# UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

PROGRAMA DE MAESTRÍA Y DOCTORADO EN INGENIERÍA
ELÉCTRICA — CONTROL

FINITE-TIME OBSERVER FOR LINEAR TIME-VARYING SYSTEMS
AND ITS APPLICATIONS

TESIS
QUE PARA OPTAR POR EL GRADO DE:
DOCTOR EN INGENIERÍA

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CIUDAD UNIVERSITARIA, CD. MX., AGOSTO 2018

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**FIRMA**

*(Segunda hoja)*

*A mis padres,*
*por todo su apoyo, consejo y cariño.*

*A mi amada Adriana,*
*por su ayuda, su amor,*
*su compañía y todo su tiempo,*
*a lo largo de mi vida.*

*A mi familia y amigos,*
*por acompañarme en este largo proceso.*

# Agradecimientos

Quiero agradecer a mi asesor, el doctor Jaime Alberto Moreno Pérez, por su tiempo, asesoría y guía durante estos últimos cinco años. Por todas las charlas y discusiones que me ayudaron a desarrollar este trabajo de tesis y demás derivados.

Agradezco también a los miembros del comité tutor, el doctor Leonid Fridman y el doctor Luis Agustin Álvarez-Icaza Longoria, por su apoyo, comentarios y reflexiones que me ayudaron a complementar y desarrollar varios temas tratados en este trabajo de tesis.

Por último, quiero agradecer al Consejo Nacional de Ciencia y Tecnología (CVU:491701) y al Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica PAPIIT-UNAM, Proyecto Número IN113617, por el financiamiento y apoyo económico otorgado para la realización de esta tesis.

# Acknowledgment

I want to acknowledge my adviser, Dr. Jaime Alberto Moreno Pérez, for his time, support and guide during the past five years. For all the talks and discussions that help me to develop this thesis and the other academic products derived from it.

I also appreciate the support, comments, and remarks that the adviser committee members, Dr. Fridman and Dr. Luis Agustin Álvarez-Icaza Longoria, gave me during the past years because they help me to complement and develop several of the topics developed in this thesis.

By last, I want to thank *Consejo Nacional de Ciencia y Tecnología* (CVU:491701) and to *Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica PAPIIT-UNAM*, Project number IN113617, for the economic and financial support given for the development of this work.

# Resumen

En este trabajo se presenta un observador con convergencia en tiempo fijo para sistemas lineales variantes en el tiempo de la forma

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)x(t), \\ y(t) &= C(t)x(t), \end{aligned}$$

donde las matrices del sistema $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$ y $C(t) \in \mathbb{R}^{r\times n}$ se asumen funciones continuas a tramos y uniformemente acotadas en magnitud. Tanto las matrices como la entrada $u(t)$ y la salida $y(t)$ se asumen conocidas. Se asume que el par $(A(t), C(t))$ es uniforme y completamente observable.

El observador propuesto está descrito por la siguiente dinámica:

$$\begin{aligned} \dot{\hat{x}}(t) = {} & A(t)\hat{x}(t) + B(t)u(t) - H(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big) \\ & - H(t)\Big(N(t)\Lambda_1\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_2}\Big), \end{aligned}$$

$$\begin{aligned} \dot{H}(t) &= H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + Q(t), \quad H(t_0) > 0, \\ \dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = 0, \\ \dot{\psi}(t) &= -\left(A(t) + Q(t)N(t)\right)^\top \psi(t) + C^\top(t)\psi(t) + N(t)B(t)u(t), \quad \psi(t_0) = 0. \end{aligned}$$

Los parámetros del observador se tienen que escoger de acuerdo a la siguiente tabla:

| Parámetros | Rango |
|---|---|
| $p_1$ | $[0, 1)$ |
| $p_2$ | $(1, \infty)$ |
| $\Lambda_i$ | $\mathrm{diag}\{\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n}\}$ |
| $\lambda_{i,j}$ | $(0, \infty)$ |
| $Q(t) = Q^\top(t)$ | $q_1\mathbb{I}_n \geq Q(t) \geq q_2\mathbb{I}_n$ |
| $q_1$ and $q_2$ | $(0, \infty)$ |
| $H(t_0)$ | Simétrica y positiva definida |
| $N(t_0)$ | $0 \in \mathbb{R}^{n\times n}$ |
| $\psi(t_0)$ | $0 \in \mathbb{R}^n$ |

Table 1: Parámetros del observador

Éste observador es capaz de proporcionar un estimado exacto del estado $x(t)$ del sistema en tiempo fijo. Esta característica es descrita en el siguiente teorema:

***Teorema.*** *Considere al sistema lineal antes descrito y al observador. Asuma que los parámetros del observador son escogidos de acuerdo a la tabla presentada. Sea el par $(A(t), C(t))$ uniforme y completamente observable en una ventana de tiempo de longitud $T$. Sea $h > 0$ y $\eta > 0$ tales que $H(t) \geq h\,\mathbb{I}_n$, para todo $t \geq t_0$, y $N(t) \geq \eta\,\mathbb{I}_n$, para todo $t \geq t_0 + T$. Entonces $\hat{x}(t)$ converge a $x(t)$ en tiempo fijo, uniformemente en el tiempo inicial. Aun más, el tiempo necesario para que $\hat{x}(t)$ converja a $x(t)$ es, a lo más*

$$T + \frac{n\,\sigma_1^{p_1}(\Lambda_1)}{h^{\frac{p_1+1}{2}}\sigma_n^{p_1+1}(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{n\,\sigma_1^{p_2}(\Lambda_2)}{h^{\frac{p_2+1}{2}}\sigma_n^{p_2+1}(\Lambda_2)\eta^{p_2+1}(p_2-1)}.$$

Como puede observarse, el estimado del tiempo de convergencia es válido para cualquier error inicial. De aquí que la convergencia no sólo sea uniforme en el tiempo inicial, sino también, en el estado inicial. Ésta es la principal característica de la convergencia en tiempo fijo. La ventaja que se obtiene de la convergencia en tiempo fijos es que pueden darse intervalos de confianza para confiar en el estimado del estado, cosa que no puede hacerse si la convergencia es asintótica y sin conocer una región donde el estado del sistema está contenido.

# Abstract


In this work an observer with fixed-time convergence for linear time-varying systems is presented. The addressed system class is as follows:

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)x(t), \\
y(t) &= C(t)x(t),
\end{aligned}
$$

where the system matrices $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $C(t) \in \mathbb{R}^{r\times n}$ are assumed piecewise continuous matrix valued functions, which are also uniformly bounded in magnitude. The matrices, the input $u(t)$, and the output $y(t)$ are assumed known. We consider that the pair $(A(t), C(t))$ is uniformly completely observable.

The proposed observer is described by the following dynamics:

$$
\begin{aligned}
\dot{\hat{x}}(t) = {} & A(t)\hat{x}(t) + B(t)u(t) - H(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big) \\
& - H(t)\Big(N(t)\Lambda_1\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_2}\Big),
\end{aligned}
$$

$$
\begin{aligned}
\dot{H}(t) &= H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + Q(t), \quad H(t_0) > 0, \\
\dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = 0, \\
\dot{\psi}(t) &= -\left(A(t) + Q(t)N(t)\right)^\top \psi(t) + C^\top(t)\psi(t) + N(t)B(t)u(t), \quad \psi(t_0) = 0.
\end{aligned}
$$

The observer parameters have to be chosen according with the next table:

| Parameter | Range |
|---|---|
| $p_1$ | $[0, 1)$ |
| $p_2$ | $(1, \infty)$ |
| $\Lambda_i$ | $\mathrm{diag}\{\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n}\}$ |
| $\lambda_{i,j}$ | $(0, \infty)$ |
| $Q(t) = Q^\top(t)$ | $q_1\mathbb{I}_n \geq Q(t) \geq q_2\mathbb{I}_n$ |
| $q_1$ and $q_2$ | $(0, \infty)$ |
| $H(t_0)$ | Symmetric, positive definite |
| $N(t_0)$ | $0 \in \mathbb{R}^{n\times n}$ |
| $\psi(t_0)$ | $0 \in \mathbb{R}^n$ |

Table 2: Observer parameters

This observer is capable of providing an exact estimate of the system state $x(t)$ in fixed time. This property is given in the next theorem:

***Theorem.*** *Consider the previous linear systems and the observer dynamics. Assume that the observer parameters are chosen following the previous table. Let the pair $(A(t), C(t))$ be uniformly completely observable over a time window of length $T$. Let $h > 0$ and $\eta > 0$ such that $H(t) \geq h\,\mathbb{I}_n$, for all $t \geq t_0$, and $N(t) \geq \eta\,\mathbb{I}_n$, for all $t \geq t_0 + T$. Then $\hat{x}(t)$ converge to $x(t)$ in fixed time. Furthermore, the convergence time is, at most*

$$T + \frac{n\,\sigma_1^{p_1}(\Lambda_1)}{h^{\frac{p_1+1}{2}}\,\sigma_n^{p_1+1}(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{n\,\sigma_1^{p_2}(\Lambda_2)}{h^{\frac{p_2+1}{2}}\,\sigma_n^{p_2+1}(\Lambda_2)\eta^{p_2+1}(p_2-1)}.$$

As can be seen, the convergence time holds for any initial error. Then, the convergence is not only uniform in the initial time, but also uniform in the initial condition. This is the main characteristic of the fixed-time convergence. This property gives us the advantage of knowing a time for which the estimate can be trusted. This is not possible if the convergences is asymptotic and one does not know a region in where the system state lives.

# List of symbols

| **Symbol** | **Notes** | **Meaning** |
|---|---|---|
| $\mathbb{R}$ | | The set of real numbers |
| $\mathbb{R}_{\geq a}$ | $a \in \mathbb{R}$ | The interval $[a, \infty)$ |
| $\mathbb{R}^n$ | $n$ a natural number | The real Euclidean space of dimension $n$ |
| $\mathbb{R}^{n\times m}$ | $n$ and $m$ natural numbers | The space of real matrices with $n$ rows and $m$ columns |
| $\mathcal{L}^2_{[t_0,t_1]}$ | $t_0$, $t_1 \in \mathbb{R}$, $t_1 > t_0$ | The set of all real squared integrable functions defined over $[t_0, t_1]$ |
| $\mathcal{L}^2_{[t_0,t_1]}, \mathbb{R}^r$ | $t_0$, $t_1 \in \mathbb{R}$, $t_1 > t_0$, and $r$ a natural number | The set of all squared integrable, $r$-vector valued, integrable functions defined over $[t_0, t_1]$ |
| $x^\top$ | $x \in \mathbb{R}^n$ | The transpose of a column vector, i.e., a row vector in $\mathbb{R}^{1\times n}$ |
| $A^\top$ | $A \in \mathbb{R}^{n\times m}$ | The transpose of the matrix $A$ |
| $x_i$, $(x)_i$ | $x \in \mathbb{R}^n$ and $i \leq n$ a natural number | The $i$-th component/coordinate of $x$ |
| $A_{i,j}$ | $A \in \mathbb{R}^{n\times m}$, $i \leq n$ and $j \leq m$ natural numbers | The component $(i, j)$ of the matrix $A$ |
| $(A)_i$ | $A \in \mathbb{R}^{n\times m}$, $i \leq m$ a natural number | The $i$-th column of the matrix $A$ |
| $\mathrm{tr}(A)$ | $A \in \mathbb{R}^{n\times n}$ | The trace of matrix $A$ |
| $\mathrm{diag}\{a_1, \cdots, a_n\}$ | $a_i \in \mathbb{R}$ | A squared diagonal matrix with diagonal elements $a_i$ |
| $\lambda_i(A)$ | $A \in \mathbb{R}^{n\times n}$, $i \leq n$ a natural number | The $i$-th eigenvalue of $A$ in some particular ordering. If $A$ is symmetric, in descending order |
| $\sigma_i(A)$ | $A \in \mathbb{R}^{n\times m}$, $i \leq \min\{n, m\}$ a natural number | The $i$-th singular value of $A$, when the singular values are ordered in descending order |
| $\mathbb{I}_n$ | $n$ a natural number | The identity matrix in $\mathbb{R}^{n\times n}$ |
| $\|x\|$ | $x \in \mathbb{R}^n$ | The Euclidean norm of $x$ equal to $\sqrt{x^\top x}$ |
| $\|x\|_p$ | $x \in \mathbb{R}^n$ and $p \in \mathbb{R}_{\geq 1}$ | The $p$-norm of $x$ defined as $(\sum_{i=1}^n \lvert x_i\rvert^p)^{1/p}$ |
| $\lceil v \rfloor^p$ | $v$, $p \in \mathbb{R}$ | The signed power of $v$: $\lvert v\rvert^p \mathrm{sign}(v)$ |
| $\lceil v \rfloor^p$ | $v \in \mathbb{R}^n$ and $p \in \mathbb{R}$ | The signed power applied to each component of $v$ |
| $A > B$ $(A \geq B)$ | $A$, $B \in \mathbb{R}^{n\times n}$, both symmetric matrices | $A - B$ is a positive (semi) definite matrix |

# Contents

# Chapter 1

# Introduction

Modern control theory relies on state feedback control. Almost all introductory textbooks in Automatic Control dedicate their content to this topic. See for example (Kwakernaak and Sivan 1972), (Slotine and Li 1991), (Khalil 2002), (Hendricks, Jannerup, and Sørensen 2008) to mention a few. However, this approach requires the availability of the full system's state at every moment. In real applications, having all the state through sensors is not possible, practical, or even affordable. That is why the design of state observers has become an intrinsic task in the development of modern control systems (Besançon 2007). An observer can be seen as an auxiliary dynamical system that uses the input-output data coming from the system that is capable of providing an accurate estimate of the system variables that are not available for measurement. Usually, those estimates get close to the system state with time, in an asymptotic fashion. In many cases, this suffices the controller requirements to achieve its objective. Yet, in some situations, to get a good performance, the degree of precision required in the estimates is high, and to obtain it, sometimes it is needed to let the observer converge exactly and fast. Then, observers with high speed of convergence are always good tools to have at hand.

The subject of study of this work is the design of observers for linear time-varying systems (LTV). Our goal is to obtain a methodology for designing observers capable of reconstructing the state of the system in a finite amount of time, that is, in a non-asymptotic way. The objective goes further: the convergence time should not exceed certain fixed amount, regardless of the initial error. This property is sometimes referred to as fixed-time convergence, and has recently attracted the attention of the control community (A. Polyakov 2012a). The importance of the LTV systems rests in different points. First, this class of systems covers the whole spectrum of linear, smooth, finite-dimensional linear systems (R. Kalman, Falb, and Arbib 1969). Second, they can be used to approximate a large class of non-linear systems. This can be done by linearization of the non-linear system along one trajectory, or by using more sophisticated and complete methods as the iteration procedure described in (Tomas-Rodriguez and Banks 2010). Also, the class of linear parameter-varying (LPV) can be represented and studied through LTV systems. At the same time, LPV systems can be used to model linear and non-linear systems (Briat 2014). Finally, LTV systems can be used to represent and analyse a broad class of classic problems in adaptive control (Narendra and Annaswamy 1989), (Ioannou and Sun 1995).

To design an observer for a general LTV system, there are not many options. Basically, one has to design a Kalman-Bucy filter. This observer was presented in (R. E. Kalman and R. S. Bucy 1961), and since then, it has become the main tool to observer time-varying systems. Over the years, several modifications of the algorithm have been proposed to improve its performance or to apply it in the observation of non-linear systems (Besançon 2007). Some examples of these modifications are presented in (Boizot et al. 2007), where the covariance matrices are adapted, or in (Deza et al. 1992), where the Kalman-Bucy filter is combined with a high gain approach. Another variation is proposed in (Besançon 2007, Sec. 1.3), where a linear term is introduced in the Riccati equation to improve the convergence. Beyond all these innovations, at the core of the procedure, the original result of Rudolf E. Kalman and Richard Bucy persists. The Kalman-Bucy filter consists of a linear time-varying output error feedback injection, which renders the observation error dynamics in an asymptotically stable LTV system. Because of the linearity, the estimation converges, at most, exponentially to the system trajectory. Additionally, the way in which the observer gain is computed makes really hard to adjust the rate of convergence, or to adapt it to particular requirements. We see these drawbacks as opportunity areas to improve the algorithm with respect to the convergence rate.

In contrast to the time varying case, for linear time-invariant systems (LTI), there are more options to design an observer. From the classical Luenberger (Hendricks, Jannerup, and Sørensen 2008, Sec. 4.6) and the stationary Kalman observer (Hendricks, Jannerup, and Sørensen 2008, Chap. 7), to new approaches through high order sliding-mode observers and differentiator (Shtessel et al. 2013). We are interested in properties mentioned above: Finite and Fixed-time convergence. The concept of finite-time convergence has more time in the literature of automatic control, and it can be tracked back to the classic sliding modes. The second property, the fixed-time convergence, is relative new. This property was first observed in homogeneous systems in the bi-limit in (Andrieu, Praly, and Astolfi 2009), and the advantages of it in the case of the robust differentiator were fist presented in (Cruz-Zavala, J. A. Moreno, and L. M. Fridman 2011). The current denominations of fixed-time stability, convergence, and others, came from (A. Polyakov 2012a). Since then, the topic has become very attractive, and several results on differentiation (Angulo, Jaime A. Moreno, and L. Fridman 2013) and observation have been obtained (J. D. Sánchez-Torres and Loukianov 2014), (J. Diego Sánchez-Torres et al. 2015), (Gutiérrez et al. 2017), (Ménard, Moulay, and Perruquetti 2017), (Lopez-Ramirez et al. 2018), (Héctor Ríos and Teel 2018). Given the interest in this topic, there has been an attempt to extend it to the case of homogeneous in the state time-varying systems (H. Ríos, Efimov, A. Polyakov, et al. 2016), but with limit applications. The previous results for time invariant systems are based on homogeneity, property that is not preserved when the time dependency is introduced, being this the main difficulty in extending the theory to time-varying systems.

Inspired by the results of the sliding-mode community, and motivated by the high interest in the topic of finite and, particularly, fixed-time convergence, we start this work with the aim of providing these properties in the observation of a particular class of homogeneous in the state time-varying systems: LTV systems. The main contribution of this work is, precisely, an observer capable of estimating the state of a general LTV system in fixed time. This observer can be seen as an improvement or modification to the Kalman-Bucy filter, where the innovation terms are responsible for the accelerated convergence. This observer has also been applied to some classic problem in adaptive control, where it provides, not only fixed-time convergence, but allows to recover parameters and states under relaxed excitation conditions, something that was not possible with the classical approaches.

## 1.1 Motivation

The initial motivation for this work came from the sliding-mode control. The achievements of this community in finite-time and fixed-time convergence in estimation of certain time-invariant systems moved us to extent these results to the time-varying case. A first attempt was made in (J. G. Rueda-Escobedo and J. A. Moreno 2015), where the problem of estimating constant parameters in finite-time was addressed. However, the proposed method in (J. G. Rueda-Escobedo and J. A. Moreno 2015) was only capable of recovering the parameters exactly under very specific circumstances. At that moment, the question on how to do the estimation in finite time remained open. This work arises in response to that question, but extending the scope to a more general class of systems.

A second motivation relies on the advantages that go along with the finite and fixed-time convergence. Academically speaking, finite-time convergence means that the state of a system can be recovered exactly, something that does not happen when the convergence is asymptotic. But not only that, finite-time convergence also means a faster recovery of the estimation in the presence of perturbations, and in some cases, disturbance rejection. On the other hand, fixed-time convergence has opened the opportunity of providing times of reliability for the estimates. Since fixed-time convergence implies that the convergence time cannot exceed,under any circumstance, certain limit, this allows to know when an estimate can be trusted. Given that these two properties are interesting and useful, it is natural to try to use them in other applications.

Finally, there are some situations where the necessary information to achieve the estimation is only available for short time intervals. That is the case of parameters estimation in adaptive control, where some kind of persistent excitation is needed. Usually, this excitation can only be kept by disturbing the nominal operation of a system. Then, it is important to have observers and estimators capable of exploiting the information more efficiently and in less time.

## 1.2 Contributions

The main contribution of this thesis is an observer for general LTV systems with fixed-time convergence. The observer and its properties are presented in Chapter 3, where the observer is introduced in (3.2). There, not only the type of convergence is given, but an upper bound for the convergence time is provided (Theorem 3.1). Also, robustness of the observer in the presence of bounded disturbances is studied (Theorem 3.2), and some conclusions about the behavior of the estimation error are obtained (Theorem 3.3). A preliminary version of the observer was presented in

- P. Oliva-Fonseca, J. G. Rueda-Escobedo, and J. A. Moreno (Sept. 2016b). "Observador con convergencia en tiempo fijo para sistemas LVT". in: *AMCA Congreso Nacional de Control Atomático*

Besides general LTV systems, there are some particular application where the observer can be simplified and where it exhibits properties that cannot be reproduced by standard or classical methods. These cases are

- Observation of LTI systems in fixed-time (Section 4.1). On this topic, the following works were presented:
  - J. G. Rueda-Escobedo, J. A. Moreno, and P. Oliva-Fonseca (June 2016a). "Finite-time state estimation for LTI systems with a First-Order Sliding Mode". In: *2016 14th International Workshop on Variable Structure Systems (VSS)*, pp. 194–199. DOI: 10.1109/VSS.2016.7506915
  - J. G. Rueda-Escobedo, J. A. Moreno, and P. Oliva-Fonseca (Oct. 2016b). "Fixed-time Convergent Unknown Input Observer for LTI Systems". In: *XVII Latin American Conference in 2016 Automatic Control*, pp. 354–359
- Estimation of constant parameters in fixed-time (Section 4.2). The results obtained in this topic were published in
  - M. Noack, J. G. Rueda-Escobedo, J. Reger, and J. A. Moreno (Dec. 2016). "Fixed-time parameter estimation in polynomial systems through modulating functions". In: *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 2067–2072. DOI: 10.1109/CDC.2016.7798568
- The design of adaptive observers for linear system with fixed-time convergence (Section 4.3). In this topic, the following publications were obtained:
  - P. Oliva-Fonseca, J. G. Rueda-Escobedo, and J. A. Moreno (Dec. 2016a). "Fixed-time adaptive observer for linear time-invariant systems". In: *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 1267–1272. DOI: 10.1109/CDC.2016.7798440

  - Juan G. Rueda-Escobedo and Jaime A. Moreno (2017). “Fixed-Time convergent Adaptive Observer for LTI Systems”. In: *IFAC-PapersOnLine* 50.1. 20th IFAC World Congress, pp. 11639–11644. ISSN: 2405-8963. DOI: `https://doi.org/10.1016/j.ifacol.2017.08.1664`

- Generalization to the estimation of time-varying parameters (Section 4.4). The study of this topic results in the following articles:

  - Juan G. Rueda-Escobedo and Jaime A. Moreno (2016). “Discontinuous gradient algorithm for finite-time estimation of time-varying parameters”. In: *International Journal of Control* 89.9, pp. 1838–1848. DOI: `10.1080/00207179.2016.1159338`
  - H. Ríos, D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo (July 2017). “Time-Varying Parameter Identification Algorithms: Finite and Fixed-Time Convergence”. In: *IEEE Transactions on Automatic Control* 62.7, pp. 3671–3678. ISSN: 0018-9286. DOI: `10.1109/TAC.2017.2673413`

Additionally to the situations exposed above, the technique developed in this thesis was extended to the observation of LTV systems with delayed measurements. The proposed observer results to be useful when the delay is time varying and unknown, but bounded. In that situation, there is a bounded observation error because the lack of information about the delay. The interesting part relies on the possibility of reaching the final bound arbitrarily fast, something that is new in this case. The results on this topic are reported in the following articles:

- J. G. Rueda-Escobedo, Rosane Ushirobira, Denis Efimov, and J. A. Moreno (June 2018a). “A Gramian-based observer with uniform convergence rate for delayed measurements”. In: *2018 European Control Conference (ECC)*, To appear
- J. G. Rueda-Escobedo, Rosane Ushirobira, Denis Efimov, and J. A. Moreno (2018b). “Gramian-based uniform convergent observer for stable LTV systems with delayed measurements”. In: *International Journal of Control*, Accepted

During the study of observer designing for LTV systems with delayed measurements, a side results for the case of constant known delay was obtained. In this case, a scheme of cascade “delayed" Kalman-Bucy filter is proposed to estimate asymptotically the state of the system allowing an arbitrarily large delay. Although similar results can be found for LTI system, this was the first time this property was obtained for LTV systems. This contribution was in part possible thanks to the research done about the Kalman-Bucy filter and presented in Section 2.3. This result was reported in:

- J. G. Rueda-Escobedo and J. A. Moreno (Sept. 2018). “Delayed Kalman-Bucy observer for a class of LTV systems with delayed measurements”. In: *9th IFAC Symposium on Robust Control Design (ROCOND)*, To appear

The contributions of this work are not limited to the observer here presented. During the development of the proposed algorithm, we have created some tools to study the properties of the observer, and also, the literature review about the Kalman-Bucy filter took us to recover some interesting properties about it. These contributions and developments are summarized in the following list:

- Lyapunov-like theorems to study finite and fixed-time stability in time-varying system were developed (Theorems 2.3 and 2.4).
- The interpretation of the Kalma-Bucy filter as the recursive solver of a linear time-varying algebraic equation (Section 2.3).

- Recovery and complementation of some properties of the Riccati differential equations (Proposition 2.10). These properties were partially studied and presented by Rudolf E. Kalman in (R. E. Kalman 1960).
- An inequality to separate the effect of disturbances when acting under non-linear terms (Lemma 3.2).

## 1.3 Thesis structure

Beside the introduction, the thesis is organized in five chapters:

- Chapter 2 is a collection of concepts and ideas that support the developments presented in the chapters after it. The considered topics are stability, linear systems, the Kalma-Bucy filter, and Riccati differential equations.
- In Chapter 3, the observer is presented and its properties are investigated. This chapter contains the core of the thesis and its main contribution.
- In Chapter 4 particular cases of LTV systems are analyzed, and is discussed how to simplify the structure of the observer for them. These cases are: LTI systems, constant parameter estimation, adaptive observer design for LTI systems, and the reconstruction of time-varying parameters. In the case of constant parameter estimation and the adaptive observer, non-uniform fixed-time convergence is also studied, resulting in a relaxation on the classical persistence of excitation condition.
- In Chapter 5, a final balance about the observer's properties is given. Its advantages, drawbacks, similarities and differences with the Kalman-Bucy filter are exposed.
- Finally, in Chapter A, some basic inequalities are given since them are used recurrently along the thesis.

# Chapter 2

# Preliminaries

The objective of this chapter is to present several concepts that support the main result of this report. Most of the concepts included in this chapter are common knowledge in the community of Automatic control; however, it is useful to have them at hand for supporting the claims made along the work. On the other hand, there are also results that are not that common, or have relevance only in certain communities. That is the case of some properties of the Kalman-Bucy filter that are developed here, or the concept of uniform finite and fixed-time stability that is even uncommon in the sliding-mode control literature.

This chapter starts with the concept of dynamical system and how to study some of its qualitative properties. In particular, the interest relies on some stability concepts and how to study these properties. After that, the study is centered in linear dynamical systems, paying attention to the form of the solution and how the internal state is reflected in the output. Finally, the chapter ends with a discussion about the Kalman-Bucy filter and its properties.

## 2.1 Dynamical systems, Lyapunov stability, and related concepts

To begin this section, the concept of dynamical system is introduced. Although the definition of dynamical system is a mathematical formalism, this definition is inspired in the work of the 19th century physicists, and it represents an attempt to abstract the properties of natural systems.

**Definition 2.1** (Dynamical system)**.** (Weiss and R. Kalman 1965, Def. 1) By a *dynamical system* we shall mean a mathematical structure denoted by the septet $\left(\Sigma, \mathcal{T}, \Omega, \mathcal{U}, \phi, \mathcal{Y}, \psi\right)$ where:

1. $\Sigma$ is an abstract space called the *state space* and $\mathcal{T}$ a set of values of time at which the behavior of the system is defined. $\mathcal{T}$ is an ordered subset of the real numbers, with the usual ordering $>$ (or $<$). If $t_1$, $t_0 \in \mathcal{T}$, the statement $t_1 > t_0$ (or $t_1 < t_0$) will mean that $t_1$ is in the future (or in the past) with respect to $t_0$; equivalently, $t_0$ is in the past (or in the future) with respect to $t_1$.

2. $\Omega$ and $\mathcal{U}$ are abstract spaces with $\Omega$ being the set of functions of time $u : \mathcal{T} \to \mathcal{U}$ which represent the admissible *inputs* to the system.

3. For any initial time $t \in \mathcal{T}$, any initial state $x \in \Sigma$, and any input $u \in \Omega$ defined for $t \geq \tau$ (or $t \leq \tau$), states at other values of time the system are determined by a given *Transition function* $\phi : \Omega \times \mathcal{T} \times \mathcal{T} \times \Sigma \to \Sigma$, which is written as $\phi_u(t, \tau, x)$. This function has the following properties:

(a) $\phi_u(\tau;\tau,x) = x$ for any $u \in \Omega$, $\tau \in \mathcal{T}$, $x \in \Sigma$.

(b) $\phi_u(t;\tau,x)$ is defined only when $t \geq \tau$ (or $t \leq \tau$).

(c) $\phi_u(t_2;t_0,x) = \phi_u(t_2;t_1,\phi_u(t_1;t_0,x))$ for all $u \in \omega$, all $t_0$, $t_1$, $t_2$ in $\mathcal{T}$ such that $t_2 \geq t_1 \geq t_0$ (or $t_2 \leq t_1 \leq t_0$), and all $x \in \Sigma$.

(d) If $u_{[\tau,t]}$ denotes the equivalent class of functions in $\Omega$ whose values agree with $u$ in the set $[\tau,t] \cap \mathcal{T}$, then

$$\phi_u(t;\tau,x) = \phi_{u_{[\tau,t]}}(t;\tau,x).$$

4. Every output of the system at time $t$ is given by the value of a real function $\psi : \mathcal{T} \times \Sigma \to \mathbb{R}$; where $\psi$ belongs to a given class $\mathcal{Y}$.

5. The functions $\phi$ and $\psi$ are continuous with respect to suitable topologies defined on $\Sigma$, $\mathcal{T}$, $\Omega$, $\mathcal{Y}$, and the reals, as well as the induced product topologies.

One way to embody the previous concept is by means of differential equations. Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $f$, $g : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ be piecewise continuous functions in $t$ and continuous in $x$. Consider the differential equation

$$\begin{aligned} &\dot{x}(t) = f(t,x(t)) + g(t,x(t))u(t), \quad x(t_0) = x_0 \\ &y(t) = h\big(t,x(t)\big) \end{aligned}, \tag{2.1}$$

and assume that its solutions are defined and are unique for any $x_0$ and for $t \geq t_0$. Then, the solutions of (2.1) satisfy all the criteria presented in Definition 2.1, and therefore (2.1) can be used to represent an abstract dynamical system where: $\Sigma = \mathbb{R}^n$, $\mathcal{T} = \{t \in \mathbb{R},\, t \geq t_0\}$, $u(t) \in \mathcal{U}$, $x(t) = \phi_u(t;t_0,x_0)$, and $h(t,x)$ can be identified with $\psi$. This kind of representation or model is common in fields likes physics, chemistry, engineering, economics, etc.

Some properties of system (2.1) can be studied by analyzing the fixed points of the differential equation for zero input. Let $x_p$ be a fixed point, then, if (2.1) starts at $x_p$ ($x_0 = x_p$) and $u(t) = 0$, the correspondent solution remains in such point: $x(t) = x_p$ for all $t \geq t_0$. This automatically means that $f(t,x_p) = 0$ for all $t \geq t_0$. Given that the derivative of the solutions is zero, these points are also called equilibrium points. This concept is an inheritance of the classic mechanic where the equilibrium is reached when the acceleration is zero. For simplicity, it is convenient to make a translation in the *state* $x$ in other to make $x_p$ coincident with zero. This can always be accomplished by the state transformation $z = x - x_p$. Then, the equilibrium solution is $z(t) = 0$. An important information to have about the equilibrium solution is to know when it is stable. Intuitively, this means that if the initial condition of (2.1) is close to $z = 0$, the solution will remain also close to the equilibrium solution, or even, the solution will approach it. In concrete, we are interested in the *Lyapunov stability* of the equilibrium point. This concept is properly defined in the next segment.

**Definition 2.2** (Lyapunov Stability)**.** (Khalil 2002, Def. 4.4) Consider a dynamical system described by (2.1), with $u(t) = 0$ and $f(t,0) = 0$ for all $t \geq t_0$. The equilibrium point $x = 0$ is Lyapunov

- stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon,t_0) > 0$ such that

$$\|x(t_0)\| < \delta \implies \|x(t)\| < \epsilon, \quad \forall\, t \geq t_0. \tag{2.2}$$

- uniformly stable if, for each $\epsilon > 0$, the is $\delta = \delta(\epsilon) > 0$, independent of $t_0$, such that (2.2) is satisfied.
- unstable if it is not stable.
- asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < 0$.

- uniformly asymptotically stable if it is uniformly stable and there is a positive constant $c$, independent of $t_0$, such that for all $\|x(t_0)\| < 0$, $x(t) \to 0$ as $t \to \infty$, uniformly in $t_0$; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|x(t)\| < 0, \quad \forall\, t \geq t_0 + T(\eta),\ \forall\, \|x(t_0)\| < c.$$

- globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$, and, for each pair of positive numbers $\eta$ and $c$, there is $T = T(\eta, c) > 0$ such that

$$\|x(t)\| < \eta, \quad \forall\, t \geq t_0 + T(\eta, c), \quad \forall\, \|x(t_0)\| < c.$$

- (Khalil 2002, Def. 4.5) exponentially stable if there exist positive constants $c$, $k$, and $\lambda$ such that

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\lambda(t - t_0)), \quad \forall\, \|x(t_0)\| < c,$$

  and globally exponentially stable if the inequality is satisfied for any initial state $x(t_0)$.

- (Haddad, Nersesov, and Du 2008) finite time stable if it is stable, and there exist an open neighborhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin, and a function $T : \mathbb{R}_{\geq t_0} \times \mathcal{D} \to [0, \infty)$, called the *settling-time function*, such that, for every $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{D} \setminus \{0\}$ $\lim_{t\to T(t_0,x_0)} x(t) = 0$ (finite-time convergent).

- (Haddad, Nersesov, and Du 2008) uniformly finite-time stable if it is uniformly stable and finite-time convergent.

- (Haddad, Nersesov, and Du 2008) globally finite-time stable (respectively, globally uniformly finite-time stable) if it is finite-time stable (respectively, uniformly finite-time stable) with $\mathcal{D} = \mathbb{R}^n$.

- (A. Polyakov 2012b) fixed-time stable (respectively, uniformly fixed-time stable) if it is globally finite-time stable (respectively, globally uniformly finite-time stable) and the settling-time function is bounded, that is, $\exists T_{\max} > 0$ such that $T(x_0) \leq T_{\max}$ for all $x_0 \in \mathbb{R}^n$ ($T(x_0, t_0) \leq T_{\max}$ for all $x_0 \in \mathbb{R}^n$ and all $t_0 \geq 0$).

In general, proving that a system equilibrium point satisfies any of the stability concepts given above by the direct application of the definition is really hard. The development of a tool to study this properties was the concern of the work of Aleksandr M. Lyapunov (Lyapunov 1992). One of the results of the Lyapunov's work was the concept of the now called Lyapunov function, a central topic in the modern Automatic Control discipline. The use of such functions allows to investigate the stability of an equilibrium point. In regards of this topic, below are presented some results that have been proved to help in the study of Lyapunov stability.

