathbf{k}_{82}^{0T}\mathbf{k}_{\psi B}^{0} & A_{612} & = & \mathbf{i}_{102}^{0T}\mathbf{k}_{\psi B}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{71} & = & \mathbf{k}_{01}^{0T}\mathbf{k}_{01}^{0} = 1 & A_{72} & = & \mathbf{j}_{21}^{0T}\mathbf{k}_{01}^{0} & A_{73} & = & \mathbf{k}_{41}^{0T}\mathbf{k}_{01}^{0} & A_{74} & = & \mathbf{k}_{61}^{0T}\mathbf{k}_{01}^{0} \\
A_{75} & = & \mathbf{k}_{81}^{0T}\mathbf{k}_{01}^{0} & A_{76} & = & \mathbf{i}_{101}^{0T}\mathbf{k}_{01}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{82} & = & \mathbf{j}_{21}^{0T}\mathbf{j}_{21}^{0} = 1 & A_{83} & = & \mathbf{k}_{41}^{0T}\mathbf{j}_{21}^{0} & A_{84} & = & \mathbf{k}_{61}^{0T}\mathbf{j}_{21}^{0} & A_{85} & = & \mathbf{k}_{81}^{0T}\mathbf{j}_{21}^{0} \\
A_{86} & = & \mathbf{i}_{101}^{0T}\mathbf{j}_{21}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{93} & = & \mathbf{k}_{41}^{0T}\mathbf{k}_{41}^{0} = 1 & A_{94} & = & \mathbf{k}_{61}^{0T}\mathbf{k}_{41}^{0} & A_{95} & = & \mathbf{k}_{81}^{0T}\mathbf{k}_{41}^{0} & A_{96} & = & \mathbf{i}_{101}^{0T}\mathbf{k}_{41}^{0}
\end{array}
$$

$$
\begin{array}{llllll}
A_{104} & = & \mathbf{k}_{61}^{0T}\mathbf{k}_{61}^{0} = 1 & A_{105} & = & \mathbf{k}_{81}^{0T}\mathbf{k}_{61}^{0} & A_{106} & = & \mathbf{i}_{101}^{0T}\mathbf{k}_{61}^{0}
\end{array}
$$

$$
\begin{array}{llllll}
A_{115} & = & \mathbf{k}_{81}^{0T}\mathbf{k}_{81}^{0} = 1 & A_{116} & = & \mathbf{i}_{101}^{0T}\mathbf{k}_{81}^{0}
\end{array}
$$

$$
A_{126} = \mathbf{i}_{101}^{0T}\mathbf{i}_{101}^{0} = 1
$$

$$
\begin{array}{llllllll}
A_{137} & = & \mathbf{k}_{02}^{0T}\mathbf{k}_{02}^{0} = 1 & A_{138} & = & \mathbf{j}_{22}^{0T}\mathbf{k}_{02}^{0} & A_{139} & = & \mathbf{k}_{42}^{0T}\mathbf{k}_{02}^{0} & A_{1310} & = & \mathbf{k}_{62}^{0T}\mathbf{k}_{02}^{0} \\
A_{1311} & = & \mathbf{k}_{82}^{0T}\mathbf{k}_{02}^{0} & A_{1312} & = & \mathbf{i}_{102}^{0T}\mathbf{k}_{02}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{148} & = & \mathbf{j}_{22}^{0T}\mathbf{j}_{22}^{0} = 1 & A_{149} & = & \mathbf{k}_{42}^{0T}\mathbf{j}_{22}^{0} & A_{1410} & = & \mathbf{k}_{62}^{0T}\mathbf{j}_{22}^{0} & A_{1411} & = & \mathbf{k}_{82}^{0T}\mathbf{j}_{22}^{0} \\
A_{1412} & = & \mathbf{i}_{102}^{0T}\mathbf{j}_{22}^{0}
\end{array}
$$

$$
\begin{array}{llllllll}
A_{159} & = & \mathbf{k}_{42}^{0T}\mathbf{k}_{42}^{0} = 1 & A_{1510} & = & \mathbf{k}_{62}^{0T}\mathbf{k}_{42}^{0} & A_{1511} & = & \mathbf{k}_{82}^{0T}\mathbf{k}_{42}^{0} & A_{1512} & = & \mathbf{i}_{102}^{0T}\mathbf{k}_{42}^{0}
\end{array}
$$

$$
\begin{array}{llllll}
A_{1610} & = & \mathbf{k}_{62}^{0T}\mathbf{k}_{62}^{0} = 1 & A_{1611} & = & \mathbf{k}_{82}^{0T}\mathbf{k}_{62}^{0} & A_{1612} & = & \mathbf{i}_{102}^{0T}\mathbf{k}_{62}^{0}
\end{array}
$$

$$
\begin{array}{llllll}
A_{1711} & = & \mathbf{k}_{82}^{0T}\mathbf{k}_{82}^{0} = 1 & A_{1712} & = & \mathbf{i}_{102}^{0T}\mathbf{k}_{82}^{0}
\end{array}
$$

$$
A_{1812} = \mathbf{i}_{102}^{0T}\mathbf{i}_{102}^{0} = 1
$$

La ecuación (4.12) se escribe en forma matricial:

$$
\delta W = \boldsymbol{\tau}^{T}\mathbf{A}_{\tau}^{T}\delta\mathbf{q}
$$

$$\delta W \quad = \quad \mathbf{Q}^T \delta \mathbf{q}$$

Con lo que se deduce el vector de fuerzas generalizadas:

$$\begin{aligned} \mathbf{Q} \quad &= \quad \left(\boldsymbol{\tau}^T \mathbf{A}_\tau^T \right)^T \\ &= \quad \mathbf{A}_\tau \boldsymbol{\tau} \end{aligned} \tag{4.13}$$

De acuerdo a la notación usada en la ecuación de Lagrange (4.1), los elementos del vector de fuerzas generalizadas se denotan como:

$$\mathbf{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_j \\ \vdots \\ Q_{18} \end{bmatrix}$$

4.6. Fuerzas de restricción

La naturaleza cíclica de la caminata de un bípedo puede describirse (Chevallereau et al., 2009) como la repetición alternada del movimiento de los eslabones de sus piernas en el lapso de dos fases de contacto entre el piso y la base de sus pies:

- Soporte sencillo (SS), durante la que el bípedo se comporta como una cadena cinemática abierta al tener sólo uno de sus pies apoyándose en el piso.

- Soporte doble (SD), en la que el movimiento del bípedo evoluciona como el de una cadena cinemática cerrada en tanto que ambos pies guardan contacto con el piso.

Estas condiciones cinemáticas implican que la porción del modelo dinámico de Lagrange correspondiente al vector de fuerzas de restricción $\boldsymbol{\Gamma}_k$ deba poseer una estructura variante en el tiempo, consistente con las ecuaciones de restricción impuestas por el patrón de contacto con el piso que se elija para los pies del robot.

Optándose por el patrón más sencillo, en el que la totalidad de la base de los pies guarda contacto con el piso, sus bordes laterales son paralelos a la recta $(\mathbf{O}; \mathbf{j}_0)$ y ambos pies permanecen equidistantes al plano sagital durante la fase de SD; la evolución temporal de la marcha del bípedo entre los instantes t_0 y t_3 se describe con la línea de tiempo en la figura 4.6. El intervalo T_1 indica la fase de SS en la que el pie izquierdo mantiene contacto con el piso, T_2 caracteriza la fase de SD y T_3 identifica la fase de SS en la que el bípedo se apoya con su pie derecho. Se asume que no existe deslizamiento ni penetración entre la base de los pies y el piso al constituir cuerpos rígidos sometidos a impactos inelásticos, que el plano del piso es horizontal y se encuentra libre de obstáculos. Vale la pena mencionar que (Chevallereau et al., 2009) expone patrones de contacto en los que se permite girar los talones del bípedo en fases transitorias entre SS y SD, lo que se considera más cercano a la dinámica de la locomoción del ser humano.
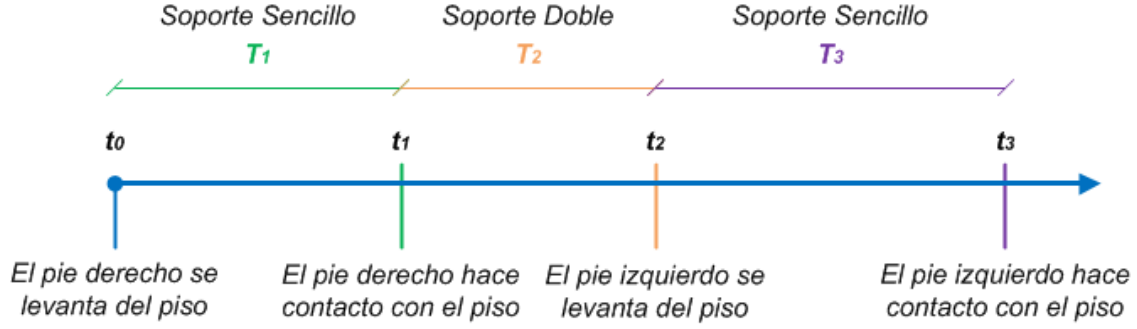
Figura 4.6: Patrón de contacto con el piso seleccionado para la deducción de las fuerzas de restricción en el modelo dinámico del bípedo.

De acuerdo a (Murray et al., 2000), el vector de fuerzas de restricción en el intervalo T_k se calcula como:

$$\boldsymbol{\Gamma}_k = \left(\frac{\partial \boldsymbol{\Phi}_k^0}{\partial \mathbf{q}}\right)^T \boldsymbol{\lambda}_k^0 \tag{4.14}$$

donde $\left(\frac{\partial \boldsymbol{\Phi}_k^0}{\partial \mathbf{q}}\right)^T$ es la matriz Jacobiana transpuesta de las ecuaciones de restricción agrupadas en el vector $\boldsymbol{\Phi}_k^0$ y $\boldsymbol{\lambda}_k^0$ es el vector de multiplicadores de Lagrange que reúne los valores relativos al marco inercial de las fuerzas y momentos de reacción del piso que deben producirse para restringir la posición de los pies. A continuación se sigue esta formulación para la deducción de los vectores $\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2, \boldsymbol{\Gamma}_3$ durante un ciclo de marcha del bípedo con el patrón de contacto propuesto.

Para el planteamiento de las restricciones cinemáticas en cada intervalo fue necesario definir los vectores:

$$\mathbf{l}_i^0 = bi_y \mathbf{j}_{14i}^0$$

donde

$$\begin{aligned} \mathbf{j}_{14i}^0 &= \mathbf{R}_{11i}^0 \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{14i} \\ &= \mathbf{j}_i^0 \end{aligned}$$

La representación gráfica de los vectores $\mathbf{l}_i^0$ se encuentra en la figura 4.7, mientras que los valores de bi_y pueden consultarse en el apéndice B.4.
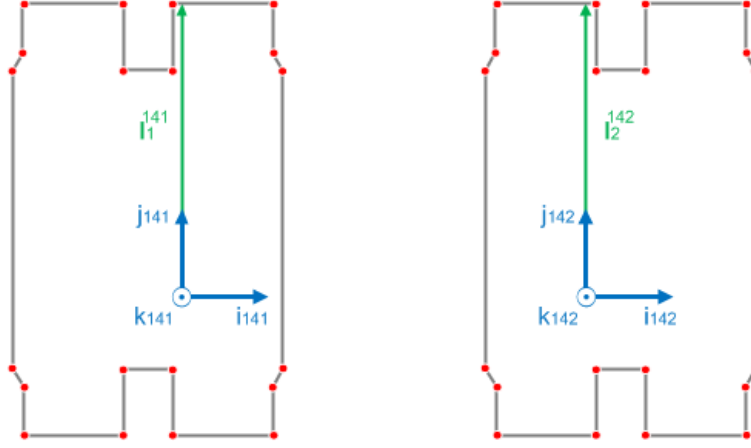
Figura 4.7: Vectores $\mathbf{l}_i^0$ en la base de los pies del bípedo para el planteamiento de las ecuaciones de restricción.

De los vectores definidos en (3.7) se recuerda:

$$\mathbf{R}_{11i}^0 = \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})$$

4.6.1. Restricciones en el intervalo T_1

Las restricciones holonómicas[2] durante la fase de SS con apoyo en el pie izquierdo son:

$$
\begin{aligned}
\varphi_1^1(q_j(t)) &= \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{k}_0 = 0 \\
\varphi_2^1(q_j(t)) &= \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{j}_0 - b1_y = 0 \\
\varphi_3^1(q_j(t)) &= \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{i}_0 = 0 \\
\varphi_4^1(q_j(t)) &= \left(\mathbf{k}_{81}^0(q_j(t))\right)^T \mathbf{j}_0 = 0 \\
\varphi_5^1(q_j(t)) &= \left(\mathbf{k}_{81}^0(q_j(t))\right)^T \mathbf{k}_0 = 0 \\
\varphi_6^1(q_j(t)) &= \left(\mathbf{k}_{111}^0(q_j(t))\right)^T \mathbf{k}_0 = 0
\end{aligned}
\tag{4.15}
$$

donde la notación $q_j(t)$ permite enfatizar que la orientación de los vectores $\mathbf{l}_1^0, \mathbf{k}_{81}^0, \mathbf{k}_{111}^0$ respecto a la base inercial depende de los valores asumidos por las coordenadas generalizadas en el tiempo t. Mientras las ecuaciones $\varphi_1^1(q_j(t)), \varphi_2^1(q_j(t)), \varphi_3^1(q_j(t))$ definen la superficie de contacto entre el pie izquierdo y el piso, $\varphi_4^1(q_j(t)), \varphi_5^1(q_j(t)), \varphi_6^1(q_j(t))$ restringen su orientación respecto al marco inercial.

Agrupando las ecuaciones (4.15) en el vector $\boldsymbol{\Phi}_1^0$, las restricciones holonómicas se escriben:

$$
\boldsymbol{\Phi}_1^0 = \begin{bmatrix}
\varphi_1^1(q_j(t)) \\
\varphi_2^1(q_j(t)) \\
\varphi_3^1(q_j(t)) \\
\varphi_4^1(q_j(t)) \\
\varphi_5^1(q_j(t)) \\
\varphi_6^1(q_j(t))
\end{bmatrix} = \mathbf{0}
$$

---

[2]Una ecuación de restricción $\varphi(q_j(t))$ se dice *holonómica*, del griego *verdaderas en conjunto*, si puede representarse algebraicamente como $\varphi(q_j(t)) = 0$. Siendo $q_j(t)$ las coordenadas generalizadas del sistema (Murray et al., 2000).

Por lo que la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_1^0}{\partial \mathbf{q}}$ queda definida como:

$$\frac{\partial \mathbf{\Phi}_1^0}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \varphi_1^1}{\partial q_1} & \cdots & \frac{\partial \varphi_1^1}{\partial q_{18}} \\ & \ddots & \\ \frac{\partial \varphi_6^1}{\partial q_1} & \cdots & \frac{\partial \varphi_6^1}{\partial q_{18}} \end{bmatrix}$$

El cálculo de las derivadas parciales comprendidas en la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_1^0}{\partial \mathbf{q}}$ se presenta para cada una de las ecuaciones de restricción en (4.15).

4.6.1.1. Cálculo de $\frac{\partial \varphi_1^1}{\partial q_j}$

$$
\begin{aligned}
\frac{\partial \varphi_1^1}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{k}_0\right) \\
&= \frac{\partial}{\partial q_j}\left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{k}_0 + \left(\mathbf{l}_1^0(q_j(t))\right)^T \frac{\partial \mathbf{k}_0}{\partial q_j} \\
&= \frac{\partial}{\partial q_j}\left(b1_y\mathbf{R}_{111}^0\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0 \\
&= \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial q_j}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141} + b1_y\mathbf{R}_{111}^0\frac{\partial \mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}}{\partial q_j}\right)^T \mathbf{k}_0 \\
&= \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial q_j}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0 \quad (4.16)
\end{aligned}
$$

Recurriendo a las matrices definidas en (4.11) se establece:

$$
\begin{aligned}
\frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} =\ & \frac{\partial \mathbf{R}_x(\theta_B)}{\partial q_j}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial q_j}\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial q_j}\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{1i}+\beta_{1i})}{\partial q_j}\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\frac{\partial \mathbf{R}_y(\theta_{2i}+\beta_{2i})}{\partial q_j}\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\frac{\partial \mathbf{R}_z(\theta_{3i}+\beta_{3i})}{\partial q_j}\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\frac{\partial \mathbf{R}_z(\theta_{4i}+\beta_{4i})}{\partial q_j}\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\frac{\partial \mathbf{R}_z(\theta_{5i}+\beta_{5i})}{\partial q_j}\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\frac{\partial \mathbf{R}_x(\theta_{6i}+\beta_{6i})}{\partial q_j} \\[6pt]
\frac{\partial \mathbf{R}_{11i}^0}{\partial q_j} =\ & \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\frac{\partial \theta_B}{\partial q_j}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B}\frac{\partial \phi_B}{\partial q_j}\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B}\frac{\partial \psi_B}{\partial q_j}\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+ \\
& \mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{1i}+\beta_{1i})}{\partial (\theta_{1i}+\beta_{1i})}\frac{\partial (\theta_{1i}+\beta_{1i})}{\partial q_j}\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+
\end{aligned}
$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\frac{\partial\mathbf{R}_y(\theta_{2i}+\beta_{2i})}{\partial(\theta_{2i}+\beta_{2i})}\frac{\partial(\theta_{2i}+\beta_{2i})}{\partial q_j}\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\frac{\partial\mathbf{R}_z(\theta_{3i}+\beta_{3i})}{\partial(\theta_{3i}+\beta_{3i})}\frac{\partial(\theta_{3i}+\beta_{3i})}{\partial q_j}\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\frac{\partial\mathbf{R}_z(\theta_{4i}+\beta_{4i})}{\partial(\theta_{4i}+\beta_{4i})}\frac{\partial(\theta_{4i}+\beta_{4i})}{\partial q_j}\mathbf{R}_z(\theta_{5i}+\beta_{5i})\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\frac{\partial\mathbf{R}_z(\theta_{5i}+\beta_{5i})}{\partial(\theta_{5i}+\beta_{5i})}\frac{\partial(\theta_{5i}+\beta_{5i})}{\partial q_j}\mathbf{R}_x(\theta_{6i}+\beta_{6i})\,+$$

$$\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{1i}+\beta_{1i})\mathbf{R}_y(\theta_{2i}+\beta_{2i})\mathbf{R}_z(\theta_{3i}+\beta_{3i})\mathbf{R}_z(\theta_{4i}+\beta_{4i})\mathbf{R}_z(\theta_{5i}+\beta_{5i})\frac{\partial\mathbf{R}_x(\theta_{6i}+\beta_{6i})}{\partial(\theta_{6i}+\beta_{6i})}\frac{\partial(\theta_{6i}+\beta_{6i})}{\partial q_j}$$

Resultado con el que se determinan:

$$\frac{\partial\varphi_1^1}{\partial q_1} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial x_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= 0$$

$$\frac{\partial\varphi_1^1}{\partial q_2} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial y_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= 0$$

$$\frac{\partial\varphi_1^1}{\partial q_3} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial z_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= 0$$

$$\frac{\partial\varphi_1^1}{\partial q_4} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial\theta_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= \left(b1_y\frac{\partial\mathbf{R}_x(\theta_B)}{\partial\theta_B}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$

$$\frac{\partial\varphi_1^1}{\partial q_5} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial\phi_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= \left(b1_y\mathbf{R}_x(\theta_B)\frac{\partial\mathbf{R}_y(\phi_B)}{\partial\phi_B}\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$

$$\frac{\partial\varphi_1^1}{\partial q_6} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial\psi_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial\mathbf{R}_z(\psi_B)}{\partial\psi_B}\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$

$$\frac{\partial\varphi_1^1}{\partial q_7} = \left(b1_y\frac{\partial\mathbf{R}_{111}^0}{\partial\theta_{11}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{k}_0$$
$$= \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial\mathbf{R}_z(\theta_{11}+\beta_{11})}{\partial(\theta_{11}+\beta_{11})}\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots\right.$$

$$\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\Big)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_8} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{21}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$= \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\frac{\partial \mathbf{R}_y(\theta_{21}+\beta_{21})}{\partial (\theta_{21}+\beta_{21})}\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots\right.$$
$$\left.\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_9} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{31}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$= \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\frac{\partial \mathbf{R}_z(\theta_{31}+\beta_{31})}{\partial (\theta_{31}+\beta_{31})}\ldots\right.$$
$$\left.\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_{10}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{41}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$= (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \frac{\partial \mathbf{R}_z(\theta_{41}+\beta_{41})}{\partial (\theta_{41}+\beta_{41})}\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\Big)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_{11}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{51}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$= (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\frac{\partial \mathbf{R}_z(\theta_{51}+\beta_{51})}{\partial (\theta_{51}+\beta_{51})}\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\Big)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_{12}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{61}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$

$$= (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\frac{\partial \mathbf{R}_x(\theta_{61}+\beta_{61})}{\partial (\theta_{61}+\beta_{61})}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\Big)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_1^1}{\partial q_{13}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{12}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_1^1}{\partial q_{14}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{22}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_1^1}{\partial q_{15}} = \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{32}}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_1^1}{\partial q_{16}} = \left(b1_y \frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{42}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_1^1}{\partial q_{17}} = \left(b1_y \frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{52}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_1^1}{\partial q_{18}} = \left(b1_y \frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{62}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{k}_0$$
$$= 0$$

4.6.1.2.   Cálculo de $\frac{\partial \varphi_2^1}{\partial q_j}$

$$\frac{\partial \varphi_2^1}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{l}_1^0(q_j(t)) \right)^T \mathbf{j}_0 - b1_y \right)$$
$$= \left(\frac{\partial}{\partial q_j} \left(\left(\mathbf{l}_1^0(q_j(t)) \right)^T \mathbf{j}_0 \right) - \frac{\partial b1_y}{\partial q_j} \right)$$
$$=$$
$$= \frac{\partial}{\partial q_j} \left(\mathbf{l}_1^0(q_j(t)) \right)^T \mathbf{j}_0 + \left(\mathbf{l}_1^0(q_j(t)) \right)^T \frac{\partial \mathbf{j}_0}{\partial q_j}$$
$$= \frac{\partial}{\partial q_j} \left(b1_y \mathbf{R}_{111}^0 \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{j}_0$$
$$= \left(b1_y \frac{\partial \mathbf{R}_{111}^0}{\partial q_j} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{j}_0 \tag{4.17}$$

Como las derivadas parciales en (4.16) y (4.17) tan sólo difieren en la presencia de $\mathbf{k}_0$ y $\mathbf{j}_0$, respectivamente; se aprovecha la deducción de los vectores transpuestos presentados en la sección anterior para establecer las derivadas parciales $\frac{\partial \varphi_2^1}{\partial q_j}$.

$$\frac{\partial \varphi_2^1}{\partial q_1} = \frac{\partial \varphi_2^1}{\partial q_2} = \frac{\partial \varphi_2^1}{\partial q_3} = 0$$

$$\frac{\partial \varphi_2^1}{\partial q_4} = \left(b1_y \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{R}_z(\theta_{51} + \beta_{51}) \mathbf{R}_x(\theta_{61} + \beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_5} = \left(b1_y \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{R}_z(\theta_{51} + \beta_{51}) \mathbf{R}_x(\theta_{61} + \beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_6} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{R}_z(\theta_{51} + \beta_{51}) \mathbf{R}_x(\theta_{61} + \beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_7} = \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{11}+\beta_{11})}{\partial(\theta_{11}+\beta_{11})}\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_8} = \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\frac{\partial \mathbf{R}_y(\theta_{21}+\beta_{21})}{\partial(\theta_{21}+\beta_{21})}\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_9} = \left(b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\frac{\partial \mathbf{R}_z(\theta_{31}+\beta_{31})}{\partial(\theta_{31}+\beta_{31})}\ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_{10}} = (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \frac{\partial \mathbf{R}_z(\theta_{41}+\beta_{41})}{\partial(\theta_{41}+\beta_{41})}\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141})^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_{11}} = (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\frac{\partial \mathbf{R}_z(\theta_{51}+\beta_{51})}{\partial(\theta_{51}+\beta_{51})}\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141})^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_{12}} = (b1_y\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots$$
$$\ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\frac{\partial \mathbf{R}_x(\theta_{61}+\beta_{61})}{\partial(\theta_{61}+\beta_{61})}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141})^T\mathbf{j}_0$$

$$\frac{\partial \varphi_2^1}{\partial q_{13}} = \frac{\partial \varphi_2^1}{\partial q_{14}} = \frac{\partial \varphi_2^1}{\partial q_{15}} = \frac{\partial \varphi_2^1}{\partial q_{16}} = \frac{\partial \varphi_2^1}{\partial q_{17}} = \frac{\partial \varphi_2^1}{\partial q_{18}} = 0$$

4.6.1.3. Cálculo de $\frac{\partial \varphi_3^1}{\partial q_j}$

$$\frac{\partial \varphi_3^1}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\left(\mathbf{l}_1^0(q_j(t))\right)^T\mathbf{i}_0\right)$$
$$= \left(b1_y\frac{\partial \mathbf{R}_{111}^0}{\partial q_j}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{i}_0$$

De manera análoga a la sección anterior:

$$\frac{\partial \varphi_3^1}{\partial q_1} = \frac{\partial \varphi_3^1}{\partial q_2} = \frac{\partial \varphi_3^1}{\partial q_3} = 0$$

$$\frac{\partial \varphi_3^1}{\partial q_4} = \left(b1_y\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{141}\right)^T\mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_5} = \left(b1_y \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_6} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_7} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{11}+\beta_{11})}{\partial (\theta_{11}+\beta_{11})} \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_8} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \frac{\partial \mathbf{R}_y(\theta_{21}+\beta_{21})}{\partial (\theta_{21}+\beta_{21})} \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_9} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \frac{\partial \mathbf{R}_z(\theta_{31}+\beta_{31})}{\partial (\theta_{31}+\beta_{31})} \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_{10}} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \frac{\partial \mathbf{R}_z(\theta_{41}+\beta_{41})}{\partial (\theta_{41}+\beta_{41})} \mathbf{R}_z(\theta_{51}+\beta_{51}) \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_{11}} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \frac{\partial \mathbf{R}_z(\theta_{51}+\beta_{51})}{\partial (\theta_{51}+\beta_{51})} \mathbf{R}_x(\theta_{61}+\beta_{61}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_{12}} = \left(b1_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11}+\beta_{11}) \mathbf{R}_y(\theta_{21}+\beta_{21}) \mathbf{R}_z(\theta_{31}+\beta_{31}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{41}+\beta_{41}) \mathbf{R}_z(\theta_{51}+\beta_{51}) \frac{\partial \mathbf{R}_x(\theta_{61}+\beta_{61})}{\partial (\theta_{61}+\beta_{61})} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{141} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_3^1}{\partial q_{13}} = \frac{\partial \varphi_3^1}{\partial q_{14}} = \frac{\partial \varphi_3^1}{\partial q_{15}} = \frac{\partial \varphi_3^1}{\partial q_{16}} = \frac{\partial \varphi_3^1}{\partial q_{17}} = \frac{\partial \varphi_3^1}{\partial q_{18}} = 0$$

4.6.1.4. Cálculo de $\frac{\partial \varphi_4^1}{\partial q_j}$

$$\frac{\partial \varphi_4^1}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \mathbf{j}_0 \right)$$
$$= \frac{\partial}{\partial q_j} \left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \mathbf{j}_0 + \left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \frac{\partial \mathbf{j}_0}{\partial q_j}$$
$$= \frac{\partial}{\partial q_j} \left(\mathbf{R}_{71}^0 \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial q_j} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

Con la definición de las matrices de rotación $\mathbf{R}_{7i}^0$ se tiene:

$$
\begin{aligned}
\frac{\partial \mathbf{R}_{7i}^0}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) \right) \\
&= \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \frac{\partial \theta_B}{\partial q_j} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \frac{\partial \phi_B}{\partial q_j} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \frac{\partial \psi_B}{\partial q_j} \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{1i} + \beta_{1i})}{\partial (\theta_{1i} + \beta_{1i})} \frac{\partial (\theta_{1i} + \beta_{1i})}{\partial q_j} \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \frac{\partial \mathbf{R}_y(\theta_{2i} + \beta_{2i})}{\partial (\theta_{2i} + \beta_{2i})} \frac{\partial (\theta_{2i} + \beta_{2i})}{\partial q_j} \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \frac{\partial \mathbf{R}_z(\theta_{3i} + \beta_{3i})}{\partial (\theta_{3i} + \beta_{3i})} \frac{\partial (\theta_{3i} + \beta_{3i})}{\partial q_j} \mathbf{R}_z(\theta_{4i} + \beta_{4i}) + \\
&\quad \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{1i} + \beta_{1i}) \mathbf{R}_y(\theta_{2i} + \beta_{2i}) \mathbf{R}_z(\theta_{3i} + \beta_{3i}) \frac{\partial \mathbf{R}_z(\theta_{4i} + \beta_{4i})}{\partial (\theta_{4i} + \beta_{4i})} \frac{\partial (\theta_{4i} + \beta_{4i})}{\partial q_j}
\end{aligned}
$$

