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el contorno S se puede probar que tal mínimo existe y que es alcanzado con una función armónica, de modo que la función deseada u es una solución del problema de Dirichlet $\Delta u = 0$, $u|_S = \phi$. El recíproco también es cierto: la solución del problema de Dirichlet da un mínimo para la integral J con respecto a todas las v que verifiquen la condición de contorno.

La demostración de la existencia de la función u, para la cual J alcanza su mínimo, y su cálculo con cualquier grado de exactitud puede hacerse, por ejemplo, de la siguiente manera (método de Ritz). Tomemos una familia infinita de funciones dos veces continuamente diferenciables $\{v_n(x, y)\}$, $n = 0, 1, 2, \cdots$, iguales a cero sobre el contorno para $n > 0$, e iguales a ϕ para $n = 0$. Consideremos J para funciones de la forma

$$v = \sum_{k=1}^{n} C_k v_k + v_0,$$

donde n es fijo y los C_k son números arbitrarios. Entonces $J(v)$ será un polinomio de segundo grado en las n variables independientes $C_1, C_2, \cdots, C_n$. Determinamos los C_k a partir de la condición de que este polinomio se haga mínimo. Esto conduce a un sistema de n ecuaciones algebraicas lineales con n incógnitas, cuyo determinante es distinto de cero. De este modo los números C_k están unívocamente determinados. Denotemos la correspondiente v por $v^n(x, y)$. Se puede probar que, si el sistema $\{v_n\}$ verifica una cierta condición de «completitud», las funciones v^n convergerán, cuando $n \to \infty$, a una función que será la solución deseada del problema.

Observemos finalmente que en este capítulo sólo hemos dado una descripción de los problemas lineales más sencillos de la mecánica y hemos ignorado muchas otras cuestiones (lejos aún de estar completamente resueltas) relacionadas con ecuaciones en derivadas parciales más generales.

Capítulo 7

CURVAS Y SUPERFICIES

§1. *Problemas y métodos en la teoría de curvas y superficies*

En los cursos de enseñanza escolar, la geometría sólo abarca las curvas más sencillas: líneas rectas, líneas quebradas, circunferencias y arcos de círculo; y en cuanto a superficies, únicamente planos, superficies de poliedros, esferas, conos y cilindros. En cursos más extensos se consideran otras curvas, principalmente las secciones cónicas: elipses, parábolas e hipérbolas. Pero el estudio de una curva o superficie arbitraria es completamente ajeno a la geometría elemental. A primera vista no está ni siquiera claro cuáles podrían ser las propiedades generales que hay que someter a investigación cuando hablamos de curvas y superficies arbitrarias. No obstante, tal investigación es completamente natural y necesaria.

En toda clase de actividad práctica y experiencia de la naturaleza encontramos constantemente curvas y superficies de muy diferentes formas. La trayectoria de un planeta en el espacio, de un barco en el mar, de un proyectil en el aire; la huella de un cincel sobre el metal, de una rueda sobre la carretera, de una pluma sobre una cinta; la forma de un árbol de levas que gobierna las válvulas de un motor; el perfil de un dibujo artístico; la forma de una cuerda colgante, sea idénticamente satisfecha por toda función continua $\Phi(x, t)$ igual a cero para $x = 0$, $x = l$ y $t = T$. Aquí, ambas funciones u y Φ deben tener derivadas primeras cuyos cuadrados sean integrables en el sentido de Lebesgue sobre el rectángulo $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$. Esta última exigencia para u significa que el valor medio con respecto al tiempo de la energía total de la cuerda

$$\frac{1}{2T}\int_0^T\int_0^l (\varrho u_t^2 + T u_x^2)\, dx\, dt$$

debe ser finito. Tal restricción sobre la función u, y también sobre sus posibles variaciones Φ, es un resultado natural del principio de Hamilton.

La identidad (29) es precisamente la condición de que la primera variación del funcional

$$\tilde{S} = \int_0^T\int_0^l \left(\frac{\varrho}{2} u_t^2 - \frac{T}{2} u_x^2\right) dx\, dt + \int_0^l \phi_1 u \,|_{t=0}\, dx$$

sea igual a cero. Así, pues, el problema de la vibración de una cuerda fija en el caso considerado se puede reducir al de hallar el mínimo del funcional $\tilde{S}$ para todas las funciones $v(x, t)$ que son continuas, verifican la condición (27) y son iguales a $u(x, T)$ para $t = T$. Además, la función deseada debe verificar la primera de las condiciones (26).

Esta modificación del principio de Hamilton nos permite, no sólo ampliar la clase de soluciones admisibles de la ecuación (24), sino también enunciar un problema de contorno bien definido para ella.

El hecho de que estas soluciones generalizadas o alguna de sus derivadas no estén definidas en todos los puntos del espacio no conduce a ninguna contradicción con la experiencia, como señaló repetidamente N. M. Gjunter, cuyas investigaciones contribuyeron fundamentalmente al establecimiento de una nueva visión del concepto de solución de una ecuación de la física matemática.

Por ejemplo, si queremos determinar el flujo de un líquido en un conducto, en la presentación clásica debemos calcular el vector velocidad y la presión en todo punto de la corriente. Sin embargo, en la práctica nunca se plantea el calcular la presión en un punto, sino más bien la presión sobre una cierta área; ni tampoco interesa el vector velocidad en un punto dado, sino la cantidad de líquido que pasa a través de cierta área en la unidad de tiempo. Así, la definición

de solución generalizada propone en esencia el cálculo de sólo aquellas cantidades que tienen un significado físico inmediato.

Con el fin de poder resolver un mayor número de problemas, debemos buscar las soluciones en la clase más amplia posible de funciones para la que los teoremas de unicidad aún se verifican. Con frecuencia tal clase viene determinada por la naturaleza física del problema. De este modo, en mecánica cuántica no es la función de estado $\psi(x)$, definida como una solución de la ecuación de Schrödinger, la que tiene significado físico, sino más bien la integral $a_v = \int_E \psi(x)\,\psi_v(x)\, dx$, donde las ψ_v son ciertas funciones para las que $\int_E \psi_v^2\, dx < \infty$. Así, pues, la solución ψ debe ser buscada, no entre las funciones dos veces continuamente diferenciables, sino entre las de cuadrado integrable. En los problemas de electrodinámica cuántica no está aún establecido entre qué clases de funciones debemos buscar soluciones para las ecuaciones consideradas en esa teoría.

Los progresos de la física matemática durante los últimos treinta años tienen mucho que ver con esta nueva formulación de los problemas y con la creación del aparato matemático necesario para su resolución. Una de las facetas centrales de este aparato es el llamado teorema de inclusión de S. L. Sobolev.

Métodos particularmente convenientes de cálculo de soluciones en una u otra de estas clases de funciones son: el método de diferencias finitas, los métodos directos del cálculo de variaciones (método de Ritz y método de Trefftz), el de Galerkin, y los métodos de operadores funcionales. Estos últimos dependen básicamente de un estudio de transformaciones originadas por estos problemas. En §5 hemos hablado ya del método de diferencias finitas y del de Galerkin. Aquí expondremos las ideas fundamentales de los métodos directos del cálculo de variaciones.

Consideremos el problema de definir la posición de una membrana uniformemente estirada y de contorno fijo. Por el principio de energía potencial mínima en una posición de equilibrio estable, la función $u(x, y)$ debe dar el menor valor de la integral

$$J(u) = \iint_G (u_x^2 + u_y^2)\, dx\, dy$$

en comparación con todas las demás funciones continuamente diferenciables $v(x, y)$ que verifican la misma condición sobre el contorno, $v_{|S} = \phi$, que la función u. Con algunas restricciones sobre ϕ y sobre propiedades de diferenciabilidad de los coeficientes de las ecuaciones. Es cierto que a primera vista este principio parece ser demasiado formal y tener un carácter puramente matemático, que no indica directamente cómo deben ser formulados los problemas de una forma análoga a los problemas clásicos.

Daremos aquí un método modificado que nos parece más apropiado físicamente, ya que está conectado directamente con el bien conocido principio de Hamilton.

Como ya es sabido, el análisis de los métodos de deducción de ecuaciones diversas de la física matemática llevó, en la primera mitad del siglo XIX, al descubrimiento de una nueva ley, conocida como principio de Hamilton. Partiendo de este principio, fue posible obtener de una manera uniforme todas las ecuaciones conocidas de la física matemática. Ilustraremos esto con el ejemplo del problema considerado en §3 para las oscilaciones de una cuerda de longitud finita con extremos fijos.

En primer lugar construyamos la llamada función de Lagrange (o lagrangiana) $L(t)$ para nuestra cuerda, la diferencia entre la energía cinética y la potencial. De lo dicho en §3 se sigue que

$$L(t) = \int_0^l \left(\frac{1}{2}\varrho u_t^2 - \frac{T}{2}u_x^2\right) dx.$$

De acuerdo con el principio de Hamilton, la integral

$$S = \int_{t_1}^{t_2} L(t)\, dt$$

toma su valor mínimo para la función $u(x, t)$ que corresponde al movimiento real de la cuerda, en contraposición con cualquier otra función $v(x, t)$ que sea igual a cero para $x = 0$ y $x = l$ y coincida con $u(x, t_1)$ y $u(x, t_2)$ para $t = t_1$ y $t = t_2$. Aquí t_1 y t_2 están fijados arbitrariamente, y las funciones v deben tener integrales finitas S. Como resultado de este principio, la llamada primera variación de S (cf. capítulo 8) debe ser igual a cero:

$$\delta S = \int_{t_1}^{t_2}\int_0^l (\varrho u_t \Phi_t - T u_x \Phi_x)\, dx\, dt = 0, \tag{25}$$

donde $\Phi(x, t)$ es una función arbitraria diferenciable con respecto a x y t e igual a cero sobre los bordes del rectángulo $0 \leq x \leq l$, $t_1 \leq t \leq t_2$.

La ecuación (25) es también la condición que debe cumplir la función deseada $u(x, t)$. Si sabemos que $u(x, t)$ tiene derivadas de segundo orden, entonces la condición (25) puede escribirse en una forma diferente. Integrando (25) por partes y aplicando el lema fundamental del cálculo de variaciones, hallamos que $u(x, t)$ debe verificar la ecuación

$$\frac{\partial}{\partial t}\left(\varrho \frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(T \frac{\partial u}{\partial x}\right) = 0, \tag{26}$$

que es idéntica a (24), si ϱ y T son constantes y $T/\varrho = a^2$.

No es difícil ver que cualquier solución $u(x, t)$ de la ecuación (26) verifica la identidad (25) para toda Φ dada. El recíproco puede ser falso, ya que $u(x, t)$ puede no tener segundas derivadas en general. Así, pues, trabajando con la ecuación (25) en lugar de la (26), lo que hacemos es ampliar la gama de problemas resolubles.

Para determinar una oscilación específica de la cuerda, debemos añadir a las condiciones de contorno

$$u(0, t) = u(l, t) = 0 \tag{27}$$

las condiciones iniciales

$$u(x, 0) = \phi_0(x),$$
$$u_t(x, 0) = \phi_1(x). \tag{28}$$

Si la solución se busca dentro de la clase de funciones continuamente diferenciables, entonces las condiciones (27 y (28) pueden desligarse de (25) como requisitos a verificar por separado. Pero si aceptamos que la solución propuesta sea «peor», entonces estas condiciones pierden su significado en la forma dada y deben ser parcial o totalmente incluidas en la identidad integral (25).

Por ejemplo, sea $u(x, t)$ continua para $0 \leq x \leq l$, $0 \leq t \leq T$, pero tal que sus derivadas primeras tengan discontinuidades. La segunda de las ecuaciones (28) pierde entonces su significado como condición limitadora. En este caso el problema puede ser enunciado como sigue: hallar una función continua u que cumpla la condición (27) y la primera de las (28) y para la cual la ecuación

$$\int_0^T\int_0^l (\varrho u_t \Phi_t - T u_x \Phi_x)\, dx\, dt + \int_0^l \phi_1 \Phi(x, 0)\, dx = 0 \tag{29}$$

3. Dando un golpe brusco a cualquier segmento pequeño de la cuerda, las oscilaciones que resultan están descritas por la ecuación

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t),$$

donde $f(x, t)$ corresponde al efecto producido y es una función discontinua, distinta de cero solamente sobre el pequeño segmento de cuerda y durante un corto intervalo de tiempo. Tal ecuación (como puede probarse con facilidad) tampoco puede tener soluciones clásicas.

Estos ejemplos muestran que exigiendo la continuidad de las derivadas para la solución deseada se restringe fuertemente la clase de problemas que podemos resolver. La búsqueda de una clase más amplia de problemas resolubles discurrió en primer lugar en el sentido de permitir discontinuidades de primera especie en las derivadas de orden máximo correspondientes a las funciones que sirven como soluciones del problema, funciones que deben verificar las ecuaciones excepto en los puntos de discontinuidad. Ocurre que las soluciones de una ecuación del tipo $\Delta u = 0$ o $\partial u/\partial t - \Delta u = 0$ no pueden tener tales discontinuidades (también llamadas débiles) en el interior del dominio de definición. Las soluciones de la ecuación de ondas no pueden tener discontinuidades débiles en las variables espaciales x, y, z y en t más que sobre superficies de una forma especial, llamadas superficies características. Si una solución $u(x, y, z, t)$ de la ecuación de ondas se considera como una función que define, para $t = t_1$, un campo escalar en el espacio (x, y, z) en el instante t_1, entonces las superficies de discontinuidad para las derivadas segundas de $u(x, y, z, t)$ se moverán a través del espacio (x, y, z) con una velocidad igual a la raíz cuadrada del coeficiente de la laplaciana en la ecuación de ondas.

El segundo ejemplo referente a la cuerda muestra que también es necesario considerar soluciones en las cuales puedan existir primeras derivadas discontinuas; y en el caso de ondas sonoras y luminosas, debemos considerar incluso soluciones que sean ellas mismas discontinuas.

La primera cuestión que surge al investigar la introducción de soluciones discontinuas consiste en aclarar exactamente qué funciones discontinuas se pueden considerar como soluciones físicamente admisibles de una ecuación o del correspondiente problema físico. Podríamos, por ejemplo, suponer que una función arbitraria constante a trozos es una «solución simple» de la ecuación de Laplace o de la ecuación

de ondas, ya que verifica la ecuación fuera de las líneas de discontinuidad.

Para aclarar esta cuestión, lo primero que se debe garantizar es que en esa clase más amplia de funciones a la cual deben pertenecer las soluciones admisibles exista un teorema de unicidad. Es evidente que si, por ejemplo, admitimos funciones arbitrarias regulares a trozos, entonces esta exigencia no se satisfará.

Históricamente, el primer principio de selección de funciones admisibles fue que éstas debían ser el límite (en un sentido o en otro) de soluciones clásicas de la misma ecuación. Así, pues, en el ejemplo 2 una solución de la ecuación (24) correspondiente a la función $\phi(x)$, que no tiene derivada en un punto angular, puede hallarse como el límite uniforme de soluciones clásicas $u_n(x, t)$ de la misma ecuación correspondientes a las condiciones iniciales $u_n|_{t=0} = \phi_n(x)$, $u_{n_t}|_{t=0} = 0$, donde las $\phi_n(x)$ son funciones dos veces continuamente diferenciables que convergen uniformemente a $\phi(x)$ para $n \to \infty$.

En lo que sigue, en vez de este principio adoptaremos el siguiente: una función admisible u debe verificar, en vez de la ecuación $Lu = f$, una identidad integral que contenga una función arbitraria Φ.

Esta identidad se halla como sigue: multiplicamos ambos miembros de la ecuación $Lu = f$ por una función arbitraria Φ que tenga derivadas continuas —con respecto a todos sus argumentos— hasta el orden de la ecuación y que sea nula fuera del dominio finito D en el que está definida la ecuación. La ecuación así obtenida se integra sobre D y se transforma, integrando por partes, hasta que no contenga derivadas de u. Como resultado obtenemos la identidad deseada. Para la ecuación (24), por ejemplo, obtenemos la forma

$$\int\int_D u \left[\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 (a^2 \Phi)}{\partial x^2} \right] dx\, dt = 0.$$

S. L. Sobolev ha demostrado que para ecuaciones con coeficientes constantes estos dos principios de selección de soluciones admisibles (o, como se denominan normalmente ahora, generalizadas), son equivalentes. Pero, para ecuaciones con coeficientes variables, el primer principio puede resultar inaplicable, ya que estas ecuaciones pueden no tener en general soluciones clásicas (cf. ejemplo 1). El segundo de estos principios proporciona la posibilidad de seleccionar soluciones generalizadas con hipótesis muy amplias sobre las aumentada debido a una más fina división del intervalo de las x, los valores de u oscilan muy rápidamente de positivos a negativos y alcanzan cotas mucho mayores que las impuestas inicialmente. Está claro que en esta tabla los valores se hallan extraordinariamente lejos de la solución verdadera.

De estos ejemplos salta a la vista que si queremos utilizar el método de redes para obtener resultados dignos de confianza y suficientemente exactos, debemos proceder con gran prudencia en la elección de intervalos en la red e investigar previamente la aplicabilidad del método.

Las soluciones que se obtienen utilizando las ecuaciones de física matemática para estos u otros problemas de las ciencias nos dan una descripción matemática del curso o carácter esperado de los sucesos físicos descritos por estas ecuaciones.

Puesto que la construcción de un modelo se lleva a cabo mediante las ecuaciones de la física matemática, nos vemos forzados a ignorar, en nuestras abstracciones, muchos aspectos de estos acontecimientos, a desechar algunos como no esenciales y a seleccionar otros como básicos, de lo que se deduce que los resultados obtenidos no son absolutamente verdaderos. Lo son para ese esquema o modelo que hemos considerado, pero deben comprobarse siempre con experimentos para asegurarnos de que nuestro modelo del fenómeno es fiel al fenómeno real y de que lo refleja con un grado suficiente de exactitud.

El criterio último de la veracidad de los resultados es, por tanto, la experiencia práctica solamente. En último término no hay más que un criterio: la experiencia práctica, aunque ésta solamente puede ser propiamente comprendida a la luz de una profunda y bien desarrollada teoría.

En el caso de la cuerda vibrante de un instrumento musical, sólo podremos entender cómo produce sus tonos si estamos familiarizados con las leyes de superposición de oscilaciones características. Las relaciones que existen entre las frecuencias sólo pueden comprenderse si estudiamos cómo vienen determinadas aquéllas por el material, por la tensión de la cuerda y por la manera de fijar los extremos. En este caso la teoría no se limita a proporcionar un método de calcular ciertas cantidades numéricas deseadas, sino que también indica precisamente cuáles de estas cantidades son de importancia fundamental, cómo tiene lugar el proceso físico exactamente y qué sería observado en él.

De esta forma, el dominio de la ciencia denominado física matemática nació de las exigencias de la práctica, pero ejerció a su vez también cierta influencia sobre esa práctica y señaló caminos para ulteriores progresos.

La física matemática está muy estrechamente conectada con otras ramas del análisis matemático. No podemos, sin embargo, discutir estas conexiones aquí, ya que nos llevaría demasiado lejos.

§6. *Soluciones generalizadas*

La gama de problemas en los cuales un proceso físico viene descrito mediante funciones continuas y diferenciables puede ser ampliada de manera esencial introduciendo en la discusión soluciones discontinuas de estas ecuaciones.

En muchos casos está claro desde el principio que el problema en consideración no puede tener soluciones que sean dos veces continuamente diferenciables; en otras palabras, desde el punto de vista del enunciado clásico del problema dado en la sección anterior, tal problema no tiene solución. No obstante, el correspondiente proceso físico sí que ocurre, si bien en la indicada clase de funciones dos veces diferenciables no existen funciones que lo describan. Consideremos algunos ejemplos sencillos.

1. Si una cuerda consiste en dos trozos de diferente densidad, entonces en la ecuación

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (24)$$

el coeficiente será igual a una constante diferente en cada uno de los correspondientes trozos y, por tanto, la ecuación (24) no tendrá, en general, soluciones clásicas (dos veces continuamente diferenciables).

2. Supongamos que el coeficiente a es constante, pero que en la posición inicial la cuerda tiene la forma de una línea quebrada dada por la ecuación $u|_{t=0} = \phi(x)$. En el vértice de la línea quebrada, la función $\phi(x)$ evidentemente no puede tener derivada primera. Se puede probar que no existe ninguna solución clásica de la ecuación (24) que verifique las condiciones iniciales

$$u|_{t=0} = \phi(x), \quad u_t|_{t=0} = 0$$

(aquí y en lo que sigue, u_t denota $\partial u/\partial t$).

esencial elegir las redes convenientemente. Para estas ecuaciones podemos obtener tanto buenos como malos resultados.

Si vamos a resolver alguna de estas ecuaciones por el método de redes, después de elegir una red para los valores de t no debemos tomar una demasiado fina para las variables espaciales, pues de otra manera obtendremos un sistema de ecuaciones sumamente insatisfactorio para los valores de la función incógnita; su resolución da un resultado que oscila rápidamente con amplitudes muy grandes y que está, por tanto, muy lejos del verdadero.

La gran variedad de resultados posibles se puede ver mejor con un sencillo ejemplo numérico. Consideremos la ecuación:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

para el flujo de calor en el caso en que la temperatura no dependa de y ni de z. Tomemos k como anchura de la malla de la red según los valores de t, y h como anchura según los valores de x:

$$\frac{\partial u}{\partial t} \approx \frac{u(t+k, x) - u(t, x)}{k},$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(t, x+h) - 2u(t, x) + u(t, x-h)}{h^2}.$$

Entonces nuestra ecuación puede ser escrita aproximadamente en la forma

$$u(t+k, x) = \frac{k}{h^2}\, u(t, x+h) + \left(1 - 2\,\frac{k}{h^2}\right) u(t, x) + \frac{k}{h^2}\, u(t, x-h).$$

Si para un valor de t correspondiente a un punto de la malla conocemos los valores de u en los puntos $x - h$, x y $x + h$, es fácil hallar el valor de u en el punto x y en el siguiente punto de la malla $t + k$. Supongamos que la constante k, es decir, la anchura de la malla con respecto a t, está ya elegida. Consideremos dos casos para la elección de h. Pongamos $h^2 = k$ en el primer caso y $h^2 = 2k$ en el segundo, y resolvamos el siguiente problema por el método de redes.

En el instante inicial, $u = 0$ para todos los valores negativos de x, y $u = 1$ para todos los valores no negativos de x. Escribiendo en una línea los valores de la función incógnita u para el instante dado, tendremos dos tablas:

TABLA 1

t \ x	$-5h$	$-4h$	$-3h$	$-2h$	$-h$	0	h	$2h$	$3h$	$4h$	$5h$
0	0	0	0	0	0	1	1	1	1	1	1
k	0	0	0	0	1	0	1	1	1	1	1
$2k$	0	0	0	1	−1	2	0	1	1	1	1
$3k$	0	0	1	−2	4	−3	3	0	1	1	1
$4k$	0	1	−3	7	−9	10	−6	4	0	1	1
$5k$	1	−4	11	−19	26	−25	20	−10	5	0	1

TABLA 2

t \ x	$-5h$	$-4h$	$-3h$	$-2h$	$-h$	0	h	$2h$	$3h$	$4h$	$5h$
0	0	0	0	0	0	1	1	1	1	1	1
k	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	1	1
$2k$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	1	1	1	1
$3k$	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{8}$	$\frac{7}{8}$	1	1	1
$4k$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	$\frac{15}{16}$	1	1
$5k$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{13}{16}$	$\frac{13}{16}$	$\frac{31}{32}$	$\frac{31}{32}$	1

En la tabla 2 obtenemos valores que, para un instante dado, varían suavemente de un punto a otro. Esta tabla da una buena aproximación a la solución de la ecuación de flujo de calor. Por otra parte, en la tabla 1, en la cual, aparentemente, la exactitud habría sido puede ser representada aproximadamente por la fórmula

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{1}{\Delta x\,\Delta y}\,[u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y) - u(x, y+\Delta y) + u(x, y)].$$

Volvamos ahora a nuestra ecuación en derivadas parciales.

Para precisar, supongamos que se trata de la ecuación de Laplace con dos variables independientes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Además, supongamos que nos dan la función incógnita u sobre la frontera S del dominio Ω. Como aproximación supongamos que

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\Delta x, y) - 2u(x, y) + u(x-\Delta x, y)}{(\Delta x)^2},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x, y+\Delta y) - 2u(x, y) + u(x, y-\Delta y)}{(\Delta y)^2}.$$

Si ponemos $\Delta x = \Delta y = h$, entonces

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{h^2}\,[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y)].$$

Recubramos ahora el dominio Ω mediante una red cuadriculada con vértices en los puntos $x = kh$, $y = bh$ (Fig. 4) y reemplacemos el dominio Ω por el polígono formado por aquellos cuadrados que caen en el interior de Ω, de modo que el contorno del dominio quede sustituido por una línea quebrada. Tomemos como valores de la función incógnita sobre esta línea quebrada los valores correspondientes al contorno de S. La ecuación de Laplace está aproximada entonces por la ecuación

$$u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y) = 0$$

para todos los puntos interiores al dominio. Esta ecuación se puede volver a escribir en la forma

$$u(x, y) = \frac{1}{4}\,[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h)].$$

Entonces el valor de u en un punto de la red, por ejemplo el punto 1 de la figura 4, es igual a la media aritmética de sus valores en los cuatro puntos adyacentes.

Supongamos que en el interior del polígono hay N puntos de nuestra red. En cada uno de tales puntos tendremos una ecuación. De esta manera obtenemos un sistema de N ecuaciones algebraicas con N incógnitas, cuya solución nos da los valores aproximados de la función u sobre el dominio Ω.

Puede demostrarse que para la ecuación de Laplace la solución se puede hallar con cualquier grado de exactitud.

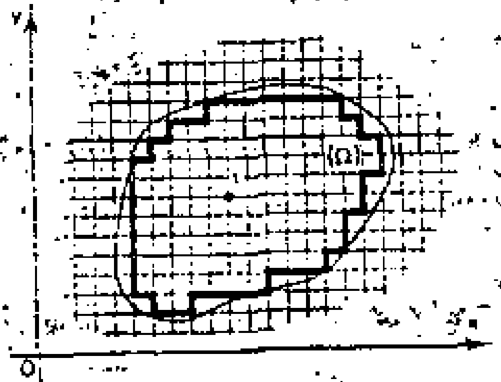

Fig. 4

El método de diferencias finitas reduce el problema a la resolución de un sistema de N ecuaciones con N incógnitas, donde éstas son los valores de la función deseada en los nudos de una red.

Además, puede demostrarse que el método de diferencias finitas es aplicable a otros problemas de física matemática: otras ecuaciones diferenciales e integrales. Sin embargo, su aplicación en muchos casos implica un gran número de dificultades.

Puede resultar que la solución del sistema de N ecuaciones algebraicas con N incógnitas, construido por el método de redes, o no exista en general o dé un resultado que sea totalmente diferente del verdadero. Esto sucede cuando la resolución del sistema de ecuaciones conduce a acumulación de errores; cuanto más pequeña sea la longitud de los lados de los cuadrados de la red, mayor será el número de ecuaciones obtenidas, de modo que el error acumulado puede hacerse más grande.

En el ejemplo dado anteriormente de la ecuación de Laplace, esto no ocurre. Los errores al resolver el sistema no se acumulan, sino que, por el contrario, disminuyen sensiblemente si resolvemos el sistema, por ejemplo, por un método de aproximaciones sucesivas. Para la ecuación del flujo de calor y para la ecuación de ondas es

2. El último de los métodos que examinaremos se denomina método de diferencias finitas o método de redes.

La derivada de la función u con respecto a la variable x se define como el límite del cociente

$$\frac{u(x+\Delta x)-u(x)}{\Delta x}.$$

Este cociente se puede representar a su vez en la forma

$$\frac{1}{\Delta x}\int_x^{x+\Delta x}\frac{\partial u}{\partial x_1}\,dx_1,$$

y utilizando el bien conocido teorema del valor medio (capítulo 2, §8), tenemos:

$$\frac{u(x+\Delta x)-u(x)}{\Delta x}=\left.\frac{\partial u}{\partial x}\right|_{x=\xi},$$

donde ξ es un punto del intervalo

$$x<\xi<x+\Delta x.$$

Todas las derivadas segundas de u —tanto las derivadas mixtas como las derivadas con respecto a una variable— se pueden representar aproximadamente en forma de un cociente de diferencias. Así, pues, el cociente de diferencias

$$\frac{u(x+\Delta x)-2u(x)+u(x-\Delta x)}{(\Delta x)^2}$$

viene representado en la forma

$$\frac{1}{\Delta x}\left[\frac{u(x+\Delta x)-u(x)}{\Delta x}-\frac{u(x)-u(x-\Delta x)}{\Delta x}\right]=$$

$$=\frac{1}{\Delta x}\left\{\left[\frac{u(x_1+\Delta x)-u(x_1)}{\Delta x}\right]\Bigg|_{x_1=x-\Delta x}^{x_1=x}\right\}.$$

Por el teorema del valor medio, el cociente de diferencias de la función

$$\phi(x_1)=\frac{u(x_1+\Delta x)-u(x_1)}{\Delta x}$$

puede reemplazarse por el valor de la derivada. En consecuencia,

$$\frac{\phi(x_1)-\phi(x_1-\Delta x)}{\Delta x}=\phi'(\xi),$$

donde ξ es un valor intermedio en el intervalo

$$x-\Delta x<\xi<x.$$

Por tanto,

$$\left(\frac{1}{\Delta x}\right)^2[u(x+\Delta x)-2u(x)+u(x-\Delta x)]=$$

$$=\frac{1}{\Delta x}[\phi(x)-\phi(x-\Delta x)]=\phi'(\xi).$$

Por otra parte

$$\phi(\xi)=\frac{u(\xi+\Delta x)-u(\xi)}{\Delta x},$$

lo cual significa que

$$\phi'(\xi)=\frac{u'(\xi+\Delta x)-u'(\xi)}{\Delta x}.$$

Usando una vez más la fórmula de los incrementos finitos, vemos que

$$\phi'(\xi)=u''(\eta),$$

donde

$$\xi<\eta<\xi+\Delta x.$$

En consecuencia,

$$\left(\frac{1}{\Delta x}\right)^2[u(x+\Delta x)-2u(x)+u(x-\Delta x)]=u''(\eta),$$

donde $x-\Delta x<\eta<x+\Delta x$.

Si la derivada $u''(x)$ es continua y el valor de Δx es suficientemente pequeño, entonces $u''(\eta)$ apenas diferirá de $u''(x)$. Por tanto nuestra segunda derivada es arbitrariamente próxima al cociente de diferencias en cuestión. Exactamente de la misma forma podemos probar, por ejemplo, que la derivada segunda mixta

$$\frac{\partial^2 u}{\partial x\,\partial y}$$

conveniente, la densidad, es decir, el valor de una función arbitraria que aparezca en él, se define de tal forma que todas las condiciones impuestas se verifiquen.

Desde un punto de vista físico, esto significa que toda función armónica puede ser representada como el potencial de una doble cara eléctrica, si distribuimos esta cara sobre una superficie S con una densidad apropiada.

Construcción aproximada de soluciones: método de Galerkin y método de redes

1. Hemos discutido dos métodos para resolver ecuaciones de física matemática: el de separación completa de variables y el de potenciales. Estos métodos fueron desarrollados por los científicos de los siglos XVIII y XIX, Fourier, Poisson, Ostrogradski, Liapunov y otros. En el siglo XX se han añadido otros muchos métodos. Examinaremos dos de ellos, el de Galerkin y el de diferencias finitas, o método de redes.

El primero fue propuesto por el académico B. G. Galerkin para resolver ecuaciones de la forma

$$\sum\sum\sum\sum A_{ijkl}\frac{\partial^4 U}{\partial x_i\,\partial x_j\,\partial x_k\,\partial x_l}+\sum\sum\sum B_{ijk}\frac{\partial^3 U}{\partial x_i\,\partial x_j\,\partial x_k}+$$
$$+\sum\sum C_{ij}\frac{\partial^2 U}{\partial x_i\,\partial x_j}+\sum D_i\frac{\partial U}{\partial x_i}+EU+\lambda U=0,$$

que contienen un parámetro desconocido λ, donde los índices i, j, k y l toman independientemente los valores 1, 2 y 3. Estas ecuaciones derivan de las ecuaciones que contienen una variable independiente t utilizando el método de separación de variables, de la misma forma que la ecuación de ondas

$$\Delta u=\frac{\partial^2 u}{\partial t^2}$$

conduce a la ecuación $\Delta U+\lambda^2 U=0$. El problema consiste en hallar aquellos valores de λ para los cuales el problema de contorno homogéneo tiene una solución no nula, y construir esta solución.

La esencia del método de Galerkin es la siguiente. La función incógnita se busca en la forma aproximada

$$U\approx\sum_{m=1}^{N}a_m\omega_m(x_1,x_2,x_3),$$

donde $\omega_m(x_1,x_2,x_3)$ son funciones arbitrarias que verifican las condiciones de contorno.

La supuesta solución se sustituye en el primer miembro de la ecuación, resultando la ecuación aproximada

$$\sum_{m=1}^{N}a_m\left\{\sum\sum\sum\sum A_{ijkl}\frac{\partial^4\omega_m}{\partial x_i\,\partial x_j\,\partial x_k\,\partial x_l}+\sum\sum\sum B_{ijk}\frac{\partial^3\omega_m}{\partial x_i\,\partial x_j\,\partial x_k}+\right.$$
$$\left.+\sum\sum C_{ij}\frac{\partial^2\omega_m}{\partial x_i\,\partial x_j}+\sum D_i\frac{\partial\omega_m}{\partial x_i}+E\omega_m\right\}+\lambda\sum_{m=1}^{N}a_m\omega_m\approx 0.$$

Por razones de brevedad denotamos la expresión dentro de los corchetes por $L\omega_m$ y escribimos la ecuación en la forma

$$\sum a_m L\omega_m+\lambda\sum a_m\omega_m\approx 0.$$

Ahora multiplicamos ambos miembros de nuestra ecuación aproximada por ω_k e integramos sobre el dominio Ω en el cual se busca la solución. Obtenemos

$$\iiint_{\Omega}\sum a_m\omega_k L\omega_m\,d\Omega+\lambda\iiint_{\Omega}\sum a_m\omega_m\omega_k\,d\Omega\approx 0,$$

la cual se puede volver a escribir en la forma

$$\sum_{m=1}^{N}a_m\iiint_{\Omega}\omega_k L\omega_m\,d\Omega+\lambda\sum_{m=1}^{N}a_m\iiint_{\Omega}\omega_m\omega_k\,d\Omega\approx 0.$$

Si nos fijamos el propósito de satisfacer estas ecuaciones exactamente, tendremos un sistema de ecuaciones algebraicas de primer grado en los coeficientes a_m. El número de ecuaciones del sistema será igual al de incógnitas, por lo cual este sistema tendrá una solución no nula sólo si su determinante es cero. Desarrollando este determinante, obtendremos una ecuación de grado N en el número desconocido λ.

Después de hallar el valor y sustituirlo en el sistema, resolvemos éste para obtener expresiones aproximadas de la función U.

El método de Galerkin es aconsejable para ecuaciones de cuarto orden, pero también se puede aplicar a ecuaciones de otros órdenes y tipos.

En vez de distribuir las masas en un volumen, podemos situar los puntos $M_1, M_2, \ldots, M_N$ sobre una superficie S. De nuevo, al crecer el número de estos puntos, obtenemos en el límite la integral

$$V = \iint_S \frac{A(Q)}{r}\,ds, \tag{22}$$

donde Q es un punto de la superficie S.

No es difícil ver que esta función es armónica en el interior y en el exterior de la superficie S. Sobre la superficie la función es continua, como se puede probar, aunque sus derivadas parciales de primer orden tengan discontinuidades finitas.

Las funciones $\partial(1/r)/\partial x_i$, $\partial(1/r)/\partial y_i$ y $\partial(1/r)/\partial z_i$ también son funciones armónicas del punto M para M_i fijo. A partir de estas funciones podemos formar las sumas

$$\sum A_i \frac{\partial \frac{1}{r}}{\partial x_i} + \sum B_i \frac{\partial \frac{1}{r}}{\partial y_i} + \sum C_i \frac{\partial \frac{1}{r}}{\partial z_i},$$

que serán funciones armónicas en todos los puntos, salvo quizá en $M_1, M_2, \ldots, M_N$.