**Theorem 2.1.** *(Khalil 2002, Theo. 4.8) Let $x = 0$ be an equilibrium point for $\dot{x}(t) = f(t, x(t))$ and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function such that*

$$W_1(\|x\|) \leq V(t, x) \leq W_2(\|x\|),$$
$$\dot{V}(t) = \frac{\partial}{\partial t} V + \frac{\partial}{\partial x} V \cdot f(t, x) \leq 0$$

*for all $t \geq t_0$ and for all $x \in \mathcal{D}$, where $W_1(\cdot)$ and $W_2(\cdot)$ are class $\mathcal{K}$ functions*[1]*. Then, $x = 0$ is uniformly stable.*
*(Khalil 2002, Theo. 4.9) If the inequality can be strengthened to*

$$\dot{V}(t) \leq -W_3(\|x(t)\|)$$

*for all $t \geq 0$ and for all $x \in \mathcal{D}$, where $W_3(\cdot)$ is a class $\mathcal{K}$ function. Then, $x = 0$ is uniformly asymptotically stable. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $W_1(\cdot)$, $W_2(\cdot)$ are class $\mathcal{K}_\infty$ functions, then $x = 0$ is globally uniformly asymptotically stable.*

[1]For a definition of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions, see (Khalil 2002, Def. 4.2).

**Theorem 2.2.** *(Khalil 2002, Theo. 4.10) Let $x = 0$ be an equilibrium point for $\dot{x}(t) = f(t, x(t))$ and $\mathcal{D} \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function such that*

$$\begin{aligned} k_1 \|x\|^a \leq & V(t, x) \leq k_2 \|x\|^a \\ \dot{V}(t) \leq & -k_3 \|x(t)\|^a \end{aligned}$$

*for all $t \geq t_0$ and for all $x \in \mathcal{D}$, where $k_1$, $k_2$, $k_2$, and $a$ are positive constants. Then, $x = 0$ is exponentially stable. If the assumptions hold globally, then $x = 0$ is globally exponentially stable.*

A positive definite function $V$ satisfying the requirements of the theorems is called a Lyapunov function. Any function $V$ that satisfies the bounds $W_2(\|x\|) \geq V(t, x) \geq W_2(\|x\|)$ for class $\mathcal{K}$ functions $W_1$ and $W_2$ is called a *candidate* Lyapunov function until its derivative, when evaluated over the system trajectories, is proven to be negative, in such case, $V$ qualifies as a Lypunov function.

The concepts of uniform stability, uniform asymptotic stability, and exponential stability are very well known in the Automatic control community. The cases of uniform finite and fixed-time stability, on the contrary, are not well established. When the system in analysis is time invariant, i.e., $\dot{x}(t) = f(x(t))$, the concepts of finite and fixed-time stability are well spread in the sliding-mode control community, where these properties have recently become a central topic (Andrey Polyakov and L. Fridman 2014). However, the way these concepts apply when the system is time-varying is unclear. To exemplify the situation, consider the system

$$\dot{x}(t) = -\alpha(t, t_0) \left(\frac{2}{3}\right)^{1/2} \lceil x(t) \rfloor^{1/4},$$

with $\alpha(t, t_0) \geq 0$ for any $t \geq t_0 \geq 0$. Also consider the candidate Lyapunov function $V(x) = \frac{2}{3}|x|^{3/2}$, which derivative along the system trajectories is

$$\dot{V}(t) = -\alpha(t, t_0) \left(\frac{2}{3}\right)^{1/2} |x(t)|^{3/4} = -\alpha(t, t_0) V^{1/2}(t).$$

To show different scenarios, we consider the following choices for $\alpha(t, t_0)$ and its integrals:

- $\alpha_1(t, t_0) = \exp(-t) \to \int_{t_0}^{t} \alpha_1(\sigma, t_0) \mathrm{d}\sigma = -\exp(-t) + \exp(-t_0)$,
- $\alpha_2(t, t_0) = \exp(-t + t_0) \to \int_{t_0}^{t} \alpha_2(\sigma, t_0) \mathrm{d}\sigma = 1 - \exp(-t + t_0)$,
- $\alpha_3(t, t_0) = 1/(1 + t) \to \int_{t_0}^{t} \alpha_3(\sigma, t_0) \mathrm{d}\sigma = \ln\left((1 + t)/(1 + t_0)\right)$,
- $\alpha_4(t, t_0) = 1/(1 + t - t_0) \to \int_{t_0}^{t} \alpha_4(\sigma, t_0) \mathrm{d}\sigma = \ln(1 + t - t0)$.

In all these cases, the solution for $V(t)$ is

$$V(t) = \left(V^{\frac{1}{2}}(t_0) - \frac{1}{2} \int_{t_0}^{t} \alpha(\sigma, t_0) \mathrm{d}\sigma\right)^2$$

if $2\sqrt{V(t_0)} > \int_{t_0}^{t} \alpha(\sigma, t_0) \mathrm{d}\sigma$, and $V(t) = 0$ otherwise. This shows that $x(t)$ can converge to zero in finite time. However, the properties of the solution are different in each case. For $\alpha_1$, the finite-time convergence is only local since $V(t_0)$ has to be less than $\exp(-2\,t_0)/4$, otherwise the integral of $\alpha_1$ will always be less than $2\sqrt{V(t_0)}$ and $V(t)$ converges to a constant. Also notice that the region for which the finite-time convergence occurs depends on the initial time. Furthermore, the (exact) convergence time is $-\ln(\exp(-t_0) - 2\sqrt{V(t_0)})$, which also depends on the initial time. Then, the convergence is not uniform. The main characteristics in each case are summarized in the following table.

| $\alpha(t,t_0)$ | Attraction region | Convergence time | Convergence |
|---|---|---|---|
| $\exp(-t)$ | $4\,V(t_0) < \exp(-2\,t_0)$ | $-\ln\big(\exp(-t_0) - 2\sqrt{V(t_0)}\big) - t_0$ | local, non-uniform |
| $\exp(-t+t_0)$ | $4\,V(t_0) < 1$ | $-\ln\big(1 - 2\sqrt{V(t_0)}\big)$ | local, uniform |
| $1/(1+t)$ | global | $\big(\exp\big(2\sqrt{V(t_0)}\big) - 1\big)\big(1+t_0\big)$ | global, non-uniform |
| $1/(1+t-t_0)$ | global | $\exp\big(2\sqrt{V(t_0)}\big) - 1$ | global, uniform |

In all cases, the function $\alpha$ is continuous and always positive. We can observe that local behavior happens when the integral of $\alpha$ is bounded, and global when the integral diverges. To obtain uniformity, we introduced dependency on $t_0$ in $\alpha$; however, this is not necessary. As an example, consider $\alpha(t) = 1 + \sin(t)$, for which there is global uniform finite-time convergence. In (Haddad, Nersesov, and Du 2008), Example 3.1, the authors claim that it is enough to have a continuous $\alpha(t)$ with $\alpha(t) > 0$ for almost all $t \in [t_0, \infty)$ to have global convergence. Clearly, this is not true, as we showed in the previous examples. This mistake is also reflected in Theorem 4.1, item (ii), (ii) and (iv), where the same property over $\alpha$ is said to guarantee not only globality, but also uniformity. The reference (Haddad, Nersesov, and Du 2008) is one of a very few works dedicated to uniform finite-time stability/convergence, and the mistake it contains reflects the lack of understanding of the topic.

Due to lack of work on Lyapunov theory in finite and fixed-time stability for time-varying systems, we are in the necessity of developing our own results to cover the requirements for this thesis. These results are encompassed in the next theorems.

**Theorem 2.3.** *Let $x(t) = 0$ be an equilibrium solution of $\dot{x}(t) = f(t, x(t))$. If there exist a continuously differentiable function $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $W_1(\cdot)$ and $W_2(\cdot)$, such that $W_1(\|x\|) \leq V(t,x) \leq W_2(\|x\|)$, a positive number $\lambda \in (0,1)$, and a positive function $\alpha : [0,\infty) \to \mathbb{R}_{\geq 0}$ such that*

$$\dot{V}(t) \leq -\alpha(t) V^{\lambda}(t), \tag{2.3}$$

*with*

$$\lim_{t\to\infty} \int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma = \infty \quad \forall\, t_0 \geq 0,$$

*then the solution $x(t) = 0$ is globally finite-time stable. Furthermore, if there exist positive constants $a_1 > 0$ and $a_0 \geq 0$ such that*

$$\int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma \geq a_1\big(t - t_0\big) - a_0,$$

*then $x(t) = 0$ is globally uniformly finite-time stable. In such case, the settling-time function can be bounded as*

$$\frac{W_2^{1-\lambda}(\|x(t_0)\|) + a_0(1-\lambda)}{a_1(1-\lambda)} \geq T\big(x(t_0)\big).$$

*Proof.* Following (Khalil 2002, Lem. 3.4), we can find the following solution for the differential inequality (2.3):

$$\alpha\big(\|x(t)\|\big) \leq V(t) \leq \left(V^{1-\lambda}(t_0) - (1-\lambda)\int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma\right)^{\frac{1}{1-\lambda}} \quad \text{for} \quad \frac{1}{1-\lambda} V^{1-\lambda}(t_0) > \int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma,$$

and $V(t)=0$ otherwise. To show the finite-time convergence, we have to find a time for which $V(t)=0$. From the previous inequality, we have that this is ensured when

$$\int_{t_0}^{t}\alpha(\sigma)\mathrm{d}\sigma \geq \frac{1}{1-\lambda}V^{1-\lambda}(t_0). \tag{2.4}$$

Since the LHS goes to infinity as $t$ does, there exist $t_1$ for every $t_0$ and $V(t_0)$ at which we have the equality. This proves the global finite time convergences, but does not ensures uniformity since $t_1$ may increase unboundedly with $t_0$. The lower bound for the growth of $\alpha(t)$ helps to establish the last property. In such case, the convergence time can be estimated by majorizing the RHS of (2.4) as

$$\int_{t_0}^{t}\alpha(\sigma)\mathrm{d}\sigma \geq a_1(t-t_0)-a_0 \geq \frac{1}{1-\lambda}V^{1-\lambda}(t_0).$$

Then, for $t$ greater than

$$t \geq \frac{V^{1-\lambda}(t_0)+a_0(1-\lambda)}{a_1(1-\lambda)}+t_0 := t_1,$$

we can guarantee that $V(t)=0$, and since $V(t)\geq W_1(\|x(t)\|)$, also $x(t)=0$. In this case, the amount of time needed to reach zero is at most

$$t_1-t_0=\frac{V^{1-\lambda}(t_0)+a_0(1-\lambda)}{a_1(1-\lambda)},$$

which does not depends on the initial time. In terms of $x(t_0)$, the convergence time can be estimated as

$$\frac{W_2^{1-\lambda}(\|x(t_0)\|)+a_0(1-\lambda)}{a_1(1-\lambda)},$$

asserting the uniformity. □

**Theorem 2.4.** *Let $x=0$ be the equilibrium point of $\dot{x}(t)=f(t,x(t))$. If there exist a continuously differentiable function $V:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}_{\geq 0}$, and class $\mathcal{K}_\infty$ functions $W_1(\cdot)$ and $W_2(\cdot)$, such that*

$$\begin{aligned} W_1(\|x\|) \leq& V(t,x) \leq W_2(\|x\|),\\ \dot{V}(t) \leq& -\alpha(t)V^p(t)-\beta(t)V^q(t), \end{aligned} \tag{2.5}$$

*with $0<p<1$, $q>1$, and for every $t_0$ there exist $t_1(t_0)<\infty$ such that*

$$\int_{t_0}^{t_1}\alpha(\sigma)\mathrm{d}\sigma \geq \frac{1}{1-p} \quad \text{and} \quad \int_{t_0}^{t_1}\beta(\sigma)\mathrm{d}\sigma \geq \frac{1}{q-1},$$

*then $x(t)=0$ is fixed-time stable. Furthermore, if there exist positive constants $a_1,b_1\in\mathbb{R}_{>0}$ and non-negative ones $a_0,b_0\in\mathbb{R}_{\geq 0}$ such that*

$$\int_{t_0}^{t}\alpha(\sigma)\mathrm{d}\sigma \geq a_1(t-t_0)-a_0 \quad \text{and} \quad \int_{t_0}^{t}\beta(\sigma)\mathrm{d}\sigma \geq b_1(t-t_0)-b_0, \tag{2.6}$$

*then $x(t)=0$ is uniformly fixed-time stable. In this case, the settling-time function can be bounded as*

$$T(x_0,t_0) \leq \frac{1+a_0(1-p)}{a_1(1-p)}+\frac{1+b_0(q-1)}{b_1(q-1)} \quad \forall\, x_0\in\mathbb{R}^n,\ t_0\geq 0.$$

*Proof.* Denote $V(t,x(t))$ by $V(t)$, in particular, $V(t_0,x_0)$ by $V(t_0)$. Without loss of generality assume that $V(t_0)>1$. From (2.5) we can see that the following two relations hold simultaneously:

$$\begin{aligned} \dot{V}(t) &\leq -\alpha(t)V^p(t),\\ \dot{V}(t) &\leq -\beta(t)V^q(t). \end{aligned}$$

The general solution of the differential equation

$$\dot{z}(t) = -a(t)z^{\alpha}(t),$$

for $\alpha \geq 0$, $\alpha \neq 1$, is

$$z(t) = \left( \left|z(t_0)\right|^{1-\alpha} - (1-\alpha) \int_{t_0}^{t} a(\sigma)\mathrm{d}\sigma \right)^{\frac{1}{1-\alpha}} \operatorname{sign}\big(z(t_0)\big).$$

Using the Comparison Lemma (Khalil 2002, Lem. 3.4), the previous solution can be specialized to our cases with $p \in [0, 1)$ and $q > 1$:

$$V(t) \leq \left( V^{1-p}(t_0) - (1-p) \int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma \right)^{\frac{1}{1-p}}, \tag{2.7}$$

$$V(t) \leq \frac{1}{\left( \frac{1}{V^{q-1}(t_0)} + (q-1) \int_{t_0}^{t} \beta(\sigma)\mathrm{d}\sigma \right)^{\frac{1}{q-1}}}, \tag{2.8}$$

when $V^{1-p}(t_0) > (1-p) \int_{t_0}^{t} \alpha(\sigma)\mathrm{d}\sigma$ in (2.7), and $V(t) = 0$ otherwise. Again, both inequalities are valid simultaneously. Consider first (2.8). Using it, let us find conditions that ensure $V(t) \leq 1$. This occurs when

$$\int_{t_0}^{t} \beta(\sigma)\mathrm{d}\sigma \geq \frac{1}{q-1} - \frac{1}{(q-1)V^{q-1}(t_0)}.$$

Noticing that

$$\frac{1}{q-1} > \frac{1}{q-1} - \frac{1}{(q-1)V^{q-1}(t_0)} \quad \forall\, V(t_0) \geq 1,$$

and given the property of the integral of $\beta(t)$, there exist $t_1(t_0)$ for which which $(q-1) \int_{t_0}^{t_1(t_0)} \beta(\sigma)\mathrm{d}\sigma \geq 1$ holds. Then, for $t \geq t_1(t_0)$ we can guarantee that $V(t) \leq 1$ independently of the initial condition. Now, we consider (2.7) to find a time that ensure $V(t) = 0$. This happens when

$$\int_{t_1(t_0)}^{t} \alpha(\sigma)\mathrm{d}\sigma \geq \frac{1}{1-p}.$$

Once more, given the integral property of $\alpha(t)$, there always exist $t_2(t_1(t_0))$ (just $t_2(t_0)$ for simplicity), such that $(1-p) \int_{t_1(t_0)}^{t_2(t_0)} \alpha(\sigma)\mathrm{d}\sigma \geq 1$. This shows the fixed-time convergence; however, the amount of time needed to converge, i.e., $t_2(t_0) - t_0$, may increase depending on $t_0$. To suppress the dependency on $t_0$, the lower bounds for the integrals in (2.6) are used. With them, we can chose

$$t_1 = t_0 + \frac{1 + b_0(q-1)}{b_1(q-1)},$$

$$t_2 = t_1 + \frac{1 + a_0(1-p)}{a_1(1-p)} = t_0 + \frac{1 + b_0(q-1)}{b_1(q-1)} + \frac{1 + a_0(1-p)}{a_1(1-p)}.$$

These relations yield

$$t_2 - t_0 = \frac{1 + b_0(q-1)}{b_1(q-1)} + \frac{1 + a_0(1-p)}{a_1(1-p)}.$$

The length of the interval $[t_0, t_2]$ does not depend on the initial time, asserting the uniformity. $\square$

These results about finite and fixed-time stability for time-varying systems are by no means the more general results on the topic that can be obtained. However, they analyze the basic properties we are looking in this class of systems. We want to remark that the bounds (2.6) not only help to establish uniformity, but they imply robustness of the stability since they exclude functions $\alpha(t)$ and $\beta(t)$ that goes to zero with time, and because of this, they can be seen as a kind of persistent of excitation condition.

To end this section, we will present how to extend the use of Lyapunov functions to analyze robustness of the stability, in particular *Input-to-state stability* (ISS). To introduce this concept, consider a dynamical system described by

$$\dot{x}(t) = f(t, x(t)) + \delta(t),$$

where $\delta(t)$ can be seen as a disturbance. Suppose that when $\delta(t) = 0$, $x(t) = 0$ is a uniformly asymptotically stable equilibrium point. One would expect that for $\delta(t) \neq 0$, but uniformly bounded $d \geq \|\delta(t)\|$, the state $x(t)$ will remain bounded, as happens in the case of linear systems. This is not true in general for non-linear systems as is shown, for example, in (Khalil 2002, pp. 175). This is precisely what ISS tries to establish. To formalize the discussion, we introduce the definition of ISS and a Lyapunov like theorem that helps to evaluate if a system is ISS or not.

**Definition 2.3.** (Edwards, Lin, and Wang 2000, Def. 2.1) Consider the system $\dot{x}(t) = f(t, x(t), u(t))$ and assume that is forward complete. The system is input-to-state stable if there exist a class $\mathcal{KL}$ function $\beta$, a class $\mathcal{K}$ function $\gamma$, such that, for each initial time $t_0 \geq 0$, each initial state $x_0$, and each input function $u(t)$, it holds that

$$\|x(t)\| \leq \beta\big(\|x_0\|, t - t_0\big) + \gamma\left(\sup_{s\in[t_0,t]} \|u(s)\|\right)$$

for all $t \geq t_0$.

**Definition 2.4.** A smooth function $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS-Lyapunov function for system $\dot{x}(t) = f(t, x(t), u(t)$ if there exist $\mathcal{K}_\infty$ functions $W_1(\cdot)$, $W_2(\cdot)$, $\rho(\cdot)$, and a continuous positive definite function $\alpha$ such that

$$W_1(\|x\|) \leq V(t, x) \leq W_2(\|x\|) \quad \forall\, t \geq 0,\ \forall\, x \in \mathbb{R}^n,$$

and

$$\|x\| \geq \rho(\|u\|) \implies \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x} f(t, x, u) \leq -\alpha(\|x\|).$$

**Theorem 2.5.** *(Edwards, Lin, and Wang 2000, Theo. 1) A forward complete time varying system $\dot{x}(t) = f(t, x(t), u(t))$ is ISS if and only if it admits a smooth ISS-Lyapunov function $V$.*

## 2.2 Linear dynamical systems

The scope of this section is to review some properties of the linear dynamical systems related to its structure, time behavior, stability, and some input-output properties. They will be used in the next section to analyze the Kalman-Bucy filter from a deterministic point of view, and it will help to develop some improvements to such observer.

To start this section, the concept of dynamical system is narrowed to finite-dimensional smooth linear systems in the next definition. This class of systems will be the main subject of study from now on.

**Definition 2.5** (Finite dimensional, smooth linear dynamical system)**.** A smooth linear dynamical system is a dynamical system in the sense of Definition 2.1 where

- $\Sigma$ is finite-dimensional vector space,
- $\mathcal{T}$ is the real line and $\phi$, $\psi$ are smooth functions of $t \in \mathcal{T}$,
- $\phi$ is linear jointly in $x \in \Sigma$ and $u \in \Omega$ and $\psi$ is linear in $x$,
- $\mathcal{U}$ is $p-$dimensional and $\mathcal{Y}$ is $r-$dimensional.

Under the previous assumptions, all finite dimensional, smooth linear dynamical systems are described as follows:

**Theorem 2.6.** *(R. Kalman, Falb, and Arbib 1969, Theo. 2.8 and 2.10) Every continuous-time, finite-dimensional, linear, smooth dynamical system obeys the relations:*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned} \tag{2.9}$$

*For some measurable matrix valued functions $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B(t) : \mathbb{R} \to \mathbb{R}^{n\times p}$ and $C(t) : \mathbb{R} \to \mathbb{R}^{r\times n}$. Furthermore, the system is reversible in time.*

The first part of Theorem 2.6 follows from the observation that linearity means $\phi(t; \tau, x, u) = \phi_1(t, \tau)x + \phi_2(t, \tau)u$. By taking the time derivative of the previous relation, the differential expression arise. The second part of the theorem comes from the solution theory of linear differential equation systems in form (2.9), as will be exposed in the next paragraphs.

From this point on, it is assumed that $u(t)$ is measurable, that $A(t)$, $B(t)$, and $C(t)$ are picewise continuous matrix valued functions of $t \in \mathbb{R}$, and that they are uniformly bounded, that is, there exist positive real constants $\mu$, $a$, $b$, and $c$ such that $\|u(t)\| \leq \mu$ $\|A(t)\| \leq a$, $\|B(t)\| \leq b$, $\|C(t)\| \leq c$ for all $t$, where $\|\cdot\|$ denote the matrix induced norm. Since the RHS of (2.9) is uniformly Lipschitz in $x$:

$$\left\|A(t)x_1(t) + B(t)u(t) - \big(A(t)x_2(t) + B(t)u(t)\big)\right\| \leq \|A(t)\| \, \|x_1(t) - x_2(t)\| \leq a\, \|x_1(t) - x_2(t)\|,$$

then, the Picard–Lindelöf theorem (Coddington and Levinson 1984, Chap. 1, Theo. 3.1) guarantees the existence and uniqueness of solutions of (2.9) for any initial condition $x(t_0) = x_0$ on the interval $[t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$. Given that the Lipschitz constant does not depend on the value of $x$, the solution can be continued on any time interval, meaning that the solutions of a linear system exist for all $t \in \mathbb{R}$. Given the uniqueness, the solutions are reversible.

Now consider the unforced system and a set of $n$ linear independent vectors $\{x_1, x_2, \cdots, x_n\}$. Define the solution of (2.9) passing through the point $(x_i, t_0)$ by $\phi_i(t) = \phi(t; t_0, x_i)$. Then the set of functions $\{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$ is linearly independent. To show it, notice that for $t = t_0$ they correspond to each $x_i$; since the functions are smooth, there is a time interval around $t_0$ for which the sum $\sum_{i=1}^{n} c_i\phi_i(t)$ is zero if and only if each $c_i$ is zero, then the functions are linear independent over any interval. Furthermore, any other solution can be expressed as a linear combination of the functions $\phi_i(t)$. To exemplify the previous assertion, suppose we want the solution of (2.9) for $u(t) = 0$ and passing through $(z, t_0)$. Since the set formed by the points $x_i$ is linear independent, there exist (unique) $c_i \in \mathbb{R}$ such that $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$. Propose $\phi(t; t_0, z) = c_1\phi_1(t) + \cdots + c_n\phi_n(t)$ as the desired solution. To test if this is correct, we only have to check the initial condition and the differential equation. It is easy to see that at $t_0$ $\phi(t_0; t_0, z) = z$; on the other hand, taking the derivative w.r.t. $t$ we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t; t_0, z) = c_1A(t)\phi_1(t) + \cdots + c_nA(t)\phi_n(t) = A(t)\phi(t; t_0, z).$$

Then $\phi(t; t_0, z)$ is, in fact, the desired solution. This also means that all solutions of (2.9) can be generated using the set $\{\phi_1(t), \phi_2(t), \cdots, \phi_n(t)\}$. Noticing that if $\phi(t; t_0, x_a)$ and $\phi(t; t_0, x_b)$ are solutions of (2.9), then

$\phi(t;t_0,x_a)+\phi(t;t_0,x_b)$ and $c\,\phi(t;t_0,x_a)$, $c\in\mathbb{R}$ also satisfy the differential equation, it is possible to conclude that the set which contains all the solutions of (2.9) is a linear vector space of dimension $n$, and a basis is given by the set $\{\phi_1(t),\, \phi_2(t), \cdots, \phi_n(t)\}$.

Let $\varepsilon_i \in \mathbb{R}^n$, for $i : 1, 2, \cdots, n$, with components $\varepsilon_{i,j} = 0$ for $j \neq i$, and $\varepsilon_{i,i} = 1$. That is, the set $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ represents the canonical basis for $\mathbb{R}^n$. The solutions $\phi_i(t,t_0) = \phi(t;t_0,\varepsilon_i)$ are of particular interest since they can be used to find any other solution easily. Consider the matrix formed by columns with these solutions $\Phi(t,t_0) = \mathrm{col}\{\phi_1(t,t_0), \cdots, \phi_n(t,t_0)\}$, this matrix can be used to express the solution to the unforced system as $x(t) = \Phi(t,t_0)x(t_0)$. This follows from the representation of the boundary condition $x(t_0)$ in the canonical base of $\mathbb{R}^n$. The matrix $\Phi(t,t_0)$ is key in the study of dynamical linear system and receive the name of *state transition matrix*. Some important properties of this matrix are given below:

**Proposition 2.1.** *(Coddington and Levinson 1984, Chap. 3), (Abou-Kandil et al. 2003, Theo. 1.1.1) Let $\Phi(t,t_0)$ be the state transition matrix associated to the homogeneous equation $\dot{x}(t) = A(t)x(t)$. Such matrix has the following properties:*

- $\Phi(t,\tau)\Phi(\tau,t_0) = \Phi(t,t_0)$ *for* $t,\, t_0,\, \tau \in \mathbb{R}$*,*
- $\Phi^{-1}(t,t_0) = \Phi(t_0,t)$ *for* $t,\, t_0 \in \mathbb{R}$*,*
- $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$ *for* $t,\, t_0 \in \mathbb{R}$*,*
- $\frac{\partial}{\partial t}\Phi^\top(t_0,t) = -A^\top(t)\Phi^\top(t_0,t)$ *for* $t,\, t_0 \in \mathbb{R}$*,*
- $\frac{\partial}{\partial t}\Phi(t_0,t) = -\Phi(t,t_0)A(t)$ *for* $t,\, t_0 \in \mathbb{R}$*,*
- $\det\Phi(t,t_0) = \exp\left(\int_{t_0}^t \mathrm{tr}\, A(s)\mathrm{d}s\right)$ *for* $t,\, t_0 \in \mathbb{R}$*,*
- *The state transition matrix is given by the Peano-Baker series*

$$\Phi(t,t_0) = \mathbb{I} + \int_{t_0}^t A(s)\mathrm{d}s + \int_{t_0}^t A(s_1)\int_{t_0}^{s_1} A(s_2)\mathrm{d}s_2\mathrm{d}s_1 + \int_{t_0}^t A(s_1)\int_{t_0}^{s_1} A(s_2)\int_{t_0}^{s_2} A(s_3)\mathrm{d}s_3\mathrm{d}s_2\mathrm{d}s_1 + \cdots$$

- *If $A$ is constant, then*

$$\Phi(t,t_0) = \mathbb{I} + A(t-t_0) + \frac{1}{2}A^2(t-t_0)^2 + \frac{1}{6}A^3(t-t_0)^3 + \cdots := \exp\big(A(t-t_0)\big),$$

- *If $A(t)A(\tau) = A(\tau)A(t)$ for all $t,\, \tau \in \mathbb{R}$, then*

$$\Phi(t,t_0) = \exp\left(\int_{t_0}^t A(s)\mathrm{d}s\right).$$

In fact, using the state transition matrix, the general solution of (2.1) can be given as (Coddington and Levinson 1984, Chap. 3, Theo. 3.1):

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^t \Phi(t,s)B(s)u(s)\mathrm{d}s. \tag{2.10}$$

This expression is commonly known as the *Variation of Constants Formula*. It can be verified by checking the boundary condition and its derivative.

Now, with the principal properties of the linear systems solutions at hand, we will discus other properties related with the input-output behavior of the systems. In particular, we are interested in computing the internal state of the system assuming that the input $u(t)$ is known and that we have an output $y(t)$ which is a linear combination of the state as in (2.9). Let us begin considering the time interval $I = [t_0,\, t_1]$. It is of interest to know when is it possible to compute the state at $t_0$ or $t_1$ using the history of $u(s)$ and $y(s)$ with $s \in I$. This reflection yields the concepts of observable and constructible systems:

**Definition 2.6** (Observable system)**.** A dynamical system is observable on $I = [t_0, t_1]$ if and only if, for all inputs and all corresponding outputs the state $x(t_0)$ is uniquely determined.

**Definition 2.7** (Constructible system)**.** A dynamical system is constructible on $I = [t_0, t_1]$ if and only if, for all inputs and all corresponding outputs the state $x(t_1)$ is uniquely determined.

The question that arise after the definition is how to investigate when a linear system is observable or constructible. To that matter, consider the output of the system at instant $s \in I$ in terms of the state at $t_0$:

$$y(s) = C(s)\left(\Phi(s,t_0)x(t_0) + \int_{t_0}^{s} \Phi(s,\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma\right).$$

Now consider two different values of $x(t_0)$, for example $\chi_1$ and $\chi_2$, and their corresponding outputs, $y_1(t)$ and $y_2(s)$, for the same input. If both *initial states* generate the exactly same output during the interval, these states are indistinguishable. This happens if the squared norm of $y_1(s) - y_2(s)$ over $I$ is identically zero:

$$\begin{aligned} y_1(s) - y_2(s) &= C(s)\Phi(s,t_0)\Big(\chi_1 - \chi_2\Big), \\ \int_{t_0}^{t_1} \|y_1(s) - y_2(s)\|^2 \mathrm{d}s &= \Big(\chi_1 - \chi_2\Big)^\top \int_{t_0}^{t_1} \Phi^\top(s,t_0)C^\top(s)C(s)\Phi(s,t_0)\mathrm{d}s\,\Big(\chi_1 - \chi_2\Big). \end{aligned}$$

Then, if the matrix represented by the integral term on the RHS of the last equation is singular on $I$, there are indistinguishable initial states, making the system unobservable. Notice that this property, in the case of linear dynamical systems, does not depend on the input, but on the matrices $A(t)$ and $C(t)$. Analogously, the output of the system on the interval $I$ can be expressed in terms of the *final* state:

$$y(s) = C(s)\left(\Phi(s,t_1)x(t_1) + \int_{t_1}^{s} \Phi(s,\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma\right).$$

Then, the squared norm of the difference between outputs generated by two different final states, $\chi_1$ and $\chi_2$, and the same input can be computed as:

$$\begin{aligned} y_1(s) - y_2(s) &= C(s)\Phi(s,t_1)\Big(\chi_1 - \chi_2\Big), \\ \int_{t_0}^{t_1} \|y_1(s) - y_2(s)\|^2 \mathrm{d}s &= \Big(\chi_1 - \chi_2\Big)^\top \int_{t_0}^{t_1} \Phi^\top(s,t_1)C^\top(s)C(s)\Phi(s,t_1)\mathrm{d}s\,\Big(\chi_1 - \chi_2\Big). \end{aligned}$$

Again, if the matrix defined by the integral term in the RHS of the last expression is singular, there would be indistinguishable final states implying that the system is unconstructible. These observations yield the following assertions[2]:

**Theorem 2.7.** *(R. Kalman, Falb, and Arbib 1969, Sec. 2.6) The linear system* (2.9) *is (completely) observable on the interval* $[t_0, t_1]$ *if and only if the gramian matrix*

$$\bar{\mathcal{N}}(t_1,t_0) := \int_{t_0}^{t_1} \Phi^\top(s,t_0)C^\top(s)C(s)\Phi(s,t_0)\,ds$$

*is nonsingular.*

[2]In (R. Kalman, Falb, and Arbib 1969) the conclusions are given in terms of unobservable/unconstructible events and not in terms of the complete system; the theorems given here are equivalent to Theorem 6.6 and 6.7 in Section 2.6 of the cited reference because if the gramian matrices are nonsingular there are no unobservable/unconstructible events.

**Theorem 2.8.** *(R. Kalman, Falb, and Arbib 1969, Sec. 2.6) The linear system* (2.9) *is (completely) constructible on the interval* $[t_0, t_1]$ *if and only if the gramian matrix*

$$\mathcal{N}(t_1, t_0) := \int_{t_0}^{t_1} \Phi^\top(s, t_1) C^\top(s) C(s) \Phi(s, t_1) ds$$

*is nonsingular.*

The gramian matrices $\bar{\mathcal{N}}(t_1, t_0)$ and $\mathcal{N}(t_1, t_0)$ are called *Observability gramian matrix* and *Constructibility gramian matrix*, respectively. Notice than in the case of *continuous time*[3] linear systems, both properties are equivalent since

$$\mathcal{N}(t_1, t_0) = \Phi^\top(t_0, t_1) \bar{\mathcal{N}}(t_1, t_0) \Phi(t_0, t_1).$$

Then, if the system is observable on the interval $I$, it is also constructible in the same interval, and vice versa.

The previous properties are tied to a specific time interval. This mean that they depend on the selection of $t_0$ and $t_1$. If a system is observable/constructible on $[t_0, t_1]$, it might not be on $[t_0 + \epsilon, t_1 + \epsilon]$. Even if it is, the magnitude of the associated gramian matrix eigenvalues may differ drastically. Let $\Delta > 0$ and suppose that $\mathcal{N}(t_0, t_0 + \Delta)$ is nonsingular for all $t_0 \in \mathbb{R}$, but its smallest eigenvalue $\lambda_n(t_0)$ decrease whenever $t_0$ increase, that is, $\lambda_n(t_0) < \bar{\lambda}$ for all $t_0 \in \mathbb{R}$ and $\lambda_n(t_0) \to 0$ as $t_0 \to \infty$. In this case, as the time evolves, there is "less" information in $\mathcal{N}(t_0, t_0 + \Delta)$ and a greater effort to recover the state will be required. These scenarios are excluded if there is *uniformity* in the observability/constructibility property. The uniformity is ensured if the gramian matrices satisfy the following:

**Definition 2.8.** (R. E. Kalman 1960, Def. 5.23) A linear system (2.9) is uniformly (completely) observable if there exist positive constants $\sigma > 0$, $\alpha_1 \geq \alpha_0 > 0$ such that

$$\alpha_1 \mathbb{I} \geq \bar{\mathcal{N}}(t + \sigma, t) \geq \alpha_0 \mathbb{I} \quad \forall\, t \in \mathbb{R}.$$

Equivalently, a definition of uniform of complete constructibility can be given:

**Definition 2.9.** A linear system (2.9) is uniformly (completely) constructible if there exist positive constants $\sigma > 0$, $\alpha_1 \geq \alpha_0 > 0$ such that

$$\alpha_1 \mathbb{I} \geq \mathcal{N}(t, t - \sigma) \geq \alpha_0 \mathbb{I} \quad \forall\, t \in \mathbb{R}.$$

Using the same argument as before, it is easy to see that a linear system that is uniformly observable, it is also uniformly constructible.

To end this section, we want to add that the uniformity on the observability/constructibility guarantee the preservation of these properties for a class of non-linear systems with a linear part and an additive *known* bounded disturbance (Sastry and C. Desoer 1982), or in the case of parameter uncertainties (Chung, Park, and Lee 1999), i.e., if the parameter variation is small enough, the system with the correct parameters will be UCO/UCC if the nominal system is UCO/UCC. The uniformity will also be essential to guarantee boundedness of the observation error when the system is affected by unknown bounded disturbances. These examples point out to the relation between uniformity and certain types of robustness.