Por lo que se desarrolla:

$$
\begin{aligned}
\frac{\partial \varphi_4^1}{\partial q_1} &= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial x_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_4^1}{\partial q_2} &= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial y_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_4^1}{\partial q_3} &= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial z_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_4^1}{\partial q_4} &= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0 \\
&= \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_4^1}{\partial q_5} &= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \phi_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0 \\
&= \left(\mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0
\end{aligned}
$$

$$\frac{\partial \varphi_4^1}{\partial q_6} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \psi_B} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_4^1}{\partial q_7} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{11}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{11} + \beta_{11})}{\partial (\theta_{11} + \beta_{11})} \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_4^1}{\partial q_8} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{21}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \frac{\partial \mathbf{R}_y(\theta_{21} + \beta_{21})}{\partial (\theta_{21} + \beta_{21})} \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_4^1}{\partial q_9} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{31}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \frac{\partial \mathbf{R}_z(\theta_{31} + \beta_{31})}{\partial (\theta_{31} + \beta_{31})} \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_4^1}{\partial q_{10}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{41}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \frac{\partial \mathbf{R}_z(\theta_{41} + \beta_{41})}{\partial (\theta_{41} + \beta_{41})} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_4^1}{\partial q_{11}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{51}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{12}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{61}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{13}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{12}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{14}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{22}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{15}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{32}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$

$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{16}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{42}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{17}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{52}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_4^1}{\partial q_{18}} = \left(\frac{\partial \mathbf{R}_{71}^0}{\partial \theta_{62}} \mathbf{k}_{81} \right)^T \mathbf{j}_0$$
$$= 0$$

4.6.1.5. Cálculo de $\frac{\partial \varphi_5^1}{\partial q_j}$

$$\frac{\partial \varphi_5^1}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \mathbf{k}_0 \right)$$
$$= \frac{\partial}{\partial q_j} \left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \mathbf{k}_0 + \left(\mathbf{k}_{81}^0(q_j(t)) \right)^T \frac{\partial \mathbf{k}_0}{\partial q_j}$$
$$= \frac{\partial}{\partial q_j} \left(\mathbf{R}_{71}^0 \mathbf{k}_{81} \right)^T \mathbf{k}_0$$
$$= \left(\frac{\partial \mathbf{R}_{71}^0}{\partial q_j} \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

Tomando en cuenta los resultados de la sección anterior:

$$\frac{\partial \varphi_5^1}{\partial q_1} = \frac{\partial \varphi_5^1}{\partial q_2} = \frac{\partial \varphi_5^1}{\partial q_3} = 0$$

$$\frac{\partial \varphi_5^1}{\partial q_4} = \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_5} = \left(\mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_6} = \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_7} = \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{11} + \beta_{11})}{\partial(\theta_{11} + \beta_{11})} \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_8} = \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \frac{\partial \mathbf{R}_y(\theta_{21} + \beta_{21})}{\partial(\theta_{21} + \beta_{21})} \mathbf{R}_z(\theta_{31} + \beta_{31}) \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_9} = \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \frac{\partial \mathbf{R}_z(\theta_{31} + \beta_{31})}{\partial(\theta_{31} + \beta_{31})} \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_{10}} = \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \frac{\partial \mathbf{R}_z(\theta_{41} + \beta_{41})}{\partial(\theta_{41} + \beta_{41})} \mathbf{k}_{81} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_5^1}{\partial q_{11}} \;=\; \frac{\partial \varphi_5^1}{\partial q_{12}} = \frac{\partial \varphi_5^1}{\partial q_{13}} = \frac{\partial \varphi_5^1}{\partial q_{14}} = \frac{\partial \varphi_5^1}{\partial q_{15}} = \frac{\partial \varphi_5^1}{\partial q_{16}} = \frac{\partial \varphi_5^1}{\partial q_{17}} = \frac{\partial \varphi_5^1}{\partial q_{18}} = 0$$

4.6.1.6. Cálculo de $\frac{\partial \varphi_6^1}{\partial q_j}$

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\left(\mathbf{k}_{111}^0(q_j(t)) \right)^T \mathbf{k}_0 \right) \\
&= \frac{\partial}{\partial q_j} \left(\mathbf{k}_{111}^0(q_j(t)) \right)^T \mathbf{k}_0 + \left(\mathbf{k}_{111}^0(q_j(t)) \right)^T \frac{\partial \mathbf{k}_0}{\partial q_j} \\
&= \frac{\partial}{\partial q_j} \left(\mathbf{R}_{111}^0 \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial q_j} \mathbf{k}_{111} \right)^T \mathbf{k}_0
\end{aligned}
$$

Con el desarrollo previo de $\frac{\partial \mathbf{R}_{111}^0}{\partial q_j}$ se tiene:

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_1} &= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial x_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_2} &= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial y_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_3} &= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial z_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_4} &= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \ldots \right. \\
&\qquad \left. \ldots \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{R}_z(\theta_{51} + \beta_{51}) \mathbf{R}_x(\theta_{61} + \beta_{61}) \mathbf{k}_{111} \right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_6^1}{\partial q_5} &= \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \phi_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0 \\
&= \left(\mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{11} + \beta_{11}) \mathbf{R}_y(\theta_{21} + \beta_{21}) \mathbf{R}_z(\theta_{31} + \beta_{31}) \ldots \right. \\
&\qquad \left. \ldots \mathbf{R}_z(\theta_{41} + \beta_{41}) \mathbf{R}_z(\theta_{51} + \beta_{51}) \mathbf{R}_x(\theta_{61} + \beta_{61}) \mathbf{k}_{111} \right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\frac{\partial \varphi_6^1}{\partial q_6} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \psi_B} \mathbf{k}_{111} \right)^T \mathbf{k}_0
$$

$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B}\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_7} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{11}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{11}+\beta_{11})}{\partial(\theta_{11}+\beta_{11})}\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_8} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{21}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\frac{\partial \mathbf{R}_y(\theta_{21}+\beta_{21})}{\partial(\theta_{21}+\beta_{21})}\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_9} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{31}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\frac{\partial \mathbf{R}_z(\theta_{31}+\beta_{31})}{\partial(\theta_{31}+\beta_{31})}\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_{10}} = \left(\frac{\partial \mathbf{R}_{11}^0}{\partial \theta_{41}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \frac{\partial \mathbf{R}_z(\theta_{41}+\beta_{41})}{\partial(\theta_{41}+\beta_{41})}\mathbf{R}_z(\theta_{51}+\beta_{51})\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_{11}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{51}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\frac{\partial \mathbf{R}_z(\theta_{51}+\beta_{51})}{\partial(\theta_{51}+\beta_{51})}\mathbf{R}_x(\theta_{61}+\beta_{61})\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_{12}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{61}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{11}+\beta_{11})\mathbf{R}_y(\theta_{21}+\beta_{21})\mathbf{R}_z(\theta_{31}+\beta_{31})\dots \right.$$
$$\left. \dots \mathbf{R}_z(\theta_{41}+\beta_{41})\mathbf{R}_z(\theta_{51}+\beta_{51})\frac{\partial \mathbf{R}_x(\theta_{61}+\beta_{61})}{\partial(\theta_{61}+\beta_{61})}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_6^1}{\partial q_{13}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{12}}\mathbf{k}_{111} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_6^1}{\partial q_{14}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{22}} \mathbf{k}_{111}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_6^1}{\partial q_{15}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{32}} \mathbf{k}_{111}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_6^1}{\partial q_{16}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{42}} \mathbf{k}_{111}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_6^1}{\partial q_{17}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{52}} \mathbf{k}_{111}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_6^1}{\partial q_{18}} = \left(\frac{\partial \mathbf{R}_{111}^0}{\partial \theta_{62}} \mathbf{k}_{111}\right)^T \mathbf{k}_0$$
$$= 0$$

Una vez desarrollados los elementos de la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_1^0}{\partial \mathbf{q}}$, el vector de las fuerzas de restricción en el intervalo T_1 se escribe de acuerdo a (4.14) como:

$$\mathbf{\Gamma}_1 = \left(\frac{\partial \mathbf{\Phi}_1^0}{\partial \mathbf{q}}\right)^T \mathbf{\lambda}_1^0 \tag{4.18}$$

donde el vector de los multiplicadores de Lagrange tiene la forma:

$$\mathbf{\lambda}_1^0 = \begin{bmatrix} \lambda_1^1 \\ \lambda_2^1 \\ \lambda_3^1 \\ \lambda_4^1 \\ \lambda_5^1 \\ \lambda_6^1 \end{bmatrix}$$

4.6.2. Restricciones en el intervalo T_2

Las restricciones holonómicas durante la fase de SD son:

$$\varphi_1^2(q_j(t)) = \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{k}_0 = 0$$
$$\varphi_2^2(q_j(t)) = \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{j}_0 - b1_y = 0$$
$$\varphi_3^2(q_j(t)) = \left(\mathbf{l}_1^0(q_j(t))\right)^T \mathbf{i}_0 = 0$$
$$\varphi_4^2(q_j(t)) = \left(\mathbf{k}_{81}^0(q_j(t))\right)^T \mathbf{j}_0 = 0$$
$$\varphi_5^2(q_j(t)) = \left(\mathbf{k}_{81}^0(q_j(t))\right)^T \mathbf{k}_0 = 0$$
$$\varphi_6^2(q_j(t)) = \left(\mathbf{k}_{111}^0(q_j(t))\right)^T \mathbf{k}_0 = 0$$

$$
\begin{aligned}
\varphi_7^2(q_j(t)) &= \left(\mathbf{l}_2^0(q_j(t))\right)^T \mathbf{k}_0 = 0 \qquad\qquad (4.19)\\
\varphi_8^2(q_j(t)) &= \left(\mathbf{l}_2^0(q_j(t))\right)^T \mathbf{j}_0 - b2_y = 0\\
\varphi_9^2(q_j(t)) &= \left(\mathbf{l}_2^0(q_j(t))\right)^T \mathbf{i}_0 = 0\\
\varphi_{10}^2(q_j(t)) &= \left(\mathbf{k}_{82}^0(q_j(t))\right)^T \mathbf{j}_0 = 0\\
\varphi_{11}^2(q_j(t)) &= \left(\mathbf{k}_{82}^0(q_j(t))\right)^T \mathbf{k}_0 = 0\\
\varphi_{12}^2(q_j(t)) &= \left(\mathbf{k}_{112}^0(q_j(t))\right)^T \mathbf{k}_0 = 0
\end{aligned}
$$

Las ecuaciones $\varphi_1^2(q_j(t)), \cdots, \varphi_6^2(q_j(t))$ representan las mismas restricciones de posición y orientación para el pie izquierdo descritas por $\varphi_1^1(q_j(t)), \cdots, \varphi_6^1(q_j(t))$, mientras que las ecuaciones $\varphi_7^2(q_j(t)), \varphi_8^2(q_j(t)), \varphi_9^2(q_j(t))$ y $\varphi_{10}^2(q_j(t)), \varphi_{11}^2(q_j(t)), \varphi_{12}^2(q_j(t))$ definen respectivamente, la superficie de contacto entre el pie derecho y el piso y su orientación respecto al marco inercial.

Agrupando las ecuaciones (4.19) en el vector $\mathbf{\Phi}_2^0$, las restricciones holonómicas se escriben:

$$
\mathbf{\Phi}_2^0 = \begin{bmatrix}
\varphi_1^2(q_j(t))\\
\varphi_2^2(q_j(t))\\
\varphi_3^2(q_j(t))\\
\varphi_4^2(q_j(t))\\
\varphi_5^2(q_j(t))\\
\varphi_6^2(q_j(t))\\
\varphi_7^2(q_j(t))\\
\varphi_8^2(q_j(t))\\
\varphi_9^2(q_j(t))\\
\varphi_{10}^2(q_j(t))\\
\varphi_{11}^2(q_j(t))\\
\varphi_{12}^2(q_j(t))
\end{bmatrix} = \mathbf{0}
$$

Por lo que la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_2^0}{\partial \mathbf{q}}$ queda definida como:

$$
\frac{\partial \mathbf{\Phi}_2^0}{\partial \mathbf{q}} = \begin{bmatrix}
\frac{\partial \varphi_1^2}{\partial q_1} & \cdots & \frac{\partial \varphi_1^2}{\partial q_{18}}\\
& \ddots &\\
\frac{\partial \varphi_{12}^2}{\partial q_1} & \cdots & \frac{\partial \varphi_{12}^2}{\partial q_{18}}
\end{bmatrix}
$$

El cálculo de las derivadas parciales comprendidas en la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_2^0}{\partial \mathbf{q}}$ se presenta para cada una de las ecuaciones de restricción en (4.19).

4.6.2.1. Cálculo de $\frac{\partial \varphi_1^2}{\partial q_j}, \frac{\partial \varphi_2^2}{\partial q_j}, \frac{\partial \varphi_3^2}{\partial q_j}, \frac{\partial \varphi_4^2}{\partial q_j}, \frac{\partial \varphi_5^2}{\partial q_j}, \frac{\partial \varphi_6^2}{\partial q_j}$

En tanto

$$
\begin{aligned}
\varphi_1^2(q_j(t)) &= \varphi_1^1(q_j(t))\\
\varphi_2^2(q_j(t)) &= \varphi_2^1(q_j(t))\\
\varphi_3^2(q_j(t)) &= \varphi_3^1(q_j(t))\\
\varphi_4^2(q_j(t)) &= \varphi_4^1(q_j(t))
\end{aligned}
$$

$$\varphi_5^2(q_j(t)) = \varphi_5^1(q_j(t))$$
$$\varphi_6^2(q_j(t)) = \varphi_6^1(q_j(t))$$

se deduce:

$$\frac{\partial \varphi_1^2}{\partial q_j} = \frac{\partial \varphi_1^1}{\partial q_j}$$

$$\frac{\partial \varphi_2^2}{\partial q_j} = \frac{\partial \varphi_2^1}{\partial q_j}$$

$$\frac{\partial \varphi_3^2}{\partial q_j} = \frac{\partial \varphi_3^1}{\partial q_j}$$

$$\frac{\partial \varphi_4^2}{\partial q_j} = \frac{\partial \varphi_4^1}{\partial q_j}$$

$$\frac{\partial \varphi_5^2}{\partial q_j} = \frac{\partial \varphi_5^1}{\partial q_j}$$

$$\frac{\partial \varphi_6^2}{\partial q_j} = \frac{\partial \varphi_6^1}{\partial q_j}$$

El cálculo de $\frac{\partial \varphi_7^2}{\partial q_j}, \frac{\partial \varphi_8^2}{\partial q_j}, \frac{\partial \varphi_9^2}{\partial q_j}, \frac{\partial \varphi_{10}^2}{\partial q_j}, \frac{\partial \varphi_{11}^2}{\partial q_j}, \frac{\partial \varphi_{12}^2}{\partial q_j}$ se desarrolla de forma análoga al de $\frac{\partial \varphi_1^1}{\partial q_j}, \frac{\partial \varphi_2^1}{\partial q_j}, \frac{\partial \varphi_3^1}{\partial q_j}, \frac{\partial \varphi_4^1}{\partial q_j}, \frac{\partial \varphi_5^1}{\partial q_j}, \frac{\partial \varphi_6^1}{\partial q_j}$ considerando las matrices de rotación $\mathbf{R}_{72}^0$ y $\mathbf{R}_{112}^0$.

4.6.2.2. Cálculo de $\frac{\partial \varphi_7^2}{\partial q_j}$

$$\frac{\partial \varphi_7^2}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\left(\mathbf{l}_2^0(q_j(t))\right)^T \mathbf{k}_0\right)$$

$$= \frac{\partial}{\partial q_j}\left(\mathbf{l}_2^0(q_j(t))\right)^T \mathbf{k}_0 + \left(\mathbf{l}_2^0(q_j(t))\right)^T \frac{\partial \mathbf{k}_0}{\partial q_j}$$

$$= \frac{\partial}{\partial q_j}\left(b2_y \mathbf{R}_{112}^0 \mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}\right)^T \mathbf{k}_0$$

$$= \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial q_j}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142} + b2_y \mathbf{R}_{112}^0 \frac{\partial \mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}}{\partial q_j}\right)^T \mathbf{k}_0$$

$$= \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial q_j}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_1} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial x_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}\right)^T \mathbf{k}_0$$

$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_2} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial y_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}\right)^T \mathbf{k}_0$$

$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_3} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial z_B}\mathbf{R}_z(\beta_7)\mathbf{R}_y(\beta_8)\mathbf{j}_{142}\right)^T \mathbf{k}_0$$

$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_4} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_B} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_5} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \phi_B} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_6} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \psi_B} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_7} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{11}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_8} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{21}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_9} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{31}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_{10}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{41}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_{11}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{51}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_{12}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{61}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_7^2}{\partial q_{13}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{12}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{12} + \beta_{12})}{\partial(\theta_{12} + \beta_{12})} \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$

$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_{14}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{22}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \frac{\partial \mathbf{R}_y(\theta_{22} + \beta_{22})}{\partial(\theta_{22} + \beta_{22})} \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$

$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_{15}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{32}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \frac{\partial \mathbf{R}_z(\theta_{32} + \beta_{32})}{\partial(\theta_{32} + \beta_{32})} \ldots \right.$$

$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_{16}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{42}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$

$$\left. \ldots \frac{\partial \mathbf{R}_z(\theta_{42} + \beta_{42})}{\partial(\theta_{42} + \beta_{42})} \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_{17}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{52}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$

$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \frac{\partial \mathbf{R}_z(\theta_{52} + \beta_{52})}{\partial(\theta_{52} + \beta_{52})} \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_7^2}{\partial q_{18}} = \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{62}} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

$$= \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$

$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \frac{\partial \mathbf{R}_x(\theta_{62} + \beta_{62})}{\partial(\theta_{62} + \beta_{62})} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{k}_0$$

4.6.2.3. Cálculo de $\frac{\partial \varphi_8^2}{\partial q_j}$

$$\frac{\partial \varphi_8^2}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{l}_2^0(q_j(t)) \right)^T \mathbf{j}_0 - b2_y \right)$$

$$= \frac{\partial}{\partial q_j} \left(\mathbf{l}_2^0(q_j(t)) \right)^T \mathbf{j}_0 + \left(\mathbf{l}_2^0(q_j(t)) \right)^T \frac{\partial \mathbf{j}_0}{\partial q_j}$$

$$= \quad \frac{\partial}{\partial q_j} \left(b2_y \mathbf{R}_{112}^0 \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$= \quad \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial q_j} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_1} = \frac{\partial \varphi_8^2}{\partial q_2} = \frac{\partial \varphi_8^2}{\partial q_3} = 0$$

$$\frac{\partial \varphi_8^2}{\partial q_4} = \left(b2_y \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_5} = \left(b2_y \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_6} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_7} = \frac{\partial \varphi_8^2}{\partial q_8} = \frac{\partial \varphi_8^2}{\partial q_9} = \frac{\partial \varphi_8^2}{\partial q_{10}} = \frac{\partial \varphi_8^2}{\partial q_{11}} = \frac{\partial \varphi_8^2}{\partial q_{12}} = 0$$

$$\frac{\partial \varphi_8^2}{\partial q_{13}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{12} + \beta_{12})}{\partial(\theta_{12} + \beta_{12})} \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_{14}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \frac{\partial \mathbf{R}_y(\theta_{22} + \beta_{22})}{\partial(\theta_{22} + \beta_{22})} \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_{15}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \frac{\partial \mathbf{R}_z(\theta_{32} + \beta_{32})}{\partial(\theta_{32} + \beta_{32})} \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_{16}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \frac{\partial \mathbf{R}_z(\theta_{42} + \beta_{42})}{\partial(\theta_{42} + \beta_{42})} \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_{17}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \ldots \right.$$
$$\left. \ldots \mathbf{R}_z(\theta_{42} + \beta_{42}) \frac{\partial \mathbf{R}_z(\theta_{52} + \beta_{52})}{\partial(\theta_{52} + \beta_{52})} \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_8^2}{\partial q_{18}} = (b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \frac{\partial \mathbf{R}_x(\theta_{62} + \beta_{62})}{\partial(\theta_{62} + \beta_{62})} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{j}_0$$

4.6.2.4. Cálculo de $\frac{\partial \varphi_9^2}{\partial q_j}$

$$\frac{\partial \varphi_9^2}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{l}_2^0(q_j(t)) \right)^T \mathbf{i}_0 \right)$$

$$= \left(b2_y \frac{\partial \mathbf{R}_{112}^0}{\partial q_j} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \right)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_1} = \frac{\partial \varphi_9^2}{\partial q_2} = \frac{\partial \varphi_9^2}{\partial q_3} = 0$$

$$\frac{\partial \varphi_9^2}{\partial q_4} = \left(b2_y \frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_5} = \left(b2_y \mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_6} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_7} = \frac{\partial \varphi_9^2}{\partial q_8} = \frac{\partial \varphi_9^2}{\partial q_9} = \frac{\partial \varphi_9^2}{\partial q_{10}} = \frac{\partial \varphi_9^2}{\partial q_{11}} = \frac{\partial \varphi_9^2}{\partial q_{12}} = 0$$

$$\frac{\partial \varphi_9^2}{\partial q_{13}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{12} + \beta_{12})}{\partial(\theta_{12} + \beta_{12})} \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_{14}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \frac{\partial \mathbf{R}_y(\theta_{22} + \beta_{22})}{\partial(\theta_{22} + \beta_{22})} \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_{15}} = \left(b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \frac{\partial \mathbf{R}_z(\theta_{32} + \beta_{32})}{\partial(\theta_{32} + \beta_{32})} \dots \right.$$

$$\dots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \Big)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_{16}} = (b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \dots$$

$$\cdots \frac{\partial \mathbf{R}_z(\theta_{42} + \beta_{42})}{\partial(\theta_{42} + \beta_{42})} \mathbf{R}_z(\theta_{52} + \beta_{52}) \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \bigg)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_{17}} = (b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \cdots$$

$$\cdots \mathbf{R}_z(\theta_{42} + \beta_{42}) \frac{\partial \mathbf{R}_z(\theta_{52} + \beta_{52})}{\partial(\theta_{52} + \beta_{52})} \mathbf{R}_x(\theta_{62} + \beta_{62}) \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \bigg)^T \mathbf{i}_0$$

$$\frac{\partial \varphi_9^2}{\partial q_{18}} = (b2_y \mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \cdots$$

$$\cdots \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{R}_z(\theta_{52} + \beta_{52}) \frac{\partial \mathbf{R}_x(\theta_{62} + \beta_{62})}{\partial(\theta_{62} + \beta_{62})} \mathbf{R}_z(\beta_7) \mathbf{R}_y(\beta_8) \mathbf{j}_{142} \bigg)^T \mathbf{i}_0$$

4.6.2.5. Cálculo de $\frac{\partial \varphi_{10}^2}{\partial q_j}$

$$\frac{\partial \varphi_{10}^2}{\partial q_j} = \frac{\partial}{\partial q_j} \left(\left(\mathbf{k}_{82}^0(q_j(t)) \right)^T \mathbf{j}_0 \right)$$

$$= \left(\frac{\partial \mathbf{R}_{72}^0}{\partial q_j} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_1} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial x_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_2} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial y_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_3} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial z_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_4} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B} \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_5} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \phi_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= \left(\mathbf{R}_x(\theta_B) \frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B} \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_6} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \psi_B} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B} \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_7} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{11}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_8} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{21}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_9} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{31}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{10}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{41}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{11}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{51}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{12}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{61}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{13}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{12}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \frac{\partial \mathbf{R}_z(\theta_{12} + \beta_{12})}{\partial (\theta_{12} + \beta_{12})} \mathbf{R}_y(\theta_{22} + \beta_{22}) \mathbf{R}_z(\theta_{32} + \beta_{32}) \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{14}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{22}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \frac{\partial \mathbf{R}_y(\theta_{22} + \beta_{22})}{\partial (\theta_{22} + \beta_{22})} \mathbf{R}_z(\theta_{32} + \beta_{32}) \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{15}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{32}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= \left(\mathbf{R}_x(\theta_B) \mathbf{R}_y(\phi_B) \mathbf{R}_z(\psi_B) \mathbf{R}_z(\theta_{12} + \beta_{12}) \mathbf{R}_y(\theta_{22} + \beta_{22}) \frac{\partial \mathbf{R}_z(\theta_{32} + \beta_{32})}{\partial (\theta_{32} + \beta_{32})} \mathbf{R}_z(\theta_{42} + \beta_{42}) \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{16}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{42}} \mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\frac{\partial \mathbf{R}_z(\theta_{42}+\beta_{42})}{\partial(\theta_{42}+\beta_{42})}\mathbf{k}_{82} \right)^T \mathbf{j}_0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{17}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{52}}\mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

$$\frac{\partial \varphi_{10}^2}{\partial q_{18}} = \left(\frac{\partial \mathbf{R}_{72}^0}{\partial \theta_{62}}\mathbf{k}_{82} \right)^T \mathbf{j}_0$$
$$= 0$$

4.6.2.6. Cálculo de $\frac{\partial \varphi_{11}^2}{\partial q_j}$

$$\frac{\partial \varphi_{11}^2}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\left(\mathbf{k}_{82}^0(q_j(t))\right)^T \mathbf{k}_0 \right)$$
$$= \left(\frac{\partial \mathbf{R}_{72}^0}{\partial q_j}\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_1} = \frac{\partial \varphi_{11}^2}{\partial q_2} = \frac{\partial \varphi_{11}^2}{\partial q_3} = 0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_4} = \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_5} = \left(\mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B}\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_6} = \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B}\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_7} = \frac{\partial \varphi_{11}^2}{\partial q_8} = \frac{\partial \varphi_{11}^2}{\partial q_9} = \frac{\partial \varphi_{11}^2}{\partial q_{10}} = \frac{\partial \varphi_{11}^2}{\partial q_{11}} = \frac{\partial \varphi_{11}^2}{\partial q_{12}} = 0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_{13}} = \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{12}+\beta_{12})}{\partial(\theta_{12}+\beta_{12})}\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_{14}} = \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\frac{\partial \mathbf{R}_y(\theta_{22}+\beta_{22})}{\partial(\theta_{22}+\beta_{22})}\mathbf{R}_z(\theta_{32}+\beta_{32})\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_{15}} = \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\frac{\partial \mathbf{R}_z(\theta_{32}+\beta_{32})}{\partial(\theta_{32}+\beta_{32})}\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_{16}} = \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\frac{\partial \mathbf{R}_z(\theta_{42}+\beta_{42})}{\partial(\theta_{42}+\beta_{42})}\mathbf{k}_{82} \right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{11}^2}{\partial q_{17}} = \frac{\partial \varphi_{11}^2}{\partial q_{18}} = 0$$