De importancia particular es la integral

$$W = \iint_S \mu(Q)\left[\frac{\partial \frac{1}{r}}{\partial x'}\cos(n,x) + \frac{\partial \frac{1}{r}}{\partial y'}\cos(n,y) + \frac{\partial \frac{1}{r}}{\partial z'}\cos(n,z)\right]ds =$$
$$= \iint_S \mu(Q)\,K(Q,M)\,ds, \tag{23}$$

en la cual x', y' y z' son las coordenadas de un punto variable Q sobre la superficie S; n es la dirección de la normal a la superficie S en el punto Q; x, y y z son las direcciones de los ejes de coordenadas, y r es la distancia de Q al punto M en el que está definido el valor de la función W.

La integral (22) se denomina *potencial de una cara simple*, y la integral (23), *potencial de una cara doble* [*]. Los potenciales de una

[*] Los nombres de estos potenciales están relacionados con el hecho físico siguiente. Suponemos que sobre la superficie S hemos introducido cargas eléctri-

cara doble y de una cara simple representan funciones armónicas dentro y fuera de la superficie S.

Muchos problemas en la teoría de funciones armónicas se pueden resolver utilizando potenciales. Utilizando el potencial de una cara doble, podemos resolver el problema de construir, en un dominio dado, una función armónica u que tome valores dados $2\pi\phi(Q)$ sobre la frontera S del dominio. Para construir tal función, sólo necesitamos elegir la función $\mu(Q)$ de una manera conveniente.

Este problema recuerda algo al problema similar de hallar los coeficientes de la serie

$$\phi = \sum a_k U_k$$

para que represente la función del primer miembro.

Una notable propiedad de la integral W consiste en el hecho de que su límite, cuando el punto M se aproxima a Q_0 desde el interior de la superficie, tiene la forma

$$\lim_{M \to Q_0} W = 2\pi\mu(Q_0) + \iint_S K(Q, Q_0)\,\mu(Q)\,ds.$$

Igualando esta expresión a la función dada $2\pi\phi(Q_0)$, obtenemos la ecuación

$$\mu(Q_0) + \frac{1}{2\pi}\iint_S K(Q, Q_0)\,\mu(Q)\,ds = \phi(Q_0).$$

Ésta es una ecuación *integral de segunda especie*. La teoría de tales ecuaciones ha sido desarrollada por muchos matemáticos. Si logramos resolver esta ecuación por algún método, obtendremos una solución de nuestro problema original.

Análogamente podemos hallar una solución a otros problemas de la teoría de funciones armónicas. Después de elegir un potencial

cas. Éstas crean en el espacio un campo eléctrico, cuyo potencial estará representado por la integral (22), que se denomina por tanto potencial de una cara simple.

Suponemos ahora que la superficie S es una delgada película no conductora. Sobre una de sus caras distribuimos, de acuerdo con cierta ley, cargas eléctricas de un signo (por ejemplo, positivo). Sobre la otra cara de S distribuimos, según la misma ley, cargas eléctricas de signo opuesto. La acción de estas dos caras eléctricas también crea en el espacio un campo eléctrico. Como se puede calcular, el potencial de este campo estará representado por la integral (23).

mite una separación completa de variables. Esta separación completa de variables puede hacerse, como fue probado por el matemático soviético V. V. Stepanov, solamente en algunos casos especiales. El método de separación de variables era conocido por los matemáticos hace mucho tiempo. Fue utilizado esencialmente por Euler, Bernoulli y d'Alembert. Fourier lo usó sistemáticamente para la solución de problemas de física matemática, particularmente en la conducción del calor. Sin embargo, como ya hemos dicho, este método es a menudo inaplicable; debemos utilizar otros procedimientos, que pasamos a discutir ahora.

El método de los potenciales

La idea esencial de este método es, como antes, la superposición de soluciones particulares para la construcción de una solución en forma general. Pero esta vez para las soluciones particulares fundamentales usaremos funciones que se hagan infinitas en un punto. Ilustrémoslo con las ecuaciones de Laplace y Poisson.

Sea M_0 un punto de nuestro espacio. Denotemos por $r(M, M_0)$ la distancia del punto M_0 al punto variable M. La función $1/r(M, M_0)$ para M_0 fijo es una función del punto variable M. Es fácil establecer el hecho de que esta función es una función armónica del punto M en todo el espacio [6], excepto, por supuesto, en el punto M_0, donde se hace infinita, junto con sus derivadas.

La suma de varias funciones de esta forma

$$\sum_{i=1}^{N} A_i \frac{1}{r(M, M_i)},$$

donde los puntos $M_1, M_2, \cdots, M_N$ son puntos cualesquiera del espacio, es también una función armónica del punto M. Esta función tendrá singularidades en todos los puntos M_i. Si elegimos los puntos $M_1, M_2, \cdots, M_N$ tan densamente distribuidos como queramos en un volumen Ω, y al mismo tiempo multiplicamos por coeficientes A_i, podemos pasar al límite en esta expresión y obtener una nueva función

$$U = \lim \sum_{i=1}^{N} \frac{A_i}{r(M, M_i)} = \iiint_{\Omega} \frac{A(M')}{r(M, M')}\, d\Omega,$$

[6] Esto es, la función verifica la ecuación de Laplace.

donde el punto M' recorre todo el volumen Ω. La integral en esta forma se denomina *potencial newtoniano*. Se puede probar, aunque no lo haremos aquí, que la función U así construida verifica la ecuación $\Delta U = -4\pi A$.

El potencial newtoniano tiene un significado físico sencillo. Para comprenderlo, comenzaremos con la función $A_i/r(M, M_i)$.

Las derivadas parciales de esta función con respecto a las coordenadas son

$$A_i \frac{x_i - x}{r^3} = X, \quad A_i \frac{y_i - y}{r^3} = Y, \quad A_i \frac{z_i - z}{r^3} = Z.$$

En el punto M_i colocamos una masa A_i, la cual atraerá a todos los cuerpos con una fuerza dirigida hacia el punto M_i e inversamente proporcional al cuadrado de la distancia desde M_i. Descomponemos esta fuerza en sus componentes según los ejes de coordenadas. Si la magnitud de la fuerza que actúa sobre un punto material de masa unidad es A_i/r^2, los cosenos de los ángulos formados por la dirección de esta fuerza y los ejes coordenados serán $(x_i - x)/r$, $(y_i - y)/r$, $(z_i - z)/r$. Por tanto, las componentes de la fuerza ejercida sobre la unidad de masa en el punto M por un centro de atracción M_i serán iguales a X, Y y Z, las derivadas parciales de la función A_i/r con respecto a las coordenadas. Si colocamos masas en los puntos $M_1, M_2, \cdots, M_N$, entonces todo punto material de masa unidad situado en M será atraído por una fuerza igual a la resultante de todas las fuerzas que actúan sobre él desde los puntos dados M_i. Dicho con otras palabras:

$$X = \frac{\partial}{\partial x} \sum \frac{A_i}{r(M, M_i)}, \quad Y = \frac{\partial}{\partial y} \sum \frac{A_i}{r(M, M_i)}, \quad Z = \frac{\partial}{\partial z} \sum \frac{A_i}{r(M, M_i)}.$$

Pasando al límite y sustituyendo la suma por una integral, obtenemos

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}, \quad \text{donde} \quad U = \iiint_{\Omega} \frac{A}{r}\, d\Omega.$$

La función U, con derivadas parciales iguales a las componentes de la fuerza que actúa sobre un punto, se llama *potencial* de la fuerza. Por tanto, la función $A_i/r(M, M_i)$ es el potencial de la atracción ejercida por el punto M_i; la función $\sum [A_i/r(M, M_i)]$ es el potencial de la atracción ejercida por el conjunto de puntos $M_1, M_2, \cdots, M_N$, y la función $U = \iiint_{\Omega} (A/r)\, d\Omega$ es el potencial de la atracción ejercida por las masas distribuidas continuamente en el volumen Ω.

Aquí

$$\frac{F_k'(t)}{F_k(t)} = -\lambda_k^2, \quad \Delta U_k + \lambda_k^2 U_k = 0.$$

La solución se obtiene en la forma

$$T = \sum_{k=1}^{\infty} e^{-\lambda_k^2 t}\, U_k(x, y, z).$$

Este método ha sido también usado con gran éxito para resolver algunas otras ecuaciones. Consideremos, por ejemplo, la ecuación de Laplace

$$\Delta u = 0$$

en el círculo

$$x^2 + y^2 \leq 1,$$

y supongamos que hemos de construir una solución que satisfaga la condición

$$u\,|_{r=1} = f(\vartheta),$$

donde r y ϑ denotan las coordenadas polares de un punto en el plano.

La ecuación de Laplace se puede transformar fácilmente en coordenadas polares, adoptando la forma

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} = 0.$$

Queremos hallar una solución de esta ecuación en la forma

$$u = \sum_{k=1}^{\infty} R_k(r)\,\Theta_k(\vartheta).$$

Si imponemos que cada término de la serie verifique individualmente la ecuación, tenemos

$$\left[R_k''(r) + \frac{1}{r} R_k'(r)\right]\Theta_k(\vartheta) + \frac{1}{r^2}\Theta_k''(\vartheta)\, R_k(r) = 0.$$

Dividiendo la ecuación por $R_k(r)\,\Theta_k(\vartheta)/r^2$, obtenemos

$$\frac{r^2\left[R_k''(r) + \frac{1}{r} R_k'(r)\right]}{R_k(r)} = -\frac{\Theta_k''(\vartheta)}{\Theta_k(\vartheta)}.$$

Poniendo nuevamente

$$\frac{\Theta_k''(\vartheta)}{\Theta_k(\vartheta)} = -\lambda_k^2,$$

tenemos

$$r^2\left[R_k'' + \frac{1}{r} R_k'\right] - \lambda_k^2 R_k = 0.$$

Es fácil ver que la función $\Theta_k(\vartheta)$ debe ser una función periódica de ϑ con período 2π. Integrando la ecuación $\Theta_k''(\vartheta) + \lambda_k^2\Theta_k(\vartheta) = 0$, obtenemos

$$\Theta_k = a_k \cos \lambda_k\vartheta + b_k \operatorname{sen} \lambda_k\vartheta.$$

Esta función será periódica con el período requerido solamente si λ_k es entero. Poniendo $\lambda_k = k$, tenemos

$$\Theta_k = a_k \cos k\vartheta + b_k \operatorname{sen} k\vartheta.$$

La ecuación para R_k tiene una solución general de la forma

$$R_k = A r^k + \frac{B}{r^k}.$$

Reteniendo solamente el término que está acotado para $r \to 0$, obtenemos la solución general de la ecuación de Laplace en la forma

$$u = a_0 + \sum_{k=1}^{\infty} (a_k \cos k\vartheta + b_k \operatorname{sen} k\vartheta)\, r^k.$$

Este método puede usarse normalmente para calcular soluciones no triviales de la ecuación $\Delta U_k + \lambda_k^2 U_k = 0$ que verifiquen condiciones de contorno homogéneas. Cuando el problema pueda reducirse al de resolver ecuaciones diferenciales ordinarias, decimos que per-

Toda solución, pues, consta de oscilaciones características, cuya amplitud y fase se pueden calcular conociendo las condiciones iniciales.

Exactamente de la misma manera podemos estudiar oscilaciones con un menor número de variables independientes. Como ejemplo consideremos la cuerda vibrante, fija por los extremos. La ecuación de la cuerda vibrante tiene la forma

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Supongamos que queremos calcular una solución del problema para una cuerda de longitud l fija por los extremos:

$$u\,|_{x=0} = u\,|_{x=l} = 0.$$

Buscamos una colección de soluciones particulares

$$u_k = T_k(t)\, U_k(x).$$

Evidentemente obtenemos, como antes,

$$T_k'' U_k = a^2 U_k'' T_k,$$

o bien

$$\frac{T_k''}{T_k} = a^2 \frac{U_k''}{U_k} = -\lambda_k^2.$$

De donde

$$T_k = A_k \cos \lambda_k t + B_k \operatorname{sen} \lambda_k t,$$

$$U_k = M_k \cos \frac{\lambda_k}{a} x + N_k \operatorname{sen} \frac{\lambda_k}{a} x.$$

Utilicemos las condiciones de contorno para calcular los valores de λ_k. Para λ_k arbitrario no es posible verificar ambas condiciones de contorno. De la condición $U_k|_{x=0} = 0$ obtenemos $M_k = 0$, y esto significa que $U_k = N_k \operatorname{sen}(\lambda_k/a)x$. Poniendo $x = l$, obtenemos $\operatorname{sen}(\lambda_k l/a) = 0$. Esto solamente puede suceder si $\lambda_k l/a = k\pi$, donde k es un entero. Esto significa que

$$\lambda_k = \frac{ak\pi}{l}.$$

La condición $\int_0^l U_k^2\, dx = 1$ prueba que $N_k = \sqrt{\frac{2}{l}}$. Finalmente

$$U_k(x) = \sqrt{\frac{2}{l}} \operatorname{sen} \frac{k\pi x}{l}, \quad T_k = A_k \cos \frac{ak\pi t}{l} + B_k \operatorname{sen} \frac{ak\pi t}{l}.$$

De esta manera las oscilaciones características de la cuerda, como vemos, tienen forma sinusoidal con un número entero de semiondas sobre el total de la cuerda. Cada oscilación tiene su propia frecuencia, y éstas se pueden clasificar en orden creciente

$$\frac{a\pi}{l},\ 2\frac{a\pi}{l},\ 3\frac{a\pi}{l},\ \ldots,\ k\frac{a\pi}{l},\ \ldots$$

Es bien conocido que estas frecuencias son precisamente las que oímos en las vibraciones de una cuerda sonora. La frecuencia $a\pi/l$ se denomina *frecuencia fundamental*, y las frecuencias restantes son *armónicos*. Las funciones características $\sqrt{2/l}\operatorname{sen}(k\pi x/l)$ sobre el intervalo $0 \leqslant x \leqslant l$ cambian de signo $k-1$ veces, ya que $k\pi x/l$ recorre todos los valores de 0 a $k\pi$, lo cual significa que su seno cambia de signo $k-1$ veces. Los puntos en que las funciones características U_k se anulan se llaman **nodos de las oscilaciones**.

Si conseguimos de alguna manera que la cuerda no se mueva en un punto correspondiente a un nodo, por ejemplo, del primer armónico, entonces el tono fundamental desaparecerá y oiremos solamente el sonido del primer armónico, que es una octava más alto. Tal artificio, llamado parada, se utiliza en instrumentos que se tocan con un arco: el violín, la viola y el violonchelo.

Hemos analizado el método de separación de variables en tanto que aplicado al problema de hallar oscilaciones características. No obstante, el método se puede aplicar con mucha mayor amplitud a los problemas de flujo de calor y a toda una serie de otros problemas.

Para la ecuación del flujo de calor

$$\Delta T = \frac{\partial T}{\partial t}$$

con la condición

$$T\,|_S = 0$$

tendremos, como antes,

$$T = \sum F_k(t)\, U_k(x, y, z).$$

Finalmente, una tercera propiedad de las oscilaciones características es que, si no excluimos ningún valor de λ_k, entonces mediante las funciones características $U_k(x, y, z)$ podemos representar, con cualquier grado de aproximación deseado, una función completamente arbitraria $f(x, y, z)$, con tal solamente de que ésta verifique la condición de contorno $f|_S = 0$ y tenga la primera y segunda derivadas continuas. Tal función $f(x, y, z)$ se puede representar mediante la serie convergente

$$f(x, y, z) = \sum_{k=1}^{\infty} C_k U_k(x, y, z). \tag{20}$$

La tercera propiedad de las funciones características nos proporciona en principio la posibilidad de representar una función $f(x, y, z)$ mediante una serie de funciones características de nuestro problema, y aplicando la segunda propiedad podemos calcular todos los coeficientes de esta serie. En efecto, si multiplicamos ambos miembros de la ecuación (20) por $U_j(x, y, z)$ e integramos sobre el dominio Ω, obtenemos

$$\iiint_{\Omega} f(x, y, z)\, U_j(x, y, z)\, dx\, dy\, dz =$$

$$= \sum_{k=1}^{\infty} C_k \iiint_{\Omega} U_k(x, y, z)\, U_j(x, y, z)\, dx\, dy\, dz.$$

En la suma del segundo miembro, todos los términos para los que $k \neq j$ desaparecen en virtud de la ortogonalidad, y el coeficiente de C_j es igual a uno. En consecuencia tenemos

$$C_j = \iiint_{\Omega} f(x, y, z)\, U_j(x, y, z)\, dx\, dy\, dz.$$

Estas propiedades de las oscilaciones características nos permiten ahora resolver el problema general de las oscilaciones para unas condiciones iniciales cualesquiera.

Para ello supongamos que tenemos una solución del problema en la forma

$$u = \sum U_k(x, y, z)\,(A_k \cos \lambda_k t + B_k \operatorname{sen} \lambda_k t) \tag{21}$$

y que tratamos de elegir las constantes A_k y B_k de tal manera que tengamos

$$u|_{t=0} = f_0(x, y, z),$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = f_1(x, y, z).$$

Poniendo $t = 0$ en el segundo miembro de (21), vemos que los términos en seno desaparecen y $\cos \lambda_k t$ se hace igual a uno, así que tendremos

$$f_0(x, y, z) = \sum_{k=1}^{\infty} A_k U_k(x, y, z).$$

Por la tercera propiedad, las oscilaciones características se pueden utilizar para tal representación; y por la segunda propiedad tenemos

$$A_k = \iiint_{\Omega} f_0(x, y, z)\, U_k(x, y, z)\, dx\, dy\, dz.$$

Análogamente, derivando la fórmula (21) con respecto a t y poniendo $t = 0$, tendremos

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = f_1(x, y, z) = \sum_{k=1}^{\infty} \lambda_k (B_k \cos \lambda_k t - A_k \operatorname{sen} \lambda_k t)|_{t=0}\, U_k(x, y, z) =$$

$$= \sum_{k=1}^{\infty} \lambda_k B_k U_k(x, y, z).$$

Así, pues, como antes, obtenemos los valores de B_k de la forma siguiente:

$$B_k = \frac{1}{\lambda_k} \iiint_{\Omega} f_1(x, y, z)\, U_k(x, y, z)\, dx\, dy\, dz.$$

Conocidos A_k y B_k conocemos las fases y las amplitudes de todas las oscilaciones características.

De esta manera hemos probado que por adición de oscilaciones características es posible obtener la solución más general del problema con condiciones de contorno homogéneas.

La primera de estas ecuaciones tiene, como sabemos, la solución

$$T = A_k \cos \lambda_k t + B_k \operatorname{sen} \lambda_k t,$$

donde A_k y B_k son constantes arbitrarias. Esta solución se puede simplificar introduciendo el argumento auxiliar ϕ. Tenemos

$$\frac{A_k}{\sqrt{A_k^2 + B_k^2}} = \operatorname{sen} \phi_k, \quad \frac{B_k}{\sqrt{A_k^2 + B_k^2}} = \cos\phi_k, \quad \sqrt{A_k^2 + B_k^2} = M_k.$$

Entonces

$$T = \sqrt{A_k^2 + B_k^2} \operatorname{sen} (\lambda_k t + \phi_k) = M_k \operatorname{sen} (\lambda_k t + \phi_k).$$

La función T representa una oscilación armónica de frecuencia λ_k y ángulo de fase ϕ_k.

Más difícil e interesante es el problema de hallar una solución de la ecuación

$$\Delta U + \lambda_k^2 U = 0 \tag{19}$$

para condiciones de contorno homogéneas dadas; por ejemplo, para la condición

$$U|_S = 0$$

(donde S es la frontera del volumen Ω en cuestión), o para cualquier otra condición homogénea. La solución de este problema no es siempre fácil de construir como combinación finita de funciones conocidas, aunque siempre existe y puede hallarse con el grado deseado de aproximación.

La ecuación $\Delta U + \lambda_k^2 U = 0$ con la condición $U|_S = 0$ tiene en principio la solución evidente $U \equiv 0$. Esta solución es trivial y completamente inútil para nuestro propósito. Si los λ_k son números elegidos arbitrariamente, entonces en general el problema no tendrá solución. Sin embargo, existen normalmente valores de λ_k para los cuales la ecuación sí tiene una solución no trivial.

Todos los valores posibles de la constante λ_k^2 están determinados por el requisito de que la ecuación (19) tenga una solución no trivial (es decir, distinta de la función idénticamente nula) que satisfaga la condición $U|_S = 0$. De esto se sigue también que los números denotados por $-\lambda_k^2$ deben ser negativos.

Para cada uno de los valores posibles de λ_k en la ecuación (19) podemos hallar al menos una función U_k. Esto nos permite construir una solución particular de la ecuación de ondas (18) en la forma

$$u_k = M_k \operatorname{sen} (\lambda_k t + \phi_k) U_k(x, y, z).$$

Tal solución se denomina *oscilación característica* (u *oscilación propia*) del volumen considerado. La constante λ_k es la frecuencia de la oscilación característica, y la función $U_k(x, y, z)$ determina su forma. Esta función se denomina por lo común *función propia* (o *función característica*, o *autofunción*). En todo instante, la función u_k, considerada como función de las variables x, y y z, diferirá de la función $U_k(x, y, z)$ solamente en la escala.

No disponemos de espacio aquí para una demostración detallada de las muchas propiedades notables de las oscilaciones y funciones características; por consiguiente nos restringiremos solamente a enumerar algunas de ellas.

La primera propiedad de las oscilaciones características consiste en el hecho de que para un volumen dado existe un conjunto numerable de frecuencias características. Estas frecuencias tienden a infinito cuando crece k.

Otra propiedad de las oscilaciones características se denomina *ortogonalidad*. Consiste en el hecho de que la integral sobre el dominio Ω del producto de las funciones características correspondientes a distintos valores de λ_k es igual a cero [4].

$$\iiint_\Omega U_k(x, y, z)\, U_j(x, y, z)\, dx\, dy\, dz = 0 \quad (j \neq k).$$

Para $j = k$ supondremos

$$\iiint_\Omega U_k(x, y, z)^2\, dx\, dy\, dz = 1.$$

Esto siempre se puede conseguir multiplicando las funciones $U_k(x, y, z)$ por una constante apropiada, cuya elección no altera el hecho de que la función satisfaga la ecuación (19) y la condición $U|_S = 0$.

[4] Si a un mismo valor de λ corresponden varias funciones U esencialmente diferentes (linealmente independientes), entonces este valor de λ se considera como si apareciese un número correspondiente de veces en el conjunto de valores característicos λ_k. La condición de ortogonalidad para funciones correspondientes al mismo valor de λ_k se puede garantizar mediante una elección adecuada de estas funciones.

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diente condición homogénea: los valores de la expresión correspondiente sobre el contorno serán iguales a cero.

Por tanto, la variedad total de las soluciones de tal ecuación, para condiciones de contorno dadas, se puede hallar tomando, por una parte, una solución particular que verifique las condiciones no homogéneas dadas, y por otra, todas las posibles soluciones de la ecuación homogénea que verifique las condiciones de contorno homogéneas (pero no, en general, las iniciales).

Las soluciones de ecuaciones homogéneas que verifican condiciones de contorno homogéneas se pueden sumar o multiplicar por constantes sin que dejen de ser soluciones.

Si una solución de una ecuación homogénea con condiciones homogéneas es función de un parámetro, integrando respecto a éste obtendremos también una tal solución. Estos hechos forman la base del método más importante de resolución de problemas lineales de todas clases para las ecuaciones de la física matemática: el método de superposición.

La solución del problema será de la forma

$$u = u_* + \sum_k u_k,$$

donde u_* es una solución particular de la ecuación, solución que verifica las condiciones de contorno, aunque no las iniciales; las u_k son soluciones de la correspondiente ecuación homogénea, y verifican las correspondientes condiciones de contorno homogéneas. Si la ecuación y las condiciones de contorno fueran originalmente homogéneas, entonces la solución del problema se habrá de buscar en la forma

$$u = \sum u_k.$$

Con el fin de poder satisfacer condiciones iniciales arbitrarias mediante la elección de soluciones particulares u_k de la ecuación homogénea, es preciso disponer de un arsenal suficientemente grande de tales soluciones.

El método de separación de variables

Para la construcción de este arsenal de soluciones existe un método denominado de *separación de variables* o *método de Fourier*.

Examinemos este método, por ejemplo, para resolver el problema

$$\Delta u = \frac{\partial^2 u}{\partial t^2}, \qquad (18)$$

$$u|_S = 0, \quad u|_{t=0} = f_0(x, y, z), \quad u'_t|_{t=0} = f_1(x, y, z).$$

Para buscar una solución particular de la ecuación, supongamos en primer lugar que la función deseada u verifica la condición de contorno $u|_S = 0$ y que puede expresarse como producto de dos funciones, una de las cuales depende solamente del tiempo t, y la otra, sólo de las variables espaciales:

$$u(x, y, z, t) = U(x, y, z)\, T(t).$$

Sustituyendo esta presunta solución en nuestra ecuación, tenemos

$$T(t)\, \Delta U = T''(t)\, U.$$

Dividiendo ambos miembros por TU se obtiene

$$\frac{T''}{T} = \frac{\Delta U}{U}.$$

El segundo miembro de esta ecuación es sólo función de las variables espaciales, mientras que el primero es independiente de las mismas. Por tanto, se deduce que la ecuación dada sólo puede verificarse si ambos miembros son iguales al mismo valor constante. Esto nos conduce a un sistema de dos ecuaciones

$$\frac{T''}{T} = -\lambda_k^2, \quad \frac{\Delta U}{U} = -\lambda_k^2.$$

La cantidad constante de la derecha se denota por $-\lambda_k^2$ para poner de relieve que es negativa (lo cual se puede probar rigurosamente). El subíndice k se introduce para hacer notar que existen infinitos valores posibles de $-\lambda_k^2$; las soluciones correspondientes a éstos forman un sistema de funciones que es completo en un sentido bien conocido.

Por multiplicación cruzada en ambas ecuaciones, obtenemos

$$T'' + \lambda_k^2 T = 0; \quad \Delta U + \lambda_k^2 U = 0.$$

Inicialmente las ondas se mueven independientemente una hacia la otra, para después comenzar a interaccionar. En el segundo caso de la figura 3 habrá un instante de completo aniquilamiento de las oscilaciones, tras lo cual las ondas se volverán a separar.

Otro ejemplo que se presta fácilmente a un estudio cualitativo es la propagación de ondas en el espacio.

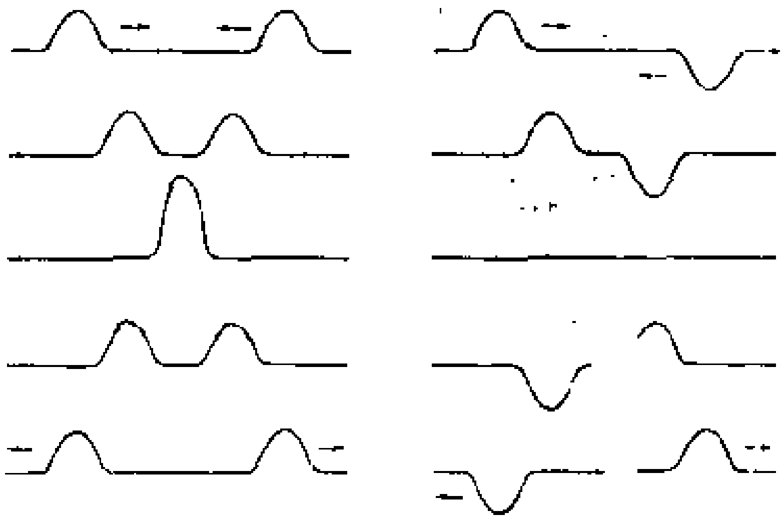

FIG. 3

La ecuación

$$\Delta u = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \qquad (16)$$

obtenida antes, tiene dos soluciones particulares de la forma

$$u_1 = \frac{1}{r}\phi_1(r - at), \quad u_2 = \frac{1}{r}\phi_2(r + at), \qquad (17)$$

donde r denota la distancia de un punto dado al origen del sistema de coordenadas, $r^2 = x^2 + y^2 + z^2$, y ϕ_1 y ϕ_2 son funciones arbitrarias, dos veces diferenciables.

La demostración de que u_1 y u_2 son soluciones sería demasiado prolija y la omitimos aquí.

La forma de las ondas descritas por estas soluciones es en general la misma que para la cuerda. Si prescindimos del factor $1/r$ que aparece en el segundo miembro, entonces la primera solución representa una onda que se propaga en la dirección de las r crecientes. Esta onda tiene simetría esférica; es idéntica en todos los puntos que tienen el mismo valor de r.

El factor $1/r$ determina el que la amplitud de la onda sea inversamente proporcional a la distancia desde el origen. Tal oscilación se denomina onda esférica divergente. Una buena imagen de ello son los círculos que se extienden sobre la superficie del agua cuando se arroja una piedra en ella, sólo que en este caso las ondas son circulares en vez de esféricas.

Esta segunda solución de (17) es también de gran interés: se denomina onda convergente y se propaga en dirección al origen. Su amplitud tiende a infinito a medida que se aproxima al origen. Vemos que una concentración tal de la perturbación en un punto puede conducir, aun cuando las oscilaciones iniciales sean pequeñas, a un choque brutal.

§5. *Métodos de construcción de soluciones*

Sobre la posibilidad de descomponer una solución en otras más simples

Las soluciones de los problemas de la física matemática formulados con anterioridad pueden ser obtenidas mediante diversos artificios, que constituyen problemas específicos distintos. No obstante, en la base de estos métodos hay una idea general. Como hemos visto, todas las ecuaciones de la física matemática son, para pequeños valores de las funciones incógnitas, lineales con respecto a éstas y a sus derivadas. Las condiciones de contorno y las iniciales son también lineales.

Si formamos la diferencia entre dos soluciones cualesquiera de la misma ecuación, dicha diferencia será también solución de la ecuación con el segundo miembro igual a cero. Tal ecuación se denomina ecuación homogénea. Por ejemplo, para la ecuación de Poisson $\Delta u = -4\pi\rho$, la homogénea correspondiente es la ecuación de Laplace $\Delta u = 0$.

Si dos soluciones de la misma ecuación verifican también las mismas condiciones de contorno, su diferencia satisfará la correspon-

En este caso, cuando calculásemos la «energía» de una tal oscilación, descrita por una función v, descubriríamos que la energía $E(v)$ es igual acero en el instante inicial. Esto significa que es siempre igual a cero y por lo tanto que la función v es idénticamente nula, con lo que las dos soluciones p_1 y p_2 son idénticas. En consecuencia, la solución del problema es única.

De esta manera llegamos al convencimiento de que los tres problemas están correctamente planteados.

Incidentalmente hemos decubierto algunas propiedades muy simples de las soluciones de estas ecuaciones. Por ejemplo, las soluciones de la ecuación de Laplace tienen la siguiente propiedad de máximo: las funciones que verifican esta ecuación alcanzan su mayor y menor valor sobre el contorno de su dominio de definición.

Las funciones que describen la distribución de calor en un medio tienen una propiedad de máximo algo diferente. Todo máximo o mínimo de temperatura en un punto se desvanece y disminuye gradualmente con el tiempo. La temperatura en un punto puede aumentar o disminuir solamente si es menor o mayor que la de los puntos contiguos. La temperatura se uniformiza con el paso del tiempo. Todas las irregularidades se nivelan por el paso del calor de los lugares más calientes a los más fríos.

Por el contrario, en la propagación de las oscilaciones aquí *no* se produce ningún proceso de nivelación. Estas oscilaciones no disminuyen *ni* se nivelan, ya que la suma de sus energías cinética y potencial debe permanecer siempre constante.

§4. *La propagación de ondas*

Las propiedades de las oscilaciones se pueden evidenciar muy claramente con ejemplos sencillísimos. Consideremos dos casos característicos.

Nuestro primer ejemplo es la ecuación de la cuerda vibrante

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}. \tag{15}$$

Esta ecuación, como se puede comprobar, tiene dos soluciones particulares de la forma

$$u_1 = \phi_1(x - at), \quad u_2 = \phi_2(x + at),$$

donde ϕ_1 y ϕ_2 son funciones arbitrarias dos veces diferenciables.

Por derivación directa es fácil probar que las funciones u_1 y u_2 verifican la ecuación (15). Se puede demostrar que

$$u = u_1 + u_2$$

es una solución general de esta ecuación.

La forma general de las oscilaciones descritas por las funciones u_1 y u_2 es de considerable interés. Para considerarla de la manera más conveniente, hagamos mentalmente el siguiente experimento. Supongamos que el observador de la cuerda vibrante no es estacionario sino que se mueve a lo largo del eje Ox con velocidad a. Para un tal observador la posición de un punto de la cuerda estará determinada, no por un sistema de coordenadas estacionario, sino móvil. Sea ξ la coordenada x de este sistema. Entonces $\xi = 0$ corresponderá evidentemente en cada instante al valor $x = at$. Por tanto está claro que

$$\xi = x - at.$$

Podemos representar una función arbitraria $u(x, t)$ en la forma

$$u(x, t) = \phi(\xi, t).$$

Para la solución u_1 tendremos

$$u_1(x, t) = \phi_1(\xi),$$

con lo que en este sistema de coordenadas la solución $u_1(x, t)$ resulta ser independiente del tiempo. En consecuencia, un observador que se mueva con velocidad a ve la cuerda como una curva estacionaria. Un observador estacionario, por el contrario, ve en la cuerda una onda que se mueve a lo largo del eje Ox con velocidad a.

Análogamente, la solución $u_2(x, t)$ puede ser considerada como una onda que discurre en dirección opuesta con velocidad a. Si la cuerda fuese infinita, ambas ondas se propagarían infinitamente lejos. Al moverse en direcciones opuestas, pueden producir, por superposición, formas muy curiosas en la cuerda. El desplazamiento resultante puede aumentar en unas ocasiones y disminuir en otras.

Si, al llegar a un punto dado desde lados opuestos, u_1 y u_2 tienen el mismo signo, entonces se refuerzan mutuamente; pero, si tienen signos opuestos, se contrarrestan. La figura 3 muestra varias posiciones sucesivas de la cuerda para dos desplazamientos particulares.

Además de energía cinética, la cuerda en su posición desplazada posee también energía potencial, creada por el incremento de longitud en comparación con la posición rectilínea. Calculemos esta energía potencial. Concentrémonos en un elemento de cuerda comprendido entre los puntos x y $x + dx$. Este elemento tiene una posición inclinada respecto al eje Ox, tal que su longitud es aproximadamente igual a

$$\sqrt{(dx)^2 + \left(\frac{\partial u}{\partial x}\,dx\right)^2}\,;$$

por tanto, su alargamiento es

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\,dx - dx \approx \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Multiplicando este alargamiento por la tensión T, hallamos la energía potencial del elemento alargado de la cuerda

$$\frac{1}{2}\,T\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

La energía total de la cuerda de longitud l se obtiene sumando la energía cinética y potencial en todos los puntos de la cuerda. Obtenemos

$$E = \frac{1}{2}\int_0^l \left[T\left(\frac{\partial u}{\partial x}\right)^2 + \varrho\left(\frac{\partial u}{\partial t}\right)^2\right] dx.$$

Si las fuerzas que actúan sobre los extremos no efectúan trabajo, en particular si los extremos de la cuerda están fijos, entonces la energía total debe ser constante:

$$E = \text{const.}$$

Nuestra expresión para la ley de conservación de la energía es un corolario matemático de las ecuaciones básicas de la mecánica y se puede derivar de ellas. Como ya hemos escrito las leyes de movimiento en la forma de la ecuación diferencial de la cuerda vibrante con condiciones sobre los extremos, podemos dar la siguiente demostración matemática de la ley de conservación de la energía en este

caso. Si derivamos E con respecto al tiempo, y haciendo uso de reglas generales básicas, tenemos:

$$\frac{dE}{dt} = \int_0^l \left(T\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial t} + \varrho\,\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t^2}\right) dx.$$

Utilizando la ecuación de ondas (6) y sustituyendo $\varrho(\partial^2 u/\partial t^2)$ por $T(\partial^2 u/\partial x^2)$, obtenemos dE/dt en la forma

$$\frac{dE}{dt} = \int_0^l T\left(\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial t} + \frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial x^2}\right) dx =$$

$$= \int_0^l T\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial t}\right) dx = T\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial t}\bigg|_{x=l} - T\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial t}\bigg|_{x=0}.$$

Si $(\partial u/\partial x)\,|_{x=0}$ o $u\,|_{x=0}$ es nulo, y también $(\partial u/\partial x)\,|_{x=l}$ o $u\,|_{x=l}$ es nulo, entonces

$$\frac{dE}{dt} = 0,$$

lo cual prueba que E es constante.

La ecuación de ondas (9) puede ser tratada exactamente igual para probar que la ley de conservación de la energía también es válida en este caso. Si p verifica la ecuación (9) y la condición

$$p\,|_S = 0 \quad \text{ó} \quad \frac{\partial p}{\partial n}\bigg|_S = 0,$$

entonces la cantidad

$$E = \iiint \left[\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2 + \left(\frac{\partial p}{\partial z}\right)^2 + \frac{1}{a^2}\left(\frac{\partial p}{\partial t}\right)^2\right] dx\,dy\,dz$$

no dependerá de t.