[3] This is not true in general for discrete time linear system where the state transition matrix can be singular or even nilpotent.

## 2.3 The Kalman-Bucy filter

The Kalman filter, for discrete-time linear systems, and the Kalman-Bucy filter, for continuous-time linear systems, are milestones in the theory of filtering and state estimators/observers. The later one will be the main topic of this section, and of great relevance along the report. The original work of Rudolf E. Kalman and Richard S. Bucy (R. E. Kalman and R. S. Bucy 1961) was presented in the framework of stochastic filtering where the objective was to recover a *message* from an *observed* signal corrupted by a white noise (stochastic) process. The original problem statement is as follows:

***Optimal Estimation Problem:****(R. E. Kalman and R. S. Bucy 1961) Let a message be a random process $x(t)$ generated by the model*

$$\dot{x}(t) = A(t)x(t) + G(t)v(t).$$

*The observed signal is*

$$y(t) = C(t)x(t) + \nu(t).$$

*The signals $v(t)$ and $\nu(t)$ are independent random processes with identically zero means and covariance matrices:*

$$\begin{array}{rcll} \operatorname{cov}\big[v(t), v(\tau)\big] & = & Q(t)\,\delta(t-\tau) & \\ \operatorname{cov}\big[\nu(t), \nu(\tau)\big] & = & R(t)\,\delta(t-\tau) & \forall\, t,\, \tau, \\ \operatorname{cov}\big[v(t), \nu(\tau)\big] & = & 0 & \end{array}$$

*where $\delta$ is the Dirac delta function, and $Q(t)$, $R(t)$ are symmetric, positive definite matrices. Given known values of $y(\tau)$ in the time-interval $t_0 \le \tau \le t$, find an estimate $\hat{x}$ with the property that the expected squared error is minimized: $\mathbf{E}\left(\|x(t) - \hat{x}(t)\|^2\right)$.*

The solution to this problem is precisely the Kalman-Bucy filter where the estimate $\hat{x}(t)$ is given by the following linear dynamical system:

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) - K(t)C^\top(t)R^{-1}(t)\Big(C(t)\hat{x}(t) - y(t)\Big), \\ \dot{K}(t) &= A(t)K(t) + K(t)A^\top(t) - K(t)C^\top(t)R^{-1}(t)C(t)K(t) + G(t)Q(t)G^\top(t), \end{aligned} \tag{2.11}$$

where the filter gain $K(t)$ is the solution of the second expression, a Riccati differential equation, with initial condition $K(t_0) = \operatorname{cov}\big[x(t_0), x(t_0)\big]$. To obtain the previous expressions, the original statement is transformed into an optimal control problem by means of a duality principle. In that representation, the solution can be derived by means of the main result in (R. E. Kalman 1960). Then, using the duality principle again, the solution to the optimal estimation problem is obtained. A key element in the solution is the Riccati differential equation that define the behavior of the filter gain $K(t)$. Given the importance of such class of differential equations, they will be reviewed latter in this section. For now, we will assume that the solution exist on the interval $[t_0, \infty)$, is unique, symmetric, and positive (semi) definite if the initial condition is a symmetric positive (semi) definite matrix.

An equivalent formulation that is free of the stochastic framework has been proposed by Jan C. Willems (Willems 2004). In the work of Willems, $v(t)$ and $\nu(t)$ are interpreted as disturbances and are assumed locally integrable functions instead of random processes. In this scenario, the observed signal $y(t)$ can be reproduced using different combinations of initial conditions and different disturbances $v(t)$, $\nu(t)$. The problem then is to find, among all the possibilities, the signals $v(t)$, $\nu(t)$ and the initial condition $x_0$ that minimize the following criteria:

$$\mathcal{J} = x_0^\top \Gamma\, x_0 + \int_{t_0}^{t} v^\top(s)Q(s)v(s)\mathrm{d}s + \int_{t_0}^{t} \nu^\top(s)R^{-1}(s)\nu(s)\mathrm{d}s.$$

Surprisingly, the solution to such problem yields, again, the Kalman-Bucy filter (2.11). In this case, by completing the squares of $\mathcal{J}$ (Brockett 1970, Chap. 3), the optimal values for $x_0$, $v(t)$, and $\nu(t)$ can be found. These values in turn yield the dynamics of $\hat{x}(t)$.

In this section, with the objective of presenting the Kalman-Bucy filter as an observer, we will follow a different path to derive the equations (2.11). To begin with, let us consider the linear system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t),\ x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \tag{2.12}$$

where both $u(t)$ and $y(t)$ are known over the interval $[t_0, t]$. Using this information, we want to find the value of $x(t)$. In this system, there are no disturbances. Let $s \in [t_0, t]$, using (2.10) we can write $y(s)$ as

$$\begin{aligned} y(s) &= C(s)\left(\Phi(s,t)x(t) + \int_t^s \Phi(s,\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma\right) \\ C(s)\Phi(s,t)x(t) &= y(s) - C(s)\int_t^s \Phi(s,\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma. \end{aligned}$$

The previous equation defines a linear map between $\mathbb{R}^n$ and $\mathcal{L}^2_{[t_0,t],\mathbb{R}^r}$. This operator accepts a left inverse if the pair $(A(t), C(t))$ is constructible for $t \geq t_1 > t_0$ for some $t_1$. We take the solution to the previous equation as an estimate. Following (Brockett 1970, Sec. 3.20, Theo. 2), the solution for this class of equations is

$$\begin{aligned} \hat{x}(t) &= \left(\int_{t_0}^t \Phi^\top(s,t)C^\top(s)C(s)\Phi(s,t)\mathrm{d}s\right)^{-1} \times \int_{t_0}^t \Phi^\top(s,t)C^\top(s)\left(y(s) - C(s)\int_t^s \Phi(s,\sigma)B(\sigma)u(\sigma)\mathrm{d}\sigma\right)\mathrm{d}s \\ &:= \mathcal{N}^{-1}(t,t_0) \times \psi(t). \end{aligned} \tag{2.13}$$

Although the previous expression solves the problem, in its current form it does not look very useful. Fortunately, the estimate can be computed *recursively*. This means that $\hat{x}(t)$ can be computed using a dynamical system. First, let us rename $\mathcal{N}(t,t_0)$ just by $N(t)$ when the initial time $t_0$ is fixed. Then, to find a recursion, we proceed in two steps. First, we look for a differential expression for $N(t)$ and $\psi(t)$. Second, such expression will be used to compute the time derivative of $\hat{x}$. Following this, we have:

$$\begin{aligned} \dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) + C^\top(t)C(t), \quad N(t_0) = 0, \\ \dot{N}^{-1}(t) &= N^{-1}(t)A^\top(t) + A(t)N^{-1}(t) - N^{-1}(t)C^\top(t)C(t)N^{-1}(t), \quad \text{for} \quad t \geq t_1, \\ \dot{\psi}(t) &= -A^\top(t)\psi(t) + C^\top(t)y(t) + N(t)B(t)u(t), \quad \psi(t_0) = 0. \end{aligned}$$

Here, the properties of the state transition matrix $\Phi(s,t)$ were used together with the relationship $\dot{N}^{-1}(t)N(t) + N^{-1}(t)\dot{N}(t) = 0$. Notice that both, $N(t)$ and $\psi(t)$, are computed using a linear differential equation. Now, the dynamics of $\hat{x}(t)$ can be derived as follows:

$$\begin{aligned} \dot{\hat{x}}(t) &= \dot{N}^{-1}(t)\psi(t) + N^{-1}(t)\dot{\psi}(t) \\ &= A(t)N^{-1}(t)\psi(t) - N^{-1}(t)C^\top(t)C(t)N^{-1}(t)\psi(t) + N^{-1}(t)C^\top(t)y(t) + B(t)u(t) \\ &= A(t)\hat{x}(t) + B(t)u(t) - N^{-1}(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big). \end{aligned}$$

This reassembles the structure of the Kalman-Bucy filter; however, there are two important differences. First, the initial condition for the constructibility gramian $N(t)$ has to be zero, and therefore, there is no proper choice to initialize its inverse $N^{-1}(t)$ and one has to wait until $t_1$ to have an estimate. Second, if the matrix $A(t)$ describes a stable motion, $-A^\top(t)$ will describe an unstable one (Callier and C. A. Desoer 1991, Com. 115). This implies that $N(t)$ and $\psi(t)$ grows unboundedly as time passes, by whereas the magnitude of $N^{-1}(t)$ will go to zero. The latter means that the observer gain is "losing" strength. To fix these problems, the differential equation for $N(t)$ is modified by adding a quadratic term and changing the initial condition:

$$\dot{N}(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) > 0, \tag{2.14}$$

with $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I}$. Now, with the introduction of the quadratic term, the differential equation for $N(t)$ becomes of the Riccati type. As mentioned before, if a Riccati differential equation is initiated in a positive (semi) definite matrix, its solution will remain positive (semi) definite. Then, the negative quadratic term will "dissipate" energy, keeping the $N(t)$ bounded (see (2.24)); additionally, if the initial condition is chosen non-singular, $N(t)$ will be non-singular over all the interval, showing that these modifications alleviate all the issues mentioned above. In addition, the null space of $N(t)$ at each $t$ is the same no matter if it is solved using the linear differential equation or the Riccati one. This means that the introduction of the quadratic term does not modify the solvability of the original problem. To show the last point, we will follow (B.D.O. Anderson 1971, Lem. 3.1). Consider the linear differential equation

$$\dot{N}_1(t) = -A^\top(t)N_1(t) - N_1(t)A(t) + C^\top(t)C(t), \quad N_1(t_0) = N_0 \geq 0,$$

and its solution (Abou-Kandil et al. 2003, Cor. 1.1.6):

$$N_1(t) = \Phi^\top(t_0,t)N_0\Phi(t_0,t) + \int_{t_0}^{t} \Phi^\top(s,t)C^\top(s)C(s)\Phi(s,t)\mathrm{d}s,$$

where $\Phi(t,t_0)$ is the state transition matrix associated to $A(t)$. Also consider $N_2(t)$, the solution of the Riccati equation

$$\dot{N}_2(t) = -A^\top(t)N_2(t) - N_2(t)A(t) - N_2(t)Q(t)N_2(t) + C^\top(t)C(t), \quad N_2(t_0) = N_0.$$

Define as $\bar{\Phi}(t,t_0)$ the state transition matrix associated to $\dot{x}(t) = (-A^\top(t) - N_2(t)Q(t))x(t)$, and for $t_1 > t_0$ consider $S(t) = \bar{\Phi}(t_1,t)N_2(t)\bar{\Phi}^\top(t_1,t)$ which satisfies

$$\dot{S}(t) = \bar{\Phi}(t_1,t)\Big(N_2(t)Q(t)N_2(t) + C^\top(t)C(t)\Big)\bar{\Phi}^\top(t_1,t).$$

To show that $N_1(t)$ and $N_2(t)$ have the same null space, consider the time interval $\mathcal{I} = [t_0,t_1]$, and suppose that $N_2(t_1)$ is singular, then for some constant nonzero vector $v \in \mathbb{R}^n$ we have that $N_2(t_1)v = 0$. From the definition of $S(t)$ we know that $S(t_1)v = N_2(t_1)v = 0$. Notice that both, $S(t)$ and $\dot{S}(t)$, are positive semi definite matrices on $\mathcal{I}$. Consider the quadratic form $q(t) = v^\top S(t)v \geq 0$. At the end point we have that $q(t_1) = 0$. $q(t)$ can be obtained from solving the differential equation $\dot{q}(t) = v^\top\dot{S}(t)v$ backwards in time. Noticing that $\dot{q}(t) \leq 0$ when the time is reversed, and that $q(t) \geq 0$ for all $t \in \mathcal{I}$, we can conclude that the only possibility is that $\dot{q}(t) = 0$ and so $q(t)$. This in tuns implies that $S(t)v = P(t)\bar{\Phi}^\top(t_1,t)v = 0$ and that $C(t)\bar{\Phi}^\top(t_1,t)v = 0$ for all $t \in \mathcal{I}$. Taking the time derivative of the product $\bar{\Phi}^\top(t_1,t)v$ we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\Phi}^\top(t_1,t)v = \Big(A(t) + Q(t)N_2(t)\Big)\bar{\Phi}^\top(t_1,t)v = A(t)\bar{\Phi}^\top(t_1,t)v.$$

This means that $\bar{\Phi}^\top(t_1,t)v = \Phi(t,t_1)v$ over $[t_0,t_1]$. Then $P(t)\bar{\Phi}^\top(t,t_1)v = P(t)\Phi(t,t_1)v = 0$ and $C(t)\bar{\Phi}^\top(t,t_1)v = C(t)\Phi(t,t_1)v = 0$. Now, the product $N_1(t_1)v$ results in

$$N_1(t_1)v = \Phi^\top(t_0,t_1)N_0\Phi(t_0,t_1)v + \int_{t_0}^{t_1} \Phi^\top(s,t_1)C^\top(s)C(s)\Phi(s,t_1)\mathrm{d}s\, v = 0.$$

This proves that $v$ belongs also to the nullspace of $N_1(t_1)$. To prove the converse, that is, $N_1(t)v = 0$ implies $N_2(t)v = 0$, we only need to reverse the procedure. Then, $N_1(t)$ and $N_2(t)$ have exactly the same nullspace at each $t$, as mentioned before. This yields the following proposition:

**Proposition 2.2.** *Let $A(t) \in \mathbb{R}^{n\times n}$, $Q(t) \in \mathbb{R}^{n\times n}$, and $C(t) \in \mathbb{R}^{m\times n}$ be two uniformly bounded, piecewise continuous matrix valued functions. Additionally, assume that $Q(t) = Q^\top(t)$ and that there exist positive constants $q_1 \geq q_2 > 0$ such that $q_1\mathbb{I}_n \geq Q(t) \geq q_2\mathbb{I}_n$. Let $N_1(t)$ and $N_2(t)$ be the respective solutions of the differential equations*

$$\begin{aligned}\dot{N}_1(t) &= -A^\top(t)N_1(t) - N_1(t)A(t) + C^\top(t)C(t),\\ \dot{N}_2(t) &= -A^\top(t)N_2(t) - N_2(t)A(t) - N_2(t)Q(t)N_2(t) + C^\top(t)C(t),\end{aligned}$$

*with $N_1(t_0) = N_2(t_0) = N_0$ and $N_0 = N_0^\top \geq 0$. Then, $N_1(t)$ and $N_2(t)$ have the same null space at each $t \geq t_0$.*

A case of interest is when the initial condition $N_0$ is set to zero. In such case $N_1(t)$ corresponds to the constructibility gramian of the pair $(A(t), C(t))$, and more importantly, for the purpose of reconstructing the state, $N_2(t)$ instead of $N_1(t)$, but with the advantages mentioned before. Now, a linear map analogue to (2.13) would be desirable. In that respect, define $\psi(t) := N_2(t)x(t)$. Differentiating $\psi(t)$ with respect to time, one gets

$$\begin{aligned}\dot{\psi}(t) &= \dot{N}_2(t)x(t) + N_2(t)\dot{x}(t)\\ &= \left(-A^\top(t)N_2(t) - N_2(t)A(t) - N_2(t)Q(t)N_2(t) + C^\top(t)C(t)\right)x(t) + N_2(t)A(t)x(t) + N_2(t)B(t)u(t)\\ &= -A^\top(t)N_2(t)x(t) - N_2(t)Q(t)N_2(t)x(t) + C^\top(t)C(t)x(t) + N_2(t)B(t)u(t)\\ &= -\left(A(t) + Q(t)N_2(t)\right)^\top \psi(t) + C^\top(t)y(t) + N_2(t)B(t)u(t). \end{aligned} \tag{2.15}$$

In order to get the equivalence $\psi(t) \equiv N_2(t)x(t)$, correct initial conditions are needed. If the initial condition of $N_2$ is $N_0$, and the initial condition of the system is $x_0$, the appropriated initial condition for $\psi$ is $\psi(t_0) = N_0x_0$. This implies the requirement of the initial condition for the system. However, if $N(t_0)$ is set to zero, then $\psi(t_0) = 0$; only in this case, there is certainty about initial condition for $\psi(t)$. This drives the following proposition:

**Proposition 2.3.** *Let $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{r\times n}$, $Q(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous, uniformly bounded matrix valued functions. Let $Q(t) = Q^\top(t)$ and $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I}$ for some positive constants $q_1 \geq q_2 > 0$. Let $x(t)$ and $N(t)$ be the solutions of the following differential equations:*

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0,\\ \dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = 0,\end{aligned}$$

*for $u(t) \in \mathbb{R}^m$ a measurable function. Define $y(t)$ as $y(t) = C(t)x(t)$. Then, $\psi(t) := N(t)x(t)$ satisfy the following differential equation:*

$$\dot{\psi}(t) = \left(-A^\top(t) - N(t)Q(t)\right)\psi(t) + C^\top(t)y(t) + N(t)B(t)u(t), \quad \psi(t_0) = 0.$$

Note that the dynamics of $\psi(t)$ depends only on input-output data, meaning that $\psi(t)$ can be computed online, despite its computation would not be necessary for the Kalman-Bucy filter. However, this characteristic will be really helpful in the developments of the next chapter. Now, considering $N(t)$ and $\psi(t)$ as in Proposition 2.3, we take as state estimate $\hat{x}(t) = N^{-1}(t)\psi(t)$, which dynamics results in

$$\begin{aligned}\dot{\hat{x}}(t) &= \dot{N}^{-1}(t)\psi(t) + N^{-1}(t)\dot{\psi}(t)\\ &= N^{-1}(t)A^\top(t)\psi(t) + A(t)N^{-1}(t)\psi(t) + Q(t)\psi(t) - N^{-1}(t)C^\top(t)C(t)N^{-1}(t)\psi(t)\\ &\quad - N^{-1}(t)A^\top(t)\psi(t) - Q(t)\psi(t) + N^{-1}(t)C^\top(t)y(t) + B(t)u(t)\\ &= A(t)\hat{x}(t) + B(t)u(t) - N^{-1}(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big),\end{aligned} \tag{2.16}$$

with

$$\dot{N}^{-1}(t) = N^{-1}(t)A^\top(t) + A(t)N^{-1}(t) - N^{-1}(t)C^\top(t)C(t)N^{-1}(t) + Q(t). \tag{2.17}$$

The previous expression is, in fact, the Kalman-Bucy filter. This shows that the Kalman-Bucy filter solves the current state from the time-varying algebraic linear equation $N(t)x(t) = \psi(t)$! This interpretation will be preferred along this thesis. Notice, however, that $N^{-1}(t)$ cannot be properly computed since there is no suitable initial condition when $N(t_0)$ is singular. If (2.14) is initiated at any positive definite matrix, said $N_0$, its inverse will exist for all $t \geq t_0$, and it correspond to the solution of (2.17) initiated at $N_0^{-1}$. From Theorem 2.10, given in the next section, we know that all solutions of (2.14) converge exponentially to the one initiating in zero when the pair $(A(t), C(t))$ is uniformly constructible, and therefore, for every initial initial condition for (2.17), its solution converges to the inverse we need. Since the transient error in $N^{-1}(t)$ disappears only exponentially, the estimate provided by the Kalman-Bucy filter converges exponentially as well.

Now, we want to analyze some properties of the Kalman-Bucy filter related to the the convergence and how it behaves when there are unknown inputs and bounded noise. To that matter, it is first necessary to develop some bounds of $N(t)$ because this matrix will be used to build a Lyapunov function. Fortunately, Richard S. Bucy in (R. Bucy 1972) already developed such bounds in terms of the bounds for the following gramian matrices:

$$\begin{aligned}\beta_1\mathbb{I} \geq &\mathcal{N}(t,t-T) := \int_{t-T}^{t} \Phi^\top(s,t)C^\top(s)C(s)\Phi(s,t)\mathrm{d}s \geq \alpha_1\mathbb{I} \geq 0,\\ \beta_2\mathbb{I} \geq &\mathcal{W}(t,t-T) := \int_{t-T}^{t} \Phi(t,s)Q(s)\Phi^\top(t,s)\mathrm{d}s \geq \alpha_2\mathbb{I} \geq 0.\end{aligned}$$

The first gramian matrix corresponds to the constructibility gramian of the pair $(A(t), C(t))$ evaluated on the "moving" interval $[t-T, T]$ of constant length $T$. The second one represents a kind of *controllability* gramian. If $\alpha_1$ is strictly positive, the pair $(A(t), C(t))$ is UCC; analogously, if $\alpha_2 > 0$, the pair $(A(t), Q^{1/2}(t))$ is uniformly completely controllable. Since $Q(t)$ is a degree of freedom, we can impose, as we have done along this section, some form of uniform positiveness, that is, $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I}$ for some constants $q_1 \geq q_2 > 0$. Given $q_1$ and $q_2$, for any $T > 0$ there always exist positive constants $\beta_2$ and $\alpha_2$ satisfying the previous inequality for $\mathcal{W}$. On the contrary, because $C^\top(t)C(t)$ is in general positive semi-definite, there is a minimum value for $T > 0$, if the pair is UCC, for which the constant $\alpha_1$ is greater than zero. Given this four constants and the gramian matrices, we have that

$$\begin{aligned} N_2(t) &\geq \left(\bar{\mathcal{N}}^{-1}(t,t-T) + n^2\tfrac{\beta_1\beta_2}{\alpha_1\alpha_2}\mathcal{W}(t,t-T)\right)^{-1} \\ &\geq \tfrac{\alpha_1\alpha_2}{\alpha_2+n^2\beta_1\beta_2^2}\mathbb{I}\end{aligned} \qquad \text{for } t \geq t_0 + T. \tag{2.18}$$

On the other hand, given $q_1$ and $q_2$, and the boundedness of $A(t)$ and $C(t)$, there always exists $\eta > 0$ such that $\eta \geq \|N_2(t)\|$ for all $t \geq t_0$(Proposition 2.5). Now, to have these properties at hand, we will summarize them in the following proposition.

**Proposition 2.4.** *Consider the Riccati differential equation*

$$\dot{N}(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) \geq 0,$$

*for some piecewise continuous, uniformly bounded matrix valued functions $A(t) \in \mathbb{R}^{n\times n}$, $C(t) \in \mathbb{R}^{r\times n}$, $Q(t) \in \mathbb{R}^{n\times n}$. Assume that the pair $(A(t), C(t))$ is uniformly completely constructible, and that $Q(t)$ satisfy $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I}$ for some constants $q_1 \geq q_2 > 0$. Then, there exist $T > 0$ and constants $\eta_1 \geq \eta_2 > 0$ such that*

$$\eta_1\mathbb{I} \geq N(t) \geq \eta_2\mathbb{I}, \quad \forall\, t \geq t_0 + T,$$

*for any $t_0$, and where $N(t)$ represents the solution of the Riccati differential equation.*

Consider $x(t)$ defined by (2.12) and $\hat{x}(t)$ defined by (2.16) and (2.17). Denote by $\tilde{x}(t) = \hat{x}(t) - x(t)$ the estimation error, which dynamics satisfy

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) - N^{-1}(t)C^\top(t)C(t)\tilde{x}(t). \tag{2.19}$$

To analyze indirectly the convergence of the observer, we will prove that $\tilde{x}(t)$ goes to zero exponentially fast. This can be done by considering the Lyapunov function candidate

$$V(\tilde{x}, t) = \tilde{x}^\top N(t)\tilde{x}, \qquad \eta_2\|\tilde{x}\|^2 \geq V(\tilde{x}, t) \geq \eta_1\|\tilde{x}\|^2,$$

with $\eta_1$ and $\eta_2$ as in Proposition 2.4. The time derivative of $V$, when evaluated over the trajectories of $\tilde{x}$, results in:

$$\begin{aligned}\dot{V}(t) &= \dot{\tilde{x}}^\top(t)N(t)\tilde{x}(t) + \tilde{x}^\top(t)N(t)\dot{\tilde{x}}(t) + \tilde{x}^\top(t)\tilde{N}(t)\tilde{x}(t)\\ &= \tilde{x}^\top(t)\left(A^\top(t) - C^\top(t)C(t)N^{-1}(t)\right)N(t)\tilde{x}(t) + \tilde{x}^\top(t)N(t)\left(A(t) - N^{-1}(t)C^\top(t)C(t)\right)\tilde{x}(t)\\ &\quad \tilde{x}^\top(t)\left(-A^\top(t)N(t) - A(t)N(t) - N(t)Q(t)N(t) + C^\top(t)C(t)\right)\tilde{x}(t)\\ &= -\tilde{x}^\top(t)\left(N(t)Q(t)N(t) + C^\top(t)C(t)\right)\tilde{x}(t) \leq -\tilde{x}^\top(t)N(t)Q(t)N(t)\tilde{x}(t)\\ &\leq -q_2\eta_2V(t).\end{aligned}$$

From the last inequality, it is easy to conclude the exponential stability (Theorem 2.2). This gives support to the assertion that $\hat{x}(t)$ converge to $x(t)$. Now, we are going to investigate what happens when there are disturbances in the system that the model do not consider. Let $\nu(t) \in \mathbb{R}^n$ and $\delta(t) \in \mathbb{R}^r$ be locally integrable, and uniformly bounded, that is, $\|\nu(t)\| \leq \Delta_1$ and $\|\delta(t)\| \leq \Delta_2$. $\nu(t)$ acts as an additive unknown disturbance, whereas $\delta(t)$ as additive noise. In the presence of these disturbances, the estimation error dynamics changes to:

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) - N^{-1}(t)C^\top(t)C(t)\tilde{x}(t) - \nu(t) + N^{-1}(t)C^\top(t)\delta(t).$$

To analyze this case, we will use the same Lyapunov function as before. In this situation, the time derivative of $V(t)$ results into

$$\begin{aligned}
\dot{V}(t) &= -\tilde{x}^\top(t)\left(N(t)Q(t)N(t) + C^\top(t)C(t)\right)\tilde{x}(t) + 2\,\tilde{x}^\top(t)N(t)\nu(t) + 2\,\tilde{x}^\top(t)C^\top(t)\delta(t) \\
&\leq -q_2\eta_2 V(t) - \tilde{x}^\top(t)C^\top(t)C(t)\tilde{x}(t) + \epsilon_1\tilde{x}^\top(t)N(t)\tilde{x}(t) + \frac{1}{\epsilon_1}\nu^\top(t)N(t)\nu(t) \\
&\quad + \tilde{x}^\top(t)C^\top(t)C(t)\tilde{x}(t) + \delta^\top(t)\delta(t) \\
&\leq -\frac{1}{2}q_2\eta_2 V(t) + \frac{1}{q_2\eta_2}\nu^\top(t)N(t)\nu(t) + \delta^\top(t)\delta(t) \\
&= -\frac{1}{2}q_2\eta_2 V(t) + \frac{\eta_1}{q_2\eta_2}\|\nu(t)\|^2 + \|\delta(t)\|^2.
\end{aligned}$$

This last inequality shows that the estimation error is Input-to-State-Stable with respect to both of the disturbances. It is possible to compute an ultimate bound of the estimation error from the last inequality. However, to obtain the ultimate bound, we will perform a different analysis. Consider now as an error measure $\varepsilon(t) = N(t)\hat{x}(t) - \psi(t)$, with $\psi(t)$ as in (2.15). The dynamics of this new variable is

$$\begin{aligned}
\dot{\varepsilon}(t) &= \dot{N}(t)\hat{x}(t) + N(t)\dot{\hat{x}}(t) - \dot{\psi}(t) \\
&= -A^\top(t)N(t)\hat{x}(t) - N(t)A(t)\hat{x}(t) - N(t)Q(t)N(t)\hat{x}(t) + C^\top(t)C(t)\hat{x}(t) + N(t)A(t)\hat{x}(t) \\
&\quad + N(t)B(t)u(t) - C^\top(t)C(t)\hat{x}(t) + C^\top(t)C(t)x(t) + C^\top(t)\delta(t) + A^\top(t)\psi(t) + N(t)Q(t)\psi(t) \\
&\quad - C^\top(t)C(t)x(t) - C^\top(t)\delta(t) - N(t)B(t)u(t) \\
&= -\big(A^\top(t) + N(t)Q(t)\big)\varepsilon(t).
\end{aligned}$$

This dynamics is uniform exponential stable, and it can be proved using $W(\varepsilon, t) = \varepsilon^\top N^{-1}(t)\varepsilon$ as Lyapunov function. Then, in the limit, when $\varepsilon(t) = 0$, we have that $\hat{x}(t) = N^{-1}(t)\psi(t)$, as expected. However, in this case, $x(t)$ is not equivalent to $N^{-1}(t)\psi(t)$ given the presence of the disturbances. If we would know the values of $\nu(t)$ and $\delta(t)$, the correct computation of $\psi(t)$ (call it $\bar{\psi}(t)$) would be

$$\dot{\bar{\psi}}(t) = \left(-A^\top(t) - N(t)Q(t)\right)\bar{\psi}(t) + C^\top(t)\big(y(t) - \delta(t)\big) + N(t)\big(B(t)u(t) + \nu(t)\big).$$

Lets denote the difference between the computed $\psi(t)$ and the correct one $\bar{\psi}(t)$ by $\psi_\Delta(t)$, which dynamics is

$$\dot{\psi}_\Delta(t) = -\left(A^\top(t) + N(t)Q(t)\right)\psi_\Delta(t) + C^\top(t)\delta(t) - N(t)\nu(t).$$

Using $W_\psi(\psi_\Delta, t) = \psi_\Delta^\top N^{-1}(t)\psi_\Delta$ as Lyapunov function, it is possible to show the boundedness of $\psi_\Delta(t)$. For the moment, the bound would not be of interest, and it would be treated later in this thesis. Using $x(t) = N^{-1}(t)\bar{\psi}(t) = N^{-1}(\psi(t) - \psi_\Delta(t))$ we can compute the size of the final error:

$$\tilde{x}(t) = \hat{x}(t) - x(t) = N^{-1}(t)\psi(t) - N^{-1}(t)\big(\psi(t) - \psi_\Delta(t)\big) = N^{-1}(t)\psi_\Delta(t). \tag{2.20}$$

This shows the intrinsic least square property of the Kalman-Bucy filter. Additionally, this relation will help us to compare the performance of the Kalman-Bucy filter and the observer presented in this work.

To finish this section, and for the sake of the self-contained, some properties of the Riccati differential equation will be discussed in the next heading.

### 2.3.1 Lyapunov and Riccati differential equations

In this section, we are going to develop some properties of the Matrix Riccati differential equations based on well known properties of Lyapunov differential equations. These two classes of equations are defined as follows:

**Definition 2.10.** Let $A(t)$ and $Q(t)$ be uniformly bounded piecewise continuous matrix valued function mapping $\mathbb{R} \to \mathbb{R}^{n\times n}$. In addition, $Q(t)$ is assumed symmetric. A Lyapunov Differential Equation (LDE) is a matrix differential equation of the form

$$\pm\dot{X}(t) = A(t)X(t) + X(t)A^\top(t) + Q(t), \quad X(t_0) = X^\top(t_0) = X_0 \geq 0 \text{ (or } \leq 0).$$

A Lyapunov Differential Inequality (LDI) is obtained when the equality sign is replaced by a inequality one, for example:

$$\pm\dot{X}(t) \geq A(t)X(t) + X(t)A^\top(t) + Q(t), \quad X(t_0) = X^\top(t_0) = X_0 \geq 0 \text{ (or } \leq 0).$$

**Definition 2.11.** Let $A(t)$, $R(t)$, and $Q(t)$ be matrix valued functions mapping $\mathbb{R} \to \mathbb{R}^{n\times n}$. The three matrices are assumed uniformly bounded in magnitude. In addition, the matrices $R(t)$ and $Q(t)$ are assumed symmetric. A Riccati Differential Equation (RDE) is a matrix differential equation of the form:

$$\dot{X}(t) = A(t)X(t) + X(t)A^\top(t) + X(t)R(t)X(t) + Q(t), \quad X(t_0) = X^\top(t_0) = X_0 \geq 0 \text{ (or } \leq 0). \tag{2.21}$$

A Riccati Differential Inequality is obtained when the equality sign is replaced by a inequality one, for example:

$$\pm\dot{X}(t) \geq A(t)X(t) + X(t)A^\top(t) + X(t)R(t)X(t) + Q(t), \quad X(t_0) = X^\top(t_0) = X_0 \geq 0 \text{ (or } \leq 0).$$

From the structure of these equations it is clear that a LDE is a RDE where $R(t)$ is zero. Furthermore, a LDE is linear and its solution can be given in term of the state transition matrix associated to $A(t)$. Such solution is given by (Abou-Kandil et al. 2003, Cor. 1.1.6):

$$X(t) = \Phi(t,t_0)X_0\Phi^\top(t,t_0) + \int_{t_0}^{t} \Phi(t,s)Q(s)\Phi^\top(t,s)\mathrm{d}s.$$

One can verify this fact by evaluating the boundary condition, and by replacing $X(t)$ in the differential equation. From the structure of the solution, the following lemma holds:

**Lemma 2.1.** *(Abou-Kandil et al. 2003, Theo. 4.1.2) Let $\mathcal{I} \subset \mathbb{R}$ be an interval with $t_0 \in \mathcal{I}$.*

- *If $X(t)$ is on $\mathcal{I}$ a solution of the homogeneous LDE:*

  $$\dot{X}(t) = A(t)X(t) + X(t)A^\top(t), \quad t \in \mathcal{I},$$

  *then $\pm X(t_0) \geq$ (or $>$) $0$ implies $\pm X(t) \geq$ (or $>$) $0$ for $t \in \mathcal{I}$.*
- *If $X(t)$ is on $\mathcal{I}$ a solution of the LDI:*

  $$\pm\dot{X}(t) \leq \pm A(t)X(t) \pm X(t)A^\top(t), \quad t \in \mathcal{I} \cap (-\infty, t_0],$$

  *then $\pm X(t_0) \geq$ (or $>$) $0$ implies $\pm X(t) \geq$ (or $>$) on $\mathcal{I} \cap (-\infty, t_0]$.*
- *If $X(t)$ is on $\mathcal{I}$ a solution of the LDI*

  $$\pm\dot{X}(t) \geq \pm A(t)X(t) \pm X(t)A^\top(t), \quad t \in \mathcal{I} \cap [t_0, \infty),$$

  *then $\pm X(t_0) \geq$ (or $>$) $0$ implies $\pm X(t) \geq 0$ (or $>$) on $\mathcal{I} \cap [t_0, \infty)$.*

- *(Abou-Kandil et al. 2003, Lem. 4.1.3)* $\dot{X}(t) \leq 0$ *(or* $<$*) for* $t \in \mathcal{I}$ *implies* $X(t_2) \leq$ *(or* $<$*)* $X(t_1)$ *for* $t_1, t_2 \in \mathcal{I}$ *and* $t_1 < t_2$.