4.6.2.7. Cálculo de $\frac{\partial \varphi_{12}^2}{\partial q_j}$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(\left(\mathbf{k}_{112}^0(q_j(t))\right)^T \mathbf{k}_0\right) \\
&= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial q_j}\mathbf{k}_{112}\right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_1} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial x_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_2} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial y_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_3} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial z_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_4} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= \left(\frac{\partial \mathbf{R}_x(\theta_B)}{\partial \theta_B}\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right. \\
&\qquad \left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_5} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \phi_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= \left(\mathbf{R}_x(\theta_B)\frac{\partial \mathbf{R}_y(\phi_B)}{\partial \phi_B}\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right. \\
&\qquad \left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_6} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \psi_B}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\frac{\partial \mathbf{R}_z(\psi_B)}{\partial \psi_B}\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right. \\
&\qquad \left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varphi_{12}^2}{\partial q_7} &= \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{11}}\mathbf{k}_{112}\right)^T \mathbf{k}_0 \\
&= 0
\end{aligned}
$$

$$\frac{\partial \varphi_{12}^2}{\partial q_8} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{21}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_9} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{31}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{10}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{41}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{11}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{51}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{12}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{61}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= 0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{13}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{12}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\frac{\partial \mathbf{R}_z(\theta_{12}+\beta_{12})}{\partial(\theta_{12}+\beta_{12})}\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{14}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{22}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\frac{\partial \mathbf{R}_y(\theta_{22}+\beta_{22})}{\partial(\theta_{22}+\beta_{22})}\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{15}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{32}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\frac{\partial \mathbf{R}_z(\theta_{32}+\beta_{32})}{\partial(\theta_{32}+\beta_{32})}\ldots\right.$$
$$\left.\ldots\mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{16}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{42}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$
$$= \left(\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots\right.$$
$$\left.\ldots\frac{\partial \mathbf{R}_z(\theta_{42}+\beta_{42})}{\partial(\theta_{42}+\beta_{42})}\mathbf{R}_z(\theta_{52}+\beta_{52})\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{17}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{52}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$= (\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots$$

$$\ldots \mathbf{R}_z(\theta_{42}+\beta_{42})\frac{\partial \mathbf{R}_z(\theta_{52}+\beta_{52})}{\partial(\theta_{52}+\beta_{52})}\mathbf{R}_x(\theta_{62}+\beta_{62})\mathbf{k}_{112}\Big)^T \mathbf{k}_0$$

$$\frac{\partial \varphi_{12}^2}{\partial q_{18}} = \left(\frac{\partial \mathbf{R}_{112}^0}{\partial \theta_{62}}\mathbf{k}_{112}\right)^T \mathbf{k}_0$$

$$= (\mathbf{R}_x(\theta_B)\mathbf{R}_y(\phi_B)\mathbf{R}_z(\psi_B)\mathbf{R}_z(\theta_{12}+\beta_{12})\mathbf{R}_y(\theta_{22}+\beta_{22})\mathbf{R}_z(\theta_{32}+\beta_{32})\ldots$$

$$\ldots \mathbf{R}_z(\theta_{42}+\beta_{42})\mathbf{R}_z(\theta_{52}+\beta_{52})\frac{\partial \mathbf{R}_x(\theta_{62}+\beta_{62})}{\partial(\theta_{62}+\beta_{62})}\mathbf{k}_{112}\Big)^T \mathbf{k}_0$$

Una vez desarrollados los elementos de la matriz Jacobiana $\frac{\partial \mathbf{\Phi}_2^0}{\partial \mathbf{q}}$, el vector de las fuerzas de restricción en el intervalo T_2 se escribe de acuerdo a (4.14) como:

$$\mathbf{\Gamma}_2 = \left(\frac{\partial \mathbf{\Phi}_2^0}{\partial \mathbf{q}}\right)^T \boldsymbol{\lambda}_2^0 \tag{4.20}$$

donde el vector de los multiplicadores de Lagrange tiene la forma:

$$\boldsymbol{\lambda}_2^0 = \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \\ \lambda_3^2 \\ \lambda_4^2 \\ \lambda_5^2 \\ \lambda_6^2 \\ \lambda_7^2 \\ \lambda_8^2 \\ \lambda_9^2 \\ \lambda_{10}^2 \\ \lambda_{11}^2 \\ \lambda_{12}^2 \end{bmatrix}$$

4.6.3. Restricciones en el intervalo T_3

Las restricciones holonómicas durante la fase de SS con apoyo en el pie derecho son:

$$\begin{aligned}
\varphi_1^3(q_j(t)) &= \mathbf{l}_2^0(q_j(t)) \cdot \mathbf{k}_0 = 0 \\
\varphi_2^3(q_j(t)) &= \mathbf{l}_2^0(q_j(t)) \cdot \mathbf{j}_0 - b2_y = 0 \\
\varphi_3^3(q_j(t)) &= \mathbf{l}_2^0(q_j(t)) \cdot \mathbf{i}_0 = 0 \\
\varphi_4^3(q_j(t)) &= \mathbf{k}_{82}^0(q_j(t)) \cdot \mathbf{j}_0 = 0 \\
\varphi_5^3(q_j(t)) &= \mathbf{k}_{82}^0(q_j(t)) \cdot \mathbf{k}_0 = 0 \\
\varphi_6^3(q_j(t)) &= \mathbf{k}_{112}^0(q_j(t)) \cdot \mathbf{k}_0 = 0
\end{aligned} \tag{4.21}$$

Las ecuaciones $\varphi_1^3(q_j(t)), \cdots, \varphi_6^3(q_j(t))$ representan las mismas restricciones de posición y orientación para

el pie derecho descritas por $\varphi_7^2(q_j(t)), \cdots, \varphi_{12}^2(q_j(t))$.

Agrupando las ecuaciones (4.21) en el vector $\boldsymbol{\Phi}_3^0$, las restricciones holonómicas se escriben:

$$
\boldsymbol{\Phi}_3^0 = \begin{bmatrix} \varphi_1^3(q_j(t)) \\ \varphi_2^3(q_j(t)) \\ \varphi_3^3(q_j(t)) \\ \varphi_4^3(q_j(t)) \\ \varphi_5^3(q_j(t)) \\ \varphi_6^3(q_j(t)) \end{bmatrix} = \boldsymbol{0}
$$

Por lo que la matriz Jacobiana $\frac{\partial \boldsymbol{\Phi}_3^0}{\partial \mathbf{q}}$ queda definida como:

$$
\frac{\partial \boldsymbol{\Phi}_3^0}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \varphi_1^3}{\partial q_1} & \cdots & \frac{\partial \varphi_1^3}{\partial q_{18}} \\ & \ddots & \\ \frac{\partial \varphi_6^3}{\partial q_1} & \cdots & \frac{\partial \varphi_6^3}{\partial q_{18}} \end{bmatrix}
$$

En tanto

$$
\begin{aligned}
\varphi_1^3(q_j(t)) &= \varphi_7^2(q_j(t)) \\
\varphi_2^3(q_j(t)) &= \varphi_8^2(q_j(t)) \\
\varphi_3^3(q_j(t)) &= \varphi_9^2(q_j(t)) \\
\varphi_4^3(q_j(t)) &= \varphi_{10}^2(q_j(t)) \\
\varphi_5^3(q_j(t)) &= \varphi_{11}^2(q_j(t)) \\
\varphi_6^3(q_j(t)) &= \varphi_{12}^2(q_j(t))
\end{aligned}
$$

se deduce:

$$
\begin{aligned}
\frac{\partial \varphi_1^3}{\partial q_j} &= \frac{\partial \varphi_7^2}{\partial q_j} \\
\frac{\partial \varphi_2^3}{\partial q_j} &= \frac{\partial \varphi_8^2}{\partial q_j} \\
\frac{\partial \varphi_3^3}{\partial q_j} &= \frac{\partial \varphi_9^2}{\partial q_j} \\
\frac{\partial \varphi_4^3}{\partial q_j} &= \frac{\partial \varphi_{10}^2}{\partial q_j} \\
\frac{\partial \varphi_5^3}{\partial q_j} &= \frac{\partial \varphi_{11}^2}{\partial q_j} \\
\frac{\partial \varphi_6^3}{\partial q_j} &= \frac{\partial \varphi_{12}^2}{\partial q_j}
\end{aligned}
$$

Por lo que el vector de las fuerzas de restricción en el intervalo T_3 se escribe de acuerdo a (4.14) como:

$$
\boldsymbol{\Gamma}_3 = \left(\frac{\partial \boldsymbol{\Phi}_3^0}{\partial \mathbf{q}} \right)^T \boldsymbol{\lambda}_3^0 \tag{4.22}
$$

donde el vector de los multiplicadores de Lagrange tiene la forma:

$$\boldsymbol{\lambda}_3^0 = \begin{bmatrix} \lambda_1^3 \\ \lambda_2^3 \\ \lambda_3^3 \\ \lambda_4^3 \\ \lambda_5^3 \\ \lambda_6^3 \end{bmatrix}$$

Es importante enfatizar que los elementos de los vectores de las fuerzas de restricción en (4.18), (4.20) y (4.22) se denotan como:

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \Gamma_1^k \\ \Gamma_2^k \\ \vdots \\ \Gamma_j^k \\ \vdots \\ \Gamma_{18}^k \end{bmatrix}$$

4.7.   Ecuaciones Dinámicas

Desarrollando las sumas de las ecuaciones (4.8) y (4.10) y factorizando los términos comunes a los vectores $\ddot{\mathbf{q}}$ y $\dot{\mathbf{q}}$ se tiene:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) &= \mathbf{D}_{Bj}\ddot{\mathbf{q}} + \mathbf{V}_{Bj}\dot{\mathbf{q}} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{D}_{nij}\ddot{\mathbf{q}} + \mathbf{V}_{nij}\dot{\mathbf{q}}\right)\right) \\
&= \mathbf{D}_{Bj}\ddot{\mathbf{q}} + \mathbf{V}_{Bj}\dot{\mathbf{q}} + \mathbf{D}_{11j}\ddot{\mathbf{q}} + \mathbf{V}_{11j}\dot{\mathbf{q}} + \mathbf{D}_{21j}\ddot{\mathbf{q}} + \mathbf{V}_{21j}\dot{\mathbf{q}} + \mathbf{D}_{31j}\ddot{\mathbf{q}} + \mathbf{V}_{31j}\dot{\mathbf{q}} + \\
&\quad \mathbf{D}_{41j}\ddot{\mathbf{q}} + \mathbf{V}_{41j}\dot{\mathbf{q}} + \mathbf{D}_{51j}\ddot{\mathbf{q}} + \mathbf{V}_{51j}\dot{\mathbf{q}} + \mathbf{D}_{61j}\ddot{\mathbf{q}} + \mathbf{V}_{61j}\dot{\mathbf{q}} + \mathbf{D}_{12j}\ddot{\mathbf{q}} + \mathbf{V}_{12j}\dot{\mathbf{q}} + \\
&\quad \mathbf{D}_{22j}\ddot{\mathbf{q}} + \mathbf{V}_{22j}\dot{\mathbf{q}} + \mathbf{D}_{32j}\ddot{\mathbf{q}} + \mathbf{V}_{32j}\dot{\mathbf{q}} + \mathbf{D}_{42j}\ddot{\mathbf{q}} + \mathbf{V}_{42j}\dot{\mathbf{q}} + \mathbf{D}_{52j}\ddot{\mathbf{q}} + \mathbf{V}_{52j}\dot{\mathbf{q}} + \mathbf{D}_{62j}\ddot{\mathbf{q}} + \mathbf{V}_{62j}\dot{\mathbf{q}} \\
&= \left(\mathbf{D}_{Bj} + \mathbf{D}_{11j} + \mathbf{D}_{21j} + \mathbf{D}_{31j} + \mathbf{D}_{41j} + \mathbf{D}_{51j} + \mathbf{D}_{61j} + \mathbf{D}_{12j} + \mathbf{D}_{22j} + \mathbf{D}_{32j} + \mathbf{D}_{42j} + \mathbf{D}_{52j} + \mathbf{D}_{62j}\right)\ddot{\mathbf{q}} + \\
&\quad \left(\mathbf{V}_{Bj} + \mathbf{V}_{11j} + \mathbf{V}_{21j} + \mathbf{V}_{31j} + \mathbf{V}_{41j} + \mathbf{V}_{51j} + \mathbf{V}_{61j} + \mathbf{V}_{12j} + \mathbf{V}_{22j} + \mathbf{V}_{32j} + \mathbf{V}_{42j} + \mathbf{V}_{52j} + \mathbf{V}_{62j}\right)\dot{\mathbf{q}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial q_j} &= \mathbf{V'}_{Bj}\dot{\mathbf{q}} + \mathbf{C'}_{Bj} + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{V'}_{nij}\dot{\mathbf{q}} + \mathbf{C'}_{nij}\right)\right) \\
&= \mathbf{V'}_{Bj}\dot{\mathbf{q}} + \mathbf{C'}_{Bj} + \mathbf{V'}_{11j}\dot{\mathbf{q}} + \mathbf{C'}_{11j} + \mathbf{V'}_{21j}\dot{\mathbf{q}} + \mathbf{C'}_{21j} + \mathbf{V'}_{31j}\dot{\mathbf{q}} + \mathbf{C'}_{31j} + \\
&\quad \mathbf{V'}_{41j}\dot{\mathbf{q}} + \mathbf{C'}_{41j} + \mathbf{V'}_{51j}\dot{\mathbf{q}} + \mathbf{C'}_{51j} + \mathbf{V'}_{61j}\dot{\mathbf{q}} + \mathbf{C'}_{61j} + \mathbf{V'}_{12j}\dot{\mathbf{q}} + \mathbf{C'}_{12j} + \\
&\quad \mathbf{V'}_{22j}\dot{\mathbf{q}} + \mathbf{C'}_{22j} + \mathbf{V'}_{32j}\dot{\mathbf{q}} + \mathbf{C'}_{32j} + \mathbf{V'}_{42j}\dot{\mathbf{q}} + \mathbf{C'}_{42j} + \mathbf{V'}_{52j}\dot{\mathbf{q}} + \mathbf{C'}_{52j} + \mathbf{V'}_{62j}\dot{\mathbf{q}} + \mathbf{C'}_{62j} \\
&= \left(\mathbf{V'}_{Bj} + \mathbf{V'}_{11j} + \mathbf{V'}_{21j} + \mathbf{V'}_{31j} + \mathbf{V'}_{41j} + \mathbf{V'}_{51j} + \mathbf{V'}_{61j} + \mathbf{V'}_{12j} + \mathbf{V'}_{22j} + \mathbf{V'}_{32j} + \mathbf{V'}_{42j} + \mathbf{V'}_{52j} + \mathbf{V'}_{62j}\right)\dot{\mathbf{q}} + \\
&\quad \left(\mathbf{C'}_{Bj} + \mathbf{C'}_{11j} + \mathbf{C'}_{21j} + \mathbf{C'}_{31j} + \mathbf{C'}_{41j} + \mathbf{C'}_{51j} + \mathbf{C'}_{61j} + \mathbf{C'}_{12j} + \mathbf{C'}_{22j} + \mathbf{C'}_{32j} + \mathbf{C'}_{42j} + \mathbf{C'}_{52j} + \mathbf{C'}_{62j}\right)
\end{aligned}
$$

Con las expresiones anteriores y de acuerdo a la ecuación de Lagrange en (4.1) se opera:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = {} & \left(\mathbf{D}_{Bj} + \mathbf{D}_{11j} + \mathbf{D}_{21j} + \mathbf{D}_{31j} + \mathbf{D}_{41j} + \mathbf{D}_{51j} + \mathbf{D}_{61j} + \mathbf{D}_{12j} + \mathbf{D}_{22j} + \mathbf{D}_{32j} + \mathbf{D}_{42j} + \mathbf{D}_{52j} + \mathbf{D}_{62j}\right) \ddot{\mathbf{q}} + \\
& \left(\mathbf{V}_{Bj} + \mathbf{V}_{11j} + \mathbf{V}_{21j} + \mathbf{V}_{31j} + \mathbf{V}_{41j} + \mathbf{V}_{51j} + \mathbf{V}_{61j} + \mathbf{V}_{12j} + \mathbf{V}_{22j} + \mathbf{V}_{32j} + \mathbf{V}_{42j} + \mathbf{V}_{52j} + \mathbf{V}_{62j}\right) \dot{\mathbf{q}} + \\
& \left(-\mathbf{V'}_{Bj} - \mathbf{V'}_{11j} - \mathbf{V'}_{21j} - \mathbf{V'}_{31j} - \mathbf{V'}_{41j} - \mathbf{V'}_{51j} - \mathbf{V'}_{61j} - \mathbf{V'}_{12j} - \mathbf{V'}_{22j} - \mathbf{V'}_{32j} - \mathbf{V'}_{42j} - \mathbf{V'}_{52j} - \mathbf{V'}_{62j}\right) \dot{\mathbf{q}} + \\
& \left(-\mathbf{C'}_{Bj} - \mathbf{C'}_{11j} - \mathbf{C'}_{21j} - \mathbf{C'}_{31j} - \mathbf{C'}_{41j} - \mathbf{C'}_{51j} - \mathbf{C'}_{61j} - \mathbf{C'}_{12j} - \mathbf{C'}_{22j} - \mathbf{C'}_{32j} - \mathbf{C'}_{42j} - \mathbf{C'}_{52j} - \mathbf{C'}_{62j}\right)
\end{aligned}
$$

Definiendo

$$
\mathbb{D}_j = \mathbf{D}_{Bj} + \mathbf{D}_{11j} + \mathbf{D}_{21j} + \mathbf{D}_{31j} + \mathbf{D}_{41j} + \mathbf{D}_{51j} + \mathbf{D}_{61j} + \mathbf{D}_{12j} + \mathbf{D}_{22j} + \mathbf{D}_{32j} + \mathbf{D}_{42j} + \mathbf{D}_{52j} + \mathbf{D}_{62j}
$$

$$
\begin{aligned}
\mathbb{V}_j = {} & \mathbf{V}_{Bj} + \mathbf{V}_{11j} + \mathbf{V}_{21j} + \mathbf{V}_{31j} + \mathbf{V}_{41j} + \mathbf{V}_{51j} + \mathbf{V}_{61j} + \mathbf{V}_{12j} + \mathbf{V}_{22j} + \mathbf{V}_{32j} + \mathbf{V}_{42j} + \mathbf{V}_{52j} + \mathbf{V}_{62j} - \\
& -\mathbf{V'}_{Bj} - \mathbf{V'}_{11j} - \mathbf{V'}_{21j} - \mathbf{V'}_{31j} - \mathbf{V'}_{41j} - \mathbf{V'}_{51j} - \mathbf{V'}_{61j} - \mathbf{V'}_{12j} - \mathbf{V'}_{22j} - \mathbf{V'}_{32j} - \mathbf{V'}_{42j} - \mathbf{V'}_{52j} - \mathbf{V'}_{62j}
\end{aligned}
$$

$$
\mathbb{C}_j = -\mathbf{C'}_{Bj} - \mathbf{C'}_{11j} - \mathbf{C'}_{21j} - \mathbf{C'}_{31j} - \mathbf{C'}_{41j} - \mathbf{C'}_{51j} - \mathbf{C'}_{61j} - \mathbf{C'}_{12j} - \mathbf{C'}_{22j} - \mathbf{C'}_{32j} - \mathbf{C'}_{42j} - \mathbf{C'}_{52j} - \mathbf{C'}_{62j}
$$

se escribe:

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_j}}\right) - \frac{\partial L}{\partial q_j} = \mathbb{D}_j \ddot{\mathbf{q}} + \mathbb{V}_j \dot{\mathbf{q}} + \mathbb{C}_j
$$

Con lo que la ecuación de Lagrange en (4.1) queda:

$$
\mathbb{D}_j \ddot{\mathbf{q}} + \mathbb{V}_j \dot{\mathbf{q}} + \mathbb{C}_j = Q_j + \Gamma_j^k
$$

Escribiendo la ecuación anterior dieciocho veces, una para cada $j = 1, 2, ..., 18$, obtenemos dieciocho ecuaciones escalares; las cuales se pueden ordenar como:

$$
\begin{bmatrix}
\mathbb{D}_1 \\ \mathbb{D}_2 \\ \mathbb{D}_3 \\ \mathbb{D}_4 \\ \mathbb{D}_5 \\ \mathbb{D}_6 \\ \mathbb{D}_7 \\ \mathbb{D}_8 \\ \mathbb{D}_9 \\ \mathbb{D}_{10} \\ \mathbb{D}_{11} \\ \mathbb{D}_{12} \\ \mathbb{D}_{13} \\ \mathbb{D}_{14} \\ \mathbb{D}_{15} \\ \mathbb{D}_{16} \\ \mathbb{D}_{17} \\ \mathbb{D}_{18}
\end{bmatrix} \ddot{\mathbf{q}} +
\begin{bmatrix}
\mathbb{V}_1 \\ \mathbb{V}_2 \\ \mathbb{V}_3 \\ \mathbb{V}_4 \\ \mathbb{V}_5 \\ \mathbb{V}_6 \\ \mathbb{V}_7 \\ \mathbb{V}_8 \\ \mathbb{V}_9 \\ \mathbb{V}_{10} \\ \mathbb{V}_{11} \\ \mathbb{V}_{12} \\ \mathbb{V}_{13} \\ \mathbb{V}_{14} \\ \mathbb{V}_{15} \\ \mathbb{V}_{16} \\ \mathbb{V}_{17} \\ \mathbb{V}_{18}
\end{bmatrix} \dot{\mathbf{q}} +
\begin{bmatrix}
\mathbb{C}_1 \\ \mathbb{C}_2 \\ \mathbb{C}_3 \\ \mathbb{C}_4 \\ \mathbb{C}_5 \\ \mathbb{C}_6 \\ \mathbb{C}_7 \\ \mathbb{C}_8 \\ \mathbb{C}_9 \\ \mathbb{C}_{10} \\ \mathbb{C}_{11} \\ \mathbb{C}_{12} \\ \mathbb{C}_{13} \\ \mathbb{C}_{14} \\ \mathbb{C}_{15} \\ \mathbb{V}_{16} \\ \mathbb{V}_{17} \\ \mathbb{V}_{18}
\end{bmatrix} =
\begin{bmatrix}
Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \\ Q_{10} \\ Q_{11} \\ Q_{12} \\ Q_{13} \\ Q_{14} \\ Q_{15} \\ Q_{16} \\ Q_{17} \\ Q_{18}
\end{bmatrix} +
\begin{bmatrix}
\Gamma_1^k \\ \Gamma_2^k \\ \Gamma_3^k \\ \Gamma_4^k \\ \Gamma_5^k \\ \Gamma_6^k \\ \Gamma_7^k \\ \Gamma_8^k \\ \Gamma_9^k \\ \Gamma_{10}^k \\ \Gamma_{11}^k \\ \Gamma_{12}^k \\ \Gamma_{13}^k \\ \Gamma_{14}^k \\ \Gamma_{15}^k \\ \Gamma_{16}^k \\ \Gamma_{17}^k \\ \Gamma_{18}^k
\end{bmatrix}
$$

Cuya escritura se simplifica a:

$$\mathbb{D}\ddot{\mathbf{q}} + \mathbb{V}\dot{\mathbf{q}} + \mathbb{C} \quad = \quad \mathbf{Q} + \mathbf{\Gamma}_k$$

Recurriendo a las expresiones de los vectores de fuerzas generalizadas y fuerzas de restricción en (4.13) y (4.14), las ecuaciones de Lagrange del bípedo son:

$$\mathbb{D}\ddot{\mathbf{q}} + \mathbb{V}\dot{\mathbf{q}} + \mathbb{C} = \mathbf{A}_\tau \boldsymbol{\tau} + \left(\frac{\partial \mathbf{\Phi}_k^0}{\partial \mathbf{q}} \right)^T \boldsymbol{\lambda}_k^0$$

Si bien la estructura de las matrices $\mathbb{D}, \mathbb{V}, \mathbb{C}, \mathbf{A}_\tau$ no depende de la fase del ciclo de marcha en que se encuentra el bípedo, las fuerzas de reacción deben ser formuladas para T_k, en tanto la solución numérica de los multiplicadores de Lagrange demanda disponer de las restricciones holonómicas específicas a cada intervalo. Como consecuencia de este hecho, la dinámica del robot en el intervalo T_k queda descrita por el sistema[3]:

$$t \in T_k, \begin{cases} \mathbb{D}\ddot{\mathbf{q}} + \mathbb{V}\dot{\mathbf{q}} + \mathbb{C} = \mathbf{A}_\tau \boldsymbol{\tau} + \left(\frac{\partial \mathbf{\Phi}_k^0}{\partial \mathbf{q}} \right)^T \boldsymbol{\lambda}_k^0 \\ \mathbf{\Phi}_k^0 = 0 \end{cases} \tag{4.23}$$

donde $k = 1, 2, 3$.

Tomando en cuenta que la solución numérica de las ecuaciones de posición en (3.4) ya ha proporcionado los valores de las coordenadas generalizadas $q_7, q_8, \ldots, q_{18}$; las ecuaciones de velocidad en (3.16) obtuvieron $\dot{q}_7, \dot{q}_8, \ldots, \dot{q}_{18}$; las ecuaciones de aceleración en (3.24) determinaron $\ddot{q}_7, \ddot{q}_8, \ldots, \ddot{q}_{18}$ y que para obtener dichos resultados se tuvieron que proporcionar $q_1, q_2, \ldots, q_6, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_6, \ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_6$; se deduce que las incógnitas del sistema en (4.23) son los pares de los actuadores τ_{ni} del bípedo y los multiplicadores de Lagrange λ_j^k.

---

[3]La notación se toma de (Chevallereau et al., 2009)

Capítulo 5

Simulación de un ciclo de marcha

El patrón de caminata empleado para la simulación del ciclo de marcha del bípedo se diseñó considerando que las bases de sus pies se conservaran paralelas al plano del piso, que el punto $\mathbf{p}_B$ se mantuviera en el plano sagital y que su coordenada z_B permaneciera constante. Bajo estas consideraciones, las posiciones de los puntos $\mathbf{p}_B, \mathbf{p}_i$ y la orientación de los marcos locales $(\mathbf{p}_B; \mathbf{i}_B, \mathbf{j}_B, \mathbf{k}_B), (\mathbf{p}_i; \mathbf{i}_i, \mathbf{j}_i, \mathbf{k}_i)$ respecto al marco inercial $(\mathbf{O}; \mathbf{i}_0, \mathbf{j}_0, \mathbf{k}_0)$ en función del tiempo t quedaron restringidas geométricamente por:

$$
\begin{aligned}
x_B(t) &= 0 \\
z_B(t) &= 234.37[mm] \\
x_1(t) &= x_2(t) = 44.54[mm] \\
\theta_B(t) &= \theta_1(t) = \theta_2(t) = 0 \\
\phi_B(t) &= \phi_1(t) = \phi_2(t) = 0 \\
\psi_B(t) &= \psi_1(t) = \psi_2(t) = 0
\end{aligned}
\tag{5.1}
$$

dejando libre la definición de las funciones $y_B(t), y_1(t), z_1(t), y_2(t), z_2(t)$.

Para la generación de dichas funciones se optó por la determinación de una secuencia arbitraria de valores y_B, y_1, z_1, y_2, z_2 que permitieran al bípedo dar un paso de $52.26\,[mm]$ en el entorno gráfico de su modelo CAD, sin violar las restricciones en (5.1). Contextualizando estas mediciones en el tiempo t se generó la tabla 5.1.