Si en el instante inicial la energía total de la oscilación es igual a cero, entonces permanecerá siempre igual a cero, y esto es posible solamente en el caso de que no haya movimiento. Si el problema de integrar la ecuación de ondas con condiciones iniciales y de contorno tuviera dos soluciones p_1 y p_2, entonces $v = p_1 - p_2$ sería una solución de la ecuación de ondas que verificaría las condiciones con cero en el segundo miembro, es decir, las condiciones homogéneas.

En el segundo miembro de esta ecuación aparece la cantidad $\partial^2 u/\partial t^2$, que expresa la aceleración de un punto arbitrario de la cuerda. El movimiento de cualquier sistema mecánico en el que las fuerzas —y en consecuencia las aceleraciones— vengan expresadas por medio de las coordenadas de los cuerpos móviles, está completamente determinado si se conocen las posiciones iniciales y las velocidades de todos los puntos del sistema. Así, pues, para la ecuación de la cuerda vibrante es natural especificar las posiciones y velocidades de todos los puntos en el instante inicial.

$$u|_{t=0} = u_0(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x).$$

Pero, como ya se apuntó antes, en los extremos de la cuerda las fórmulas que expresan las leyes de la mecánica para puntos interiores dejan de aplicarse. Así, pues, en ambos extremos debemos imponer condiciones suplementarias. Si, por ejemplo, la cuerda está fijada en una posición de equilibrio en ambos extremos, entonces tendremos

$$u|_{x=0} = u|_{x=l} = 0.$$

Estas condiciones pueden reemplazarse a veces por otras más generales, pero un cambio de esta clase no es de importancia trascendental.

El problema de hallar las soluciones necesarias de la ecuación (9) es análogo. Para que tal solución esté bien definida, es habitual imponer las condiciones

$$p|_{t=0} = \phi_0(x, y, z),$$

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} = \phi_1(x, y, z), \qquad (13)$$

y también una de las «condiciones de contorno»

$$p|_S = \phi(Q), \qquad (14)$$

$$\left.\frac{\partial p}{\partial n}\right|_S = \psi(Q), \qquad (14')$$

$$\alpha \left.\frac{\partial p}{\partial n}\right|_S + \beta p|_S = \chi(Q)\,[3]. \qquad (14'')$$

[3] Si los segundos miembros de las condiciones (13) y (14) son nulos, tales condiciones se llaman «homogéneas».

La diferencia con respecto al caso anterior es simplemente que en vez de una sola condición inicial, la ecuación (11), tenemos ahora las dos condiciones (13).

La ecuaciones (14) expresan evidentemente las leyes físicas para las partículas sobre el contorno del volumen en cuestión.

La demostración de que en el caso general las condiciones (13), junto con una cualquiera de las condiciones (14), determinan unívocamente una solución del problema será omitida. Probaremos solamente que la solución puede ser única para una de las condiciones (14).

Supongamos que una función u verifica la ecuación

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2},$$

con condiciones iniciales

$$u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

y condición de contorno

$$\left.\frac{\partial u}{\partial n}\right|_S = 0.$$

(Sería igual de fácil discutir el caso en que $u|_S = 0$.)

Demostraremos que bajo estas condiciones la función u debe ser idénticamente nula.

Para probar esta propiedad no bastan los argumentos introducidos con anterioridad para establecer la unicidad de la solución del primero de ambos problemas. En cambio aquí podemos hacer uso de la interpretación física.

Sólo precisamos de una ley física, la ley de «conservación de la energía». Nos restringiremos de nuevo, por razones de simplicidad, al caso de la cuerda vibrante: el desplazamiento $u(x, t)$ de sus puntos verifica la ecuación

$$T\frac{\partial^2 u}{\partial x^2} = \varrho\frac{\partial^2 u}{\partial t^2}.$$

La energía cinética de cada partícula de la cuerda, al oscilar de μ a $\mu + d\mu$, viene expresada por:

$$\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 \varrho\, dx.$$

De esta manera, el problema puede enunciarse como sigue. Buscamos una solución de la ecuación (8) bajo la condición

$$T|_{t=0} = T_0(x, y, z) \tag{11}$$

y una de las tres condiciones siguientes:

$$T|_S = \phi(Q), \tag{12}$$

$$\left.\frac{\partial T}{\partial n}\right|_S = \psi(Q), \tag{12'}$$

$$\alpha \left.\frac{\partial T}{\partial n}\right|_S + \beta T|_S = \chi(Q), \tag{12''}$$

donde Q es un punto cualquiera de la superficie S.

La condición (11) se denomina una *condición inicial*, mientras que las (12) son *condiciones de contorno*.

No probaremos con detalle que todo problema de este tipo tiene una única solución; *sólo* estableceremos este hecho para el primero de estos problemas; además, consideraremos solamente el caso en que no existen fuentes de calor en el interior del medio. Demostraremos que la ecuación

$$\Delta T = \frac{1}{a^2}\frac{\partial T}{\partial t}$$

bajo las condiciones

$$T|_{t=0} = T_0(x, y, z),$$

$$T|_S = \phi(Q)$$

solamente puede tener una solución.

La demostración de este enunciado es muy similar a la demostración de la unicidad de la solución de la ecuación de Laplace. Probamos en primer lugar que si

$$\Delta T - \frac{1}{a^2}\frac{\partial T}{\partial t} < 0,$$

entonces T, como función de cuatro variables, x, y, z y $t (0 \leqslant t \leqslant t_0)$, alcanza un mínimo sobre la frontera del dominio Ω o en el interior, si bien en este último caso necesariamente ha de ser en el instante inicial, $t = 0$.

Si no fuera así, el mínimo lo alcanzaría en un punto interior.

En este punto las derivadas primeras, incluida $\partial T/\partial t$, serían iguales a cero, y si este mínimo se diera para $t = t_0$, entonces $\partial T/\partial t$ sería no positiva. Asimismo, en este punto las derivadas segundas respecto a las variables x, y y z serían no negativas. En consecuencia, $\Delta T - (1/a^2)(\partial T/\partial t)$ sería no negativo, lo que en nuestro caso es imposible.

Exactamente de la misma forma podemos establecer que, si $\Delta T - (1/a^2)(\partial T/\partial t) > 0$, entonces en el interior de Ω, para $0 < t \leqslant t_0$, no puede existir un máximo para la función T.

Finalmente, si $\Delta T - (1/a^2)(\partial T/\partial t) = 0$, entonces en el interior de Ω, para $0 < t \leqslant t_0$, la función T no puede alcanzar un máximo ni un mínimo absolutos, ya que si T alcanzase, por ejemplo, un mínimo, entonces sumándole el término $\eta(t - t_0)$ y considerando la función $T_1 = T + \eta(t - t_0)$, no destruiríamos el mínimo absoluto con tal de que η fuera suficientemente pequeño, y entonces $\Delta T_1 - (1/a^2)(\partial T_1/\partial t)$ sería negativo, lo cual es imposible.

De la misma forma, podemos también probar la ausencia de un máximo absoluto de T en el dominio considerado.

Sin embargo, un mínimo, o un máximo, absoluto de temperatura sí puede alcanzarse en el instante inicial $t = 0$ o sobre la frontera S del medio. Si $T = 0$ en el instante inicial y sobre la frontera, entonces tenemos la identidad $T = 0$ en todo el interior del dominio para todo $t \leqslant t_0$. Si dos distribuciones de temperatura T_1 y T_2 tienen idénticos valores para $t = 0$ y sobre el contorno, entonces su diferencia $T_1 - T_2 = T$ satisfará la ecuación del calor y se anulará para $t = 0$ y sobre el contorno. Esto implica que $T_1 - T_2$ será igual a cero en todo punto, por lo que las dos distribuciones de temperatura T_1 y T_2 serán idénticas.

En el estudio que haremos más adelante de los métodos para resolver las ecuaciones de la física matemática veremos que el valor de T para $t = 0$ y el segundo miembro de una de las ecuaciones (12) se pueden dar arbitrariamente, es decir, que la solución de tal problema existe.

La energía de las oscilaciones y el problema de contorno para la ecuación de oscilación

Consideremos las condiciones bajo las que la tercera de las ecuaciones diferenciales básicas, la ecuación (9), tiene una única solución.

Por sencillez consideraremos la ecuación de la cuerda vibrante $\partial^2 u/\partial x^2 = (1/a^2)(\partial^2 u/\partial t^2)$, que es muy similar a la ecuación (9), diferenciándose únicamente de ella en el número de variables espaciales.

especificar una función arbitraria sobre la frontera del dominio*. Examinemos la ecuación de Laplace con un poco más detalle. Probaremos que una función armónica u, es decir, una función que verifica la ecuación de Laplace, está completamente determinada si conocemos sus valores sobre la frontera del dominio.

En primer lugar, sentemos el hecho de que una función armónica no puede tomar en el interior del dominio valores mayores que el máximo valor sobre el contorno. Más concretamente, el máximo absoluto, así como el mínimo absoluto, de una función armónica se alcanzan en la frontera del dominio.

De aquí se deducirá que, si una función armónica es constante sobre el contorno de un dominio Ω, entonces en su interior también valdrá la misma constante. Ya que si el máximo y el mínimo de una función coinciden, entonces dicha función será en todo punto igual a este valor común.

Establezcamos ahora el hecho de que el máximo y el mínimo absolutos de una función armónica no pueden alcanzarse en el interior del dominio. En primer lugar, observemos que si la laplaciana Δu de la función $u(x, y, z)$ es positiva en todo el dominio, la función no puede tener un máximo en el interior, y si es negativa, entonces no puede tener un mínimo en el interior. En un punto donde la función alcance un máximo absoluto debe tener también un máximo como función de cada una de las variables por separado para valores fijos de las demás. Así, pues, se deduce que todas las derivadas parciales de segundo orden respecto a cada una de las variables deben ser no positivas. Esto significa que su suma será no positiva, mientras que la laplaciana es positiva, lo cual es imposible. Análogamente se puede probar que si la función tiene un mínimo en algún punto interior, entonces la laplaciana no puede ser negativa en él. Esto significa que si la laplaciana es negativa en todo el dominio, la función no puede tener un mínimo en él.

Una función armónica se puede siempre transformar mediante una cantidad arbitrariamente pequeña de tal manera que tenga una laplaciana positiva o negativa; para ello basta con sumar la cantidad

$$\pm \eta r^2 = \pm \eta(x^2 + y^2 + z^2),$$

donde η es una constante arbitrariamente pequeña.

* La expresión «función arbitraria» significa aquí y en lo que sigue que no se impone a las funciones ninguna condición especial, salvo ciertos requisitos de regularidad.

La adición de una cantidad suficientemente pequeña no puede alterar la propiedad de que la función tenga un máximo o un mínimo absolutos dentro del dominio. Si una función armónica tiene un máximo en el interior del dominio, entonces añadiéndole $+\eta r^2$ obtendríamos una función con laplaciana positiva que, como probamos antes, no podría tener máximo en el interior. Esto significa que una función armónica no puede tener un máximo absoluto interior al dominio. Análogamente se puede probar que tampoco puede tener un mínimo en el interior.

Este teorema tiene un importante corolario. Dos funciones armónicas que coincidan sobre la frontera de un dominio deben coincidir en el interior. Ya que entonces la diferencia de estas funciones (que será una función armónica) se anula sobre el contorno y por tanto es igual a cero en todo punto del interior.

Vemos así que los valores de una función armónica sobre el contorno determinan completamente la función. Se puede probar (aunque no podemos hacerlo aquí con detalle) que para valores arbitrariamente prefijados sobre el contorno se puede hallar siempre una función armónica que tome dichos valores.

Algo más complicado es probar que la temperatura estacionaria establecida en un cuerpo viene completamente determinada si conocemos las variación de flujo de calor a través de cada elemento de superficie del cuerpo o una ley que relacione el flujo de calor con la temperatura. Volveremos a algunos aspectos de esta cuestión cuando discutamos los métodos de resolución de problemas de la física matemática.

El problema de contorno para la ecuación del calor

Una situación completamente diferente la tenemos en el problema de la ecuación del calor en el caso no estacionario. Físicamente está claro que los valores de la temperatura sobre el contorno o la variación de flujo de calor a través de la frontera no son suficientes para definir la unidad de la solución del problema. Pero, si además conocemos la distribución de la temperatura en un instante inicial, entonces el problema sí está unívocamente determinado. Por tanto, para determinar la solución de la ecuación de conducción de calor (8) es normalmente necesario y suficiente designar una función arbitraria $T_0(x, y, z)$ que describa la distribución inicial de temperatura y también una función arbitraria sobre la frontera del dominio. Como antes, esta función puede ser, o la temperatura sobre la superficie del cuerpo, o la variación de flujo de calor a través de cada elemento de superficie, o una ley que relacione el flujo de calor con la temperatura.

Las ecuaciones de Laplace y Poisson; funciones armónicas y unicidad de solución de los problemas de contorno relativos a ellas

Analicemos estos problemas con un poco más de detalle. Comencemos con las ecuaciones de Laplace y Poisson. La *ecuación de Poisson* es [1]

$$\Delta u = -4\pi\varrho,$$

donde ϱ es usualmente la densidad. En particular, ϱ puede ser nula. Para $\varrho = 0$ obtenemos la *ecuación de Laplace*

$$\Delta u = 0.$$

No es difícil ver que la diferencia entre dos soluciones particulares u_1 y u_2 de la ecuación de Poisson es una función que verifica la ecuación de Laplace, o, dicho con otras palabras, que es una *función armónica*. El espacio total de soluciones de la ecuación de Poisson se reduce, por tanto, al espacio de funciones armónicas.

Una vez obtenida una solución particular u_0 de la ecuación de Poisson, si definimos una nueva función desconocida w por

$$u = u_0 + w,$$

vemos que w debe verificar la ecuación de Laplace; y exactamente de la misma manera determinamos las correspondientes condiciones de contorno para w. Así, pues, es particularmente importante estudiar los problemas de contorno para la ecuación de Laplace.

El caso más frecuente en los problemas matemáticos es que el planteamiento del problema para una ecuación de la física matemática venga inmediatamente sugerido por la situación práctica. Las condiciones suplementarias que aparecen en la solución de la ecuación de Laplace provienen del enunciado físico del problema.

Consideremos, por ejemplo, el establecimiento de una temperatura estacionaria en un medio, es decir, la propagación del calor en un medio en que las fuentes de calor son constantes y están situadas dentro o fuera del medio. En estas condiciones, con el transcurso del tiempo la temperatura alcanzada en un punto de dicho medio será independiente del tiempo. Así, pues, para hallar la temperatura T en cada punto debemos calcular aquella solución de la ecuación

$$\frac{\partial T}{\partial t} = \Delta T + q,$$

[1] El símbolo Δu es una abreviatura de la expresión $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 + \partial^2u/\partial z^2$ y se denomina *laplaciana* de la función u.

(donde q es la densidad de fuentes de distribución de calor) que es independiente de t. Obtenemos

$$\Delta T + q = 0.$$

Por tanto, la temperatura en nuestro medio satisface la ecuación de Poisson. Si la densidad q de fuentes de calor es cero, entonces la ecuación de Poisson se convierte en la de Laplace.

Para calcular la temperatura en el interior del medio es necesario, por simples consideraciones físicas, conocer también lo que ocurre en la frontera del mismo.

Evidentemente, las leyes físicas que consideramos anteriormente para puntos interiores de un cuerpo necesitan formulación bien distinta para puntos del contorno.

En el problema de establecer un estado estacionario de temperatura podemos imponer, o la distribución de la misma en el contorno, o la variación de flujo de calor a través de una unidad de área de la superficie, o, finalmente, una ley que relacione la temperatura con el flujo de calor.

Considerando la temperatura en un volumen Ω limitado por la superficie S, podemos escribir estas tres condiciones así:

$$T|_S = \varphi(Q), \tag{10}$$

o bien

$$\frac{\partial T}{\partial n}\Big|_S = \psi(Q), \tag{10'}$$

o, finalmente, en el caso más general

$$\alpha \frac{\partial T}{\partial n}\Big|_S + \beta T|_S = \chi(Q), \tag{10''}$$

donde Q denota un punto arbitrario de la superficie S. Las condiciones de la forma (10) se denominan *condiciones de contorno*. El estudio de las ecuaciones de Laplace y Poisson bajo condiciones de contorno de uno de estos tipos demostrará que, por regla general, la solución está unívocamente determinada.

Por tanto, en nuestra búsqueda de una solución de la ecuación de Laplace o de la de Poisson, será normalmente necesario y suficiente cuerpos calientes a los fríos. Así, pues, el vector flujo de calor es de dirección opuesta al llamado vector gradiente de temperatura. También es natural suponer, como justifica la experiencia, que en una primera aproximación la longitud de dicho vector es directamente proporcional al gradiente de temperatura.

Las componentes del gradiente de temperatura son

$$\frac{\partial T}{\partial x},\ \frac{\partial T}{\partial y},\ \frac{\partial T}{\partial z}.$$

Tomando k como coeficiente de proporcionalidad, obtenemos tres ecuaciones

$$\tau_x = -k\frac{\partial T}{\partial x},\quad \tau_y = -k\frac{\partial T}{\partial y},\quad \tau_z = -k\frac{\partial T}{\partial z}$$

Estas ecuaciones hay que resolverlas junto con la ecuación de conservación de la energía calorífica

$$C\frac{\partial T}{\partial t} + \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_z}{\partial z} = q.$$

Sustituyendo τ_x, τ_y y τ_z por sus valores en función de T, obtenemos

$$C\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + q. \qquad (8)$$

3. Finalmente, para pequeñas vibraciones en un medio gaseoso, por ejemplo las vibraciones sonoras, la ecuación

$$\frac{\partial^2 \varrho}{\partial t^2} + \varrho\frac{\partial}{\partial x}\left(\frac{dv_x}{dt}\right) + \varrho\frac{\partial}{\partial y}\left(\frac{dv_y}{dt}\right) + \varrho\frac{\partial}{\partial z}\left(\frac{dv_z}{dt}\right) = 0$$

y las ecuaciones de la dinámica (5) dan

$$\varrho\frac{dv_x}{dt} + \frac{\partial p}{\partial x} = F_x,\quad \varrho\frac{dv_y}{dt} + \frac{\partial p}{\partial y} = F_y,\quad \varrho\frac{dv_z}{dt} + \frac{\partial p}{\partial z} = F_z,$$

y, suponiendo la ausencia de fuerzas externas ($F_x = F_y = F_z = 0$), obtenemos

$$\frac{\partial^2 p}{\partial t^2} = a^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right) \qquad (9)$$

(para obtener esta ecuación basta sustituir la expresión de las aceleraciones en la ecuación de continuidad y eliminar la densidad utilizando la ley de Boyle-Mariotte: $p = a^2\varrho$).

Las ecuaciones (7), (8) y (9) son típicas de muchos problemas de física matemática aparte de los considerados aquí. El hecho de que hayan sido estudiados con detalle nos permite tener un conocimiento de muchas situaciones físicas.

§3. *Problemas iniciales y de contorno; unicidad de una solución*

Tanto en las ecuaciones diferenciales en derivadas parciales como en las ordinarias ocurre, salvo raras excepciones, que toda ecuación tiene infinitas soluciones particulares. Así, pues, para resolver un problema físico concreto, es decir, para calcular una función desconocida que verifique una ecuación dada, debemos saber cómo elegir, en el conjunto infinito de soluciones, la que nos interesa. Para este propósito suele ser necesario conocer no sólo la ecuación, sino también un cierto número de condiciones suplementarias. Como vimos anteriormente, las ecuaciones en derivadas parciales expresan las leyes elementales de la mecánica o la física referentes a pequeñas partículas situadas en un medio. Pero, si queremos predecir el curso de un proceso, no basta con conocer las leyes de la mecánica. Por ejemplo, para predecir el movimiento de los astros es preciso conocer no sólo la formulación general de las leyes de Newton, sino también —suponiendo que las masas de estos cuerpos son conocidas— el estado inicial del sistema, es decir, la posición de los astros y sus velocidades en algún instante inicial. Condiciones suplementarias de esta clase se encuentran siempre en la resolución de problemas de física matemática.

Así, pues, los problemas de física matemática consisten en el cálculo de soluciones de ecuaciones en derivadas parciales que satisfagan ciertas condiciones suplementarias.

Las ecuaciones (7), (8) y (9) difieren en su estructura; de ahí que también sean diferentes los problemas físicos que pueden ser resueltos por medio de estas ecuaciones.

Esta es la ecuación usual que expresa la segunda ley de la mecánica en forma integral. Es fácil transformarla en forma diferencial. Es evidente que:

$$\varrho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(T \operatorname{sen} \phi).$$

Recurriendo a teoremas bien conocidos del cálculo diferencial, es fácil relacionar $T \operatorname{sen} \phi$ con la función desconocida u. Obtenemos

$$\operatorname{tg} \phi = \frac{\partial u}{\partial x}, \quad \operatorname{sen} \phi = \frac{\operatorname{tg} \phi}{\sqrt{1 + \operatorname{tg}^2 \phi}} = \frac{\partial u/\partial x}{\sqrt{1 + (\partial u/\partial x)^2}},$$

y bajo la hipótesis de que $(\partial u/\partial x)^2$ es pequeño, tenemos

$$\operatorname{sen} \phi \approx \frac{\partial u}{\partial x}.$$

Entonces

$$T \frac{\partial^2 u}{\partial x^2} = \varrho \frac{\partial^2 u}{\partial t^2}. \tag{6}$$

Esta última es la *ecuación de la cuerda vibrante* en forma diferencial.

Formas básicas de las ecuaciones de la física matemática

Como ya mencionamos anteriormente, las distintas ecuaciones en derivadas parciales que describen fenómenos físicos forman normalmente un sistema de ecuaciones en varias variables desconocidas. Pero en la gran mayoría de los casos es posible reemplazar este sistema por una sola ecuación, como se puede probar fácilmente con ejemplos muy sencillos.

Por ejemplo, volvamos a las ecuaciones del movimiento consideradas en la sección anterior y tratemos de resolverlas junto con la ecuación de continuidad. Los métodos efectivos de resolución los consideraremos más adelante.

1. Comencemos con la ecuación del flujo uniforme de un fluido ideal.

Los movimientos posibles de un fluido se pueden dividir en rotacionales e irrotacionales, estos últimos también llamados *potenciales*. Aunque los irrotacionales son sólo casos especiales del movimiento

y, por lo general, el movimiento de un líquido o gas es siempre más o menos rotacional, sin embargo la experiencia prueba que en muchos casos el movimiento es prácticamente irrotacional. Por otra parte, se puede demostrar mediante consideraciones teóricas que en un fluido con viscosidad nula un movimiento que es inicialmente irrotacional persiste en este estado.

Para un movimiento potencial, existe una función escalar $U(x, y, z, t)$, llamada *potencial de velocidades*, tal que el vector velocidad viene expresado en términos de esta función mediante las fórmulas

$$v_x = \frac{\partial U}{\partial x}, \quad v_y = \frac{\partial U}{\partial y}, \quad v_z = \frac{\partial U}{\partial z}.$$

En todos los casos estudiados hasta ahora hemos tratado con sistemas de cuatro ecuaciones con cuatro funciones desconocidas o, dicho con otras palabras, con una ecuación escalar y una ecuación vectorial que contenían un campo escalar y un campo vectorial desconocidos. Normalmente estas ecuaciones se pueden combinar en una sola con una función desconocida, aunque esta ecuación será de segundo orden. Hagámoslo, empezando por el caso más simple.

Para un movimiento potencial de un fluido incompresible, para el que $\partial \varrho/\partial t = 0$, tenemos dos sistemas de ecuaciones: la ecuación de continuidad

$$\varrho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

y las ecuaciones del movimiento potencial

$$v_x = \frac{\partial U}{\partial x}, \quad v_y = \frac{\partial U}{\partial y}, \quad v_z = \frac{\partial U}{\partial z}.$$

Sustituyendo en la primera los valores de la velocidad que tenemos en la segunda, queda

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{7}$$

2. El campo vectorial de «flujo de calor» puede también expresarse, mediante ecuaciones diferenciales, en términos de una magnitud escalar, la temperatura. Es bien sabido que el calor «fluye» de los

Sumando sobre todas las partículas del volumen Ω, obtenemos que el ritmo de variación de la cantidad de movimiento es igual a

$$\iiint_{\Omega} \varrho \frac{dv}{dt}\, d\Omega,$$

o, lo que es lo mismo,

$$\iiint_{\Omega} \varrho \frac{dv_x}{dt}\, d\Omega, \quad \iiint_{\Omega} \varrho \frac{dv_y}{dt}\, d\Omega, \quad \iiint_{\Omega} \varrho \frac{dv_z}{dt}\, d\Omega.$$

[Aquí las derivadas dv_x/dt, dv_y/dt y dv_z/dt denotan la variación de las componentes de v, no en un punto dado del espacio, sino para una partícula dada. Este es el significado de utilizar la notación d/dt en vez de $\partial/\partial t$. Como es bien sabido, $d/dt = \partial/\partial t + v_x(\partial/\partial x) + v_y(\partial/\partial y) + v_z(\partial/\partial z)$.]

Las fuerzas que actúan sobre el volumen pueden ser de dos clases: fuerzas de volumen que actúan sobre cada partícula del cuerpo y fuerzas de superficie o tensiones sobre la superficie S que limita el volumen. Aquéllas son fuerzas de largo alcance, las últimas de corto alcance.

Para ilustrar estas consideraciones, supongamos que el medio en cuestión es un fluido. Las fuerzas de superficie que actúan sobre un elemento de área ds tendrán en este caso el valor $p\,ds$, donde p es la presión sobre el fluido, y se ejercerán en dirección opuesta a la de la normal exterior.

Si denotamos por n el vector unitario en la dirección de la normal a la superficie, entonces la fuerza total que actúa sobre la sección ds será igual a

$$-pn\, ds.$$

Si denotamos por F el vector de las fuerzas exteriores que actúan sobre una unidad de volumen, nuestra ecuación toma la forma

$$\iiint_{\Omega} \varrho \frac{dv}{dt}\, d\Omega = \iiint_{\Omega} F\, d\Omega - \iint_{S} pn\, ds.$$

Esta es la ecuación del movimiento en forma integral. Como la de continuidad, también se puede transformar en forma diferencial.

Obtenemos el sistema

$$\varrho \frac{dv_x}{dt} + \frac{\partial p}{\partial x} = F_x, \quad \varrho \frac{dv_y}{dt} + \frac{\partial p}{\partial y} = F_y, \quad \varrho \frac{dv_z}{dt} + \frac{\partial p}{\partial z} = F_z. \qquad (5)$$

Este sistema es la forma diferencial de la segunda ley de Newton.

2. Otro ejemplo característico de la aplicación de las leyes de la mecánica en forma diferencial es la ecuación de una cuerda vibrante. Una cuerda es un trozo de material elástico, alargado, muy delgado, que es flexible debido a su extrema delgadez, y que normalmente se mantiene tensado. Si imaginamos la cuerda dividida por el punto x en dos partes, sobre cada una de ellas se ejerce una fuerza igual a la tensión en la dirección de la tangente a la curva formada por la cuerda.

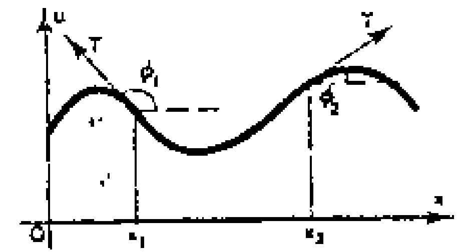

FIG. 2

Examinemos un corto segmento de la cuerda. Denotemos por $u(x, t)$ el desplazamiento de un punto de la misma desde su posición de equilibrio. Supongamos que la cuerda oscila en un plano y que tal oscilación consiste en desplazamientos perpendiculares al eje Ox. Representemos el desplazamiento $u(x, t)$ gráficamente en un instante dado (Fig. 2), y estudiemos el comportamiento del segmento de cuerda comprendido entre los puntos x_1 y x_2. En estos puntos actúan dos fuerzas iguales a la tensión T en la dirección de la correspondiente tangente a $u(x, t)$.

Si el segmento es curvo, la resultante de estas dos fuerzas no será nula. Esta resultante, por las leyes de la mecánica, debe ser igual a la variación de la cantidad de movimiento del segmento.

Sea ϱ la masa contenida en cada centímetro de longitud de la cuerda. Entonces la variación de la cantidad de movimiento será

$$\varrho \int_{x_1}^{x_2} \frac{\partial^2 u}{\partial t^2}\, dx.$$

Si el ángulo formado por la tangente a la cuerda y el eje Ox se denota por ϕ, tendremos

$$T \operatorname{sen} \phi_2 - T \operatorname{sen} \phi_1 = \varrho \int_{x_1}^{x_2} \frac{\partial^2 u}{\partial t^2}\, dx.$$

forma que $q_n\,ds$ expresaba la cantidad de materia que pasaba por segundo a través de un área ds. En lugar del flujo de líquido $q = \varrho v$ tenemos el vector flujo de calor τ.

Análogamente a como hemos obtenido la ecuación de continuidad, que expresa, para el movimiento de un líquido, la ley de conservación de la masa, podemos deducir una nueva ecuación en derivadas parciales que exprese la ley de conservación de la energía; veámoslo.

La densidad de energía calorífica Q en un punto dado puede expresarse mediante la fórmula

$$Q = CT,$$

donde C es la capacidad calorífica y T la temperatura.

Aquí es fácil obtener la ecuación

$$C\frac{\partial T}{\partial t} + \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_z}{\partial z} = 0. \qquad (2)$$

La deducción de esta ecuación es idéntica a la de la ecuación de continuidad, sustituyendo «densidad» por «densidad de energía calorífica» y flujo de masa por flujo de calor. Aquí hemos supuesto que la energía calorífica en el medio nunca aumenta. Pero, si existe una fuente de calor en él, la ecuación (2) del equilibrio de energía calorífica debe ser modificada. Si q es la densidad de productividad de la fuente, es decir, la cantidad de energía calorífica producida por unidad de volumen en un segundo, entonces la ecuación de conservación de la energía calorífica adopta la siguiente forma, algo más complicada:

$$C\frac{\partial T}{\partial t} + \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y} + \frac{\partial \tau_z}{\partial z} = q. \qquad (3)$$

3. Aun otra ecuación del mismo tipo que la de continuidad se puede obtener derivando (1) respecto al tiempo. Hagámoslo para la ecuación de pequeñas oscilaciones de un gas próximo a una posición de equilibrio. Supondremos que para tales oscilaciones los cambios de densidad no son grandes y que las cantidades $\partial\varrho/\partial x$, $\partial\varrho/\partial y$, $\partial\varrho/\partial z$ y $\partial\varrho/\partial t$ son suficientemente pequeñas para que sus productos por v_x, v_y y v_z puedan despreciarse. Entonces

$$\frac{\partial \varrho}{\partial t} + \varrho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0.$$

Derivando esta ecuación con respecto al tiempo y despreciando los productos de $\partial\varrho/\partial t$ con $\partial v_x/\partial x$, $\partial v_y/\partial y$ y $\partial v_z/\partial z$, obtenemos

$$\frac{\partial^2 \varrho}{\partial t^2} + \varrho\left[\frac{\partial\left(\frac{dv_x}{dt}\right)}{\partial x} + \frac{\partial\left(\frac{dv_y}{dt}\right)}{\partial y} + \frac{\partial\left(\frac{dv_z}{dt}\right)}{\partial z}\right] = 0. \qquad (4)$$

Ecuación del movimiento

1. Un importante ejemplo de la expresión de una ley física mediante una ecuación diferencial se da en las ecuaciones de equilibrio o de movimiento de un medio. Sea un medio formado por partículas materiales moviéndose con distintas velocidades. Como en el primer ejemplo, delimitemos mentalmente en el espacio un volumen Ω encerrado por la superficie S y lleno de partículas de materia del medio, y escribamos la segunda ley de Newton para las partículas de este volumen. Dicha ley establece que la variación de la cantidad de movimiento en el volumen, sumada para todas las partículas, es igual a la suma de todas las fuerzas que actúan sobre el volumen. La cantidad de movimiento, como se sabe por la mecánica, está representada por el vector

$$P = \iiint_{\Omega} \varrho v\, d\Omega.$$

Las partículas que ocupan un pequeño volumen $d\Omega$ con densidad ϱ llenarán, al cabo de un tiempo Δt, un nuevo volumen $d\Omega'$ con densidad ϱ', si bien la masa será la misma

$$\varrho'\, d\Omega' = \varrho\, d\Omega.$$

Si la velocidad v se transforma durante ese tiempo en una nueva velocidad v', es decir, varía en el incremento $\Delta v = v' - v$, el correspondiente incremento de la cantidad de movimiento será

$$\varrho' v'\, d\Omega' - \varrho v\, d\Omega = \varrho v'\, d\Omega - \varrho v\, d\Omega = \varrho\, \Delta v\, d\Omega,$$

o en la unidad de tiempo:

$$\varrho\frac{\Delta v}{\Delta t}\, d\Omega \approx \varrho\frac{dv}{dt}\, d\Omega.$$

Para calcular la variación de flujo de materia hacia el exterior del volumen Ω basta dividir esta expresión por dt, de modo que para la variación de flujo tenemos

$$\iint_S \varrho v_n \, ds = \iint_S q_n \, ds,$$

donde

$$v_n = v \cos(n, v), \quad q_n = q \cos(n, q).$$

La componente normal del vector v puede reemplazarse por su expresión en términos de las componentes de los vectores v y n sobre los ejes de coordenadas. Por geometría analítica sabemos que

$$v_n = v \cos(n, v) = v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z);$$

así, pues, podemos volver a escribir la expresión de la variación de flujo en la forma

$$\iint_S \varrho [v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z)] \, ds.$$

Por la ley de conservación de la materia, estos dos métodos de calcular la variación de la cantidad de materia deben dar el mismo resultado, ya que todo cambio en la masa contenida en Ω sólo puede ser debido a la entrada o salida de masa a través de la superficie S.

Entonces, igualando la variación de la cantidad de materia contenida en el volumen a la variación de flujo de materia dentro del volumen, obtenemos

$$\iiint_\Omega \frac{\partial \varrho}{\partial t} \, dx\, dy\, dz =$$

$$= -\iint_S [\varrho v_x \cos(n, x) + \varrho v_y \cos(n, y) + \varrho v_z \cos(n, z)] \, ds =$$

$$= -\iint_S [q_x \cos(n, x) + q_y \cos(n, y) + q_z \cos(n, z)] \, ds.$$

Esta relación integral, como hemos dicho, es válida para todo volumen Ω. Se denomina «ecuación de continuidad».

La integral del segundo miembro de la última ecuación se puede transformar en una integral de volumen mediante la fórmula de Ostrogradski. Esta fórmula, deducida en el capítulo 2, nos da

$$\iint_S [\varrho v_x \cos(n, x) + \varrho v_y \cos(n, y) + \varrho v_z \cos(n, z)] \, ds =$$

$$= \iiint_\Omega \left[\frac{\partial(\varrho v_x)}{\partial x} + \frac{\partial(\varrho v_y)}{\partial y} + \frac{\partial(\varrho v_z)}{\partial z} \right] d\Omega.$$

De aquí se deduce que

$$\iiint_\Omega \left[\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_x)}{\partial x} + \frac{\partial(\varrho v_y)}{\partial y} + \frac{\partial(\varrho v_z)}{\partial z} \right] d\Omega = 0.$$

De esta forma obtenemos el siguiente resultado: la integral de la función

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_x)}{\partial x} + \frac{\partial(\varrho v_y)}{\partial y} + \frac{\partial(\varrho v_z)}{\partial z} \text{ ó } \frac{\partial \varrho}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

sobre cualquier volumen Ω es cero. Pero esto sólo es posible si la función es idénticamente nula. De este modo obtenemos la ecuación de continuidad en forma diferencial

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho v_x)}{\partial x} + \frac{\partial(\varrho v_y)}{\partial y} + \frac{\partial(\varrho v_z)}{\partial z} = 0 \qquad (1)$$

La ecuación (1) es un típico ejemplo de la formulación de una ley física en el lenguaje de las ecuaciones en derivadas parciales.