These ordering properties for the solutions of homogeneous LDI will help us to establish analogue properties for RDE. This in turn allows to proof the existence of solutions of RDE in certain cases. The ordering property for the solution of RDE is given in the next theorem:

**Theorem 2.9** (Comparison Theorem)**.** *(Abou-Kandil et al. 2003, Theo. 4.1.1) Let* $\mathcal{I} \subset \mathbb{R}$ *be some interval,* $t_0 \in \mathcal{I}$*, and let the matrix functions* $A_i(t)$*,* $R_i(t) = R_i^\top(t)$*,* $Q_i(t) = Q_i^\top(t)$*, for* $i = 1, 2$*, be piecewise continuous and locally bounded. Consider two RDE of the form*

$$\dot{X}_i(t) = -A_i^\top(t)X_i(t) - X_i(t)A(t) + Q_i(t) - X_i(t)R_i(t)X_i(t), \quad X_i(t_0) = X_i^\top(t_0).$$

*Define an associated Hamiltonian block matrix* $H_i(t)$ *and the "imaginary" matrix unit* $J$ *as*

$$H_i(t) := \begin{bmatrix} A_i(t) & R_i(t) \\ Q_i(t) & -A_i^\top(t) \end{bmatrix}, \qquad J := \begin{bmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix}.$$

*Spouse that* $X_2(t_0) \geq$ *(or* $>$*)* $X_1(t_0)$*, then*

$$J\,H_1(t) \geq J\,H_2(t) \quad \text{for} \quad t \in \mathcal{I}$$

*implies* $X_2(t) \geq$ *(or* $>$*)* $X_1(t)$ *for* $t \in \mathcal{I} \cap [t_0, \infty)$*; i.e., the solutions of the RDE* $X_i(t)$ *depend monotonically on* $J\,H_i(t)$ *(and in particular on* $Q_i(t)$ *and* $-R_i(t)$*) and on the initial value of* $X_i(t_0)$*. This statement remains valid if we replace therein everywhere* $\geq$ *(or* $>$*) by* $\leq$ *(or* $<$*) and simultaneously* $[t_0, \infty)$ *by* $(-\infty, t_0]$*.*

*Proof.* Define $\tilde{X}(t) = X_2(t) - X_1(t)$ for which $\tilde{X}(t_0) \geq$ (or $>$) 0. Let us drop the time dependency for the sake of brevity; the time derivative of $\tilde{X}(t)$ results in

$$\begin{aligned}
\dot{\tilde{X}} &= -A_2^\top X_2 - X_2A_2 + Q_2 - X_2R_2X_2 + A_1^\top X_1 + X_1A_1 - Q_1 + X_1R_1X_1 \\
&= -A_2^\top X_2 - X_2A_2 - \big(X_2 - X_1\big)R_2\big(X_2 - X_1\big) - X_2R_2X_1 - X_1R_2X_2 + X_1R_2X_1 + Q_2 - Q_1 \\
&\quad + A_1^\top X_1 + X_1A_1 + X_1R_1X_1 \\
&= -A_2^\top\big(X_2 - X_1\big) - \big(X_2 - X_1\big)A_2 - \big(X_2 - X_1\big)R_2\big(X_2 - X_1\big) + Q_2 - Q_1 + 2X_1R_2X_1 \\
&\quad - X_2R_2X_1 - X_1R_2X_2 - \big(A_2^\top - A_1^\top\big)X_1 - X_1\big(A_2 - A_1\big) - X_1\big(R_2 - R_1\big)X_1 \\
&= \big(-A_2^\top - X_1R_2\big)\tilde{X} + \tilde{X}\big(-A_2 - R_2X_1\big) - \tilde{X}R_2\tilde{X} \\
&\quad \big(A_1^\top - A_2^\top\big)X_1 + X_1\big(A_1 - A_2\big) + X_1\big(R_1 - R_2\big)X_1 + Q_2 - Q_1 \\
&= \big(-A_2^\top - X_1R_2 - \frac{1}{2}\tilde{X}R_2\big)\tilde{X} + \tilde{X}\big(-A_2 - R_2X_1 - \frac{1}{2}R_2\tilde{X}\big) \\
&\quad \big(A_1^\top - A_2^\top\big)X_1 + X_1\big(A_1 - A_2\big) + X_1\big(R_1 - R_2\big)X_1 + Q_2 - Q_1 \\
&= \tilde{A}\tilde{X} + \tilde{A}^\top\tilde{X} + \begin{bmatrix}\mathbb{I} & X_1\end{bmatrix} J\left(H_1 - H_2\right)\begin{bmatrix}\mathbb{I} \\ X_1\end{bmatrix} = \tilde{A}\tilde{X} + \tilde{A}^\top\tilde{X} + \tilde{Q}(t).
\end{aligned}$$

Then, the dynamics of $\tilde{X}(t)$ satisfy a LDE. In account of $J\,(H_1(t) - H_2(t)) \geq 0$ we have

$$\dot{\tilde{X}}(t) \geq \tilde{A}(t)\tilde{X}(t) + \tilde{X}(t)\tilde{A}^\top(t),$$

then, by Lemma 2.1, $\tilde{X}(t_0) = X_2(t_0) - X_1(t) \geq 0$ (or $> 0$) implies that $\tilde{X}(t) \geq 0$ (or $> 0$) on $\mathcal{I} \cap [t_0, \infty)$. Equivalently, we have that $X_2(t) \geq$ (or $>$) $X_1(t)$ over the same interval. If we reverse everywhere the inequalities, the statement is true over $\mathcal{I} \cap (\infty, t_0]$. □

Now, we are going to consider a specific class of RDE involved in the problem of state estimation. With the help of the Comparison Theorem, the existence of solution for the RDE over $[t_0, \infty)$ can be guarantee. First, we consider the LDE for the constructibility gramian and the associated RDE:

$$\begin{aligned} \dot{N}_1(t) &= -A^\top(t)N_1(t) - N_1(t)A(t) - N_1Q(t)N_1(t) + C^\top(t)C(t), \\ \dot{N}_2(t) &= -A^\top(t)N_2(t) - N_2(t)A(t) + C^\top(t)C(t), \end{aligned} \tag{2.22}$$

with $Q(t) \geq 0$ and $N_2(t_0) \geq N_1(t_0)$. The associated Hamiltonian matrices for these equations are

$$H_1(t) = \begin{bmatrix} A(t) & Q(t) \\ C^\top(t)C(t) & -A^\top(t) \end{bmatrix} \qquad H_2(t) = \begin{bmatrix} A(t) & 0 \\ C^\top(t)C(t) & -A^\top(t) \end{bmatrix}.$$

In this case, the criteria $J(H_1(t) - H_2(t))$ results in

$$J(H_1(t) - H_2(t)) = \begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & Q(t) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & Q(t) \end{bmatrix} \geq 0,$$

or $JH_1(t) \geq JH_2(t)$. Then, by Theorem 2.9, $N_2(t) \geq N_1(t)$ for $t \geq t_0$. In other words, $N_2(t)$ (the constructibility gramian of the pair $(A(t), C(t))$) represent an upper bound for the solution of the first RDE, asserting the existence of solutions for all $t \geq t_0$. Notice that here we did not need the invertibility of $N_2(t)$ at any point, then, the existence of the solution is independent of the pair $(A(t), C(t))$ being constructible or not. The only requirement is that $Q(t)$ be a positive semi-definite matrix.

An other interesting property of the RDE (2.22) is that its solution remains positive semi-definite if the initial condition is a positive semi-definite matrix. To see this, consider a third RDE:

$$\dot{N}_3(t) = -A^\top(t)N_3(t) - N_3(t)A(t) - N_3(t)Q(t)N_3(t), \quad N_3(t_0) = 0.$$

Then, for any $N_1(t_0)$, we have $N_1(t_0) \geq N_3(t_0)$. The criteria $J(H_3(t) - H_1(t))$ in this case yields

$$\begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -C^\top(t)C(t) & 0 \end{bmatrix} = \begin{bmatrix} C^\top(t)C(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

This means that $N_1(t) \geq N_3(t)$ for all $t \geq t_0$. Since $N_3(t) = 0$, we can conclude that $N_1(t) \geq 0$ for $t \geq t_0$. Furthermore, if the initial condition $N_1(t_0)$ is positive definite, the solution $N_1(t)$ remains positive definite. If the initial condition is positive definite then, by continuity, the solution remains positive at least for a short period of time. In such time interval the inverse of $N_1(t)$ exist. Denote it by $K(t)$. It satisfy $K(t_0) = N_1^{-1}(t_0)$ and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}K(t)N_1(t) &= \dot{K}(t)N_1(t) + K(t)N_1(t) = 0, \\ \dot{K}(t) &= -K(t)N_1(t)K(t) = K(t)A^\top(t) + A(t)K(t) - K(t)C^\top(t)C(t)K(t) + Q(t). \end{aligned} \tag{2.23}$$

The existence of solutions for $K(t)$ can be proved analogously to the case of $N_1(t)$. Given that $N_1(t)$ and $K(t)$ exist, and because they satisfy the relation $N_1(t)K(t) = K(t)N_1(t) = \mathbb{I}$, is possible to assert that both of them are positive definite. Also, notice that $K(t)$ has the same dynamics as the gain for the Kalman-Bucy filter.

Now that we know that $N_1(t)$ exist for all $t \geq t_0$, and that it remains positive (semi) definite if $N(t_0)$ is positive (semi) definite, we can proceed to proof that $N_1(t)$ is uniformly bounded if there exist positive constants $q_1 \geq q_2$, $a$, and $c$ such that

$$q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I}, \qquad a \geq \|A(t)\|, \qquad c \geq \|C^\top(t)C(t)\|.$$

Consider the positive definite function $V(N_1) = \frac{1}{2}\mathrm{tr}\big(N_1^2\big)$, which derivative along the dynamics of $N_1(t)$ results in

$$\begin{aligned} \dot{V}(t) &= \mathrm{tr}\left(-N_1(t)\big(A^\top(t) + A(t)\big)N_1(t) - N_1^{3/2}(t)Q(t)N_1^{3/2}(t) + C^\top(t)C(t)N_1(t)\right) \\ &\leq \sigma_1\big(A^\top(t) + A(t)\big)\mathrm{tr}\big(N_1^2(t)\big) - \sigma_n\big(Q(t)\big)\mathrm{tr}\Big(\big(N_1^2(t)\big)^{3/2}\Big) + \frac{1}{2}\mathrm{tr}\Big(\big(C^\top(t)C(t)\big)^2\Big) + \frac{1}{2}\mathrm{tr}\big(N_1^2(t)\big) \\ &\leq (4\,a + 1)V(t) + \frac{1}{2}c\,\mathrm{tr}\big(C^\top(t)C(t)\big) - q_2\mathrm{tr}\Big(\big(N_1^2(t)\big)^{3/2}\Big). \end{aligned}$$

Here, the relationship $\mathrm{tr}(XMX) \geq \sigma_n(M)\mathrm{tr}(X^2)$, for $X, M \in \mathbb{R}^{n\times n}$ and $M = M^\top$, and $\|M\| = \|M^\top\| = \sigma_1(M)$, for $M \in \mathbb{R}^{n\times n}$, where used. Now, consider the following properties:

$$\mathrm{tr}\big(M^p\big) = \sum_{i=1}^{n} \lambda_i^p\big(M\big), \quad 0 \leq M = M^\top \in \mathbb{R}^{n\times n},\ p > 0,$$
$$\sum_{i=1}^{n} r_i^p \geq \frac{1}{n^{p-1}}\Big(\sum_{i=1}^{n} r_i\Big)^p, \quad p \geq 1,\ r_i \geq 0.$$

Using these relations, we can arrive to the inequality

$$\begin{aligned}
\dot V(t) &\leq \frac{n}{2}c^2 + \big(4\,a+1\big)V(t) - q_2\sum_{i=1}^{n}\lambda_i^{3/2}\big(N_1^2(t)\big) \\
&\leq \frac{n}{2}c^2 + \big(4\,a+1\big)V(t) - \frac{q_2}{\sqrt{n}}\left(\sum_{i=1}^{n}\lambda_i\big(N_1^2(t)\big)\right)^{3/2} \\
&\leq \frac{n}{2}c^2 + \big(4\,a+1\big)V(t) - 2^{3/2}\frac{q_2}{\sqrt{n}}V^{3/2}(t).
\end{aligned} \tag{2.24}$$

Denote by $\nu > 0$ the unique root of the polynomial

$$2^{3/2}\frac{q_2}{\sqrt{n}}x^{3/2} - (4\,a+1)x - \frac{n}{2}c^2 = 0.$$

Then, for $V(t) > \nu$, $\dot V(t) < 0$. Then, recalling Theorem 2.5, we can claim that $N_1(t)$ remains bounded. This is summarized in the following proposition.

**Proposition 2.5.** *Let $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{r\times n}$ be piecewise continuous matrix valued function. Assume that there exist positive constants $a$ and $c$ such that $a \geq \|A(t)\|$, $c \geq \|C(t)\|$ for all $t \geq 0$. Let $Q(t) \in \mathbb{R}^{n\times n}$ be a symmetric piecewise continuous matrix function, accepting the bounds $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I} > 0$. Then, any solution of the Riccati differential equation*

$$\dot N(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) \geq 0,$$

*remains bounded, that is, $\|N(t)\| \leq \eta$ for all $t \geq t_0$ and for some $\eta > 0$.*

The final point to discuss is the dependency of the solution of (2.22) on the parameters $A(t)$, $Q(t)$, and $C(t)$, rather than on the initial condition, when the pair $(A(t), C(t))$ is UCC. What we found is that all the solutions starting in any positive positive semi-definite matrix converge to the solution initiated in zero. This was already known by R. E. Kalman 1960. However, the proof presented in (R. E. Kalman 1960, Theo. 7.2) has some omissions and mistakes, which do not change the result, but make it difficult of reproducing. In the proof presented below, we not only reconstructed the proof of Kalman, but we investigate some convergence properties that were left aside by him. More precisely, the difference between two ordered solutions of the RDE converge to a fix bounded value in fixed-time.

**Theorem 2.10.** *Let $A(t) \in \mathbb{R}^{n\times n}$, $C(t) \in \mathbb{R}^{r\times n}$, and $Q(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous, uniformly bounded matrix valued functions. Let $Q(t) = Q^\top(t)$ with $q_1\mathbb{I} \geq Q(t) \geq q_2\mathbb{I} > 0$. Consider the Riccati differential equation*

$$\dot N(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = N^\top(t_0) \geq 0.$$

*Assume that the pair $(A(t), C(t))$ is UCC. Consider $N_a(t)$ and $N_b(t)$ two solutions of the RDE starting at $N_a(t_0) \geq 0$ and $N_b(t_0) \geq 0$ with $N_a(t_0) \geq N_b(t_0)$. Then $N_a(t) \to N_b(t)$ as $t \to \infty$. In particular this means that all the solutions of the RDE starting at any positive semi-definite matrix converge to the solution started in zero.*

*Denote by $\bar{N}(t)$ the solution starting at zero and by $N(t)$ any other solution with positive semi-definite initial condition. Let $\eta_1 \mathbb{I}_n \geq \bar{N}(t) \geq \eta_2 \mathbb{I}_n$ for all $t \geq t_0 + T$. Then, we have that $r \geq \|N(t) - \bar{N}(t)\|$ for*

$$t \geq t_0 + T + \frac{\eta_1}{q_2 \eta_2^2} \ln\left(\frac{\eta_1^2}{r} + 1\right)$$

*for given given $r > 0$ and $N(t_0) \geq 0$.*

*Proof.* Given the ordering property of the RDE we have that $N_a(t) \geq N_b(t)$ for all $t \geq t_0$, and the difference $E(t) = N_a(t) - N_b(t)$ also satisfy $E(t) \geq 0$ for all $t_0$. The dynamics of the difference satisfy

$$\begin{aligned}
\dot{E}(t) &= -A^\top(t)E(t) - E(t)A(t) - N_a(t)Q(t)N_a(t) + N_b(t)Q(t)N_b(t) \\
&= -A^\top(t)E(t) - E(t)A(t) - E(t)Q(t)E(t) - N_a(t)Q(t)N_b(t) - N_b(t)Q(t)N_a(t) + 2\,N_b(t)Q(t)N_b(t) \\
&= -\Big(A^\top(t) + N_b(t)Q(t)\Big)E(t) - E(t)\Big(A(t) + Q(t)N_b(t)\Big) - E(t)Q(t)E(t).
\end{aligned}$$

Since the pair $(A(t), C(t))$ is UCC, for $t \geq t_0 + T$ we have the bounds $\eta_1 \mathbb{I}_n \geq N_b(t) \geq \eta_2 \mathbb{I}_n > 0$. Denote by $H(t)$ the inverse of $N_b(t)$ for $t \geq t_0 + T$. It satisfies the bounds $\frac{1}{\eta_2}\mathbb{I}_n \geq H(t) \geq \frac{1}{\eta_1}\mathbb{I}_n$, and the dynamics:

$$\dot{H}(t) = H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + Q(t),\ H(t_0+T) = N_b^{-1}(t_0+T).$$

For $t \geq t_0 + T$, consider the candidate Lyapunov function

$$\begin{aligned}
V(E,t) = \frac{1}{2}\mathrm{tr}\Big(\big(E\,H(t)\big)^2\Big) &= \frac{1}{2}\mathrm{tr}\Big(H^{1/2}(t)E\,H(t)E\,H^{1/2}(t)\Big), \\
\frac{1}{2\,\eta_2^2}\mathrm{tr}\big(E^2\big) \geq V(E,t) &\geq \frac{1}{2\,\eta_1^2}\mathrm{tr}\big(E^2\big).
\end{aligned}$$

To keep the equations short, the time dependency of the matrices is dropped. The derivative of $V(t)$ along the dynamics of $E(t)$ yields

$$\begin{aligned}
\dot{V} &= \mathrm{tr}\left(E\,H\Big(\dot{E}H + E\,\dot{H}\Big)\right) \\
&= \mathrm{tr}\big(-(E\,H)^2A^\top - E\,H\,E\,Q - (H\,E)^2A - E\,H\,E\,Q - (H\,E)^2Q\,E + (E\,H)^2A^\top + (H\,E)^2A \\
&\quad -(E\,H)^2C^\top C\,H + E\,H\,E\,Q\big) \\
&= \mathrm{tr}\left(-E\,H\,E\,Q - (H\,E)^2Q\,E - H^{1/2}E\,H\,C^\top C\,H\,E\,H^{1/2}\right) \\
&\leq \mathrm{tr}\left(-H^{1/2}E\,Q\,E\,H^{1/2} - E^{1/2}H\,E\,Q\,E\,H\,E^{1/2}\right) \\
&\leq -\sigma_n(Q)\mathrm{tr}\Big(E\,H\,E\Big) - \sigma_n(Q)\mathrm{tr}\Big(E\big(E\,H\big)^2\Big).
\end{aligned}$$

Knowing that $\mathrm{tr}(A^2B^2) \geq \mathrm{tr}((A\,B)^2)$ (Bernstein 2009, Fact 8.12.22), it follows that

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{q_2}{\eta_1}\mathrm{tr}\Big(E^2(t)\Big) - q_2\mathrm{tr}\Big(H^{1/2}(t)E^{3/2}(t)H(t)E^{3/2}(t)H^{1/2}(t)\Big) \\
&\leq -\frac{q_2}{\eta_1}\mathrm{tr}\Big(E^2(t)\Big) - \frac{q_2}{\eta_1^2}\mathrm{tr}\Big(E^3(t)\Big) \\
&\leq -\frac{q_2}{\eta_1}\mathrm{tr}\Big(E^2(t)\Big) - \frac{q_2}{n^2\eta_1^2}\mathrm{tr}^{3/2}\Big(E^2(t)\Big) \\
&\leq -2\,q_2\frac{\eta_2^2}{\eta_1}V(t) - 2^{3/2}q_2\frac{\eta_2^2}{\eta_1^2}V^{3/2}(t) < 0.
\end{aligned} \tag{2.25}$$

Then $E(t) = 0$ is uniformly asymptotically stable. Now, we are interested in investigate the time it takes for $\|E(t)\|$ to be less than or equal to some $r > 0$ for any given initial error. In order to proceed, we have to solve the differential inequality (2.25), which is of separable variables. Knowing that

$$\int \frac{1}{k_1 v + k_2 v^p} \mathrm{d}v = \frac{p\, \ln(v) - \ln(k_1 v + k_2 v^p)}{k_1(p-1)} \quad \text{and} \quad \begin{array}{c} f(v) = \frac{v^p}{k_1 v + k_2 v^p} \\ f^{-1}(v) = \left(\frac{1}{k_1 v} - \frac{k_2}{k_1}\right)^{1/(1-p)} \end{array},$$

the differential equation can be solved. Using the solution together with the Comparison Lemma (Khalil 2002, Lem. 3.4), we have

$$V(t) \leq \left( \left( \frac{1}{V^{1/2}(t_0+T)} + \frac{2^{1/2}}{\eta_1} \right) \exp\left( q_2 \frac{\eta_2^2}{\eta_1}(t - t_0 - T) \right) - \frac{2^{1/2}}{\eta_1} \right)^{-2} \quad \text{for} \quad t \geq t_0 + T. \tag{2.26}$$

Since $\|E(t)\| \leq \mathrm{tr}^{1/2}(E^2(t))$ and $2\,\eta_1^2 V(t) \geq \mathrm{tr}(E^2(t))$, if $2^{1/2}\eta_1 V^{1/2}(t) \leq r$ we can ensure the $\|E(t)\| \leq r$. Then, we need to estimate a time for which $V(t) \leq r^2/(2\,\eta_1^2)$. From (2.26) we have

$$V(t) \leq \left( \left( \frac{1}{V^{1/2}(t_0+T)} + \frac{2^{1/2}}{\eta_1} \right) \exp\left( q_2 \frac{\eta_2^2}{\eta_1}(t_0 + T) \right) - \frac{2^{1/2}}{\eta_1} \right)^{-2} \leq \frac{r^2}{2\,\eta_1^2}.$$

Solving for $t - t_0 - T$ we get

$$t - t_0 - T \geq \frac{\eta_1}{q_2 \eta_2^2} \ln \left( \frac{2^{1/2}(\eta_1^2 + r)}{r\left(2^{1/2} + \frac{\eta_1}{V^{1/2}(t_0+T)}\right)} \right).$$

The RHS represent an amount of time that ensure $\|E(t)\| \leq r$. Taking the limit for $V(t_0+T) \to \infty$ we have

$$t - t_0 - T \geq \frac{\eta_1}{q_2 \eta_2^2} \ln \big( \frac{\eta_1^2}{r} + 1 \big).$$

Then, the time needed to reach the region defined by $\|E(t)\| \leq r$ is bounded for any initial condition. □

# Chapter 3

# Fixed-time observer for linear time-varying systems

In this chapter, we present the main result of this work: An observer capable of estimating the internal state of a linear time-varying system in fixed-time. First, the observer is introduced and its properties exposed under ideal conditions. These ideal conditions consist on having the exact model of the linear system. Under such assumptions, the observer is capable of giving an exact estimate of the system's state, and such estimate is obtained in a fixed amount of time. The second part of the chapter is dedicated to study how the observer and the estimate given by it behave when there are some deviations on the model. The deviations are modelled as unknown "inputs" and additive noise in the measurements. The presence of these disturbances do not allow the exact convergence; however, the error committed by the observer is not arbitrary large, but it depends on the size of the disturbances. This shows that the observer is robust.

## 3.1 Observer in the unperturbed case

The objective of this work is to design an observer for a general linear time-varying system. The aim is to have an accurate estimate of the internal state of the system in a finite amount of time. To characterize the class of systems for which the observer will be designed, consider piecewise continuous matrix valued functions $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, and $C(t) \in \mathbb{R}^{r\times n}$, defined for all $t \in [0,\infty)$, for which there exist positive constants $a$, $b$, and $c$ such that $a \geq \|A(t)\|$, $b \geq \|B(t)\|$, $c \geq \|C(t)\|$ for all $t \geq 0$, that is, the matrices are uniformly bounded. Using these matrices, we will consider a $n$-dimensional linear system described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0, \\ y(t) &= C(t)x(t), \end{aligned} \tag{3.1}$$

where $u(t) \in \mathbb{R}^m$, the input, is assumed to be a measurable function, and the output $y(t) \in \mathbb{R}^r$ represents the available information about the state. Both functions, $u(t)$ and $y(t)$, are assumed known for all $t \geq t_0$.

The objective is then to estimate $x(t)$ using only the input-output information, and the knowledge of the system matrices. The estimation will be handled by a dynamical system, also known as observer. As we showed in Section 2.2, this estimation task is not always possible, and it is conditioned by the matrices $A(t)$ and $C(t)$. So, in order to recover the internal state, we have to assume the following:

**Assumption 3.1.** *The pair $\big(A(t), C(t)\big)$ is uniformly completely constructible (see Definition 2.9).*

This in turn implies Proposition 2.2, 2.4 and 2.10 which will help us to study the properties of the observer.

Now, with the system specified, and the main assumption posed, we are in position of introducing the observer. Defining $\hat{x}(t)$ as the estimate of $x(t)$, the observer is described by the following set of equations

$$\begin{aligned}\dot{\hat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) - H(t)C^\top(t)\Big(C(t)\hat{x}(t) - y(t)\Big)\\ &\quad - H(t)\Big(N(t)\Lambda_1\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_2}\Big),\end{aligned} \tag{3.2}$$

$$\dot{H}(t) = H(t)A^\top(t) + A(t)H(t) - H(t)C^\top(t)C(t)H(t) + Q(t), \quad H(t_0) > 0, \tag{3.3}$$

$$\dot{N}(t) = -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = 0, \tag{3.4}$$

$$\dot{\psi}(t) = -\left(A(t) + Q(t)N(t)\right)^\top \psi(t) + C^\top(t)\psi(t) + N(t)B(t)u(t), \quad \psi(t_0) = 0. \tag{3.5}$$

The observer's parameters are: $0 \leq p_1 < 1$, $p_2 > 1$, $\Lambda_i = \text{diag}\{\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n}\}$, $\lambda_{i,j} > 0$, $Q(t) = Q^\top(t)$ and $q_1\mathbb{I}_n \geq Q(t) \geq q_2\mathbb{I}_n$ for all $t \geq t_0$ for some $q_1 \geq q_2 > 0$. These parameters and their ranges are shown in the next table.

| Parameter | Range |
|---|---|
| $p_1$ | $[0, 1)$ |
| $p_2$ | $(1, \infty)$ |
| $\Lambda_i$ | $\text{diag}\{\lambda_{i,1}, \lambda_{i,2}, \cdots, \lambda_{i,n}\}$ |
| $\lambda_{i,j}$ | $(0, \infty)$ |
| $Q(t) = Q^\top(t)$ | $q_1\mathbb{I}_n \geq Q(t) \geq q_2\mathbb{I}_n$ |
| $q_1$ and $q_2$ | $(0, \infty)$ |
| $H(t_0)$ | Symmetric positive definite |
| $N(t_0)$ | $0 \in \mathbb{R}^{n\times n}$ |
| $\psi(t_0)$ | $0 \in \mathbb{R}^n$ |

Table 3.1: Observer parameters

First of all, notice that observer (3.2) has the same linear feedback term we found in the Kalman-Bucy filter. The novelty lies in the non-linear feedback terms depending on exponents $p_1$ and $p_2$. Notice also that the non-linear terms depend on $N(t)$ and $\psi(t)$. As it has been shown in the previous chapter, in (2.15), this functions are related by the equivalence $\psi(t) = N(t)x(t)$ when the initial conditions $N(t_0) = 0$ and $\psi(t_0) = 0$ are used (Proposition 2.3). Then, the terms inside the semi-brackets can be rephrased as $N(t)(\hat{x}(t) - x(t))$. Since $N(t)$ becomes non-singular for any positive semi-definite initial condition if the pair $(A(t), C(t))$ is constructible (Proposition 2.4), the non-linear terms represent a direct feedback of the estimation error. Second, the idea of using different powers is motivated by the sliding-mode control theory. It is expected that the term depending on $p_1$ provides finite-time attraction when it is chosen in the proposed interval. In the case of $p_2$, since it is proposed to be greater than one, one would expect that it provides uniform attraction w.r.t. the initial condition, to a neighborhood of the origin. With both terms, the observer convergence should be accelerated to the point of requiring only a finite amount of time to provide an exact estimation of the state. Such amount of time is expected to be bounded for *any* initial condition and *any* initial time. In other words, we are speaking of *uniform fixed-time convergence*. This, in fact, happens, as will be shown latter. Finally, the role of the linear term is merely the stabilization of the estimation error. This task can also be accomplished using only the non-linear terms, but the gains $\Lambda_1$ and $\Lambda_2$ would have to be selected large enough to dominate the linear part of the system's dynamics.

With respect to the other parameters, $Q(t)$ can be chosen arbitrarily inside the imposed boundaries. However, it has to be noticed that $Q(t)$ affects directly the eigenvalues of $H(t)$ and $N(t)$, and then, the convergence speed. Nevertheless, the qualitative properties of the observer are not affected by a particular choice of $Q(t)$. In the case of the gain matrices $\Lambda_1$ and $\Lambda_2$, they help to weight the nonlinear terms and their coordinates. These matrices have a direct impact on the convergence time; increasing they will reduce such time, although, the convergence time has a lower bound imposed by the size of the constructibility window of the system.

Now, with the brief introduction to the observer, we proceed to present its main properties.

**Theorem 3.1.** *Consider systems* (3.1) *and* (3.2)*. Assume that the observer parameters are chosen as in Table 3.1. Let the pair $(A(t), C(t))$ be uniformly completely constructible with a time window of length $T$. Let $h > 0$ and $\eta > 0$ such that $H(t) \geq h\,\mathbb{I}_n$, for all $t \geq t_0$, and $N(t) \geq \eta\,\mathbb{I}_n$, for all $t \geq t_0 + T$. Then, $\hat{x}(t)$ converges to $x(t)$ in fixed time, uniformly in the initial time. Furthermore, the amount of time needed by $\hat{x}(t)$ to reach $x(t)$ is, at most,*

$$T + \frac{n\,\sigma_1^{p_1}(\Lambda_1)}{h^{\frac{p_1+1}{2}}\sigma_n^{p_1+1}(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{n\,\sigma_1^{p_2}(\Lambda_2)}{h^{\frac{p_2+1}{2}}\sigma_n^{p_2+1}(\Lambda_2)\eta^{p_2+1}(p_2-1)}. \tag{3.6}$$

*Proof.* To start, we have to derive the dynamics of the estimation error. Using the definition of $e(t)$ and the relationship $\psi(t) = N(t)x(t)$ developed in Proposition 2.3, the error dynamics results in

$$\dot{e}(t) = \Big(A(t) - H(t)C^\top(t)C(t)\Big)e(t) - H(t)\Big(N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2}\Big). \tag{3.7}$$

To establish the result we have to prove that $e(t) = 0$ is an uniformly fixed-time stable equilibrium point, and give a bound for the settling time function. To that matter, we propose the following Lyapunov candidate function:

$$V(t,e) = e^\top H^{-1}(t)e.$$

Notice that the dynamics of $H^{-1}(t)$ has the same structure as $N(t)$. $N(t)$ is not the inverse of $H(t)$ given that $N(t_0)$ does not correspond to $H^{-1}(t_0)$, being this the only reason. On the other hand, by Proposition 2.10 we know that $H^{-1}(t)$ converge to $N(t)$ exponentially fast. Then, using Proposition 2.4 we have

$$\big(\eta_1 + \bar{\epsilon}\big)\mathbb{I}_n \geq H^{-1}(t) \geq \big(\eta_2 - \underline{\epsilon}\big)\mathbb{I}_n > 0 \quad \forall\, t \geq t_0 + T,$$

for positive constants $\bar{\epsilon}$, $\underline{\epsilon}$. Since $H(t_0) > 0$ and $Q(t)$ is positive definite, uniformly in $t$, $H^{-1}(t)$ is bounded from above and below for all $t \geq t_0$(Proposition 2.5). This makes possible to bound $V$ as

$$\big(\eta_1 + \bar{\epsilon}\big)\|e\|^2 \geq V(t,e) \geq \big(\eta_2 - \underline{\epsilon}\big)\|e\|^2 \quad \forall\, t \geq t_0 + T,$$

making it a valid Lyapunov function. To proceed, we have to compute the time derivative of $V$ when evaluated over the trajectories of $e(t)$. This results in

$$\begin{aligned}
\dot{V}(t) = {} & e^\top(t)H^{-1}(t)\Big(\big(A(t) - H(t)C^\top(t)C(t)\big)e(t) - H(t)\big(N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} + N(t)\Lambda_2\big)\lceil N(t)e(t)\rfloor^{p_2}\Big) \\
& \Big(e^\top(t)\big(A^\top(t) - C^\top(t)C(t)H(t)\big) - \big(\lceil e^\top(t)N(t)\rfloor^{p_1}\Lambda_1 N(t) + \lceil e^\top(t)N(t)\rfloor^{p_2}\Lambda_2 N(t)\big)H(t)\Big)H^{-1}(t)e(t) \\
& e^\top(t)\Big(-A^\top(t)H^{-1}(t) - H^{-1}(t)A(t) - H^{-1}(t)Q(t)H^{-1}(t) + C^\top(t)C(t)\Big)e(t) \\
= {} & -e^\top(t)H^{-1}(t)Q(t)H^{-1}(t)e(t) - e^\top(t)C^\top(t)C(t)e(t) - 2\,e^\top(t)N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} \\
& -2\,e^\top(t)N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2} \\
\leq {} & -q_2\big(\eta_2 - \underline{\epsilon}\big)e^\top(t)H^{-1}(t)e(t) - e^\top(t)C^\top(t)C(t)e(t) - 2\sum_{j=1}^{n}\lambda_{1,j}\big|\big(N(t)e(t)\big)_j\big|^{p_1+1} \\
& -2\sum_{j=1}^{n}\lambda_{2,j}\big|\big(N(t)e(t)\big)_j\big|^{p_2+1}.
\end{aligned}$$

By using the Jensen's inequality we have:

$$\frac{\sum_{j=1}^{n}\lambda_{i,j}\left|\left(N(t)e(t)\right)_i\right|^{p_i+1}}{\sum_{j=1}^{n}\lambda_{i,j}} \geq \left(\frac{\sum_{j=1}^{n}\lambda_{i,j}\left|\left(N(t)e(t)\right)_i\right|}{\sum_{j=1}^{n}\lambda_{i,j}}\right)^{p_i+1}.$$

Then

$$\begin{aligned}\dot V(t) &\leq -q_2\left(\eta_2-\underline{\epsilon}\right)V(t)-\left\|C(t)e(t)\right\|^2-2\frac{\sigma_n^{p_1+1}(\Lambda_1)}{n\,\sigma_1^{p_1}(\Lambda_1)}\left\|N(t)e(t)\right\|_1^{p_1+1}-2\frac{\sigma_n^{p_2+1}(\Lambda_2)}{n\,\sigma_1^{p_2}(\Lambda_2)}\left\|N(t)e(t)\right\|_1^{p_2+1}\\ &\leq -q_2\left(\eta_2-\underline{\epsilon}\right)V(t)-\left\|C(t)e(t)\right\|^2-2\frac{\sigma_n^{p_1+1}(\Lambda_1)}{n\,\sigma_1^{p_1}(\Lambda_1)}\left\|N(t)e(t)\right\|^{p_1+1}-2\frac{\sigma_n^{p_2+1}(\Lambda_2)}{n\,\sigma_1^{p_2}(\Lambda_2)}\left\|N(t)e(t)\right\|^{p_2+1}.\end{aligned}$$

By Theorem 2.1 we have that $e(t)=0$ is uniformly asymptotically stable. For $t\geq t_0+T$ we have that $N(t)\geq \eta_2\mathbb{I}_n$. Given that

$$\left\|N(t)e(t)\right\|^{p_i+1}\geq \eta_2^{p_i+1}\|e(t)\|^{p_i+1}, \qquad \|e(t)\|\geq \frac{1}{\sqrt{\eta_1+\bar\epsilon}}V^{1/2}(t),$$

then

$$\left\|N(t)e(t)\right\|^{p_i+1}\geq \frac{\eta_2^{p_i+1}}{(\eta_1+\bar\epsilon)^{\frac{p_i+1}{2}}}V^{\frac{p_i+1}{2}}(t).$$

This expression yields the following inequality for the derivative of $V$:

$$\begin{aligned}\dot V(t) &\leq -q_2\left(\eta_2-\underline{\epsilon}\right)V(t)-2\frac{\sigma_n^{p_1+1}(\Lambda_1)\eta_2^{p_1+1}}{n\,\sigma_1^{p_1}(\Lambda_1)(\eta_1+\bar\epsilon)^{\frac{p_1+1}{2}}}V^{\frac{p_1+1}{2}}(t)-2\frac{\sigma_n^{p_2+1}(\Lambda_2)\eta_2^{p_2+1}}{n\,\sigma_1^{p_2}(\Lambda_2)(\eta_1+\bar\epsilon)^{\frac{p_2+1}{2}}}V^{\frac{p_2+2}{2}}(t)\\ &\leq -2\frac{\sigma_n^{p_1+1}(\Lambda_1)\eta_2^{p_1+1}}{n\,\sigma_1^{p_1}(\Lambda_1)(\eta_1+\bar\epsilon)^{\frac{p_1+1}{2}}}V^{\frac{p_1+1}{2}}(t)-2\frac{\sigma_{n,2}^{p_2+1}(\Lambda_2)\eta_2^{p_2+1}}{n\,\sigma_1^{p_2}(\Lambda_2)(\eta_1+\bar\epsilon)^{\frac{p_2+1}{2}}}V^{\frac{p_2+2}{2}}(t).\end{aligned}$$

Since $\frac{1}{2}\left(p_1+1\right)<1$ and $\frac{1}{2}\left(p_2+1\right)>1$, by Theorem 2.4, we can conclude the uniform fixed-time stability of $e(t)=0$. Furthermore, from the theorem, we have the following bound for the convergence time:

$$T+\frac{n\,\sigma_1^{p_1}(\Lambda_1)(\eta_1+\bar\epsilon)^{\frac{p_i+1}{2}}}{\sigma_n^{p_1+1}(\Lambda_1)\eta_2^{p_1+1}(1-p_1)}+\frac{n\,\sigma_1^{p_2}(\Lambda_2)(\eta_1+\bar\epsilon)^{\frac{p_2+1}{2}}}{\sigma_n^{p_2+1}(\Lambda_2)\eta_2^{p_2+1}(p_2-1)}.$$

To get rid off $\bar\epsilon$ we have to ask for a lower bound for $H(t)$ of the form $H(t)\geq h\mathbb{I}_n>0$. In such case we have $\frac{1}{h}\mathbb{I}_n\geq H^{-1}(t)$. We know that such bounds exists by Proposition 2.2 and 2.5. Then, we can replace the constant $\eta_1+\bar\epsilon$ by $1/h$. □

From the bound to the convergence time (3.6) it appears a natural lower bound over it, the value of $T$. This is because the system may not be constructible on the open interval $[t_0,t_0+T)$. On the other hand, the bound also tell us how to reduce the convergence time. There are two basic mechanism for it; first, one of the exponents can be chosen zero and the other very large to increase the difference $p_2-1$. This mechanism only allows to decrease the second terms in (3.6) arbitrarily, the one depending on $p_2$; the first one reaches a minimum when $p_1$ is set to zero. The other mechanism is by modifying the values of $\sigma_n(\Lambda_i)$ and $\lambda_1(\Lambda_i)$. If the difference between $\sigma_n(\Lambda_i)$ and $\sigma_1(\Lambda_i)$ is fixed, then, by increasing $\sigma_n(\Lambda_i)$ one can make (3.6) as close to $T$ as desired; however, this is traduced in an arbitrarily gain increase. An indirect way to modify (3.6) is by means of $h$ and $\eta$. These constants depend manly on the constructibility properties of the system given by $A(t)$ and $C(t)$, but they are also affected by the matrix $Q(t)$. The effect of $Q(t)$ on these values is really complex, but it can be observed that increasing $q_1$ makes $\eta$ decrease, whereas increasing $q_2$ also increases $h$.