Cuadro 5.1: Datos para la construcción de las splines que rigen el movimiento del torso y los pies del bípedo durante el ciclo de marcha simulado.

t	$y_B\,[mm]$	$y_1\,[mm]$	$z_1\,[mm]$	$y_2\,[mm]$	$z_2\,[mm]$
0	38.5	0	0	0	0
5	-	0	0	9	20
10	-	0	0	19	30
15	72	0	0	29	20
18	-	0	0	45	-
20	84	0	0	52.26	0
24	90.39	-	-	52.26	0
25	90.76	9	20	52.26	0
30	90.76	19	30	52.26	0
35	90.76	29	20	52.26	0
38	90.76	45	-	52.26	0
40	90.76	52.26	0	52.26	0

Con esta información y siguiendo el procedimiento que se describe en el apéndice C, se obtuvieron las splines (5.2)-(5.6) para definir las configuraciones espaciales asumidas por el bípedo en $t \in [0, 40]$. Donde t se mide en potencias de segundo y las coordenadas y_B, y_1, z_1, y_2, z_2 se calculan en $[mm]$. En la figura 5.1 se proporciona su representación gráfica.

$$y_B\,(t) \;=\; \begin{cases} 2.2333t + 38.5; & 0 \le t < 15 \\ 0.0333t^2 + 1.2333t + 46; & 15 \le t < 20 \\ -0.2423t^2 + 12.2583t - 64.25; & 20 \le t < 25 \\ 90.76; & 25 \le t < 40 \end{cases} \tag{5.2}$$

$$y_1\,(t) \;=\; \begin{cases} 0; & 0 \le t < 20 \\ 1.9333t - 38.6667; & 20 \le t < 35 \\ 1.1333t^2 - 77.4t + 1349.67; & 35 \le t < 38 \\ -2.5517t^2 + 202.66t - 3971.47; & 38 \le t < 40 \end{cases} \tag{5.3}$$

$$z_1\,(t) \;=\; \begin{cases} 0; & 0 \le t < 20 \\ 4t - 80; & 20 \le t < 25 \\ -\frac{2}{5}t^2 + 24t - 330; & 25 \le t < 35 \\ -4t + 160; & 35 \le t \le 40 \end{cases} \tag{5.4}$$

$$y_2\,(t) \;=\; \begin{cases} 1.9333t; & 0 \le t < 15 \\ 1.1333t^2 - 32.0667t + 255; & 15 \le t < 18 \\ -2.5517t^2 + 100.593t - 938.94; & 18 \le t < 20 \\ 52.26; & 20 \le t < 40 \end{cases} \tag{5.5}$$

$$z_2\left(t\right) \quad = \quad \begin{cases} 4t; & 0 \leq t < 5 \\ -\frac{2}{5}t^2 + 8t - 10; & 5 \leq t < 15 \\ -4t + 80; & 15 \leq t \leq 20 \\ 0; & 20 \leq t < 40 \end{cases} \tag{5.6}$$

Escapa a los alcances de este trabajo asignar las unidades de tiempo que definan la duración del ciclo de marcha.
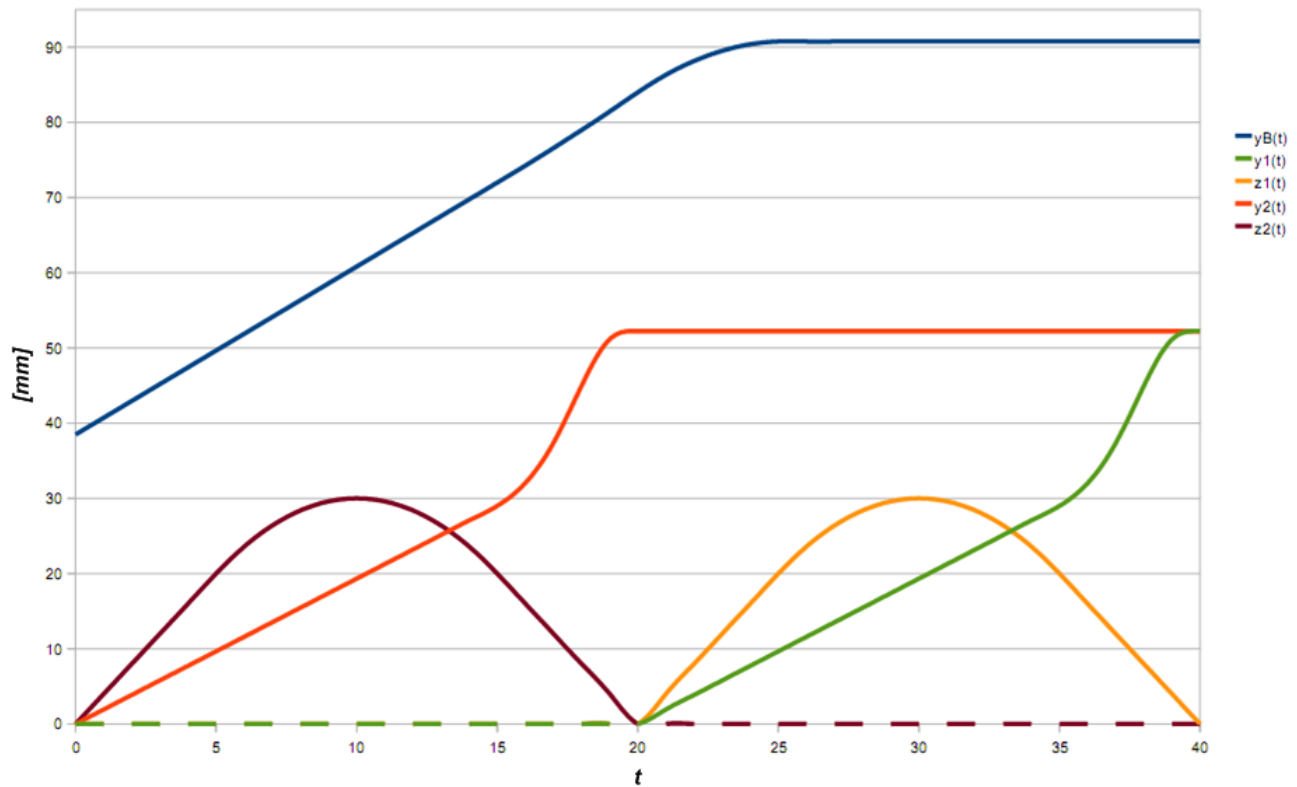
Figura 5.1: Gráfica de las splines en (5.2)-(5.6) para definir la configuración espacial de los pies y el torso del bípedo en $t \in [0, 40]$.

Con la solución numérica de las ecuaciones cinemáticas de posición en (3.4) para cada una de las configuraciones espaciales de los pies y el torso definidas por las funciones en (5.2-5.6) con $t = 1, 2, 3, \ldots, 40$; el programa en Mathematica presentado en el apéndice D arroja los valores ilustrados en las gráficas de la figura 5.2 para los ángulos $\theta_{1i}, \theta_{2i}, \theta_{3i}, \theta_{4i}, \theta_{5i}, \theta_{6i}$. En tanto que $\theta_{1i}, \theta_{2i}, \theta_{6i}$ permanecen constantes e iguales a 0, se omite su presentación.

Disponiendo de todos los ángulos anteriores, se logró animar exitosamente el ciclo de marcha propuesto en este capítulo en la sección *Simulación* del programa transcrito en el apéndice D, con lo que se considera haber demostrado la utilidad del modelo cinemático desarrollado en este trabajo. En la figura 5.3 se ofrecen algunos de los cuadros de dicha animación para ilustrar la transición del modelo simplificado del bípedo entre sus periodos de soporte doble y sencillo durante el ciclo de marcha.
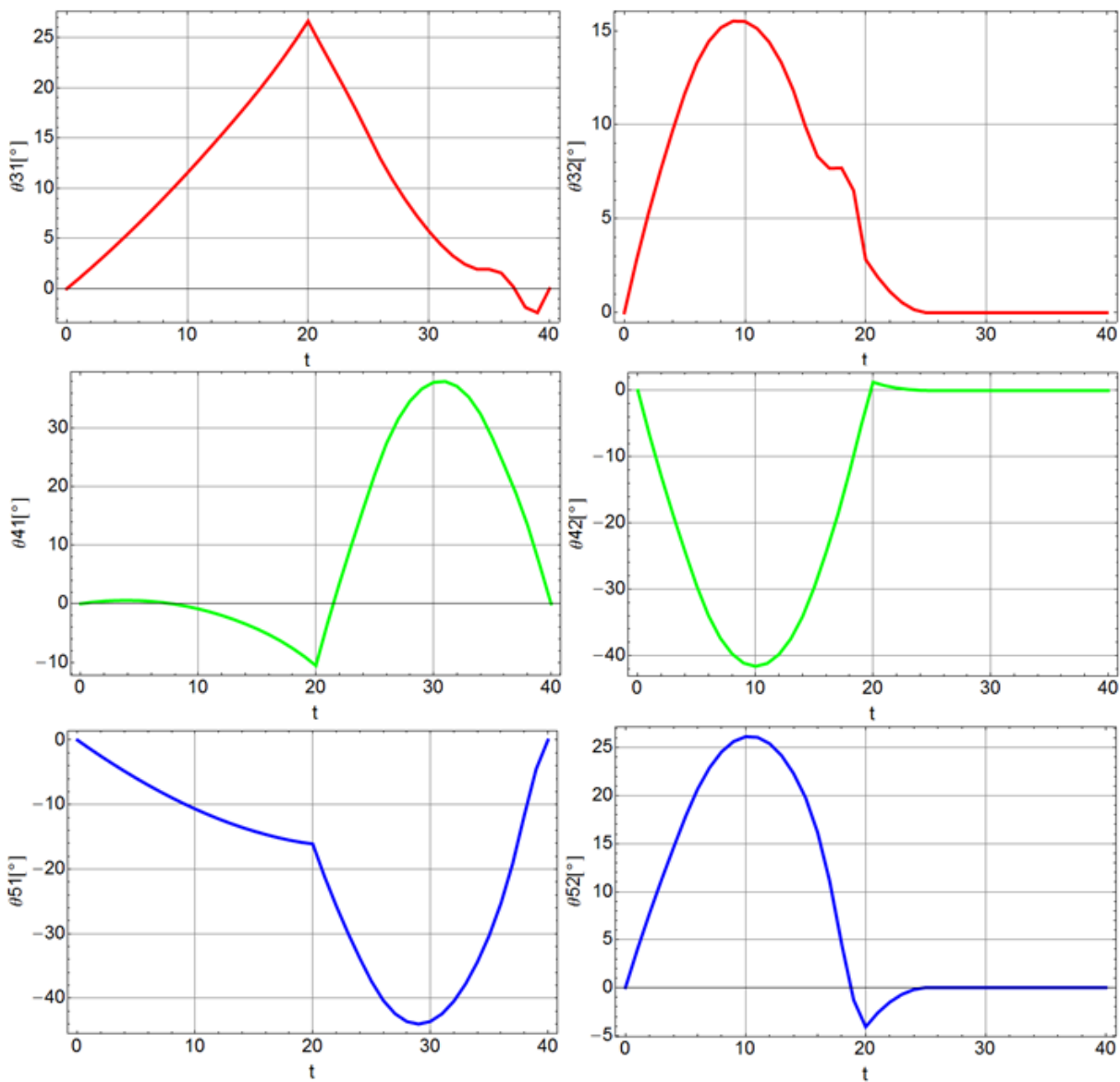
Figura 5.2: Valores adoptados por los ángulos $\theta_{31}, \theta_{41}, \theta_{51}, \theta_{32}, \theta_{42}, \theta_{52}$ durante el ciclo de marcha propuesto. Los ángulos $\theta_{11}, \theta_{21}, \theta_{61}, \theta_{12}, \theta_{22}, \theta_{62}$ permanecen constantes e iguales a 0, por lo que se omiten sus gráficas.
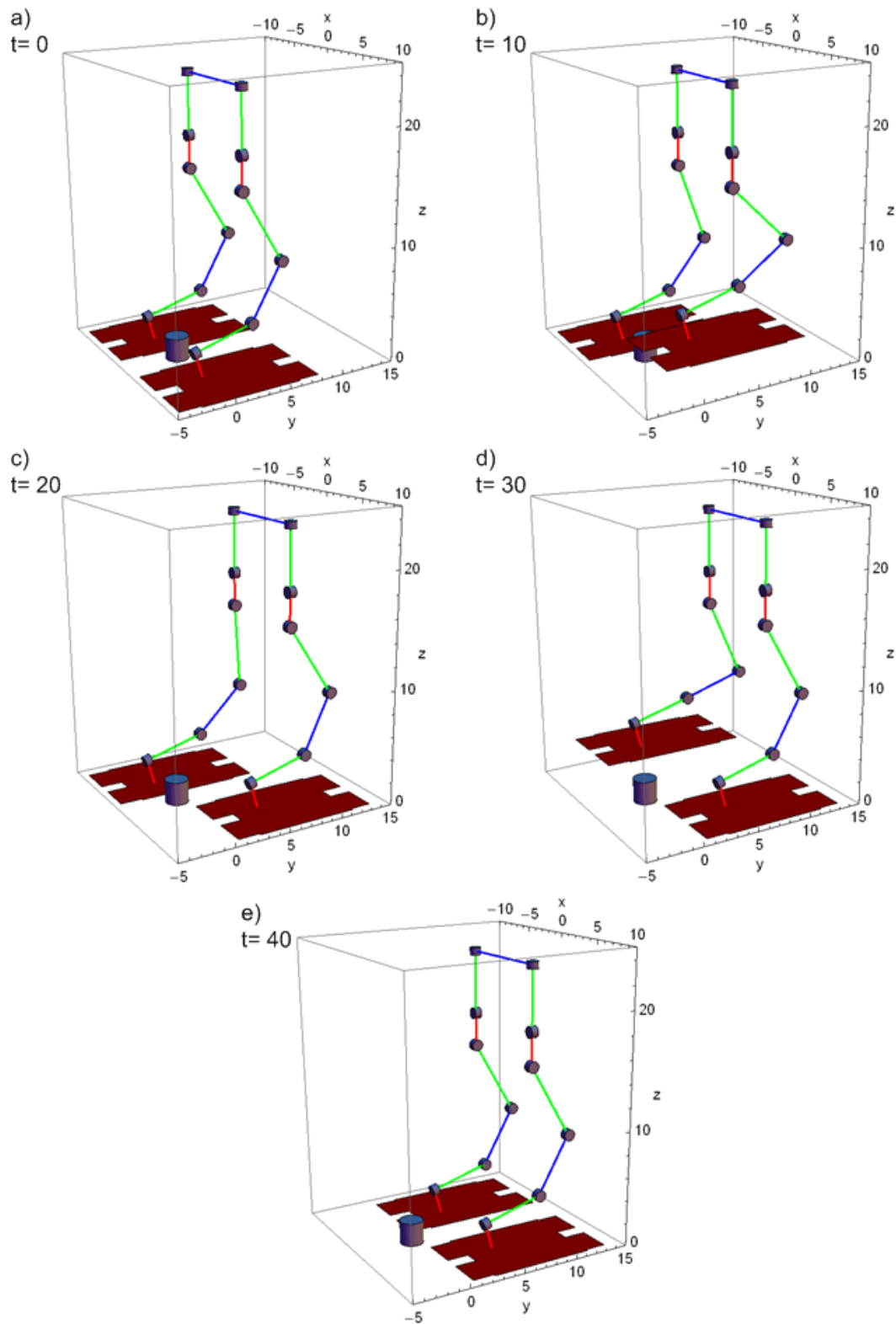
Figura 5.3: Configuraciones adoptadas por el bípedo durante la simulación del ciclo de marcha propuesto; a),c) y e) corresponden a periodos de soporte doble, mientras que b) y d) son de soporte sencillo. Se grafica un cilindro cuyo eje coincide con el vector del marco inercial $\mathbf{k}_0$ como referencia para la percepción del movimiento.

Capítulo 6

Conclusiones y trabajo futuro

6.1. Conclusiones

Se considera que con las ecuaciones cinemáticas en (3.4), (3.16), (3.24) y los sistemas de ecuaciones dinámicas en (4.23), junto con los marcos de referencia y vectores definidos para su deducción a lo largo de esta tesis, se han aportado los modelos cinemático y dinámico con las características planteadas en los objetivos de la introducción.

Las precauciones que se deben tener en cuenta al usarlos son:

- Las bases de los pies son los únicos elementos mecánicos del bípedo que se reproducen con exactitud en el entorno gráfico para la simulación de su movimiento; por lo que se pueden obtener configuraciones espaciales que el choque entre los eslabones reales del robot no admita, sin que este fenómeno sea evidente en términos de la visualización.

- En contraste con el modelo cinemático, que puede proporcionar las características de posición, velocidad y aceleración angulares de los eslabones para cualquiera de las configuraciones espaciales del bípedo que sean consistentes con las limitaciones de sus parámetros geométricos, sin ser reformulado; el modelo dinámico debe ser replanteado de acuerdo al patrón de contacto con el piso elegido para un ciclo de marcha, debido a la interdependencia de los multiplicadores de Lagrange y las ecuaciones de restricción impuestas por dicho patrón.

- El modelo dinámico desprecia los efectos de las fuerzas disipativas en las juntas rotacionales del bípedo.

- La simulación del ciclo de marcha admite el uso de cualquier técnica de interpolación para la definición de las trayectorias del torso y los pies. Se debe notar que la programación de un patrón de caminata que involucre el movimiento de todos los eslabones del bípedo, demandará establecer 18 curvas para la definición de las configuraciones espaciales en el tiempo, de acuerdo a los 18 GDL del bípedo.

Se debe agregar que los modelos cinemático y dinámico en esta tesis proporcionan una excelente plataforma de cálculo que no se encuentra restringida al estudio ni programación de la caminata del bípedo Scout. En tanto que la generalidad de su formulación permite describir el movimiento de bípedos con trece eslabones rígidos unidos por medio de juntas rotacionales actuadas y la base de sus pies constituida por polígonos; los modelos pueden brindar información valiosa en la solución de los problemas de diseño de este tipo de robots, tales como las dimensiones con las que deben contar sus eslabones para tener un rango de movilidad

determinado y los pares necesarios en los actuadores para realizar una buena gama de movimientos, con lo que a su vez podría escalarse hacia el diseño de una órtesis de miembro inferior que restituya la caminata a una persona.

6.2. Trabajo futuro

Una vez programado el modelo dinámico bajo el patrón de contacto expuesto en (4.6), se pretende extender la formulación del vector de fuerzas de restricción para la consideración de fases transitorias entre SS y SD que permitan aproximar el patrón de caminta del bípedo al de un ser humano, de acuerdo a lo expuesto en (Chevallereau et al., 2009); quedando pendiente la introducción de modelos de impacto con el piso y la consideración de las fuerzas disipativas en las juntas, para disponer de una descripción detallada de los fenómenos físicos que pueden alterar la estabilidad de la marcha.

A grandes rasgos, el criterio de estabilidad dinámica de la caminata bípeda basado en el principio del punto de momento cero $\mathbf{p}_{ZMP}$ refiere que éste debe permanecer en el polígono de soporte constituido por el conjunto convexo de los puntos de contacto entre los pies del robot y el piso, para garantizar que los pies guarden contacto completo con éste y que un patrón de caminata pueda llevarse a cabo tal y como fue planeado (Westervelt et al., 2007).

Con el método documentado en (Kajita and Espiau, 2008), se determinan los vectores del centro de masa del bípedo $\mathbf{b}_T^0$, su momento lineal total $\mathcal{P}$ y el momento angular total $\mathcal{L}$ respecto al marco inercial de acuerdo a:

$$
\begin{aligned}
\mathbf{b}_T^0 &= \frac{m_B}{m_T}\mathbf{b}_B^0 + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}\frac{m_{ni}}{m_T}\mathbf{b}_{ni}^0\right) \\
\mathcal{P} &= m_B\dot{\mathbf{b}}_B^0 + \sum_{i=1}^{2}\left(\sum_{n=1}^{6}m_{ni}\dot{\mathbf{b}}_{ni}^0\right) \\
\mathcal{L} &= \mathbf{b}_B^0\times\left(m_B\dot{\mathbf{b}}_B^0\right)+\mathbf{I}_B^0\boldsymbol{\omega}_B^0+\sum_{i=1}^{2}\left(\sum_{n=1}^{6}\left(\mathbf{b}_{ni}^0\times\left(m_{ni}\dot{\mathbf{b}}_{ni}^0\right)+\mathbf{I}_{ni}^0\boldsymbol{\omega}_{ni}^0\right)\right)
\end{aligned}
$$

Refiriéndose a los elementos de los vectores anteriores como:

$$
\mathcal{P}=\left[\begin{array}{c}\mathcal{P}_x\\\mathcal{P}_y\\\mathcal{P}_z\end{array}\right],\quad \mathcal{L}=\left[\begin{array}{c}\mathcal{L}_x\\\mathcal{L}_y\\\mathcal{L}_z\end{array}\right],\quad \mathbf{b}_T^0=\left[\begin{array}{c}b_{Tx}\\b_{Ty}\\b_{Tz}\end{array}\right]
$$

El vector asociado a la posición del punto de momento cero $\mathbf{p}_{ZMP}$ en el marco inercial es:

$$
\mathbf{p}_{ZMP}=\left[\begin{array}{c}ZMP_x\\ZMP_y\\ZMP_z\end{array}\right]
$$

donde

$$
ZMP_x = \frac{m_T g b_{Tx}+ZMP_z\dot{\mathcal{P}}_x-\dot{\mathcal{L}}_y}{m_T g+\dot{\mathcal{P}}_z}
$$

$$ZMP_y \quad = \quad \frac{m_T g b_{Ty} + ZMP_z \dot{\mathcal{P}}_y + \dot{\mathcal{L}}_x}{m_T g + \dot{\mathcal{P}}_z} \tag{6.1}$$

y ZMP_z es la altura del piso respecto al marco inercial.

Se puede verificar que cuando el bípedo permanece estacionario, el punto de momento cero es la proyección del centro de masa total $\mathbf{b}_T^0$ sobre el plano del piso:

$$ZMP_x \quad = \quad b_{Tx}$$
$$ZMP_y \quad = \quad b_{Ty}$$

Habiendo identificado en la revisión del estado del arte al principio del ZMP como el criterio que rige el desarrollo de los bípedos más versátiles de la actualidad, el siguiente paso rumbo a la síntesis de esquemas de control para la caminata estable de Scout es el diseño de un algoritmo que valide los patrones de caminata generados con la asistencia de los modelos cinemático y dinámico de esta tesis, determinando en cada instante del ciclo de marcha el conjunto de puntos que pertenecen al polígono de soporte del robot y verificando con las ecuaciones en (6.1) que $\mathbf{p}_{ZMP}$ se encuentre incluido en éste.

Apéndice A

Población discapacitada de México de acuerdo a datos del INEGI

A.1. Distribución porcentual de la población con discapacidad según causa de la discapacidad para cada entidad federativa, 2000

Entidad federativa	Población con discapacidad	Nacimiento	Enfermedad	Accidente	Edad avanzada	Otra causa	No especificado
Estados Unidos Mexicanos	**2 241 193**	**19.4**	**31.6**	**17.7**	**22.7**	**1.9**	**6.7**
Aguascalientes	23 147	23.1	28.5	15.0	23.3	2.1	8.0
Baja California	44 707	19.6	30.8	20.4	14.6	3.0	11.5
Baja California Sur	8 338	22.5	32.6	19.9	14.5	2.7	7.8
Campeche	21 012	18.1	29.8	17.2	25.8	2.8	6.4
Coahuila de Zaragoza	50 299	19.9	33.2	19.0	18.7	2.3	7.0
Colima	17 742	19.2	31.1	20.6	20.7	1.8	6.7
Chiapas	61 597	21.7	31.8	14.6	20.3	1.7	9.9
Chihuahua	68 137	18.9	30.5	20.1	21.6	1.4	7.6
Distrito Federal	188 443	18.8	32.5	19.5	20.9	2.2	6.1
Durango	41 905	17.2	32.7	18.1	21.8	1.4	8.8
Guanajuato	115 001	19.3	30.3	16.4	25.7	2.0	6.4
Guerrero	67 936	17.3	29.5	16.8	28.6	1.4	6.4
Hidalgo	59 845	19.7	31.3	16.9	25.0	1.8	5.3
Jalisco	162 257	19.7	30.7	18.9	22.6	1.7	6.4
México	240 498	21.9	30.2	20.5	18.3	2.0	7.1
Michoacán de Ocampo	104 555	18.3	31.5	16.6	25.0	1.6	7.1
Morelos	41 153	19.0	29.1	18.3	24.8	1.8	7.0
Nayarit	29 551	18.3	33.8	15.2	25.3	1.7	5.7
Nuevo León	81 463	19.0	35.0	17.4	19.5	2.0	7.1
Oaxaca	85 357	17.1	30.0	15.0	30.8	1.6	5.5
Puebla	110 922	20.3	29.6	16.9	25.5	1.4	6.3
Querétaro Arteaga	28 187	19.2	30.5	17.4	24.6	1.7	6.6
Quintana Roo	14 473	24.8	28.6	15.3	22.6	3.1	5.7
San Luis Potosí	62 698	18.5	30.8	16.7	26.4	2.0	5.5
Sinaloa	59 035	20.5	35.3	16.1	18.7	1.8	7.6
Sonora	51 073	20.6	35.3	16.1	16.1	1.7	10.3
Tabasco	58 609	16.2	33.4	14.0	26.2	4.4	5.8
Tamaulipas	59 974	19.2	33.1	19.2	20.5	1.9	6.1
Tlaxcala	18 652	19.8	30.3	19.1	24.3	1.2	5.3
Veracruz de Ignacio de la Llave	172 835	19.2	32.9	17.6	22.7	2.0	5.7
Yucatán	55 836	17.8	33.4	14.2	27.0	2.2	5.3
Zacatecas	35 956	18.5	32.9	15.8	24.9	0.9	6.9

NOTA: La información de discapacidad que se captó en el cuestionario ampliado, se aplicó solamente a una muestra de la población al momento del censo, es decir, las cifras presentadas surgen de una estimación y, por lo tanto, pueden no coincidir con las del censo (cuestionario básico). Debido al redondeo de las cifras, la suma de los decimales puede no coincidir con el 100.0%. Cifras correspondientes al 14 de febrero.
FUENTE: INEGI. *XII Censo General de Población y Vivienda 2000.* Base de datos de la muestra censal.

A.2. Porcentaje de la población con discapacidad según tipo de discapacidad para cada entidad federativa, 2000

Entidad federativa	Motriz	Auditiva	Del lenguaje	Visual	Mental	Otra
Estados Unidos Mexicanos	45.3	15.7	4.9	26.0	16.1	0.7
Aguascalientes	49.5	14.7	3.6	21.5	18.1	1.1
Baja California	55.7	12.0	3.4	16.3	17.6	0.7
Baja California Sur	48.0	13.9	4.3	22.2	18.9	0.6
Campeche	40.5	15.0	5.3	37.7	13.9	0.6
Coahuila de Zaragoza	51.3	13.8	3.4	21.4	16.1	0.6
Colima	46.0	15.4	3.9	29.2	14.6	1.1
Chiapas	40.3	14.9	8.7	28.0	15.7	0.5
Chihuahua	51.8	15.2	3.6	20.7	15.6	0.5
Distrito Federal	50.3	16.2	3.1	19.8	17.2	0.9
Durango	51.3	14.1	3.7	23.9	14.7	0.5
Guanajuato	47.6	15.4	4.1	26.1	15.2	0.8
Guerrero	42.1	16.6	7.8	27.6	15.4	0.3
Hidalgo	38.8	19.0	6.4	31.2	14.5	0.8
Jalisco	48.5	14.6	3.4	22.3	18.2	0.9
México	45.2	15.4	4.5	23.8	17.5	1.2
Michoacán de Ocampo	44.9	17.2	4.8	26.8	14.9	0.8
Morelos	43.5	17.8	4.8	28.1	15.1	1.1
Nayarit	43.1	16.5	4.6	28.1	16.9	0.7
Nuevo León	50.9	13.1	3.6	21.8	17.3	0.6
Oaxaca	37.8	18.7	7.4	31.2	13.8	0.4
Puebla	43.1	17.6	6.5	26.8	14.6	0.6
Querétaro Arteaga	45.6	15.6	4.5	25.5	16.1	1.2
Quintana Roo	37.8	14.5	6.5	34.6	15.6	0.6
San Luis Potosí	42.4	17.8	5.4	29.5	15.1	0.8
Sinaloa	45.7	13.9	5.2	23.0	19.8	0.6
Sonora	50.1	13.9	4.1	21.7	17.4	0.6
Tabasco	33.3	12.8	6.0	43.5	15.4	0.4
Tamaulipas	48.0	14.0	4.9	24.9	15.9	0.5
Tlaxcala	45.5	17.4	5.7	25.9	14.0	0.8
Veracruz de Ignacio de la Llave	38.3	16.9	6.5	32.7	15.0	0.4
Yucatán	41.3	15.1	4.5	37.0	14.1	0.7
Zacatecas	46.1	17.0	4.3	26.3	15.3	0.4

NOTA: La suma de los distintos tipos de discapacidad puede ser mayor a cien por la población que presenta más de una discapacidad. Cifras correspondientes al 14 de febrero.
FUENTE: INEGI. *XII Censo General de Población y Vivienda 2000.* Base de datos.

Apéndice B

Especificaciones del Bípedo Scout

Todos los dibujos, mediciones y cálculos relacionados a los modelos CAD del bípedo fueron elaborados en Solid Edge Academic V16.