2. Consideremos otro problema análogo: el problema de la conducción del calor.

En un medio cuyas partículas están en movimiento, el calor fluye de unos puntos a otros. Este flujo de calor tiene lugar a través de cada elemento de superficie ds del medio considerado. Es posible probar que el proceso se puede describir numéricamente por medio de una única magnitud vectorial, el vector conducción de calor, que se denota por τ. Entonces la cantidad de calor que fluye por segundo a través de un elemento de área ds se podrá expresar por $\tau_n\, ds$, de la misma

§2. *Las ecuaciones más simples de la física matemática*

Las conexiones y relaciones elementales entre magnitudes físicas están expresadas por las leyes de la mecánica y de la física. Si bien estas relaciones son extremadamente variadas en su carácter, dan lugar a otras más complicadas que se derivan de ellas mediante argumentos matemáticos y que son aún más variadas. Las leyes de la mecánica y la física se pueden escribir en lenguaje matemático en forma de ecuaciones en derivadas parciales, o quizá ecuaciones integrales, que relacionan entre sí funciones desconocidas. Para comprender lo que esto significa, consideremos algunos ejemplos de ecuaciones de la física matemática.

Ecuaciones de la conservación de la masa y de la energía calorífica

Expresemos en forma matemática las leyes físicas básicas que gobiernan los movimientos de un medio.

1. En primer lugar expresemos la ley de conservación de la materia contenida en un volumen Ω que delimitamos mentalmente en el espacio y mantenemos fijo. Para este propósito debemos calcular la masa de la materia contenida en dicho volumen. La masa $M_\Omega(t)$ viene expresada por la integral

$$M_\Omega(t) = \iiint_\Omega \varrho(x, y, z, t)\,dx\,dy\,dz.$$

Esta masa no será, por supuesto, constante; en un proceso oscilatorio la densidad en cada punto variará, puesto que, en sus oscilaciones, las partículas de materia estarán unas veces dentro del volumen y otras fuera. La variación de la masa se puede hallar derivando respecto al tiempo, y viene dada por la integral

$$\frac{dM_\Omega}{dt} = \iiint_\Omega \frac{\partial\varrho}{\partial t}\,dx\,dy\,dz.$$

Esta tasa de variación de la masa contenida en el volumen se puede calcular también de otra manera. Podemos expresar la cantidad de materia que pasa cada segundo a través de la superficie S que limita nuestro volumen, afectando a la materia que sale de Ω de un signo

menos. Con este fin consideremos en la superficie S un elemento ds suficientemente pequeño para que se pueda suponer plano y para que tenga el mismo desplazamiento en todos sus puntos. Observamos el desplazamiento de los puntos de un segmento de la superficie durante el intervalo de tiempo de t a $t + dt$. Primeramente calculamos el vector

$$\mathbf{v} = \frac{du}{dt},$$

que representa la velocidad de cada partícula. En el tiempo dt las partículas que están sobre ds se mueven según el vector $\mathbf{v}\,dt$ y tocan la posición ds_1, mientras que la posición ds estará ocupada ahora por las partículas que anteriormente estaban en ds_2 (Fig. 1). Por tanto, durante este tiempo la columna de materia que sale del volumen Ω será la que antes estaba contenida entre ds_2 y ds. La altura de esta columna es igual a $v\,dt\cos(n, v)$, donde n es la normal exterior a la superficie; el volumen de la pequeña columna será pues igual a

$$v\cos(n, v)\,ds\,dt,$$

y la masa igual a

$$\varrho v\cos(n, v)\,ds\,dt.$$

Uniendo todos estos pequeños trozos obtenemos, para la cantidad de materia que sale del volumen durante el tiempo dt, la expresión

$$\iint_S \varrho v\cos(n, v)\,ds\,dt.$$

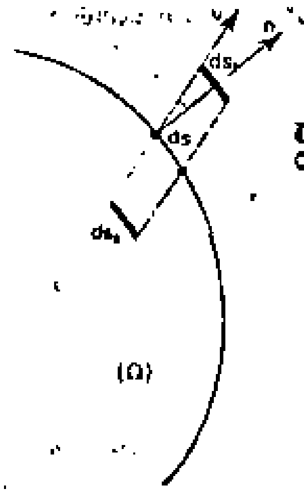

Fig. 1

En aquellos puntos en que la velocidad está dirigida hacia el interior de Ω el signo del coseno será negativo, lo que significa que en esta integral la materia que entra en Ω va afectada de un signo menos. El producto de la velocidad del movimiento del medio por su densidad se denomina flujo. El vector flujo de la masa es $\mathbf{q} = \varrho\mathbf{v}$.

un volumen arbitrariamente elegido del cuerpo ejerce el resto del mismo.

Los sucesos y procesos físicos que ocurren en el espacio y el tiempo siempre consisten en variaciones, a lo largo del tiempo, de ciertas magnitudes físicas relativas a los puntos del espacio. Como hemos visto en el capítulo 2, estas cantidades pueden ser descritas por funciones de cuatro variables independientes, x, y, z y t, donde x, y y z son las coordenadas de un punto del espacio, y t es el tiempo.

Las magnitudes físicas pueden ser de diferentes clases. Algunas están completamente caracterizadas por sus valores numéricos, como la temperatura, la densidad y otras análogas, y se denominan escalares. Otras tienen dirección y son por lo tanto magnitudes vectoriales: velocidad, aceleración, intensidad de un campo eléctrico, etcétera. Las magnitudes vectoriales se pueden expresar por la longitud del vector y su dirección, pero también por sus «componentes», una vez descompuesto el vector en suma de tres vectores perpendiculares entre sí, por ejemplo, paralelos a los ejes de coordenadas.

En física matemática una magnitud escalar o un campo escalar se representa por una función de cuatro variables independientes, mientras que una magnitud vectorial definida sobre todo el espacio o, como se suele decir, un campo vectorial está descrito por tres funciones de estas variables. Podemos escribir tal magnitud en la forma

$$\mathbf{u}(x, y, z, t)$$

donde el tipo negrita indica que $\mathbf{u}$ es un vector, o en forma de tres funciones

$$u_x(x, y, z, t), \quad u_y(x, y, z, t), \quad u_z(x, y, z, t),$$

donde u_x, u_y y u_z son las proyecciones del vector sobre los ejes de coordenadas.

Además de las magnitudes vectoriales y escalares, aparecen en física entes aún más complicados, como es, por ejemplo, la expresión del estado de tensión de un cuerpo en un punto dado. Tales magnitudes se llaman tensores; una vez fijada la elección de los ejes de coordenadas, se pueden caracterizar por un conjunto de funciones de las cuatro variables independientes.

De esta forma, la descripción de clases muy distintas de fenómenos físicos viene normalmente dada por medio de varias funciones de varias variables. Por supuesto, tal descripción no puede ser absolutamente exacta.

Por ejemplo, cuando describimos la densidad de un medio a través de una función de cuatro variables independientes, ignoramos el hecho de que en un punto concreto podemos no tener densidad alguna. Los cuerpos que estudiamos tienen una estructura molecular, y las moléculas no están contiguas, sino que existen distancias finitas entre una y otra. Las distancias entre las moléculas son en general considerablemente más grandes que las dimensiones de las mismas. Así, pues, la densidad en cuestión es la razón de la masa contenida en un pequeño (aunque no extremadamente pequeño) volumen a este mismo volumen. La densidad en un punto podemos definirla como el límite de tales razones para volúmenes decrecientes. Una simplificación e idealización aún mayores se introducen en el concepto de temperatura de un medio. El calor en un cuerpo es debido al movimiento aleatorio de sus moléculas. La energía de las distintas moléculas es también distinta, pero si consideramos un volumen que contenga un gran número de ellas, entonces la energía media de sus movimientos aleatorios definirá lo que se denomina temperatura.

Análogamente, cuando hablamos de presión de un gas o un líquido sobre la pared de un recipiente, no pensamos en ella como si una partícula del líquido o gas estuviera realmente presionando contra la pared de dicho recipiente. En realidad, estas partículas, en su movimiento aleatorio, chocan contra la pared y rebotan en ella. Por tanto, entendemos por presión sobre la pared el efecto total de un gran número de impulsos recibidos por una parte de la misma; parte que es pequeña desde cualquier punto de vista macroscópico, pero extremadamente grande en comparación con las distancias entre las moléculas del líquido o gas. Sería fácil citar docenas de ejemplos de naturaleza similar. La mayoría de las magnitudes estudiadas en la física tienen exactamente el mismo carácter. La física matemática trabaja con magnitudes ideales, obtenidas mediante abstracción de las propiedades concretas de los entes físicos correspondientes y considerando solamente los valores medios de estas magnitudes.

Tal idealización puede parecer algo burda, pero, como veremos, es muy útil, ya que permite hacer un excelente análisis de muchas materias complicadas, en las cuales consideramos solamente los elementos esenciales y omitimos esas características que son secundarias desde nuestro punto de vista.

El objeto de la física matemática es el estudio de las relaciones que existen entre estos elementos idealizados; dichas relaciones se describen por medio de conjuntos de funciones de varias variables independientes.

Capítulo 6

ECUACIONES DIFERENCIALES EN DERIVADAS PARCIALES

§1. *Introducción*

En el estudio de los fenómenos naturales, las ecuaciones diferenciales en derivadas parciales aparecen con tanta frecuencia como las ordinarias. Por regla general esto ocurre en casos en que un suceso viene descrito por una función de varias variables.

Del estudio de la naturaleza surgió el tipo de ecuaciones en derivadas parciales, que en el momento presente está siendo investigado más a fondo y que probablemente es el más importante en la estructura general del saber humano: las ecuaciones de la física matemática.

Consideremos en primer lugar oscilaciones en un medio cualquiera. En virtud de tales oscilaciones todo punto del medio, que en equilibrio ocupa la posición (x, y, z), se habrá desplazado al cabo de un tiempo t según el vector $u(x, y, z, t)$, que depende de la posición inicial del punto (x, y, z) y del tiempo t. En este caso el proceso en cuestión estará descrito por un campo vectorial. Pero es fácil ver que el conocimiento de este campo vectorial, esto es, el campo de desplazamiento de los puntos del medio, no es suficiente para una descripción completa de la oscilación. También es necesario conocer, por ejemplo, la densidad $\rho(x, y, z, t)$ en cada punto del medio, la temperatura $T(x, y, z, t)$ y la tensión interna, por ejemplo, la fuerza que sobre

For the continuous initial conditions u and $\partial u/\partial n$ (and u in the second case), the solutions will be regular or satisfy the differential equation everywhere in the rectangles described above. Discontinuities in u and $\partial u/\partial n$ will propagate along the characteristics in the rectangle.

A great deal of work was done in the second half of the century on the existence of eigenvalues for $\Delta u + k^2 u = 0$ considered in a domain D. The main result is that for a given domain and under any one of the three boundary conditions $u = 0$, $\partial u/\partial n = 0$, $\partial u/\partial n + hu = 0$ ($h > 0$ when the positive normal is directed outside the domain), there are always an infinite number of discrete values of k^2, for each of which there is a solution. In two dimensions the vibrations of a membrane fixed along its boundary illustrate this theorem. The values of k are the frequencies of the infinitely many purely harmonic vibrations. The corresponding solutions give the deformation of the membrane in carrying out its characteristic oscillations.

The first major step was the proof by Schwarz[57] of the existence of the first eigenfunction of

$$\Delta u + \xi f(x, y)u = 0,$$

that is, the existence of a U_1 such that

$$\Delta U_1 + k_1^2 f(x, y)U_1 = 0$$

and $U_1 = 0$ on the boundary of the domain considered. His method gave a procedure for finding the solution and permitted the calculation of k_1^2. Picard[58] then established the existence of the second eigenvalue k_2^2.

Schwarz also showed in the 1885 paper that when the domain varies continuously, the value of k_1^2, the first characteristic number, also varies continuously; and as the domain becomes smaller, k_1^2 increases unboundedly. Thus a smaller membrane gives off a higher first harmonic.

In 1894 Poincaré[59] demonstrated the existence and the essential properties of all the eigenvalues of

$$\Delta u + \lambda u = f, \tag{60}$$

λ complex, in a bounded, three-dimensional domain, with $u = 0$ on the boundary. The existence of u was demonstrated by a generalization of Schwarz's method. He proved next that $u(\lambda)$ is a meromorphic function of the complex variable λ and that the poles are real; these are just the eigenvalues λ_n. Then he obtained the characteristic solutions U_i, that is,

$$\Delta U_i + k_i^2 U_i = 0 \quad \text{(in the interior)}$$
$$U_i = 0 \quad \text{(on the boundary)}.$$

57. *Acta Soc. Fennicae*, 15, 1885, 315–62 = *Ges. Math. Abh.*, 1, 223–69.
58. *Comp. Rend.*, 117, 1893, 502–7.
59. *Rendiconti del Circolo Matematico di Palermo*, 8, 1894, 57–155 = *Œuvres*, 9, 123–96.

The k_i^2 are the characteristic numbers (eigenvalues) and determine the frequencies of the respective characteristic solutions.

Physically Poincaré's result has the following significance. The function f in (60) can be thought of as an applied force. The free oscillations of a mechanical system are those at which the forced oscillations degenerate and become infinite. In fact, (60) is the equation of an oscillating system excited by a periodically varying force of amplitude f; and the characteristic solutions are the free oscillations of the system, which, once excited, continue indefinitely. The frequencies of the free oscillations, which are proportional to the k_i, are calculated by Poincaré's method as the values of $\sqrt{\lambda}$ for which the forced oscillation u becomes infinite.

At the end of the century the systematic theory of boundary- and initial-value problems for partial differential equations, which dates from Schwarz's fundamental 1885 paper, was still young. The work in this area expanded rapidly in the twentieth century.

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Langer, R. E.: "Fourier Series, the Genesis and Evolution of a Theory," *Amer. Math. Monthly*, 54, Part II, 1947, 1–86.

Pockels, Friedrich: *Über die partielle Differentialgleichung $\Delta u + k^2 u = 0$*, B. G. Teubner, 1891.

problem in two dimensions (but not of the Dirichlet principle of minimizing the Dirichlet integral) was given by Hermann Amandus Schwarz (1843–1921), a pupil of Weierstrass, whom he succeeded at Berlin in 1892 and who suggested the problem to him. Under general assumptions about the bounding curve, and by using a process called the alternating procedure,[46] he demonstrated the existence of a solution.[47]

In the same year, 1870, Carl G. Neumann gave another proof of the existence of a solution of the Dirichlet problem in three dimensions[48] by using the method of arithmetic means, though he too did not use the Dirichlet principle.[49] The principal exposition of his ideas is in his *Vorlesungen über Riemann's Theorie der Abel'schen Integrale*.[50]

Then Henri Poincaré[51] used the *méthode de balayage*, the method of "sweeping out," which approaches the problem by building a succession of functions not harmonic in the domain R but taking on the correct boundary values, the functions becoming more and more harmonic.

Finally, David Hilbert reconstructed the calculus of variations method of Thomson and Dirichlet and established the Dirichlet *principle* as a method for proving the existence of a solution of the Dirichlet problem. In 1899[52] Hilbert showed that under proper conditions on the region, boundary values, and the admissible functions U, the Dirichlet principle does hold. He made the Dirichlet principle a powerful tool in function theory. In another publication, of work done in 1901,[53] Hilbert gave more general conditions.

The history of the Dirichlet principle is remarkable. Green, Dirichlet, Thomson, and others of their time regarded it as a completely sound method and used it freely. Then Riemann in his complex function theory showed it to be extraordinarily instrumental in leading to major results. All of these men were aware that the fundamental existence question was not settled, even before Weierstrass announced his critique in 1870, which discredited the method for several decades. The principle was then rescued by Hilbert and was used and extended in this century. Had the progress made with the use of the principle awaited Hilbert's work, a large segment

46. Schwarz's method is sketched in Felix Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Chelsea (reprint), 1950, 1, p. 265, and given fully in A. R. Forsyth, *Theory of Functions*, Dover (reprint), 1965, 2, Chap. 17. Many references are given in the latter source.

47. *Monatsber. Berliner Akad.*, 1870, 767–95 = *Ges. Math. Abh.*, 2, 144–71.

48. *Königlich Sächsischen Ges. der Wiss. zu Leipzig*, 1870, 49–56, 264–321.

49. The method is described in O. D. Kellogg, *Foundations of Potential Theory*, Julius Springer, 1929, 281 ff.

50. Second ed., 1884, 238 ff.

51. *Amer. Jour. of Math.*, 12, 1890, 211–94 = *Œuvres*, 9, 28–113.

52. *Jahres. der Deut. Math.-Verein.*, 8, 1900, 184–88 = *Ges. Abh.*, 3, 10–14.

53. *Math. Ann.*, 59, 1904, 161–85 = *Ges. Abh.*, 3, 15–37.

of nineteenth-century work on potential theory and function theory would have been lost.

The Laplace equation $\Delta V = 0$ is the basic form of elliptic differential equations. Many more existence theorems were established for more general elliptic differential equations, such as

$$(59) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0.$$

The variety of such theorems is vast. We shall mention just one key result. The existence and uniqueness of a solution of this equation (the value of the solution is prescribed on the boundary) was demonstrated by Picard[54] for domains of sufficiently small area. The result has been extended to more variables, large domains, and in other respects by Picard and others. Picard also established[55] that every equation of the above form (and even slightly more general) whose coefficients are analytic functions possesses only analytic solutions inside the domain in which the solution is sought, even though the solution assumes non-analytic boundary values.

The theorems discussed thus far have generally dealt with analytic differential equations and analytic initial or boundary data. However, such conditions are too restrictive for applications because the given physical data may not be analytic. Another major class of theorems deals with less stringent conditions. We shall give just one example. Riemann's method, which applies to the hyperbolic equation, relies upon the existence of his characteristic function v and, as we pointed out, the existence of v was not established by Riemann.

For this hyperbolic case (see [40]), Du Bois-Reymond in 1889 sought the proper conditions and obtained results[56] which, expressed for the case where x = const. and y = const. are the characteristics, read thus: Given the continuous functions u and $\partial u/\partial n$ along a curve AB that is cut not more than once by any characteristic line, then there exists one and only one solution u of the differential equation which takes on the given values of u and $\partial u/\partial n$ along AB. This solution is defined in the rectangle determined by the characteristics through A and through B. If instead the values of a continuous function u on two segments of characteristics which abut one another are given, then u is again uniquely determined in the rectangle determined by the characteristics. In terms of x, y, and u as spatial coordinates, the first result states that the surface $u(x, y)$ goes through a given space curve and with a given inclination. The second result means that the solution or surface is enclosed in the space determined by two intersecting space curves.

54. *Comp. Rend.*, 107, 1888, 939–41, *Jour. de Math.*, (4), 6, 1890, 145–210, *Jour. de Math.*, (5), 2, 1896, 295–304.

55. *Jour. de l'École Poly.*, 60, 1890, 89–105.

56. *Jour. für Math.*, 104, 1889, 241–301.

where, as usual, $r = \partial^2 z/\partial x^2$ and where f is analytic in its variables. In this case one must specify on the initial line $x = 0$ that

$$z(0, y) = z_0(y), \qquad \frac{\partial z}{\partial x}(0, y) = z_1(y),$$

where z_0 and z_1 are analytic. (The initial line may be replaced by a curve, in which case $\partial z/\partial x$ must be replaced by $\partial z/\partial n$.) If the above conditions are fulfilled, then the solution $z = z(x, y)$ exists and is unique and analytic in some domain starting at the initial line.

Cauchy's work on systems was done independently and in somewhat improved form by Sophie Kowalewsky (1850–91),[41] who was a pupil of Weierstrass and who pursued his ideas. Kowalewsky is one of the few women mathematicians of distinction. In 1816 Sophie Germain (1776–1831) had won a prize awarded by the French Academy for a paper on elasticity. Kowalewsky, too, won the Paris Academy's prize, for a work of 1888 on the integration of the equations of motion for a solid body rotating around a fixed point; in 1889 she became a professor of mathematics in Stockholm. The proofs of Cauchy and Kowalewsky were later improved by Goursat.[42]

If, instead of (55), the given second order equation is in the form

$$G(z, x, y, p, q, r, s, t) = 0, \tag{56}$$

it is necessary to solve for r before it can be put in the form (55). To consider a simple but vital case, if the equation is

$$G = A\frac{\partial^2 z}{\partial x^2} + 2B\frac{\partial^2 z}{\partial x\,\partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz = 0,$$

where $A, B, \ldots, F$ are functions of x and y, then $\partial G/\partial r$ must not be 0 to solve for r. In case $\partial G/\partial r = 0$, the solution of the Cauchy problem need not exist, and when it does is not unique. In the case of three or more independent variables (let us consider three), and if the equation is written as

$$\sum_{i,k} A_{ik}\frac{\partial^2 u}{\partial x_i\,\partial x_k} + \sum_i B_i\frac{\partial u}{\partial x_i} + Cu = f, \tag{57}$$

where the coefficients are functions of the independent variables x_1, x_2, and x_3, then the exceptional case occurs when the initial surface S satisfies the first order partial differential equation

$$\sum A_{ik}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_k} = 0. \tag{58}$$

41. *Jour. für Math.*, 80, 1875, 1–32.
42. *Bull. Soc. Math. de France*, 26, 1898, 129–34.

Along such surfaces two solutions of (57) can be tangent and even have higher-order contact. This property is the same as for the characteristic curves of the first order equation $f(x, y, u, p, q) = 0$ (Chap. 22, sec. 5), and so these surfaces S are also called characteristics. Physically the surfaces S are wave fronts.

This theory of characteristics for the case of two independent variables was known to Monge and to André-Marie Ampère (1775–1836). Its extension to the case of second order equations in more than two independent variables was first made by Albert Victor Bäcklund (1845–1922),[43] but was not widely known until it was redone by Jules Beudon.[44]

In his *Leçons sur la propagation des ondes* (1903), Jacques Hadamard (1865–1963), the leading French mathematician of this century, generalized the theory of characteristics to partial differential equations of any order. As an example, let us consider a system of three partial differential equations of the second order in the dependent variables ξ, η, and ζ and the independent variables $x_1, x_2, \ldots, x_n$. The Cauchy problem for this system is: Given the values of ξ, η, ζ, and $\partial\xi/\partial x_n$, $\partial\eta/\partial x_n$, and $\partial\zeta/\partial x_n$ on a "surface" M_{n-1} of $n - 1$ dimensions, to find the functions ξ, η, and ζ. The values of the second and higher derivatives of ξ, η, and ζ may then be computed unless M_{n-1} satisfies a first order partial differential equation of the sixth degree, say $H = 0$. All "surfaces" satisfying $H = 0$ are characteristic "surfaces." According to the theory of first order partial differential equations, the differential equation $H = 0$ has characteristic lines (curves) defined by

$$\frac{dx_1}{\partial H/\partial P_1} = \frac{dx_2}{\partial H/\partial P_2} = \cdots = \frac{dx_{n-1}}{\partial H/\partial P_{n-1}},$$

where $P_1, P_2, \ldots, P_{n-1}$ are the partial derivatives of x_n with respect to $x_1, x_2, \ldots, x_{n-1}$ taken along the "surface" M_{n-1}. These lines are called the bicharacteristics of the original second order system. In the theory of light they are the rays.

The characteristics now play a vital role in the theory of partial differential equations. For example, Darboux[45] has given a powerful method of integrating second order partial differential equations in two independent variables that rests on the theory of characteristics. It converts the problem to the integration of one or more ordinary differential equations and embraces the methods of Monge, Laplace, and others.

Another class of existence theorems dealt with the Dirichlet problem, that is, establishing the existence of a solution of $\Delta V = 0$ either directly or by means of the Dirichlet principle. The first existence proof of the Dirichlet

43. *Math. Ann.*, 13, 1878, 411–28.
44. *Bull. Soc. Math. de France*, 25, 1897, 108–20.
45. *Ann. de l'École Norm. Sup.*, (1), 7, 1870, 175–80.

proof of the existence of a solution would at least insure that a search for a solution would not be attempting the impossible. The proof of existence would also answer the question: What must we know about a given physical situation, that is, what initial and boundary conditions insure a solution and preferably a unique one? Other objectives, perhaps not envisaged at the beginning of the work on existence theorems, were soon recognized. Does the solution change continuously with the initial conditions, or does some totally new phenomenon enter when the initial or boundary conditions are varied slightly? Thus a parabolic orbit that obtains for one value of the initial velocity of a planet may change to an elliptic orbit as a consequence of a slight change in the initial velocity. Such a difference in orbit is physically most significant. Further, some of the methodologies of solution, such as the use of the Dirichlet principle or Green's theorem, presupposed the existence of a particular solution. The existence of these particular solutions had not been established.

Before we give some brief indications of the work on existence theorems, it may be helpful to note a classification of partial differential equations that was actually made rather late in the century. Though some efforts toward classification by reducing these equations to normal or standard forms had been made by Laplace and Poisson, the classification introduced by Du Bois-Reymond has now become standard. In 1889[39] he classified the most general homogeneous linear equation of second order

$$(54)\qquad R\frac{\partial^2 u}{\partial x^2} + S\frac{\partial^2 u}{\partial x\,\partial y} + T\frac{\partial^2 u}{\partial y^2} + P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + Zu = 0,$$

where the coefficients are functions of x and y and they and their first and second derivatives are continuous, by means of the characteristics (Chap. 22, sec. 7). The projections of the characteristic curves onto the xy-plane (these projections are also called characteristics) satisfy

$$T\,dx^2 - S\,dx\,dy + R\,dy^2 = 0.$$

The characteristics are imaginary, real and distinct, or real and coincident according as

$$TR - S^2 > 0, \qquad TR - S^2 < 0, \qquad TR - S^2 = 0.$$

Du Bois-Reymond called these cases elliptic, hyperbolic, and parabolic, respectively. He then pointed out that by introducing new real independent variables

$$\xi = \phi(x, y), \qquad \eta = \psi(x, y),$$

39. *Jour. für Math.*, 104, 1889, 241–301.

the above equation can always be transformed into one of the three types of normal forms

$$\text{(a)}\quad R'\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2}\right) + P'\frac{\partial u}{\partial \xi} + Q'\frac{\partial u}{\partial \eta} + Zu = 0$$

$$\text{(b)}\quad S'\frac{\partial^2 u}{\partial \xi\,\partial \eta} + P'\frac{\partial u}{\partial \xi} + Q'\frac{\partial u}{\partial \eta} + Zu = 0$$

$$\text{(c)}\quad R'\frac{\partial^2 u}{\partial \xi^2} + P'\frac{\partial u}{\partial \xi} + Q'\frac{\partial u}{\partial \eta} + Zu = 0$$

respectively. The two families $\phi(x, y) =$ const. and $\psi(x, y) =$ const. are the equations of two families of characteristics.

The supplementary conditions that can be imposed differ for the three types of equations. In the elliptic case (a) one considers a bounded domain of the xy-plane and specifies the value of u on the boundary (or an equivalent condition) and asks for the value of u in the domain. For the initial-value problem of the hyperbolic differential equation (b) one must specify u and $\partial u/\partial n$ on some initial curve. There may also be boundary conditions. The proper initial conditions for the parabolic case (c) were not specified at this time, though it is now known that one initial condition and boundary conditions can be imposed. This classification of partial differential equations was extended to equations in more independent variables, higher-order equations and to systems. Though the classification and the supplementary conditions were not known early in the century, the mathematicians gradually became aware of the distinctions and these figured in the existence theorems they were able to prove.

The work on existence theorems became a major activity with Cauchy, who emphasized that existence can often be established where an explicit solution is not available. In a series of papers[40] Cauchy noted that any partial differential equation of order greater than one can be reduced to a system of partial differential equations, and he treated the existence of a solution for the system. He called his method the *calcul des limites* but it is known today as the method of majorant functions. The essence of the method is to show that a power series in the independent variables with a definite domain of convergence does satisfy the system of equations. We shall illustrate the method in connection with Cauchy's work on ordinary differential equations (Chap. 29, sec. 4). His theorem covers only the case of analytic coefficients in the equations and analytic initial conditions.

To obtain some concrete idea about Cauchy's work, we shall consider what it implies for the second order equation in two independent variables

$$(55)\qquad r = f(z, x, y, p, q, s, t)$$

40. *Comp. Rend.*, 14, 1842, 1020–25 = *Œuvres*, (1), 6, 461–67, and *Comp. Rend.*, 15, 1842, 44–59, 85–101, 131–38 = *Œuvres*, (1), 7, 17–33, 33–49, 52–58.

pitfalls because little was known of the internal or molecular structure of matter; hence it was difficult to grasp any physical principles. The assumptions made as to solid bodies, the air, and ether varied from one writer to another and were disputed. In the case of ether, which presumably penetrated solid bodies because light passed through some and was absorbed by others, the relationship of the ether molecules to the molecules of the solid body also posed great difficulties. We do not intend to follow the physical theories of elastic bodies, nor is our understanding complete even today.

Navier[34] was the first (1821) to investigate the general equations of equilibrium and vibrations of elastic solids. The material was assumed to be isotropic, and the equations contained a single constant representing the nature of the solid. By 1822, stimulated by Fresnel's work, Cauchy had created another approach to the theory of elasticity.[35] Cauchy's equations contain two constants to represent the material of the body, and for an isotropic body are

$$
(51)\qquad
\begin{aligned}
(\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\Delta u &= \rho\frac{\partial^2 u}{\partial t^2}\\
(\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\Delta v &= \rho\frac{\partial^2 v}{\partial t^2}\\
(\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\Delta w &= \rho\frac{\partial^2 w}{\partial t^2}\\
\theta = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}&+\frac{\partial w}{\partial z}.
\end{aligned}
$$

Here u, v, and w are components of displacements, θ is called the dilatation, and λ and μ are constants of the body or medium. For general anisotropic media the equations are quite complicated, and it may be pointless to write them in all generality. The equations are given by Cauchy.[36]

The most spectacular triumph of the nineteenth century, with an enormous impact on science and technology, was Maxwell's derivation in 1864 of the laws of electromagnetism.[37] Maxwell, utilizing the electrical and magnetic researches of numerous predecessors, notably Faraday, introduced the notion of a displacement current—radio waves are one form of displacement current—and with this notion formulated the laws of electromagnetic wave propagation. His equations, which are most conveniently stated in the vector form adopted later by Oliver Heaviside, are four in number and involve the electric field intensity E, the magnetic field

34. *Mém. de l'Acad. des Sci., Paris*, (2), 7, 1827, 375–94.
35. *Exercices de math.*, 1828 = *Œuvres*, (2), 8, 195–226.
36. *Exercices de math.*, 1828 = *Œuvres*, (2), 8, 253–77.
37. *Phil. Trans.*, 155, 1865, 459–512 = *Scientific Papers*, 1, 526–97.

intensity H, the dielectric constant ϵ of the medium, the magnetic permeability μ of the medium, and the charge density ρ. The equations are

$$
(52)\qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial(\epsilon\mathbf{E})}{\partial t},\qquad \operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial(\mu\mathbf{H})}{\partial t}
$$

$$
(53)\qquad \operatorname{div}\epsilon\mathbf{E} = \rho \quad\text{and}\quad \operatorname{div}\mu\mathbf{H} = 0.
$$

[38]

The first two equations are the primary ones and amount to six scalar (non-vectorial) partial differential equations. The displacement current is the term $\partial(\epsilon\mathbf{E})/\partial t$.

By working with just these equations, Maxwell predicted that electromagnetic waves travel through space and at the speed of light. On the basis of the identity of the two speeds, he dared to assert that light is an electromagnetic phenomenon, a prediction that has been amply confirmed since his time.

No general methods for solving any of the above systems of equations are known. However, the nineteenth-century men gradually realized that in the case of partial differential equations, whether single equations or systems, general solutions are not nearly so useful as the solutions for specific problems where the initial and boundary conditions are given, and where experimental work might also aid one in making useful simplifying assumptions. The writings of Fourier, Cauchy, and Riemann furthered this realization. The work on the solution of the many initial- and boundary-value problems to which specializations of these systems gave rise is enormous, and almost all of the mathematicians of the century undertook such problems.

36

8. *Existence Theorems*

As the eighteenth- and nineteenth-century mathematicians created a vast number of types of differential equations, they found that methods of solving many of these equations were not available. Somewhat as in the case of polynomial equations, where efforts to solve equations of degree higher than four failed and Gauss turned to the proof of existence of a root (Chap. 25, sec. 2), so in the work on differential equations the failure to find explicit solutions, which of course *ipso facto* demonstrate existence, caused the mathematicians to turn to proof of the existence of solutions. Such proofs, even though they do not exhibit a solution or exhibit it in a useful form, serve several purposes. The differential equations were in nearly all cases the mathematical formulation of physical problems. No guarantees were available that the mathematical equations could be solved; hence a

38. For the meaning of curl and div see Chap. 32, sec. 5.

circle. Then the generalized Green's theorem gives, when the circle is contracted to (ξ, η):

$$(49)\quad 2\pi u(\xi, \eta) = -\iint vL(u)\,dx\,dy + \int\left[v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} + (a\cos(n, x) + b\cos(n, y))\cdot uv\right]ds$$

where n is positive if directed to the outside of the domain; the single integral is taken counterclockwise over the boundary. Since u satisfies $L(u) = 0$, if we know v and if u and $\partial u/\partial n$ are given (both are not arbitrary) on the boundary, then we have u expressed as a single integral. The function v is called a Green's function, though often the condition that v vanish on the boundary of R is added in the definition of the Green's function. Various specializations and generalizations of this use of Green's theorem have been developed.

7. *Systems of Partial Differential Equations*

In the eighteenth century the differential equations of fluid motion presented the first important system of partial differential equations. In the nineteenth century three more fundamental systems, the fluid dynamical equations for viscous media, the equations of elastic media, and the equations of electromagnetic theory, were created.

The acquisition of the equations of fluid motion when viscosity is present (as it always is) took a tortuous path. Euler had given the equations of motion of a fluid that is nonviscous. Since the time of Lagrange the essential difference between the motion of a fluid when a velocity potential exists and when it does not had been recognized. Led by a formal analogy with the theory of elasticity and by the hypothesis of molecules animated by repulsive forces, Claude L. M. H. Navier (1785–1836), professor of mechanics at the Ecole Polytechnique and at the Ecole des Ponts et Chaussées, obtained the basic equations in 1821.[31] The Navier-Stokes equations, as they are now identified, are

$$(50)\quad \begin{aligned} \rho\frac{Du}{Dt} &= \rho X - \frac{\partial p}{\partial x} + \frac{1}{3}\mu\frac{\partial\theta}{\partial x} + \mu\Delta u \\ \rho\frac{Dv}{Dt} &= \rho Y - \frac{\partial p}{\partial y} + \frac{1}{3}\mu\frac{\partial\theta}{\partial y} + \mu\Delta v \\ \rho\frac{Dw}{Dt} &= \rho Z - \frac{\partial p}{\partial z} + \frac{1}{3}\mu\frac{\partial\theta}{\partial z} + \mu\Delta w \\ \theta &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \end{aligned}$$

31. *Mém. de l'Acad. des Sci., Paris*, (2), 1827, 375–94.

where Δ has the usual meaning; ρ is the density of the fluid; p, the pressure; u, v, and w are the components of velocity of the fluid at any x, y, z, and t; X, Y, and Z are the components of an external force; the constant μ, which depends on the nature of the fluid, is called the coefficient of viscosity; and the derivative D/Dt has the meaning explained in Chapter 22, section 8. For an incompressible fluid, $\theta = 0$.

The equations were also obtained in 1829 by Poisson.[32] They were then rederived in 1845 on the basis of the mechanics of continua by George Gabriel Stokes (1819–1903), professor of mathematics at Cambridge University, in his essay "On the Theories of the Internal Friction of Fluids in Motion."[33] Stokes endeavored to account for the frictional action in all known liquids, which causes the motion to subside by converting kinetic energy into heat. Fluids, by virtue of their viscosity, stick to the surfaces of solids and thus exert tangential forces on them.

The subject of elasticity was founded by Galileo, Hooke, and Mariotte and was cultivated by the Bernoullis and Euler. But these men dealt with specific problems. To solve them they concocted *ad hoc* hypotheses on how beams, rods, and plates behaved under stresses, pressures, or loads. The theory proper is the creation of the nineteenth century. From the beginning of the nineteenth century on a number of great men worked persistently to obtain the equations that govern the behavior of elastic media, which includes the air. These men were primarily engineers and physicists. Cauchy and Poisson are the great mathematicians among them, though Cauchy was an engineer by training.

The problems of elasticity include the behavior of bodies under stress wherein one considers what equilibrium position they will assume, the vibrations of bodies when set in motion by an initial disturbance or by a continuously applied force, and, in the case of air and solid bodies, the propagation of waves through them. The interest in elasticity in the nineteenth century was heightened by the appearance about 1820 of a wave theory of light, initiated by the physician Thomas Young (1773–1829) and by Augustin-Jean Fresnel (1788–1827), an engineer. Light was regarded as a wave motion in ether, and ether was believed to be an elastic medium. Hence the propagation of light through ether became a basic problem. Another stimulus to a strong interest in elasticity in the early nineteenth century was Ernst F. F. Chladni's (1756–1827) experiments (1787) on the vibrations of glass and metal, in which he showed the nodal lines. These should be related to the sounds given off by, for example, a vibrating drumhead.