As a final comment, we want to emphasize that other non-linear terms of the form $\lceil N(t)\hat x(t)-\psi(t)\rfloor^{p_i}$ can be added to the algorithm, whenever the exponents $p_i$ are selected as non-negative numbers. However, introducing further terms does not result in new or different properties for the scheme. Nevertheless, they might help to shape the reach trajectory. In contrast, removing one of the non-linearities results in a different property. If one keeps the non-linear term linked to $p_1$, the convergence is in finite time, uniformly in $t_0$. On the other hand, by keeping the term pending on $p_2$, the convergence is exponential, uniformly in $t_0$, but there is also a kind of uniformity w.r.t. the initial conditions since the time needed to reach any bounded region of $e(t) = 0$ accept a constant upper bound that does not depend on how the initial error was.

## 3.2 Robustness of the observer

Now, in this section, we are going to investigate how the observer (3.2) behaves when the system does not follow the model exactly. This means that the observer is designed for the same system (3.1), but "in reality" the system is described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + \nu(t), \\ y(t) &= C(t)x(t) + \delta(t), \end{aligned} \tag{3.8}$$

where $\nu(t) \in \mathbb{R}^n$, $\delta(t) \in \mathbb{R}^r$ are unknown signals. We assume that both $\nu(t)$ and $\delta(t)$ are measurable functions, and that they are uniformly bounded in the following manner: $v \geq \|\nu(t)\|$, $d \geq \|\delta(t)\|$ for all $t \geq 0$.

Before analysing the algorithm, we have to study what happens with the auxiliary signals described in (3.3)-(3.5). The computation of $N(t)$ and $H(t)$ is not affected by the presence of the disturbances, that is no the case of $\psi(t)$. Let us denote by $\Psi(t)$ the signal that result of the relation $\Psi(t) = N(t)x(t)$. The dynamics of this signal is given by

$$\begin{aligned} \dot{\Psi}(t) &= \dot{N}(t)x(t) + N(t)\dot{x}(t) \\ &= \Big( - A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top(t)C(t)\Big)x(t) + N(t)\Big(A(t)x(t) + B(t)u(t) + \nu(t)\Big) \\ &= -A^\top(t)\Psi(t) - N(t)Q(t)\Psi(t) + N(t)B(t)u(t) + N(t)\nu(t) + C^\top(t)\Big(C(t)x(t) + \delta(t) - \delta(t)\Big) \\ &= -\Big(A(t) + Q(t)N(t)\Big)^\top \Psi(t) + N(t)B(t)u(t) + C^\top(t)y(t) + N(t)\nu(t) - C^\top(t)\delta(t). \end{aligned} \tag{3.9}$$

As can be seen, to keep the relation $\Psi(t) = N(t)x(t)$, it is necessary to known the value of the disturbances. Since this signals are not available, we can only compute $\psi(t)$ as in (3.5). Denote by $\zeta(t)$ the difference between $\Psi(t)$ and $\psi(t)$. The dynamics of $\zeta(t)$ results into

$$\dot{\zeta}(t) = \dot{\Psi}(t) - \dot{\psi}(t) = -\Big(A(t) + Q(t)N(t)\Big)^\top \zeta(t) + N(t)\nu(t) - C^\top(t)\delta(t). \tag{3.10}$$

It is important to show that $\zeta(t)$ remains bounded. In that matter, the following bound will be of great importance:

**Lemma 3.1.** *Let $\zeta(t) := \Psi(t) - \psi(t)$ where $\Psi(t)$ and $\psi(t)$ are computed following* (3.9) *and* (3.5) *respectively. Let $\dot{\zeta}(t)$ be given by* (3.10)*, and $\nu(t)$ and $\delta(t)$ satisfy the bounds $v \geq \|\nu(t)\|$, $d \geq \|\delta(t)\|$. Let $N(t)$ be computed as in* (3.4)*, and for $t_0 + T$ let it satisfy the bounds $\eta_1\mathbb{I} \geq N(t) \geq \eta_2\mathbb{I}$. Then,*

$$\lim_{t\to\infty} \|\zeta(t)\| \leq \sqrt{\frac{2\,\eta_1}{q_2\eta_2}}\sqrt{\frac{2}{q_2}v^2 + d^2}.$$

*Proof.* For $t \geq t_0 + T$ we have that $\eta_1\mathbb{I} \geq N(t) \geq \eta_2\mathbb{I}$, and $\frac{1}{\eta_2}\mathbb{I} \geq N^{-1}(t) \geq \frac{1}{\eta_1}\mathbb{I}$. To compute an ultimate bound for $\zeta(t)$ we propose the following Lyapunov like function $V(t,\zeta) = \zeta^\top N^{-1}(t)\zeta$. The derivative of $V(t)$

along the trajectories of (3.10) is

$$\begin{aligned}\dot V(t) &= \zeta^\top(t)N^{-1}(t)\left[-A^\top(t)\zeta(t) - N(t)Q(t)\zeta(t) + N(t)\nu(t) - C^\top(t)\delta(t)\right] + \left[-\zeta^\top(t)A(t) - \zeta^\top(t)Q(t)N(t)\right.\\ &\quad \left.+\nu^\top(t)N(t) - \delta^\top(t)C(t)\right]N^{-1}(t)\zeta(t) + \zeta^\top(t)\left[N^{-1}(t)A^\top(t) + A(t)N^{-1}(t) + Q(t)\right.\\ &\quad \left.-N^{-1}(t)C^\top(t)C(t)N^{-1}(t)\right]\zeta(t)\\ &= -\zeta^\top(t)Q(t)\zeta(t) + 2\,\nu^\top(t)\zeta(t) - 2\,\delta^\top(t)C(t)N^{-1}(t)\zeta(t) - \left\|C(t)N^{-1}(t)\zeta(t)\right\|^2\\ &\le -q_2\|\zeta(t)\|^2 + 2\,\|\nu(t)\|^2\|\zeta(t)\|^2 + 2\,\|\delta(t)\|^2\left\|C(t)N^{-1}(t)\zeta(t)\right\| - \left\|C(t)N^{-1}(t)\zeta(t)\right\|^2.\end{aligned}$$

Using the Young's inequality, the derivative of $V(t)$ can be bounded as

$$\dot V(t) \le -\frac{q_2}{2}\|\zeta(t)\|^2 + \frac{2}{q_2}\|\nu(t)\|^2 + \|\delta(t)\|^2.$$

Using the bounds $\frac{1}{\eta_2}\|\zeta(t)\|^2 \ge V(t)$ and $V(t) \ge \frac{1}{\eta_1}\|\zeta(t)\|^2$, the previous inequality can be transformed into the following differential inequality:

$$\begin{aligned}\dot V(t) &\le -\frac{1}{2}q_2\eta_2 V(t) + \frac{2}{q_2}\|\nu(t)\|^2 + \|\delta(t)\|^2\\ &\le -\frac{1}{2}q_2\eta_2 V(t) + \frac{2}{q_2}v^2 + d^2.\end{aligned}$$

The solution to the differential inequality yields

$$\begin{aligned}V(t) &\le V(t_0+T)\exp\left(-\frac{1}{2}q_2\eta_2\,(t-t_0-T)\right) + \frac{2\,d^2}{q_2\eta_2} + \frac{4\,v^2}{q_2^2\eta}\\ \|\zeta(t)\|^2 &\le \frac{\eta_1}{\eta_2}\|\zeta(t_0+T)\|^2\exp\left(-\frac{1}{2}q_2\eta_2\,(t-t_0-T)\right) + \frac{2\,d^2\eta_1}{q_2\eta_2} + \frac{4\,v^2\eta_1}{q_2^2\eta_2}.\end{aligned}$$

Taking the limit when $t\to\infty$ we have

$$\lim_{t\to\infty}\|\zeta(t)\| \le \sqrt{\frac{2\,\eta_1}{q_2\eta_2}}\sqrt{\frac{2}{q_2}v^2 + d^2}.$$

□

With the bound given in Lemma 3.1 we are ready to study what happens with the observer and the observation error under the effect of the disturbances.

To begin the analysis, we have to modify (3.7) to include the effect of $\nu(t)$ and $\delta(t)$. This yields

$$\begin{aligned}\dot e(t) &= \Big(A(t) - H(t)C^\top(t)C(t)\Big)e(t) - \nu(t) + H(t)C^\top(t)\delta(t) - H(t)\Big[N(t)\Lambda_1\lceil N(t)e(t)+\zeta(t)\rfloor^{p_1}\\ &\quad + N(t)\Lambda_2\lceil N(t)e(t)+\zeta(t)\rfloor^{p_2}\Big].\end{aligned} \tag{3.11}$$

Here, the relation $\psi(t) = \Psi(t) - \zeta(t)$ was used together with $\Psi(t) = N(t)x(t)$. What we found about $e(t)$ is that it is Input State Stable (ISS) w.r.t. $\nu(t)$, $\delta(t)$ and $\zeta(t)$. The disturbances $\nu(t)$ and $\delta(t)$ are bounded by assumption, and their boundedness ensure the boundedness of $\zeta(t)$ as can be seen in the proof of Lemma 3.1. We summarize our findings in the following two theorems:

**Theorem 3.2.** *The dynamical system defined by* (3.11) *is ISS w.r.t.* $\nu(t)$, $\delta(t)$, *and* $\zeta(t)$.

**Theorem 3.3.** *Consider the error dynamics* (3.11) *and assume that the pair* $(A(t), C(t))$ *is UCC such that, for* $t \ge t_0 + T$, *we have* $\eta_1\mathbb{I}_n \ge N(t) \ge \eta_2\mathbb{I}_n > 0$ *and* $h\,\mathbb{I}_n \ge H(t)$ *with* $N(t)$ *and* $H(t)$ *as in* (3.4) *and* (3.3). *Let* $\Delta$ *be*

$$\Delta := \sup_{t\in[t_0+T,\infty)}\left\|C^\top(t)\big(C(t)e(t) - \delta(t)\big)\right\|,$$

*and $k = 1$ if $p_2 \leq 2$ or $k = n^{\frac{2-p_2}{2}}$ if $p_2 > 2$. Then, the observation error $e(t)$ converges asymptotically to the trajectory $-N^{-1}(t)\zeta(t)$; furthermore, for $p_1 = 0$, the trajectory is reached in fixed-time, in an amount of time that does not exceed*

$$T + \frac{\eta_1}{q_2\eta_2^2}\max\left\{\ln\left(1+\frac{2\,\eta_1^2(h\,\Delta+\eta_2 n^{1/2}\sigma_1(\Lambda_1))}{\eta_2\sigma_n(\Lambda_1)}\right), \ln\left(1+\frac{2\,\eta_1^2 h\,k\,\sigma_1(\Lambda_2)}{n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)}\right)\right\}$$
$$+\frac{2}{\eta_2^{1/2}\sigma_n(\Lambda_1)}+\frac{2}{\eta_2^{\frac{p_2+1}{2}}n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)(p_2-1)}.$$

Theorem 3.2 tells us that the observer is not fragile. Whenever the disturbances are not large, the generated error will be “proportional" to the size of the disturbances. Not only that, if the disturbances disappear, the convergence to zero is recovered. It is important to remark that in such case, the convergence will be exponential since the dynamics of $\zeta(t)$ is linear and it cannot reach zero in finite time. The only way to recover the finite-time convergence is by resetting the values of $N(t)$ and $\psi(t)$ in order to forget “the corrupted information". Also, it is important to remark that, in general, the presence of $\nu(t)$ and in particular of $\delta(t)$ makes impossible to recover the system's state since the constructibility of the system breaks under such circumstances. The second theorem, Theorem 3.3, gives us the “exact" behaviour of the error. The final trajectory is the same as the one reached by the Kalman-Bucy observer (see (2.20)), the difference is that the Kalman-Bucy observer approaches the trajectory in an exponential fashion, whereas the proposed observer can reached it in fixed-time for $p_1 = 0$. The other information that Theorem 3.3 gives us is an ultimate bound for the observation error:

$$\|e(t)\| \to \left\|N^{-1}(t)\zeta(t)\right\| \leq \frac{1}{\eta_2}\|\zeta(t)\|,$$

then

$$\lim_{t\to\infty}\|e(t)\| \leq \sqrt{\frac{2\,\eta_1}{q_2\eta_2^3}\left(\frac{2}{q_2}v^2+d^2\right)}.$$

This bound follows from Lemma 3.1.

Before presenting the proofs of Theorem 3.2 and 3.3, we need the following result.

**Lemma 3.2.** *Let $x$, $\delta \in \mathbb{R}$ and $p \geq 0$. Then, for any $\kappa_1 \in (0,1)$ there exist $\kappa_2 > 0$ such that*

$$x\lceil x+\delta\rfloor^p \geq \kappa_1\,|x|^{p+1} - \kappa_2\,|\delta|^{p+1}.$$

*In particular, for $p > 0$, one can select $\kappa_2 = \max\left\{1+\kappa_1,\ \kappa_1/\left(1-\kappa_1^{(1/p)}\right)^p\right\}$. For $p = 0$ and $\kappa_1 \in (0,1]$, $\kappa_2$ can be selected as $1+\kappa_1$.*

*Proof.* For $x = 0$, the inequality is satisfied trivially with any $\kappa_2 \geq 0$. Now, by homogeneity we have:

$$x\lceil x+\delta\rfloor^p = \frac{1}{\epsilon^{p+1}}\Big((\epsilon\,x)\lceil(\epsilon\,x)+(\epsilon\,\delta)\rfloor^p\Big)$$
$$\kappa_1\,|x|^{p+1}-\kappa_2\,|\delta|^{p+1} = \frac{1}{\epsilon^{p+1}}\Big(\kappa_1\,|\epsilon\,x|^{p+1}-\kappa_2\,|\epsilon\,\delta|^{p+1}\Big),$$

for any $\epsilon > 0$. Set $\epsilon = 1/|\delta|$ and define $z = x/|\delta|$. The inequality then is equivalent to $z\lceil z+\lceil\delta\rfloor^0\rfloor^p \geq \kappa_1\,|z|^{p+1}-\kappa_2$. We will only consider the case $\lceil\delta\rfloor^0 = 1$ since the other one is analogous. For $\lceil\delta\rfloor^0 = 1$ we have $z\lceil z+1\rfloor^p \geq \kappa_1\,|z|^{p+1}-\kappa_2$, or

$$z\lceil z+1\rfloor^p - \kappa_1\,|z|^{p+1} \geq -\,\kappa_2. \tag{3.12}$$

This reduces the problem to prove that the LHS has a lower bound. For $z \geq 0$, we have that $\lceil z+1 \rfloor^0 = 1$ and $|z+1| > |z|$, then $z\lceil z+1\rfloor^p > |z|^{p+1}$. Since $\kappa_1 < 1$, $z\lceil z+1\rfloor^p - \kappa_1 |z|^{p+1} > 0$ and (3.12) holds for $\kappa_2 \geq 0$ on this interval. Now, for $z \in (-1, 0)$, (3.12) becomes $-|z|\,|z+1|^p - \kappa_1 |z|^{p+1} \geq -\kappa_2$. In this interval we have that $|z| < 1$ and $|z+1| < 1$, then $-|z|\,|z+1|^p - \kappa_1 |z|^{p+1} \geq -1 - \kappa_1$, which implies that (3.12) holds with $\kappa_2 \geq 1+\kappa_1$. Last, we consider the interval $z \in (-\infty, -1]$, where (3.12) reduces to $|z|\,|z+1|^p - \kappa_1|z|^{p+1} \geq -\kappa_2$. For $p = 0$, the inequality holds trivially since

$$|z|\,|z+1|^p - \kappa_1|z|^{p+1} = |z|\big(1-\kappa_1\big) \geq 0.$$

For $p > 0$, we have that $|z|^p > |z+1|^p$. To find a lower bound, consider the following auxiliary function:

$$|z|\,|z+1|^p - \kappa_1|z|^{p+1} \geq |z+1|^{p+1} - \kappa_1|z|^{p+1} := g(z).$$

Now, we proceed to look for the minimum of $g(z)$. Taking its derivative, this results in $g'(z) = (p+1)\big(\kappa_1|z|^p - |z+1|^p\big)$, which has a unique zero at $z_0 = -1/(1-\kappa_1^{1/p})$. The second derivative of $g(z)$ evaluated at $z_0$ is positive, reveling that $g(z)$ has a minimum at this point. Then $g(z_0)$ can be taken as $\kappa_2$. This gives us $\kappa_2 \geq \kappa_1/(1-\kappa_1^{1/p})^p$. Finally, looking at the three conditions we get that

$$\kappa_2 \geq \max\left\{\kappa_1 + 1,\ \frac{\kappa_1}{\big(1-\kappa_1^{1/p}\big)^p}\right\} \quad \text{and} \quad p > 0.$$

For $p = 0$, we only need to verify one region, when $z \in (0, 1)$. In this region, it is sufficient that $\kappa_2 \geq 1+\kappa_1$. □

*Proof of Theorem 3.2.* To analyse (3.11) we propose as ISS-Lyapunov function the same as before $V(t,e) = e^\top H^{-1}(t)e$. The derivative of $V(t)$ along the dynamics if (3.11) is

$$\begin{aligned}
\dot V(t) &= e^\top(t)H^{-1}(t)\dot e(t) + \dot e^\top(t)H^{-1}(t)e(t) + e^\top(t)\dot H(t)e(t)\\
&= -e^\top(t)C^\top(t)C(t)e(t) - e^\top(t)H^{-1}(t)Q(t)H^{-1}(t)e(t) - 2\,e^\top(t)H^{-1}(t)\nu(t) + 2\,e^\top(t)C^\top(t)\delta(t)\\
&\quad - 2\,e^\top(t)N(t)\Lambda_1\big\lceil N(t)e(t)+\zeta(t)\big\rfloor^{p_1} - 2\,e^\top(t)N(t)\Lambda_2\big\lceil N(t)e(t)+\zeta(t)\big\rfloor^{p_2}.
\end{aligned}$$

By using the Young's inequality, the derivative can be bounded as

$$\begin{aligned}
\dot V(t) \leq{}& \|\delta(t)\|^2 + \frac{2}{q_2}\|\nu(t)\|^2 - \frac{q_2}{2}\big\|H^{-1}(t)e(t)\big\|^2 - 2\sum_{j=1}^n \lambda_{1,j}\big(N(t)e(t)\big)_j\big\lceil\big(N(t)e(t)\big)_j + \zeta_j(t)\big\rfloor^{p_1}\\
&- 2\sum_{j=1}^n \lambda_{2,j}\big(N(t)e(t)\big)_j\big\lceil\big(N(t)e(t)\big)_j + \zeta_j(t)\big\rfloor^{p_2}.
\end{aligned}$$

Using Lemma 3.2 we have

$$\begin{aligned}
\dot V(t) \leq{}& d^2 + \frac{2}{q_2}v^2 - \frac{q_2}{2}\big\|H^{-1}(t)e(t)\big\|^2 - 2\sum_{j=1}^n \lambda_{1,j}\Big(\kappa_{1,p_1}\big|\big(N(t)e(t)\big)_j\big|^{p_1+1} - \kappa_{2,p_1}\big|\zeta_j(t)\big|^{p_1+1}\Big)\\
&- 2\sum_{j=1}^n \lambda_{2,j}\Big(\kappa_{1,p_2}\big|\big(N(t)e(t)\big)_j\big|^{p_2+1} - \kappa_{2,p_2}\big|\zeta_j(t)\big|^{p_2+1}\Big)\\
\leq{}& d^2 + \frac{2}{q_2}v^2 - \frac{q_2}{2}\big\|H^{-1}(t)e(t)\big\|^2 - 2\,\sigma_n(\Lambda_1)\kappa_{1,p_1}\big\|N(t)e(t)\big\|_{p_1+1}^{p_1+1} - 2\,\sigma_n(\Lambda_2)\kappa_{1,p_2}\big\|N(t)e(t)\big\|_{p_2+1}^{p_2+1}\\
&+ 2\,\sigma_1(\Lambda_1)\kappa_{2,p_1}\big\|\zeta(t)\big\|_{p_1+1}^{p_1+1} + 2\,\sigma_1(\Lambda_2)\kappa_{2,p_2}\big\|\zeta(t)\big\|_{p_2+1}^{p_2+1}.
\end{aligned}$$

Using the equivalence between norms the inequality results in

$$\begin{aligned}
\dot V(t) \leq{}& d^2 + \frac{2}{q_2}v^2 - \frac{q_2}{2}\big\|H^{-1}(t)e(t)\big\|^2 - 2\,\sigma_n(\Lambda_1)\kappa_{1,p_1}\big\|N(t)e(t)\big\|^{p_1+1} - 2\,n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\kappa_{1,p_2}\big\|N(t)e(t)\big\|^{p_2+1}\\
&+ 2\,n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)\kappa_{2,p_1}\big\|\zeta(t)\big\|^{p_1+1} + 2\,\sigma_1(\Lambda_2)\kappa_{2,p_2}\big\|\zeta(t)\big\|^{p_2+1}.
\end{aligned}$$

Now, for $t \geq t_0 + T$ we have $h\,\mathbb{I} \geq H(t)$ and $N(t) \geq \eta\,\mathbb{I}$. Using these lower bounds, the inequality for $\dot{V}(t)$ is changed into

$$\begin{aligned}\dot{V}(t) \leq d^2 + \frac{2}{q_2}v^2 + 2\,n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)\kappa_{2,p_1}\big\|\zeta(t)\big\|^{p_1+1} + 2\,\sigma_1(\Lambda_2)\kappa_{2,p_2}\big\|\zeta(t)\big\|^{p_2+1} - \frac{q_2}{2\,h^2}\|e(t)\|^2 \\ - 2\eta^{p_1+1}\sigma_n(\Lambda_1)\kappa_{1,p_1}\|e(t)\|^{p_1+1} - 2\eta^{p_2+1}\sigma_n(\Lambda_2)\kappa_{1,p_2}\|e(t)\|^{p_2+1}.\end{aligned}$$

In the RHS of the previous inequality we found three positive terms. Each of them is bounded and we can group them into one constant, say $\Delta$. On the other hand, we can see the negative terms as a polynomial in $\|e(t)\|$, which we denote as $P(\|e(t)\|)$. To proof the ISS of $e(t)$, let us introduce an extra parameter, $\theta \in (0,1)$. Using this parameter, we can rewrite the inequality as

$$\dot{V}(t) \leq -(1-\theta)P(\|e(t)\|) - \big(\theta P(\|e(t)\|) - \Delta\big).$$

For

$$\begin{aligned} P\big(\|e(t)\|\big) &\geq \frac{1}{\theta}\Delta \\ \frac{q_2}{2\,h^2}\|e(t)\|^2 + 2\eta^{p_1+1}\sigma_n(\Lambda_1)\kappa_{1,p_1}\|e(t)\|^{p_1+1} + 2\eta^{p_2+1}\sigma_n(\Lambda_2)\kappa_{1,p_2}\|e(t)\|^{p_2+1} &\geq \frac{1}{\theta}\Delta, \end{aligned} \tag{3.13}$$

we have that

$$\dot{V}(t) \leq -(1-\theta)P(\|e(t)\|) < 0.$$

Given that $P(\|e(t)\|)$ is a monotone function of $\|e(t)\|$, there is a value $\mu > 0$ for which $\|e(t)\| \geq \mu$ implies $\theta\,P(\|e(t)\|) \geq \Delta$. Then, by Theorem 2.5, we can conclude the ISS. □

*Proof of Theorem 3.3.* To study the the convergence of $e(t)$ to the trajectory $-N^{-1}(t)\zeta(t)$, let us define the auxiliary variable $\chi(t) = N(t)e(t) + \zeta(t)$. The dynamic of this variable is

$$\dot{\chi}(t) = -\big(A^\top(t) + N(t)Q(t)\big)\chi(t) - N(t)H(t)N(t)\sum_{i=1}^{2}\Lambda_i\lceil\chi(t)\rfloor^{p_i} + \big(\mathbb{I}_n - N(t)H(t)\big)C^\top(t)\big(C(t)e(t) - \delta(t)\big).$$

To study the stability of $\chi(t) = 0$ assume $t \geq t_0 + T$ such that $\eta_1\mathbb{I}_n \geq N(t) \geq \eta_2\mathbb{I}_n$. To keep the equations short, we dropped the time dependency. Consider the candidate Lyapunov function $V(\chi, t) = \chi^\top N^{-1}(t)\chi$, which derivative along the dynamics of $\chi(t)$ results in

$$\begin{aligned}\dot{V}(t) &= -\chi^\top Q\,\chi - 2\,\chi^\top H\,N\sum_{i=1}^{2}\Lambda_i\lceil\chi\rfloor^{p_i} + 2\,\chi^\top N^{-1}(\mathbb{I}_n - N\,H)C^\top(C\,e - \delta) \\ &= -\chi^\top Q\,\chi - 2\,\chi^\top\sum_{i=1}^{2}\Lambda_i\lceil\chi\rfloor^{p_i} + 2\,\chi^\top N^{-1}(\mathbb{I}_n - N\,H)C^\top(C\,e - \delta) + 2\,\chi^\top(\mathbb{I}_n - H\,N))\sum_{i=1}^{2}\Lambda_i\lceil\chi\rfloor^{p_i}.\end{aligned}$$

Now, we proceed to find a bound for $\dot{V}(t)$:

$$\begin{aligned}\dot{V}(t) &\leq -q_2\big\|\chi\big\|^2 - 2\,\sigma_n(\Lambda_1)\big\|\chi\big\|_{p_1+1}^{p_1+1} - 2\,\sigma_n(\Lambda_2)\big\|\chi\big\|_{p_2+1}^{p_2+1} + \frac{2}{\eta_2}\big\|\mathbb{I}_n - N\,H\big\|\,\big\|\chi\big\|\,\big\|C^\top(C\,e - \delta)\big\| \\ &\quad + 2\,\big\|\mathbb{I}_n - H\,N\,\big\|\left(n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)\big\|\chi\big\|^{p_1+1} + \sigma_1(\Lambda_2)k\big\|\chi\big\|^{p_2+1}\right) \\ &\leq -\frac{q_2}{2}\big\|\chi\big\|^2 - 2\,\sigma_n(\Lambda_1)\big\|\chi\big\|^{p_1+1} - 2\,n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\big\|\chi\big\|^{p_2+1} + \frac{2}{q_2\eta_2}\big\|\mathbb{I}_n - N\,H\big\|^2\big\|C^\top(C\,e - \delta)\big\|^2 \\ &\quad + 2\,\big\|\mathbb{I}_n - H\,N\,\big\|\left(n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)\big\|\chi\big\|^{p_1+1} + \sigma_1(\Lambda_2)k\big\|\chi\big\|^{p_2+1}\right), \\ &\leq -\frac{q_2}{2}\big\|\chi\big\|^2 - 2\,\sigma_n(\Lambda_1)\big\|\chi\big\|^{p_1+1} - 2\,n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\big\|\chi\big\|^{p_2+1} + \frac{2\,h}{q_2\eta_2}\big\|H^{-1} - N\big\|^2\big\|C^\top(C\,e - \delta)\big\|^2 \\ &\quad + 2\,h\big\|H^{-1} - N\big\|\left(n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)\big\|\chi\big\|^{p_1+1} + \sigma_1(\Lambda_2)k\big\|\chi\big\|^{p_2+1}\right),\end{aligned} \tag{3.14}$$

with $k = 1$ if $p_2 \leq 2$ or $k = n^{\frac{2-p_2}{2}}$ if $p_2 > 2$. From Theorem 2.10 we know that $H^{-1}(t)$ converge to $N(t)$, then the term $\|H^{-1}(t) - N(t)\|$ tends to zero. From the estimate of the reaching time in Theorem 2.10 we have that

$$\|H^{-1}(t) - N(t)\| \leq \min\left\{\frac{\sigma_n(\Lambda_1)}{2\,h\,n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)}, \frac{n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)}{2\,h\,k\,\sigma_1(\Lambda_2)}\right\}$$

for

$$t \geq t_0 + T + \frac{\eta_1}{q_2\eta_2^2}\max\left\{\ln\left(1 + \frac{2\,\eta_1^2 h\,n^{\frac{1-p_1}{2}}\sigma_1(\Lambda_1)}{\sigma_n(\Lambda_1)}\right), \ln\left(1 + \frac{2\,\eta_1^2 h\,k\,\sigma_1(\Lambda_2)}{n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)}\right)\right\},$$

and then

$$\begin{aligned}\dot{V}(t) \leq & -\frac{q_2}{2}\|\chi(t)\|^2 - \sigma_n(\Lambda_1)\|\chi(t)\|^{p_1+1} - n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\|\chi(t)\|^{p_2+1} \\ & + \frac{2\,h}{q_2\eta_2}\|H^{-1}(t) - N(t)\|^2\|C^\top(t)(C(t)e(t) - \delta(t))\|^2.\end{aligned}$$

Since $\delta(t)$ is bounded and so it is $e(t)$ by Theorem 3.2, the term

$$\frac{2\,h}{q_2\eta_2}\|H^{-1}(t) - N(t)\|^2\|C^\top(t)(C(t)e(t) - \delta(t))\|^2$$

fades out as $H^{-1}(t)$ converge to $N(t)$, and $V(t) \to 0$ as $t \to \infty$, asserting the convergence to $\chi(t) = 0$ or $e(t) = N^{-1}(t)\zeta(t)$.

For $p_1 = 0$, the analysis can be done slightly different. Consider again (3.14), from which we have

$$\begin{aligned}\dot{V}(t) \leq & -q_2\|\chi\|^2 - \left(2\,\sigma_n(\Lambda_1) - \frac{2}{\eta_2}\|\mathbb{I}_n - N\,H\|\,\|C^\top(C\,e - \delta)\| - 2\,n^{1/2}\sigma_1(\Lambda_1)\|\mathbb{I}_n - H\,N\,\|\right)\|\chi\| \\ & - 2\,\sigma_n(\Lambda_2)\|\chi\|_{p_2+1}^{p_2+1} + 2\,\sigma_1(\Lambda_2)k\|\mathbb{I}_n - H\,N\,\|\|\chi\|^{p_2+1} \\ \leq & -q_2\|\chi\|^2 - \left(2\,\sigma_n(\Lambda_1) - \frac{2\,h}{\eta_2}\|H^{-1} - N\|\|C^\top(C\,e - \delta)\| - 2\,n^{1/2}\sigma_1(\Lambda_1)\|H^{-1} - N\,\|\right)\|\chi\| \\ & - 2\,n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\|\chi\|^{p_2+1} + 2\,h\sigma_1(\Lambda_2)k\|H^{-1} - N\,\|\|\chi\|^{p_2+1}.\end{aligned}$$

Let $\Delta$ be

$$\Delta := \sup_{t\in[t_0+T,\infty)}\|C^\top(t)\big(C(t)e(t) - \delta(t)\big)\|,$$

then, for $t$ such that

$$t \geq t_0 + T + \frac{\eta_1}{q_2\eta_2^2}\max\left\{\ln\left(1 + \frac{2\,\eta_1^2(h\,\Delta + \eta_2 n^{1/2}\sigma_1(\Lambda_1))}{\eta_2\sigma_n(\Lambda_1)}\right), \ln\left(1 + \frac{2\,\eta_1^2 h\,k\,\sigma_1(\Lambda_2)}{n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)}\right)\right\}$$

we have

$$\begin{aligned}\dot{V}(t) & \leq -q_2\|\chi(t)\|^2 - \sigma_n(\Lambda_1)\|\chi(t)\| - n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)\|\chi(t)\|^{p_2+1} < 0 \\ & \leq -\eta_2^{1/2}\sigma_n(\Lambda_1)V^{1/2}(t) - \eta_2^{\frac{p_2+1}{2}}n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)V^{\frac{p_2+1}{2}}(t).\end{aligned}$$

By Theorem 2.4 we can conclude the fixed-time convergence of $\chi(t)$ to zero, and estimate the convergence time as

$$\begin{aligned}T + \frac{\eta_1}{q_2\eta_2^2}\max\left\{\ln\left(1 + \frac{2\,\eta_1^2(h\,\Delta + \eta_2 n^{1/2}\sigma_1(\Lambda_1))}{\eta_2\sigma_n(\Lambda_1)}\right), \ln\left(1 + \frac{2\,\eta_1^2 h\,k\,\sigma_1(\Lambda_2)}{n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)}\right)\right\} + \frac{2}{\eta_2^{1/2}\sigma_n(\Lambda_1)} \\ + \frac{2}{\eta_2^{\frac{p_2+1}{2}}n^{\frac{1-p_2}{2}}\sigma_n(\Lambda_2)(p_2 - 1)}.\end{aligned}$$

□

## Summary

In this chapter, the observer was presented together with its convergence properties and an analysis of robustness was developed. Although the proposed observer is applicable to the same class of systems as the Kalman-Bucy filter, it provides an improvement in the rate of convergence, and exact convergence in absence of disturbances. When the systems is affected by unmodeled inputs and noise, the observation error remains bounded, and the size of this error is comparable to the error exhibit by the Kalman-Bucy filter under the same circumstances. The reason for this behavior is that the estimation error generated in any of the observers converge to the same trajectory, being the only difference the speed at which the trajectory is reached.