B.1.   Lista de Elementos

Elemento	Descripción	Cantidad	Material
	Soporte L	2	Aluminio ρ= 2.6865 [g/cm^3]
	Separador hexagonal corto 4-40 x 0.750"	12	Nylon ρ= 1.1756 [g/cm^3]
	Separador hexagonal largo 4-40 x 1.000"	4	Nylon
	Separador hexagonal M/F 4-40 x 0.375"	4	Nylon
	Remache plástico	16	Nylon
	Buje	12	Acero ρ= 7.75 [g/cm^3]
	Tornillo corto 4-40 x 0.250"	8	Acero
	Tornillo largo 4-40 x 0.375"	24	Acero
	Servomotor programable Hitec HS-5475HB Velocidad de operación: 4.55[rad/s] @ 4.8[V] Torque: 0.432 [N.m] @ 4.8[V] Masa: 40 [g] Tamaño: 38.8 x 19.8 x 36 [mm]	12	Varios

Elemento	Descripción	Cantidad	Material
	Lexan 01	1	Lexan ρ= 1.1789 [g/cm^3]
	Lexan 02	1	Lexan
	Lexan 03	1	Lexan
	Lexan 04	1	Lexan
	Lexan 05	1	Lexan
	Soporte para servomotor	2	Nylon ρ= 1.1756 [g/cm^3]
	Soporte C corto	10	Aluminio ρ= 2.6865 [g/cm^3]
	Soporte C largo	2	Aluminio
	Soporte multipropósito	4	Aluminio
	Soporte dual	2	Aluminio
	Soporte para tobillo	2	Aluminio
	Pie	2	Aluminio

ocnaar	ESCALA 1:4	mm
	Bípedo Scout	

ocnaar

ESCALA 1:4 | mm

Simplificación del Bípedo Scout
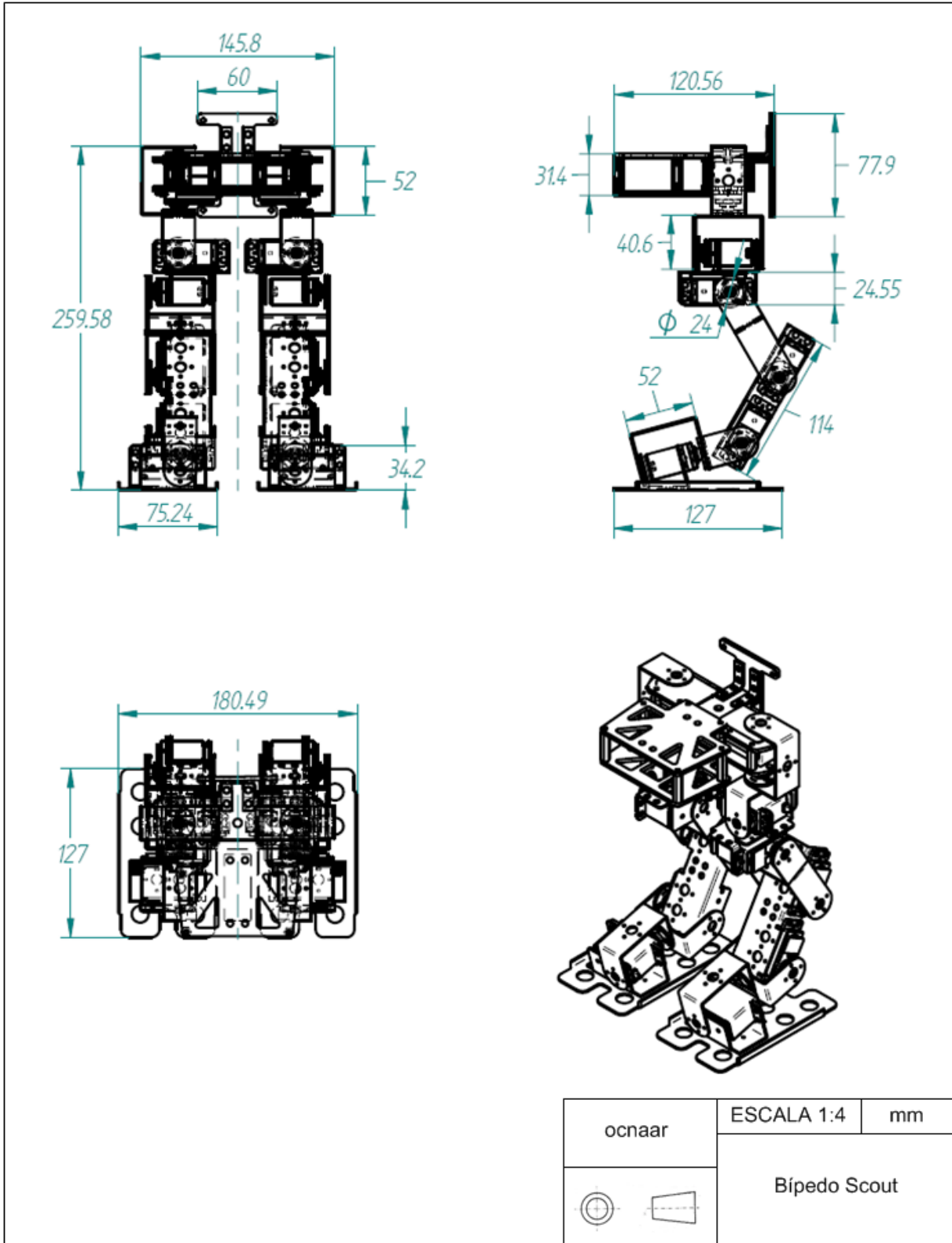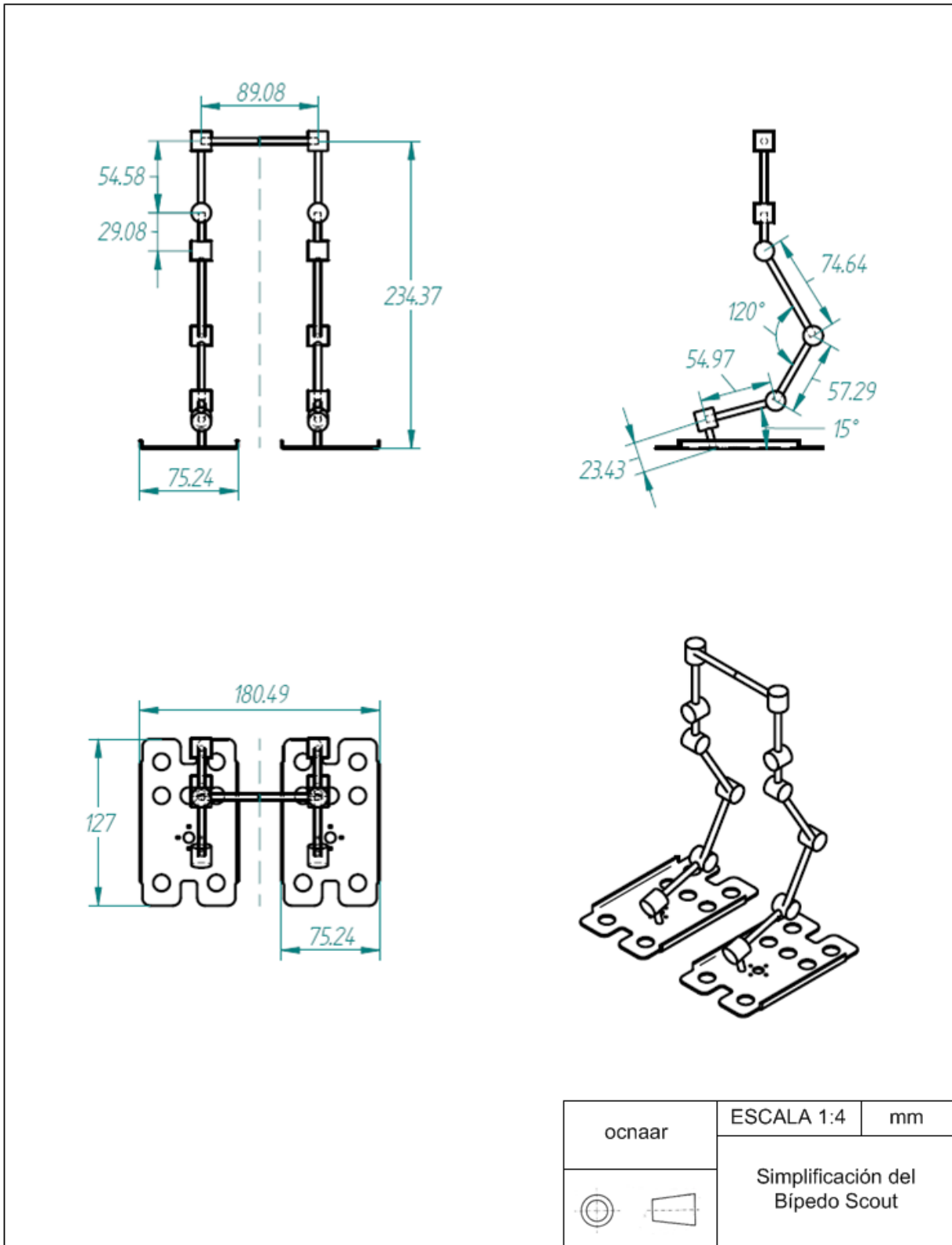
B.2. Parámetros geométricos

$$
\begin{aligned}
x_{0i} &= 44.54 \times 10^{-3}[m] \\
z_{1i} &= 54.58 \times 10^{-3}[m] \\
x_{2i} &= 29.08 \times 10^{-3}[m] \\
x_{3i} &= 74.60 \times 10^{-3}[m] \\
x_{4i} &= 57.29 \times 10^{-3}[m] \\
x_{5i} &= 54.97 \times 10^{-3}[m] \\
y_{6i} &= 23.06 \times 10^{-3}[m] \\
l_1 &= 75.24 \times 10^{-3}[m] \\
l_2 &= 127 \times 10^{-3}[m]
\end{aligned}
$$

B.3. Ángulos β_{ni}

$$
\begin{aligned}
\beta_{11} &= 180° \\
\beta_{21} &= 90° \\
\beta_{31} &= 330° \\
\beta_{41} &= 60° \\
\beta_{51} &= 45° \\
\beta_{61} &= 180° \\
\beta_{12} &= 0° \\
\beta_{22} &= 90° \\
\beta_{32} &= 30° \\
\beta_{42} &= 300° \\
\beta_{52} &= 315° \\
\beta_{62} &= 0° \\
\beta_7 &= 75° \\
\beta_8 &= 270°
\end{aligned}
$$

B.4. Vectores de posición para el trazo de los pies

Todas las unidades están en $[m]$.

Pie izquierdo:

$$
\mathbf{r}_{a1}^1 = -2.45 \times 10^{-3}\mathbf{i}_1 + 64.34 \times 10^{-3}\mathbf{j}_1
$$

$$\mathbf{r}_{b1}^1 = -2.45 \times 10^{-3}\mathbf{i}_1 + 83.33 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{c1}^1 = 26.18 \times 10^{-3}\mathbf{i}_1 + 83.33 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{d1}^1 = 26.18 \times 10^{-3}\mathbf{i}_1 + 64.34 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{e1}^1 = 28.18 \times 10^{-3}\mathbf{i}_1 + 64.81 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{f1}^1 = 28.18 \times 10^{-3}\mathbf{i}_1 - 24.20 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{g1}^1 = 26.18 \times 10^{-3}\mathbf{i}_1 - 24.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{h1}^1 = 26.18 \times 10^{-3}\mathbf{i}_1 - 43.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{i1}^1 = -2.45 \times 10^{-3}\mathbf{i}_1 - 43.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{j1}^1 = -2.45 \times 10^{-3}\mathbf{i}_1 - 24.7 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{k1}^1 = -15.12 \times 10^{-3}\mathbf{i}_1 - 24.7 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{l1}^1 = -15.12 \times 10^{-3}\mathbf{i}_1 - 43.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{m1}^1 = -44.33 \times 10^{-3}\mathbf{i}_1 - 43.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{n1}^1 = -44.33 \times 10^{-3}\mathbf{i}_1 - 24.70 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{o1}^1 = -46.33 \times 10^{-3}\mathbf{i}_1 - 24.20 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{p1}^1 = -46.33 \times 10^{-3}\mathbf{i}_1 + 64.81 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{q1}^1 = -44.33 \times 10^{-3}\mathbf{i}_1 + 64.34 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{r1}^1 = -44.33 \times 10^{-3}\mathbf{i}_1 + 83.33 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{s1}^1 = -15.12 \times 10^{-3}\mathbf{i}_1 + 83.33 \times 10^{-3}\mathbf{j}_1$$
$$\mathbf{r}_{t1}^1 = -15.12 \times 10^{-3}\mathbf{i}_1 + 64.34 \times 10^{-3}\mathbf{j}_1$$

Pie derecho:

$$\mathbf{r}_{a2}^2 = 15.70 \times 10^{-3}\mathbf{i}_2 + 64.34 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{b2}^2 = 15.70 \times 10^{-3}\mathbf{i}_2 + 83.33 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{c2}^2 = 44.33 \times 10^{-3}\mathbf{i}_2 + 83.33 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{d2}^2 = 44.33 \times 10^{-3}\mathbf{i}_2 + 64.34 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{e2}^2 = 46.33 \times 10^{-3}\mathbf{i}_2 + 64.81 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{f2}^2 = 46.33 \times 10^{-3}\mathbf{i}_2 - 24.20 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{g2}^2 = 44.33 \times 10^{-3}\mathbf{i}_2 - 24.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{h2}^2 = 44.33 \times 10^{-3}\mathbf{i}_2 - 43.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{i2}^2 = 15.70 \times 10^{-3}\mathbf{i}_2 - 43.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{j2}^2 = 15.70 \times 10^{-3}\mathbf{i}_2 - 24.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{k2}^2 = 3.03 \times 10^{-3}\mathbf{i}_2 - 24.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{l2}^2 = 3.03 \times 10^{-3}\mathbf{i}_2 - 43.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{m2}^2 = -26.18 \times 10^{-3}\mathbf{i}_2 - 43.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{n2}^2 = -26.18 \times 10^{-3}\mathbf{i}_2 - 24.70 \times 10^{-3}\mathbf{j}_2$$
$$\mathbf{r}_{o2}^2 = -28.18 \times 10^{-3}\mathbf{i}_2 - 24.20 \times 10^{-3}\mathbf{j}_2$$

$$\begin{aligned}
\mathbf{r}_{p2}^2 &= -28.18 \times 10^{-3}\mathbf{i}_2 + 64.81 \times 10^{-3}\mathbf{j}_2 \\
\mathbf{r}_{q2}^2 &= -26.18 \times 10^{-3}\mathbf{i}_2 + 64.34 \times 10^{-3}\mathbf{j}_2 \\
\mathbf{r}_{r2}^2 &= -26.18 \times 10^{-3}\mathbf{i}_2 + 83.33 \times 10^{-3}\mathbf{j}_2 \\
\mathbf{r}_{s2}^2 &= 3.03 \times 10^{-3}\mathbf{i}_2 + 83.33 \times 10^{-3}\mathbf{j}_2 \\
\mathbf{r}_{t2}^2 &= 3.03 \times 10^{-3}\mathbf{i}_2 + 64.34 \times 10^{-3}\mathbf{j}_2
\end{aligned}$$

B.5. Masa de los eslabones

$$\begin{aligned}
m_B &= 162.08 \times 10^{-3}[kg] \\
m_{1i} &= 23.12 \times 10^{-3}[kg] \\
m_{2i} &= 102.27 \times 10^{-3}[kg] \\
m_{3i} &= 26.67 \times 10^{-3}[kg] \\
m_{4i} &= 103.57 \times 10^{-3}[kg] \\
m_{5i} &= 23.12 \times 10^{-3}[kg] \\
m_{6i} &= 89.48 \times 10^{-3}[kg]
\end{aligned}$$

La masa total del bípedo es $m_T = 898.54 \times 10^{-3}[kg]$.

B.6. Momento de Inercia de los eslabones

Todas las unidades son $\left[kg \cdot m^2\right]$.

Torso:

$$\mathbf{I}_B^B = \begin{bmatrix}
152.08 \times 10^{-6} & -0.09 \times 10^{-6} & -0.02 \times 10^{-6} \\
-0.09 \times 10^{-6} & 214.21 \times 10^{-6} & 1.63 \times 10^{-6} \\
-0.02 \times 10^{-6} & 1.63 \times 10^{-6} & 306.38 \times 10^{-6}
\end{bmatrix}$$

Pierna izquierda:

$$\mathbf{I}_{11}^{11} = \begin{bmatrix}
29.40 \times 10^{-6} & 0.04 \times 10^{-6} & -0.04 \times 10^{-6} \\
0.04 \times 10^{-6} & 29.04 \times 10^{-6} & -0.09 \times 10^{-6} \\
-0.04 \times 10^{-6} & -0.09 \times 10^{-6} & 11.63 \times 10^{-6}
\end{bmatrix}$$

$$\mathbf{I}_{21}^{31} = \begin{bmatrix}
38.74 \times 10^{-6} & 9.88 \times 10^{-6} & -5.06 \times 10^{-6} \\
9.88 \times 10^{-6} & 63.10 \times 10^{-6} & -0.27 \times 10^{-6} \\
-5.06 \times 10^{-6} & -0.27 \times 10^{-6} & 61.51 \times 10^{-6}
\end{bmatrix}$$

$$\mathbf{I}_{31}^{51} = \begin{bmatrix}
14.47 \times 10^{-6} & 0.08 \times 10^{-6} & -1.09 \times 10^{-6} \\
0.08 \times 10^{-6} & 57.65 \times 10^{-6} & -0.00 \times 10^{-6} \\
-1.09 \times 10^{-6} & -0.00 \times 10^{-6} & 45.90 \times 10^{-6}
\end{bmatrix}$$

$$\mathbf{I}_{41}^{71} = \left[\begin{array}{ccc} 21.97 \times 10^{-6} & 4.37 \times 10^{-6} & -6.21 \times 10^{-6} \\ 4.37 \times 10^{-6} & 155.35 \times 10^{-6} & -1.82 \times 10^{-6} \\ -6.21 \times 10^{-6} & -1.82 \times 10^{-6} & 144.59 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{51}^{91} = \left[\begin{array}{ccc} 11.62 \times 10^{-6} & 10.53 \times 10^{-6} & -0.98 \times 10^{-6} \\ 10.53 \times 10^{-6} & 50.49 \times 10^{-6} & -0.23 \times 10^{-6} \\ -0.98 \times 10^{-6} & -0.23 \times 10^{-6} & 50.05 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{61}^{111} = \left[\begin{array}{ccc} 63.40 \times 10^{-6} & -26.31 \times 10^{-6} & 6.48 \times 10^{-6} \\ -26.31 \times 10^{-6} & 93.08 \times 10^{-6} & -8.23 \times 10^{-6} \\ 6.48 \times 10^{-6} & -8.23 \times 10^{-6} & 86.73 \times 10^{-6} \end{array} \right]$$

Pierna derecha:

$$\mathbf{I}_{12}^{12} = \left[\begin{array}{ccc} 29.40 \times 10^{-6} & 0.04 \times 10^{-6} & -0.04 \times 10^{-6} \\ 0.04 \times 10^{-6} & 29.04 \times 10^{-6} & -0.09 \times 10^{-6} \\ -0.04 \times 10^{-6} & -0.09 \times 10^{-6} & 11.63 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{22}^{32} = \left[\begin{array}{ccc} 38.74 \times 10^{-6} & -9.88 \times 10^{-6} & -5.06 \times 10^{-6} \\ -9.88 \times 10^{-6} & 63.10 \times 10^{-6} & 0.27 \times 10^{-6} \\ -5.06 \times 10^{-6} & 0.27 \times 10^{-6} & 61.51 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{32}^{52} = \left[\begin{array}{ccc} 14.47 \times 10^{-6} & -0.08 \times 10^{-6} & -1.09 \times 10^{-6} \\ -0.08 \times 10^{-6} & 57.65 \times 10^{-6} & 0.00 \times 10^{-6} \\ -1.09 \times 10^{-6} & 0.00 \times 10^{-6} & 45.90 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{42}^{72} = \left[\begin{array}{ccc} 21.97 \times 10^{-6} & -4.37 \times 10^{-6} & -6.21 \times 10^{-6} \\ -4.37 \times 10^{-6} & 155.35 \times 10^{-6} & 1.82 \times 10^{-6} \\ -6.21 \times 10^{-6} & 1.82 \times 10^{-6} & 144.59 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{52}^{92} = \left[\begin{array}{ccc} 11.62 \times 10^{-6} & -10.53 \times 10^{-6} & -0.98 \times 10^{-6} \\ -10.53 \times 10^{-6} & 50.49 \times 10^{-6} & 0.23 \times 10^{-6} \\ -0.98 \times 10^{-6} & 0.23 \times 10^{-6} & 50.05 \times 10^{-6} \end{array} \right]$$

$$\mathbf{I}_{62}^{112} = \left[\begin{array}{ccc} 63.40 \times 10^{-6} & -26.31 \times 10^{-6} & -6.48 \times 10^{-6} \\ -26.31 \times 10^{-6} & 93.08 \times 10^{-6} & 8.23 \times 10^{-6} \\ -6.48 \times 10^{-6} & 8.23 \times 10^{-6} & 86.73 \times 10^{-6} \end{array} \right]$$

B.7. Vectores para determinar la posición de los centros de gravedad

Todas las unidades están en $[m]$.

Torso:

$$\mathbf{r}_{GB}^{B} = -7.69 \times 10^{-3}\mathbf{j}_B - 0.62 \times 10^{-3}\mathbf{k}_B$$

Pierna izquierda:

$$
\begin{aligned}
\mathbf{r}_{G11}^{11} &= 8.49 \times 10^{-3}\mathbf{i}_{11} + 0.22 \times 10^{-3}\mathbf{j}_{11} - 18.77 \times 10^{-3}\mathbf{k}_{11} \\
\mathbf{r}_{G21}^{31} &= 14.48 \times 10^{-3}\mathbf{i}_{31} + 2.60 \times 10^{-3}\mathbf{j}_{31} - 6.24 \times 10^{-3}\mathbf{k}_{31} \\
\mathbf{r}_{G31}^{51} &= 34.33 \times 10^{-3}\mathbf{i}_{51} + 0.08 \times 10^{-3}\mathbf{j}_{51} - 1.21 \times 10^{-3}\mathbf{k}_{51} \\
\mathbf{r}_{G41}^{71} &= 19.52 \times 10^{-3}\mathbf{i}_{71} + 2.13 \times 10^{-3}\mathbf{j}_{71} - 3.44 \times 10^{-3}\mathbf{k}_{71} \\
\mathbf{r}_{G51}^{91} &= 35.49 \times 10^{-3}\mathbf{i}_{91} + 8.37 \times 10^{-3}\mathbf{j}_{91} - 1.21 \times 10^{-3}\mathbf{k}_{91} \\
\mathbf{r}_{G61}^{111} &= -9.58 \times 10^{-3}\mathbf{i}_{111} + 11.39 \times 10^{-3}\mathbf{j}_{111} - 7.92 \times 10^{-3}\mathbf{k}_{111}
\end{aligned}
$$

Pierna derecha:

$$
\begin{aligned}
\mathbf{r}_{G12}^{12} &= 8.49 \times 10^{-3}\mathbf{i}_{12} + 0.22 \times 10^{-3}\mathbf{j}_{12} - 18.77 \times 10^{-3}\mathbf{k}_{12} \\
\mathbf{r}_{G22}^{32} &= 14.48 \times 10^{-3}\mathbf{i}_{32} - 2.60 \times 10^{-3}\mathbf{j}_{32} - 6.24 \times 10^{-3}\mathbf{k}_{32} \\
\mathbf{r}_{G32}^{52} &= 34.33 \times 10^{-3}\mathbf{i}_{52} - 0.08 \times 10^{-3}\mathbf{j}_{52} - 1.21 \times 10^{-3}\mathbf{k}_{52} \\
\mathbf{r}_{G42}^{72} &= 19.52 \times 10^{-3}\mathbf{i}_{72} - 2.13 \times 10^{-3}\mathbf{j}_{72} - 3.44 \times 10^{-3}\mathbf{k}_{72} \\
\mathbf{r}_{G52}^{92} &= 35.49 \times 10^{-3}\mathbf{i}_{92} - 8.37 \times 10^{-3}\mathbf{j}_{92} - 1.21 \times 10^{-3}\mathbf{k}_{92} \\
\mathbf{r}_{G62}^{112} &= -9.58 \times 10^{-3}\mathbf{i}_{112} + 11.39 \times 10^{-3}\mathbf{j}_{112} + 7.92 \times 10^{-3}\mathbf{k}_{112}
\end{aligned}
$$

Apéndice C

Splines

De acuerdo al diccionario inglés Merriam Webster Online (Merriam-Webster, 2008), la palabra spline, pronunciada /´splin/, fue acotada en 1756 para describir un instrumento compuesto de bandas metálicas y pesos variables empleado por los ingenieros en el trazo de curvas suaves en planos de construcción, especialmente de barcos. El parecido de las curvas generadas mediante este dispositivo y las gráficas de los polinomios de interpolación usados en este trabajo para aproximar funciones $f(t) : \Re \to \Re$ dadas en forma tabular como $(t_0, f(t_0)), (t_1, f(t_1)), \ldots, (t_m, f(t_m))$ en el intervalo $[a, b]$, derivó en el uso del término spline en el campo de los métodos numéricos.

En tanto se conoce el intervalo sobre el cual se va aproximar $f(t)$, es posible dividir $[a, b]$ en m subintervalos y aproximar $f(t)$ en cada uno de estos por medio de un polinomio $S_i(t)$ adecuado, de tal manera que $i = 1, 2, \ldots m$. Este proceso da como aproximación de $f(t)$ una función $S(t)$ definida a tramos:

$$S(t) = \begin{cases} S_1(t); & t_0 \le t < t_1 \\ S_2(t); & t_1 \le t < t_2 \\ S_3(t); & t_2 \le t < t_3 \\ \vdots & \vdots \\ S_m(t); & t_{m-1} \le t < t_m \end{cases} \tag{C.1}$$

Obsérvese que la función $f(t)$ y la spline $S(t)$ coinciden en los nodos $t_0, t_1, t_2, \ldots, t_m$; de tal manera que $t_0 = a$, $t_m = b$ y el error en cualquier punto $(t, f(t))$ de $[t_0, t_m]$ queda acotado por la función $S_i(t)$ definida en su intervalo correspondiente.

Es importante resaltar que la interpolación a través de splines también recibe el nombre de *aproximación polinomial segmentaria.* En este apéndice se presenta un método para la construcción de splines de segundo grado basado en la información de (Nieves and Domínguez, 2004) y (Prenter, 1975), elaborándose como ejemplo el cálculo de la spline en (5.2) que define el desplazamiento del torso en la dirección del vector $\mathbf{j}_0$ durante el ciclo de marcha propuesto en el capítulo 5.

C.1.   Construcción de splines de segundo grado

Una función $S(t)$ con la estructura presentada en (C.1) se considera una spline cuadrática o de grado dos si presenta las siguientes propiedades:

1. El máximo grado de los polinomios $S_i(t)$ es dos.

2. $S(t)$ es continua en $[a, b]$.

3. $\dot{S}(t)$ es continua en $[a, b]$.

4. $S(t)$ es igual a $S_1(t)$ cuando $x < a$.

5. $S(t)$ es igual a $S_m(t)$ cuando $x > b$.

De acuerdo a la propiedad 1, una spline cuadrática tiene la forma general:

$$
S(t) = \begin{cases}
S_1(t) = a_1 t^2 + b_1 t + c_1; & t_0 \leq t < t_1 \\
S_2(t) = a_2 t^2 + b_2 t + c_2; & t_1 \leq t < t_2 \\
S_3(t) = a_3 t^2 + b_3 t + c_3; & t_2 \leq t < t_3 \\
\quad \vdots & \quad \vdots \\
S_m(t) = a_m t^2 + b_m t + c_m; & t_{m-1} \leq t < t_m
\end{cases}
\tag{C.2}
$$

Por lo que se aprecia que el problema de construcción de una spline de segundo grado consiste en la determinación de los valores a_i, b_i, c_i con $i = 1, 2, \ldots, m$.