The work to obtain basic equations of elasticity was long and full of

32. *Jour. de l'Ecole Poly.*, 13, 1831, 1–74.
33. *Trans. Camb. Phil. Soc.*, 8, 1849, 287–319 = *Math. and Phys. Papers*, 1, 75–129.

of air in a tube (organ pipe) with an open end, gave the first general investigation of its solutions.[27] He was concerned with the acoustical problem in which u is the velocity potential of a harmonically moving air mass, k is a constant determined by the elasticity of the air and the oscillation frequency, and λ, which equals $2\pi/k$, is the wavelength. By applying Green's theorem he showed that any solution of $\Delta w + k^2 w = 0$ that is continuous in a given domain can be represented as the effect of single and double layers of excitation points on the surface of the domain. Using $e^{-ikr}/4\pi r$ as one of the functions in Green's theorem, he obtained

$$(46) \qquad w(P) = -\frac{1}{4\pi}\iint \frac{e^{-ikr}}{r}\frac{\partial w}{\partial n}\,dS + \frac{1}{4\pi}\iint w\frac{\partial}{\partial n}\left(\frac{e^{-ikr}}{r}\right)dS$$

wherein r denotes the distance from P to a variable point on the boundary. Thus w at any point P within the domain in which the solution is sought is given in terms of the values of w and $\partial w/\partial n$ on the boundary S.

The work of Helmholtz was used by Gustav R. Kirchhoff (1824–87), one of the great German nineteenth-century mathematical physicists, to obtain another solution of the initial-value problem for the wave equation. Let us suppose that $\Delta w + k^2 w = 0$ comes from

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

wherein we have let $u = we^{i\sigma t}$ so that $k = \sigma/c$. Then (46) may be written as

$$(47) \quad u(P,t) = -\frac{1}{4\pi}\iint \frac{e^{i\sigma(t-(r/c))}}{r}\frac{\partial u}{\partial n}\,dS + \frac{1}{4\pi}\iint u\frac{\partial}{\partial n}\left(\frac{e^{i\sigma(t-(r/c))}}{r}\right)dS.$$

This formula was generalized by Kirchhoff. If we let $\phi(t)$ be the value of u at any point (x, y, z) of the boundary at the instant t and let $f(t)$ be the corresponding value of $\partial u/\partial n$, then Kirchhoff showed[28] that

$$(48) \quad u(P,t) = -\frac{1}{4\pi}\iint \frac{f[t-(r/c)]}{r}\,dS + \frac{1}{4\pi}\iint \frac{\partial}{\partial n}\left(\frac{\phi[t-(r/c)]}{r}\right)dS,$$

provided that in the last term the differentiation with respect to n applies to r only insofar as it appears explicitly in both numerator and denominator. Thus u is obtained at P in terms of values of u and $\partial u/\partial n$ at earlier times at points of the closed surface surrounding P. This result is called Huygens's principle of acoustics and it is a generalization of Poisson's formula.

We have noted that Riemann used a somewhat generalized Green's theorem. The full generalization of Green's theorem that employs the adjoint differential equation and which is also called Green's theorem, comes from a paper by Paul Du Bois-Reymond (1831–89)[29] and from

27. *Jour. für Math.*, 57, 1860, 1–72 = *Wissenschaftliche Abhandlungen*, 1, 303–82.
28. *Sitzungsber. Akad. Wiss. zu Berlin*, 1882, 641–69 = *Ges. Abh.*, 2, 22 ff.
29. *Jour. für Math.*, 104, 1889, 241–301.

Darboux in his *Théorie générale des surfaces*;[30] both cite Riemann's paper of 1858/59. If the given equation is

$$L(u) = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0,$$

wherein the coefficients are functions of x and y, one integrates the product $vL(u)$ over an arbitrary domain R of the xy-plane under the assumption that u, v, and their first and second partial derivatives are continuous. Then integration by parts yields the generalized Green's theorem, which states that

$$\iint uM(v)\,dx\,dy = -\iint vL(u)\,dx\,dy - \int (Q\,dy - P\,dx),$$

where the double integrals are over the interior of R and the single integrals over the boundary of R,

$$M(v) = \frac{\partial^2(Av)}{\partial x^2} + 2\frac{\partial^2(Bv)}{\partial x\,\partial y} + \frac{\partial^2(Cv)}{\partial y^2} - \frac{\partial(Dv)}{\partial x} - \frac{\partial(Ev)}{\partial y} + Fv$$

$$P = B\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + C\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(E - \frac{\partial B}{\partial x} - \frac{\partial C}{\partial y}\right)uv$$

$$Q = A\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + B\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(D - \frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}\right)uv.$$

$M(v)$ is the adjoint expression of $L(u)$ and $M(v) = 0$ is the adjoint differential equation. Conversely $L(u)$ is the adjoint of $M(v)$.

The significance of Green's theorem is that it can be used to obtain solutions of some partial differential equations. Thus, since the elliptic equation can always be put in the form

$$L(u) = \Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0,$$

then

$$M(v) = \Delta v - \frac{\partial(av)}{\partial x} - \frac{\partial(bv)}{\partial y} + cv = 0.$$

Let v be a solution of the adjoint equation that becomes logarithmically infinite at an arbitrary point (ξ, η); that is, it behaves like

$$v = U \log r + V,$$

where r is the distance from (ξ, η) to (x, y); U and V are continuous in the domain R being considered; and U is normalized so that $U(\xi, \eta) = 1$. Now exclude (ξ, η) from the domain of integration by enclosing it in a

30. Vol. 2, Book IV, Chap. 4, 2nd ed., 1915.

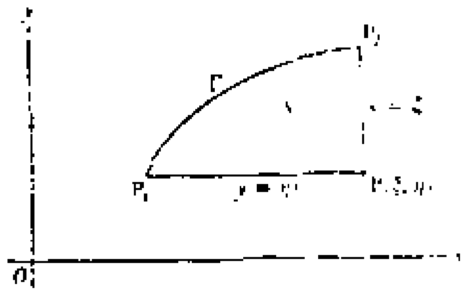

Figure 28.3

where D, E, and F are continuous and differentiable to second order functions of x and y. The problem calls for finding u at an arbitrary point P (Fig. 28.3) when one knows u and $\partial u/\partial n$ (which means knowing $\partial u/\partial x$ and $\partial u/\partial y$) along a curve Γ. His method depends on finding a function v (called a Riemann function or characteristic function)[25] that satisfies what is now called the adjoint equation

$$(41)\qquad M(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial(Dv)}{\partial x} - \frac{\partial(Ev)}{\partial y} + Fv = 0$$

and other conditions we shall specify shortly.

Riemann introduced the segments PP_1 and PP_2 of the characteristics (he did not use the term) $x = \xi$ and $y = \eta$ through P. Now a generalized Green's theorem (in two dimensions) is applied to the differential expression $L(u)$. To express the theorem compactly, let us introduce

$$X = \frac{1}{2}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + Duv$$

$$Y = \frac{1}{2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + Euv.$$

Then Green's theorem states

$$(42)\qquad \iint_S [vL(u) - uM(v)]\,dS = \iint_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) dS = \int_C \{X\cos(n, x) + Y\cos(n, y)\}\,ds,$$

where S is the area in the figure, C is the entire boundary of S, and $\cos(n, x)$ is the cosine of the angle between the normal to C and the x-axis.

25. v is not the same as a fundamental solution or a Green's function.

Beyond satisfying (41), Riemann requires of v that

$$(43)\qquad \begin{aligned} &(a)\ v = 1 \quad \text{at} \quad P, \\ &(b)\ \frac{\partial v}{\partial y} - Dv = 0 \quad \text{on} \quad x = \xi, \\ &(c)\ \frac{\partial v}{\partial x} - Ev = 0 \quad \text{on} \quad y = \eta.\end{aligned}$$ [26]

By using the condition that $M(v) = 0$ and the conditions (43), and by evaluation of the curvilinear integral over C, Riemann obtains

$$(44)\qquad u(\xi, \eta) = \int_\Gamma \{X\cos(n, x) + Y\cos(n, y)\}\,ds + \frac{1}{2}\{(uv)_{P_1} + (uv)_{P_2}\}.$$

Thus the value of u at any arbitrary point P is given in terms of the values of u, $\partial u/\partial n$, v, and $\partial v/\partial n$ on Γ and the values of u and v at P_1 and P_2.

Now u is given at P_1 and P_2. The function v must itself be found by solving $M(v) = 0$ and meeting the conditions in (43). Hence what Riemann's method achieves is to change the original initial-value problem for u to another kind of initial-value problem, the one for v. The second problem is usually easier to solve: In Riemann's physical problem it was especially easy to find v. However the existence of such a v generally was not established by Riemann.

The Riemann method as just described is useful only for the type of equation exemplified by the wave equation (hyperbolic equations) in two independent variables and cannot be extended directly. The extension of the method to more than two independent variables meets with the difficulty that the Riemann function becomes singular on the boundary of the domain of integration and the integrals diverge. The method has been extended at the cost of increased complication.

Progress in the solution of the wave equation by other methods is intimately connected with what are called steady-state problems, which lead to the reduced wave equation. The wave equation, by its very form, involves the time variable. In many physical problems, where one is interested in simple harmonic waves, one assumes that $u = w(x, y, z)e^{ikt}$, and by substituting this into the wave equation one obtains

$$(45)\qquad \Delta w + k^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} + k^2 w = 0.$$

This is the reduced wave equation or the Helmholtz equation. The equation $\Delta w + k^2 w = 0$ represents all harmonic, acoustic, elastic, and electromagnetic waves. While the older authors were satisfied to find particular integrals, Hermann von Helmholtz (1821–94), in his work on the oscillations

26. For two-dimensional problems v is a function of four variables, ξ, η, x, and y. It satisfies $M(v) = 0$ as a function of x and y.

6. *The Wave Equation and the Reduced Wave Equation*

Perhaps the most important type of partial differential equation is the wave equation. In three spatial dimensions the basic form is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial^2 u}{\partial t^2}. \tag{36}$$

As we know, this equation had already been introduced in the eighteenth century and had also been expressed in spherical coordinates. During the nineteenth century new uses of the wave equation were found, especially in the burgeoning field of elasticity. The vibrations of solid bodies of a variety of shapes with different initial and boundary conditions and the propagation of waves in elastic bodies produced a host of problems. Further work in the propagation of sound and light raised hundreds of additional problems.

Where separation of variables is possible, the technique of solving (37) is no different from what Fourier did with the heat equation or Lamé did after expressing the potential equation in some system of curvilinear coordinates. Mathieu's use of curvilinear coordinates to solve the wave equation by separation of variables is typical of hundreds of papers.

Quite another and important class of results dealing with the wave equation was obtained by treating the equation as an entirety. The first of such major results deals with initial-value problems and goes back to Poisson, who worked on this equation during the years 1808 to 1819. His principal achievement[23] was a formula for the propagation of a wave $u(x, y, z, t)$ whose initial state is described by the initial conditions

$$u(x, y, z, 0) = \phi_0(x, y, z), \qquad u_t(x, y, z, 0) = \phi_1(x, y, z) \tag{37}$$

and which satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \tag{38}$$

wherein a is a constant. The solution u is given by

$$\begin{aligned} u(x, y, z, t) &= \frac{1}{4\pi a} \int_0^{\pi} \int_0^{2\pi} \phi_1(x + at \sin\phi \cos\theta, y + at \sin\phi \sin\theta, \\ &\qquad\qquad z + at \cos\phi)\, at \sin\phi \, d\theta \, d\phi \\ &+ \frac{1}{4\pi a} \frac{\partial}{\partial t} \int_0^{\pi} \int_0^{2\pi} \phi_0(x + at \sin\phi \cos\theta, y + at \sin\phi \sin\theta, \\ &\qquad\qquad z + at \cos\phi)\, at \sin\phi \, d\theta \, d\phi \end{aligned} \tag{39}$$

wherein θ and ϕ are the usual spherical coordinates. The domain of integration is the surface of a sphere S_{at} with radius at about the point P with coordinates x, y, and z.

23. *Mém. de l'Acad. des Sci., Paris*, (2), 3, 1818, 121–76.

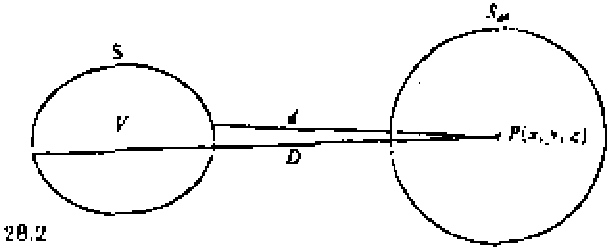

Figure 28.2

To obtain some indication of what Poisson's result means, let us consider a physical example. Suppose that the initial disturbance is set up by a body V (Fig. 28.2) with boundary S so that ϕ_0 and ϕ_1 are defined on V and are 0 outside of V. We say that the initial disturbance is localized in V. Physically a wave sets out from V and spreads out into space. Poisson's formula tells us what happens at any point $P(x, y, z)$ outside of V. Let d and D represent the minimum and maximum distances of P to the points of V. When $t < d/a$, the integrals in (39) are 0 because the domain of integration is the surface of the sphere S_{at} with radius at and center at P. Since ϕ_0 and ϕ_1 are 0 on S_{at}, then the function u is 0 at P. This means that the wave spreading out from S has not reached P. At $t = d/a$, the sphere S_{at} just touches S so the leading front of the wave emanating from S arrives at P. Between $t = d/a$ and $t = D/a$ the sphere S_{at} cuts V and so $u(P, t) \neq 0$. Finally for $t > D/a$, the sphere S_{at} will not cut S (the entire region V lies inside S_{at}); that is, the initial disturbance has passed through P. Hence again $u(P, t) = 0$. The instant $t = D/a$ corresponds to the passage of the trailing edge of the wave through P. At any given time t the leading edge of the wave takes the form of a surface which separates points not reached by the disturbance from those reached. This leading edge is the envelope of the family of spheres with centers on S and with radii at. The terminating edge of the wave at time t is a surface separating points at which the disturbance exists from those which the disturbance has passed. We see, then, that the disturbance which is localized in space gives rise at each point P to an effect that lasts only for a finite time. Moreover the wave (disturbance) has a leading and a terminating edge. This entire phenomenon is called Huygens's principle.

A quite different method of solving the initial-value problem for the wave equation was created by Riemann in the course of his work on the propagation of sound waves of finite amplitude.[24] Riemann considers a linear differential equation of second order that can be put in the form

$$L(u) = \frac{\partial^2 u}{\partial x \partial y} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0, \tag{40}$$

24. *Abh. der Ges. der Wiss. zu Gött.*, 8, 1858/59, 43–65 = *Werke*, 156

Lamé introduced several new coordinate systems for the express purpose of solving the heat equation in these systems.[15] His chief system was the three families of surfaces given by the equations

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = 0$$

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - c^2} - 1 = 0$$

$$\frac{x^2}{\nu^2} + \frac{y^2}{\nu^2 - b^2} + \frac{z^2}{\nu^2 - c^2} - 1 = 0,$$

where $\lambda^2 > c^2 > \mu^2 > b^2 > \nu^2$. These three families are ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets, all of which possess the same foci. Any surface of one family cuts all the surfaces of the other two orthogonally, and in fact cuts them in lines of curvature (Chap. 23, sec. 7). Any point in space accordingly has coordinates (λ, μ, ν), namely the λ, μ, and ν of the surfaces, one from each family, which go through that point. This new coordinate system is called ellipsoidal, though Lamé called it elliptical, a term now used for another system.

Lamé transformed the heat equation for the steady-state case (temperature independent of time), that is, the potential equation, to these coordinates, and showed that he could use separation of variables to reduce the partial differential equation to three ordinary differential equations. Of course these equations must be solved subject to appropriate boundary conditions. In a paper of 1839[16] Lamé studied further the steady-state temperature distribution in a three-axis ellipsoid and gave a complete solution of the problem treated in his 1833 paper. In this 1839 paper he also introduced another curvilinear coordinate system, now called the spheroconal system, wherein the coordinate surfaces are a family of spheres and two families of cones. This system too Lamé used to solve heat conduction problems. Lamé wrote many other papers on heat conduction using ellipsoidal coordinates, including a second one of 1839 in the same volume of the *Journal de Mathématiques*, in which he treats special cases of the ellipsoid.[17]

The subject of mutually orthogonal families of surfaces had such obvious importance in the solution of partial differential equations that it became a subject of investigation in and for itself. In a paper of 1834[18] Lamé considered the general properties of any three families of mutually orthogonal surfaces and gave a procedure for expressing a partial differen-

15. *Annales de Chimie et Physique*, (2), 53, 1833, 190–204.
16. *Jour. de Math.*, 4, 1839, 126–63.
17. *Jour. de Math.*, 4, 1839, 351–85.
18. *Jour. de l'École Poly.*, 14, 1834, 191–288.

tial equation in any orthogonal coordinate system, a technique used continually ever since.

(Heinrich) Eduard Heine (1821–81) followed in Lamé's tracks. Heine in his doctoral dissertation of 1842[19] determined the potential (steady-state temperature) not merely for the interior of an ellipsoid of revolution when the value of the potential is given at the surface, but also for the exterior of such an ellipsoid and for the shell between confocal ellipsoids of revolution.

Lamé was so much impressed with what he and others accomplished by the use of triply orthogonal coordinate systems that he thought all partial differential equations could be solved by finding a suitable system. Later he realized that this was a mistake. In 1859 he published a book on the whole subject, *Leçons sur les coordonnées curvilignes.*

Though the use of mutually orthogonal families of surfaces as the coordinate surfaces did not solve all partial differential equations, it did open up a new technique that could be exploited to advantage in many problems. The use of curvilinear coordinates was carried over to other partial differential equations. Thus Émile-Léonard Mathieu (1835–1900), in a paper of 1868,[20] treated the vibrations of an elliptic membrane, which involves the wave equation, and here introduced elliptic cylinder coordinates and functions appropriate to these coordinates, now called Mathieu functions (Chap. 29, sec. 2). In the same year Heinrich Weber (1842–1913), working with the equation $\partial^2u/\partial x^2 + \partial^2u/\partial y^2 + k^2u = 0$, solved it[21] for a domain bounded by a complete ellipse and also for the region bounded by two arcs of confocal ellipses and two arcs of hyperbolas confocal with the ellipses. The special case in which the ellipses and hyperbolas become confocal parabolas was also considered, and here Weber introduced functions appropriate to expansions in this coordinate system, now called Weber functions or parabolic cylinder functions. In his *Cours de physique mathématique* (1873), Mathieu took up new problems involving the ellipsoid and introduced still other new functions.

Our discussion of the idea initiated by Lamé, the use of curvilinear coordinates, describes just the beginning of this work. Many other coordinate systems have been introduced; corresponding special functions that result from solving the ordinary differential equations, which arise from separation of variables, have also been studied.[22] Most of this theory of special functions was created by physicists as they needed the functions and their properties in concrete problems (see also Chap. 29).

19. *Jour. für Math.*, 26, 1843, 185–216.
20. *Jour. de Math.*, (2), 13, 1868, 137–203.
21. *Math. Ann.*, 1, 1869, 1–36.
22. See William E. Byerly, *An Elementary Treatise on Fourier Series*, Dover (reprint), 1959, and E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea (reprint), 1955.

Riemann's work on complex functions gave a new importance to the Dirichlet problem and the principle itself. Riemann's "proof" of the existence of V in his doctoral thesis used the two-dimensional case of the Dirichlet principle, but it was not rigorous, as he himself realized.

When Weierstrass in his paper of 1870[13] presented a critique of the Dirichlet principle, he showed that the *a priori* existence of a minimizing U was not supported by proper arguments. It was correct that for all continuous differentiable functions U that go continuously from the interior onto the prescribed boundary values the integral has a *lower bound*. But whether there is a function U_0 in the class of continuous, differentiable functions that furnishes the lower bound was not established.

Another technique for the solution of the potential equation employs complex function theory. Though d'Alembert in his work of 1752 (Chap. 27, sec. 2) and Euler in special problems had used this technique to solve the potential equation, it was not until the middle of the nineteenth century that complex function theory was vitally employed in potential theory. The relevance of function theory to potential theory rests on the fact that if $u + iv$ is an analytic function of z, then both u and v satisfy Laplace's equation. Moreover, if u satisfies Laplace's equation, then the conjugate function v such that $u + iv$ is analytic necessarily exists (Chap. 27, sec. 8).

Where the equation $\Delta u = 0$ is used in fluid flow, the function $u(x, y)$ is what Helmholtz called the velocity potential, and then $\partial u/\partial x$ and $\partial u/\partial y$ represent the components of the velocity of the fluid at any point (x, y). In the case of static electricity, u is the electrostatic potential and $\partial u/\partial x$ and $\partial u/\partial y$ are the components of electric force. In both cases the curves $u =$ const. are equipotential lines and the curves $v =$ const., which are orthogonal to $u =$ const., are the flow or stream lines (lines of force for electricity). The function $v(x, y)$ is called the stream function. The introduction of this function is clearly helpful because of its physical significance.

One advantage of the use of complex function theory in solving the potential equation derives from the fact that if $F(z) = F(x + iy)$ is an analytic function, so that its real and imaginary parts satisfy $\Delta V = 0$, then the transformation of x and y to ξ and η by

$$\xi = f(x, y), \qquad \eta = g(x, y)$$

where

$$\zeta = \xi + i\eta$$

produces another analytic function $G(\zeta) = G(\xi + i\eta)$, and its real and imaginary parts also satisfy $\Delta V(\xi, \eta) = 0$. Now if the original potential

13. Chap. 27, sec. 9.

problem $\Delta V = 0$ has to be solved in some domain D, then by proper choice of the transformation the domain D' in which the transformed $\Delta V = 0$ has to be solved can be much simpler. Here the use of conformal transformations, such as the Schwarz-Christoffel transformation, is of great service.

We shall not pursue the uses of complex function theory in potential theory because the details of its use go far beyond any basic methodology in the solution of partial differential equations. It is, however, again worth noting that many mathematicians resisted the use of complex functions because they were still not reconciled to complex numbers. At Cambridge University, even in 1850, cumbrous devices were used to avoid involving complex functions. Horace Lamb's *Treatise on the Mathematical Theory of the Motion of Fluids*, published in 1879 and still a classic (now known as *Hydrodynamics*), was the first book to acknowledge the acceptance of function theory at Cambridge.

5. *Curvilinear Coordinates*

Green introduced a number of major ideas whose significance extended far beyond the potential equation. Gabriel Lamé (1795–1870), a mathematician and engineer concerned primarily with the heat equation, introduced another major technique, the use of curvilinear coordinate systems, which could also be used for many types of equations. Lamé pointed out in 1833[14] that the heat equation had been solved only for conducting bodies whose surfaces are normal to the coordinate planes $x =$ const., $y =$ const., and $z =$ const. Lamé's idea was to introduce new systems of coordinates and the corresponding coordinate surfaces. To a very limited extent this had been done by Euler and Laplace, both of whom used spherical coordinates ρ, θ, and ϕ, in which case the coordinate surfaces $\rho =$ const., $\theta =$ const., and $\phi =$ const. are spheres, planes, and cones respectively. Knowing the equations that transform from rectangular to spherical coordinates, one can, as Euler and Laplace did, transform the potential equation from rectangular to spherical coordinates.

The value of the new coordinate systems and surfaces is twofold. First, a partial differential equation in rectangular coordinates might not be separable into ordinary differential equations in this system but might be separable in some other system. Secondly, the physical problem might call for a boundary condition on, say, an ellipsoid. Such a boundary is represented simply in a coordinate system wherein one family of surfaces consists of ellipsoids, whereas in the rectangular system a relatively complicated equation must be used. Moreover, after separation of variables in the proper coordinate system is employed, this boundary condition becomes applicable to just one of the resulting ordinary differential equations.

14. *Jour. de l'École Poly.*, 14, 1833, 194–251.

where the integral extends over the surface, and $\partial U/\partial n$ is the derivative of U in the direction perpendicular to the surface and into the body. It is understood that the coordinates of P are contained in $\partial U/\partial n$ and are the arguments at P. This function U, introduced by Green, which Riemann later called the Green's function, became a fundamental concept of partial differential equations. Green himself used the term "potential function" for this special function U as well as for V. His method of obtaining solutions of the potential equation, as opposed to the method of using series of special functions, is called the method of singularities. There is unfortunately no general expression for the function U, nor is there a general method for finding it. Green was content in this matter to give the physical meaning of U for the case of the potential created by electric charges.

Green applied his theorem and concepts to electrical and magnetic problems. He also took up in 1833[10] the problem of the gravitational potential of ellipsoids of variable densities. In this work Green showed that when V is given on the boundary of a body, there is just one function that satisfies $\Delta V = 0$ throughout the body, has no singularities, and has the given boundary values. To make his proof, Green assumed the existence of a function that minimizes

$$(33) \qquad \iiint \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right] dv.$$

This is the first use of the Dirichlet principle (cf. Chap. 27, sec. 8).

In this 1835 paper Green did much of the work in n dimensions instead of three and also gave important results on what are now called ultraspherical functions, which are a generalization to n variables of Laplace's spherical surface harmonics. Because Green's work did not become well known for some time, other men did some of this work independently.

Green is the first great English mathematician to take up the threads of the work done on the Continent after the introduction of analysis to England. His work inspired the great Cambridge school of mathematical physicists which included Sir William Thomson, Sir Gabriel Stokes, Lord Rayleigh, and Clerk Maxwell.

Green's achievements were followed by Gauss's masterful work of 1839,[11] "Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-kräfte" (General Theorems on Attractive and Repulsive Forces Which Act According to the Inverse Square of the Distance). Gauss proved rigorously Poisson's result, namely, that $\Delta V = -4\pi\rho$ at a point inside the acting mass, under the condition that ρ is continuous at that point and in a

10. *Trans. Camb. Phil. Soc.*, 5_3, 1835, 395–429 = *Mathematical Papers*, 187–222.
11. *Resultate aus den Beobachtungen des magnetischen Vereins*, Vol. 4, 1840 = *Werke*, 5, 197–242.

small domain around it. This condition is not fulfilled on the surface of the acting mass. On the surface the quantities $\partial^2V/\partial x^2$, $\partial^2V/\partial y^2$, and $\partial^2V/\partial z^2$ have jumps.

The work thus far on the potential equation and on Poisson's equation assumed the existence of a solution. Green's proof of the existence of a Green's function rested entirely on a physical argument. From the existence standpoint the fundamental problem of potential theory was to show the existence of a potential function V, which William Thomson about 1850 called a harmonic function, whose values are given on the boundary of a region and which satisfies $\Delta V = 0$ in the region. One might establish this directly, or establish the existence of a Green's function U and from that obtain V. The problem of establishing the existence of the Green's function or of V itself is known as the Dirichlet problem or the first boundary-value problem of potential theory, the most basic and oldest existence problem of the subject. The problem of finding a V satisfying $\Delta V = 0$ in a region when the normal derivative of V is specified on the boundary is called the Neumann problem, after Carl G. Neumann (1832–1925), a professor at Leipzig. This problem is called the second fundamental problem of potential theory.

One approach to the problem of establishing the existence of a solution of $\Delta V = 0$, which Green had already used (see (33)), was brought into prominence by William Thomson. In 1847[12] Thomson announced the theorem or principle which in England is named after him and on the Continent is called the Dirichlet *principle* because Riemann so named it. Though Thomson stated it in a somewhat more general form, the essence of the principle may be put thus: Consider the class of all functions U that have continuous derivatives of the second order in the interior and exterior domains T and T' respectively separated by a surface S. The U's are to be continuous everywhere and assume on S the values of a continuous function f. The function V that minimizes the Dirichlet integral

$$(34) \qquad I = \iiint_{T+T'} \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2\right] dv$$

is the one that satisfies $\Delta V = 0$ and takes on the value f on the boundary S. The connection between (34) and ΔV is that the first variation of I in the sense of the calculus of variations is ΔV, and this must be 0 for a minimizing V. Since for real U, I cannot be negative, it seemed clear that a minimizing function V must exist, and it is then not difficult to prove it is unique. The Dirichlet principle is then *one* approach to the Dirichlet problem of potential theory.

12. *Jour. de Math.*, 12, 1847, 493–96 = *Cambridge and Dublin Math. Jour.*, 3, 1848, 84–87 = *Math. and Physical Papers*, 1, 93–96.

distribution in a body, though varying from point to point, remains the same as time varies, or is in the steady state, then T in (1) is independent of time and the heat equation reduces to the potential equation. The emphasis on the potential equation for the calculation of gravitational attraction continued in the early nineteenth century but was accentuated by a new class of applications to electrostatics and magnetostatics. Here too the attraction of ellipsoids was a key problem.

One correction in the theory of gravitational attraction as expressed by the potential equation was made by Poisson.[7] Laplace (Chap. 22, sec. 4) had assumed that the potential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{30}$$

wherein V is a function of x, y, and z, holds at any point (x, y, z) whether inside or outside of the body that exerts the gravitational attraction. Poisson showed that if (x, y, z) lies inside the attracting body, then V satisfies

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho, \tag{31}$$

where ρ is the density of the attracting body and is also a function of x, y, and z. Though (31) is still called Poisson's equation, his proof that it holds was not rigorous, as he himself recognized, even by the standards of that time.

In this same paper Poisson called attention to the utility of this function V in electrical investigations, remarking that its value over the surface of any conductor must be constant when electrical charge is allowed to distribute itself over the surface. In other papers he solved a number of problems calling for the distribution of charge on the surfaces of conducting bodies when the bodies are near each other. His basic principle was that the resultant electrostatic force in the interior of any one of the conductors must be zero.

Despite the work of Laplace, Poisson, Gauss, and others, almost nothing was known in the 1820s about the general properties of solutions of the potential equation. It was believed that the general integral must contain two arbitrary functions, of which one gives the value of the solution on the boundary and the other, the derivative of the solution on the boundary. Yet it was known in the case of steady-state heat conduction, in which the temperature satisfies the potential equation, that the temperature distribution throughout the three-dimensional body is determined when the temperature alone is specified on the surface. Hence one of the

7. *Nouv. [illegible] de la Soc. Philo.*, 3, 1813, 383–92.

arbitrary functions in the supposed general solution of the potential equation must somehow be fixed by some other condition.

At this point George Green (1793–1841), a self-taught English mathematician, undertook to treat static electricity and magnetism in a thoroughly mathematical fashion. In 1828 Green published a privately printed booklet, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*. This was neglected until Sir William Thomson (Lord Kelvin, 1824–1907) discovered it, recognized its great value, and had it published in the *Journal für Mathematik*.[8] Green, who learned much from Poisson's papers, also carried over the notion of the potential function to electricity and magnetism.

He started with (30) and proved the following theorems. Let U and V be any two continuous functions of x, y, and z whose derivatives are not infinite at any point of an arbitrary body. The major theorem asserts that (we shall used ΔV for the left hand side of (30), though it was not used by Green)

$$\iiint U\,\Delta V\,dv + \iint U\frac{\partial V}{\partial n}\,d\sigma = \iiint V\,\Delta U\,dv + \iint V\frac{\partial U}{\partial n}\,d\sigma, \tag{32}$$

where n is the surface normal of the body directed inward and $d\sigma$ is a surface element. Theorem (32), incidentally, was also proved by Michel Ostrogradsky (1801–61), a Russian mathematician, who presented it to the St. Petersburg Academy of Sciences in 1828.[9]

Green then showed that the requirement that V and each of its first derivatives be continuous in the interior of the body can be imposed instead of a boundary condition on the derivatives of V. In light of this fact, Green represented V in the interior of the body in terms of its value $\bar{V}$ on the boundary (which function would be given) and in terms of another function U which has the properties: (a) U must be 0 on the surface; (b) at a fixed but undetermined point P in the interior, U becomes infinite as $1/r$ where r is the distance of any other point from P; (c) U must satisfy the potential equation (30) in the interior. If U is known, and it might be found more readily because it satisfies simpler conditions than V, then V can be represented at every interior point by

$$4\pi V = -\iint \bar{V}\frac{\partial U}{\partial n}\,d\sigma,$$

8. *Jour. für Math.*, 39, 1850, 73–89; 44, 1852, 356–74; and 47, 1854, 161–221 = *Green's Mathematical Papers*, 1871, 3–115.
9. *Mém. Acad. Sci. St. Peters.*, (6), 1, 1831, 39–53.

where $\phi(x)$ would usually be the initial function. By a "limiting process" that replaces a_n by Q and n by q, he obtains

$$(23) \qquad Q = \frac{2}{\pi}\int_0^\infty F(x)\cos qx \, dx,$$

where $F(x)$, an even function, is the given initial temperature in the infinite domain. Then by using (23) in (22) and by an interchange of the order of integration, which Fourier does not bother to question, he has

$$u = \frac{2}{\pi}\int_0^\infty F(\alpha)\, d\alpha \int_0^\infty e^{-kq^2t}\cos qx \cos q\alpha \, dq.$$

Fourier then does the analogous thing for an odd $F(x)$, and so finally obtains

$$(24) \qquad u = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\alpha)\, d\alpha \int_0^\infty e^{-kq^2t}\cos q(x-\alpha)\, dq.$$

Thus the solution is expressed in closed form. Now for $t = 0$, u is $F(x)$, which could be any given function. Hence Fourier asserts that, for an *arbitrary* $F(x)$,

$$(25) \qquad F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(\alpha)\, d\alpha \int_0^\infty \cos q(x-\alpha)\, dq,$$

and this is one form of the Fourier double-integral representation of an arbitrary function. In his book Fourier showed how to solve many types of differential equations with this integral. One use lies in the fact that if (24) is obtained by any process, then (25) shows that u satisfies the initial condition at $t = 0$. Another use is more evident if one writes the Fourier integral in exponential form, using the Euler relation, $e^{ix} = \cos x + i\sin x$. Then (25) becomes

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iqx}\, dq \int_{-\infty}^{\infty} F(\alpha)e^{-iq\alpha}\, d\alpha.$$

This form shows that $F(x)$ can be resolved into an infinite number of harmonic components with continuously varying frequency $q/2\pi$ and with amplitude $(1/2\pi)\int_{-\infty}^{\infty} F(\alpha)e^{-iq\alpha}\, d\alpha$, whereas the ordinary Fourier series resolves a given function into an infinite but discrete set of harmonic components.

Cauchy's derivation of the Fourier integral is somewhat similar. The paper in which it appeared, "Théorie de la propagation des ondes," received the prize of the Paris Academy for 1816.[5] This paper is the first large investigation of waves on the surface of a fluid, a subject initiated by

5. *Mém. divers savans*, 1, 1827, 3–312 = *Œuvres*, (1), 1, 5–318; see also Cauchy, *Nouv. Bull. de la Soc. Phil.*, 1817, 121–24 = *Œuvres*, (2), 2, 223–27.

Laplace in 1778. Though Cauchy sets up the general hydrodynamical equations he limits himself almost at once to special cases. In particular he considers the equation

$$\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0$$

in which q is what was later called a velocity potential and x and y are spatial coordinates. He writes down without explanation the solution (cf. [22])

$$(26) \qquad q = \int_0^\infty \cos mx \, e^{-my} f(m)\, dm,$$

wherein $f(m)$ is arbitrary thus far. Since $y = 0$ on the surface, q reduces to a given $F(x)$,

$$(27) \qquad F(x) = \int_0^\infty \cos mx \, f(m)\, dm.$$

Then Cauchy shows that

$$(28) \qquad f(m) = \frac{2}{\pi}\int_0^\infty \cos mu \, F(u)\, du.$$

With this value of $f(m)$

$$(29) \qquad F(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \cos mx \cos mu \, F(u)\, du\, dm.$$

Cauchy thus obtains not only the Fourier double-integral representation of $F(x)$, but he also has the Fourier transform from $f(m)$ to $F(x)$ and the inverse transform. Given $F(x)$, $f(m)$ is determined by (28) and can be used in (26).

Shortly after Cauchy turned in his prize paper, Poisson, who could not compete for the prize because he was a member of the Academy, published a major work on water waves, "Mémoire sur la théorie des ondes."[6] In this work he derives the Fourier integral in about the same manner as Cauchy.

4. *The Potential Equation and Green's Theorem*

The next significant development centered about the potential equation, though the principal result, Green's theorem, has application to many other types of differential equations. The potential equation had figured in the eighteenth-century work on gravitation and had also appeared in the nineteenth-century work on heat conduction, for when the temperature

6. *Mém. de l'Acad. des Sci., Paris*, (2), 1, 1816, 71–186.

represents a function over an entire interval, whereas a Taylor series represents a function only in a neighborhood of a point at which a function is analytic, though in special cases the radius of convergence may be infinite.