The main backwards of the proposed observer in relation to the Kalman-Bucy filter is the number of differential equations to compute. For the proposed observer, $n(n+2)/2$ ($n$ the dimension of the system) additional equation have to be computed. This equations correspond to the dynamics of $N(t)$ and $\psi(t)$. However, this does not have to be the case. In the next chapter, particular classes of LTV systems are studied. For this systems, the computation of $H(t)$ is not needed. Then, only $n$ extra equation have to be computed in relation to the Kalman-Bucy filter. Although this does not alleviate the problem completely, it represent a significant saving since for these systems, the number of equations to compute grows linearly with the system's dimension and not quadratically.

# Chapter 4

# Applications of the observer

In this chapter we are going to revisit some basic problems that appear in observer design and adaptive control. These problems are usually explained in very different frameworks. To address these topics, we have to put them in the context of observation of a linear time-varying system. Fortunately, these problems, with a correct interpretation, are precisely that. Then, the technique developed in the previous chapter can handle them. However, the specific structure of each problem makes possible to adjust the observer in order to simplify it without losing any of its properties. The problems we address in this chapter are:

- Observer design for linear time-invariant systems.
- Design of a parameter estimator for constant parameters that are present in a linear regression model.
- Design of an adaptive observer for linear systems affected by parametrized disturbances.
- Design of a parameter estimator for smooth time-varying parameters with bounded derivative.

## 4.1 Linear time invariant systems

We consider the class of linear systems with constant system matrices. Although the observer presented in the previous chapter is fully applicable in this case, some simplifications can be done. The system considered in this section is as follows:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t), \end{aligned} \tag{4.1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{r\times n}$ constant matrices. The main difference in this case is that we do not need to compute $H(t)$, and it can be replaced by a constant matrix. Let $L \in \mathbb{R}^{n\times r}$ is such that $A - L\,C$ be a Hurwitz matrix. In such case, for every $R = R^\top \in \mathbb{R}^{n\times n}$, $R > 0$, there exist $P = P^\top \in \mathbb{R}^{n\times n}$, $P > 0$, such that

$$P\big(A - L\,C\big) + \big(A - L\,C\big)^\top P = -R.$$

Consider the matrices $L$ and $P$. The observer (3.2) for system (4.1) can be modified into:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\,\hat{x}(t) + B\,u(t) - L\big(C\,\hat{x}(t) - y(t)\big) \\ &\quad - P^{-1}\big(N(t)\Lambda_1\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_2}\big). \end{aligned} \tag{4.2}$$

$$\dot{N}(t) = -A^\top N(t) - N(t)A - N(t)Q\,N(t) + C^\top C, \quad N(0) = 0,$$
$$\dot{\psi}(t) = -\big(A^\top + N(t)Q\big)\psi(t) + C^\top y(t) + N(t)B\,u(t), \quad \psi(0) = 0.$$

The replacement of $H(t)$ by constant matrices does not affect the properties of the observer. By this we mean that the fixed-time convergence is kept, in the unperturbed case, as well as the robustness properties investigated in Section 3.2 in the presence of disturbances. These properties are condensed in the following theorems:

**Theorem 4.1.** *Consider systems (4.1) and (4.2) with $p_1 \in [0,1)$ and $p_2 > 1$. Let $N(t) \geq \eta\,\mathbb{I}_n > 0$. Then $\hat{x}(t)$ converges to $x(t)$ in a fixed time, that is, the amount of time needed for the converge does not exceed*

$$\frac{\sigma_1^{\frac{p_1+1}{2}}(P)}{\sigma_n(\Lambda_1)\eta^{p_1+1)}(1-p_1)} + \frac{\sigma_1^{\frac{p_2+1}{2}}(P)}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2-1)}.$$

*for any given initial error $\hat{x}(0) - x(0)$.*

*Proof.* To analyze observer convergence, the estimation error dynamics is analyzed. The dynamics of such error results in

$$\dot{e}(t) = \big(A - L\,C\big)e(t) - P^{-1}\big(N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} + N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2}\big). \tag{4.3}$$

The stability of the solution $e(t) = 0$ can be analysed using the Lyapunov function candidate $V(e) = e^\top P\,e$. Evaluating the derivative of $V$ along the trajectories of $e(t)$ one gets

$$\begin{aligned}
\dot{V}(t) &= e^\top(t)\big(P\big(A - L\,C\big) + \big(A - L\,C\big)^\top P\big)e(t) - 2\,e^\top(t)N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} - 2\,e^\top(t)N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2}\\
&= -e^\top(t)Q(t)e(t) - 2\,e^\top(t)N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} - 2\,e^\top(t)N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2}\\
&\leq -\sigma_n(Q)\|e(t)\|^2 - 2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|_{p_1+1}^{p_1+1} - 2\,\sigma_n(\Lambda_2)\big\|N(t)e(t)\big\|_{p_2+1}^{p_2+1}\\
&\leq -\sigma_n(Q)\|e(t)\|^2 - 2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|^{p_1+1} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\big\|N(t)e(t)\big\|^{p_2+1}\\
&\leq -\sigma_n(Q)\|e(t)\|^2 - 2\,\sigma_n(\Lambda_1)\eta^{p_1+1}\|e(t)\|^{p_1+1} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}\|e(t)\|^{p_2+1}.
\end{aligned}$$

Using the relation $\sigma_1(P)\|e(t)\|^2 \geq V(t)$, we obtain a differential inequality for $V(t)$:

$$\dot{V}(t) \leq -\frac{\sigma_n(Q)}{\sigma_1(P)}V(t) - \frac{2}{\sigma_1^{\frac{p_1+1}{2}}(P)}\sigma_n(\Lambda_1)\eta^{p_1+1}V^{\frac{p_1+1}{2}}(t) - \frac{2}{\sigma_1^{\frac{p_2+1}{2}}(P)}\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}V^{\frac{p_2+1}{2}}(t).$$

This differential inequality automatically tell us that $e(t) = 0$ is reached in fixed-time. This is asserted by Theorem 2.4. The theorem also gives us the bound for the convergence time:

$$\frac{\sigma_1^{\frac{p_1+1}{2}}(P)}{\sigma_n(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{\sigma_1^{\frac{p_2+1}{2}}(P)}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2-1)}.$$

□

In the case of LTI systems, where the observability is instantaneous, the lower bound for the convergence time is zero. However, this limit cannot be reached. This is reflected by the fact that the convergence time can be made arbitrarily small by increasing the components of $\Lambda_1$ and $\Lambda_2$.

The ISS w.r.t. bounded disturbances showed for the general observer in the previous chapter can also be obtained for the observer (4.2). To that matter, $V(e) = e^\top P\,e$ can be considered as an ISS-Lyapunov function candidate. By repeating the procedure showed in the proof of Theorem 3.2, the ISS property can be concluded. The difference appears when analyzing the limit behavior of the observation error. To recover the result of Theorem 3.3 we need a particular selection of $P$ and $L$: they should correspond to a design of a time-invariant Kalman filter. This is showed in the next theorem:

**Theorem 4.2.** *Consider the perturbed LTI system*

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + B\,u(t) + \nu(t),\\ y(t) &= C\,x(t) + \delta(t),\end{aligned}$$

*where $\nu(t)$ and $\delta(t)$ are uniformly bounded, integrable vector valued functions acting as unknown disturbances. Consider the observer* (4.2) *with $P$ the unique positive definite solution of the Riccati algebraic equation:*

$$-A^\top P - P\,A - P\,Q\,P + C^\top C = 0,$$

*and set $L = P^{-1}C^\top$. Let $\zeta(t)$ be defined as in* (3.10)*. Then, the observation error $e(t) = \hat{x}(t) - x(t)$ reach the trajectory $-N^{-1}(t)\zeta(t)$ asymptotically. Furthermore, for $p_1 = 0$ the trajectory is reached in fixed-time.*

*Proof.* This proof follows the one for Theorem 3.3. Given the presence of $\nu(t)$ and $\delta(t)$ the observation error dynamics is modified and results in

$$\dot{e}(t) = \big(A - P^{-1}C^\top C\big)e(t) - \nu(t) + P^{-1}C^\top\delta(t) - P^{-1}\sum_{i=1}^{2} N(t)\Lambda_i\lceil N(t)e(t) + \zeta(t)\rfloor^{p_i}.$$

In this situation, the dynamics of $\zeta(t)$ is

$$\dot{\zeta}(t) = -\big(A^\top + N(t)Q\big)\zeta(t) + N(t)\nu(t) - C^\top\delta(t).$$

Consider the auxiliary error function $\chi(t) = N(t)e(t) + \zeta(t)$, which dynamics results in

$$\dot{\chi}(t) = -\big(A^\top + N(t)Q\big)\chi(t) - N(t)P^{-1}\sum_{i=1}^{2} N(t)\Lambda_i\lceil\chi(t)\rfloor^{p_i} + \big(\mathbb{I}_n - N(t)P^{-1}\big)C^\top\big(C\,e(t) - \delta\big).$$

Since $P$ represent the final value of $N(t)$, i.e., $\lim_{t\to\infty} N(t) = P$, the difference $\mathbb{I}_n - N(t)P^{-1}$ converge to zero. Since the estimation error remains bounded, and $\delta(t)$ is bounded by assumption, the dynamics of $\chi(t)$ tends asymptotically to

$$\dot{\chi}(t) = -\big(A^\top + N(t)Q\big)\chi(t) - \sum_{i=1}^{2} N(t)\Lambda_i\lceil\chi(t)\rfloor^{p_i}.$$

Using $V(\chi, t) = \chi^\top N^{-1}(t)\chi$ as Lyapunov function, it is possible to show that $\chi(t) = 0$ is asymptotically stable. Then the relation $N(t)e(t)+\zeta(t) = 0$ is reached asymptotically as well. This means that $e(t)$ converge to $-N^{-1}(t)\zeta(t)$. When $p_1 = 0$, the discontinuous term is capable of rejecting the term depending on the difference $\mathbb{I}_n - N(t)P^{-1}$. Then, for $p_1 = 0$, the trajectory $e(t) - N^{-1}(t)\zeta(t)$ can be reached in fixed-time. □

Theorem 4.1 and 4.2 show that the replace of $H(t)$ by $P$ and $L$ does not affect the properties of the algorithm, that is, the fixed-time convergence is kept in the unperturbed case. In the presence of bounded disturbances, the ISS property is also guarantee for any chose of $P$ and $L$. However, to be able of telling something about the size of the observation error and its behavior, a particular choice of $P$ and $L$ is needed.

### 4.1.1 Numerical Example

To exemplify the application of the observer in the case of linear time-invariant systems, we are going to use the next parametrization:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 1\\ -3 & -1\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t),\ x(0) = \begin{bmatrix}5\\0\end{bmatrix},\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}x(t).\end{aligned}$$

This system can be related to a mechanical system with a mass, a spring, and a damper. As input we propose the function $u(t) = \sin(3\,t)$. In this example, the problem of estimating $x_2(t)$ is equivalent to "computing" the velocity from the position measurements. To configure the observer, we chose the following parameters:

$$L = \frac{1}{3}\begin{bmatrix} 1 \\ -9 \end{bmatrix}, \quad P = \frac{1}{8}\begin{bmatrix} 12 & 9 \\ 9 & 13 \end{bmatrix}, \quad \Lambda_1 = \Lambda_2 = 10\,\mathbb{I}_2$$
$$p_1 = \frac{1}{2}, \quad p_2 = \frac{3}{2}, \quad Q(t) = \mathbb{I}_2, \quad \hat{x}(0) = 0.$$

The numerical solver was set as indicated in Table 4.1. The results of the simulation are presented in

| **Parameter** | **Value** |
|---|---|
| Method | Backward differentiation formula |
| Precision goal | $10^{-6}$ |
| Accuracy goal | $10^{-6}$ |
| Max step size | 0.01 |

Table 4.1: Parameters of the numerical simulation: LTI system.

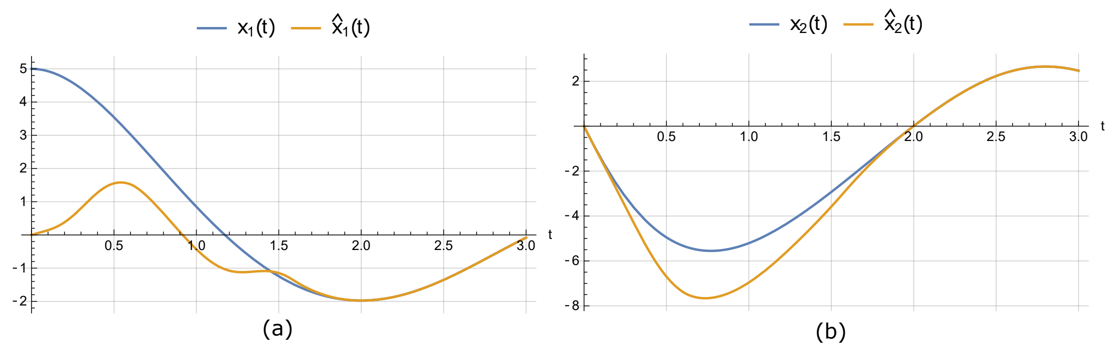


Figure 4.1: Response of the system and the observer. First state (a). Second state (b). The convergence of the estimate can be observed in both cases.

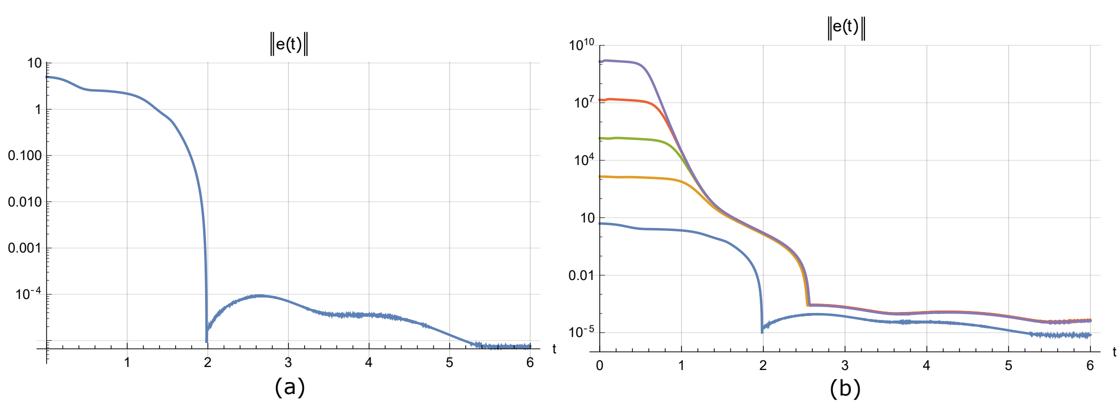


Figure 4.2: Logarithmic plots of the estimation error norm. Plot (a) exhibit finite-time convergence. Plot (b) shows the advantages of fixed-time convergence.

figures 4.1 and 4.2. In Figure 4.1 it can be observed how the estimated state follows and reach the trajectory described by the system state after 2.5 seconds. To be sure of the exact convergence, the norm of the estimation error is plotted in a logarithmic scale in Figure 4.2 (a). It is observed an acceleration of the decrease in the error norm before 2 seconds. That kind of behaviour is evidence of a rate of convergence that exceed the exponential convergence. As in any numerical simulation, the numerical error is present. The size of the numerical error can be appreciated after 5 seconds, and it has a size of about $10^{-5}$, which is close to the expected error given the numerical solver setting.

Now, to make evident the effect and the implications of the fixed-time convergence, the initial error was intentionally increase to $10^3$, $10^5$, $10^7$, and $10^9$. The behaviour of the error norm for these initial errors is shown in Figure 4.2 (b). It can be observed that the increase in the initial error does not have an appreciable impact in the convergence time. This fact contrast with the behaviour of the exponential convergence where an increase in the initial error always means a increase in the time needed to get close to zero. It also can be seen that the four trajectories practically converge at the same time, about 2.6 seconds. In Figure 4.2 (b) the acceleration of the convergence can also be appreciated. For this simulation, the numerical error increases a little, from $10^{-5}$ to $10^{-4}$.

Figure 4.2 as a whole present the two main features of the proposed observer. It provides an exact convergence and more importantly, it can guarantee a time of trustiness on the estimate that does not depend on the initial error.

## 4.2 Constant parameters estimation

The classic problem of estimating parameters in a linear regression model can be posed as follows (Narendra and Annaswamy 1989)[Sec. 3.2]: Let $\theta \in \mathbb{R}^n$ represent $n$ constant unknown parameters. Let $\omega(t) \in \mathbb{R}^{m\times n}$ a piece-wise continuous, uniformly bounded, matrix valued function that will be denotes as regressor. It is assumed that the regressor is known at each $t$. Consider the linear regression model

$$y(t) = \omega(t)\theta, \quad y(t) \in \mathbb{R}^m. \tag{4.4}$$

The problem consists in determining $\theta$ using the information of $y(t)$ and $\omega(t)$.

It is well known in the adaptive control theory that persistency of excitation is needed to recovery the unknown parameters (Narendra and Annaswamy 1989)[Sec. 3.2]. This property depends exclusively on $\omega(t)$ and is defined as follows:

**Definition 4.1** (Persistency of excitation)**.** (Ioannou and Sun 1995)[Def. 4.3.1] A picewise continuous signal vector function $\omega(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times n}$ is of persistent excitation in $\mathbb{R}^n$ with a level of excitation $\alpha_0 > 0$ if there exist positive constants $\alpha_1$, $T$ such that

$$\alpha_1 \mathbb{I}_n \geq \frac{1}{T}\int_t^{t+T} \omega^\top(s)\omega(s)\mathrm{d}s \geq \alpha_0 \mathbb{I}_b, \quad t \geq 0. \tag{4.5}$$

The persistency of excitation is required for all the classical parameter estimation schemes to ensure the convergence.

Although the previous description is the usual way to present the problem of parameter estimation, it leaves aside the fact that the parameter estimation problem is an observation problem. To see it, (4.4) can be written as follows:

$$\begin{aligned} \dot{x}(t) &= 0, \\ y(t) &= C(t)x(t). \end{aligned} \tag{4.6}$$

Here, $\theta$ was replaced by $x(t)$ and $\omega(t)$ by $C(t)$. Given that $\dot{x}(t) = 0$, $x(t)$ is constant. Then, (4.4) and (4.6) represent the same system, but (4.6) in a more familiar fashion. Notice that for system (4.6), the observability gramian and the constructibility gramian are the same. Also, notice that such gramians coincide with the description of the persistency of excitation when there is uniformity. This has been pointed out by B. Anderson 1977.

System (4.6) is a linear time-varying system with the peculiarity that matrix $A$ is the null matrix. The absence of the linear part in the dynamics allows us to leave out the linear term in the observer, and to omit the computation of $H(t)$. To estimate $x(t)$ in this scenario, we propose the following simplification of (3.2):

$$\dot{\hat{x}}(t) = -N(t)\Lambda_1\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_1} - N(t)\Lambda_2\lceil N(t)\hat{x}(t) - \psi(t)\rfloor^{p_2}, \tag{4.7}$$

$$\begin{aligned} \dot{N}(t) &= -N(t)Q(t)N(t) + C^\top(t)C(t), \quad N(t_0) = 0, \\ \dot{\psi}(t) &= -N(t)Q(t)\psi(t) + C^\top(t)y(t), \quad \psi(t_0) = 0. \end{aligned}$$

Again, the proposed simplifications do not change the properties of the observer. In this scenario, we have the following result:

**Theorem 4.3.** *Consider system* (4.6) *and observer* (4.7)*. Assume that the pair* $(0, C(t))$ *is UCC over a time window of length* $T$*. Equivalently,* $C(t)$ *is of persistent excitation. Given the UCC, we have that* $N(t) \geq \eta\,\mathbb{I}_n$ *for* $t \geq t_0 + T$ *and for some positive* $\eta > 0$*. Under this circumstances,* $\hat{x}(t)$ *converge to* $x(t)$ *in fixed-time, uniformly in* $t_0$*. The convergence time does not exceed*

$$T + \frac{1}{\sigma_n(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{1}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2-1)}.$$

*Proof.* Define $e(t) = \hat{x}(t) - x(t)$ as the parameters estimation error. The dynamics of $e(t)$ is

$$\dot{e}(t) = -N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} - N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2}. \tag{4.8}$$

Consider as Lyapunv function candidate $V(e) = e^\top e$. The derivative of $V(t)$ along the solutions of (4.8) is

$$\begin{aligned} \dot{V}(t) &= -2\,e^\top(t)N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} - 2\,e^\top(t)N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2} \\ &\leq -2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|_{p_1+1}^{p_1+1} - 2\,\sigma_n(\Lambda_2)\big\|N(t)e(t)\big\|_{p_2+1}^{p_2+1} \\ &\leq -2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|^{p_1+2} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\big\|N(t)e(t)\big\|^{p_2+1}. \end{aligned} \tag{4.9}$$

For $t \geq t_0 + T$, we have that $N(t) \geq \eta\,\mathbb{I}_n$, and

$$\begin{aligned} \dot{V}(t) &\leq -2\,\sigma_n(\Lambda_1)\eta^{p_1+1}\|e(t)\|^{p_1+2} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}\|e(t)\|^{p_2+1} \\ &\leq -2\,\sigma_n(\Lambda_1)\eta^{p_1+1}V^{\frac{p_1+1}{2}}(t) - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}V^{\frac{p_2+1}{2}}(t). \end{aligned}$$

Again, by Theorem 2.4, the uniform fixed-time convergence can be asserted. From the same theorem, we obtain the following upper bound for the convergence time:

$$T + \frac{1}{\sigma_n(\Lambda_1)\eta^{p_1+1}(1-p_1)} + \frac{1}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2-1)}.$$

□

Notice that in this case, the convergence time cannot be less than $T$. By increasing $\sigma_n(\Lambda_1)$ and $\sigma_n(\Lambda_2)$ the convergence time can be made arbitrarily close to $T$, but not equal to it.

In contrast to the general case, in the constant parameter estimation we can investigate non-uniform fixed-time convergence. The main reason is that here the Lyapunov function can be chosen time invariant and it accept upper and lower bound in terms of the error norm. To see this, consider again (4.9). Instead of assuming the bound $N(t) \geq \eta\,\mathbb{I}_n$, let us consider directly $\sigma_n(N(t))$ as a function of $t$. Then, the time derivative of $V(t)$ can be bounded as

$$\begin{aligned}\dot{V}(t) &\leq -2\,\sigma_n(\Lambda_1)\sigma_n^{p_1+1}\big(N(t)\big)\|e(t)\|^{p_1+1} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\sigma_n^{p_2+1}\big(N(t)\big)\|e(t)\|^{p_2+1} \\ &\leq -2\,\sigma_n(\Lambda_1)\sigma_n^{p_1+1}\big(N(t)\big)V^{\frac{p_1+1}{2}}(t) - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\sigma_n^{p_2+1}\big(N(t)\big)V^{\frac{p_2+1}{2}}(t)\end{aligned}$$

Now, from Theorem 2.4, we know that if there exist $t_1$ for each $t_0 \geq 0$, $t_1 > t_0$, such that

$$\int_{t_0}^{t_1} \sigma_n^{p_1+1}\big(N(s)\big)\mathrm{d}s \geq \frac{1}{\sigma_n(\Lambda_1)(1-p_1)} \quad \text{and} \quad \int_{t_0}^{t_1} \sigma_n^{p_2+1}\big(N(s)\big)\mathrm{d}s \geq \frac{n^{\frac{p_2-1}{2}}}{\sigma_n(\Lambda_2)(p_2-1)},$$

then, $e(t) = 0$ is reached in fixed-time, but not necessarily with uniformity in $t_0$. The restriction over the integrals means that some information is needed in order to make the estimation. This restriction cannot be avoided without violating the constructibility. However, uniform constructibility is not necessary. In other words, this means that the classical requirement of persistent of excitation can be relaxed since the integral of $\sigma_n(N(t))$ can be finite. Furthermore, notice that the requisite for the integrals can be lowered by increasing the gains $\sigma_n(\Lambda_i)$, or by adjusting the exponents $p_i$. Beside this interesting property of the observer (4.7), it is important to remark that without uniform constructibility, i.e., without persistence of excitation, the estimation error may grow unbounded if there is, for example, persistent noise in the measurements.

In relation to the robustness of the observer, the result of Theorem 3.2 can be recovered by using $V(e) = e^\top e$ as ISS-Lyapunov function and assuming UCC. On the other hand, recovering the result of Theorem 3.3 is more complicated. Without the linear term depending on $H(t)$, the dynamics of $\chi(t) = N(t)e(t) + \zeta(t)$, with $\zeta(t)$ as in (3.10), results in

$$\dot{\chi}(t) = -N(t)Q(t)\chi(t) - N^2(t)\sum_{i=1}^{2}\Lambda_i\lceil\chi(t)\rfloor^{p_i} + C^\top(t)\big(C(t)e(t) - \delta(t)\big).$$

Here, the term depending on $e(t)$ and $\delta(t)$ is bounded, but does not disappear. This term prevents the convergence of $\chi(t)$ to zero. One may thing that the disturbance can be compensated by using a discontinuous (setting $p_1 = 0$) with a large gain. However, increasing the gain might result in an increase of the size of $e(t)$, making the increment in the gain insufficient. To recover the convergence of $\chi(t)$ to zero, it seems that a linear term of the form $-N^{-1}(t)C^\top(t)\big(C(t)\hat{x}(t) - y(t)\big)$ is needed in the observer.

### 4.2.1 Numerical Example

To illustrate the advantages given by the proposed observer, we are going to use the following system:

$$\begin{aligned}y(t) &= \omega(t)\theta \\ &= \begin{bmatrix}\cos(t) & 1\end{bmatrix}\begin{bmatrix}12 \\ -3\end{bmatrix}.\end{aligned}$$

To estimate the parameters $\theta$, the observer was configured using the next settings:

$$\Lambda_1 = \Lambda_2 = 10\,\mathbb{I}_2, \quad p_1 = \frac{1}{2}, \quad p_2 = \frac{3}{2}, \quad Q(t) = \mathbb{I}_2.$$

In Figure 4.3 we show the results of the simulation. The simulation was computed using the parameters show in Table 4.2 for the numerical method. In Figure 4.3 (a), the convergence of the estimate to the true parameters is shown. It can be seen that after approximately 3.5 seconds the estimation is exact. To confirm the fixed-time convergence, the initial error was intentionally increased to $10^3$, $10^5$, $10^7$, and $10^9$. The convergence in the error norm is illustrated in a logarithmic plot in Figure 4.3 (b). As it can be appreciated,

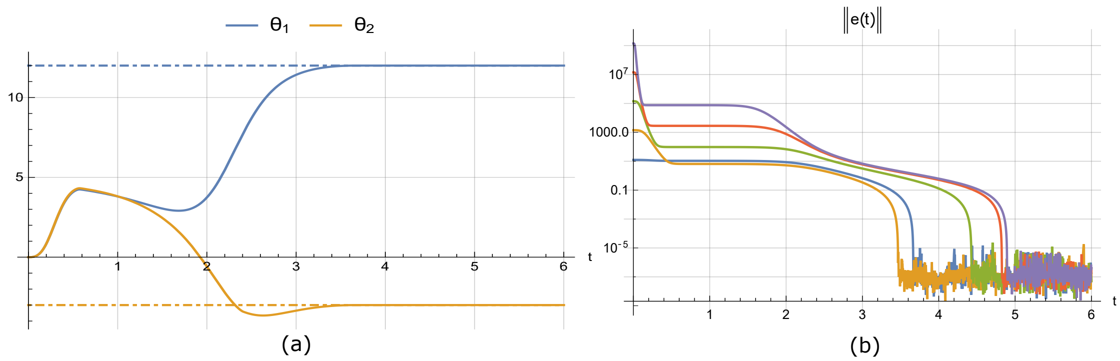


Figure 4.3: Results of the parameters estimation process. Convergence of the estimates (a). Logarithmic plot of the error norm for different initial conditions (b).

| **Parameter** | **Value** |
|---|---|
| Method | Backward differentiation formula |
| Precision goal | $10^{-7}$ |
| Accuracy goal | $10^{-7}$ |
| Max step size | 0.01 |

Table 4.2: Parameters of the numerical simulation: Constant parameters estimation.

the convergence is accelerated at the beginning when the error is really large. This makes that in all the cases the convergence occurs at almost the same time. Also, it is important to remark the strict decrease at the end of the trajectory, which revels the finite-time convergence. As expected in a numerical simulation, there is numerical error, which in this case results with an order of $10^{-7}$.

Now, to show a case when there is not persistency of excitation, we change the regressor for

$$\omega(t) = \frac{1}{1+t}\begin{bmatrix}\cos(t) & 1\end{bmatrix}.$$

In this situation, the information present in the regressor is lost with time. Clearly, a regressor with this characteristic cannot be of persistent excitation. To help the observer in this situation, its parameter were changed to

$$\Lambda_1 = \Lambda_2 = 50\,\mathbb{I}_2, \quad Q(t) = \frac{1}{5}\,\mathbb{I}_2,$$

leaving the rest intact. The results of the simulation under the exposed circumstances are displayed in Figure 4.4. As can be seen in Figure 4.4 (a), the convergence to the true parameters is not loss for this particular regressor. Figure 4.4 (b) illustrates that the fixed-time convergence is kept. It is important to remark that in this situation the starting time matters, so that we cannot speak about uniformity. Different initial times may result in very different convergence times. To illustrated this phenomena, the simulation was repeated taking the starting time as 5 and 15 seconds, instead of zero. The results of this experiment is shown in Figure 4.5. In this figure, the numerical solution of the three situation are compared. Notice that the initial condition was the same in each case. As can be seen, the time needed for the convergence increase when the starting time increases, an such increase is not proportional. Other effect due to the lack of uniformity is the lost of robustness against bounded disturbances in the dynamics and the measurements. To illustrate this, the function $0.1\,\sin(3\,t) + 0.1$ was added to $y(t)$ and it is not considered for the observer. Even though this disturbance is bounded, it generates a derive in the estimates. This is shown in Figure 4.6.

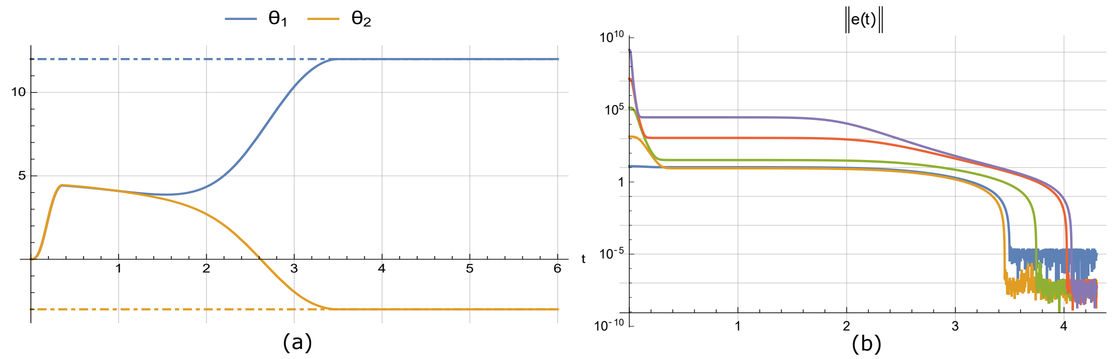


Figure 4.4: Results of the parameters estimation process with loss of information. Convergence of the estimates (a). Logarithmic plot of the error norm for different initial conditions (b).

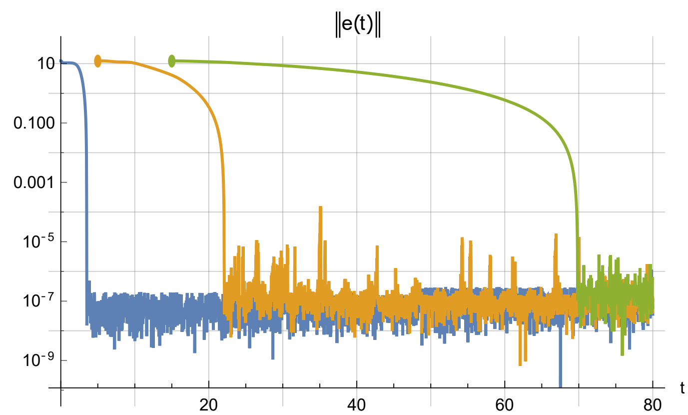


Figure 4.5: Convergence of the estimation error for different starting times in the absence of uniformity.

## 4.3 Adaptive observer

An adaptive observe should be capable of estimating the state of the system even in the presence of partially unknown disturbances (Besançon 2007, Chap. 7). By partially unknown disturbances we mean disturbances for which a model is available. One common model is a linear parametrization, as shown in the next system structure:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + b\,\omega(t)\theta, \\ y(t) &= C_0 x(t), \end{aligned} \tag{4.10}$$

with $x(t) \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $\omega(t) \in \mathbb{R}^{1\times m}$ a piece-wise continuous, uniformly bounded matrix valued function. The system matrices are defined as follows:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}, \qquad C_0^\top = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

with $b_1 + b_2\,s + \cdots + b_n s^n$ a Hurwitz polynomial. This system class is taken from (Marino and Tomei 1992). Although the structure of (4.10) seems restrictive, it appears when estimating the state of a single-input-single-output (SISO) LTI systems together with the $2\,n$ parameters of its minimal realization (Narendra and Annaswamy 1989)[Sec. 4.3.2](Ioannou and Sun 1995)[Sec. 5.4]. Structure (4.10) can also appear when

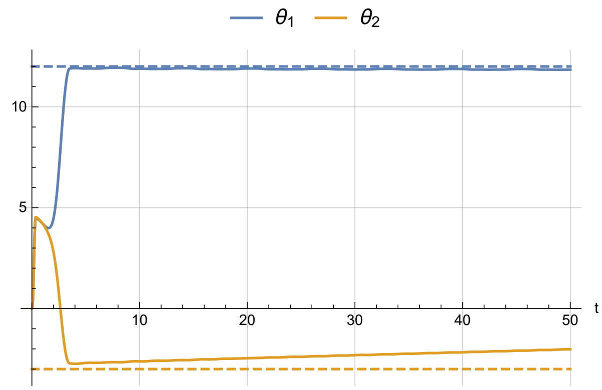


Figure 4.6: Derive of the estimates due to the presence of noise and lack of persistency in the excitation.

designing an adaptive observer for non-linear system of the form

$$\dot{x}(t) = f\big(x(t)\big) + q_0\big(x(t), u(t)\big) + \sum_{i=1}^{m} q_1\big(x(t), u(t)\big)\theta_i,$$
$$y(t) = h\big(x(t)\big).$$

To be able of designing the observer, the system has to be put in the form

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + g_0\big(y(t), u(t)\big) + G\big(y(t), u(t)\big)\theta, \\ y(t) &= C_0 z(t), \end{aligned} \tag{4.11}$$

by a parameter independent transformation. Conditions for the existence of such transformation are given in (Marino and Tomei 1992)[Lem. 1.1]. Once the system is in form (4.11), by the parameter dependent transformation (filtered transformation) (Marino and Tomei 1992)

$$\xi(t) = z(t) - M(t)\theta,$$
$$\dot{M}(t) = \big(A_0 - b\, C_0 A_0\big) M(t) + \big(\mathbb{I}_n - b\, C_0\big) G\big(y(t), u(t)\big),$$

the dynamics of $\xi(t)$ results in form (4.10). Using this procedure, it is possible to choose $b$. However, to recover $z(t)$ and then $x(t)$, the parameters are needed in order to invert the filtered transformation.