Puesto que las funciones $f(t)$ y $S(t)$ coinciden en los nodos $t_0, t_1, t_2, \ldots, t_m$ y la función $S(t)$ debe ser continua de acuerdo a la propiedad 2, el primer paso para la síntesis de una spline cuadrática es el planteamiento de las $2m$ ecuaciones consistentes con estas condiciones, esto es:

$$
\begin{aligned}
a_1 t_0^2 + b_1 t_0 + c_1 &= f(t_0) \\
a_1 t_1^2 + b_1 t_1 + c_1 &= f(t_1) \\
a_2 t_1^2 + b_2 t_1 + c_2 &= f(t_1) \\
a_2 t_2^2 + b_2 t_2 + c_2 &= f(t_2) \\
&\vdots \\
a_i t_{i-1}^2 + b_i t_{i-1} + c_i &= f(t_{i-1}) \\
a_i t_i^2 + b_i t_i + c_i &= f(t_i) \\
&\vdots \\
a_m t_{m-1}^2 + b_m t_{m-1} + c_m &= f(t_{m-1}) \\
a_m t_m^2 + b_m t_m + c_m &= f(t_m)
\end{aligned}
\tag{C.3}
$$

Además, las primeras derivadas de las splines $S_1(t)$ y $S_2(t)$ son:

$$
\begin{aligned}
\dot{S}_1(t) &= 2a_1 t + b_1 \\
\dot{S}_2(t) &= 2a_2 t + b_2
\end{aligned}
$$

De acuerdo a la propiedad 3, $\dot{S}_1(t)$ y $\dot{S}_2(t)$ deben ser iguales en $t = t_1$, por lo tanto:

$$
\begin{aligned}
2a_1 t_1 + b_1 &= 2a_2 t_1 + b_2 \\
2a_1 t_1 + b_1 - 2a_2 t_1 - b_2 &= 0
\end{aligned}
$$

Extendiendo este razonamiento a los demás intervalos de $S(t)$ se obtienen $m-1$ ecuaciones de la forma:

$$
\begin{aligned}
2a_1 t_1 + b_1 - 2a_2 t_1 - b_2 &= 0 \\
2a_2 t_2 + b_2 - 2a_3 t_2 - b_3 &= 0 \\
&\vdots \\
2a_i t_i + b_i - 2a_{i+1} t_i - b_{i+1} &= 0 \\
&\vdots \\
2a_{m-1} t_{m-1} + b_{m-1} - 2a_m t_{m-1} - b_m &= 0
\end{aligned}
\tag{C.4}
$$

De acuerdo a (C.3) y (C.4) el número total de ecuaciones disponibles para resolver el sistema de $3m$ incógnitas en (C.2) es $(2m) + (m-1) = 3m - 1$; por lo que se suele suponer que la spline $S_1(t)$ es lineal, asignando:

$$
a_1 = 0 \tag{C.5}
$$

Esta consideración permite tener $3m$ ecuaciones; por lo que se procede a encontrar la solución del sistema de ecuaciones formado por (C.3), (C.4) y (C.5) para determinar los $3m$ valores de a_i, b_i, c_i con $i = 1, 2, \ldots, m$. Una vez establecidos estos coeficientes será posible aproximar cualquier valor de $f(t)$ en $[a, b]$, usando $S(t)$. Para propósitos de extrapolación se puede hacer uso de las propiedades 4 y 5.

C.2. Determinación de la spline $y_B(t)$

El programa escrito en Mathematica 7.0 para obtener la solución del sistema de $3m$ ecuaciones descrito en la sección anterior se presenta en la figura C.1. De acuerdo a los datos de la tabla 5.1, se desea determinar la spline $y_B(t)$ con los nodos:

$$
\begin{aligned}
(t_0 &= 0, y_B(t_0) = 38.5) \\
(t_1 &= 15, y_B(t_1) = 72) \\
(t_2 &= 20, y_B(t_2) = 84) \\
(t_3 &= 24, y_B(t_3) = 90.39)
\end{aligned}
$$

Introduciendo estos valores expresados en $[mm]$ en la sección *Datos* y seleccionando las $3(3) = 9$ ecuaciones correspondientes a las incógnitas a_i, b_i, c_i con $i = 1, 2, 3$. en la sección *Cálculo*; se puede ejecutar el programa para obtener la información necesaria en la construcción de la spline de segundo grado.

El cálculo de las splines $y_1(t), z_1(t), y_2(t), z_2(t)$ se elaboró con el mismo programa considerando los datos de la tabla 5.1 para los valores de cada una de las coordenadas durante el ciclo de marcha. Las ecuaciones en *Cálculo* deben elegirse de acuerdo al número de nodos con los que se cuenta en las interpolaciones.

Cálculo de splines de segundo grado

Datos

```
(*Definición de los nodos (t,y) para la construcción de la spline*)
t0 = 0; y0 = 38.5;
t1 = 15; y1 = 72;
t2 = 20; y2 = 84;
t3 = 24; y3 = 90.39;
(*
t4=30; y4=90.76;
t5=; y5=;
t6=; y6=;
*)
```

Cálculo

```
(*Solución numérica del sistema de ecuaciones para la construcción de la spline*)

NSolve[
 {
   (*Condiciones para la continuidad de la spline*)

   a1 * (t0)^2 + b1 * t0 + c1 == y0,
   a1 * (t1)^2 + b1 * t1 + c1 == y1,
   a2 * (t1)^2 + b2 * t1 + c2 == y1,
   a2 * (t2)^2 + b2 * t2 + c2 == y2,
   a3 * (t2)^2 + b3 * t2 + c3 == y2,
   a3 * (t3)^2 + b3 * t3 + c3 == y3,
   (*
   a4*(t3)^2+ b4*t3+c4==y3,
   a4*(t4)^2+ b4*t4+c4==y4,
   a5*(t4)^2+ b5*t4+c5==y4,
   a5*(t5)^2+ b5*t5+c5==y5,
   a6*(t5)^2+ b6*t5+c6==y5,
   a6*(t6)^2+ b6*t6+c6==y6,
   *)

   (*Condiciones para la continuidad de la derivada de la spline*)

   2 * a1 * t1 + b1 == 2 * a2 * t1 + b2,
   2 * a2 * t2 + b2 == 2 * a3 * t2 + b3,
   (*
   2*a3*t3+ b3==2*a4*t3+ b4,
   2*a4*t4+ b4==2*a5*t4+ b5,
   2*a5*t5+ b5==2*a6*t5+ b6,
   *)

   (*Linearización de la primera porción de la spline*)
   a1 == 0
 },
 {a1, b1, c1, a2, b2, c2, a3, b3, c3(*,a4,b4,c4,a5,b5,c5,a6,b6,c6*)}
]

{{a1 → -2.84217 × 10^-16, b1 → 2.23333, c1 → 38.5, a2 → 0.0333333,
   b2 → 1.23333, c2 → 46., a3 → -0.242292, b3 → 12.2583, c3 → -64.25}}
```

Figura C.1: Programa escrito en Mathematica 7.0 para el cálculo de splines de segundo grado. Se presentan los resultados obtenidos con su ejecución para la definición de la spline $y_B(t)$, donde $a_1 = 0, b_1 = 2.2333, c_1 = 38.5, a_2 = 0.0333, b_2 = 1.2333, c_2 = 46, a_3 = -0.2423, b_3 = 12.2583, c_3 = -64.25$.

Apéndice D

Programa en Mathematica 7.0 para la simulación de un ciclo de marcha

Funciones

```
In[1]:= (*Matrices de transformaciones homogéneas*)

(*Traslaciones*)

(*En x*)
Tz1[x_] := {{1, 0, 0, x}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
(*En y*)
Tz2[y_] := {{1, 0, 0, 0}, {0, 1, 0, y}, {0, 0, 1, 0}, {0, 0, 0, 1}};
(*En z*)
Tz3[z_] := {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, z}, {0, 0, 0, 1}};


(*Rotaciones*)

(*En x*)
Tz4[θx_] :=
   {{1, 0, 0, 0}, {0, Cos[θx], -Sin[θx], 0}, {0, Sin[θx], Cos[θx], 0}, {0, 0, 0, 1}};
(*En y*)
Tz5[θy_] :=
   {{Cos[θy], 0, Sin[θy], 0}, {0, 1, 0, 0}, {-Sin[θy], 0, Cos[θy], 0}, {0, 0, 0, 1}};
(*En z*)
Tz6[θz_] :=
   {{Cos[θz], -Sin[θz], 0, 0}, {Sin[θz], Cos[θz], 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};


(*Matrices de Rotación Vectorial*)

(*Rotación en x*)
Rx[γ_] := {{1, 0, 0}, {0, Cos[γ], -Sin[γ]}, {0, Sin[γ], Cos[γ]}};
(*Rotación en y*)
Ry[β_] := {{Cos[β], 0, Sin[β]}, {0, 1, 0}, {-Sin[β], 0, Cos[β]}};
(*Rotación en z*)
Rz[α_] := {{Cos[θz], -Sin[θz], 0}, {Sin[θz], Cos[θz], 0}, {0, 0, 1}};


(*Función para el trazo de gráficas a partir de tablas*)
Grafica[tabla_, X_, Y_, rojo_, verde_, azul_] := ListPlot[tabla,
    ImageSize → 800, BaseStyle → {24, FontFamily → "Century Gothic"},
    Frame → True, FrameLabel → {X, Y}, GridLines → Automatic, Joined → True,
    PlotStyle → {RGBColor[rojo, verde, azul], AbsoluteThickness[4]}];

(*Transforma vectores a 3D*)
T3D[P_] := {P[[1]], P[[2]], P[[3]]};

(*Función para graficar eslabones*)
Barra[espesor_, rojo_, verde_, azul_, linea_] :=
  Graphics3D[{AbsoluteThickness[espesor], RGBColor[rojo, verde, azul], linea}];

(*Función para graficar polígonos*)
Poligono[rojo_, verde_, azul_, poligono_] :=
  Graphics3D[{RGBColor[rojo, verde, azul], poligono}];
```

Datos

```mathematica
In[14]:= (*Dimensiones de los eslabones del bípedo*)
     esc = 1 / 10;
     (*Las medidas originales se reportan en [mm] y se modifican con el
      factor de escala, principalmente con propósitos de graficación.*)
     x01 = esc * 44.54;
     z11 = esc * 54.58;
     x21 = esc * 29.08;
     x31 = esc * 74.60;
     x41 = esc * 57.29;
     x51 = esc * 54.97;
     y61 = esc * 23.06;

     x02 = x01;
     z12 = z11;
     x22 = x21;
     x32 = x31;
     x42 = x41;
     x52 = x51;
     y62 = y61;

     (*Ángulos de rotación constantes para las articulaciones*)
     β11 = 90 °;
     β21 = 330 °;
     β31 = 60 °;
     β41 = 45 °;

     β12 = β11;
     β22 = 30 °;
     β32 = 300 °;
     β42 = 315 °;

     (*Coordenadas de los puntos para el trazo del pie derecho*)

     pd1x = esc * 44.33;
     pd1y = esc * 83.33;
     pd2x = esc * 15.7;
     pd2y = esc * 83.33;
     pd3x = esc * 15.7;
     pd3y = esc * 64.34;
     pd4x = esc * 3.03;
     pd4y = esc * 64.34;
     pd5x = esc * 3.03;
     pd5y = esc * 83.33;
```

```
pd6x = esc * - 26.18;
pd6y = esc * 83.33;
pd7x = esc * - 26.18;
pd7y = esc * 64.34;
pd8x = esc * - 28.18;
pd8y = esc * 64.81;
pd9x = esc * - 28.18;
pd9y = esc * - 24.20;
pd10x = esc * - 26.18;
pd10y = esc * - 24.70;
pd11x = esc * - 26.18;
pd11y = esc * - 43.70;
pd12x = esc * 3.03;
pd12y = esc * - 43.70;
pd13x = esc * 3.03;
pd13y = esc * - 24.7;
pd14x = esc * 15.7;
pd14y = esc * - 24.7;
pd15x = esc * 15.7;
pd15y = esc * - 43.7;
pd16x = esc * 44.33;
pd16y = esc * - 43.7;
pd17x = esc * 44.33;
pd17y = esc * - 24.7;
pd18x = esc * 46.33;
pd18y = esc * - 24.2;
pd19x = esc * 46.33;
pd19y = esc * 64.81;
pd20x = esc * 44.33;
pd20y = esc * 64.34;

(*Coordenadas de los puntos para el trazo del pie izquierdo*)
(*Se obtienen desplazando el sistema
 local del pie derecho 18.15[mm] sobre el eje i*)

pi1x = esc * (44.33 - 18.15);
pi1y = esc * 83.33;
pi2x = esc * (15.7 - 18.15);
pi2y = esc * 83.33;
pi3x = esc * (15.7 - 18.15);
pi3y = esc * 64.34;
pi4x = esc * (3.03 - 18.15);
pi4y = esc * 64.34;
pi5x = esc * (3.03 - 18.15);
pi5y = esc * 83.33;
pi6x = esc * (- 26.18 - 18.15);
```

```
pi6y = esc * 83.33;
pi7x = esc * (-26.18 - 18.15);
pi7y = esc * 64.34;
pi8x = esc * (-28.18 - 18.15);
pi8y = esc * 64.81;
pi9x = esc * (-28.18 - 18.15);
pi9y = esc * -24.20;
pi10x = esc * (-26.18 - 18.15);
pi10y = esc * -24.70;
pi11x = esc * (-26.18 - 18.15);
pi11y = esc * -43.70;
pi12x = esc * (3.03 - 18.15);
pi12y = esc * -43.70;
pi13x = esc * (3.03 - 18.15);
pi13y = esc * -24.7;
pi14x = esc * (15.7 - 18.15);
pi14y = esc * -24.7;
pi15x = esc * (15.7 - 18.15);
pi15y = esc * -43.7;
pi16x = esc * (44.33 - 18.15);
pi16y = esc * -43.7;
pi17x = esc * (44.33 - 18.15);
pi17y = esc * -24.7;
pi18x = esc * (46.33 - 18.15);
pi18y = esc * -24.2;
pi19x = esc * (46.33 - 18.15);
pi19y = esc * 64.81;
pi20x = esc * (44.33 - 18.15);
pi20y = esc * 64.34;
```

Ecuaciones de Posición

```
In[117]:=  (*Limpiar variables del bípedo*)
           Clear[θ11, θ21, θ31, θ41, θ51, θ61, θ12, θ22, θ32, θ42, θ52, θ62];

           (*Limpiar parámetros del bípedo*)
           Clear[xB, yB, zB, ψB, θB, φB, x1, y1, z1, ψ1, θ1, φ1, x2, y2, z2, ψ2, θ2, φ2]

           (*Transformaciones homogéneas aplicadas al bípedo*)

           (*Pierna Izquierda*)

           T0B = Tz1[xB].Tz2[yB].Tz3[zB].Tz4[θB].Tz5[φB].Tz6[ψB];
           TB11 = Tz1[- x01].Tz6[θ11 + 180 °];
           T1121 = Tz3[-z11].Tz5[θ21 + β11];
           T2131 = Tz1[x21].Tz6[θ31 + β21];
           T3141 = Tz1[x31].Tz6[θ41 + β31];
           T4151 = Tz1[x41].Tz6[θ51 + β41];
           T5161 = Tz1[x51].Tz4[θ61 + 180 °];
           T611 = Tz2[y61].Tz6[75 °].Tz5[270 °];
           T01 = Tz1[-x1].Tz2[y1].Tz3[z1].Tz4[θ1].Tz5[φ1].Tz6[ψ1];

           PosI1 = T0B.TB11.T1121.T2131.T3141.T4151.T5161.T611;
           PosI2 = T01;

           (*Pierna Derecha*)

           TB12 = Tz1[x02].Tz6[θ12];
           T1222 = Tz3[-z12].Tz5[θ22 + β12];
           T2232 = Tz1[x22].Tz6[θ32 + β22];
           T3242 = Tz1[x32].Tz6[θ42 + β32];
           T4252 = Tz1[x42].Tz6[θ52 + β42];
           T5262 = Tz1[x52].Tz4[θ62];
           T622 = Tz2[y62].Tz6[75 °].Tz5[270 °];
           T02 = Tz1[ x2].Tz2[y2].Tz3[z2].Tz4[θ2].Tz5[φ2].Tz6[ψ2];

           PosD1 = T0B.TB12.T1222.T2232.T3242.T4252.T5262.T622;
           PosD2 = T02;
```

Solución de la Posición Inicial

```
In[140]:=  (*Limpiar variables de las juntas*)
       Clear[θ11, θ21, θ31, θ41, θ51, θ61, θ12, θ22, θ32, θ42, θ52, θ62];

       (*Proponer valores para las posiciones de los puntos B, 1 y 2*)

       (*Posiciones deseadas para el torso del bípedo B*)
       (*Para posición erecta se deben dar los valores: xB=esc*0;
       yB=esc*38.5; zB=esc*234.37; ψB= 0 °; θB=0 °; φB=0 °;*)
       xB = esc * 0;
       yB = esc * 38.5;
       zB = esc * 234.37;
       θB = 0 °;
       φB = 0 °;
       ψB = 0 °;

       (*Posiciones deseadas para el pie izquierdo del bípedo 1*)
       (*Para posición erecta se deben dar los valores: x1=esc*44.54;
       y1=z1=ψ1=θ1=φ1=0;*)
       x1 = esc * 44.54;
       y1 = esc * 0;
       z1 = esc * 0;
       θ1 = 0 °;
       φ1 = 0 °;
       ψ1 = 0 °;

       (*Posiciones deseadas para el pie derecho del bípedo 2*)
       (*Para posición erecta se deben dar los valores: x2=esc*44.54;
       y2=z2=ψ2=θ2=φ2=0;*)
       x2 = esc * 44.54;
       y2 = esc * 0;
       z2 = esc * 0;
       θ2 = 0 °;
       φ2 = 0 °;
       ψ2 = 0 °;

       (*Ejecución del método numérico seleccionado
        para obtener las posiciones iniciales del mecanismo*)

       SolInicialI = FindRoot[{
           PosI1[[1, 4]] == PosI2[[1, 4]],
           PosI1[[2, 4]] == PosI2[[2, 4]],
           PosI1[[3, 4]] == PosI2[[3, 4]],
```

```
        PosI1[[1, 1]] == PosI2[[1, 1]],
        PosI1[[2, 2]] == PosI2[[2, 2]],
        PosI1[[3, 3]] == PosI2[[3, 3]]
       },
       {θ11, 0 °},
       {θ21, 0 °},
       {θ31, 0 °},
       {θ41, 0 °},
       {θ51, 0 °},
       {θ61, 0 °},
       MaxIterations → 30];

    SolInicialD = FindRoot[{
        PosD1[[1, 4]] == PosD2[[1, 4]],
        PosD1[[2, 4]] == PosD2[[2, 4]],
        PosD1[[3, 4]] == PosD2[[3, 4]],

        PosD1[[1, 1]] == PosD2[[1, 1]],
        PosD1[[2, 2]] == PosD2[[2, 2]],
        PosD1[[3, 3]] == PosD2[[3, 3]]
       },
       {θ12, 0 °},
       {θ22, 0 °},
       {θ32, 0 °},
       {θ42, 0 °},
       {θ52, 0 °},
       {θ62, 0 °},
       MaxIterations → 30];

    ({θ11, θ21, θ31, θ41, θ51, θ61} /. SolInicialI) / °
    ({θ12, θ22, θ32, θ42, θ52, θ62} /. SolInicialD) / °
```

Out[161]= {0., 0., 0.00279586, 0.0240831, -0.0268796, 0.}

Out[162]= {0., 0., -0.00279586, -0.0240831, 0.0268796, 0.}

Visualizacion de la Solución Inicial

```
In[163]:= (*Definición del origen*)
    cero = {0, 0, 0};

    (*Vector para la extracción de la posición
     a partir de las matrices de transformación*)
```

```
n = {0, 0, 0, 1};

(*Transformación para el torso*)

T0B = Tz1[xB].Tz2[yB].Tz3[zB].Tz4[θB].Tz5[ϕB].Tz6[ψB];

(*Transformaciones para la pierna izquierda*)

TB11 = Tz1[- x01].Tz6[θ11 + 180 °];
T1121 = Tz3[-z11].Tz5[θ21 + β11];
T2131 = Tz1[x21].Tz6[θ31 + β21];
T3141 = Tz1[x31].Tz6[θ41 + β31];
T4151 = Tz1[x41].Tz6[θ51 + β41];
T5161 = Tz1[x51].Tz4[θ61 + 180 °];
T611 = Tz2[y61].Tz6[75 °].Tz5[270 °];
T01 = Tz1[-x1].Tz2[y1].Tz3[z1].Tz4[θ1].Tz5[ϕ1].Tz6[ψ1];

(*Transformaciones para la pierna derecha*)

TB12 = Tz1[x02].Tz6[θ12];
T1222 = Tz3[- z12].Tz5[θ22 + β12];
T2232 = Tz1[x22].Tz6[θ32 + β22];
T3242 = Tz1[x32].Tz6[θ42 + β32];
T4252 = Tz1[x42].Tz6[θ52 + β42];
T5262 = Tz1[x52].Tz4[θ62];
T622 = Tz2[y62].Tz6[75 °].Tz5[270 °];
T02 = Tz1[ x2].Tz2[y2].Tz3[z2].Tz4[θ2].Tz5[ϕ2].Tz6[ψ2];

(*Puntos P para la graficación de la solución
 inicial a partir de las matrices de transformación*)

(*Torso*)

PB = T3D[T0B.n];

(*Pierna Izquierda*)

P11 = T3D[T0B.TB11.n];
P21 = T3D[T0B.TB11.T1121.n];
P31 = T3D[T0B.TB11.T1121.T2131.n];
P41 = T3D[T0B.TB11.T1121.T2131.T3141.n];
P51 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.n];
P61 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.n];
P1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.T611.n];

(*Pierna Derecha*)
```

```
P12 = T3D[T0B.TB12.n];
P22 = T3D[T0B.TB12.T1222.n];
P32 = T3D[T0B.TB12.T1222.T2232.n];
P42 = T3D[T0B.TB12.T1222.T2232.T3242.n];
P52 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.n];
P62 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.n];
P2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.T622.n];

(*Pie Izquierdo*)

PI1 = T3D[T01.Tz1[pi1x].Tz2[pi1y].n];
PI2 = T3D[T01.Tz1[pi2x].Tz2[pi2y].n];
PI3 = T3D[T01.Tz1[pi3x].Tz2[pi3y].n];
PI4 = T3D[T01.Tz1[pi4x].Tz2[pi4y].n];
PI5 = T3D[T01.Tz1[pi5x].Tz2[pi5y].n];
PI6 = T3D[T01.Tz1[pi6x].Tz2[pi6y].n];
PI7 = T3D[T01.Tz1[pi7x].Tz2[pi7y].n];
PI8 = T3D[T01.Tz1[pi8x].Tz2[pi8y].n];
PI9 = T3D[T01.Tz1[pi9x].Tz2[pi9y].n];
PI10 = T3D[T01.Tz1[pi10x].Tz2[pi10y].n];
PI11 = T3D[T01.Tz1[pi11x].Tz2[pi11y].n];
PI12 = T3D[T01.Tz1[pi12x].Tz2[pi12y].n];
PI13 = T3D[T01.Tz1[pi13x].Tz2[pi13y].n];
PI14 = T3D[T01.Tz1[pi14x].Tz2[pi14y].n];
PI15 = T3D[T01.Tz1[pi15x].Tz2[pi15y].n];
PI16 = T3D[T01.Tz1[pi16x].Tz2[pi16y].n];
PI17 = T3D[T01.Tz1[pi17x].Tz2[pi17y].n];
PI18 = T3D[T01.Tz1[pi18x].Tz2[pi18y].n];
PI19 = T3D[T01.Tz1[pi19x].Tz2[pi19y].n];
PI20 = T3D[T01.Tz1[pi20x].Tz2[pi20y].n];

(*Pie Derecho*)

PD1 = T3D[T02.Tz1[pd1x].Tz2[pd1y].n];
PD2 = T3D[T02.Tz1[pd2x].Tz2[pd2y].n];
PD3 = T3D[T02.Tz1[pd3x].Tz2[pd3y].n];
PD4 = T3D[T02.Tz1[pd4x].Tz2[pd4y].n];
PD5 = T3D[T02.Tz1[pd5x].Tz2[pd5y].n];
PD6 = T3D[T02.Tz1[pd6x].Tz2[pd6y].n];
PD7 = T3D[T02.Tz1[pd7x].Tz2[pd7y].n];
PD8 = T3D[T02.Tz1[pd8x].Tz2[pd8y].n];
PD9 = T3D[T02.Tz1[pd9x].Tz2[pd9y].n];
PD10 = T3D[T02.Tz1[pd10x].Tz2[pd10y].n];
PD11 = T3D[T02.Tz1[pd11x].Tz2[pd11y].n];
PD12 = T3D[T02.Tz1[pd12x].Tz2[pd12y].n];
```

```
PD13 = T3D[T02.Tz1[pd13x].Tz2[pd13y].n];
PD14 = T3D[T02.Tz1[pd14x].Tz2[pd14y].n];
PD15 = T3D[T02.Tz1[pd15x].Tz2[pd15y].n];
PD16 = T3D[T02.Tz1[pd16x].Tz2[pd16y].n];
PD17 = T3D[T02.Tz1[pd17x].Tz2[pd17y].n];
PD18 = T3D[T02.Tz1[pd18x].Tz2[pd18y].n];
PD19 = T3D[T02.Tz1[pd19x].Tz2[pd19y].n];
PD20 = T3D[T02.Tz1[pd20x].Tz2[pd20y].n];

(*Articulaciones pierna izquierda*)

P11A1 = T3D[T0B.TB11.Tz3[esc * 3].n];
P11A2 = T3D[T0B.TB11.Tz3[- esc * 3].n];
P21A1 = T3D[T0B.TB11.T1121.Tz2[esc * 3].n];
P21A2 = T3D[T0B.TB11.T1121.Tz2[- esc * 3].n];
P31A1 = T3D[T0B.TB11.T1121.T2131.Tz3[esc * 3].n];
P31A2 = T3D[T0B.TB11.T1121.T2131.Tz3[-esc * 3].n];
P41A1 = T3D[T0B.TB11.T1121.T2131.T3141.Tz3[esc * 3].n];
P41A2 = T3D[T0B.TB11.T1121.T2131.T3141.Tz3[- esc * 3].n];
P51A1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.Tz3[esc * 3].n];
P51A2 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.Tz3[- esc * 3].n];
P61A1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.Tz1[esc * 3].n];
P61A2 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.Tz1[- esc * 3].n];

(*Articulaciones pierna derecha*)

P12A1 = T3D[T0B.TB12.Tz3[esc * 3].n];
P12A2 = T3D[T0B.TB12.Tz3[- esc * 3].n];
P22A1 = T3D[T0B.TB12.T1222.Tz2[esc * 3].n];
P22A2 = T3D[T0B.TB12.T1222.Tz2[- esc * 3].n];
P32A1 = T3D[T0B.TB12.T1222.T2232.Tz3[esc * 3].n];
P32A2 = T3D[T0B.TB12.T1222.T2232.Tz3[- esc * 3].n];
P42A1 = T3D[T0B.TB12.T1222.T2232.T3242.Tz3[esc * 3].n];
P42A2 = T3D[T0B.TB12.T1222.T2232.T3242.Tz3[- esc * 3].n];
P52A1 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.Tz3[esc * 3].n];
P52A2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.Tz3[- esc * 3].n];
P62A1 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.Tz1[esc * 3].n];
P62A2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.Tz1[- esc * 3].n];

(*Líneas trazadas de acuerdo a los puntos anteriores*)

(*Pierna Izquierda*)

Linea0I = Line[{PB, P11}] /. SolInicialI;
Linea1I = Line[{P11, P21}] /. SolInicialI;
Linea2I = Line[{P21, P31}] /. SolInicialI;
```

```
Linea3I = Line[{P31, P41}] /. SolInicialI;
Linea4I = Line[{P41, P51}] /. SolInicialI;
Linea5I = Line[{P51, P61}] /. SolInicialI;
Linea6I = Line[{P61, P1}] /. SolInicialI;

(*Pierna Derecha*)

Linea0D = Line[{PB, P12}] /. SolInicialD;
Linea1D = Line[{P12, P22}] /. SolInicialD;
Linea2D = Line[{P22, P32}] /. SolInicialD;
Linea3D = Line[{P32, P42}] /. SolInicialD;
Linea4D = Line[{P42, P52}] /. SolInicialD;
Linea5D = Line[{P52, P62}] /. SolInicialD;
Linea6D = Line[{P62, P2}] /. SolInicialD;

(*Articulaciones pierna izquierda*)

Cilindro11 = Cylinder[{P11A1, P11A2}, 0.5] /. SolInicialI;
Cilindro21 = Cylinder[{P21A1, P21A2}, 0.5] /. SolInicialI;
Cilindro31 = Cylinder[{P31A1, P31A2}, 0.5] /. SolInicialI;
Cilindro41 = Cylinder[{P41A1, P41A2}, 0.5] /. SolInicialI;
Cilindro51 = Cylinder[{P51A1, P51A2}, 0.5] /. SolInicialI;
Cilindro61 = Cylinder[{P61A1, P61A2}, 0.5] /. SolInicialI;

(*Articulaciones pierna derecha*)

Cilindro12 = Cylinder[{P12A1, P12A2}, 0.5] /. SolInicialD;
Cilindro22 = Cylinder[{P22A1, P22A2}, 0.5] /. SolInicialD;
Cilindro32 = Cylinder[{P32A1, P32A2}, 0.5] /. SolInicialD;
Cilindro42 = Cylinder[{P42A1, P42A2}, 0.5] /. SolInicialD;
Cilindro52 = Cylinder[{P52A1, P52A2}, 0.5] /. SolInicialD;
Cilindro62 = Cylinder[{P62A1, P62A2}, 0.5] /. SolInicialD;

(*Barras para el dibujo de los eslabones*)

Barra0I = Barra[3, 0, 0, 1, Linea0I];
Barra1I = Barra[3, 0, 1, 0, Linea1I];
Barra2I = Barra[3, 1, 0, 0, Linea2I];
Barra3I = Barra[3, 0, 1, 0, Linea3I];
Barra4I = Barra[3, 0, 0, 1, Linea4I];
Barra5I = Barra[3, 0, 1, 0, Linea5I];
Barra6I = Barra[3, 1, 0, 0, Linea6I];

Barra0D = Barra[3, 0, 0, 1, Linea0D];
Barra1D = Barra[3, 0, 1, 0, Linea1D];
Barra2D = Barra[3, 1, 0, 0, Linea2D];
```

```
Barra3D = Barra[3, 0, 1, 0, Linea3D];
Barra4D = Barra[3, 0, 0, 1, Linea4D];
Barra5D = Barra[3, 0, 1, 0, Linea5D];
Barra6D = Barra[3, 1, 0, 0, Linea6D];

Articulacion11 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro11];
Articulacion21 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro21];
Articulacion31 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro31];
Articulacion41 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro41];
Articulacion51 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro51];
Articulacion61 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro61];

Articulacion12 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro12];
Articulacion22 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro22];
Articulacion32 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro32];
Articulacion42 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro42];
Articulacion52 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro52];
Articulacion62 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro62];

(*Polígonos para el trazo de los pies*)

PoligonoI = Polygon[{PI1, PI2, PI3, PI4, PI5, PI6, PI7, PI8, PI9,
     PI10, PI11, PI12, PI13, PI14, PI15, PI16, PI17, PI18, PI19, PI20}];
PoligonoD = Polygon[{PD1, PD2, PD3, PD4, PD5, PD6, PD7, PD8, PD9, PD10,
     PD11, PD12, PD13, PD14, PD15, PD16, PD17, PD18, PD19, PD20}];

PieI = Poligono[1, 0, 0, PoligonoI];
PieD = Poligono[1, 0, 0, PoligonoD];

Show[Barra0I, Barra1I, Barra2I, Barra3I, Barra4I, Barra5I, Barra6I, PieI, Barra0D,
 Barra1D, Barra2D, Barra3D, Barra4D, Barra5D, Barra6D, PieD, Articulacion11,
 Articulacion21, Articulacion31, Articulacion41, Articulacion51, Articulacion61,
 Articulacion12, Articulacion22, Articulacion32, Articulacion42,
 Articulacion52, Articulacion62, Axes → True, AxesLabel → {"X", "Y", "Z"},
 BaseStyle → {12, FontFamily -> "Arial"}, ImageSize → 300, PlotRange → All]
```