We have already noted that Fourier's paper of 1807, in which he had maintained that an arbitrary function can be expanded in a trigonometric series, was not well received by the Academy of Sciences of Paris. Lagrange in particular denied firmly the possibility of such expansions. Though he criticized only the lack of rigor in the paper, he was certainly disturbed by the generality of the functions that Fourier entertained, because Lagrange still believed that a function was determined by its values in any arbitrarily small interval, which is true of analytic functions. In fact Lagrange returned to the vibrating-string problem and, with no better insight than he had shown in earlier work, insisted on defending Euler's contention that an arbitrary function cannot be represented by a trigonometric series. Poisson did assert later that Lagrange had shown that an arbitrary function can be represented by a Fourier series but Poisson, who was envious of Fourier, said this to rob Fourier of the credit and give it to Lagrange.

Fourier's work made explicit another fact that was also implicit in the eighteenth-century work of Euler and Laplace. These men had expanded functions in series of Bessel functions and Legendre polynomials in order to solve specific problems. The general fact that a function might be expanded in a series of functions such as the trigonometric functions, Bessel functions, and Legendre polynomials was thrust into the light by Fourier's work. He showed, further, how the initial condition imposed on the solution of a partial differential equation could be met, and so advanced the technique of solving such equations. Fourier's paper of 1811, though not published until 1824–26, was accessible to other men in the meantime, and his ideas, at first grudgingly accepted, finally won favor.

Fourier's method was taken up immediately by Siméon-Denis Poisson (1781–1840), one of the greatest of nineteenth-century analysts and a first-class mathematical physicist. Though his father had wanted him to study medicine, he became a student and then professor at the fountainhead of nineteenth-century French mathematics, the Ecole Polytechnique. He worked in the theory of heat, was one of the founders of the mathematical theory of elasticity, and was one of the first to suggest that the theory of the gravitational potential be carried over to static electricity and magnetism.

Poisson was so much impressed with Fourier's evidence that arbitrary functions can be expanded in a series of functions that he believed all partial differential equations could be solved by series expansions; each term of the series would itself be a product of functions (cf. [10]), one for each independe[nt] variable. These expansions, he thought, embraced the most general soluti[on]. He also believed that if an expansion diverged, this meant that one should seek an expansion in terms of other functions. Of course Poisson was far too optimistic.

From about 1815 on he himself solved a number of heat conduction problems and used expansions in trigonometric functions, Legendre polynomials, and Laplace surface harmonics. We shall encounter some of this work later. Much of Poisson's work on heat conduction was presented in his *Théorie mathématique de la chaleur* (1835).

3. *Closed Solutions; the Fourier Integral*

Despite the success and impact of Fourier's series solutions of partial differential equations, one of the major efforts during the nineteenth century was to find solutions in closed form, that is, in terms of elementary functions and integrals of such functions. Such solutions, at least of the kind known in the eighteenth and early nineteenth centuries, were more manageable, more perspicuous, and more readily used for calculation.

The most significant method of solving partial differential equations in closed form, which arose from work initiated by Laplace, was the Fourier integral. The idea is due to Fourier, Cauchy, and Poisson. It is impossible to assign priority for this important discovery, because all three presented papers orally to the Academy of Sciences that were not published until some time afterward. But each heard the others' papers, and one cannot tell from the publications what each may have taken from the verbal accounts.

In the last section of his prize paper of 1811, Fourier treated the propagation of heat in domains that extend to infinity in one direction. To obtain an answer for such problems, he starts with the general form of the solution of the heat equation for a bounded domain, namely (cf. [10]),

$$u = \sum_{n=1}^{\infty} a_n e^{-kq_n^2 t} \cos q_n x, \tag{21}$$

where the q_n are determined by the boundary conditions and the a_n are determined by the initial conditions. Fourier now regards the q_n as abscissas of a curve and the a_n as the ordinates of that curve. Then $a_n = Q(q_n)$ where Q is some function of q. He then replaces (21) by

$$u = \int_0^{\infty} Q(q) e^{-kq^2 t} \cos qx \, dq, \tag{22}$$

and seeks to determine Q. He goes back to the formula for the coefficients

$$a_n = \frac{2}{\pi} \int_0^{\pi} \phi(x) \cos nx \, dx,$$

only graphically. Hence Fourier concluded that *every* function $f(x)$ could be represented as

$$f(x) = \sum_{\nu=1}^{\infty} b_\nu \sin \nu x, \qquad \text{for } 0 < x < \pi. \tag{18}$$

This possibility had, of course, been rejected by the eighteenth-century masters except for Daniel Bernoulli.

How much Fourier knew of the work of his predecessors is not clear. In a paper of 1825 he says that Lacroix had informed him of Euler's work but he does not say when this happened. In any case Fourier was not deterred by the opinions of his predecessors. He took a great variety of functions $f(x)$, calculated the first few b_ν for each function, and plotted the sum of the first few terms of the sine series (18) for each one. From this graphical evidence he concluded that the series always represents $f(x)$ over $0 < x < \pi$, whether or not the representation holds outside this interval. He points out in his book (p. 198) that two functions may agree in a given interval but not necessarily outside that interval. The failure to see this explains why earlier mathematicians could not accept that an arbitrary function can be expanded in a trigonometric series. What the series does give is the function in the domain 0 to π, in the present case, and periodic repetitions of it outside.

Once Fourier obtained the above simple result for the b_ν, he, like Euler, realized that each b_ν can be obtained by multiplying the series (18) by $\sin \nu x$ and integrating from 0 to π. He also points out that this procedure is applicable to the representation

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x. \tag{19}$$

He considers next the representation of any $f(x)$ in the interval $(-\pi, \pi)$. The series (18) represents an odd function $[f(x) = -f(-x)]$ and the series (19) an even function $[f(x) = f(-x)]$. But any function can be represented as the sum of an odd function $f_0(x)$ and an even function $f_e(x)$ where

$$f_0(x) = \frac{1}{2}[f(x) - f(-x)], \qquad f_e = \frac{1}{2}[f(x) + f(-x)].$$

Then any $f(x)$ can be represented in the interval $(-\pi, \pi)$ by

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{20}$$

and the coefficients can be determined by multiplying through by $\cos \nu x$ or $\sin \nu x$ and integrating from $-\pi$ to π, which yields (17).

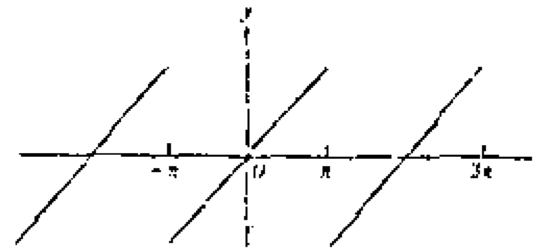

Figure 28.1

Fourier never gave any complete proof that an "arbitrary" function can be represented by a series such as (20). In the book he gives some loose arguments, and in his final discussion of this point (paragraphs 415, 416, and 423) he gives a sketch of a proof. But even there Fourier does not state the conditions that a function must satisfy to be expansible in a trigonometric series. Nevertheless Fourier's conviction that this was possible is expressed throughout the book. He also says[4] that his series are convergent no matter what $f(x)$ may be, whether or not one can assign an analytic expression to $f(x)$ and whether or not the function follows any regular law. Fourier's conviction that any function can be expanded in a Fourier series rested on the geometrical evidence described above. About this he says in his book (p. 206), "Nothing has appeared to us more suitable than geometrical constructions to demonstrate the truth of the new results and to render intelligible the forms which analysis employs for their expressions."

Fourier's work incorporated several major advances. Beyond furthering the theory of partial differential equations, he forced a revision in the notion of function. Suppose the function $y = x$ is represented by a Fourier series (20) in the interval $(-\pi, \pi)$. The series repeats its behavior in each interval of length 2π. Hence the function given by the series looks as shown in Figure 28.1. Such functions cannot be represented by a single (finite) analytic expression, whereas Fourier's predecessors had insisted that a function must be representable by a single expression. Since the entire function $y = x$ for all x is not represented by the series, they could not see how an arbitrary function, which is not periodic, could be represented by the series, though Euler and Lagrange had actually done so for particular nonperiodic functions. Fourier is explicit that his series can represent functions that also have different analytical expressions in different parts of the interval $(0, \pi)$ or $(-\pi, \pi)$, whether or not the expressions join one another continuously. He points out, finally, that his work settles the arguments on solutions of the vibrating-string problem in favor of Daniel Bernoulli. Fourier's work marked the break from analytic functions or functions developable in Taylor's series. It is also significant that a Fourier series

4. Page 196 = *Œuvres*, 1, 210.

Since the general solution of (7) is an exponential function but λ is now limited to the λ_ν, then, in view of (5), Fourier had, so far, that

$$T_\nu(x, t) = b_\nu e^{-(\nu^2\pi^2 k^2/l^2)t} \sin \frac{\nu\pi x}{l},$$

where b_ν now denotes the constant in place of b and $\nu = 1, 2, 3, \ldots$. However the equation (2) is linear, so that a sum of solutions is a solution. Hence one can assert that

$$(10) \qquad T(x, t) = \sum_{\nu=1}^{\infty} b_\nu e^{-(\nu^2\pi^2 k^2/l^2)t} \sin \frac{\nu\pi x}{l}.$$

To satisfy the initial condition (4), one must have for $t = 0$

$$(11) \qquad f(x) = \sum_{\nu=1}^{\infty} b_\nu \sin \frac{\nu\pi x}{l}.$$

Fourier then faced the question, Can $f(x)$ be represented as a trigonometric series? In particular, can the b_ν be determined?

Fourier proceeded to answer these questions. Though by this time he was somewhat conscious of the problem of rigor, he proceeded formally in the eighteenth-century spirit. To follow Fourier's work we shall, for simplicity, let $l = \pi$. Thus we consider

$$(12) \qquad f(x) = \sum_{\nu=1}^{\infty} b_\nu \sin \nu x, \qquad \text{for } 0 < x < \pi.$$

Fourier takes each sine function and expands it by Maclaurin's theorem into a power series; that is, he uses

$$(13) \qquad \sin \nu x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\nu^{2n-1}}{(2n-1)!} x^{2n-1}$$

to replace $\sin \nu x$ in (12). Then, by interchanging the order of the summations, an operation unquestioned at the time, he obtains

$$(14) \qquad f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} \left(\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu \right) x^{2n-1}.$$

Thus $f(x)$ is expressed as a power series in x, which implies a strong restriction on the admissible $f(x)$ that was not presupposed for the $f(x)$ Fourier treats. This power series must be the Maclaurin series for $f(x)$, so that

$$(15) \qquad f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k.$$

By equating coefficients of like powers of x in (14) and (15), Fourier finds that $f^{(k)}(0) = 0$ for even k and beyond that

$$\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu = (-1)^{n-1} f^{(2n-1)}(0), \qquad n = 1, 2, 3, \ldots.$$

Now the derivatives of $f(x)$ are known, because $f(x)$ is a given initial condition. Hence the b_ν are an infinite set of unknowns in an infinite system of linear algebraic equations.

In a previous problem, wherein he faced this same kind of system, Fourier took the first k terms and the right-hand constant of the first k equations, solved these, and by obtaining a general expression for $b_{\nu,k}$, which denotes the approximate value of b_ν obtained from the first k equations, he boldly concluded that $b_\nu = \lim_{k\to\infty} b_{\nu,k}$. However, this time he had much difficulty in determining the b_ν. He took several different $f(x)$'s and showed how to determine the b_ν by very complicated procedures that involved divergent expressions. Using these special cases as a guide he obtained an expression for b_ν involving infinite products and infinite sums. Fourier realized that this expression was rather useless, and by further bold and ingenious, though again often questionable, steps finally arrived at the formula

$$(16) \qquad b_\nu = \frac{2}{\pi} \int_0^\pi f(s) \sin \nu s \, ds.$$

The conclusion was to an extent not new. We have already related (Chap. 20, sec. 5) how Clairaut and Euler had expanded some functions in Fourier series and had obtained the formulas

$$(17) \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \qquad n \geq 1.$$

Moreover, Fourier's results as derived thus far were limited, because he assumed his $f(x)$ had a Maclaurin expansion, which means an infinite number of derivatives. Finally, Fourier's method was certainly not rigorous and was more complicated than Euler's. Whereas Fourier had to use an infinite system of linear algebraic equations, Euler proceeded more simply by using the properties of trigonometric functions.

But Fourier now made some remarkable observations. He noted that each b_ν can be interpreted as the area under the curve of $y = (2/\pi) f(x) \cdot \sin \nu x$ for x between 0 and π. Such an area makes sense even for very arbitrary functions. The functions need not be continuous [illegible]uld be known as a young student of mathematics but had set his heart on becoming an army officer. Denied a commission because he was the son of a tailor, he turned to the priesthood. When he was offered a professorship at the military school he had attended he accepted and mathematics became his life interest.

Like other scientists of his time, Fourier took up the flow of heat. The flow was of interest as a practical problem in the handling of metals in industry and as a scientific problem in attempts to determine the temperature in the interior of the earth, the variation of that temperature with time, and other such questions. He submitted a basic paper on heat conduction to the Academy of Sciences of Paris in 1807.[1] The paper was judged by Lagrange, Laplace, and Legendre and was rejected. But the Academy did wish to encourage Fourier to develop his ideas, and so made the problem of the propagation of heat the subject of a grand prize to be awarded in 1812. Fourier submitted a revised paper in 1811, which was judged by the men already mentioned and others. It won the prize but was criticized for its lack of rigor and so not published at that time in the *Mémoires* of the Academy. Fourier resented the treatment he received. He continued to work on the subject of heat and, in 1822, published one of the classics of mathematics, *Théorie analytique de la chaleur*.[2] It incorporated the first part of his 1811 paper practically without change. This book is the main source for Fourier's ideas. Two years later he became secretary of the Academy and was able to have his 1811 paper published in its original form in the *Mémoires*.[3]

In the interior of a body that is gaining or losing heat, the temperature is generally distributed nonuniformly and changes at any one place with time. Thus the temperature T is a function of space and time. The precise form of the function depends upon the shape of the body, the density, the specific heat of the material, the initial distribution of T, that is, the distribution at time $t = 0$, and the conditions maintained at the surface of the body. The first major problem Fourier considered in his book was the determination of the temperature T in a homogeneous and isotropic body as a function of x, y, z, and t. He proved on the basis of physical principles that T must satisfy the partial differential equation called the heat equation in three space dimensions,

$$(1)\qquad \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = k^2 \frac{\partial T}{\partial t},$$

where k^2 is a constant whose value depends on the material of the body.

1. The manuscript is in the library of the *École des Ponts et Chaussées*.
2. *Œuvres*, 1.
3. *Mém. de l'Acad. des Sci., Paris*, (2), 4, 1819/20, 185–555, pub. 1824, and 5, 1821/22, 153–246, pub. 1826; only the second part is reproduced in Fourier's *Œuvres*, 2, 3–94.

Fourier then solved specific heat conduction problems. We shall consider a case that is typical of his method, the problem of solving equation (1) for the cylindrical rod whose ends are kept at 0° temperature and whose lateral surface is insulated so that no heat flows through it. Since this rod involves only one space dimension, (1) becomes

$$(2)\qquad \frac{\partial^2 T}{\partial x^2} = k^2 \frac{\partial T}{\partial t}$$

subject to the boundary conditions

$$(3)\qquad T(0, t) = 0, \qquad T(l, t) = 0, \qquad \text{for } t > 0,$$

and the initial condition

$$(4)\qquad T(x, 0) = f(x) \qquad \text{for } 0 < x < l.$$

To solve this problem Fourier used the method of separation of variables. He let

$$(5)\qquad T(x, t) = \phi(x)\psi(t).$$

On substitution in the differential equation, he obtained

$$\frac{\phi''(x)}{k^2\phi(x)} = \frac{\psi'(t)}{\psi(t)}.$$

He then argued (*cf.* [30] of Chap. 22) that each of these ratios must be a constant, $-\lambda$ say, so that

$$(6)\qquad \phi''(x) + \lambda k^2 \phi(x) = 0$$

and

$$(7)\qquad \psi'(t) + \lambda\psi(t) = 0.$$

However the boundary conditions (3), in view of (5), imply that

$$(8)\qquad \phi(0) = 0 \quad \text{and} \quad \phi(l) = 0.$$

The general solution of (6) is

$$\phi(x) = b \sin(\sqrt{\lambda}kx + c).$$

The condition that $\phi(0) = 0$ implies that $c = 0$. The condition $\phi(l) = 0$ imposes a limitation on λ, namely that $\sqrt{\lambda}$ must be an integral multiple of π/kl. Hence there are an infinite number of admissible values λ_ν of λ or

$$(9)\qquad \lambda_\nu = \left(\frac{\nu\pi}{kl}\right)^2, \qquad \nu \text{ integral}.$$

These λ_ν are what we now call the eigenvalues or characteristic values.

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28 Partial Differential Equations in the Nineteenth Century

The profound study of nature is the most fertile source of mathematical discoveries.
JOSEPH FOURIER

1. *Introduction*

The subject of partial differential equations, which had its beginnings in the eighteenth century, burgeoned in the nineteenth. As physical science expanded, in both the variety and depth of the phenomena investigated, the number of new types of differential equations increased; and even the types already known, the wave equation and the potential equation, were applied to new areas of physics. Partial differential equations became and remain the heart of mathematics. Their importance for physical science is only one of the reasons for assigning them this central place. From the standpoint of mathematics itself, the solution of partial differential equations created the need for mathematical developments in the theory of functions, the calculus of variations, series expansions, ordinary differential equations, algebra, and differential geometry. The subject has become so extensive that in this chapter we can give only a few of the major results.

We are accustomed today to classifying partial differential equations according to types. At the beginning of the nineteenth century, so little was known about the subject that the idea of distinguishing the various types could not have occurred. The physical problems dictated which equations were to be pursued and the mathematicians passed freely from one type to another without recognizing some differences among them that we now consider fundamental. The physical world was and still is indifferent to the mathematicians' classification.

2. *The Heat Equation and Fourier Series*

The first big nineteenth-century step, and indeed one of enormous importance, was made by Joseph Fourier (1768–1830). Fourier [illegible] very well basic equations and generalized them. Here he gave credit to d'Alembert but none to Euler. He too says the equations of fluid motion are too difficult to be handled by analysis. Only the cases of infinitely small movements are susceptible of rigorous calculation.

In the area of systems of partial differential equations, the equations of hydrodynamics were, in the eighteenth century, the main inspiration for mathematical research on this subject. Actually the eighteenth century accomplished little in the solution of systems.

9. *The Rise of the Mathematical Subject*

Up to 1765 partial differential equations appeared only in the solution of physical problems. The first paper devoted to purely mathematical work on partial differential equations is by Euler: "Recherches sur l'intégration de l'équation $\left(\frac{d\,dz}{dt^2}\right) = aa\left(\frac{d\,dz}{dx^2}\right) + \frac{b}{x}\left(\frac{dz}{dx}\right)$."[51] Shortly thereafter Euler published a treatise on the subject in the third volume of his *Institutiones Calculi Integralis*.[52]

Before d'Alembert's work of 1747 on the vibrating string, partial differential equations were known as equations of condition and only special solutions were sought. After this work and d'Alembert's book on the general causes of winds (1746), the mathematicians realized the difference between special and general solutions. However, once aware of this distinction, they seemed to believe that general solutions would be more important. In the first volume of his *Mécanique céleste* (1799), Laplace still complained that the potential equation in spherical coordinates could not be integrated in general form. Appreciation of the fact that general solutions such as Euler and d'Alembert obtained for the vibrating string were not as useful as particular ones satisfying initial and boundary conditions was not attained in that century.

The mathematicians did realize that partial differential equations involved no new operational techniques but differed from ordinary differential equations in that arbitrary functions might appear in the solution. These they expected to determine by reducing partial differential equations to ordinary ones. Laplace (1773) and Lagrange (1785) say clearly that they regard a partial differential equation as integrated when it is reduced to a problem of ordinary differential equations. An alternative, such as Daniel Bernoulli used for the wave equation and Laplace for the potential equation, was to seek expansions in series of special functions.

The major achievement of the eighteenth-century work on partial differential equations was to reveal their importance for problems of elasticity,

51. *Misc. Taur.*, 3_2, 1762/65, 60–91, pub. 1766 = *Opera*, (1), 23, 42–73.
52. 1770 = *Opera*, (1), 13.

hydrodynamics, and gravitational attraction. Except in the case of Lagrange's work on first order equations, no broad methods were developed, nor were the potentialities in the method of expansion in special functions appreciated. The efforts were directed to solving the special equations that arose in physical problems. The theory of the solution of partial differential equations remained to be fashioned and the subject as a whole was still in its infancy.

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not take time to cover. Monge's integration of the equation for minimal surfaces was one of his claims to glory.

For the equation (87) also Monge introduced the theory of characteristics. The total differential equation of the characteristics is (92), that is,

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0.$$

This equation, which appears in his work as early as 1784,[48] defines at each point of an integral surface two directions that are the characteristic directions at that point. Through each point on an integral surface there pass two characteristic curves, along each of which two consecutive integral surfaces touch each other.

8. *Systems of First Order Partial Differential Equations*

Systems of partial differential equations arose first in the eighteenth-century work on fluid dynamics or hydrodynamics. The work on incompressible fluids, for example, water, was motivated by such practical problems as designing the hulls of ships to reduce resistance to motion in water and calculating the tides, the flow of rivers, the flow of water from jets, and the pressure of water on the sides of a ship. The work on compressible fluids, air in particular, sought to analyze the action of air on sails of ships, the design of windmill vanes, and the propagation of sound. The work we studied earlier on the propagation of sound was historically an application of the work on hydrodynamics specialized to waves of small amplitude.

After having treated incompressible fluids in 1752 in a paper entitled "Principles of the Motion of Fluids,"[49] Euler generalized this work in a paper of 1755, entitled "General Principles of the Motion of Fluids."[50] Here he gave the still-famous equations of fluid flow for perfect (nonviscous) compressible and incompressible fluids. The fluid is regarded as a continuum and the particles are mathematical points. He considers the force acting on a small volume of the fluid subject to the pressure p, density ρ, and external forces with components P, Q, and R per unit mass.

In one of the two approaches to fluid dynamics that Euler created, known in the literature as the spatial description, the components u, v, and w of the fluid velocity are given at every point in the fluid by

$$(97)\qquad u = u(x, y, z, t), \qquad v = v(x, y, z, t), \qquad w = w(x, y, z, t).$$

Now

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt.$$

48. *Hist. de l'Acad. des Sci., Paris*, 1784, 118–92, pub. 1787.
49. *Novi Comm. Acad. Sci. Petrop.*, 6, 1756/57, 271–311, pub. 1761 = *Opera*, (2), 12, 133–68.
50. *Hist. de l'Acad. de Berlin*, 11, 1755, 274–315, pub. 1757 = *Opera*, (2), 12, 54–91.

In time dt, the particle at (x, y, z) will travel a distance $u\,dt$ in the x-direction, $v\,dt$ in the y-direction, and $w\,dt$ in the z-direction. Then the actual changes dx, dy, and dz in the expression for du are given by these quantities, so that

$$du = \frac{\partial u}{\partial x}u\,dt + \frac{\partial u}{\partial y}v\,dt + \frac{\partial u}{\partial z}w\,dt + \frac{\partial u}{\partial t}dt$$

or

$$(98)\qquad \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

and there are the corresponding expressions for dv/dt and dw/dt. These quantities give what is called now the convective rate of change of the velocity at (x, y, z) or the convective acceleration. By calculating the forces acting on the particle at (x, y, z) and applying Newton's second law, Euler obtains the system of differential equations

$$(99)\qquad \begin{aligned} P - \frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{du}{dt} \\ Q - \frac{1}{\rho}\frac{\partial p}{\partial y} &= \frac{dv}{dt} \\ R - \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{dw}{dt}. \end{aligned}$$

Euler also generalized d'Alembert's differential equation of continuity (45) and obtained for compressible flow the equation

$$(100)\qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0.$$

There are four equations and five unknowns, but the pressure p as a function of the density, the equation of state, must be specified.

In the 1755 paper Euler says, "And if it is not permitted to us to penetrate to a complete knowledge concerning the motion of fluids, it is not to mechanics, or to the insufficiency of the known principles of motion, that we must attribute the cause. It is analysis itself which abandons us here, since all the theory of the motion of fluids has just been reduced to the solution of analytic formulas." Unfortunately analysis was still too weak to do much with these equations. He then proceeds to discuss some special solutions. He also wrote other papers on the subject and dealt with the resistance encountered by ships and ship propulsion. Euler's equations are not the final ones for hydrodynamics. He neglected viscosity, which was introduced seventy years later by Navier and Stokes (Chap. 28, sec. 7).

Lagrange, too, worked on fluid motion. In the first edition of his *Mécanique analytique*, which contains some of the work, he gave Euler's be a tangent plane of the Monge cone at (x, y, z). A curve C on an integral surface S is called a characteristic curve if at each point of C the tangent is a generator of the Monge cone at that point. These characteristic curves are the same as those Monge deduced from the complete integral illustrated by (80) and are the solutions of the simultaneous equations (79). He also points out in a paper of 1802 which he incorporated in his *Application de l'analyse à la géométrie*[46] (1807) that each integral surface is a locus of characteristic curves and only one characteristic curve passes through each point of the integral surface.

The significance of the characteristic curves lies in the following. If one chooses a space curve $x(t), y(t), z(t)$ (for some interval of t-values) that is not a characteristic curve, then there is just one integral surface of $F = 0$ that passes through this curve; that is, there is just one $z = g(x, y)$ such that $z(t) = g(x(t), y(t))$ (for the range of t-values). On the other hand, as Monge noted in lectures of 1806, through any characteristic curve one can pass an infinity of integral surfaces. Moreover, the infinity of integral surfaces that pass through the curve are tangent to each other on that curve.

7. *Monge and Nonlinear Second Order Equations*

In addition to the second order linear equations we have already reviewed, the eighteenth-century mathematicians had occasion to consider more general linear second order equations in two independent variables and even nonlinear ones. Thus they studied the linear equation

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + D\frac{\partial z}{\partial x} + E\frac{\partial z}{\partial y} + Fz + G = 0,$$

where $A, B, \cdots, G$ are functions of x and y. This equation is commonly written as

$$Ar + Bs + Ct + Dp + Eq + Fz + G = 0, \tag{85}$$

where the letters r, s, t, p, and q have the obvious meanings. Laplace showed in 1773[47] that equation (85) can, by a change of variables, be reduced to the form

$$s + ap + bq + cz + g = 0, \tag{86}$$

where a, b, c, and g are functions of x and y only, provided that $B^2 - 4AC \neq 0$. He then solved the equation in terms of an infinite series.

In his *Feuilles d'analyse*, Monge considered the nonlinear equation

$$Rr + Ss + Tt = V \tag{87}$$

46. This is the title of the third edition of his *Feuilles d'analyse*.
47. *Hist. de l'Acad. des Sci., Paris*, 1773, pub. 1777 = *Œuvres*, 9, 5–68.

in which R, S, T, and V are functions of x, y, z, p, and q, so that the equation is linear only in the second derivatives r, s, and t. This type of equation arose in Lagrange's work on minimal surfaces, that is, surfaces of least area bounded by given space curves, wherein the specific differential equation is $(1 + q^2)r - 2pqs + (1 + p^2)t = 0$. (See also Chap. 24, sec. 4). Though Monge had already done some work on equation (87), in the present work (1795) he was able to solve it elegantly by the method we shall sketch.

By using the immediate facts

$$dz = p\,dx + q\,dy \tag{88}$$

$$dp = r\,dx + s\,dy \tag{89}$$

$$dq = s\,dx + t\,dy \tag{90}$$

and eliminating r and t from (87), (89), and (90) he obtained the equation

$$s(R\,dy^2 - S\,dx\,dy + T\,dx^2) - (R\,dy\,dp + T\,dx\,dq - V\,dx\,dy) = 0. \tag{91}$$

His argument then was that whenever it is possible to solve simultaneously

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \tag{92}$$

and

$$R\,dy\,dp + T\,dx\,dq - V\,dx\,dy = 0 \tag{93}$$

then (91) will be satisfied and so will (87).

Equation (92) is equivalent to two first order equations

$$dy - W_1(x, y, z, p, q)\,dx = 0 \quad \text{and} \quad dy - W_2(x, y, z, p, q) = 0. \tag{94}$$

Equations (88) and (93), together with either one of (94), constitute a system of three total differential equations in the five variables x, y, z, p, and q. When these three equations can be solved it is possible to find two solutions

$$u_1(x, y, z, p, q) = C_1 \quad \text{and} \quad u_2(x, y, z, p, q) = C_2$$

and then

$$u_1 = \phi(u_2), \tag{95}$$

where ϕ is arbitrary, is a first order partial differential equation. The equation (95) is called an intermediate integral. Its general solution is the solution of (87). If the other equation in (94) can be used together with (88) and (93), we get another function

$$u_3 = \phi(u_4). \tag{96}$$

In this case (95) and (96) can be solved simultaneously for p and q and these values are substituted in (88); then this total differential equation can be solved. This at least is the general scheme, though there are details we shall

6. Monge and the Theory of Characteristics

Lagrange worked purely analytically. Gaspard Monge (1746–1818) introduced the language of geometry. His work in partial differential equations was not as great as that of Euler, Lagrange, and Legendre, but he started the movement to interpret the analytic work geometrically and introduced thereby many fruitful ideas. He saw that just as problems involving curves led to ordinary differential equations, so problems involving surfaces lead to partial differential equations. More generally, for Monge geometry and analysis were one subject, whereas for the other mathematicians of the century the two branches were distinct, with just points of contact. Monge began his work in 1770 but did not publish until much later.

It was primarily in the subject of nonlinear first order equations that Monge not only introduced the geometric interpretation but emphasized a new concept, that of characteristic curves.[45] His ideas on characteristics and on integrals as envelopes were not understood and were called a metaphysical principle by his contemporaries, but the theory of characteristics was to become a very significant theme in later work. Monge developed his ideas more fully in his lectures and in subsequent publications, notably the *Feuilles d'analyse appliquée à la géométrie* (1795). The ideas are best illustrated by his own example.

Consider the two-parameter family of spheres, all of constant radius R and centers anywhere in the XY-plane. The equation of this family is

$$(80)\qquad (x-a)^2+(y-b)^2+z^2=R^2.$$

This equation is the complete integral of the nonlinear first order partial differential equation

$$(81)\qquad z^2(p^2+q^2+1)=R^2,$$

because (80) contains two arbitrary constants a and b and clearly satisfies (81). Any subfamily of spheres introduced by letting $b=\phi(a)$ is a family of spheres with centers on a curve, the curve $y=\phi(x)$ in the XY-plane. The envelope of this one-parameter family of spheres (a tubular surface) is also a solution of (81). This particular solution is obtained by eliminating a from

$$(82)\qquad (x-a)^2+(y-\phi(a))^2+z^2=R^2$$

and the partial derivative of (82) with respect to a, namely,

$$(83)\qquad (x-a)+(y-\phi(a))\phi'(a)=0.$$

For each particular choice of a, equations (82) and (83) represent two particular surfaces, and therefore the two considered simultaneously represent

45. *Hist. de l'Acad. des Sci., Paris*, 1784, 85–117, 118–92, pub. 1787.

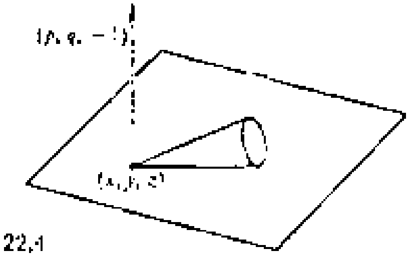

Figure 22.4

sent a curve called a characteristic curve. This curve is also the curve of intersection of two "consecutive" members of the subfamily. The set of characteristic curves fills out the envelope; that is, the envelope touches each member of the subfamily along a characteristic. The general integral is the aggregate of surfaces (envelopes of one-parameter families), each generated by a set of characteristic curves. The envelope of all the solutions of (80), that is, the singular solution, obtained by eliminating a and b from (80) and the partial derivatives of (80) with respect to a and b respectively, is $z=\pm R$.

The characteristic curve appears in another way. Consider two subfamilies of spheres whose envelopes are tangential along any one sphere. We might call such envelopes consecutive envelopes. The curve of intersection of these two consecutive envelopes is the same characteristic curve on the sphere as the one obtained by considering consecutive members of either subfamily. Any one sphere may belong to an infinity of different subfamilies whose envelopes are all different, and so there will be different characteristic curves on the same sphere. All are great circles in vertical planes.

Monge gave the analytical form of the differential equations of the characteristic curves, which amounts to the fact that equations (79) determine the characteristic curves of (69). (Monge used total differential equations to express the equations of the characteristic curves.)

Monge also introduced (1784) the notion of a characteristic cone. At any point (x, y, z) of space (Fig. 22.4) one may consider a plane whose normal has the direction numbers $p, q, -1$. For a fixed (x, y, z), the set of p and q which satisfy

$$(84)\qquad F(x,y,z,p,q)=0$$

determines a one-parameter family of planes all passing through (x, y, z). This set of planes envelopes a cone with vertex at (x, y, z). This is the characteristic cone or Monge cone at (x, y, z). If we now consider a surface S whose equation is $z=g(x,y)$, then the surface has a tangent plane at each (x, y, z). A necessary and sufficient condition that such a surface be an integral surface of $F=0$ is that at each point (x, y, z) the tangent plane of F

If one puts into the last of these three equations the values of $\partial M/\partial x$ and $\partial M/\partial y$ from the first two, one obtains

$$(73)\qquad \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} - p\frac{\partial q}{\partial z} + q\frac{\partial p}{\partial z} = 0.$$

This is the condition (68) for the integrability of (72), a condition known before, as Lagrange remarks. In (73) q can be taken as the given function Q of x, y, z, and p so that explicitly the equation becomes

$$(74)\qquad -Q_p\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} + (Q - pQ_p)\frac{\partial p}{\partial z} - Q_x - pQ_z = 0.$$

Lagrange's plan now is to find a solution $p = P$ of this first order equation, which is *linear* in the derivatives of p and whose solution contains an arbitrary constant α. Having obtained this, he integrates the two equations

$$(75)\qquad q - Q(x, y, z, p) = 0, \qquad p - P(x, y, z, \alpha) = 0,$$

which represent $\partial z/\partial x$ and $\partial z/\partial y$ as functions of x, y, z, and finds a family of ∞^2 integral surfaces of the original equation (70); that is, he finds the complete solution. Thus far, then, Lagrange has replaced the problem of solving the nonlinear equation (70) by the problem of solving the *linear* equation (74).

In 1779 Lagrange gave his method of solving linear first order partial differential equations.[42] He considers the equation still called Lagrange's linear equation

$$(76)\qquad Pp + Qq = R,$$

where P, Q, and R are functions of x, y, and z; the equation is called nonhomogeneous because of the presence of the R term. This equation is intimately related to the homogeneous equation in three independent variables

$$(77)\qquad P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = 0.$$

What Lagrange shows readily is that if $u(x, y, z) = c$ is a solution of (76), then $f = u(x, y, z)$ is a solution of (77) and conversely. Hence the problem of solving (76) is equivalent to that of solving (77). The equation (77) in turn is related to the system of ordinary differential equations

$$(78)\qquad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q}{P} \quad \text{and} \quad \frac{dz}{dx} = \frac{R}{P}.$$

In fact, if $f = u(x, y, z)$ and $f = v(x, y, z)$ are two independent solutions of (77), then $u = c_1$ and $v = c_2$ are a solution of (78) and conversely. Hence, if we can find the solutions $u = c_1$ and $v = c_2$ of (78), $f = u$ and $f = v$ will be solutions of (77) and $u = c$ and $v = c$ will be solutions of (76). Moreover,

42. *Nouv. Mém. de l'Acad. de Berlin*, 1779 = *Œuvres*, 4, 585–634.

one can show readily that $f = \phi(u, v)$ where ϕ is an arbitrary function of u and v also satisfies (77). Then $\phi(u, v) = c$ or, since ϕ is arbitrary, $\phi(u, v) = 0$ is a general solution of (76). Lagrange gave the above scheme in 1779 and gave a proof in 1785.[43] It is perhaps worth noting incidentally that Euler was aware that the solution of (77) can be reduced to the solution of (78).

If one takes this work on linear equations in connection with the 1772 work on nonlinear equations, one sees that Lagrange had succeeded in reducing an arbitrary first order equation in x, y, and z to a system of simultaneous ordinary differential equations. He does not state the result explicitly but it follows from the above work. Curiously, in 1785 he had to solve a particular first order partial differential equation and said it was impossible with present methods; he had forgotten his earlier (1772) work.