In (4.10), the nominal part of the system is represented by $A_0 x(t)$, whereas the term $b\,\omega(t)\theta$ represents a disturbance with parameters $\theta$ as the unknown part. The problem of designing an adaptive observer for system (4.10) consists in finding an algorithm to recover the internal state in the presence of the disturbance, and, if possible, the value of $\theta$.

To put this problem in the framework that has been proposed in this work, we need to extend the system state to include the parameters as part of it. To that matter, define $\chi(t) \in \mathbb{R}^{n+m}$ as $\chi(t) = [x^\top(t), \theta^\top]^\top$. The dynamics of the system can be represented through this new variable as

$$\begin{aligned} \dot{\chi}(t) &= A(t)\chi(t), \\ y(t) &= C\,\chi(t), \end{aligned} \tag{4.12}$$

with

$$A(t) = \begin{bmatrix} A_0 & b\,\omega(t) \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \big[C_0 \; 0\big].$$

Using this representation it is clear that the problem of designing an adaptive observer for system (4.10) is equivalent to designing an observer for the linear time-varying system (4.12). To be able of recovering the state and parameters, constructibility of the pair $(A(t), C)$ is needed. Investigate this property with the only knowledge of $A_0$, $C_0$, $b$, and $\omega(t)$ is a hard task. Fortunately, for a smooth $\omega(t)$, the UCC of the pair $(A(t), C)$ is equivalent to the observability of the pair $(A_0, C_0)$, which holds, and the persistent excitation of $\omega(t)$. Additionally, for system (4.12) it has been shown in (Juan G. Rueda-Escobedo and Jaime A. Moreno 2017) that the state can always been recovered regardless of the properties of $\omega(t)$.

For system (4.12), the observer structure proposed in the last chapter can also be simplified. As in the previous cases studied in this chapter, the main simplification is related to the computation of $H(t)$. The advantage given by the structure of the matrices $A_0$ and $C_0$ is that one can always find $L_0 \in \mathbb{R}^n$, $P_0 \in \mathbb{R}^{n\times n}$, $P_0 = P_0^\top > 0$, and $R \in \mathbb{R}^{n\times n}$, $R = R^\top > 0$, such that

$$P_0\big(A_0 - L_0C_0\big) + \big(A_0 - L_0C_0\big)^\top P_0 = -R \quad \text{and} \quad P_0 b = C_0^\top .$$

This possibility allows the design of an observer for (4.12) of the form

$$\dot{\hat{\chi}}(t) = A(t)\hat{\chi}(t) - L(t)\big(C\,\hat{\chi}(t) - y(t)\big), \tag{4.13}$$

with

$$L(t) = \begin{bmatrix} L_0 \\ \Gamma\,\omega^\top(t) \end{bmatrix}, \quad \text{and} \quad \Gamma \in \mathbb{R}^{m\times m},\ \Gamma = \Gamma^\top > 0.$$

The solution showed in (4.13) is the classic way of solving the problem (Marino and Tomei 1992). The introduction of $L(t)$ allows to stabilize the error dynamics induced by (4.13). The same $L(t)$ can be used to replace the necessity of $H(t)$ in the observer (3.2). Using that modification, the resulting observer is

$$\dot{\hat{\chi}}(t) = A(t)\hat{\chi}(t) - L(t)\big(C\,\hat{\chi}(t) - y(t)\big) - P^{-1}N(t)\sum_{i=1}^{2}\Lambda_i\big\lceil N(t)\hat{\chi}(t) - \psi(t)\big\rfloor^{p_i}, \tag{4.14}$$

$$\begin{aligned}
\dot{N}(t) &= -A^\top(t)N(t) - N(t)A(t) - N(t)Q(t)N(t) + C^\top C, \quad N(t_0) = 0,\\
\dot{\psi}(t) &= -\Big(A^\top(t) + N(t)Q(t)\Big)\psi(t) + C^\top y(t), \quad \psi(t_0) = 0,
\end{aligned}$$

where

$$P \in \mathbb{R}^{(n+m)\times(n+m)}, \quad P = \begin{bmatrix} P_0 & 0 \\ 0 & \Gamma^{-1} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} P_0^{-1} & 0 \\ 0 & \Gamma \end{bmatrix}.$$

The main difference between the classic approach and the proposed observer is the type of convergence. The classic approach can recover $x(t)$ asymptotically; if, in addition, $\omega(t)$ is of persistent excitation, both, $x(t)$ and $\theta$, are recovered exponentially, but the speed of convergence cannot be freely specified (Narendra and Kudva 1974), (Narendra and Annaswamy 1989). In the case of (4.14), $x(t)$ can also be recovered asymptotically. If in addition, $\omega(t)$ is of persistent excitation, both, $x(t)$ and $\theta$, can be recovered in fixed-time, uniformly in $t_0$. Furthermore, the speed of convergence can be adjusted by means of $\Lambda_1$ and $\Lambda_2$. More precisely, we have

**Theorem 4.4.** *Consider the systems* (4.12) *and* (4.14) *with* $\omega(t)$ *uniformly bounded and piecewise continuous. Then, the overall estimation error* $e(t) = \hat{\chi}(t) - \chi(t)$ *remains bounded. Furthermore, the estimation error in the state* $\tilde{x}(t) = \hat{x}(t) - x(t)$ *tends to zero as* $t \to \infty$.

*Proof.* Define the observation error as $e(t) = \hat{\chi}(t) - \chi(t)$. Its dynamics is

$$\dot{e}(t) = \Big(A(t) - L(t)C\Big)e(t) - P^{-1}N(t)\sum_{i=1}^{2}\Lambda_i\big\lceil N(t)e(t)\big\rfloor^{p_i}.$$

To analyze the the stability of the solution $e(t) = 0$, we propose the candidate Lyapunov function $V(e) = e^\top P\, e(t)$. The derivative of $V$ along the dynamics of $e(t)$ results in

$$\dot{V}(t) = e^\top(t)\Big(P\big(A(t) - L(t)C\big) + \big(A(t) - L(t)C\big)^\top P\Big)e(t) - 2\, e^\top(t)N(t)\sum_{i=1}^{2}\Lambda_i\lceil N(t)e(t)\rfloor^{p_i}.$$

To continue, the matrix in the quadratic term, $P\big(A(t) - L(t)C\big) + \big(A(t) - L(t)C\big)^\top P$, has to be analysed. For it, each term has to be expressed by blocks

$$\begin{bmatrix} P_0 & 0 \\ 0 & \Gamma^{-1} \end{bmatrix}\begin{bmatrix} A_0 - L_0C_0 & b\,\omega(t) \\ -\Gamma\,\omega^\top(t)C_0 & 0 \end{bmatrix} + \begin{bmatrix} (A_0 - L_0C_0)^\top & -C_0^\top\omega(t)\Gamma \\ \omega^\top(t)b^\top & 0 \end{bmatrix}\begin{bmatrix} P_0 & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} =$$
$$\begin{bmatrix} P_0\big(A_0 - L_0C_0\big) + \big(A_0 - L_0C_0\big)^\top P_0 & P_0 b\,\omega(t) - C_0^\top\omega(t) \\ -\omega^\top(t)C_0 + \omega^\top(t)b^\top P_0 & 0 \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 0 & 0 \end{bmatrix}.$$

Define $\tilde{x}(t) = \hat{x}(t) - x(t)$ and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. Then

$$\begin{aligned}
\dot{V}(t) &= -\tilde{x}^\top(t)R\,\tilde{x}(t) - 2\, e^\top(t)N(t)\Lambda_1\lceil N(t)e(t)\rfloor^{p_1} - 2\, e^\top(t)N(t)\Lambda_2\lceil N(t)e(t)\rfloor^{p_2} \\
&\leq -\sigma_n(R)\|\tilde{x}(t)\|^2 - 2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|_{p_1+1}^{p_1+1} - 2\,\sigma_n(\Lambda_2)\big\|N(t)e(t)\big\|_{p_2+1}^{p_2+1} \\
&\leq -\sigma_n(R)\|\tilde{x}(t)\|^2 - 2\,\sigma_n(\Lambda_1)\big\|N(t)e(t)\big\|^{p_1+1} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\big\|N(t)e(t)\big\|^{p_2+1} \leq 0.
\end{aligned} \tag{4.15}$$

Then, $e(t) = 0$ is uniformly stable by Theorem 2.1. Furthermore, by the Barbalat's Lemma (Narendra and Annaswamy 1989)[Lem. 2.12], $\|x(t)\| \to 0$ as $t \to \infty$. □

**Theorem 4.5.** *Consider the systems* (4.12) *and* (4.14). *Assume that $\omega(t)$ is uniformly bounded, smooth, and of persistent excitation with $T$ the length of the time window for which* (4.5) *holds. Then, there exist $\eta > 0$ such that $N(t) \geq \eta\,\mathbb{I}_n$ for all $t \geq t_0 + T$ and $\hat{\chi}(t)$ converges to $\chi(t)$ in fixed-time, uniformly in $t_0$. The convergence time does not exceed*

$$T + \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_1)\eta^{p_1+1}(1 - p_1)} + \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2 - 1)}$$

*for any initial error $\hat{\chi}(t_0) - \chi(t_0)$.*

*Proof.* Consider the same Lyapunov function as in the proof of Theorem 4.5. We retake the procedure from (4.15). Since $\omega(t)$ is of persistent excitation, there exist $T > 0$ and $\eta > 0$ such that $N(t) \geq \eta\,\mathbb{I}_n$ for $t \geq t_0 + T$. Given that $e(t) = 0$ is uniformly stable, the error does not grow too much in the interval $[t_0, t_0 + T]$. Then, for $t \geq t_0 + T$, inequality (4.15) becomes strict, and we have

$$\begin{aligned}
\dot{V}(t) &\leq -2\,\sigma_n(\Lambda_1)\eta^{p_1+1}\|e(t)\|^{p_1+1} - 2\,\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}\|e(t)\|^{p_2+1} \\
&\leq -2\frac{\sigma_n(\Lambda_1)\eta^{p_1+1}}{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}V^{\frac{p_1+1}{2}}(t) - 2\frac{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}}{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}V^{\frac{p_2+1}{2}}(t).
\end{aligned} \tag{4.16}$$

Then, by Theorem 2.4, we can conclude the uniform fixed-time stability of $e(t) = 0$. Furthermore, this theorem provide us with a bound for the convergence time:

$$T + \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_1)\eta^{p_1+1}(1 - p_1)} + \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_2)n^{\frac{1-p_2}{2}}\eta^{p_2+1}(p_2 - 1)}.$$

□

As in the case of constant parameters estimation, here, we can also analyze non-uniform fixed-time convergence. The reason, as before, is that $V$ is time invariant and its lower bound does not degenerate to zero.

To analyze this, consider again (4.15) and replace $\eta$ by $\sigma_n(N(t))$ as a function of time. Then, if for every $t_0 > 0$ there exist $t_1$ such that

$$\int_{t_0}^{t} \sigma_n^{p_1+1}(N(s))\mathrm{d}s \geq \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_1)(1-p_1)} \quad \text{and} \int_{t_0}^{t} \sigma_n^{p_2+1}(N(s))\mathrm{d}s \geq \frac{\max\{\sigma_1(P_0), \sigma_1(\Gamma^{-1})\}}{\sigma_n(\Lambda_2) n^{\frac{1-p_2}{2}}(p_2-1)},$$

$\hat{\chi}(t)$ converges to $\chi(t)$ in fixed-time. As before, the lack of uniformity makes the convergence time dependent of the initial time. Also, in the presence of bounded disturbances, the estimation error may diverge if the disturbances are persistent. On the other hand, if $\omega(t)$ is of persistent excitation, the ISS property showed in Theorem 3.2 can be recovered by using $V(e) = e^\top P\, e$ as ISS-Lyapunov function. In the case of Theorem 3.3, something similar to the case of constant parameter estimation happens. Because the use of $H(t)$ is avoided, the error dynamic of the variable $\xi(t) = N(t)e(t) + \zeta(t)$ is perturbed:

$$\dot{\xi}(t) = -\big(A^\top(t) + N(t)Q(t)\big)\xi(t) - N(t)P^{-1}N(t)\sum_{i=1}^{2} \Lambda_i \lceil \xi(t) \rfloor^{p_i} + \big(N(t)L(t) + C^\top\big)\big(\delta(t) - C\, e(t)\big).$$

Then, it cannot be guarantee that $\xi(t)$ converge to zero. Due to this, the estimation error does not reach the limit trajectory $-N^{-1}(t)\zeta(t)$.

### 4.3.1 Numerical Example

To illustrate the properties of the adaptive observer, we will use it to observer the following system:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + b\,\omega(t)\theta \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} \cos(3t) & 1 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \\ y(t) &= C_0\, x(t) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

For this system, $L_0$ and $P_0$ can be chosen as

$$L_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad\qquad P_0 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}.$$

Using these matrices, we obtain

$$P_0\big(A_0 - L_0 C_0\big) + \big(A_0 - L_0 C_0\big)^\top P_0 = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}, \qquad P_0\, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now, to configure the observer, the following parameters were chosen:

$$\Gamma = 10\,\mathbb{I}_2, \quad \Lambda_1 = \Lambda_2 = 10\,\mathbb{I}_4, \quad Q(t) = \mathbb{I}_4, \quad p_1 = \frac{1}{2}, \quad p_2 = \frac{3}{2}.$$

In figures 4.7 and 4.8 the convergence of the adaptive observer is shown. In Figures 4.7 (a) and (b) the trajectories of the system's state and the estimates provided by the observer are compared. It can be appreciated how the estimates reach the target. In Figure 4.8 (a) the parametric convergence can be appreciated. On the other hand, Figure 4.8 (b) shows a logarithmic plot of the error norm for the state and parameter estimation. This plot helps us to make evident the exact convergence. This is reflected in the high slope that the graphs exhibit after 8 seconds. Now, to illustrate the fixed-time part, the simulation was repeated with an intentional increment in the initial error. The initial error was set in $10^3$, $10^5$, and $10^7$. The results of this process is shown in Figure 4.9. It can be seen that the increments in the initial error are not traduced in a significant increment in the convergence time. The high slope at the end of the

| Parameter | Value |
|---|---|
| Method | Implicit Runge-Kutta 4-5 |
| Precision goal | $10^{-6}$ |
| Accuracy goal | $10^{-6}$ |
| Max step size | 0.01 |

Table 4.3: Parameters of the numerical simulation: Adaptive observer.

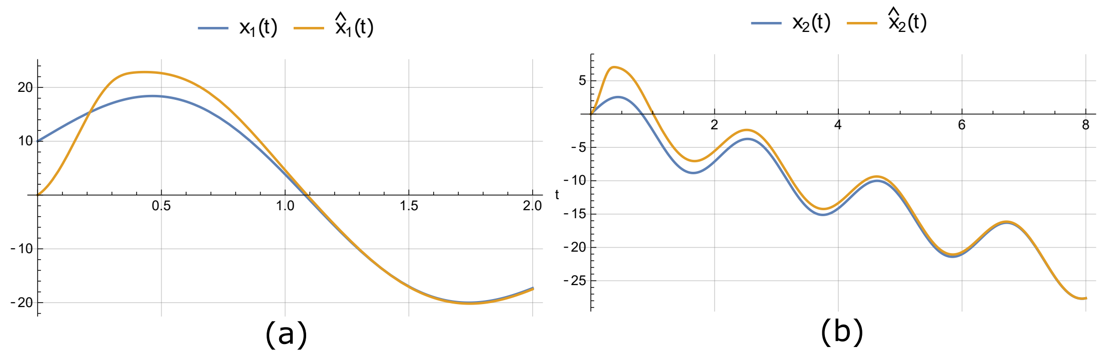


Figure 4.7: Comparison of the estimated state and the state of the system. It is shown how the estimate reach the trajectory of the real system.

convergence also testifies to the exact convergence. As in the previous cases, a numeric error can be seen after the reaching phase. The numerical error in this example has an order around $10^{-5}$.

Finally, we proceed to show the behavior of the observer when $\omega(t)$ lack of excitation. For this, consider

$$\omega(t) = \frac{1}{t+1}\begin{bmatrix}\cos(3\,t) & 1\end{bmatrix}.$$

Using the same parameters for the observer and the simulation we obtained the plots shown in Figures 4.10 and 4.11. As can be seen in Figure 4.11, the observer is capable of recovering the state and the parameters exactly for arbitrarily large initial error, that is, the fixed-time convergence is preserved. However, if we change the starting time, the convergence time increases for the same initial conditions. This is illustrated in Figure 4.12 (a). Now, to exhibit some of the problems that appear when there is no uniformity, we added the function $0.2\sin(3\,t) + 0.2$ to the output. The result is shown in Figure 4.12 (b). As can be seen, the estimation error in the case of the state remains bounded. However, in the case of the parameters, the estimation error diverges.

## 4.4 Time-varying parameters estimation

The problem formulation of estimating time-varying parameters is quite similar, in structure, to the one of estimating constant parameters. However, the similarities end there. Estimating time-varying parameters involves the reconstruction of functions instead of constant vectors. The problem statement is as follows: Let $f(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a measurable uniformly bounded vector valued function, with bound $\Delta \geq \|f(t)\|$ for all $t \geq 0$. Let $C(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times n}$ be a uniformly bounded, piecewise smooth matrix valued function, with a uniformly bounded derivative where defined, and for which the intervals in where is smooth, does not shrink to zero. Consider the following system

$$\begin{aligned}\dot{\theta}(t) &= f(t), \quad \theta(t_0) = \theta_0,\\ y(t) &= C(t)\theta(t).\end{aligned} \tag{4.17}$$

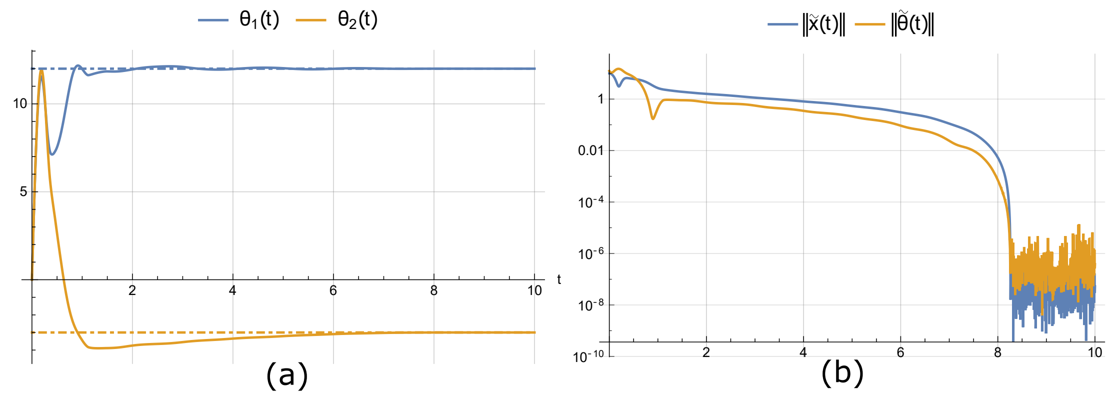


Figure 4.8: Parametric convergence (a). Logarithmic plot of the error norm in the state and parameters (b).

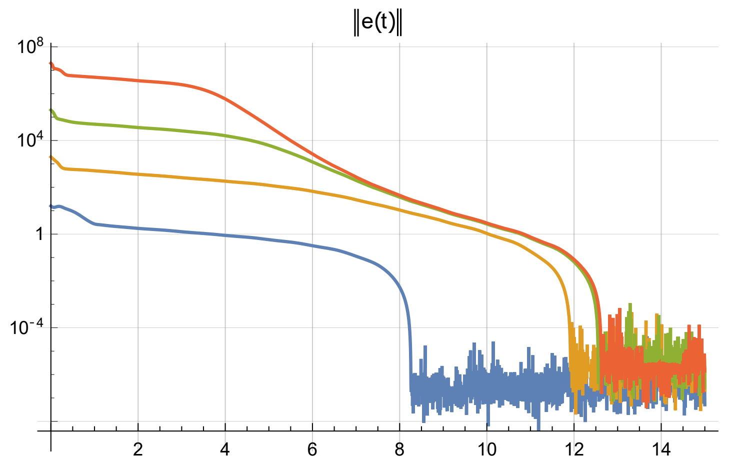


Figure 4.9: Logarithmic plot of the error norm combining the state estimation error and the parametric error for different magnitudes of initial error.

The problem consists in determining $\theta(t)$ from the knowledge of the output $y(t)$, the regressor $C(t)$, and the bound $\Delta$.

To observe $\theta(t)$, the algorithm developed in the previous chapter cannot be applied since (4.17) corresponds to a linear time-varying system with unknown input. Therefore, given that $f(t)$ is unknown, the dynamics of $\theta(t)$ cannot be replicated. However, this problem can be put in a linear operator framework, and the ideas of chapters 2 and 3 can be extended to this scenario.

Notice that $\theta(t)$ as the solution of (4.17) is a Lipschitz function since $f(t)$ is uniformly bounded. Since $C(t)$ is assumed piecewise smooth, $y(t)$ results piecewise Lipschitz. Then, the output equation in (4.17) can be seen as a linear map between the space of vector valued Lipschitz functions of $n$ components and the space of piecewise Lipschitz functions of $m$ components, where the operator is $C(t)$. As we saw in Chapter 2, the observability conditions are related to the injectivity of the operator, whereas an observer can be obtained by looking at the left inverse. This principle is the basic mechanism behind the Kalman-Bucy filter and the proposed observer, and can be extended to this situation.

With respect to the injectivity of $C(t)$ and the identifiability of $\theta(t)$, we have the following result:

**Theorem 4.6.** *(Juan G. Rueda-Escobedo and Jaime A. Moreno 2016) Let* $C(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times n}$ *and Lip*$(\mathbb{R}^n)$*,*

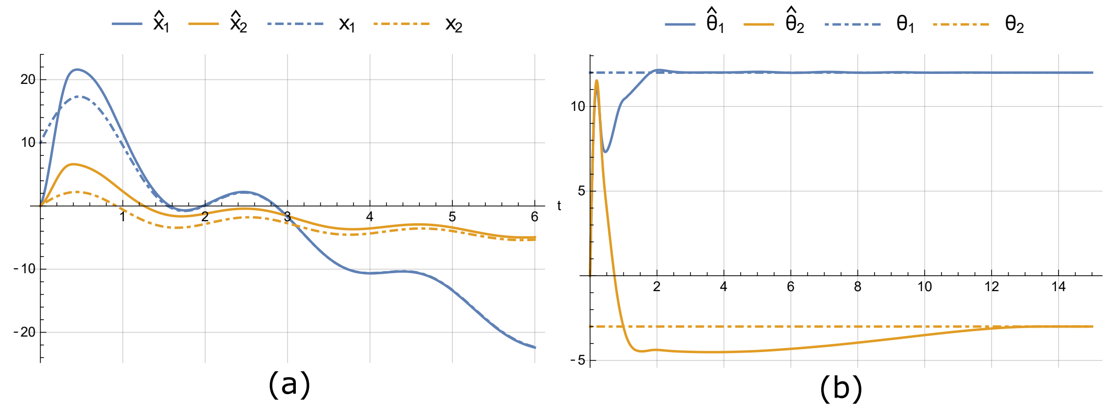


Figure 4.10: State and parameter estimation in absence of uniformity. Convergence for the state (a). Convergence for the parameters (b).

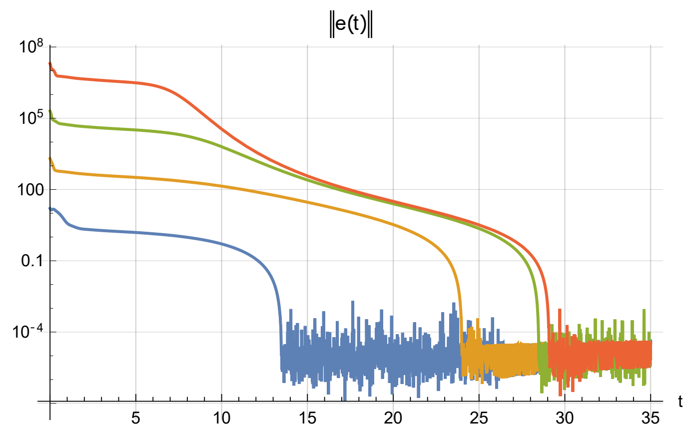


Figure 4.11: Fixed-time convergence for state and parameters without persistence of excitation.

*the set of all bounded Lipschitz continuous functions in* $\mathbb{R}^n$. *Let* $\mathcal{S}$ *be the range of* $C(t)\big(Lip(\mathbb{R}^n)\big)$. *Consider* $y(t) \in \mathcal{S}$; *on the interval* $[t_0, t_1]$ *there exist a unique* $\theta(t) \in Lip(\mathbb{R}^n)$ *such that* $y(t) = C(t)\theta(t)$ *iff* $\operatorname{rank}(C(t)) = n$ *for all* $t \in [t_0, t_1]$.

*Proof.* **Sufficiency**. Since $C(t)$ has rank $n$ for all $t$, $C^\top(t)C(t)$ is always invertible, and the *unique* solution can be computed as

$$\theta(t) = \left(C^\top(t)C(t)\right)^{-1} C^\top(t)y(t). \tag{4.18}$$

**Necessity**. Let $C^\dagger(t)$ be the (unique) Moore-Penrose pseudoinverse of $C(t)$ for each time instant. Then all possible solutions can be expressed as follows

$$\theta(t) = C^\dagger(t)y(t) + \left(\mathbb{I}_n - C^\dagger(t)C(t)\right)v(t), \tag{4.19}$$

where $v(t)$ is an arbitrary function. This is possible since $\mathbb{I} - C^\dagger(t)C(t)$ projects onto the orthogonal space of $C(t)$ for each $t$. However, for any $v(t)$, the solution might not be Lipschitz. Notice also that if $\operatorname{Rank}(C(t)) = n$ then $C^\dagger(t)C(t) = \mathbb{I}$. On the other hand, since it is assumed that $y(t) \in \mathcal{S}$, there exist $\theta_0(t) \in \operatorname{Lip}(\mathbb{R}^n)$ with $y(t) = C(t)\theta_0(t)$, and $v(t)$, such that (4.19) is Lipschitz continuous, in fact $v(t) = \theta_0(t)$. Furthermore, any other solution can be expressed as $v(t) = \theta_0(t) + w(t)$. Before constructing a suitable $w(t)$, let us analyses

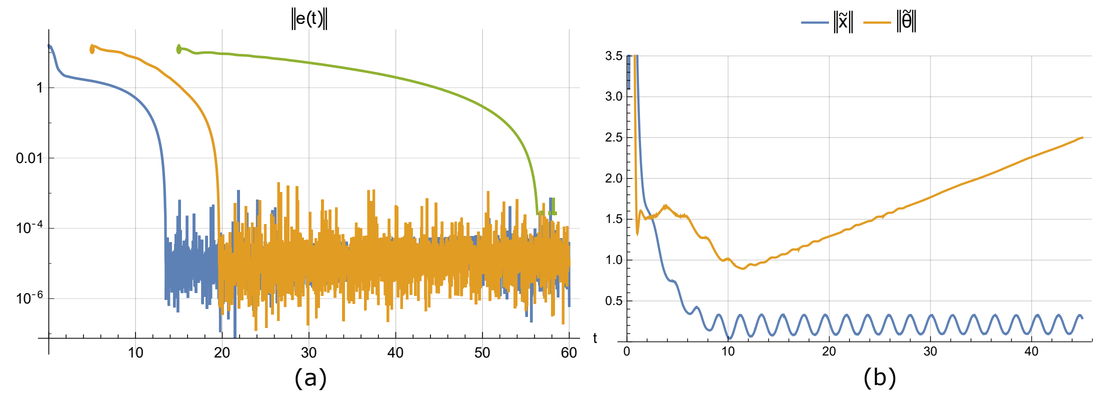


Figure 4.12: Effects of the lack of uniformity. Time convergence depending on $t_0$ (a). Derive of the estimates due to noise (b).

the term $C^\dagger(t)C(t)$.
$C^\dagger(t)C(t)$ is the orthogonal projector onto the range of $C^\top(t)$, then its induced norm is one, which implies that it is bounded element-wise. The term $C^\dagger(t)C(t)$ has discontinuities whenever $C(t)$ has it or $C^\dagger(t)$. The first one has a finite number of discontinuities in any bounded interval given the properties asked for $C(t)$. The second one has discontinuities when $C(t)$ has it, or when the rank of $C(t)$ changes (Campbell and Meyer 2009, Theo. 10.5.1). Then the set of points $\ell \in [t_0, t_1]$ where $C^\dagger(t)C(t)$ has discontinuities is closed and has no interior (Campbell and Meyer 2009, Theo. 10.5.2). Now consider $a$ and $b$, with $b > a$, any two consecutive elements of $\ell$; on the interval $(a, b)$ the regressor is differentiable, and so it is $C^\dagger(t)$ (Campbell and Meyer 2009, Theo. 10.5.3). The product $C^\dagger(t)C(t)$ is differentiable on $(a, b)$.
To construct $w(t)$, consider again $a$ and $b$ as before. Define $\epsilon = \frac{1}{3}(b-a)$, and the interval $l_1 = [a, a+\epsilon]$, $l_2 = (a+\epsilon, b-\epsilon)$, and $l_3 = [b-\epsilon, b]$. Define each component of $w(t)$, $w_i(t)$ with $i = \{1, \cdots, n\}$, as

$$w_i(t) = \begin{cases} 0 & t \in l_1 \text{ or } t \in l_3 \\ k \sin\left(\frac{\pi}{\epsilon}(t-a-\epsilon)\right) & t \in l_2 \end{cases},$$

with $k \in \mathbb{R}$, $k \neq 0$. Then $w(t)$ is Lipschitz continuous on $(a, b)$. Since $C^\dagger(t)C(t)$ is differenciable on $(a, b)$, its derivative exist and it is bounded on $l_2$, then the product $C^\dagger(t)C(t)w(t)$ is going to be Lipschitz. In a similar way, $w(t)$ can be defined for each pair $a, b \in \ell$, and (4.19) with $v(t) = \theta_0(t) + w(t)$ represent a different solution. Then, the system is not identifiable. □

The previous result give us the conditions that $C(t)$ has to satisfy in order of having identifiability of $\theta(t)$. The first consequence of Theorem 4.6 is that $m \geq n$, otherwise the rank conditions cannot be fulfilled. This means that at least the same number of measurements and parameters is needed. Now, if the rank condition is met, the left inverse of the operator results in $(C^\top(t)C(t))^{-1}C^\top(t)$, as shown in the Proof of Theorem 4.6. However, the on-line calculation of this function might be computationally expensive or cumbersome when $n$ or $m$ are large. Instead, we propose a sort of gradient algorithm to minimize the following instantaneous cost function

$$\mathcal{J} = \left\| C(t)\hat{\theta}(t) - y(t) \right\|_1.$$

The negative gradient of $\mathcal{J}$ yields

$$\dot{\hat{\theta}}(t) = -C^\top(t)\left\lceil C(t)\hat{\theta}(t) - y(t)\right\rfloor^0.$$

To have a degree of freedom, a positive definite matrix $\Gamma$ is introduced, then the algorithm results in

$$\dot{\hat{\theta}}(t) = -\Gamma\, C^\top(t)\left\lceil C(t)\hat{\theta}(t) - y(t)\right\rfloor^0. \tag{4.20}$$

It is important to remark that (4.20) has a discontinuous RHS. And because of this, the solutions of system (4.20) have to be understood in the sense of Filipov (Filippov 2013). This is crucial since the discontinuity will allow to compensate the dynamics of $\theta(t)$ without knowing $f(t)$, but by imposing a gain $\Gamma$ large enough. Other modification to (4.20) can be proposed in order to enhance the convergence, for example, including correction terms of higher degree:

$$\dot{\hat{\theta}}(t) = -\Gamma\, C^\top(t)\lceil C(t)\hat{\theta}(t) - y(t)\rfloor^0 - \Gamma\, C^\top(t)\lceil C(t)\hat{\theta}(t) - y(t)\rfloor^p,\ p > 1. \tag{4.21}$$

The conditions given in Theorem 4.6 are necessary and sufficient to reconstruct $\theta(t)$ using the left inverse. However, they might not be enough for (4.20) or (4.21) to work. In general, it would be necessary to introduce the following uniformity condition on $C(t)$:

$$\sigma_n\Big(C(t)\Big) \geq \alpha > 0 \quad \forall\, t \geq 0.$$

Knowing the value of $\alpha$ and $\Delta$, then it is possible to design $\Gamma$ such that (4.20) can work properly. This yields $\sigma_n\big(\Gamma\big) \geq \Delta/\alpha + \epsilon$, with $\epsilon > 0$ a free parameter. Under this condition, it is possible to prove that $\hat{\theta}(t) \to \theta(t)$ in finite time when using (4.20), and in fixed-time if (4.21) is used. To complete this part, the previous assertion is proven for (4.21) since it encompasses the one for (4.20).