Out[317]=
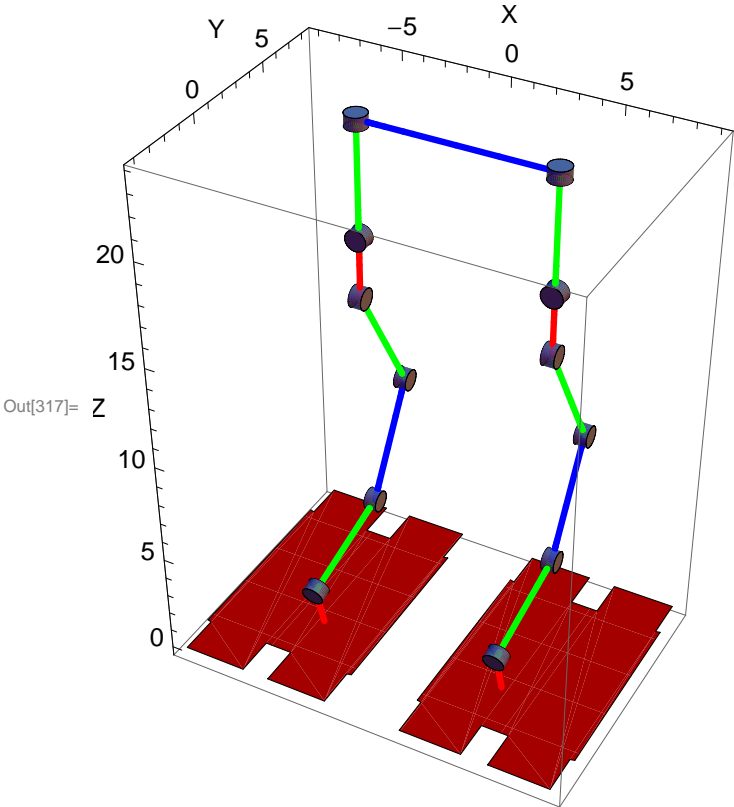
Solución total de la posición

```
In[318]:= (*Limpiar variables de las juntas*)
      Clear[θ11, θ21, θ31, θ41, θ51, θ61, θ12, θ22, θ32, θ42, θ52, θ62];
      Clear[xB, yB, zB, θB, ϕB, ψB, x1, y1, z1, θ1, ϕ1, ψ1, x2, y2, z2, θ2, ϕ2, ψ2, i];

      θ11i = θ11 /. SolInicialI;
      θ21i = θ21 /. SolInicialI;
      θ31i = θ31 /. SolInicialI;
      θ41i = θ41 /. SolInicialI;
      θ51i = θ51 /. SolInicialI;
      θ61i = θ61 /. SolInicialI;

      θ12i = θ12 /. SolInicialD;
      θ22i = θ22 /. SolInicialD;
      θ32i = θ32 /. SolInicialD;
      θ42i = θ42 /. SolInicialD;
      θ52i = θ52 /. SolInicialD;
      θ62i = θ62 /. SolInicialD;

      (*Valores constantes para las posiciones del torso del bípedo B*)
      xB = esc * 0;
      zB = esc * 234.37;
      θB = 0 °;
      ϕB = 0 °;
      ψB = 0 °;

      (*Valores constantes para las posiciones del pie izquierdo del bípedo 1*)
      x1 = esc * 44.54;
      θ1 = 0 °;
      ϕ1 = 0 °;
      ψ1 = 0 °;

      (*Valores constantes para las posiciones del pie derecho del bípedo 2*)
      x2 = esc * 44.54;
      θ2 = 0 °;
      ϕ2 = 0 °;
      ψ2 = 0 °;


      For[i = 0, i ≤ 40, i += 1,

         (*Calcular valores para las posiciones de los puntos B,
         1 y 2 de acuerdo a las splines de segundo grado*)
```

```
If[i ≥  0 && i < 15, yB = esc * (2.2333 * i + 38.5),
 If[i ≥  15 && i < 20, yB = esc * (0.0333 * (i^2) + 1.2333 * i + 46),
  If[i ≥  20 && i < 25, yB = esc * (-0.2427 * (i^2) + 12.275 * i - 64.4167),
   yB = esc * 90.76]]];

If[i ≥  0 && i < 20, y1 = 0,
 If[i ≥  20 && i < 35, y1 = esc * (1.9333 * i - 38.6667),
  If[i ≥  35 && i < 38, y1 = esc * (1.1333 * (i^2) - 77.4 * i + 1349.67),
   If[i ≥  38 && i < 40,
    y1 = esc * (-2.5517 * (i^2) + 202.66 * i - 3971.47), y1 = esc * 52.26]]]];

If[i ≥  0 && i < 20, z1 = 0,
 If[i ≥  20 && i < 25, z1 = esc * (4 * i - 80),
  If[i ≥  25 && i < 35, z1 = esc * (-(2 / 5) * (i^2) + 24 * i - 330),
   If[i ≥  35 && i < 40, z1 = esc * (-4 * i + 160), z1 = 0]]]];

If[i ≥  0 && i < 15, y2 = esc * (1.9333 * i),
 If[i ≥  15 && i < 18, y2 = esc * (1.1333 * (i^2) - 32.0667 * i + 255),
  If[i ≥  18 && i < 20,
   y2 = esc * (-2.5517 * (i^2) + 100.593 * i - 938.94), y2 = esc * 52.26]]];

If[i ≥  0 && i < 5, z2 = esc * (4 * i),
 If[i ≥  5 && i < 15, z2 = esc * (-(2 / 5) * (i^2) + 8 * i - 10),
  If[i ≥  15 && i < 20, z2 = esc * (-4 * i + 80), z2 = 0]]];

(*Ejecución del método numérico seleccionado
 para obtener las posiciones angulares del bípedo*)

SolPosI[i] = FindRoot[{
    PosI1[[1, 4]] == PosI2[[1, 4]],
    PosI1[[2, 4]] == PosI2[[2, 4]],
    PosI1[[3, 4]] == PosI2[[3, 4]],

    PosI1[[1, 1]] == PosI2[[1, 1]],
    PosI1[[2, 2]] == PosI2[[2, 2]],
    PosI1[[3, 3]] == PosI2[[3, 3]]
   },
   {θ11, θ11i},
   {θ21, θ21i},
   {θ31, θ31i},
   {θ41, θ41i},
   {θ51, θ51i},
   {θ61, θ61i},
   MaxIterations → 30];
θ11i = θ11 /. SolPosI[i];
θ21i = θ21 /. SolPosI[i];
```

```
      θ31i = θ31 /. SolPosI[i];
      θ41i = θ41 /. SolPosI[i];
      θ51i = θ51 /. SolPosI[i];
      θ61i = θ61 /. SolPosI[i];

      SolPosD[i] = FindRoot[{
          PosD1[[1, 4]] == PosD2[[1, 4]],
          PosD1[[2, 4]] == PosD2[[2, 4]],
          PosD1[[3, 4]] == PosD2[[3, 4]],

          PosD1[[1, 1]] == PosD2[[1, 1]],
          PosD1[[2, 2]] == PosD2[[2, 2]],
          PosD1[[3, 3]] == PosD2[[3, 3]]
          },
          {θ12, θ12i},
          {θ22, θ22i},
          {θ32, θ32i},
          {θ42, θ42i},
          {θ52, θ52i},
          {θ62, θ62i},
          MaxIterations → 30];
       θ12i = θ12 /. SolPosD[i];
       θ22i = θ22 /. SolPosD[i];
       θ32i = θ32 /. SolPosD[i];
       θ42i = θ42 /. SolPosD[i];
       θ52i = θ52 /. SolPosD[i];
       θ62i = θ62 /. SolPosD[i];
      ];
```

Graficación de las Posiciones Angulares

```
In[346]:= (*Limpiar contadores*)
      Clear[i];
      (*Construcción de las tablas de
       resultados para la graficación de la posición*)

      tabla1 = Table[{i, (θ11 /. SolPosI[i]) / °}, {i, 0, 40, 1}]
      tabla2 = Table[{i, (θ21 /. SolPosI[i]) / °}, {i, 0, 40, 1}]
      tabla3 = Table[{i, (θ31 /. SolPosI[i]) / °}, {i, 0, 40, 1}]
      tabla4 = Table[{i, (θ41 /. SolPosI[i]) / °}, {i, 0, 40, 1}]
      tabla5 = Table[{i, (θ51 /. SolPosI[i]) / °}, {i, 0, 40, 1}]
      tabla6 = Table[{i, (θ61 /. SolPosI[i]) / °}, {i, 0, 40, 1}]

      tabla7 = Table[{i, (θ12 /. SolPosD[i]) / °}, {i, 0, 40, 1}]
```

```mathematica
tabla8 = Table[{i, (θ22 /. SolPosD[i]) / °}, {i, 0, 40, 1}]
tabla9 = Table[{i, (θ32 /. SolPosD[i]) / °}, {i, 0, 40, 1}]
tabla10 = Table[{i, (θ42 /. SolPosD[i]) / °}, {i, 0, 40, 1}]
tabla11 = Table[{i, (θ52 /. SolPosD[i]) / °}, {i, 0, 40, 1}]
tabla12 = Table[{i, (θ62 /. SolPosD[i]) / °}, {i, 0, 40, 1}]


(*Construcción en memoria de los objetos gráficos a desplegar*)

fig1 = Grafica[tabla1, "t", "θ11[°]", 0.976471, 0.486275, 0];
fig2 = Grafica[tabla2, "t", "θ21[°]", 0.0196078, 0.313725, 0.603922];
fig3 = Grafica[tabla3, "t", "θ31[°]", 1, 0, 0];
fig4 = Grafica[tabla4, "t", "θ41[°]", 0, 1, 0];
fig5 = Grafica[tabla5, "t", "θ51[°]", 0, 0, 1];
fig6 = Grafica[tabla6, "t", "θ61[°]", 0.0196078, 0.313725, 0.603922];


fig7 = Grafica[tabla7, "t", "θ12[°]", 0.976471, 0.486275, 0];
fig8 = Grafica[tabla8, "t", "θ22[°]", 0.0196078, 0.313725, 0.603922];
fig9 = Grafica[tabla9, "t", "θ32[°]", 1, 0, 0];
fig10 = Grafica[tabla10, "t", "θ42[°]", 0, 1, 0];
fig11 = Grafica[tabla11, "t", "θ52[°]", 0, 0, 1];
fig12 = Grafica[tabla12, "t", "θ62[°]", 0.0196078, 0.313725, 0.603922];


Show[fig1, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig2, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig3, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig4, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig5, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig6, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]


Show[fig7, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig8, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig9, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig10, ImageSize → 300, Frame → True, GridLines → Automatic,
 BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
Show[fig11, ImageSize → 300, Frame → True, GridLines → Automatic,
```

```
        BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
       Show[fig12, ImageSize → 300, Frame → True, GridLines → Automatic,
        BaseStyle → {12, FontFamily → "Arial"}, PlotRange → Automatic]
```

Out[347]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
         {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
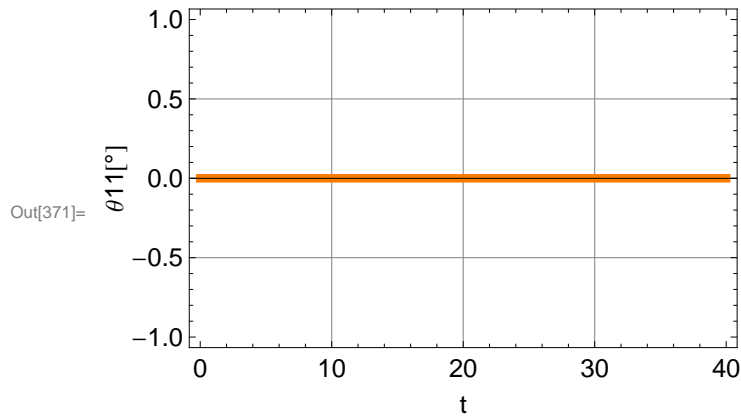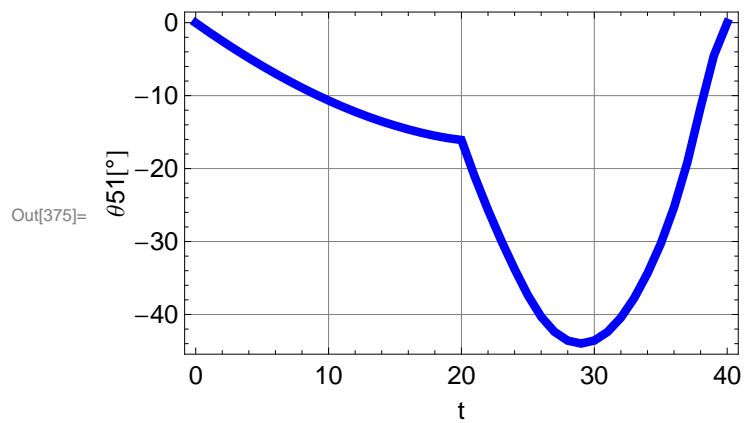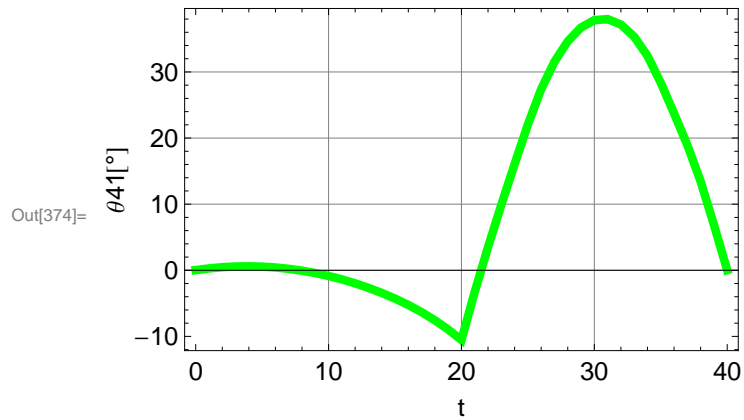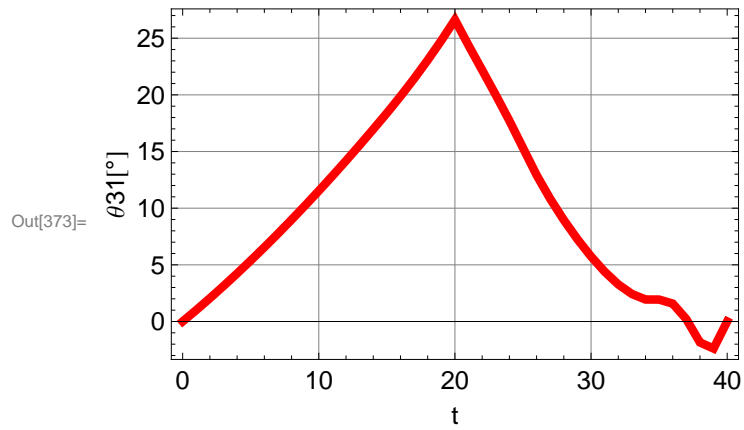   {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
         {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
         {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[348]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
         {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
         {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
         {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
         {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[349]= {{0, 0.00279586}, {1, 1.01127}, {2, 2.05422}, {3, 3.13057}, {4, 4.23936},
         {5, 5.3797}, {6, 6.55079}, {7, 7.75192}, {8, 8.98249}, {9, 10.242}, {10, 11.5301},
         {11, 12.8464}, {12, 14.191}, {13, 15.5638}, {14, 16.9651}, {15, 18.3906},
         {16, 19.8717}, {17, 21.4294}, {18, 23.0681}, {19, 24.7932}, {20, 26.6244},
         {21, 24.3652}, {22, 22.1851}, {23, 19.9914}, {24, 17.7202}, {25, 15.3139},
         {26, 12.9208}, {27, 10.8259}, {28, 8.95369}, {29, 7.2622}, {30, 5.7383},
         {31, 4.39555}, {32, 3.27204}, {33, 2.42777}, {34, 1.94177}, {35, 1.93484},
         {36, 1.5788}, {37, 0.207796}, {38, -1.84708}, {39, -2.38525}, {40, 0.00279622}}

Out[350]= {{0, 0.0240831}, {1, 0.283382}, {2, 0.465191}, {3, 0.569932},
         {4, 0.597842}, {5, 0.548986}, {6, 0.423252}, {7, 0.220354}, {8, -0.060184},
         {9, -0.419027}, {10, -0.857053}, {11, -1.37537}, {12, -1.97536},
         {13, -2.65868}, {14, -3.42735}, {15, -4.28074}, {16, -5.24186},
         {17, -6.33246}, {18, -7.56627}, {19, -8.95933}, {20, -10.5423},
         {21, -3.25609}, {22, 3.49443}, {23, 9.89451}, {24, 16.0559}, {25, 22.0542},
         {26, 27.3939}, {27, 31.5488}, {28, 34.6362}, {29, 36.7223}, {30, 37.8388},
         {31, 37.9924}, {32, 37.169}, {33, 35.335}, {34, 32.4338}, {35, 28.3702},
         {36, 23.7385}, {37, 18.9477}, {38, 13.4509}, {39, 6.89032}, {40, 0.0240815}}

Out[351]= {{0, -0.0268796}, {1, -1.29466}, {2, -2.51941}, {3, -3.7005}, {4, -4.8372},
         {5, -5.92869}, {6, -6.97404}, {7, -7.97228}, {8, -8.92231}, {9, -9.82297},
         {10, -10.673}, {11, -11.471}, {12, -12.2156}, {13, -12.9051}, {14, -13.5378},
         {15, -14.1098}, {16, -14.6299}, {17, -15.097}, {18, -15.5019}, {19, -15.8339},
         {20, -16.0821}, {21, -21.1091}, {22, -25.6795}, {23, -29.8859},
         {24, -33.7761}, {25, -37.3681}, {26, -40.3147}, {27, -42.3747},
         {28, -43.5899}, {29, -43.9845}, {30, -43.5771}, {31, -42.3879},
         {32, -40.4411}, {33, -37.7628}, {34, -34.3756}, {35, -30.305}, {36, -25.3173},
         {37, -19.1555}, {38, -11.6039}, {39, -4.50507}, {40, -0.0268774}}

Out[352]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
         {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
         {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
         {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
         {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[353]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
         {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
         {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
         {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
         {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[354]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
         {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
         {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
         {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
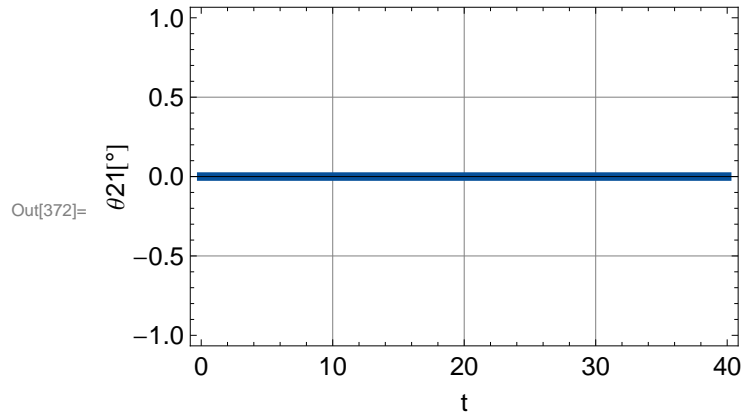   {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[355]= {{0, -0.00279586}, {1, 2.77358}, {2, 5.2807}, {3, 7.5745}, {4, 9.69094},
        {5, 11.654}, {6, 13.2852}, {7, 14.4397}, {8, 15.1706}, {9, 15.5146},
        {10, 15.4957}, {11, 15.1258}, {12, 14.4045}, {13, 13.3183}, {14, 11.8391},
        {15, 9.92028}, {16, 8.31776}, {17, 7.67758}, {18, 7.69896}, {19, 6.49842},
        {20, 2.82635}, {21, 1.89207}, {22, 1.12357}, {23, 0.541847}, {24, 0.161714},
        {25, -0.00279611}, {26, -0.00279619}, {27, -0.00279615}, {28, -0.00279615},
        {29, -0.00279608}, {30, -0.00279596}, {31, -0.002796}, {32, -0.002796},
        {33, -0.002796}, {34, -0.002796}, {35, -0.002796}, {36, -0.002796},
        {37, -0.002796}, {38, -0.002796}, {39, -0.002796}, {40, -0.002796}}

Out[356]= {{0, -0.0240831}, {1, -6.79502}, {2, -13.0119}, {3, -18.8086}, {4, -24.2742},
        {5, -29.4721}, {6, -33.9632}, {7, -37.3557}, {8, -39.7355}, {9, -41.1543},
        {10, -41.6398}, {11, -41.2005}, {12, -39.8274}, {13, -37.4927}, {14, -34.1448},
        {15, -29.6985}, {16, -24.4823}, {17, -18.7872}, {18, -12.2551}, {19, -5.22378},
        {20, 1.23961}, {21, 0.722669}, {22, 0.375104}, {23, 0.153564}, {24, 0.0266649},
        {25, -0.024082}, {26, -0.0240816}, {27, -0.0240818}, {28, -0.0240818},
        {29, -0.0240821}, {30, -0.0240826}, {31, -0.0240825}, {32, -0.0240825},
        {33, -0.0240825}, {34, -0.0240825}, {35, -0.0240825}, {36, -0.0240825},
        {37, -0.0240825}, {38, -0.0240825}, {39, -0.0240825}, {40, -0.0240825}}

Out[357]= {{0, 0.0268796}, {1, 4.02144}, {2, 7.73124}, {3, 11.2341}, {4, 14.5833},
        {5, 17.8181}, {6, 20.6781}, {7, 22.916}, {8, 24.565}, {9, 25.6397}, {10, 26.1441},
        {11, 26.0747}, {12, 25.4229}, {13, 24.1744}, {14, 22.3057}, {15, 19.7782},
        {16, 16.1645}, {17, 11.1096}, {18, 4.55615}, {19, -1.27464}, {20, -4.06596},
        {21, -2.61474}, {22, -1.49868}, {23, -0.695411}, {24, -0.188379},
        {25, 0.0268781}, {26, 0.0268776}, {27, 0.0268778}, {28, 0.0268778},
        {29, 0.0268782}, {30, 0.026879}, {31, 0.0268787}, {32, 0.0268787},
        {33, 0.0268787}, {34, 0.0268787}, {35, 0.0268787}, {36, 0.0268787},
        {37, 0.0268787}, {38, 0.0268787}, {39, 0.0268787}, {40, 0.0268787}}

Out[358]= {{0, 0.}, {1, 0.}, {2, 0.}, {3, 0.}, {4, 0.}, {5, 0.}, {6, 0.}, {7, 0.}, {8, 0.},
        {9, 0.}, {10, 0.}, {11, 0.}, {12, 0.}, {13, 0.}, {14, 0.}, {15, 0.}, {16, 0.},
        {17, 0.}, {18, 0.}, {19, 0.}, {20, 0.}, {21, 0.}, {22, 0.}, {23, 0.}, {24, 0.},
        {25, 0.}, {26, 0.}, {27, 0.}, {28, 0.}, {29, 0.}, {30, 0.}, {31, 0.}, {32, 0.},
        {33, 0.}, {34, 0.}, {35, 0.}, {36, 0.}, {37, 0.}, {38, 0.}, {39, 0.}, {40, 0.}}