Then Paul Charpit (d. 1784) in 1784 presumably combined the methods for nonlinear and linear equations to reduce any $f(x, y, z, p, q) = 0$ to a system of ordinary differential equations. Lacroix said in 1798 that Charpit had submitted a paper in 1784 (which was not published) in which he reduced first order partial differential equations to systems of ordinary differential equations. Jacobi found Lacroix's statement striking and expressed the wish that Charpit's work be published. But this was never done and we do not know whether Lacroix's statement is correct. Actually Lagrange had done the full job and Charpit could have added nothing. The method given in modern texts, called the Lagrange, Lagrange-Charpit, or Charpit method, is the fusion of the ideas Lagrange presented in his 1772 and 1779 papers. It states that to solve the general first order partial differential equation $f(x, y, z, p, q) = 0$ one must solve the system of ordinary differential equations (the characteristic equations of $f = 0$),

$$(79)\qquad \begin{aligned} &\frac{dx}{dt} = f_p, \qquad \frac{dy}{dt} = f_q, \qquad \frac{dz}{dt} = pf_p + qf_q \\ &\frac{dp}{dt} = -f_x - f_z p, \qquad \frac{dq}{dt} = -f_y - f_z q. \end{aligned}$$

The solution is effected by finding any one integral of (79), say $u(x, y, z, p, q) = A$. One solves this and $f = 0$ simultaneously for p and q and substitutes for p and q in $dz = p\,dx + q\,dy$ (see [72]). Then one integrates by the method used for (65).

Lagrange's method is often called Cauchy's method of characteristics because the generalization to n variables of the method of arriving at (79) used by Lagrange and Charpit for a differential equation in two independent variables presents difficulties which were surmounted by Cauchy in 1819.[44]

43. *Nouv. Mém. de l'Acad. de Berlin*, 1785 = *Œuvres*, 5, 543–62.
44. *Bull. de la Société Philomathique*, 1819, 10–21; see also *Exercices d'analyse et de phys. math.*, 2, 238–72 = *Œuvres*, (2), 12, 272–309.

primary attention because the physical problems led directly to them. A few special first order equations had been solved, but these were either readily integrated or integrated by tricks. There was one exception, the equation usually called today a total differential equation and having the form

$$(65) \qquad P\,dx + Q\,dy + R\,dz = 0,$$

where P, Q, and R are functions of x, y, and z. Such an equation, if integrable, defines z as a function of x and y. Clairaut encountered such equations in 1739 in his work on the shape of the earth.[38] If the expression on the left side of (65) is an exact differential, that is, if there is a function $u(x, y, z) = C$ such that

$$(66) \qquad du = P\,dx + Q\,dy + R\,dz,$$

then Clairaut points out that

$$(67) \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Clairaut showed how to solve (65) by a method still used in modern texts. The interest in equation (65) stemmed from the fact that if P, Q, R are components of velocity in fluid motion, then (65) has to be an exact differential.

If (65) is not an exact differential, then Clairaut also showed that it may be possible to find an integrating factor, that is, a function $\mu(x, y, z)$ such that when multiplied into (65), it makes the new left side an exact differential. Clairaut[39] and later d'Alembert (*Traité de l'équilibre et du mouvement des fluides*, 1744) gave a necessary condition that it be integrable (with the aid of an integrating factor). This condition (which is also sufficient) is

$$(68) \qquad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

The *general* first order partial differential equation in two independent variables is of the form

$$(69) \qquad f(x, y, z, p, q) = 0,$$

where $p = \partial x/\partial z$ and $q = \partial z/\partial y$. If the equation is linear in p and q then it is a linear partial differential equation and if not, then nonlinear. The significant theory was contributed by Lagrange.

Lagrange's terminology, which is still current, must be noted first to understand his work. He classified solutions of nonlinear first order equations as follows. Any solution $V(x, y, z, a, b) = 0$ containing two arbitrary constants is the *complete solution or complete integral*. By letting $b = \phi(a)$ where ϕ

38. *Mém. de l'Acad. des Sci., Paris*, 1739, 425–36.
39. *Mém. de l'Acad. des Sci., Paris*, 1740, 293–323.

is arbitrary, we obtain a one-parameter family of solutions. When $\phi(a)$ is arbitrary, the envelope of this family is called a *general integral*. When a definite $\phi(a)$ is used, the envelope is a *particular case* of the general integral. The envelope of all the solutions in the complete integral is called the *singular integral*. We shall see later what these solutions are geometrically. The complete integral is not unique in that there can be many different ones, which are not obtainable from each other by a simple change in the arbitrary constants. But from any one we can get all the solutions given by another through the particular cases and the singular integral.

Between two important papers of 1772 and 1779, which we shall take up shortly, Lagrange wrote a paper in 1774 discussing the relationships among complete, general, and singular solutions of first order partial differential equations. The general integral is obtained by eliminating a from $V(x, y, z, a, \phi(a)) = 0$ and $\partial V/\partial a = 0$ where $\phi(a)$ is arbitrary.[40] (For a particular $\phi(a)$ we get a particular solution.) The singular solution is obtained by eliminating a and b from $V(x, y, z, a, b) = 0$, $\partial V/\partial a = 0$ and $\partial V/\partial b = 0$.

Lagrange gave first the general theory of *nonlinear* first order equations. In the paper of 1772[41] he considered a general first order equation with two independent variables x and y and the dependent variable z. Here he improved on and generalized what Euler did earlier. He regarded equation (69) as given in the form where q is a function of x, y, z, and p, namely,

$$(70) \qquad q - Q(x, y, z, p) = 0$$

and sought to determine p as a function P of x, y, and z so that the two equations

$$(71) \qquad q - Q(x, y, z, p) = 0 \quad \text{and} \quad p - P(x, y, z) = 0$$

have a single infinity of common integral surfaces, or, as Lagrange put it analytically, so that the expression

$$(72) \qquad dz - p\,dx - q\,dy$$

by multiplication by a suitable factor $M(x, y, z)$ becomes an exact differential dN of $N(x, y, z) = 0$. For this to be so he must have

$$\frac{\partial N}{\partial z} = M, \qquad \frac{\partial N}{\partial x} = -Mp, \qquad \frac{\partial N}{\partial y} = -Mq.$$

For these equations the integrability conditions (67) imply

$$\frac{\partial M}{\partial x} = -\frac{\partial (Mp)}{\partial z}, \qquad \frac{\partial M}{\partial y} = -\frac{\partial (Mq)}{\partial z}, \qquad \frac{\partial (Mp)}{\partial y} = \frac{\partial (Mq)}{\partial x}.$$

40. For arbitrary ϕ it is not generally possible to actually carry out the elimination of a. The general integral is a concept and amounts to a collection of particular solutions.
41. *Nouv. Mém. de l'Acad. de Berlin*, 1772 = *Œuvres*, 3, 549–75.

He deals with rational integral functions of μ, $\sqrt{1-\mu^2}\cos\phi$, and $\sqrt{1-\mu^2}\sin\phi$, and so his need for the general result is limited. He does prove the basic orthogonality property

$$(59)\qquad \int_{-1}^{1}\int_{0}^{2\pi} U_n(\mu,\phi)U_m(\mu,\phi) = 0, \qquad m \neq n.$$

However in his *Mécanique céleste*, Volume 2, he shows that an arbitrary function of θ and ϕ can be expanded in a series of the U_n (or the Y_n) and shows that (59) implies uniqueness of the expansion.

Laplace wrote several more papers on the attraction of spheroids and on the shape of the earth (e.g. 1783, pub. 1786; 1787, pub. 1789); and in these papers uses expansions in spherical functions. In the last paper, in which Laplace gave the rectangular coordinate form of the potential equation, he made one mistake of consequence. He assumed that this equation holds when the point mass being attracted by a body lies inside the body. This error was corrected by Poisson (Chap. 28, sec. 4).

During the 1780s Legendre continued his investigations. His fourth paper, written in 1790,[56] introduced the $P_n(x)$ for *odd* n. The expression for $P_n(x)$ given in (47) is the correct one for all n. Legendre proves that for any positive integral m and n

$$(60)\qquad \int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 \text{ for } m \neq n, \\ \dfrac{2}{2n+1} \text{ for } m = n. \end{cases}$$

Then he too introduces the spherical functions. That is, he lets Y_n be the coefficient of z^n in the expansion of $(1 - 2zt + z^2)^{-1/2}$ where $t = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$. Then, letting $\mu = \cos\theta$, $\mu' = \cos\theta'$ and $\psi = \phi - \phi'$, he shows that

$$Y_n(t) = P_n(\mu)P_n(\mu') + \frac{2}{n(n+1)}\frac{dP_n(\mu)}{d\mu}\frac{dP_n(\mu')}{d\mu'}\sin\theta\sin\theta'\cos\psi$$
$$+ \frac{2}{(n-1)(n)(n+1)(n+2)}\frac{d^2P_n(\mu)}{d\mu^2}\frac{d^2P_n(\mu')}{d\mu'^2}\sin^2\theta\sin^2\theta'\cos 2\psi + \cdots,$$

the higher terms containing higher derivatives of P_n. This equation is equivalent to

$$P_n(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')) = \sum_{m=1}^{n} P_n^m(\cos\theta)P_n^m(\cos\theta')\cos m(\phi - \phi'),$$

56. *Mém. de l'Acad. des Sci., Paris*, 1789, 372–454, pub. 1793.

wherein the m is a superscript and not an exponent. The $P_n^m(x)$ satisfy

$$(61)\qquad \frac{d}{dx}\left[(1-x^2)\frac{dP_n^m}{dx}\right] + \left[n(n+1) - \frac{\nu^2}{1-x^2}\right]P_n^m = 0$$

and the $P_n^m(x)$ agree up to a constant factor with

$$(1-x^2)^{m/2}\frac{d^mP_n(x)}{dx^m}.$$

The $P_n^m(x)$ thus introduced are now called the associated Legendre polynomials. Then Legendre proves that

$$(62)\qquad \int_{-1}^{1}\int_{0}^{2\pi} U_n(\mu',\phi')P_n(\mu,\phi,\mu',\phi')\,d\mu'\,d\phi' = \frac{4\pi}{2n+1}U_n(\mu,\phi)$$

and that

$$(63)\qquad \int_{-1}^{1}\int_{0}^{2\pi}(P_n(\mu))^2\,d\mu\,d\phi = \frac{4\pi}{2n+1}.$$

The fact that the $P_n(x)$ satisfy Legendre's differential equation is used in this paper.

Many other special results involving the Legendre polynomials and the spherical harmonics were obtained by Legendre, Laplace, and others. A basic result is the formula of Olinde Rodrigues (1794–1851), given in 1816,[57]

$$(64)\qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}.$$

Laplace's work on the solution of the potential equation for the attracting force of spheroids was the beginning of a vast amount of work on this subject. Equally important was his and Legendre's work on the Legendre polynomials $P_n(x)$, the associated Legendre polynomials $P_n^m(x)$, and the spherical (surface) harmonics $Y_n(\mu,\phi)$, because rather arbitrary functions can be expressed in terms of infinite series of the P_n, P_n^m and the Y_n. These series of functions are analogous to the trigonometric functions, which Daniel Bernoulli claimed could also be used to represent arbitrary functions. The choice of class of functions depends on the differential equation being solved and on the initial and boundary conditions. Of course far more had to be done and was to be done with these functions to render them more useful in the solution of partial differential equations.

5. *First Order Partial Differential Equations*

Up to the time of Lagrange there was very little systematic work on first order partial differential equations. The second order equations received

57. *Corresp. sur l'École Poly.*, 3, 1816, 361–85.

He also proved that the zeros of each of the P_{2n} are real, different from each other, symmetric with respect to 0, and in absolute value less than 1. Also for $0 < x < 1$, $P_{2n}(x) < 1$.

Then, with the help of the orthogonality condition (50), he proves (by integration of the series term by term) that a given function of x^2 can be expressed in only one way in a series of functions $P_{2n}(x)$.

Finally, using these and other properties of his polynomials, Legendre returns to the main problem of gravitational attraction and, using the expression (43) for the potential and the condition for equilibrium of a rotating fluid mass, he obtains the equation for the meridian curve of such a mass in the form of a series of his polynomials. He believed that this equation included all possible equilibrium figures for a spheroid of revolution.

Now Laplace enters the picture. He had written several papers on the force of attraction exerted by volumes of revolution (1772, pub. 1776; 1773, pub. 1776; and 1775, pub. 1778), in which he worked with the components of the force but not the potential function. The article by Legendre of 1782, published in 1785, inspired a famous and remarkable fourth paper by Laplace, "Théorie des attractions des sphéroïdes et de la figure des planètes."[33] Without mentioning Legendre, Laplace took up the problem of the attraction exerted by an arbitrary spheroid as opposed to Legendre's figures of revolution. By a spheroid Laplace meant any surface given by one equation in r, θ, and ϕ.

He starts with the theorem that the potential V of the force an arbitrary body exerts on an external point, expressed in spherical coordinates r, θ, ϕ with $\mu = \cos\theta$, satisfies the potential equation

$$(51)\qquad \frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2 V}{\partial\phi^2} + r\frac{\partial^2(rV)}{\partial r^2} = 0.$$

Laplace does not say here how he obtained the equation. In a later paper[34] he gives the rectangular coordinate form (34). One may be fairly sure that he possessed the rectangular form first and derived the spherical coordinate form from it. In fact, both forms had already been given by Euler and Lagrange, but Laplace does not mention them. He may not have known their work, though this is doubtful.

In the 1782 paper Laplace sets

$$(52)\qquad V(r,\theta,\phi) = \frac{U_0}{r} + \frac{U_1}{r^2} + \frac{U_2}{r^3} + \cdots,$$

33. *Mém. de l'Acad. des Sci., Paris*, 1782, 113–196, pub. 1785 = *Œuvres*, 10, 339–419.
34. *Mém. de l'Acad. des Sci., Paris*, 1787, 249–267, pub. 1789 = *Œuvres*, 11, 275–92.

where $U_n = U_n(\theta, \phi)$, and substitutes this in (51). Then the individual U_n satisfy[35]

$$(53)\qquad \frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial U}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2 U}{\partial\phi^2} + n(n+1)U = 0.$$

With the help of Legendre's P_{2n} he is able to show that

$$(54)\qquad U_n(\theta,\phi) = \iiint r'^{n+2} P_{2n}(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'))\cdot \sin\theta'\, d\theta'\, d\phi'\, dr'.$$

Now Laplace uses this result and (52) to calculate the potential of a spheroid differing little from a sphere. He writes the equation of the surface of the spheroid as

$$(55)\qquad r = a(1+\alpha y),$$

where α is small and y on the spheroid is a function of θ' and ϕ'. Laplace assumes that $y(\theta, \phi)$ can be expanded in a series of functions

$$(56)\qquad y = Y_0 + Y_1 + Y_2 + \cdots,$$

where the Y_n are functions of θ and ϕ and satisfy the differential equation (53). The result he obtains here is first that

$$(57)\qquad Y_n = \frac{2n+1}{4\pi\alpha a^{n+3}}\, U_n.$$

These Y_n, then, may be used in (52). Also the expansion (56) may be recast as

$$(58)\qquad y(\mu,\phi) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)\int_{-1}^{1}\int_{0}^{2\pi} Y_n(\mu',\phi')P_n(\mu,\phi,\mu',\phi')\,d\mu'\,d\phi',$$

where $\mu' = \cos\theta'$. With the value of y he now has an expression for r in (55) and with this and the U_n in (54) he obtains V in (52).

Laplace does not consider here the general problem of the development of *any* function of θ and ϕ into a series of the Y_n. In what he does do here and in later papers he presumes that such an expression is possible and is unique.

35. If we ignore the middle term (ϕ is absent) the resulting ordinary differential equation is what we now call Legendre's differential equation,

$$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0.$$

The $P_n(x)$ satisfy this equation. On the other hand the U_n (and the Y_n in (57)) regarded as functions of the two variables $\mu = \cos\theta$ and ϕ satisfy (53). The U_n and Y_n are called by the Germans spherical functions and by Lord Kelvin spherical harmonics or spherical surface harmonics.

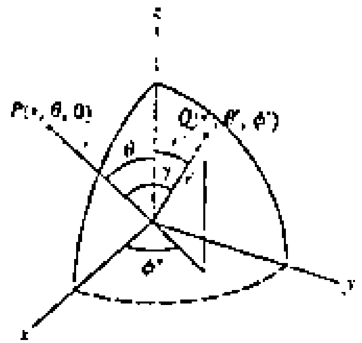

Figure 22.3

where (Fig. 22.3) r is the radius vector to the attracted point, r' is the radius vector to any point of the attracting body, and γ is the angle formed at the center of the body by the two radii vectors. The ϕ coordinate of the external point can be taken to be 0 because the solid is a figure of revolution around the z-axis. Then he expanded the integrand in powers of r'/r. This is done by writing the denominator as

$$r^3\left[1 - \left(2\frac{r'}{r}\cos\gamma - \frac{r'^2}{r^2}\right)\right]^{3/2}.$$

The quantity in the brackets can be put into the numerator and then expanded by the binomial theorem with the quantity in parentheses as the second term of the binomial. Legendre obtained for the integrand, apart from the volume element, the series

$$\frac{1}{r^2}\left\{1 + 3P_2(\cos\gamma)\frac{r'^2}{r^2} + 5P_4(\cos\gamma)\frac{r'^4}{r^4} + 7P_6(\cos\gamma)\frac{r'^6}{r^6} + \cdots\right\}.$$

The coefficients $P_2, P_4, \ldots$ are rational integral functions of $\cos\gamma$. These functions are what we now call the Legendre polynomials or Laplace coefficients or zonal harmonics. Legendre gave the form of the functions so that the general P_n, namely,

$$(47)\qquad P_n(x) = \frac{(2n-1)(2n-3)\cdots 1}{n!}\cdot$$
$$\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} + \cdots\right],$$

could be inferred.

He could now integrate with respect to r', and he obtained

$$\frac{2}{r^3}\iint\left\{\frac{R^3}{3} + \frac{3}{5}P_2(\cos\gamma)\frac{R^5}{r^2} + \frac{5}{7}P_4(\cos\gamma)\frac{R^7}{r^4} + \cdots\right\}\sin\theta'\,d\theta'\,d\phi',$$

where $R = f(\theta')$ is the value of r' at a given θ' (it is independent of ϕ'). He then had to integrate with respect to ϕ'. For this he used[31]

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'.$$

Having established the subsidiary result

$$\frac{1}{\pi}\int_0^{\pi} P_{2n}(\cos\gamma)\,d\phi' = P_{2n}(\cos\theta)P_{2n}(\cos\theta'),$$

he obtained finally

$$P(r, \theta, 0) = \frac{3M}{r^2}\sum_{n=0}^{\infty}\frac{2n-3}{2n-1}P_{2n}(\cos\theta)\frac{\alpha_n}{r^{2n}},$$

where

$$\alpha_n = \frac{4\pi}{3M}\int_0^{\pi/2} R^{2n+3}P_{2n}(\cos\theta')\sin\theta'\,d\theta'.$$

The value of this integral depends upon the shape of the meridian curves $R = f(\theta')$.

From the above result, and on the basis of a communication from Laplace, Legendre then obtained the expression for the potential function for this problem, and from the potential derived the component of the force of attraction perpendicular to the radius vector.

In a second paper written in 1784,[32] Legendre derived some properties of the functions P_{2n}. Thus

$$(48)\qquad \int_0^1 f(x^2)P_{2n}(x)\,dx = 0$$

for each rational integral function of x^2 whose degree in x^2 is less than n. If n is any positive integer,

$$(49)\qquad \int_0^1 x^n P_{2m}\,dx = \frac{n(n-2)\cdots(n-2m+2)}{(n+1)(n+3)\cdots(n+2m+1)}.$$

If m and n are positive integers,

$$(50)\qquad \int_0^1 P_{2n}(x)P_{2m}(x)\,dx = \begin{cases} 0 & \text{for } m \gtrless n, \\ \dfrac{1}{4m+1} & \text{for } m = n. \end{cases}$$

31. This expression is derived as follows: In view of the equations of transformation from spherical to rectangular coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the rectangular coordinates of P are $(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ and the rectangular coordinates of Q are $(r'\sin\theta'\cos\phi', r'\sin\theta'\sin\phi', r'\cos\theta')$. Then using the distance formula we can express PQ. But by the law of cosines $PQ = r^2 + r'^2 + 2rr'\cos\gamma$. Equating the two expressions for PQ gives the above expression for $\cos\gamma$.

32. *Mém. de l'Acad. des Sci., Paris*, 1784, 370–89, pub. 1787.

The force exerted by the entire body on the unit mass at P has the components

$$f_x = -k\iiint \rho\,\frac{x-\xi}{r^3}\,d\xi\,d\eta\,d\zeta$$

(42)
$$f_y = -k\iiint \rho\,\frac{y-\eta}{r^3}\,d\xi\,d\eta\,d\zeta$$

$$f_z = -k\iiint \rho\,\frac{z-\zeta}{r^3}\,d\xi\,d\eta\,d\zeta,$$

wherein the integral is extended over the entire attracting body. These integrals are finite and correct also when P is inside the attracting body.

Instead of treating each component of the force separately, it is possible to introduce one function $V(x, y, z)$ whose partial derivatives with respect to x, y, and z respectively are the three components of the force. This function is

(43)
$$V(x, y, z) = \iiint \frac{\rho}{r}\,d\xi\,d\eta\,d\zeta.$$

By differentiating under the integral sign with respect to x, y, and z which are involved in r, one obtains

$$\frac{\partial V}{\partial x} = \frac{1}{k}f_x, \qquad \frac{\partial V}{\partial y} = \frac{1}{k}f_y, \qquad \frac{\partial V}{\partial z} = \frac{1}{k}f_z$$

and these equations also hold when P is inside the attracting body. The function V is called a potential function. When problems involving the three components f_x, f_y, and f_z can be reduced to the problem of working with V, there is the advantage of working with one function instead of three.

If one knows the distribution of mass inside the body, which means knowing ρ as a function of ξ, η, and ζ, and if one knows the precise shape of the body, one can sometimes calculate V by actually evaluating the integral. However, for most shapes of bodies this triple integral is not integrable in terms of simple functions. Moreover, we do not know the true distribution of mass inside the earth and other bodies. Hence V must be determined in other ways. The principal fact about V is that for points (x, y, z) *outside* the attracting body, it satisfies the partial differential equation

(44)
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

in which we note that ρ does not appear. This differential equation is known as the potential equation and as Laplace's equation.

The idea that a force can be derived from a potential function, and even the term "potential function," were used by Daniel Bernoulli in

Hydrodynamica (1738). The potential equation itself appears for the first time in one of Euler's major papers composed in 1752, "Principles of the Motion of Fluids."[28] In dealing with the components u, v, and w of the velocity of any point in a fluid, Euler had shown that $u\,dx + v\,dy + w\,dz$ must be an exact differential. He introduces the function S such that $dS = u\,dx + v\,dy + w\,dz$. Then

$$u = \frac{\partial S}{\partial x}, \qquad v = \frac{\partial S}{\partial y}, \qquad w = \frac{\partial S}{\partial z}.$$

But the motion of incompressible fluids is subject to what is called the law of continuity, namely,

(45)
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

which expresses mathematically the fact that no matter is destroyed or created during the motion. Then it follows that

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} = 0.$$

How to solve this equation generally, Euler says, is not known; so he considers just special cases where S is a polynomial in x, y, and z. The function S was later (1868) called by Helmholtz the velocity potential. In a paper published in 1762[29] Lagrange reproduced all of these quantities, which he took over from Euler without acknowledgment, though he did improve the order of the ideas and the expressions.

Before we can investigate the work done to solve the potential equation in behalf of gravitational attraction, we must review some efforts to evaluate this attraction directly by means of the integrals (42) or the equivalents in other coordinate systems.

In a paper written in 1782 but published in 1785, entitled "Recherches sur l'attraction des sphéroïdes,"[30] Legendre, interested in the attraction exerted by solids of revolution, proved the theorem: If the attraction of a solid of revolution is known for every external point on the prolongation of its axis, then it is known for every external point. He first expressed the component of the force of attraction in the direction of the radius vector r by means of

(46)
$$P(r, \theta, 0) = \iiint \frac{(r - r')\cos\gamma}{(r^2 - 2rr'\cos\gamma + r'^2)^{3/2}}\,r'^2 \sin\theta'\,d\theta'\,d\phi'\,dr',$$

28. *Novi Comm. Acad. Sci. Petrop.*, 6, 1756/57, 271–311, pub. 1761 = *Opera*, (2), 12, 133–68.
29. *Misc. Taur.*, 2₂, 1760/61, 196–298, pub. 1762 = *Œuvres*, 1, 365–468.
30. *Mém. des sav. étrangers*, 10, 1785, 411–34.

Euler, too, studied cylindrical pipes and noncylindrical figures of revolution[27] and considered reflection at open and closed ends. The efforts of these men were directed toward understanding flutes; organ pipes; all sorts of horns of hyperboloidal, conical, and cylindrical shape; trumpets; bugles; and other wind instruments.

On the whole these efforts to solve partial differential equations in three and four variables were limited, mainly because the solutions were expressed in series involving several variables as opposed to simpler trigonometric series in x and t separately (cf. [20]). But the mathematicians knew too little about the functions that appeared in these more complicated series and about methods of determining the coefficients. Such methods were soon developed.

It is worth mentioning that Euler, in considering the sound of a bell and reconsidering some of the problems of the vibrations of rods, was led to fourth order partial differential equations. However, he was unable to do much with them and in fact, for the rest of the century no progress was made with them.

4. *Potential Theory*

The development of the subject of partial differential equations was furthered by another class of physical investigations. One of the major problems of the eighteenth century was the determination of the amount of gravitational attraction one mass exerts on another, the prime cases being the attraction of the sun on a planet, of the earth on a particle exterior or interior to it, and of the earth on another extended mass. When the two masses are very far apart compared to their sizes, it is possible to treat them as point masses; but in other cases, notably the earth attracting a particle, the extent of the earth must be taken into account. Clearly the shape of the earth must be known if one is to calculate the gravitational attraction its distributed mass exerts on a particle or another distributed mass. Although the precise shape remained a subject for investigation (Chap. 21, sec. 1), it was already clear by 1700 that it must be some form of ellipsoid, perhaps an oblate spheroid (an ellipsoid generated by revolving an ellipse around the minor axis). For the solid oblate spheroid the force of attraction both on an external and on an internal particle cannot be calculated as though the mass were concentrated at the center.

In a prize paper of 1740 on the tides, and in his *Treatise of Fluxions* (1742), Maclaurin proved that, for a fluid of uniform density under constant angular rotation, the oblate spheroid is an equilibrium shape. Then Maclaurin proved synthetically that, given two confocal homogeneous ellipsoids

27. *Novi Comm. Acad. Sci. Petrop.*, 16, 1771, 281–425, pub. 1772 = *Opera*, (2), 13, 262–369.

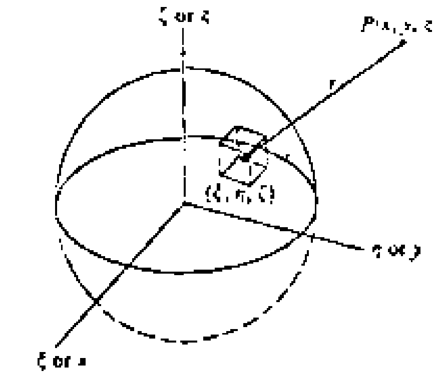

Figure 22.2

of revolution, the attractions of the two bodies on the same particle external to both, provided the particle be on the prolongation of the axis of revolution or in the plane of the equator, will be proportional to the volumes. Some other limited results were also established geometrically in the nineteenth century by James Ivory (1765–1842) and Michel Chasles.

The geometrical approach to the problem of gravitational attraction used by Newton, Maclaurin, and others is good only for special bodies and special locations of the attracted masses. This approach soon gave way to analytical methods, which one finds first in papers by Clairaut before 1743 and especially in his famous book *Théorie de la figure de la terre* (1743), in which he considers both the shape of the earth and gravitational attraction.

Let us first note some facts about the analytical formulation. The gravitational force exerted by an extended body on a unit mass P regarded as a particle is the sum of the forces exerted by all the small masses that make up the body. If $d\xi\, d\eta\, d\zeta$ is a small volume of the body (Fig. 22.2), so small that it may be regarded as a particle centered at the point (ξ, η, ζ), and if P has the coordinates (x, y, z), the attraction exerted by the small mass of density ρ on the unit particle is a vector directed from P to the small mass and, in view of the Newtonian law of gravitation, the components of this vector are (Chap. 21, sec. 7)

$$-k\rho\frac{x-\xi}{r^3}\,d\xi\,d\eta\,d\zeta, \qquad -k\rho\frac{y-\eta}{r^3}\,d\xi\,d\eta\,d\zeta, \qquad -k\rho\frac{z-\zeta}{r^3}\,d\xi\,d\eta\,d\zeta,$$

where k is the constant in the Newtonian law and

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}.$$

Of course ρ may be a function of ξ, η, and ζ or, in the case of a homogeneous body, a constant.

years old (1727) and established this field as a branch of mathematical physics. His best work on the subject followed his major papers of the 1750s on hydrodynamics. Air is a compressible fluid, and the theory of the propagation of sound is part of fluid mechanics (and of elasticity, because air is also an elastic medium). However, to treat the propagation of sound he made reasonable simplifications of the general hydrodynamical equations.

Three fine and definitive papers were read to the Berlin Academy in 1759. In the first, "On the Propagation of Sound,"[24] Euler considers the propagation of sound in one space dimension. After some approximations, which amount to considering waves of small amplitude, he is led to the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = 2gh\,\frac{\partial^2 y}{\partial x^2},$$

where y is the amplitude of the wave at the point x and at time t, g is the acceleration of gravity and h is a constant relating the pressure and density. This equation, as Euler of course recognized, is the same as that for the vibrating string and he did nothing mathematically new in solving it.

In his second paper[25] Euler gives the two-dimensional equation of propagation in the form

$$(38)\qquad \frac{\partial^2 x}{\partial t^2} = c^2\frac{\partial^2 x}{\partial X^2} + c^2\frac{\partial^2 y}{\partial X\,\partial Y}, \qquad \frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial Y^2} + c^2\frac{\partial^2 x}{\partial X\,\partial Y},$$

where x and y are the wave amplitudes in the X-direction and Y-direction respectively, or the components of the displacement, and $c = \sqrt{2gh}$. He gives the plane wave solution

$$x = \alpha\phi(\alpha X + \beta Y + c\sqrt{\alpha^2 + \beta^2}\,t),\ y = \beta\phi(\alpha X + \beta Y + c\sqrt{\alpha^2 + \beta^2}\,t),$$

where ϕ is an arbitrary function and α and β are arbitrary constants. Then letting

$$(39)\qquad v = \frac{\partial x}{\partial X} + \frac{\partial y}{\partial Y}$$

(v is called the divergence of the displacement), he gets the two-dimensional wave equation

$$(40)\qquad \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2}.$$

He also states the need for the superposition of solutions to obtain the most general solution of the problem in order to meet some initial condition, that is, the value of v or of x and y at $t = 0$.

24. *Mém. de l'Acad. de Berlin*, 15, 1759, 185–209, pub. 1766 = *Opera*, (3), 1, 428–51.
25. *Mém. de l'Acad. de Berlin*, 15, 1759, 210–40, pub. 1766 = *Opera*, (3), 1, 452–83.

Euler then shows how he can get the differential equation whose solutions are called cylindrical waves because the wave spreads out like an expanding cylinder. He lets $Z = \sqrt{X^2 + Y^2}$ and introduces $v = f(Z, t)$ where f is arbitrary. By letting $x = vX$ and $y = vY$, he obtains from (40)

$$\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{3}{Z}\frac{\partial v}{\partial Z} + \frac{\partial^2 v}{\partial Z^2}.$$

He also obtains in this paper in a similar manner the three-dimensional wave equation

$$(41)\qquad \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2} + \frac{\partial^2 v}{\partial Z^2},$$

where v is again the divergence of the displacement (x, y, z). Euler gives plane wave and spherical wave solutions using the kind of substitution just indicated for cylindrical waves. The basic equation for spherical waves is

$$\frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} = \frac{4}{V}\frac{\partial s}{\partial V} + \frac{\partial^2 s}{\partial V^2}, \qquad V = \sqrt{X^2 + Y^2 + Z^2}.$$

Much of the above work on spherical and cylindrical waves was also done independently by Lagrange at the end of the year 1759. Each communicated his results to the other. Though there are many details in which Lagrange's work differs from Euler's, there are no major mathematical points worthy of being related here.

From the propagation of sound waves in air it was but a step to the study of the sounds given off by musical instruments that employ air motion. This study was initiated by Daniel Bernoulli in 1739. Bernoulli, Euler, and Lagrange wrote numerous papers on the tones given off by an almost incredible variety of such instruments. In a publication of 1762 Daniel Bernoulli showed that at the open end of a cylindrical tube (organ pipe) no condensation of air can take place.[26] At a closed end the air particles must be at rest. He concluded from this that a tube closed at both ends or open at both ends has the same fundamental mode as a tube of half the length but open at one end and closed at the other. He also discovered the theorem that for closed organ pipes the frequencies of the overtones are odd multiples of the frequency of the fundamental. In the same paper Bernoulli took up pipes of other than cylindrical form, in particular the conical pipe, for which he obtained expressions for the individual tones (modes) but recognized that these hold only for infinite cones and not for the truncated one. For the (infinite) conical pipe the overtones proved to be harmonic to the fundamental. Bernoulli confirmed many of his theoretical results by experiments.

26. *Mém. de l'Acad. des Sci., Paris*, 1762, 431–85, pub. 1764.

As we shall see in a moment, Euler had introduced all the Bessel functions of the first kind in a paper on the vibrating drum (see also Chap. 21, secs. 4 and 6) and in this 1781 paper he remarks that it is possible to express any motion by a series of Bessel functions (despite the fact that he had argued against Daniel Bernoulli's claim, in the vibrating-string problem, that any function can be represented as a series of trigonometric functions).

Papers on the vibrating string and the hanging chain, of which the above are just samples, were published by many other men up to the end of the century. The authors continued to disagree, correct each other, and make all sorts of errors in doing so, including contradicting what they themselves had previously said and even proven. They made assertions, contentions, and rebuttals on the basis of loose arguments and often just personal predilections and convictions. Their references to papers to prove their contentions did not prove what they claimed. They also resorted to sarcasm, irony, invective, and self-praise. Mingled with these attacks were seeming agreements expressed in order to curry favor, particularly with d'Alembert, who had considerable influence with Frederick II of Prussia and as director of the Berlin Academy of Sciences.

The second order partial differential equation problems described thus far involved only one space variable and time. The eighteenth century did not go much beyond this. In a paper of 1759[22] Euler took up the vibration of a rectangular drum, thus considering a two-dimensional body. He obtained for the vertical displacement z of the surface of the drum

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}, \tag{33}$$

wherein x and y represent the coordinates of any point on the drum and c is determined by the mass and tension. Euler tried

$$z = v(x, y)\sin(\omega t + \alpha)$$

and found that

$$0 = \frac{\omega^2 v}{c^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

This equation has sinusoidal solutions of the form

$$v = \sin\left(\frac{\beta x}{a} + B\right)\sin\left(\frac{\gamma y}{b} + C\right)$$

where

$$\frac{\omega^2}{c^2} = \frac{\beta^2}{a^2} + \frac{\gamma^2}{b^2}.$$

22. *Novi Comm. Acad. Sci. Petrop.*, 10, 1764, 243–60, pub. 1766 = *Opera*, (2), 10, 344–59.

The dimensions of the drum are a and b, so that $0 \le x \le a$ and $0 \le y \le b$. When the initial velocity is 0, B and C may be taken to be 0. If the boundaries are fixed, then $\beta = m\pi$ and $\gamma = n\pi$ where m and n are integers. Then, since $\omega = 2\pi\nu$ where ν is the frequency per second, he obtains readily that the frequencies are

$$\nu = \frac{1}{2}c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

He then considers a circular drum and transforms (33) to polar coordinates (a highly original step), obtaining

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}. \tag{34}$$

He now tries solutions of the form

$$z = u(r)\sin(\omega t + A)\sin(\beta\phi + B) \tag{35}$$

so that $u(r)$ satisfies

$$u'' + \frac{1}{r}u' + \left(\frac{\omega^2}{c^2} - \frac{\beta^2}{r^2}\right)u = 0. \tag{36}$$

Here Bessel's equation appears in the current form (Cf. Chap. 21, sec. 6). Euler then calculates a power series solution

$$u\left(\frac{\omega}{c}r\right) = r^\beta\left\{1 - \frac{1}{1(\beta+1)}\left(\frac{\omega}{c}\frac{r}{2}\right)^2 + \frac{1}{1\cdot 2(\beta+1)(\beta+2)}\left(\frac{\omega}{c}\frac{r}{2}\right)^4 + \cdots\right\},$$

which we would write now as

$$u\left(\frac{\omega}{c}r\right) = \left(\frac{c}{\omega}\right)^\beta 2^\beta\Gamma(\beta+1)J_\beta\left(\frac{\omega}{c}r\right).$$

Since the edge $r = a$ must remain fixed,

$$J_\beta\left(\frac{\omega}{c}a\right) = 0. \tag{37}$$

It also follows from (35), since z must be of period 2π in ϕ, that β is an integer. Euler asserts that for a fixed β there are infinitely many roots ω so that infinitely many simple sounds result. However, he did not calculate these roots. He did attempt to find a second solution of (36) but failed to do so. The theory of the vibrating membrane was derived independently by Poisson[23] and is often credited solely to him.