**Theorem 4.7.** *Consider the systems* (4.17) *and* (4.21)*. Let* $\|f(t)\| \leq \Delta$*,* $\sigma_n(C(t)) \geq \alpha > 0$*, and* $\sigma_n\big(\Gamma\big) \geq \Delta/\alpha + \epsilon$*,* $\epsilon > 0$*. Then* $\hat{\theta}(t)$ *convergence to* $\theta(t)$ *in fixed-time. The convergence time does not exceed*

$$\frac{\sigma_n^{1/2}(\Gamma)}{\alpha\,\epsilon} + \frac{1}{\alpha^{p+1} n^{\frac{1-p}{2}} \sigma_n^{\frac{p+1}{2}}\big(\Gamma\big)(p-1)}.$$

*for any initial error* $\tilde{\theta}(t_0) = \hat{\theta}(t_0) - \theta(t_0)$*.*

*Proof.* To begin with, the error dynamics $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t)$ induced by (4.21) is derived:

$$\dot{\tilde{\theta}}(t) = -\Gamma\, C^\top(t)\lceil C(t)\tilde{\theta}(t)\rfloor^0 - \Gamma\, C^\top(t)\lceil C(t)\tilde{\theta}(t)\rfloor^p - f(t).$$

To analyse the stability of $\tilde{\theta}(t) = 0$, we propose as candidate Lyapunov function $V(\tilde{\theta}) = \tilde{\theta}^\top \Gamma^{-1}\tilde{\theta}$, which derivative along the trajectories of (4.21) yields

$$\begin{aligned}
\dot{V}(t) &= -2\left\|C(t)\tilde{\theta}(t)\right\|_1 - 2\left\|C(t)\tilde{\theta}(t)\right\|_{p+1}^{p+1} - 2\, f^\top(t)\Gamma^{-1}\tilde{\theta}(t)\\
&\leq -2\,\alpha\|\tilde{\theta}(t)\| - 2\, n^{\frac{1-p}{2}}\alpha^{p+1}\|\tilde{\theta}(t)\|^{p+1} + \frac{2\,\Delta}{\sigma_n(\Gamma)}\|\tilde{\theta}(t)\|\\
&\leq -\frac{2}{\sigma_n(\Gamma)}\big(\alpha\,\sigma_n(\Gamma) - \Delta\big)\|\tilde{\theta}(t)\| - 2\, n^{\frac{1-p}{2}}\alpha^{p+1}\|\tilde{\theta}(t)\|^{p+1}
\end{aligned}$$

Given that $\sigma_n(\Gamma) \geq \Delta/\alpha + \epsilon$, we can conclude

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{2\,\alpha\,\epsilon}{\sigma_n(\Gamma)}\|\tilde{\theta}(t)\| - 2\, n^{\frac{1-p}{2}}\alpha^{p+1}\|\tilde{\theta}(t)\|^{p+1}\\
&\leq -\frac{2\,\alpha\,\epsilon}{\sigma_n^{1/2}(\Gamma)} V^{1/2}(t) - 2\, n^{\frac{1-p}{2}}\alpha^{p+1}\sigma_n^{\frac{p+1}{2}}\big(\Gamma\big)V^{\frac{p+1}{2}}(t).
\end{aligned}$$

Then, by Theorem 2.4, the fixed-time stability of $\tilde{\theta}(t)$ is asserted. Also, from this theorem the bound for the convergence times:

$$\frac{\sigma_n^{1/2}(\Gamma)}{\alpha\,\epsilon} + \frac{1}{\alpha^{p+1} n^{\frac{1-p}{2}} \sigma_n^{\frac{p+1}{2}}\big(\Gamma\big)(p-1)}.$$

□

Now, as in the previous sections, it is interesting to see what happens when there are disturbances in the model. Consider $\delta(t) \in \mathbb{R}^m$ a uniformly bounded integrable vector valued function such that $\|\delta(t)\| \leq d$. Consider now the perturbed system

$$\begin{aligned} \dot{\theta}(t) &= f(t), \\ y(t) &= C(t)\theta(t) + \delta(t), \end{aligned} \tag{4.22}$$

we want to investigate the effect of $\delta(t)$ in the observer when this functions is unknown. As expected, the presence of $\delta(t)$ makes impossible to recover $\theta(t)$ exactly. However, the error made by the observer remains bounded, in other words, the error dynamic induced by the observer is ISS w.r.t. $\delta(t)$.

**Theorem 4.8.** *Consider systems* (4.22) *and* (4.21) *with* $\|\delta(t)\| \leq d$, $\sigma_n(C(t)) \geq \alpha$, *and* $\sigma_n(\Gamma) \geq \Delta/\alpha + \epsilon$. *The error dynamics which results of applying the observer* (4.21) *under this situation is*

$$\dot{\tilde{\theta}}(t) = -\Gamma\, C^\top(t)\big\lceil C(t)\tilde{\theta}(t) - \delta(t)\big\rfloor^0 - \Gamma\, C^\top(t)\big\lceil C(t)\tilde{\theta}(t) - \delta(t)\big\rfloor^p - f(t).$$

*This dynamics is ISS w.r.t.* $\delta(t)$.

*Proof.* Consider the candidate ISS-Lyapunov function $V(\tilde{\theta}) = \tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$, which derivative along the error dynamics is

$$\begin{aligned} \dot{V}(t) &= -2\,\tilde{\theta}^\top(t)C^\top(t)\big\lceil C(t)\tilde{\theta}(t) - \delta(t)\big\rfloor^0 - 2\,\tilde{\theta}^\top(t)C^\top(t)\big\lceil C(t)\tilde{\theta}(t) - \delta(t)\big\rfloor^p - 2\,\tilde{\theta}^\top(t)\Gamma^{-1}f(t) \\ &\leq -2\sum_{i=1}^{m}\big(C(t)\tilde{\theta}(t)\big)_i\big\lceil\big(C(t)\tilde{\theta}(t)\big)_i - \delta_i(t)\big\rfloor^0 - 2\sum_{i=1}^{m}\big(C(t)\tilde{\theta}(t)\big)_i\big\lceil\big(C(t)\tilde{\theta}(t)\big)_i - \delta_i(t)\big\rfloor^p + \frac{2\,\Delta}{\sigma_n(\Gamma)}\|\tilde{\theta}(t)\|. \end{aligned}$$

By Lemma 3.2, $\dot{V}(t)$ can be bounded as

$$\begin{aligned} \dot{V}(t) &\leq -2\sum_{i=1}^{m}\big|\big(C(t)\tilde{\theta}(t)\big)_i\big| - 2\,\kappa_1\sum_{i=1}^{m}\big|\big(C(t)\tilde{\theta}(t)\big)_i\big|^{p+1} + 4\sum_{i=1}^{m}\big|\delta_i(t)\big| + 2\,\kappa_2\sum_{i=1}^{m}\big|\delta_i(t)\big|^{p+1} \\ &\quad + \frac{2\,\Delta}{\sigma_n(\Gamma)}\|\tilde{\theta}(t)\| \\ &\leq -2\,\big\|C(t)\tilde{\theta}(t)\big\| - 2\,n^{\frac{1-p}{2}}\kappa_1\big\|C(t)\tilde{\theta}(t)\big\|^{p+1} + 4\,n^{\frac{1}{2}}\big\|\delta(t)\big\| + 2\,\kappa_2\big\|\delta(t)\big\|^{p+1} + \frac{2\,\Delta}{\sigma_n(\Gamma)}\|\tilde{\theta}(t)\| \\ &\leq -\frac{2}{\sigma_n(\Gamma)}\big(\alpha\,\sigma_n(\Gamma) - \Delta\big)\big\|\tilde{\theta}(t)\big\| - 2\,n^{\frac{1-p}{2}}\alpha^{p+1}\big\|\tilde{\theta}(t)\big\|^{p+1} + 4\,n^{\frac{1}{2}}d + 2\,\kappa_2 d^{p+1} \\ &\leq -\frac{2\,\alpha\,\epsilon}{\sigma_n^{1/2}(\Gamma)}V^{1/2}(t) - 2\,n^{\frac{1-p}{2}}\alpha^{p+1}\sigma_n^{\frac{p+1}{2}}(\Gamma)V^{\frac{p+1}{2}}(t) + 4\,n^{\frac{1}{2}}d + 2\,\kappa_2 d^{p+1}. \end{aligned}$$

Consider $q \in (0, 1)$. For $V(t)$ such that

$$\frac{2\,\alpha\,\epsilon}{\sigma_n^{1/2}(\Gamma)}V^{1/2}(t) + 2\,n^{\frac{1-p}{2}}\alpha^{p+1}\sigma_n^{\frac{p+1}{2}}(\Gamma)V^{\frac{p+1}{2}}(t) \geq \frac{4\,n^{\frac{1}{2}}}{q}d + \frac{2\,\kappa_2}{q}d^{p+1}, \tag{4.23}$$

we have

$$\dot{V}(t) \leq -(1-q)\frac{2\,\alpha\,\epsilon}{\sigma_n^{1/2}(\Gamma)}V^{1/2}(t) - 2(1-q)n^{\frac{1-p}{2}}\alpha^{p+1}\sigma_n^{\frac{p+1}{2}}(\Gamma)V^{\frac{p+1}{2}}(t) < 0, \tag{4.24}$$

then, the error dynamics is ISS w.r.t. $\delta(t)$ □

As in the previous cases, we would like to investigate the behavior of the estimation error and its bounds. In the presence of $\delta(t)$ one would expect that the convergence of $\chi(t) = C(t)\tilde{\theta}(t) - \delta(t)$ to zero, in other words, that $\tilde{\theta}(t)$ tends to $C^{\dagger}(t)\delta(t)$. However, since there is not a specific model for $C(t)$ or $\delta(t)$, we cannot investigate the dynamics of $\chi(t)$. If an ultimate bound for the estimation error is wanted, one way to proceed is by finding the minimum value of $V$ (or $\|\tilde{\theta}\|$) for which (4.23) holds. That value represent a region of attraction. Given (4.24), this region is reached in fixed-time.

Before ending this section, one final remark is mandatory. Persistency of excitation is not a sufficient condition for estimating time-varying parameters (see (Juan G. Rueda-Escobedo and Jaime A. Moreno 2016, Sec. 5.2)). Even if $C(t)$ is of persistence of excitation, the may be multiple constant and time-varying parameters that explain the output $y(t)$. Then, if (4.21) is used in an attempt of estimating constant parameters, the algorithm will fail, and it might generate "equivalent" time-varying ones while making the output error $C(t)\tilde{\theta}(t) - y(t)$ zero. That is why (4.21) is not suitable for constant parameter estimation under the condition that $C(t)$ is of persistence excitation.

### 4.4.1 Numerical example

To exemplify the estimator, we choose a van der Pol oscillator as the model for the parameters:

$$\begin{aligned} \dot{\theta}_1(t) &= \theta_2(t), \ \theta_1(t_0) = 3, \\ \dot{\theta}_2(t) &= \frac{1}{2}\Big(2 - \theta_1^2(t)\Big)\theta_2(t) - \theta_1(t), \ \theta_2(t_0) = 0. \end{aligned}$$

The regressor is taken as

$$C^{\top}(t) = \begin{bmatrix} 2\cos(t) & -\cos(t+1) & 3\,cos(2\,t+1/2) & 2\cos(t/3+1) \\ \cos(2\,t) & \cos(t/2) & 2\cos(3\,t/2+3/4) & -3\cos(4\,t/3) \end{bmatrix}.$$

For this configuration we have $\Delta = 5.4$ and $\alpha = 0.4$. To specify the estimator, we set

$$\Gamma = \begin{bmatrix} 16 & 4 \\ 4 & 20 \end{bmatrix}, \quad p = 2, \quad \hat{\theta}_1(t_0) = \hat{\theta}_1(t_0) = 0.$$

The results of this simulation are presented in figures 4.13 and 4.14. In Table 4.4 the setting for the numerical

| Parameter | Value |
|---|---|
| Method | Backward differentiation formula |
| Precision goal | $10^{-3}$ |
| Accuracy goal | $10^{-3}$ |
| Max step size | 0.01 |

Table 4.4: Parameters of the numerical simulation: Time-varying parameters.

method is shown. In the Figure 4.13 the trajectory of both, the parameters and the estimates, are displayed. It can be appreciated how the estimates can track the trajectory of the actual parameters. The reach phase cannot be seen in this figure. In Figure 4.14 (a), the logarithmic plot of the error norm is shown. This plot has two objectives, it illustrates the reaching phase and also helps to confirm the finite-time convergence. In Figure 4.14 (b), the error norm it is also displayed in a logarithmic plot, but for different initial errors of magnitude $10^3$, $10^5$, and $10^7$. In this plot, it can be appreciated how the convergence time reaches a limit and at the same time, the acceleration part corresponding to the finite-time convergence can be seen. This shows how the fixed-time convergence is achieved by the estimator.

## Summary

In this chapter, the algorithm presented in Chapter 3 was specialized to solve estimation and observation problems that arise in areas such that LTI systems and adaptive control. In the cases of study, we proposed

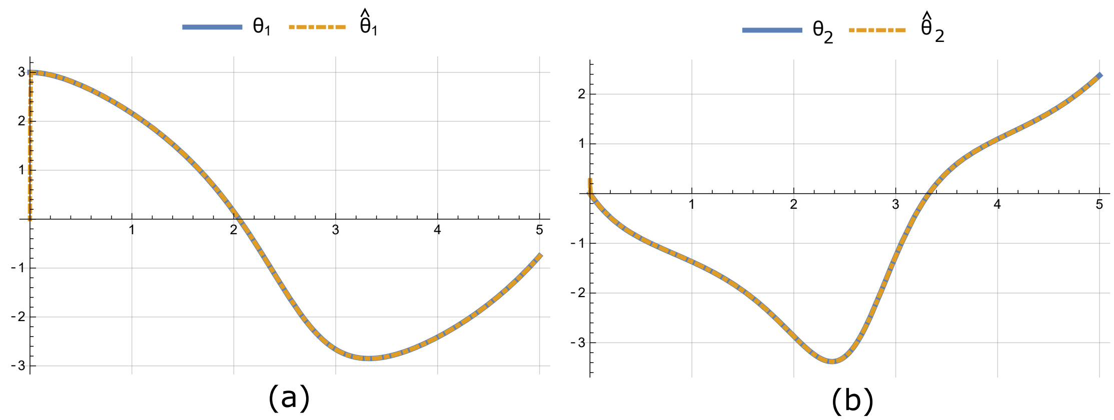

Figure 4.13: Time behaviour of the parameters and the estimates. First parameters (a). Second parameter (b).

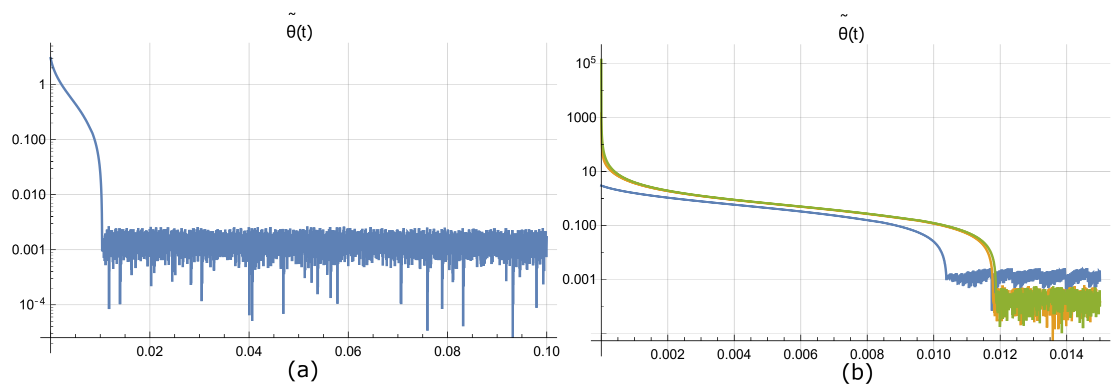

Figure 4.14: Logarithmic plot of the error norm for the initial simulation (a), and for different initial errors (b).

strategies to overcome the computation of $H(t)$, reducing the dimension of the observer. By doing this, the properties of fixed-time convergences, in the unperturbed case, and the ISS property, in the presence of disturbances, are kept. However, in the cases of constant parameter estimation and the adaptive observer, the final bound of the estimation error, when there are disturbances, does not converge to $-N^{-1}(t)\zeta(t)$ as in Theorem 3.3. This is an indicative of the "optimality" loss by neglecting the computation of $H(t)$.

# Chapter 5

# Conclusions

The purpose of this work was to find an strategy to reconstruct withing a fixed time the internal state of linear time-varying systems. The study of this class of systems took us to the idea of using the constructibility equation $\psi(t) = \mathcal{N}(t, t_0)x(t)$ as a source of valuable information to correct the estimate. Using only the instantaneous information provided by the output to achieve exact convergence of the observer proved to be a hard task, which for the moment is unsolved. On the contrary, if part of that information was stored, as it is done in $\psi(t)$, it can be used to achieve the goal. It is important to remark that for single-input single-output observable linear time-invariant systems, exact convergence can be achieved by sliding-modes differentiators without storing any information. However, such approach does not seem to work for general uniformly-observable linear time-varying systems. From the first moment it was obvious that the direct use of the constructibility gramian $\mathcal{N}(t, t_0)$ has several drawbacks given its potential unbounded growth. Then, the study was centred in obtaining something equivalent to the gramian, but with a better behaviour. This takes us to modify the gramian dynamics to incorporate stabilizing linear terms, and ended in the Riccati differential equation (RDE) that describe the dynamics of $N(t)$. The change to a RDE contributed several benefices to the observer in terms of generality, compactness, and robustness. It also pointed out to the relation with the Kalman-Bucy filter. This relationship has been used to introduce the observer and to contrast it. At the end, the general structure of the observer can be seen as an extension or a modification to the Kalman-Bucy filter. However, for particular problems and systems, the observer can be modified to avoid the terms inherited from the Kalman-Bucy filter, making it to stand for its own. Even if it is wanted to classify the observer as a variation of the Kalman-Bucy filter, it does something that for the Kalman-Bucy filter is not possible, recovering the system state in fixed-time. This property is provided by the non-linear innovation terms,what constitutes, the main contribution of this work.

The fixed-time convergence has proved to have some advantages. This property can be described as the synergy of uniformity w.r.t. the initial conditions and finite-time convergence. The finite-time convergence means, in theory, exact recovery of the system state, something that had not been achieved for general linear time-varying systems. This, however, can be debated when a real application is in the middle. In contrast, the uniformity w.r.t. the initial error is preserved in any circumstance. From both properties, the last one proved to be the more useful. The uniformity implies that a time ensuring the reliability of the estimate can be given, being this time independent from the initial estimation error. This information can be very valuable in applications that evolve fast, where quick decision making is mandatory. The fixed-time convergence makes also possible to perform some estimation task under non-classical conditions. This is the case of constant parameter estimation and the design of adaptive observers for linear time-invariant systems, where the estimation can be performed in fixed-time with a regressor that is not of persistent excitation, but carrying enough information. Under these circumstances, the convergence is non-uniform in time.

Although the idea of storing information allowed us to achieve our goals, it has a price to pay: the effect of noise and other disturbance is also stored. Since the variable recording this information has a linear dynamics, it can only “forget" the corrupted information exponentially. The only way to get rid of the incorrect information is by resetting $\psi(t)$ and $N(t)$ to zero, and start the information collection again. Whenever there is corrupted information in $\psi(t)$, the exact convergence is lost. This makes the finite-time convergence fragile because in the hypothetical case of a disturbance that disappear completely, the exact convergence is not recovered after the system returns to the nominal scenario, except if the estimation process is reinitiated after the perturbation has disappeared. In contrast, the uniformity is kept in this situation, and even in the case of a persistent disturbance, making it a robust property of the observer.

The main properties provided by the observer can be summarized in the following points:

- Under ideal conditions, the estimated state converges to the system state's exactly, in a time that is bounded by a constant which is independent of the initial guess (Theorem 3.1).
- In the presence of disturbances, the error committed by the observer remains bounded and retain a relation with the size of the disturbances (Theorem 3.2).
- The error committed by the observer when disturbances are present, is comparable to the one committed by the Kalman-Bucy filter (Theorem 3.3).

## 5.1 Ideas for the future

During the development of this work, many questions arose but could not be addressed. Some of them could lead to interesting contributions. We include here some of them:

- As shown in Section 2.3.1, the solutions of symmetric RDE for different initial conditions converge to the solution started in zero. This convergence is only asymptotic. It would be interesting to find a way to make them reach the final trajectory in finite time. This can be used to make $H^{-1}(t)$ converge to $N(t)$ exactly.
- As mentioned, the computation of $\psi(t)$ is susceptible of the disturbances. It would be very useful to make $\psi(t)$ only store the information of a finite-length time window without imposing an infinite-dimensional dynamic to it.
- In general, the presence of disturbances make impossible to recover the system state exactly. However, under particular assumptions, it is possible. For example, if a linear time-invariant systems is strongly detectable$^\star$, an unknown input observer can be designed. Other case is the design of a differentiator when assuming the boundedness of the second derivative. In such case, a sliding-mode differentiator is capable of reconstructing the state. For linear time-varying systems it is not clear what assumptions or system's properties are necessary to recover the state exactly in the presence of unknown inputs, or other kind of disturbances.
- It is unknown if it is possible to achieve finite-time convergence when observing a linear time-varying system without storing information. If possible, how to do it?

# Appendix A

# Classic inequalities

In this brief appendix we present three classic inequalities that are at the base of functional analysis. These inequalities are used repeatedly along this thesis, and they have been our main tool to prove the negative definiteness of the derivative of all the Lyapunov functions at the core of the results of this work.

**Proposition A.1** (Young's inequality). *(Beckenbach and Bellman 1961)[Eq. 1.15.2] Let $a$, $b \in \mathbb{R}_{>0}$, and $p$, $q \in \mathbb{R}_{>1}$ with $1/p + 1/q = 1$. Then*

$$a\,b \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

*Furthermore, for any $\epsilon > 0$ we have*

$$\left(\epsilon^{1/p} a\right)\left(\epsilon^{-1/p} b\right) \leq \epsilon \frac{a^p}{p} + \frac{b^q}{\epsilon^{q/p} q}.$$

**Proposition A.2** (Jensen's inequality). *(Beckenbach and Bellman 1961)[Eq. 1.16.5] Let $f(x) : [a, b] \to \mathbb{R}$ be a convex function over its interval of definition. Consider $x_i \in [a, b]$ and $\alpha_i > 0$ for $i : \{1, 2, \cdots, n\}$, with $\sum_{i=1}^{n} \alpha_1 = 1$. Then*

$$\sum_{i=1}^{n} \alpha_i f(x_i) \geq f\left(\sum_{i=1}^{n} \alpha_i x_i\right).$$

*In particular, for $f(x) = x^{p+1}$, $x \in \mathbb{R}_{\geq 0}$ and $p \geq 0$, we have*

$$\sum_{i=1}^{n} \alpha_i x_i^{p+1} \geq \left(\sum_{i=1}^{n} \alpha_i x_i\right)^{p+1}.$$

**Proposition A.3** (Equivalence between norms). *(Hardy, Littlewood, and Pólya 1952)[Eq. 2.10.3] Let $q \geq p \geq 1$. Then, for $x \in \mathbb{R}^n$, we have*

$$\|x\|_p \geq \|x\|_q,$$
$$n^{\frac{q-p}{p\,q}} \|x\|_q \geq \|x\|_p.$$

# Bibliography


Abou-Kandil, Hisham, Gerhard Freiling, Vlad Ionescu, and Gerhard Jank (2003). *Matrix Riccati Equations in Control and Systems Theory*. 1st ed. Birkhäuser Basell. ISBN: 9783764300852. DOI: 10.1007/978-3-0348-8081-7.

Anderson, B. (Feb. 1977). "Exponential stability of linear equations arising in adaptive identification". In: *IEEE Transactions on Automatic Control* 22.1, pp. 83–88. ISSN: 0018-9286. DOI: 10.1109/TAC.1977.1101406.

Anderson, B.D.O. (1971). "Stability properties of Kalman-Bucy filters". In: *Journal of the Franklin Institute* 291.2, pp. 137–144. ISSN: 0016-0032. DOI: https://doi.org/10.1016/0016-0032(71)90016-0.

Andrieu, V., L. Praly, and A. Astolfi (Dec. 2009). "Homogeneity in the bi-limit as a tool for observer and feedback design". In: *Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference*, pp. 1050–1055. DOI: 10.1109/CDC.2009.5400263.

Angulo, Marco Tulio, Jaime A. Moreno, and Leonid Fridman (2013). "Robust exact uniformly convergent arbitrary order differentiator". In: *Automatica* 49.8, pp. 2489–2495. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2013.04.034.

Beckenbach, Edwin F. and Richard Bellman (1961). *Inequalities*. Berlin-Göttingen-Heidelberg: Springer-Verlagl. ISBN: 9783642649738.

Bernstein, Dennis S. (2009). *Matrix Mathematics: Theory, Facts, and Formulas: Second Edition*. 2nd ed. Princeton University Press. ISBN: 9780691140391.

Besançon, Gildas (2007). *Nonlinear Observers and Applications*. Springer Berlin Heidelberg.

Boizot, N., E. Busvelle, J. P. Gauthier, and J. Sachau (2007). "Adaptive-gain extended Kalman Filter: Application to a series-connected DC motor". In: *Conference on System and Control (CSC'2007)*. Marrakech.

Briat, C. (2014). *Linear Parameter-Varying and Time-Delay Systems: Analysis, Observation, Filtering & Control*. Advances in Delays and Dynamics. Springer Berlin Heidelberg. ISBN: 9783662440506.

Brockett, Roger W. (1970). *Finite Dimensional Linear Systems (Decision & Control)*. Series in decision and control. New York, London, Sydney, Toronto: John Wiley and Sons. ISBN: 9780471105855.

Bucy, R.S. (1972). "The Riccati equation and its bounds". In: *Journal of Computer and System Sciences* 6.4, pp. 343–353. ISSN: 0022-0000. DOI: https://doi.org/10.1016/S0022-0000(72)80026-6.

Callier, Frank M. and Charles A. Desoer (1991). *Linear System Theory*. Springer Texts in Electrical Engineering. New York: Springer-Velarg. ISBN: 9781461269618. DOI: 10.1007/978-1-4612-0957-7.

Campbell, S. and C. Meyer (2009). *Generalized Inverses of Linear Transformations*. Society for Industrial and Applied Mathematics.

Chung, Dohyoung, Chan Gook Park, and Jang Gyu Lee (Oct. 1999). "Robustness of controllability and observability of continuous linear time-varying systems with parameter perturbations". In: *IEEE Transactions on Automatic Control* 44.10, pp. 1919–1923. ISSN: 0018-9286. DOI: 10.1109/9.793737.

Coddington, Earl A. and Norman Levinson (1984). *Theory of ordinary differential equations*. 9th reprint of the 1972 TMH edition. McGraw-Hill. ISBN: 9780070992566.

Cruz-Zavala, E., J. A. Moreno, and L. M. Fridman (Nov. 2011). "Uniform Robust Exact Differentiator". In: *IEEE Transactions on Automatic Control* 56.11, pp. 2727–2733. ISSN: 0018-9286. DOI: 10.1109/TAC.2011.2160030.

Deza, F., E. Busvelle, J.P. Gauthier, and D. Rakotopara (1992). “High gain estimation for nonlinear systems”. In: *Systems & Control Letters* 18.4, pp. 295–299. ISSN: 0167-6911. DOI: https://doi.org/10.1016/0167-6911(92)90059-2.

Edwards, H. A., Y. Lin, and Y. Wang (2000). “On input-to-state stability for time varying nonlinear systems”. In: *Proceedings of the 39th IEEE Conference on Decision and Control*. Vol. 4, 3501–3506 vol.4. DOI: 10.1109/CDC.2000.912246.

Filippov, A. F. (2013). *Differential Equations with Discontinuous Righthand Sides: Control Systems*. Mathematics and its Applications. Springer Netherlands. ISBN: 9789401577939.

Gutiérrez, D., H. Ríos, J.A. Rosales, and R. Galván-Guerra (2017). “Finite and Fixed-Time State Estimation: Pendulum-Cart System”. In: *IFAC-PapersOnLine* 50.1. 20th IFAC World Congress, pp. 7139–7144.

Haddad, W. M., S. G. Nersesov, and Liang Du (June 2008). “Finite-time stability for time-varying nonlinear dynamical systems”. In: *2008 American Control Conference*, pp. 4135–4139. DOI: 10.1109/ACC.2008.4587141.

Hardy, G. H., J. E. Littlewood, and G. Pólya (1952). *Inequalities*. The Edinburgh building, Cambridge CB2 8RU, UK: Cambrifge University Press. ISBN: 9780521052061.

Hendricks, E., O. Jannerup, and P.H. Sørensen (2008). *Linear Systems Control: Deterministic and Stochastic Methods*. Springer Berlin Heidelberg. ISBN: 9783540784869.

Ioannou, Petros A. and Jing Sun (1995). *Robust Adaptive Control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc.

Kalman, R. E. (1960). “Contributions to the Theory of Optimal Control”. In: *Boletin de la Sociedad Matemática Méxicana* 5, pp. 102–119.

Kalman, R. E. and R. S. Bucy (1961). “New results in linear filtering and prediction theory”. In: *ASME Journal of Basic Engineering* 83.1, pp. 95–108.

Kalman, R.E., P.L. Falb, and M.A. Arbib (1969). *Topics in mathematical system theory*. International series in pure and applied mathematics. New York, San Francisco, St. Louis, Toronto, London, Sydney: McGraw-Hill.

Khalil, Hassan K. (2002). *Nonlinear systems*. Upper Saddle River, NJ 07458: Prentice Hall. ISBN: 9780130673893.

Kwakernaak, H. and R. Sivan (1972). *Linear optimal control systems*. Wiley Interscience. ISBN: 9780471511106.

Lopez-Ramirez, Francisco, Andrey Polyakov, Denis Efimov, and Wilfrid Perruquetti (2018). “Finite-time and fixed-time observer design: Implicit Lyapunov function approach”. In: *Automatica* 87, pp. 52–60.

Lyapunov, A. M. (1992). “The general problem of the stability of motion”. In: *International Journal of Control* 55.3, pp. 531–534. DOI: 10.1080/00207179208934253.

Marino, R. and P. Tomei (Aug. 1992). “Global adaptive observers for nonlinear systems via filtered transformations”. In: *IEEE Trans. Autom. Control* 37.8, pp. 1239–1245.

Ménard, Tomas, Emmanuel Moulay, and Wilfrid Perruquetti (2017). “Fixed-time observer with simple gains for uncertain systems”. In: *Automatica* 81, pp. 438–446.

Narendra, K. S. and A. M. Annaswamy (1989). *Stable Adaptive Systems*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc.

Narendra, K. S. and P. Kudva (Nov. 1974). “Stable Adaptive Schemes for System Identification and Control - Part II”. In: *IEEE Transactions on Systems, Man, and Cybernetics* SMC-4.6, pp. 552–560.

Noack, M., J. G. Rueda-Escobedo, J. Reger, and J. A. Moreno (Dec. 2016). “Fixed-time parameter estimation in polynomial systems through modulating functions”. In: *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 2067–2072. DOI: 10.1109/CDC.2016.7798568.

Oliva-Fonseca, P., J. G. Rueda-Escobedo, and J. A. Moreno (Dec. 2016a). “Fixed-time adaptive observer for linear time-invariant systems”. In: *2016 IEEE 55th Conference on Decision and Control (CDC)*, pp. 1267–1272. DOI: 10.1109/CDC.2016.7798440.

– (Sept. 2016b). “Observador con convergencia en tiempo fijo para sistemas LVT”. In: *AMCA Congreso Nacional de Control Atomático*.

Polyakov, A. (Aug. 2012a). “Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems”. In: *IEEE Transactions on Automatic Control* 57.8, pp. 2106–2110.

– (Aug. 2012b). “Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems”. In: *IEEE Transactions on Automatic Control* 57.8, pp. 2106–2110. ISSN: 0018-9286. DOI: 10.1109/TAC.2011.2179869.

Polyakov, Andrey and Leonid Fridman (2014). "Stability notions and Lyapunov functions for sliding mode control systems". In: *Journal of the Franklin Institute* 351.4. Special Issue on 2010-2012 Advances in Variable Structure Systems and Sliding Mode Algorithms, pp. 1831–1865. ISSN: 0016-0032. DOI: https://doi.org/10.1016/j.jfranklin.2014.01.002.

Ríos, H., D. Efimov, J. A. Moreno, W. Perruquetti, and J. G. Rueda-Escobedo (July 2017). "Time-Varying Parameter Identification Algorithms: Finite and Fixed-Time Convergence". In: *IEEE Transactions on Automatic Control* 62.7, pp. 3671–3678. ISSN: 0018-9286. DOI: 10.1109/TAC.2017.2673413.

Ríos, H., D. Efimov, A. Polyakov, and W. Perruquetti (Dec. 2016). "Homogeneous Time-Varying Systems: Robustness Analysis". In: *IEEE Transactions on Automatic Control* 61.12, pp. 4075–4080. ISSN: 0018-9286. DOI: 10.1109/TAC.2016.2540806.

Ríos, Héctor and Andrew R. Teel (2018). "A hybrid fixed-time observer for state estimation of linear systems". In: *Automatica* 87, pp. 103–112.

Rueda-Escobedo, J. G. and J. A. Moreno (Dec. 2015). "Non-linear gradient algorithm for parameter estimation". In: *2015 54th IEEE Conference on Decision and Control (CDC)*, pp. 4086–4091. DOI: 10.1109/CDC.2015.7402855.

– (Sept. 2018). "Delayed Kalman-Bucy observer for a class of LTV systems with delayed measurements". In: *9th IFAC Symposium on Robust Control Design (ROCOND)*, To appear.

Rueda-Escobedo, J. G., J. A. Moreno, and P. Oliva-Fonseca (June 2016a). "Finite-time state estimation for LTI systems with a First-Order Sliding Mode". In: *2016 14th International Workshop on Variable Structure Systems (VSS)*, pp. 194–199. DOI: 10.1109/VSS.2016.7506915.

– (Oct. 2016b). "Fixed-time Convergent Unknown Input Observer for LTI Systems". In: *XVII Latin American Conference in 2016 Automatic Control*, pp. 354–359.

Rueda-Escobedo, J. G., Rosane Ushirobira, Denis Efimov, and J. A. Moreno (June 2018a). "A Gramian-based observer with uniform convergence rate for delayed measurements". In: *2018 European Control Conference (ECC)*, To appear.

– (2018b). "Gramian-based uniform convergent observer for stable LTV systems with delayed measurements". In: *International Journal of Control*, Accepted.

Rueda-Escobedo, Juan G. and Jaime A. Moreno (2016). "Discontinuous gradient algorithm for finite-time estimation of time-varying parameters". In: *International Journal of Control* 89.9, pp. 1838–1848. DOI: 10.1080/00207179.2016.1159338.

– (2017). "Fixed-Time convergent Adaptive Observer for LTI Systems". In: *IFAC-PapersOnLine* 50.1. 20th IFAC World Congress, pp. 11639–11644. ISSN: 2405-8963. DOI: https://doi.org/10.1016/j.ifacol.2017.08.1664.

Sánchez-Torres, J. D. and A. G. Loukianov (June 2014). "A fixed-time second order sliding mode observer for a class of nonlinear systems". In: *2014 13th International Workshop on Variable Structure Systems (VSS)*, pp. 1–4.

Sánchez-Torres, J. Diego, J. A. Isaza, O. Jaramillo, E. Jiménez, and H. Botero (Oct. 2015). "A fixed-time convergent observer for biomass in a batch process". In: *2015 IEEE 2nd Colombian Conference on Automatic Control (CCAC)*, pp. 1–5.

Sastry, S. and C. Desoer (Aug. 1982). "The robustness of controllability and observability of linear time-varying systems". In: *IEEE Transactions on Automatic Control* 27.4, pp. 933–939. ISSN: 0018-9286. DOI: 10.1109/TAC.1982.1103026.

Shtessel, Y., C. Edwards, L. Fridman, and A. Levant (2013). *Sliding Mode Control and Observation*. Control Engineering. Springer New York. ISBN: 9780817648930.

Slotine, J.J.E. and W. Li (1991). *Applied Nonlinear Control*. Prentice-Hall. ISBN: 9780130400499.

Tomas-Rodriguez, M. and S.P. Banks (2010). *Linear, Time-varying Approximations to Nonlinear Dynamical Systems: with Applications in Control and Optimization*. Lecture Notes in Control and Information Sciences. Springer London. ISBN: 9781849961004.

Weiss, Leonard and R.E. Kalman (1965). "Contributions to linear system theory". In: *International Journal of Engineering Science* 3.2, pp. 141–171. ISSN: 0020-7225. DOI: 10.1016/0020-7225(65)90042-X.

Willems, J.C. (2004). "Deterministic least squares filtering". In: *Journal of Econometrics* 118.1. Contributions to econometrics, time series analysis, and systems identification: a Festschrift in honor of Manfred Deistler, pp. 341–373. ISSN: 0304-4076. DOI: 10.1016/S0304-4076(03)00146-5.