Apéndice A

Ecuaciones de Navier - Stokes en diferentes sistemas coordenados

a) Sistema de coordenadas cartesianas

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] = \rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right]$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right]$$

La ecuación de continuidad es:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

b) Sistema de coordenadas cilíndricas

$$\rho\left[\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right] = F_r - \frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2}\right)$$

$$\rho\left[\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right] = F_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{\partial^2 V_\theta}{\partial z^2}\right)$$

$$\rho\left[\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right] = F_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2}\right)$$

La ecuación de continuidad es:

$$\nabla \cdot \overrightarrow{V} = \frac{1}{r}\frac{\partial (rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V\theta}{\partial \theta} + \frac{\partial V z}{\partial z} = 0$$

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c) Sistema de coordenadas esféricas

$$\rho\left(\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}+\frac{V_\phi}{r\sin\theta}\frac{\partial V_r}{\partial \phi}-\frac{V_\theta^2}{r}-\frac{V_\phi^2}{r}\right)$$

$$=F_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 V_r}{\partial r^2}+\frac{2}{r}\frac{\partial V_r}{\partial r}-\frac{2V_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V_r}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V_r}{\partial \phi^2}-\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}-\frac{2V_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right]$$

$$\rho\left(\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_rV_\theta}{r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_\phi}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi}-\frac{V_\phi^2\cot\theta}{r}\right)$$

$$=F_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial^2 V_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V_\theta}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial V_r}{\partial V_\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right]$$

$$\rho\left(\frac{\partial V_\phi}{\partial t}+V_r\frac{\partial V_\phi}{\partial r}+\frac{V_rV_\phi}{r}+\frac{V_\theta}{r}\frac{\partial V_\phi}{\partial \theta}+\frac{V_\theta V_\phi\cot\theta}{r}+\frac{V_\phi}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right)$$

$$=F_\phi-\frac{1}{r\sin\phi}\frac{\partial p}{\partial \phi}+\mu\left[\frac{\partial^2 V_\phi}{\partial r^2}+\frac{2}{r}\frac{\partial V_\phi}{\partial r}-\frac{V_\phi}{r^2\sin^2\theta}+\frac{1}{r^2}\frac{\partial^2 V_\phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V_\phi}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial V_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial V_\theta}{\partial \phi}\right]$$

La ecuación de continuidad es:

$$\frac{\partial V_r}{\partial r}+\frac{2V_r}{r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_\theta\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}=0$$

Tabla A.1: *Ecuaciones de Navier - Stokes en coordenadas cartesianas, cilíndricas y esféricas [26]*

Apéndice B

Ecuaciones Euler en diferentes sistemas coordenados

a) Sistema de coordenadas cartesianas

$$\frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla) u = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + (\mathbf{u} \cdot \nabla) v = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$
$$\frac{\partial w}{\partial t} + (\mathbf{u} \cdot \nabla) w = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

donde

$$(\mathbf{u} \cdot \nabla) f = u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}$$

La ecuación de continuidad es:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

b) Sistema de coordenadas cilíndricas

$$\frac{\partial V_r}{\partial t} + (\mathbf{u} \cdot \nabla) V_r - \frac{V_\phi^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r}$$
$$\frac{\partial V_\phi}{\partial t} + (\mathbf{u} \cdot \nabla) V_\phi + \frac{V_r V_\phi}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \phi}$$
$$\frac{\partial V_z}{\partial t} + (\mathbf{u} \cdot \nabla) V_z = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

donde

$$(\mathbf{u} \cdot \nabla) f = V_r \frac{\partial f}{\partial r} + \frac{V_\phi}{r} \frac{\partial f}{\partial \phi} + V_z \frac{\partial f}{\partial z}$$

La ecuación de continuidad es:

$$\frac{1}{r} \frac{\partial (r V_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z} = 0$$

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c) Sistema de coordenadas esféricas
$$\frac{\partial V_r}{\partial t} + (\mathbf{u}\cdot\nabla)V_r - \frac{V_\theta^2 + V_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$ $$\frac{\partial V_\theta}{\partial t} + (\mathbf{u}\cdot\nabla)V_\theta + \frac{V_r V_\theta}{r} - \frac{V_\phi^2 cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$ $$\frac{\partial V_\phi}{\partial t} + (\mathbf{u}\cdot\nabla)V_\phi + \frac{V_r V_\phi}{r} + \frac{V_\theta V_\phi cot\theta}{r} = -\frac{1}{\rho r sen\theta}\frac{\partial p}{\partial \phi}$$ donde $$(\mathbf{u}\cdot\nabla)f = V_r\frac{\partial f}{\partial r} + \frac{V_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{V_\phi}{r sen\theta}\frac{\partial f}{\partial \phi}$$ La ecuación de continuidad es: $$\frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r sen\theta}\frac{\partial(V_\theta sen\theta)}{\partial \theta} + \frac{1}{r sen\theta}\frac{\partial V_\phi}{\partial \phi} = 0$$

Tabla B.1: *Ecuaciones Euler en coordenadas cartesianas cilíndricas y esféricas [29]*