Euler, Lagrange, and others worked on the propagation of sound in air. Euler wrote on the subject of sound frequently from the time he was twenty

23. *Mém. de l'Acad. des Sci., Paris*, (2), 8, 1829, 357–570.

Thus the ratio of two successive frequencies is the same as for a string of uniform thickness, but the fundamental frequency is no longer inversely proportional to the length.

In this paper of 1762/63 Euler also considered the vibrations of a string composed of two lengths, a and b, of different thicknesses m and n. He derived the equation for the frequencies ω of the modes. These turn out to be solutions of

$$m \tan \frac{\omega a}{m} + n \tan \frac{\omega b}{n} = 0, \tag{29}$$

and he solves for ω in special cases. The solutions of (29) are called the characteristic values or eigenvalues of the problem. These values are, as we shall see, of prime importance in the theory of partial differential equations. It is almost evident from (29) that the characteristic frequencies are not integral multiples of the fundamental one.

However, Euler took up this question again in another paper on the vibrating string of variable thickness,[18] and starting with (28) he shows that there are functions $c(x)$ for which the frequencies of the higher modes are not integral multiples of the fundamental.

D'Alembert, too, took up the string of variable thickness.[19] Here he used a significant method of solution that he had introduced earlier for the string of constant density. In this earlier attempt at the vibrating-string problem d'Alembert had introduced the idea of separation of variables, which is now a basic method of solution in partial differential equations.[20] To solve

$$\frac{\partial^2 y(t,x)}{\partial t^2} = a^2 \frac{\partial^2 y(t,x)}{\partial x^2}$$

d'Alembert sets

$$y = h(t)\, g(x),$$

substitutes this in the differential equation, and obtains

$$\frac{1}{a^2} \frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)}. \tag{30}$$

He then argues, as we do now, that since g''/g does not vary when t does, it must be a constant, and by the like argument applied to h''/h, this expression too must be a constant. The two constants are equal and are denoted by A. Thus he gets the two separate ordinary differential equations

$$\begin{aligned} h''(t) - a^2 A h(t) &= 0 \\ g''(x) - A g(x) &= 0. \end{aligned} \tag{31}$$

18. *Misc. Taur.*, 3, 1762/65, 25–59, pub. 1766 = *Opera*, (2), 10, 397–425.
19. *Hist. de l'Acad. de Berlin*, 19, 1763, 242 ff., pub. 1770.
20. *Hist. de l'Acad. de Berlin*, 6, 1750, 335–[illegible], pub. 1752.

Since a and A are constants, each of these equations is readily solvable, and d'Alembert gets

$$y(t,x) = h(t)\, g(x) = [Me^{a\sqrt{A}\,t} + Ne^{-a\sqrt{A}\,t}][Pe^{\sqrt{A}\,x} + Qe^{-\sqrt{A}\,x}].$$

The end-conditions, $y(t,0) = 0$ and $y(t,l) = 0$, led d'Alembert to assert that $g(x)$ must be of the form $k \sin Rx$ and that $h(t)$ must be of the same form because $y(t,x)$ must be periodic in t. He left the matter there. Daniel Bernoulli had used the idea of separation of variables in 1732 in his treatment of the vibrations of a chain suspended from one end, but d'Alembert was more explicit, despite the fact that he did not complete the solution.

In his 1763 paper d'Alembert wrote the wave equation as

$$\frac{\partial^2 y}{\partial t^2} = X(x) \frac{\partial^2 y}{\partial x^2}$$

and sought solutions of the form

$$u = \zeta(x) \cos \lambda \pi t.$$

He obtained for ζ the equation

$$\frac{d^2\zeta}{dx^2} = \frac{-\lambda^2 \pi^2 \zeta}{X(x)}. \tag{32}$$

D'Alembert now had to determine ζ so that it was 0 at both ends of the string. By a detailed analysis he showed that there are values of λ for which ζ meets this condition. He did not appreciate in this work that there are infinitely many values of λ. The significance of the investigation is that it is another step in the direction of boundary value or eigenvalue problems for ordinary differential equations.

The transverse oscillation of a *heavy* continuous horizontal cord was taken up by Euler. In the paper, "On the Modifying Effect of Their Own Weight on the Motion of Strings,"[21] Euler obtains the differential equation

$$\frac{1}{c^2} y_{tt} = \frac{g}{c^2} + y_{xx}.$$

For constant c and with fixed ends at $x = 0$ and $x = l$, Euler finds that

$$y = -\frac{(1/2)gx(x-l)}{c^2} + \phi(ct + x) + \psi(ct - x).$$

Thus the results are the same as for the "weightless" string (where the gravitational force is ignored), except that the oscillation takes place about the parabolic figure of equilibrium

$$y = -\frac{(1/2)gx(x-l)}{c^2}.$$

21. *Acta Acad. Sci. Petrop.*, 1, 1781, 178–90, pub. 1784 = *Opera*, (2), 11, 3[illegible]44, but dating from 1774.

must be periodic in x. However, he failed to realize that, given any arbitrary function in, say, $0 \leq x \leq l$, this function could be repeated in every interval $[nl, (n + 1)l]$ for integral n and so be periodic. Of course, such a periodic function might not be representable by one (closed) formula. Euler and Lagrange were, at least in their time, justified in believing that not all "discontinuous" functions could be represented by Fourier series, yet equally right in believing (though they did not have proof) that the initial curve can be very general. It need not be analytic, nor need it be periodic. Bernoulli did adopt the correct position on physical grounds but could not back it up with the mathematics.

One of the very curious features of the debate on the trigonometric series representation of functions is that all the men involved knew that non-periodic functions can be represented (in an interval) by trigonometric series. Reference to Chapter 20 (sec. 5) will show that Clairaut, Euler, Daniel Bernoulli, and others had actually produced such representations; many of their papers also had the formulas for the coefficients of the trigonometric series. Practically all of this work was in print by 1759, the year in which Lagrange presented his basic paper on the vibrating string. He could then have inferred that any function has a trigonometric expansion and could have read off the formulas for the coefficients, but failed to do so. Only in 1773, when the heat of the controversy was past, did Daniel Bernoulli notice that the sum of a trigonometric series may represent different algebraic expressions in different intervals. Why did all these results have no influence on the controversy concerning the vibrating string? It may be explained in several ways. Many of the results on the representation of quite general functions by trigonometric series were in papers on astronomy, and Daniel Bernoulli may not have read these and so could not point to them in defense of his position. Euler and d'Alembert, who must have known Clairaut's work of 1757 (Chap. 20, sec. 5), were probably not inclined to study it, since it refuted their own arguments. Also, this astronomical work by Clairaut was soon superseded and forgotten. On the other hand, whereas Euler used trigonometric series, as in his work on interpolation theory, to represent polynomial expressions, he did not accept the general fact that quite arbitrary functions could be so represented; the existence of such a series representation, where he used it, was assured by other means.

Another issue, how a partial differential equation with analytic coefficients (e.g. constants) could have a non-analytic solution, was not really clarified. In the case of ordinary differential equations, if the coefficients are analytical, the solutions must be. However, this is not true for partial differential equations. Though Euler was correct in saying that solutions with corners are admissible (and he did insist on it), determination of the singularities that are admissible in the solution of partial differential equations was still far in the future.

3. *Extensions of the Wave Equation*

While the controversy over the vibrating string was being carried on, the interest in musical instruments prompted further work, not only on vibrations of physical structures but also on hydrodynamical questions which concern the propagation of sound in air. Mathematically, these involve extensions of the wave equation.

In 1762 Euler took up the problem of the vibrating string with variable thickness. He had been stimulated by one of the principal questions of musical aesthetics. Jean-Philippe Rameau (1683–1764) had explained (1726) that the consonance of a musical sound is due to the fact that the component tones of any one sound are harmonics of the fundamental tone; that is, their frequencies are integral multiples of the fundamental frequency. But Euler, in his *Tentamen Novae Theoriae Musicae* (1739)[16] maintained that only in proper musical instruments were the overtones harmonics of the fundamental tone. He therefore undertook to show that the string of variable thickness or nonuniform density $\sigma(x)$ and tension T gives off inharmonic overtones.

The partial differential equation becomes

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \tag{28}$$

with c now a function of x. The first substantial results were obtained by Euler in a paper, "On the Vibratory Motion of Non-Uniformly Thick Strings."[17] The general solution he declares to be beyond the power of analysis. He obtains a solution in the special case where the mass distribution σ is given by

$$\sigma = \frac{\sigma_0}{\left(1 + \frac{x}{\alpha}\right)^4},$$

wherein σ_0 and α are constants. Then

$$y = \left(1 + \frac{x}{\alpha}\right)\left[\phi\left(\frac{x}{1 + \frac{x}{\alpha}} + c_0 t\right) + \psi\left(\frac{x}{1 + \frac{x}{\alpha}} - c_0 t\right)\right],$$

where $c_0 = \sqrt{T/\sigma_0}$. The frequencies of the modes or harmonics are given by

$$\nu_k = \frac{k}{2l}\left(1 + \frac{l}{\alpha}\right)\sqrt{T/\sigma_0}, \qquad k = 1, 2, 3, \ldots.$$

16. *Opera*, (3), 1, 197–427.
17. *Novi Comm. Acad. Sci. Petrop.*, 9, 1762/63, 246–304, pub. 1764 = *Opera*, (2), 10, 293–343.

> other for the most part, and those which remain are so disfigured and altered as to become absolutely unrecognizable. It is truly annoying that so ingenious a theory . . . is shown false in the principal case, to which all the small reciprocal motions occurring in nature may be related.

All of this is almost total nonsense.

Lagrange's main basis for contending that his solution did not require the initial curve $Y(x)$ and initial velocity $V(x)$ to be restricted is that he did not apply differentiation to them. But if one were to rigorize what he did do, restrictions would have to be made.

Euler and d'Alembert criticized Lagrange's work but actually did not hit at the main failings; they picked on details in his "prodigious calculations," as Euler put it. Lagrange tried to answer these criticisms. The replies and rebuttals on both sides are too extensive to relate here, though many are revealing of the thinking of the times. For example, Lagrange replaced $\sin \pi/m$ for $m = \infty$ by π/m and $\sin v\pi/2m$ by $v\pi/2m$ for $m = \infty$. D'Alembert allowed the first but not the second, because the values of v involved were comparable with m. The objection that a series of the form

$$\cos x + \cos 2x + \cos 3x + \cdots$$

might be divergent was also raised by d'Alembert and answered by Lagrange with the argument, common at that time, that the value of the series is the value of the function from which the series comes.

Though Euler did criticize mathematical details, his overall response to Lagrange's paper, communicated in a letter of October 23, 1759,[12] was to commend Lagrange's mathematical skill and to state that it put the whole discussion beyond all quibbling, and that everyone must now recognize the use of irregular and discontinuous (in Euler's sense) functions in this class of problems.

On October 2, 1759, Euler wrote Lagrange: "I am delighted to learn that you approve my solution . . . which d'Alembert has tried to undermine by various cavils, and that for the sole reason that he did not get it himself. He has threatened to publish a weighty refutation; whether he really will I do not know. He thinks he can deceive the semi-learned by his eloquence. I doubt whether he is serious, unless perhaps he is thoroughly blinded by self-love."[13]

In 1760/61 Lagrange, seeking to answer criticisms of d'Alembert and Bernoulli that had been communicated by letter, gave a different solution of the vibrating-string problem.[14] This time he starts directly with the wave

12. Lagrange, *Œuvres*, 14, 164–70.
13. *Œuvres*, 14, 162–64.
14. *Misc. Taur.*, 2_2, 1760/61, 11–172, pub. 1762 = *Œuvres*, 1, 151–316.

equation (with $c = 1$), and by multiplying by an unknown function and further steps he reduces the partial differential equation to the solution of two ordinary differential equations. Then by still further steps, not all correct, Lagrange obtains the solution

$$y(t, x) = \frac{1}{2}f(x + t) + \frac{1}{2}f(x - t) - \frac{1}{2}\int_0^{x+t} g\,dx + \frac{1}{2}\int_0^{x-t} g\,dx,$$

wherein $f(x) = y(0, x)$ and $g(x) = \partial y/\partial t$ at $t = 0$ are the given initial data. As Lagrange shows, this agrees with d'Alembert's result. But then, without reference to his own work, he tries to convince his readers that he did not use any law of continuity (analyticity) for the initial curve. It is true that he did not use any direct operation of differentiation on the initial function. But, in this paper also, to justify rigorously his limit procedures, one cannot avoid assumptions about the continuity and differentiability of the initial functions.

The debate raged throughout the 1760s and 1770s. Even Laplace entered the fray in 1779,[15] and sided with d'Alembert. D'Alembert continued it in a series of booklets, entitled *Opuscules*, which began to appear in 1768. He argued against Euler, on the ground that Euler admitted too general initial curves, and against Daniel Bernoulli, on the ground that his (d'Alembert's) solutions could not be represented as a sum of sine curves, so that Bernoulli's solutions were not general enough. The idea that the infinite series of trigonometric functions $\sum_{n=1}^{\infty} a_n \sin nx$ might be made to fit any initial curve because there is an infinity of a_n's to be determined (Daniel Bernoulli had so contended) was rejected by Euler as impossible to execute. He also raised the question of how a trigonometric series could represent the initial curve when only a part of the string is disturbed initially. Euler, d'Alembert, and Lagrange continued throughout to deny that a trigonometric series could represent any analytic function, to say nothing of more arbitrary functions.

Many of the arguments each presented were grossly incorrect; and the results, in the eighteenth century, were inconclusive. One major issue, the representability of an arbitrary function by trigonometric series, was not settled until Fourier took it up. Euler, d'Alembert, and Lagrange were on the threshold of discovering the significance of Fourier series but did not appreciate what lay before them. Judging by the knowledge of the times, all three men and Bernoulli were correct in their main contentions. D'Alembert, following a tradition established since Leibniz's time, insisted that functions must be analytical, so that any problem not solvable in such terms was unsolvable. He was also correct in the argument he gave that $y(t, x)$

15. *Mém. de l'Acad. des Sci.*, Paris, 1779, 207–309, pub. 1782 = *Œuvres*, 10, 1–89.

f is arbitrary (and discontinuous, in Euler's sense) and so cannot be expressed as a sum of sine functions. In fact, he says, f could be a combination of arcs spread out over the infinite x domain and be odd and periodic; yet because it is discontinuous (in Euler's sense), it cannot be expressed as a sum of sine curves. His own solution, he affirms, is not limited in any respect. The initial curve, in fact, need not be expressible by an equation (single analytic expression).

Euler also pointed, in this instance rightly, to the Maclaurin series and said this could not represent any arbitrary function; hence neither could an infinite sine series. All he would grant was that Bernoulli's trigonometric series represented special solutions; and indeed he (Euler) had obtained such solutions in his 1749 paper (see (17) and (18)).

D'Alembert, in his article "Fondamental" in Volume 7 (1757) of the *Encyclopédie*, also attacked Bernoulli. He did not believe that all odd and periodic functions could be represented by a series such as (19), because the series is twice differentiable but all odd and periodic functions need not be so. However, even when the initial curve is sufficiently differentiable—and d'Alembert did require that it be twice differentiable in his 1746 paper—it need not be representable in Bernoulli's form. On the same ground d'Alembert objected to Euler's discontinuous curves. Actually d'Alembert's requirement that the initial curve $y = f(x)$ must be twice differentiable was correct, because a solution derived from an $f(x)$ that does not have a second derivative at some value or values of x must satisfy special conditions at such singular points.

Bernoulli did not retreat from his position. In a letter of 1758[19] he repeats that he had in the a_n an infinite number of coefficients at his disposal, and so by choosing them properly could make the series in (19) agree with any function $f(x)$ at an infinite number of points. In any case he insisted that (20) was the most general solution. The argument between d'Alembert, Euler, and Bernoulli continued for a decade with no agreement reached. The essence of the problem was the extent of the class of functions that could be represented by the sine series, or, more generally, by Fourier series.

In 1759, Lagrange, then young and unknown, entered the controversy. In his paper, which dealt with the nature and propagation of sound,[20] he gave some results on that subject and then applied his method to the vibrating string. He proceeded as though he were tackling a new problem but repeated much that Euler and Daniel Bernoulli had done before. Lagrange, too, started with a string loaded with a finite number of equal and equally spaced masses and then passed to the limit of an infinite number of masses. Though he criticized Euler's method as restricting the results to continuous (analytic) curves, Lagrange said he would prove that Euler's

19. *Jour. des Sçavans*, March 1758, 157–66.
20. *Misc. Taur.*, 1_2, 1759, i–x, 1–112 = *Œuvres*, 1, 39–148.

conclusion, that any initial curve can serve, is correct. We shall pass at once to Lagrange's conclusion for the continuous string. He had obtained

$$(23)\quad y(x,t) = \frac{2}{l}\sum_{r=1}^{\infty}\sin\frac{r\pi x}{l}\sum_{q=1}^{\infty}\sin\frac{r\pi x}{l}\,dx\left[Y_q\cos\frac{r\pi ct}{l}+\frac{l}{r\pi c}V_q\sin\frac{r\pi ct}{l}\right].$$

Here Y_q and V_q are the initial displacement and initial velocity of the qth mass. He then replaced Y_q and V_q by $Y(x)$ and $V(x)$ respectively. Lagrange regarded the quantities

$$(24)\qquad \sum_{q=1}^{\infty}\sin\frac{r\pi x}{l}\,Y(x)\,dx \quad\text{and}\quad \sum_{q=1}^{\infty}\sin\frac{r\pi x}{l}\,V(x)\,dx$$

as integrals and he took the integration operation outside the summation $\sum_{r=1}^{\infty}$. From these moves there resulted

$$(25)\qquad y(x,t) = \left(\frac{2}{l}\int_0^l Y(x)\sum_{r=1}^{\infty}\sin\frac{r\pi x}{l}\,dx\right)\sin\frac{r\pi x}{l}\cos\frac{r\pi ct}{l}$$
$$+\left(\frac{2}{\pi c}\int_0^l V(x)\sum_{r=1}^{\infty}\frac{1}{r}\sin\frac{r\pi x}{l}\,dx\right)\sin\frac{r\pi x}{l}\sin\frac{r\pi ct}{l}.$$

The interchange of summation and integration not only introduced divergent series but spoiled whatever chance Lagrange might have had to recognize

$$(26)\qquad \int_0^l Y(x)\sin\frac{r\pi x}{l}\,dx$$

as a Fourier coefficient. After other long, difficult, and dubious steps, Lagrange obtained Euler's and d'Alembert's result

$$(27)\qquad y = \phi(ct+x) + \psi(ct-x).$$

He concluded that the above derivation put the theory of this great geometer [Euler]

> beyond all doubt and established on direct and clear principles which rest in no way on the law of continuity [analyticity] which Mr. d'Alembert requires; this, moreover, is how it can happen that the same formula that has served to support and prove the theory of Mr. Bernoulli on the mixture of isochronous vibrations when the number of bodies is . . . finite shows us its insufficiency . . . when the number of these bodies becomes infinite. In fact, the change that this formula undergoes in passing from one case to the other is such that the simple motions which made up the absolute motions of the whole system annul each

mechanical theory of heat (as opposed to heat as a substance) and gives many results on the theory of gases.

In his paper of 1732/33 cited in the preceding chapter, Bernoulli had expressly stated that the vibrating string could have higher modes of oscillations. In a later paper[7] on composite oscillation of weights on a loaded vertical flexible string, he made the following remark:

> Similarly, a taut musical string can produce its isochronous tremblings in many ways and even according to theory infinitely many, . . . and moreover in each mode it emits a higher or lower note. The first and most natural mode occurs when the string in its oscillations produces a single arch; then it makes the slowest oscillations and gives out the deepest of all its possible tones, fundamental to all the rest. The next mode demands that the string produce two arches on the opposite sides [of the string's rest position] and then the oscillations are twice as fast, and now it gives out the octave of the fundamental sound.

Then he describes the higher modes. He does not give the mathematics but it seems evident that he had it.

In a paper on the vibrations of a bar and the sounds given off by the vibrating bar,[8] Bernoulli not only gives the separate modes in which the bar can vibrate but says distinctly that both sounds (the fundamental and a higher harmonic) can exist together. This is the first statement of the coexistence of small harmonic oscillations. Bernoulli based it on his physical understanding of how the bar and the sounds can act but gave no mathematical evidence that the sum of two modes is a solution.

When he read d'Alembert's first paper of 1746 and Euler's paper of 1749 on the vibrating string, he hastened to publish the ideas he had had for many years.[9] After indulging in sarcasm about the abstractness of d'Alembert's and Euler's works, he reasserts that many modes of a vibrating string can exist simultaneously (the string then responds to the sum or superposition of all the modes) and claims that this is all that Euler and d'Alembert have shown. Then comes a major point. He insists that *all* possible initial curves are representable as

$$(19) \qquad f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

because there are enough constants a_n to make the series fit any curve. Hence, he asserts, *all* subsequent motions would be

$$(20) \qquad y(t, x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}.$$

7. *Comm. Acad. Sci. Petrop.*, 12, 1740, 97–108, pub. 1750.
8. *Comm. Acad. Sci. Petrop.*, 13, 1741/43, 167–96, pub. 1751.
9. *Hist. de l'Acad. de Berlin*, 9, 1753, 147–72 and 173–95, pub. 1755.

Thus every motion corresponding to any initial curve is no more than a sum of sinusoidal periodic modes, and the combination has the frequency of the fundamental. However, he gives no mathematical arguments to back up his contentions; he relies on the physics. In this paper of 1753 Bernoulli states:

> My conclusion is that all sonorous bodies include an infinity of sounds with a corresponding infinity of regular vibrations. . . . But it is not of this multitude of sounds that Messrs. d'Alembert and Euler claim to speak. . . . Each kind [each fundamental mode generated by some initial curve] multiplies an infinite number of times to accord to each interval an infinite number of curves, such that each point starts, and achieves at the same instant, these vibrations while, following the theory of Mr. Taylor, each interval between two nodes should assume the form of the companion of the cycloid [sine function] extremely elongated.
>
> We then remark that the chord *AB* cannot make vibrations only conforming to the first figure [fundamental] or second [second harmonic] or third and so forth to infinity but that it can make a combination of these vibrations in all possible combinations, and that all new curves given by d'Alembert and Euler are only combinations of the Taylor vibrations.

In this last remark Bernoulli ascribes knowledge to Taylor that Taylor never displayed. This apart, however, Bernoulli's contentions were enormously significant.

Euler objected at once to Bernoulli's last assertion. In fact Euler's 1753 paper presented to the Berlin Academy (already referred to above) was in part a reply to Bernoulli's two papers. Euler emphasizes the importance of the wave equation as the starting point for the treatment of the vibrating string. He praises Bernoulli's recognition that many modes can exist simultaneously so that the string can emit many harmonics in one motion, but denies, as did d'Alembert, that all possible motions can be expressed by (20). He admits that an initial curve such as

$$(21) \qquad f(x) = \frac{c \sin(\alpha x/l)}{1 - a \cos(\alpha x/l)}, \qquad |a| < 1,$$

can be expressed by a series such as (19). Bernoulli would be borne out if every function could be represented by an infinite trigonometric series, but this Euler regards as impossible. A sum of sine functions is, Euler says, an odd periodic function. But in his solution (see [16]),

$$(22) \qquad y(t, x) = \frac{1}{2} f(x + ct) + \frac{1}{2} f(x - ct),$$

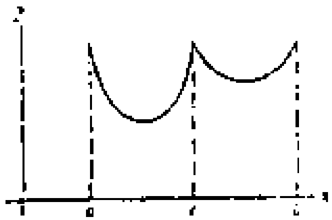

Figure 22.1

this concept. Euler called such curves discontinuous, though in modern terminology they are continuous with discontinuous derivatives. In his text, the *Introductio* of 1748, he stuck to the notion that was standard in the eighteenth century, that a function must be given by a single analytical expression. However, the physics of the vibrating-string problem seems to have been his compelling reason for bringing his new concept of function to the fore. He accepted any function defined by a formula $\phi(x)$ in $-l \leq x \leq l$, and regarded $\phi(x + 2l) = \phi(x)$ to be the definition of the curve outside $(-l, l)$. In a later paper[4] he goes further; he says that

$$y = \phi(ct + x) + \psi(ct - x), \tag{14}$$

with arbitrary ϕ and ψ, is a solution of

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}. \tag{15}$$

This follows by substitution in the differential equation. But the initial curve is equally satisfactory, whether it is expressed by some equation, or whether it is traced in any fashion not expressible by an equation. Of the initial curve, only the part in $0 \leq x \leq l$ is relevant. The continuation of this part is not to be taken into consideration. The different parts of this curve are thus not joined to each other by any law of continuity (single analytic expression); it is only by the description that they are joined together. For this reason it may be impossible to comprise the entire curve in one equation, except when by chance the curve is some sine function.

In 1755 Euler gave as a new definition of function, "If some quantities depend on others in such a way as to undergo variation when the latter are varied, then the former are called functions of the latter." And in another paper[5] he says that the parts of a "discontinuous" function do not belong to one another and are not determined by one single equation for the whole extent of the function. Moreover, given the initial shape in $0 \leq x \leq l$, one repeats it in reverse order in $-l \leq x \leq 0$ (so as to make it odd) and conceives the continual repetition of this curve in each interval of length $2l$ to infinity.

4. *Hist. de l'Acad. de Berlin*, 9, 1753, 196–222, pub. 1755 = *Opera*, (2), 10, 232–54.
5. *Novi Comm. Acad. Sci. Petrop.*, 11, 1765, 67–102, pub. 1767 = *Opera*, (1), 23, 74–91.

Then, if this curve $[y = f(x)]$ is used to represent the initial function, after the time t the ordinate that will answer to the abscissa x of the string in vibration will be (cf. [13] and [12])

$$y = \frac{1}{2}f(x + ct) + \frac{1}{2}f(x - ct). \tag{16}$$

In his basic 1749 paper Euler points out that all possible motions of the vibrating string are periodic in time whatever the shape of the string; that is, the period is (usually) the period of what we now call the fundamental. He also realized that individual modes whose periods are one half, one third, and so on of the basic (fundamental) period can occur as the vibrating figure. He gives such special solutions as

$$y(t, x) = \sum A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \tag{17}$$

when the initial shape is

$$y(0, x) = \sum A_n \sin\frac{n\pi x}{l}, \tag{18}$$

but does not say whether the summation covers a finite or infinite number of terms. Nevertheless he has the idea of superposition of modes. Thus Euler's main point of disagreement with d'Alembert is that he would admit all kinds of initial curves, and therefore non-analytic solutions, whereas d'Alembert accepted only analytic initial curves and solutions.

In introducing his "discontinuous" functions, Euler appreciated that he had taken a big step forward. He wrote to d'Alembert on December 20, 1763, that "considering such functions as are subject to no law of continuity [analyticity] opens to us a wholly new range of analysis."[6]

The solution of the vibrating-string problem was given in entirely different form by Daniel Bernoulli; this work stirred up another ground for controversy about the allowable solutions. Daniel Bernoulli (1700–82), the son of John Bernoulli, was a professor of mathematics at St. Petersburg from 1725 to 1733 and then, successively, professor of medicine, metaphysics, and natural philosophy at Basle. His chief work was in hydrodynamics and elasticity. In the former area, he won a prize for a paper on the flow of the tides; he also contemplated the application of the theory of the flow of liquids to the flow of blood in human blood vessels. He was a skilled experimentalist and through experimental work discovered the law of attraction of static electric charges before 1760. This law is usually credited to Charles Coulomb. Bernoulli's *Hydrodynamica* (1738), which contains studies that appeared in a number of papers, is the first major text in its field. It has a chapter on the

6. *Opera* (2), 11, sec. 1, 2.

He now observed that as n becomes infinite so that Δx approaches 0, the bracketed expression becomes $\partial^2 y/\partial x^2$. Hence

$$\frac{\partial^2 y(t,x)}{\partial t^2} = a^2 \frac{\partial^2 y(t,x)}{\partial x^2}, \tag{1}$$

where a^2 is now T/σ, σ being the mass per unit length. Thus what is now called the wave equation in one spatial dimension appears for the first time.

Since the string is fixed at the endpoints $x = 0$ and $x = l$, the solution must satisfy the boundary conditions

$$y(t, 0) = 0, \qquad y(t, l) = 0. \tag{2}$$

At $t = 0$ the string is displaced into some shape $y = f(x)$ and then released, which means that each particle starts with zero initial velocity. These initial conditions are expressed mathematically as

$$y(0, x) = f(x), \qquad \left.\frac{\partial y(t,x)}{\partial t}\right|_{t=0} = 0 \tag{3}$$

and they must also be satisfied by the solution.

This problem was solved by d'Alembert in so clever a manner that it is often reproduced in modern texts. We shall not take space for all the details. He proved first that

$$y(t,x) = \frac{1}{2}\phi(at + x) + \frac{1}{2}\psi(at - x), \tag{4}$$

where ϕ and ψ are as yet unknown functions.

Thus far d'Alembert had deduced that *every* solution of the partial differential equation (1) is the sum of a function of $(at + x)$ and a function of $(at - x)$. The converse is easy to show by direct substitution of (4) into (1). Of course d'Alembert had yet to satisfy the boundary and initial conditions. The condition $y(t, 0) = 0$ applied to (4) gives, for all t,

$$\frac{1}{2}\phi(at) + \frac{1}{2}\psi(at) = 0. \tag{5}$$

Since for any x, $ax + t = at'$ for some value of t', we may say that for any x and t

$$\phi(x + at) = -\psi(x + at). \tag{6}$$

Then the condition $y(t, l) = 0$, becomes, in view of (4),

$$\frac{1}{2}\phi(at + l) = \frac{1}{2}\phi(at - l); \tag{7}$$

and since this is an identity in t, it shows that ϕ must be periodic in $at + x$ with period $2l$.

The condition

$$\left.\frac{\partial y(t,x)}{\partial t}\right|_{t=0} = 0 \tag{8}$$

yields, from (4) and the fact that $\phi = -\psi$,

$$\phi'(x) = \phi'(-x). \tag{9}$$

On integration this becomes

$$\phi(x) = -\phi(-x), \tag{10}$$

and thus ϕ is an odd function of x. If we now use the fact that $\phi = -\psi$ in (4), form $y(0, x)$ and use (10), we have that

$$y(0, x) = \phi(x), \tag{11}$$

and since the initial condition is $y(0, x) = f(x)$ we have

$$\phi(x) = f(x) \quad \text{for} \quad 0 \le x \le l. \tag{12}$$

To sum up,

$$y(t,x) = \frac{1}{2}\phi(at + x) - \frac{1}{2}\phi(at - x), \tag{13}$$

where ϕ is subject to the above conditions of periodicity and oddness. Moreover, if the initial state is $y(0, x) = f(x)$, then (12) must hold between 0 and l. Thus there would be just one solution for a given $f(x)$. Now d'Alembert regarded functions as analytic expressions formed by the processes of algebra and the calculus. Hence if two such functions agree in one interval of x-values, they must agree for every value of x. Since $\phi(x) = f(x)$ in $0 \le x \le l$ and ϕ had to be odd and periodic, then $f(x)$ had to meet the same conditions. Finally, since $y(t, x)$ had to satisfy the differential equation, it had to be twice differentiable. But $y(0, x) = f(x)$, and so $f(x)$ had to be twice differentiable.

Within a few months of seeing d'Alembert's 1746 papers, Euler wrote his own paper, "On the Vibration of Strings," which was presented on May 16, 1748.[3] Though in method of solution he followed d'Alembert, Euler by this time had a totally different idea as to what functions could be admitted as initial curves and therefore as solutions of partial differential equations. Even before the debate on the vibrating-string problem, in fact in a work of 1734, he allowed functions formed from parts of different well-known curves and even formed by drawing curves freehand. Thus the curve (Fig. 22.1) formed by an arc of a parabola in the interval (a, c) and by an arc of a third degree curve in the interval (c, b) constituted one curve or one function under

3. *Nova Acta Erud.*, 1749, 512–27 = *Opera*, (2), 10, 50–62; also in French by Euler, *Hist. de l'Acad. de Berlin*, 4, 1748, 69–85 = *Opera*, (2), 10, 63–77.

22 Partial Differential Equations in the Eighteenth Century

Mathematical Analysis is as extensive as nature herself.
JOSEPH FOURIER

1. *Introduction*

As in the case of ordinary differential equations, the mathematicians did not consciously create the subject of partial differential equations. They continued to explore the same physical problems that had led to the former subject; and as they secured a better grasp of the physical principles underlying the phenomena, they formulated mathematical statements that are now comprised in partial differential equations. Thus, whereas the displacement of a vibrating string had been studied separately as a function of time and as a function of the distance of a point on the string from one end, the study of the displacement as a function of both variables and the attempt to comprehend all the possible motions led to a partial differential equation. The natural continuation of this study, namely, the investigation of the sounds created by the string as they propagate in air, introduced additional partial differential equations. After studying these sounds the mathematicians took up the sounds given off by horns of all shapes, organ pipes, bells, drums, and other instruments.

Air is one type of fluid, as the term is used in physics, and happens to be compressible. Liquids are (virtually) incompressible fluids. The laws of motion of such fluids and, in particular, the waves that can propagate in both became a broad field of investigation that now constitutes the subject of hydrodynamics. This field, too, gave rise to partial differential equations.

Throughout the eighteenth century, mathematicians continued to work on the problem of the gravitational attraction exerted by bodies of various shapes, notably the ellipsoid. While basically this is a problem of triple integration, it was converted by Laplace into a problem of partial differential equations in a manner we shall examine shortly.

2. *The Wave Equation*

Though specific partial differential equations appear as early as 1734 in the work of Euler[1] and in 1743 in d'Alembert's *Traité de dynamique*, nothing worth noting was done with them. The first real success with partial differential equations came in renewed attacks on the vibrating-string problem, typified by the violin string. The approximation that the vibrations are small was imposed to make the partial differential equation tractable. Jean Le Rond d'Alembert (1717–83), in his papers of 1746[2] entitled "Researches on the Curve Formed by a Stretched String Set into Vibrations," says he proposes to show that infinitely many curves other than the sine curve are modes of vibration.

We may recall from the preceding chapter that in the first approaches to the vibrating string, it was regarded as a "string of beads." That is, the string was considered to contain n discrete equal and equally spaced weights joined to each other by pieces of weightless, flexible, and elastic thread. To treat the continuous string, the number of weights was allowed to become infinite while the size and mass of each was decreased, so that the total mass of the increasing number of individual "beads" approached the mass of the continuous string. There were mathematical difficulties in passing to the limit, but these subtleties were ignored.

The case of a discrete number of masses had been treated by John Bernoulli in 1727 (Chap. 21, sec. 4). If the string is of length l and lies along $0 \le x \le l$, and if x_k is the abscissa of the kth mass, $k = 1, 2, \cdots, n$ (the nth mass at $x = l$ is motionless), then

$$x_k = k\frac{l}{n}, \qquad k = 1, 2, \cdots, n.$$

By analyzing the force on the kth mass, Bernoulli had shown that if y_k is the displacement of the kth mass, then

$$\frac{d^2 y_k}{dt^2} = \left(\frac{na}{l}\right)^2 (y_{k+1} - 2y_k + y_{k-1}), \qquad k = 1, 2, \cdots, n-1,$$

where $a^2 = lT/M$, T is the tension in the string (which is taken to be constant as the string vibrates), and M the total mass. D'Alembert replaced y_k by $y(t, x)$ and l/n by Δx. Then

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2\left[\frac{y(t, x + \Delta x) - 2y(t, x) + y(t, x - \Delta x)}{(\Delta x)^2}\right].$$

1. *Comm. Acad. Sci. Petrop.*, 7, 1734/35, 184–200, pub. 1740 = *Opera*, (1), 22, 57–75.
2. *Hist. de l'Acad. de Berlin*, 3, 1747, 214–19 and 220–49, pub. 1749.