Out[371]=

Out[372]=

Out[373]=

Out[374]=

Out[375]=

Out[376]=

Out[377]=

Out[378]=

Out[379]=

Out[380]=

Out[381]=

Out[382]=
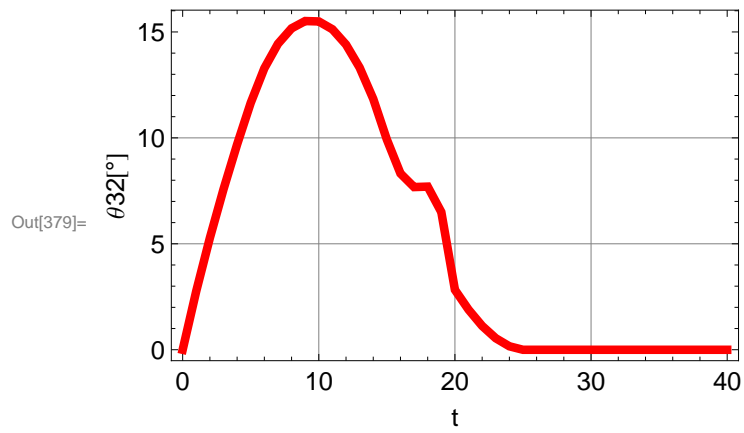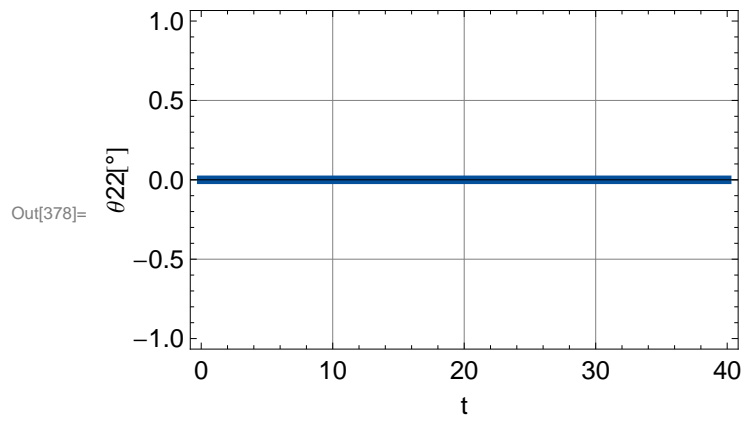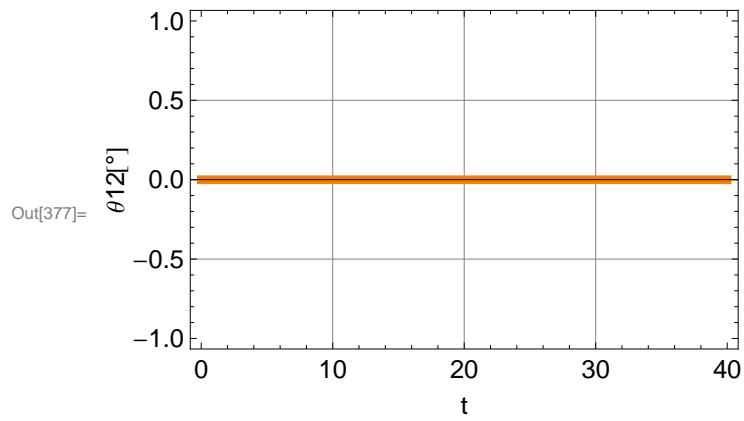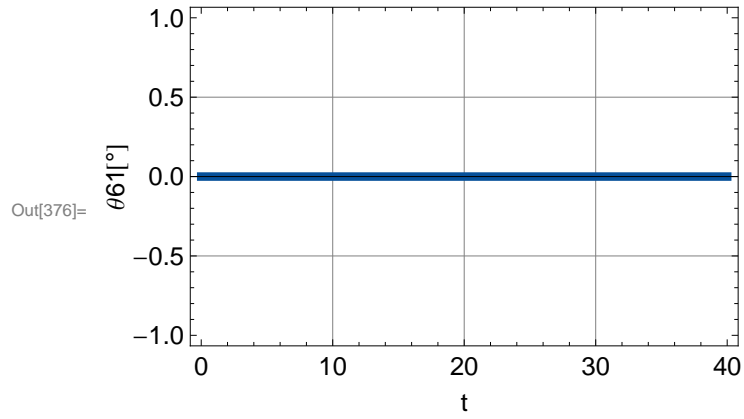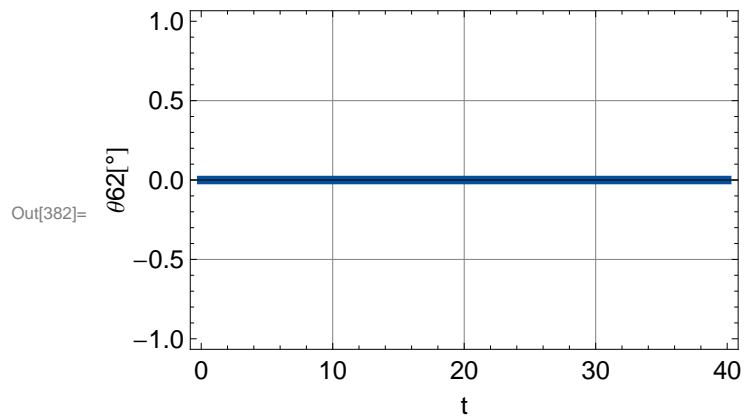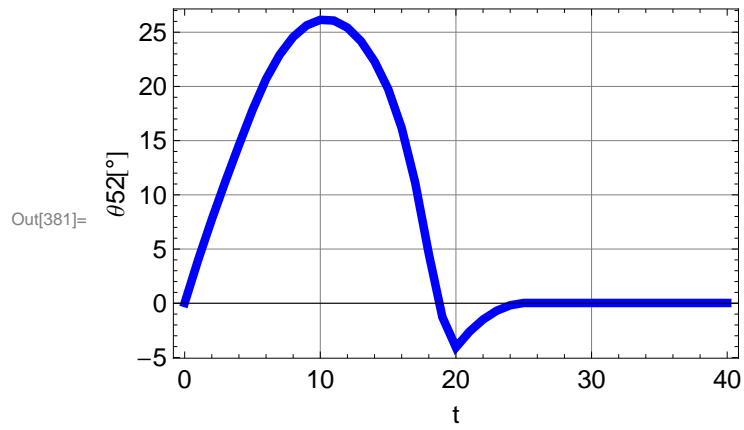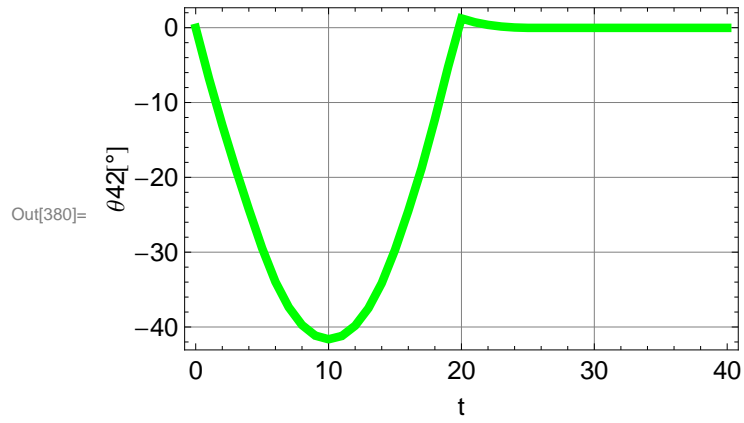
Simulación

In[383]:=

```
Clear[i];

(*Valores constantes para las posiciones del torso del bípedo B*)
xB = esc * 0;
zB = esc * 234.37;
θB = 0 °;
φB = 0 °;
ψB = 0 °;

(*Valores constantes para las posiciones del pie izquierdo del bípedo 1*)
x1 = esc * 44.54;
θ1 = 0 °;
φ1 = 0 °;
ψ1 = 0 °;

(*Valores constantes para las posiciones del pie derecho del bípedo 2*)
x2 = esc * 44.54;
θ2 = 0 °;
φ2 = 0 °;
ψ2 = 0 °;

Animate[
 (*Definición del origen*)
 cero = {0, 0, 0};

 (*Calcular valores para las posiciones de los puntos B,
 1 y 2 de acuerdo a las splines de segundo grado*)

 If[i ≥  0 && i < 15, yB = esc * (2.2333 * i + 38.5),
  If[i ≥  15 && i < 20, yB = esc * (0.0333 * (i^2) + 1.2333 * i + 46),
    If[i ≥  20 && i < 25, yB = esc * (-0.2427 * (i^2) + 12.275 * i - 64.4167),
     yB = esc * 90.76]]];

 If[i ≥  0 && i < 20, y1 = 0,
  If[i ≥  20 && i < 35, y1 = esc * (1.9333 * i - 38.6667),
    If[i ≥  35 && i < 38, y1 = esc * (1.1333 * (i^2) - 77.4 * i + 1349.67),
      If[i ≥  38 && i < 40,
       y1 = esc * (-2.5517 * (i^2) + 202.66 * i - 3971.47), y1 = esc * 52.26]]]];

 If[i ≥  0 && i < 20, z1 = 0,
  If[i ≥  20 && i < 25, z1 = esc * (4 * i - 80),
```

```
   If[i ≥  25 && i < 35, z1 = esc * (- (2 / 5) * (i^2) + 24 * i - 330),
    If[i ≥  35 && i < 40, z1 = esc * (- 4 * i + 160), z1 = 0]]]];

If[i ≥  0 && i < 15, y2 = esc * (1.9333 * i),
 If[i ≥  15 && i < 18, y2 = esc * (1.1333 * (i^2) - 32.0667 * i + 255),
  If[i ≥  18 && i < 20,
   y2 = esc * (- 2.5517 * (i^2) + 100.593 * i - 938.94), y2 = esc * 52.26]]];

If[i ≥  0 && i < 5, z2 = esc * (4 * i),
 If[i ≥  5 && i < 15, z2 = esc * (- (2 / 5) * (i^2) + 8 * i - 10),
  If[i ≥  15 && i < 20, z2 = esc * (- 4 * i + 80), z2 = 0]]];


(*Vector para la extracción de la
 posición a partir de las matrices de transformación*)
n = {0, 0, 0, 1};

(*Transformación para el torso*)

T0B = Tz1[xB].Tz2[yB].Tz3[zB].Tz4[θB].Tz5[ϕB].Tz6[ψB];

(*Transformaciones para la pierna izquierda*)

TB11 = Tz1[- x01].Tz6[θ11 + 180 °];
T1121 = Tz3[-z11].Tz5[θ21 + β11];
T2131 = Tz1[x21].Tz6[θ31 + β21];
T3141 = Tz1[x31].Tz6[θ41 + β31];
T4151 = Tz1[x41].Tz6[θ51 + β41];
T5161 = Tz1[x51].Tz4[θ61 + 180 °];
T611 = Tz2[y61].Tz6[75 °].Tz5[270 °];
T01 = Tz1[-x1].Tz2[y1].Tz3[z1].Tz4[θ1].Tz5[ϕ1].Tz6[ψ1];

(*Transformaciones para la pierna derecha*)

TB12 = Tz1[x02].Tz6[θ12];
T1222 = Tz3[-z12].Tz5[θ22 + β12];
T2232 = Tz1[x22].Tz6[θ32 + β22];
T3242 = Tz1[x32].Tz6[θ42 + β32];
T4252 = Tz1[x42].Tz6[θ52 + β42];
T5262 = Tz1[x52].Tz4[θ62];
T622 = Tz2[y62].Tz6[75 °].Tz5[270 °];
T02 = Tz1[ x2].Tz2[y2].Tz3[z2].Tz4[θ2].Tz5[ϕ2].Tz6[ψ2];

(*Puntos P para la graficación de la solución
 inicial a partir de las matrices de transformación*)
```

```
(*Torso*)

PB = T3D[T0B.n];

(*Pierna Izquierda*)

P11 = T3D[T0B.TB11.n];
P21 = T3D[T0B.TB11.T1121.n];
P31 = T3D[T0B.TB11.T1121.T2131.n];
P41 = T3D[T0B.TB11.T1121.T2131.T3141.n];
P51 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.n];
P61 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.n];
P1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.T611.n];

(*Pierna Derecha*)

P12 = T3D[T0B.TB12.n];
P22 = T3D[T0B.TB12.T1222.n];
P32 = T3D[T0B.TB12.T1222.T2232.n];
P42 = T3D[T0B.TB12.T1222.T2232.T3242.n];
P52 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.n];
P62 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.n];
P2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.T622.n];

(*Pie Izquierdo*)

PI1 = T3D[T01.Tz1[pi1x].Tz2[pi1y].n];
PI2 = T3D[T01.Tz1[pi2x].Tz2[pi2y].n];
PI3 = T3D[T01.Tz1[pi3x].Tz2[pi3y].n];
PI4 = T3D[T01.Tz1[pi4x].Tz2[pi4y].n];
PI5 = T3D[T01.Tz1[pi5x].Tz2[pi5y].n];
PI6 = T3D[T01.Tz1[pi6x].Tz2[pi6y].n];
PI7 = T3D[T01.Tz1[pi7x].Tz2[pi7y].n];
PI8 = T3D[T01.Tz1[pi8x].Tz2[pi8y].n];
PI9 = T3D[T01.Tz1[pi9x].Tz2[pi9y].n];
PI10 = T3D[T01.Tz1[pi10x].Tz2[pi10y].n];
PI11 = T3D[T01.Tz1[pi11x].Tz2[pi11y].n];
PI12 = T3D[T01.Tz1[pi12x].Tz2[pi12y].n];
PI13 = T3D[T01.Tz1[pi13x].Tz2[pi13y].n];
PI14 = T3D[T01.Tz1[pi14x].Tz2[pi14y].n];
PI15 = T3D[T01.Tz1[pi15x].Tz2[pi15y].n];
PI16 = T3D[T01.Tz1[pi16x].Tz2[pi16y].n];
PI17 = T3D[T01.Tz1[pi17x].Tz2[pi17y].n];
PI18 = T3D[T01.Tz1[pi18x].Tz2[pi18y].n];
PI19 = T3D[T01.Tz1[pi19x].Tz2[pi19y].n];
PI20 = T3D[T01.Tz1[pi20x].Tz2[pi20y].n];
```

```
(*Pie Derecho*)

PD1 = T3D[T02.Tz1[pd1x].Tz2[pd1y].n];
PD2 = T3D[T02.Tz1[pd2x].Tz2[pd2y].n];
PD3 = T3D[T02.Tz1[pd3x].Tz2[pd3y].n];
PD4 = T3D[T02.Tz1[pd4x].Tz2[pd4y].n];
PD5 = T3D[T02.Tz1[pd5x].Tz2[pd5y].n];
PD6 = T3D[T02.Tz1[pd6x].Tz2[pd6y].n];
PD7 = T3D[T02.Tz1[pd7x].Tz2[pd7y].n];
PD8 = T3D[T02.Tz1[pd8x].Tz2[pd8y].n];
PD9 = T3D[T02.Tz1[pd9x].Tz2[pd9y].n];
PD10 = T3D[T02.Tz1[pd10x].Tz2[pd10y].n];
PD11 = T3D[T02.Tz1[pd11x].Tz2[pd11y].n];
PD12 = T3D[T02.Tz1[pd12x].Tz2[pd12y].n];
PD13 = T3D[T02.Tz1[pd13x].Tz2[pd13y].n];
PD14 = T3D[T02.Tz1[pd14x].Tz2[pd14y].n];
PD15 = T3D[T02.Tz1[pd15x].Tz2[pd15y].n];
PD16 = T3D[T02.Tz1[pd16x].Tz2[pd16y].n];
PD17 = T3D[T02.Tz1[pd17x].Tz2[pd17y].n];
PD18 = T3D[T02.Tz1[pd18x].Tz2[pd18y].n];
PD19 = T3D[T02.Tz1[pd19x].Tz2[pd19y].n];
PD20 = T3D[T02.Tz1[pd20x].Tz2[pd20y].n];

(*Articulaciones pierna izquierda*)

P11A1 = T3D[T0B.TB11.Tz3[esc * 3].n];
P11A2 = T3D[T0B.TB11.Tz3[- esc * 3].n];
P21A1 = T3D[T0B.TB11.T1121.Tz2[esc * 3].n];
P21A2 = T3D[T0B.TB11.T1121.Tz2[- esc * 3].n];
P31A1 = T3D[T0B.TB11.T1121.T2131.Tz3[esc * 3].n];
P31A2 = T3D[T0B.TB11.T1121.T2131.Tz3[-esc * 3].n];
P41A1 = T3D[T0B.TB11.T1121.T2131.T3141.Tz3[esc * 3].n];
P41A2 = T3D[T0B.TB11.T1121.T2131.T3141.Tz3[- esc * 3].n];
P51A1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.Tz3[esc * 3].n];
P51A2 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.Tz3[- esc * 3].n];
P61A1 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.Tz1[esc * 3].n];
P61A2 = T3D[T0B.TB11.T1121.T2131.T3141.T4151.T5161.Tz1[- esc * 3].n];

(*Articulaciones pierna derecha*)

P12A1 = T3D[T0B.TB12.Tz3[esc * 3].n];
P12A2 = T3D[T0B.TB12.Tz3[- esc * 3].n];
P22A1 = T3D[T0B.TB12.T1222.Tz2[esc * 3].n];
P22A2 = T3D[T0B.TB12.T1222.Tz2[- esc * 3].n];
P32A1 = T3D[T0B.TB12.T1222.T2232.Tz3[esc * 3].n];
```

```
P32A2 = T3D[T0B.TB12.T1222.T2232.Tz3[- esc * 3].n];
P42A1 = T3D[T0B.TB12.T1222.T2232.T3242.Tz3[esc * 3].n];
P42A2 = T3D[T0B.TB12.T1222.T2232.T3242.Tz3[- esc * 3].n];
P52A1 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.Tz3[esc * 3].n];
P52A2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.Tz3[- esc * 3].n];
P62A1 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.Tz1[esc * 3].n];
P62A2 = T3D[T0B.TB12.T1222.T2232.T3242.T4252.T5262.Tz1[- esc * 3].n];


(*Líneas trazadas de acuerdo a los puntos anteriores*)

(*Pierna Izquierda*)

Linea0I = Line[{PB, P11}] /. SolPosI[i];
Linea1I = Line[{P11, P21}] /. SolPosI[i];
Linea2I = Line[{P21, P31}] /. SolPosI[i];
Linea3I = Line[{P31, P41}] /. SolPosI[i];
Linea4I = Line[{P41, P51}] /. SolPosI[i];
Linea5I = Line[{P51, P61}] /. SolPosI[i];
Linea6I = Line[{P61, P1}] /. SolPosI[i];

(*Pierna Derecha*)

Linea0D = Line[{PB, P12}] /. SolPosD[i];
Linea1D = Line[{P12, P22}] /. SolPosD[i];
Linea2D = Line[{P22, P32}] /. SolPosD[i];
Linea3D = Line[{P32, P42}] /. SolPosD[i];
Linea4D = Line[{P42, P52}] /. SolPosD[i];
Linea5D = Line[{P52, P62}] /. SolPosD[i];
Linea6D = Line[{P62, P2}] /. SolPosD[i];

(*Cilindro de Referencia*)
Cilindro0 = Cylinder[{{0, 0, 0}, {0, 0, 2}}, 1];

(*Articulaciones pierna izquierda*)

Cilindro11 = Cylinder[{P11A1, P11A2}, 0.5] /. SolPosI[i];
Cilindro21 = Cylinder[{P21A1, P21A2}, 0.5] /. SolPosI[i];
Cilindro31 = Cylinder[{P31A1, P31A2}, 0.5] /. SolPosI[i];
Cilindro41 = Cylinder[{P41A1, P41A2}, 0.5] /. SolPosI[i];
Cilindro51 = Cylinder[{P51A1, P51A2}, 0.5] /. SolPosI[i];
Cilindro61 = Cylinder[{P61A1, P61A2}, 0.5] /. SolPosI[i];

(*Articulaciones pierna derecha*)

Cilindro12 = Cylinder[{P12A1, P12A2}, 0.5] /. SolPosD[i];
```

```
Cilindro22 = Cylinder[{P22A1, P22A2}, 0.5] /. SolPosD[i];
Cilindro32 = Cylinder[{P32A1, P32A2}, 0.5] /. SolPosD[i];
Cilindro42 = Cylinder[{P42A1, P42A2}, 0.5] /. SolPosD[i];
Cilindro52 = Cylinder[{P52A1, P52A2}, 0.5] /. SolPosD[i];
Cilindro62 = Cylinder[{P62A1, P62A2}, 0.5] /. SolPosD[i];

(*Barras para el dibujo de los eslabones*)

Barra0I = Barra[3, 0, 0, 1, Linea0I];
Barra1I = Barra[3, 0, 1, 0, Linea1I];
Barra2I = Barra[3, 1, 0, 0, Linea2I];
Barra3I = Barra[3, 0, 1, 0, Linea3I];
Barra4I = Barra[3, 0, 0, 1, Linea4I];
Barra5I = Barra[3, 0, 1, 0, Linea5I];
Barra6I = Barra[3, 1, 0, 0, Linea6I];

Barra0D = Barra[3, 0, 0, 1, Linea0D];
Barra1D = Barra[3, 0, 1, 0, Linea1D];
Barra2D = Barra[3, 1, 0, 0, Linea2D];
Barra3D = Barra[3, 0, 1, 0, Linea3D];
Barra4D = Barra[3, 0, 0, 1, Linea4D];
Barra5D = Barra[3, 0, 1, 0, Linea5D];
Barra6D = Barra[3, 1, 0, 0, Linea6D];

Articulacion11 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro11];
Articulacion21 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro21];
Articulacion31 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro31];
Articulacion41 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro41];
Articulacion51 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro51];
Articulacion61 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro61];

Articulacion12 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro12];
Articulacion22 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro22];
Articulacion32 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro32];
Articulacion42 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro42];
Articulacion52 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro52];
Articulacion62 = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro62];

Base = Barra[15, 0.427451, 0.466667, 0.529412, Cilindro0];

(*Polígonos para el trazo de los pies*)

PoligonoI = Polygon[{PI1, PI2, PI3, PI4, PI5, PI6, PI7, PI8, PI9,
    PI10, PI11, PI12, PI13, PI14, PI15, PI16, PI17, PI18, PI19, PI20}];
PoligonoD = Polygon[{PD1, PD2, PD3, PD4, PD5, PD6, PD7, PD8, PD9, PD10,
    PD11, PD12, PD13, PD14, PD15, PD16, PD17, PD18, PD19, PD20}];
```

```
PieI = Poligono[1, 0, 0, PoligonoI];
PieD = Poligono[1, 0, 0, PoligonoD];

Show[Base, Barra0I, Barra1I, Barra2I, Barra3I, Barra4I, Barra5I, Barra6I,
 PieI, Barra0D, Barra1D, Barra2D, Barra3D, Barra4D, Barra5D, Barra6D,
 PieD, Articulacion11, Articulacion21, Articulacion31, Articulacion41,
 Articulacion51, Articulacion61, Articulacion12, Articulacion22,
 Articulacion32, Articulacion42, Articulacion52, Articulacion62, Axes → True,
 AxesLabel → {"x", "y", "z"}, BaseStyle → {12, FontFamily -> "Arial"},
 ImageSize → 190, PlotRange → {{-10, 10}, {-5, 15}, {0, 25}}],
{i, 0, 40, 1}]
```

Out[397]=
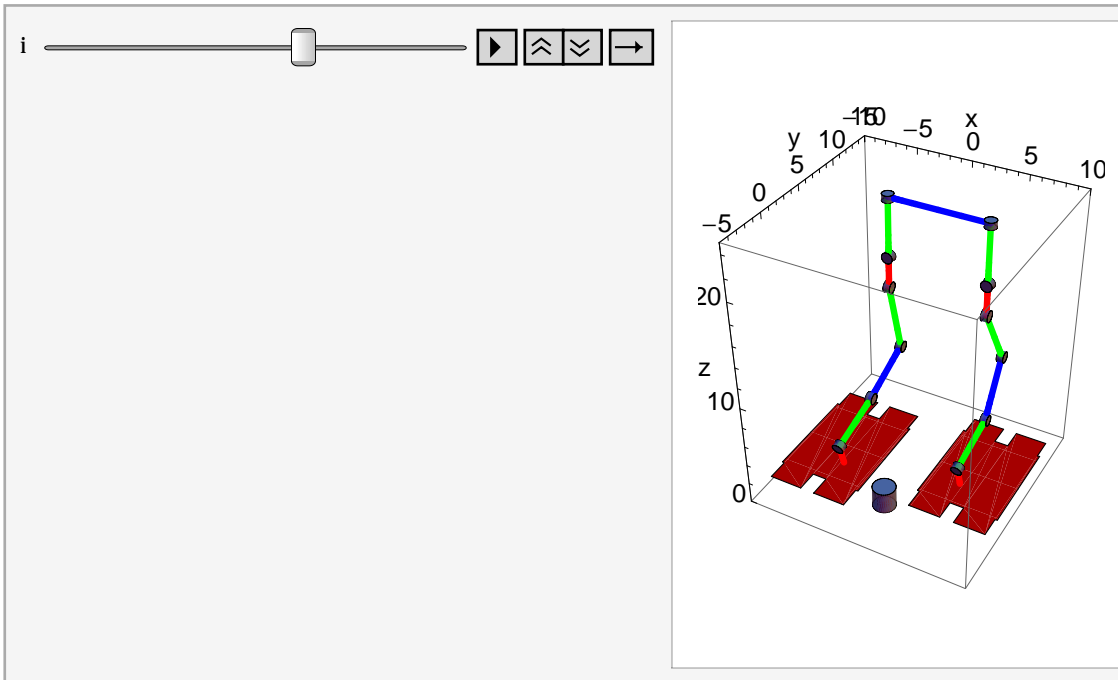
Bibliografía

Aldebaran-Robotics. Nao academics datasheet, Marzo 2010. URL `http://www.aldebaran-robotics.com/en/node/1166`.

Atsuo-Takanishi-Lab. Biped humanoid robot wabian-2r, Noviembre 2009a. URL `http://www.takanishi.mech.waseda.ac.jp/top/research/wabian/index.htm`.

Atsuo-Takanishi-Lab. Multi purpose biped locomotor wl-16rv, Noviembre 2009b. URL `http://www.takanishi.mech.waseda.ac.jp/top/research/wabian/index.htm`.

Ryan Block. Sony killed aibo and qrio?, Marzo 2010. URL `http://www.engadget.com/2006/01/26/sony-killed-aibo-and-qrio/`.

C. Chevallereau, Guy Bessonnet, G. Abba, and Y. Aoustin. *Bipedal Robots: Modeling, Design and Walking Synthesis*. John Wiley & Sons, Gran Bretaña, 2009.

Francisco Cuenca-Jiménez. Notas de clase: Dinámica espacial. 2010.

Aaron M. Dollar and Hugh Herr. Lower extremity exoskeletons and active orthoses: Challenges and state of the art. *IEEE Transactions on Robotics*, 24(1):144–158, 2008.

Kamalettin Erbatur and Okan Kurt. Natural zmp trajectories for biped robot reference generation. *IEEE Transactions on industrial electronics*, 56(3):835–845, 2009.

Donald T. Greenwood. *Principles of Dynamics*. Prentice-Hall International, London, 1965.

Erico Guizzo and Harry Goldstein. The rise of the body bots. *IEEE Spectrum Magazine*, 42(1):50–56, 2005.

Olli Haavisto and Heikki Hyotyniemi. Simulation tool of a biped walking robot model. Technical report, Helsinki University of Technology Control Engineering Laboratory, 2004.

Honda-Worldwide. Asimo, Marzo 2010. URL `http://world.honda.com/ASIMO/`.

Humanoid-Robot-Research-Center. Albert hubo, Noviembre 2009a. URL `http://hubolab.co.kr/AlbertHUBO_Specification.php`.

Humanoid-Robot-Research-Center. Hubo (fx-1), Noviembre 2009b. URL `http://hubolab.co.kr/HUBOFX-1_Specification.php`.

Shuuji Kajita and Bernard Espiau. Legged robots. In Bruno Siciliano and Oussama Khatib, editors, *Springer Handbook of Robotics*, pages 361–378. Springer, EUA, 2008.

Shuuji Kajita, Mitsuharu Morisawa, Kensuke Harada, Kenji Kaneko, Fumio Kanehiro, Kiyoshi Fujiwara, and Hirohisa Hirukawa. Biped walking pattern generator allowing auxiliary zmp control. *Proceedings of the 2006 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 2993–2999, Octubre 2006.

Kawada-Industries-Inc. Humanoid robot hrp-3 promet mk-ii, Noviembre 2009. URL `http://global.kawada.jp/mechatronics/hrp3.html`.

A. D. Kuo. Choosing your steps carefully. *IEEE Robotics & Automation Magazine*, 14(2):18–29, 2007.

Jinsu Liu and Manuela Veloso. Online zmp sampling search for biped walking planning. *Proceedings of 2005 5th IEEE-RAS International Conference on Humanoid Robots*, 53:10–15, 2005.

Merriam-Webster. Internet, 11 2008. URL `http://www.merriam-webster.com/dictionary/spline`.

Richard M. Murray, Zexiang Li, and S. Shankar Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, EUA, 2000.

Antonio Nieves and Federico Domínguez. *Métodos Numéricos Aplicados a la Ingeniería*. CECSA, México, 2004.

P.M. Prenter. *Splines and Variational Methods*. Wiley-Interscience Publication, Estados Unidos de América, 1975.

Ching-Long Shih, William A. Gruver, and Tsu-Tian Lee. Dynamics for control of a biped walking machine. *Journal of Robotic Systems*, 10(4):531–555, 1993.

Sony-Global. World's first running humanoid robot, Marzo 2010. URL `http://www.sony.net/SonyInfo/News/Press_Archive/200312/03-060E/`.

Mark W. Spong and Mathukumalli Vidyasagar. *Robot Dynamics and Control*. John Wiley & Sons, EUA, 1989.

Mark W. Spong, Seth Hutchinson, and Mathukumalli Vidyasagar. *Robot Modeling and Control*. John Wiley & Sons, EUA, 2006.

Atsuo Takanishi. In memoriam: Professor ichiro kato. *Autonomous Robots*, 2(1):7–10, 1995.

Tomotaka-Takahashi. Robo garage robots, Marzo 2010. URL `http://www.robo-garage.com/en/prd/p_00/index.html`.

Eric R. Westervelt and Carlos Canudas de Wit. Walking and running robots. *IEEE Robotics & Automation Magazine*, 14(2):6–7, 2007.

Eric R. Westervelt, Jessy W. Grizzle, Christine Chevallereau, Jun Ho Choi, and Benjamin Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. Control and Automation. CRC Press, EUA, 2007.

Martijn Wisse, Guillaume Feliksdal, Jan Van Frankenhuyzen, and Brian Moyer. Passive-based walking robot. *IEEE Robotics & Automation Magazine*, 14(2):52–62, 2007.